Recursion

Contents

Introduction

Recursion is:

- A way of thinking about problems.
- A method for solving problems.
- Related to mathematical induction.

A method is **recursive** if it can call itself, either directly:

```
void f() {
   ... f() ...
}
```

or indirectly:

```
void f() {
    ... g() ...
}

void g() {
   ... f() ...
}
```

You might wonder about the following issues:

> Q: Does using recursion usually make your code ***faster***?
> A: No.
>
> Q: Does using recursion usually use ***less memory***?
> A: No.
>
> Q: Then ***why*** use recursion?
> A: It sometimes makes your code much ***simpler***!

One way to think about recursion:

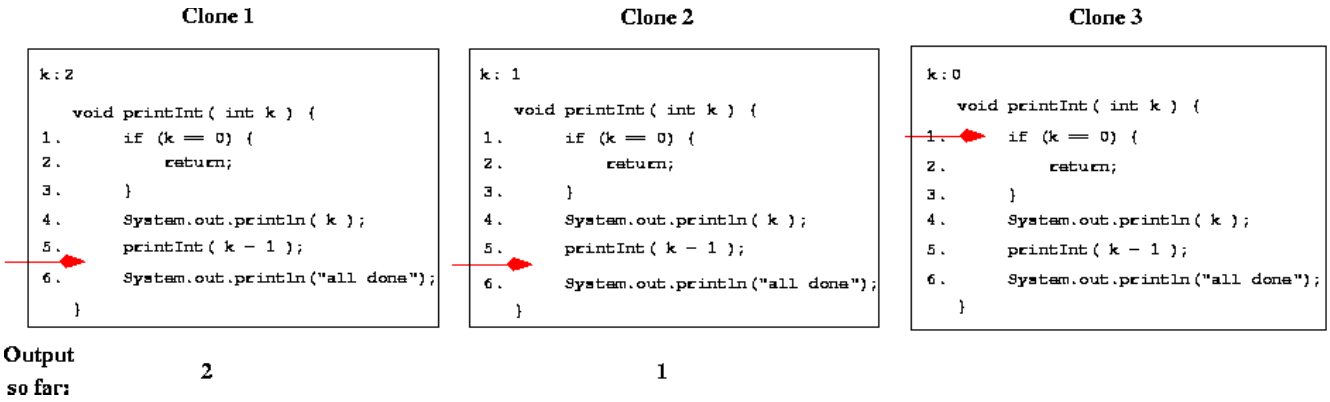
- When a recursive call is made, the method ***clones itself***, making new copies of:

  - the code,
  - the local variables (with their initial values),
  - the parameters
- Each copy of the code includes a marker indicating the current position. When a recursive call is made, the marker in the old copy of the code is just after the call; the marker in the "cloned" copy is at the beginning of the method.
- When the method returns, ***that*** clone goes away, but the previous ones are still there and know what to execute next because their current position in the code was saved (indicated by the marker).

Here's an example of a simple recursive method:

```
    void printInt( int k ) {
1.      if (k == 0) {
2.          return;
3.      }
4.      System.out.println( k );
5.      printInt( k - 1 );
6.      System.out.println( "all done" );
    }
```

If the call `printInt(2)` is made, three "clones" are created, as illustrated below:



The original call causes 2 to be output, and then a recursive call is made, creating a clone with `k` = 1. That clone executes line 1 (the `if` condition is false), line 4 (prints 1), and line 5 (makes another recursive call, creating a clone with `k` = 0). That clone just returns (goes away) because the `if` condition is true. The previous clone executes line 6 (the line after its "marker") then returns, and similarly for the original clone.

Now let's think about what we have to do to make sure that recursive methods work correctly. First, consider the following recursive method:

```
void badPrint(int k) {
    System.out.println(k);
    badPrint(k + 1);
}
```

Note that a runtime error will occur when the call `badPrint(2)` is made (in particular, an error message like "`java.lang.StackOverflowError`" will be printed, and the program will stop). This is because there is no code that prevents the recursive call from being made again and again and .... and eventually the program runs out of memory (to store all the clones). This is an example of an **infinite recursion**. It inspires:

***** RECURSION RULE #1 *****

> Every recursive method must have a **base case** -- a condition under which no recursive call is made -- to prevent infinite recursion.

