Problem 1

(a).

$$Q(u) = \frac{1}{6}[u^3 \ u^2 \ u \ 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$Q'(u) = \frac{1}{6}[3u^2 \ \ 2u \ \ 1 \ \ 0]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$Q''(u) = \frac{1}{6}[6u \ \ 2 \ \ 0 \ \ 0]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$Q(u) = \frac{1}{6}[u^3 \quad u^2 \quad u \quad 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

B-spline matrix

(from Slides)

(b). For $P_0P_1P_2P_3$

$$\Rightarrow Q_1(u) = \frac{1}{6}\left((1-u)^3 P_0 + (3u^3 - 6u^2 + 4) P_1 + (-3u^3 + 3u^2 + 3u + 1) P_2 + u^3 P_3\right)$$

$$Q_1'(u) = \frac{1}{6}\left(-3(1-u)^2 P_0 + (9u^2 - 12u) P_1 + (-9u^2 + 6u + 3) P_2 + 3u^2 P_3\right)$$

$$Q_1''(u) = \frac{1}{6}\left(6(1-u) P_0 + (18u - 12) P_1 + (-18u + 6) P_2 + 6u P_3\right)$$

At joint, $u = 1$,

$$Q_1(1) = \frac{1}{6}(P_1 + 4P_2 + P_3)$$
$$Q_1'(1) = \frac{1}{6}(-3P_1 + 3P_3)$$
$$Q_1''(1) = \frac{1}{6}(6P_1 - 12P_2 + 6P_3)$$

As for $P_1P_2P_3P_4$, we substitute $P_0P_1P_2P_3$ in the above equations by $P_1P_2P_3P_4$ and use $u=0$ instead: $(1-0)^3$ $0-0+4$ $0+0+0+1$

$$Q_2(0) = \frac{1}{6}(P_1 + 4P_2 + P_3)$$
$$Q_2'(0) = Q_1'(1)$$
$$Q_2''(0) = Q_1''(1)$$ by simple substitutions.

And it is obvious that the two curves' joint is C^2 continuous.

c(2)

Consider $u=1$

$$① \begin{cases} N_0(u) = \frac{1}{6}(1-u)^3 = 0 \\ N_0'(u) = -\frac{1}{2}(1-u)^2 = 0 \\ N_0''(u) = (1-u) = 0 \end{cases}$$

$$② \begin{cases} N_1(u) = \frac{1}{6}(3u^3-6u^2+4) = \frac{1}{6} \\ N_1'(u) = \frac{1}{6}(9u^2-12u) = -\frac{1}{2} \\ N_1''(u) = \frac{1}{6}(18u-12) = 1 \end{cases}$$

$$③ \begin{cases} N_2(u) = \frac{1}{6}(-3u^3+3u^2+3u+1) = \frac{2}{3} \\ N_2'(u) = \frac{1}{6}(-9u^2+6u+3) = 0 \\ N_2''(u) = \frac{1}{6}(-18u+6) = -2 \end{cases}$$

$$④ \begin{cases} N_3(u) = \frac{1}{6}u^3 = \frac{1}{6} \\ N_3'(u) = \frac{1}{2}u^2 = \frac{1}{2} \\ N_3''(u) = u = 1 \end{cases}$$

Consider $u=0$

$$ⓐ \begin{cases} N_0(u) = \frac{1}{6} \\ N_0'(u) = -\frac{1}{2} \\ N_0''(u) = 1 \end{cases} \quad ⓑ \begin{cases} N_1(u) = \frac{2}{3} \\ N_1'(u) = 0 \\ N_1''(u) = -2 \end{cases} \quad ⓒ \begin{cases} N_2(u) = \frac{1}{6} \\ N_2'(u) = \frac{1}{2} \\ N_2''(u) = 1 \end{cases}$$

$$ⓓ \begin{cases} N_3(u) = 0 \\ N_3'(u) = 0 \\ N_3''(u) = 0 \end{cases}$$

And we Notice that:

at $u=1$,

function set ① = function set ⓓ

② = ⓐ

③ = ⓑ

④ = ⓒ

up to second derivative of the basis functions.