Here's another example; this version does have a base case, but the call `badPrint2(2)` will still cause an infinite recursion:

```
void badPrint2(int k) {
    if (k < 0) {
        return;
    }
```

```
        System.out.println(k);
        badPrint2(k + 1);
    }
```

This inspires:

***** RECURSION RULE #2 *****

Every recursive method must **make progress** toward the base case to prevent infinite recursion.

TEST YOURSELF #1

Consider the method `printInt`, repeated below.

```
void printInt(int k) {
    if (k == 0) {
        return;
    }
    System.out.println(k);
    printInt(k - 1);
}
```

Does it obey recursion rules 1 and 2? Are there calls that will lead to an infinite recursion? If yes, how could it be fixed?

solution

How Recursion Really Works

This is how method calls (recursive and non-recursive) really work:

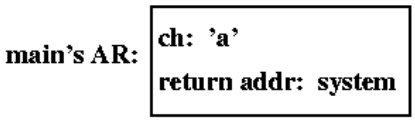
- At runtime, a stack of **activation records** (ARs) is maintained: one AR for each active method, where "active" means: has been called, has not yet returned. This stack is also referred to as the **call stack**.
- Each AR includes space for:
  - the method's parameters,
  - the method's local variables,
  - the return address -- where (in the code) to start executing after the method returns.
- When a method is called, its AR is pushed onto the stack. The return address in that AR is the place in the code just after the call (so the return address is the "marker" for the previous "clone").
- When a method is about to return, the return address in its AR is saved, its AR is popped from the stack, and control is transferred to the place in the code referred to by the return address.

Example (no recursion):

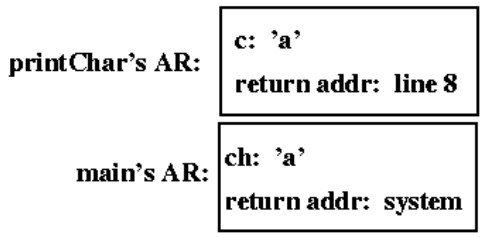
```
1.  void printChar (char  c) {
2.      System.out.print(c);
3.  }
4.
5.  void main ( ... ) {
6.      char ch = 'a';
7.      printChar(ch);
8.      ch = 'b';
9.      printChar(ch);
10. }
```

The runtime stack of activation records is shown below, first as it would be just before the first call to `printChar` (just after line 6) and then as it would be while `printChar` is executing (lines 1 - 3). Note that the return address stored in `printChar`'s AR is the place in `main`'s code where execution will resume when `printChar` returns.

STACK BEFORE 1ST CALL TO printChar

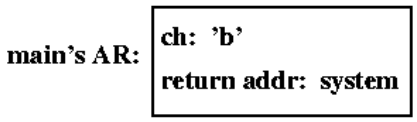


STACK WHEN printChar HAS BEEN CALLED

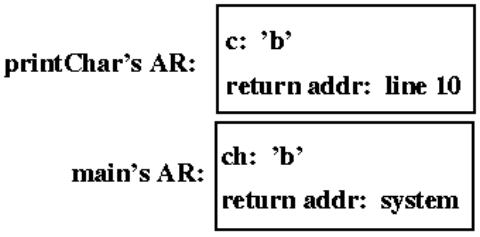


When the first call to `printChar` returns, the top activation record is popped from the stack and the `main` method begins executing again at line 8. After executing the assignment at line 8, a second call to `printChar` is made (line 9), as illustrated below:

STACK BEFORE 2ND CALL TO printChar



STACK WHEN printChar HAS BEEN CALLED



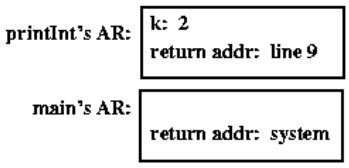
The two calls to `printChar` return to different places in `main` because different return addresses are stored in the ARs that are pushed onto the stack when the calls are made.