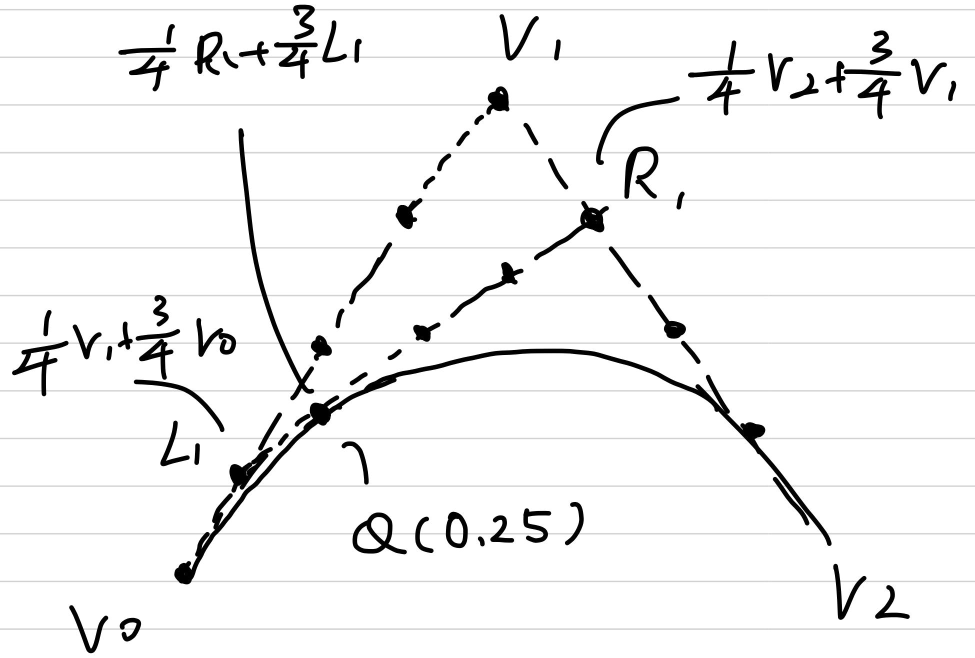
$\therefore$ The basis functions are C^2 continuous at knots following

$$\left.\begin{array}{l} N_i(1) \\ N_i'(1) \\ N_i''(1) \end{array}\right\} = \left\{\begin{array}{l} N_{i+1}(0) \\ N_{i+1}'(0) \\ N_{i+1}''(0) \end{array}\right.$$

(If $i=0$, RHS is 3, and all 3 values 0)

Problem 2.

(a).



$$B_0^2(u) = \binom{2}{0} x^0 (1-x)^2$$

$$B_1^2(u) = \binom{2}{1} x^1 (1-x)^1$$

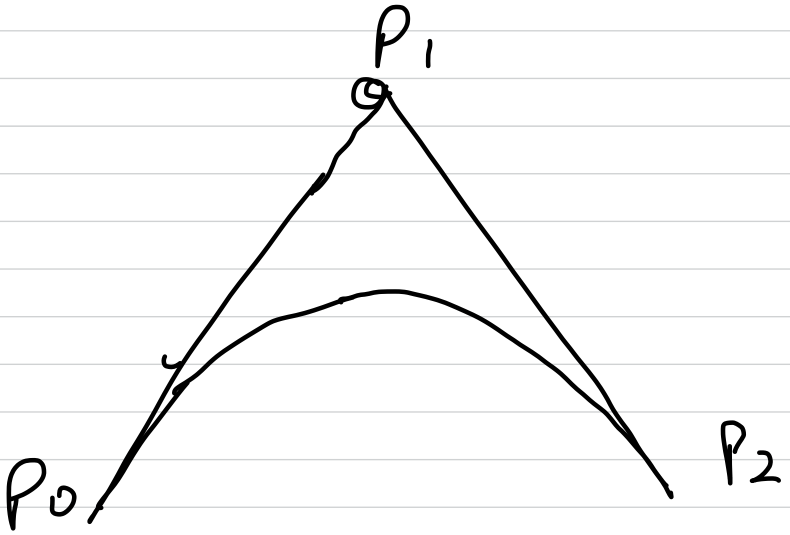
$$B_2^2(u) = \binom{2}{2} x^2 (1-x)^0$$

$$\hookrightarrow (1-x)^2 P_0 + 2(x-x^2) P_1 + x^2 P_2$$

$$\text{or} \quad P_1 + (1-t)^2 (P_0 - P_1) + t^2 (P_2 - P_1)$$

Side Note

(b). $\sum_{i=0}^{2} B_i(u) P_i$



$$L(u) = P_0 + u(P_1 - P_0) = (1-u)P_0 + uP_1$$
$$R(u) = P_1 + u(P_2 - P_1) = (1-u)P_1 + uP_2$$

$$\begin{aligned}
Q(u) &= L(u) + u(R(u) - L(u)) \\
&= (1-u)P_0 + uP_1 + u((1-u)P_1 + uP_2 - \\
&\qquad ((1-u)P_0 + uP_1)) \\
&= (1-u)P_0 + uP_1 + u((1-2u)P_1 + uP_2 + (u-1)P_0) \\
&= ((1-u) + (u^2 - u))P_0 + (u + u - 2u^2)P_1 + u^2 P_2 \\
&= (u-1)^2 P_0 + 2u(1-u)P_1 + u^2 P_2
\end{aligned}$$

$\therefore$ $B_0(u) = (u-1)^2$

$B_1(u) = 2u(1-u)$

$B_2(u) = u^2$

(C). Suppose the first segment is controlled by P_0, P_1, P_2 and second segment controlled by V_0, V_1, V_2.

Since they are joined, we have $P_2 = V_0$ which represents $Q_p(1) = Q_v(0)$ already.

$\because Q'(u) = 2(u-1) P_0 + 2(1-2u) P_1 + 2u P_2$

We need to ensure $Q_p'(1) = Q_v'(0)$ in order to achieve C'- continuity

$\Rightarrow \quad 2(-1) P_1 + 2P_2 = -2V_0 + 2V_1$

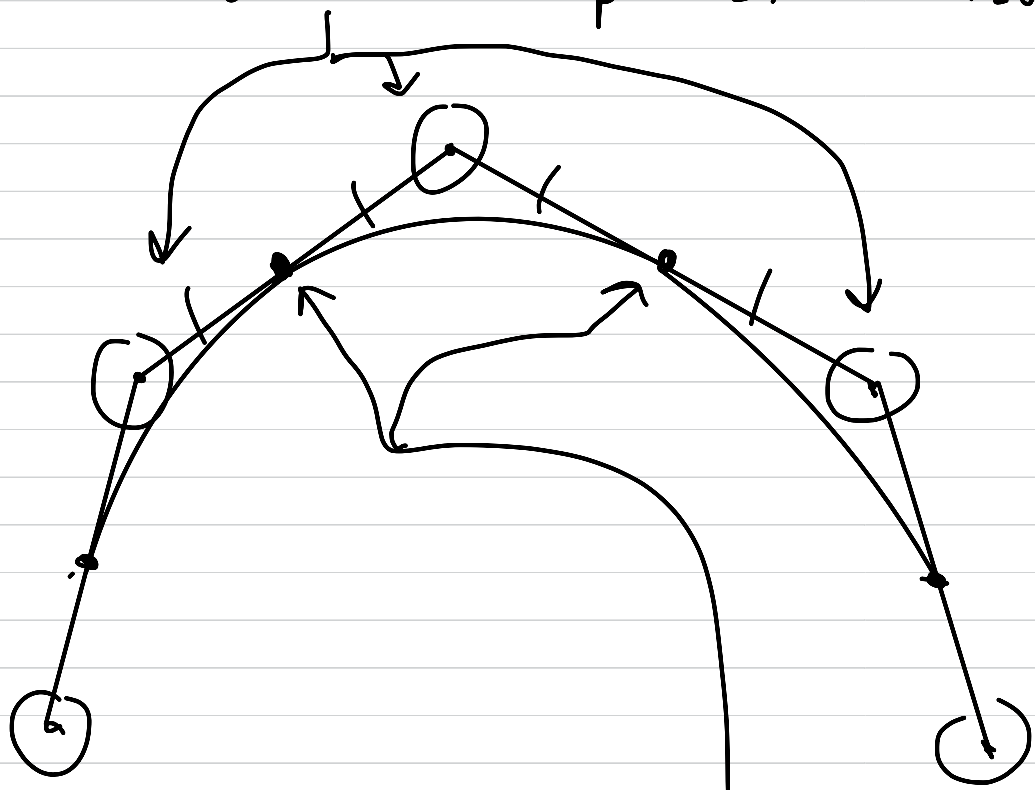
$\Leftrightarrow \quad -P_1 + P_2 = -V_0 + V_1$

$\Leftrightarrow$ Condition $\underline{2P_2 - P_1 = V_1}$ needed

Apart from $V_0 = P_2$

latter previous

(d). From (c), we know we need $V_1 - V_0 = P_2 - P_1$ suppose we start with 5 points circled in the below image, we connect them first and get the connecting lines' midpoints, the midpoints, together with the middle 3 control points, constitute the



3 Bezier curves with C^1 continuity guaranteed by the midpoints we picked.

(e). Firstly, we have the basis functions for cubic Bezier curves as:

$$B_0(u) = (1-u)^3$$
$$B_1(u) = 3u(1-u)^2$$
$$B_2(u) = 3u^2(1-u)$$
$$B_3(u) = u^3$$

from the Bernstein Basis.