Now let's consider recursive calls:

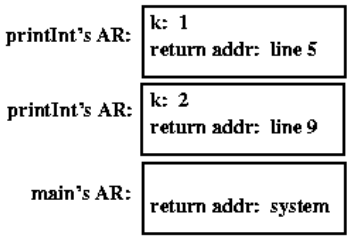
```
1.  void printInt( int k ) {
2.      if (k <= 0) return;
3.      System.out.println( k );
4.      printInt( k - 1 );
5.  }
6.
7.  void main( ... ) {
8.      printInt( 2 );
9.  }
```

The following pictures illustrate the runtime stack as this program executes.

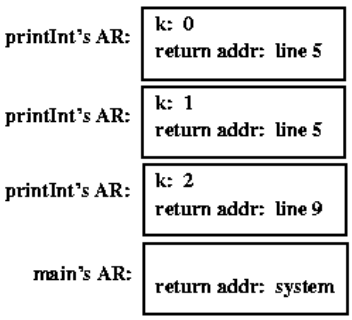
STACK AFTER CALL TO printInt FROM main



STACK AFTER 1ST RECURSIVE CALL



STACK AFTER 2ND RECURSIVE CALL



Note that all of the recursive calls have the same return address -- after a recursive call returns, the previous call to `printInt` starts executing again at line 5. In this case, there is nothing more to do at that point, so the method would, in turn, return.

Note also that each call to `printInt` causes the current value of `k` to be printed, so the output of the program is: `2 1`

Now consider a slightly different version of the `printInt` method:

```
1.  void printInt(int k) {
2.      if (k <= 0) return;
3.      printInt(k - 1);
4.      System.out.println(k);
5.  }
```

Now what is printed as a result of the call `printInt(2)`? Because the print statement comes ***after*** the recursive call, it is not executed until the recursive call finishes (i.e., `printInt`'s activation record will have line 4 -- the print statement -- as its return address, so that line will be executed only after the recursive call finishes). In this case, that means that the output is: `1 2` (instead of `2 1`, as it was when the print statement came before the recursive call). This leads to the following insight:

***** UNDERSTANDING RECURSION *****

> Sometimes a method has more to do following a recursive call; it gets done only ***after*** the recursive call (and all calls it makes) are finished.

TEST YOURSELF #2

```
void printTwoInts(int k) {
    if (k == 0) {
```

```
            return;
        }
        System.out.println("From before recursion: " + k);
        printTwoInts(k - 1);
        System.out.println("From after recursion: " + k);
    }
```

What is printed as a result of the call `printTwoInts(3)`?

solution

Recursion vs Iteration

Now let's think about when it is a good idea to use recursion and why. In many cases there will be a choice: many methods can be written either with or without using recursion.

> Q: Is the recursive version usually faster?
> A: No -- it's usually slower (due to the overhead of maintaining the stack)

> Q: Does the recursive version usually use less memory?
> A: No -- it usually uses ***more*** memory (for the stack).

> Q: Then ***why*** use recursion?
> A: Sometimes it is much simpler to write the recursive version.

Here are a few examples where recursion makes things a little bit clearer, though in the second case, the efficiency problem makes it a bad choice.

Factorial

Factorial can be defined as follows:

- factorial of 0 is 1
- factorial of N (for N > 0) is N*N-1* ... *3*2*1

or:

- factorial of 0 is 1
- factorial of N (for N > 0) is N * factorial (N-1)

(Note that factorial is undefined for negative numbers.)

The first definition leads to an **iterative** version of factorial:

```
int factorial(int N) {
    if (N == 0) {
        return 1;
    }
    int tmp = N;
    for (int k = N-1; k >= 1; k--) {
        tmp = tmp * k;
    }
    return (tmp);
}
```

The second definition leads to a **recursive** version:

```
int factorial(int N) {
   if (N == 0) {
       return 1;
   } else {
       return (N * factorial( N-1 ));
   }
}
```

The recursive version is

- a little shorter,
- a little clearer (closer to the mathematical definition),
- a little slower

Because the recursive version causes an activation record to be pushed onto the runtime stack for every call, it is also more limited than the iterative version (it will fail, with a "stack overflow" error), for large values of N.

TEST YOURSELF #3

Question 1: Draw the runtime stack, showing the activation records that would be pushed as a result of the call `factorial(3)` (using the recursive version of factorial). Just show the value of N in each AR (don't worry about the return address). Also indicate the value that is ***returned*** as the result of each call.

Question 2: Recall that (mathematically) factorial is undefined for a negative number. What happens as a result of the call `factorial(-1)` for each of the two versions (iterative and recursive)? What do you think really should happen when the call `factorial(-1)` is made? How could you write the method to make that happen?

solution

Fibonacci

Fibonacci can be defined as follows:

- fibonacci of 1 or 2 is 1
- fibonacci of N (for N>2) is fibonacci of (N-1) + fibonacci of (N-2)

Fibonacci can be programmed either using iteration or using recursion. Here are the two versions (iterative first):

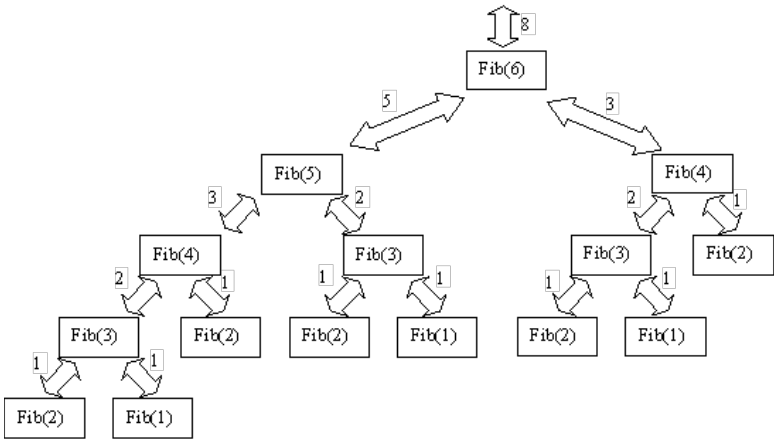
```
// iterative version
int fib( int N ) {
    int k1, k2, k3;
    k1 = k2 = k3 = 1;
    for (int j = 3; j <= N; j++) {
        k3 = k1 + k2;
        k1 = k2;
        k2 = k3;
    }
    return k3;
}

// recursive version
int fib( int N ) {
    if ((N == 1) || (N == 2)) {
        return 1;
    } else {
        return (fib( N-1 ) + fib( N-2 ));
    }
}
```

For fibonacci, the recursive version is:

- shorter,
- clearer,
- but ***much*** slower

Here's a picture of a **trace** of the recursive version of fibonacci, for the initial call `fib(6)`:



Note: one reason the recursive version is so slow is that it is repeating computations. For example, `fib(4)` is computed twice and `fib(3)` is computed 3 times.

Given that the recursive version of fibonacci is slower than the iterative version, is there a good reason for using it? The answer may be yes: because the recursive solution is so much simpler, it is likely to take much less time to write, debug, and maintain. If those costs (the cost for programming time) are more important than the cost of having a slow program, then the advantages of using the recursive solution outweigh the disadvantage and you should use it! If the speed of the final code is of vital importance, then you should not use the recursive fibonacci.

Recursive Data Structures

So far, we've been doing all our recursion based on the value of some int parameter; e.g., for factorial, we start with some value N and each recursive call passes a smaller value (N - 1) until we get down to 0. A more common way to use recursion in programming is to base the recursion on a **recursive data structure**. A data structure is recursive if you can define it in terms of itself. For example, we can give a recursive definition of a string:

> A string is either empty, or it is a character followed by a string.

We can give a similar, recursive definition of a linked list:

> A linked list is either empty, or it is a listnode followed by a linked list.

Below are two methods that use recursive data structures; the first one prints each character of its `String` parameter and the second prints each value in its linked list parameter.