$$\therefore \; Q(u) = (1-u)^3 P_0 + 3u(1-u)^2 P_1 + 3u^2(1-u) P_2 + u^3 P_3$$

$$Q'(u) = -\cancel{3}(1-u)^2 P_0 + \cancel{3}(3u^2-4u+1)P_1 + \cancel{3}(2u-3u^2)P_2 + \cancel{3}u^2 P_3$$

$$Q''(u) = \cancel{3}\cdot 2(1-u)P_0 + \cancel{3}(6u-4)P_1 + \cancel{3}(2-6u)P_2 + \overset{2}{\cancel{6}}uP_3$$

In order to satisfy: (Suppose $P_0P_1P_2P_3 + V_0V_1V_2V_3$)

C^0 continuity $\Rightarrow P_3 = V_0$

C^1 continuity $\Rightarrow Q'_p(1) = Q'_v(0)$

$\Longleftrightarrow P_3 - P_2 = V_1 - V_0$

C^2 continuity $\Rightarrow Q''_p(1) = Q''_r(0)$

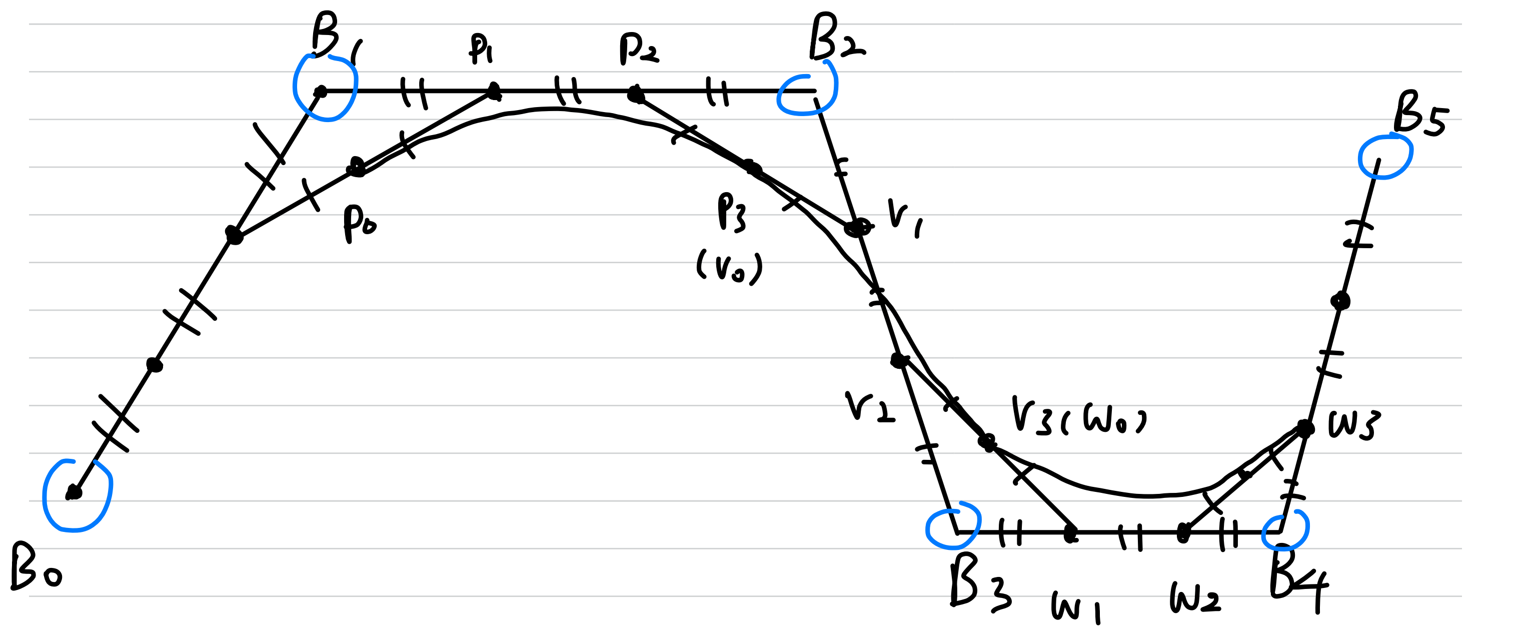
$\Longleftrightarrow 2P_1 - 4P_2 + 2P_3$