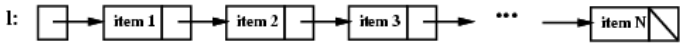
Example 1: Use a `String` parameter

```
printStr( String S ) {
    if (S.equals("")) {
        return;
    }
    System.out.print( S.substring(0, 1) + " " );
    printStr( S.substring(1) );
}
```

Example 2: Use a linked list parameter

```
printList( Listnode<E> l ) {
    if (l == null) {
        return;
    }
    System.out.println(l.getData());
    printList(l.getNext());
}
```

Now suppose we want to print the values in a linked list ***backwards***; i.e., if the list looks like this:



we want to print:

```
itemN
itemN-1
 ...
item2
item1
```

It is easy to do this using recursion; all we have to do is swap the order of the print statement and the recursive call in the Example 2 code given above. Here's the new version:

```
printList(Listnode<E> llist) {
    if (llist == null) {
        return;
    }
    printList(llist.getNext());
    System.out.println(llist.getData());
}
```

In order to print the list backwards ***without*** using recursion, you'd have to use some auxiliary data structure. For example, you could traverse the list from left to right, pushing each value into a stack, then pop the stack, printing each value. However, it's much simpler just to use recursion!

TEST YOURSELF #4

Question 1: Write a recursive method called `sum` with the following header:

```
public static int sum(Listnode<Integer> node)
```

Assume that `node` points to the first node in a linked list of `Integer`s. The method `sum` should return the sum of the values in the list.

Question 2: Write a recursive method called `vowels` with the following header:

```
public static String vowels(String str)
```

The method `vowels` should return a `String` containing just the vowels (a, e, i, o, u) in `str`, in the order in which they occur in `str`. For example, `vowels("hooligan")` should return `"ooia"`.

solution

Analyzing Runtime for Recursive Methods

There are two ways to determine how much time is required for a call to a recursive method:

1. Informal Reasoning
2. Using Recurrence Equations

Informal Reasoning

To reason about the time required for a call to a recursive method, you must figure out how many times total the method will be called and how much work (in terms of the problem size, N) is done on each call in addition to the recursive calls. The total time is the product of those two values.

For example, consider recursive method `printInt`:

```
public static void printInt(int k) {
    if (k == 0) {
        return;
    }
    System.out.println( k );
    printInt( k - 1 );
}
```

It is easy to see that if `printInt` is first called with the value N, then a total of N+1 calls will be made (for N, N-1, N-2, ... 0).

Each time `printInt` is called, a constant amount of work is done (to check and print the value of `k`) in addition to the recursive call. Therefore, the total time is O(N).

Using Recurrence Equations

A more formal way to analyze the running time of a recursive method is to use **recurrence equations**. The recurrence equations for a method include an equation for the base case (or several equations if there are several base cases) and an equation for the recursive case.

For example, for the `printInt` method above, the recurrence equations are:

T(0) = 1
T(N) = 1 + T(N-1)

The first equation says that a problem of size 0 (i.e., when parameter `k` = 0) requires a constant amount of work (test `k` and return). The second equation says that a problem of size N (i.e., when parameter `k` is not 0) requires a constant amount of work (test and print `k`) plus the time required for solving a problem of size N-1.

In general, the recurrence equation for the recursive case will be of the following form:

T(N) = _____ + _____ * T(_____)

The first blank is the **amount of time outside the recursive call(s)**. In the `printInt` method above, that is constant, so we fill in the first blank with a 1.

The second blank is the **number of recursive calls**. Not the total number of recursive calls that will be made, just the number made here. In the `printInt` method above, there is just one recursive call, so we fill in the second blank with a 1.

The third blank is the **problem size passed to the recursive call**. It better be less than the current problem size, or the code will not make progress toward the base case! In our example, the problem size (the value of `k`) decreases by one, so we fill in the third blank with N-1.