$= 2V_0 - 4V_1 + 2V_2$

$\Longleftrightarrow (P_1-P_2)+(P_3-P_2) = (V_0-V_1)+(V_2-V_1)$

$\Longleftrightarrow (P_2-P_1)+(P_2-P_3) = (V_1-V_0)+(V_1-V_2)$

Just like in class, firstly we connect all 6 de Boor points given, then cut these connecting lines into three equal parts, and construct letter "A" by connecting the $\frac{2}{3}$ point from the previous connecting line to the $\frac{1}{3}$ point from the next connecting line, and then halve the "$\frac{2}{3}$ - $\frac{1}{3}$" line to produce our Bezier curves' starting/end control points, "$\frac{1}{3}$", "$\frac{2}{3}$" points will be the middle control points for our Bezier curves. And done like below:



Problem 3.

Vertex A : $K_A = K_{ABD} + K_{ABC} + K_{ACD}$

Vertex B : $K_B = K_{BAC} + K_{BAD} + \underline{K_{BCD}}$

Vertex C : $K_C = K_{CAB} + K_{CAD} + \underline{K_{CBD}}$

Vertex D : $K_D = K_{DAB} + K_{DAC} + \underline{K_{DBC}}$

Normals to planes are:

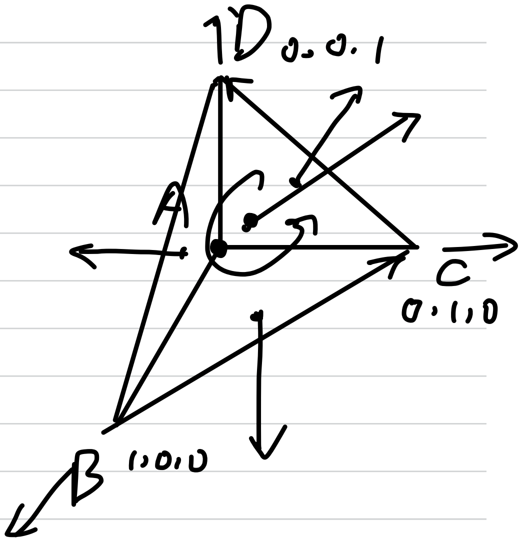
$n_{ABC} = (0, 0, -1)$

$n_{ABD} = (0, -1, 0)$

$n_{ACD} = (-1, 0, 0)$

$n_{BCD} = (\overrightarrow{BC} \times \overrightarrow{CD})_{normalize}$

$= \frac{1}{\sqrt{3}}(1, 1, 1)$



We then compute offset ds:

$(0,0,0)$
$$\begin{cases} d_{A-ABC} = 0 \\ d_{A-ABD} = 0 \\ d_{A-ACD} = 0 \end{cases}$$

$(0,1,0)$
$$\begin{cases} d_{C-ABC} = 0 \\ d_{C-ACD} = 0 \\ d_{C-BCD} = \frac{1}{\sqrt{3}} \end{cases}$$

$(1,0,0)$
$$\begin{cases} d_{B-ABC} = 0 \\ d_{B-ABD} = 0 \\ d_{B-BCD} = \frac{1}{\sqrt{3}} \end{cases}$$

$(0,0,1)$
$$\begin{cases} d_{D-ABD} = 0 \\ d_{D-ACD} = 0 \\ d_{D-BCD} = \frac{1}{\sqrt{3}} \end{cases}$$

Now we notice:

K_{ABC}

K_{ABD}

K_{ACD}

K_{BCD}

All have same offsets at different vertices, so we can compute K_{A-D} using subtraction after $K_{sum} = K_{ABC} + K_{ABD} + K_{ACD} + K_{BCD}$ are computed, because for example, K_{ABC} at A is same to K_{ABC} at rest of the vertices.

$$K_{ABC} = (0,0,-1,0)\begin{pmatrix}0\\0\\-1\\0\end{pmatrix} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

$$K_{ABD} = (0,-1,0,0)\begin{pmatrix}0\\-1\\0\\0\end{pmatrix} = \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$K_{ACD} = (-1,0,0,0)\begin{pmatrix}-1\\0\\0\\0\end{pmatrix} = \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$K_{BCD} = \frac{1}{3}(1,1,1,1)\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{3}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

$$\therefore K_{sum} = \frac{1}{3}\begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

$$K_A = K_{sum} - K_{BCD} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$K_B = K_{sum} - K_{ACD} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

$$K_C = K_{sum} - K_{ABD} = \frac{1}{3}\begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

$$K_D = K_{sum} - K_{ABC} = \frac{1}{3}\begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$