The final equation for the recursive case is

T(N) = 1 + 1 * T(N-1)

Which is the same as the equation given above (just with an explicit "times 1"). Now consider a different version of `printInt`:

```
public static void printInt(int k) {
  if (k == 0) {
      return;
  }
  printInt(k/2);
  for (int j = 0; j < k; j++) {
      System.out.println(j);
  }
  printInt(k/2);
}
```

For this version, the amount of time outside of the recursive calls is proportional to the problem size, there are ***two*** recursive calls, and the problem size passed to each of those calls is half the original problem size. Therefore, the recurrence equation for this new version of `printInt` is

T(N) = N + 2 * T(N/2)

Now let's consider how to solve recurrence equations. To find the worst-case time for a problem of size N, we need to find a **closed-form solution** to these equations, i.e., a solution for T(N) that doesn't involve any T(...) on the right-hand side. To do that we perform three steps:

1. Expand: find the times required for a problem of size 1, 2, 3, etc.
2. Look for a pattern and guess a solution for T(N).
3. Plug in your solution and see if you got it right.

Below are those three steps for the original `printInt` equations, which were:

T(0) = 1
T(N) = 1 + T(N-1)

Step 1: Expand

T(0) = 1

T(1) = 1 + T(0) = 1 + 1 = 2

T(2) = 1 + T(1) = 1 + 2 = 3

T(3) = 1 + T(2) = 1 + 3 = 4

Step 2: Look for a pattern and guess a solution

We'll guess that T(N) = N+1

Step 3: Verify the guessed solution

T(N)	=	1 + T(N-1)	// given
N+1	=?=	1 + ((N-1)+1)	// replace T(...) on both sides with the guessed solution (N+1) in which each N is replaced by ...
N+1	=?=	1 + N - 1 + 1	// simplify
N+1	=	1 + N	// yes the two sides are equal!

TEST YOURSELF #5

Suppose we have the following recurrence equations:

T(1) = 1
T(N) = 1 + 2 * T(N/2)

Do the steps nexcessary to solve those equations. First, fill in the following table:

N	T(N)
1	
2	
4	
8	

Next, look for a pattern and guess a solution.

Finally, verify your solution using the recursive equation.

solution

Using Mathematical Induction to Prove the Correctness of Recursive Code

If you like math, you may enjoy thinking about how to use mathematical induction to prove that recursive code is correct. If not, feel free to skip this section.

When using induction to prove a theorem, you need to show:

1. that the base case (usually n=0 or n=1) is true

2. that case k implies case k+1

It is sometimes straightforward to use induction to prove that recursive code is correct. Let's consider how to do that for the recursive version of factorial. First we need to prove the base case: factorial(0) = 0! (which, by definition is 1). The correctness of the factorial method for N=0 is obvious from the code: when N==0 it returns 1.

Now we need to show that if factorial(k) = k!, then factorial(k+1) = (k+1)!.
Looking at the code we see that for N != 0, factorial(N) = (N)*factorial(N-1).So factorial(k+1) = (k+1)*factorial(k).
By assumption, factorial(k) = k!, so factorial(k+1) = (k+1)*(k!).
By definition, k! = k*(k-1)*(k-2)*...*3*2*1. So (k+1)*(k!) = (k+1)*k*(k-1)*(k-2)*...*3*2*1, which is (by definition) (k+1)!, and the proof is done.

Note that we've shown that the factorial method is correct for all values of N greater than or equal to zero (because our base case was N=0). We haven't shown anything about factorial for negative numbers.

Summary

- Use recursion for ***clarity*** and (sometimes) for a reduction in the time needed to write and debug code, not for space savings or speed of execution.
- Remember that every recursive method must have a **base case** (rule #1).
- Also remember that every recursive method must ***make progress*** towards its base case (rule #2).
- Sometimes a recursive method has ***more to do*** following a recursive call. It gets done only ***after*** the recursive call (and all calls it makes) finishes.
- Use **informal reasoning** or **recurrence equations** to determine how much time a recursive call will require.
- Recursion is often simple and elegant, can be efficient, and tends to be underutilized. Consider using recursion when solving a problem!

Back

Next: Continue with Searching