Assignment*

$$\begin{aligned}\phi(\vec{x}) &= \int \frac{dp^3}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\big[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\big] = \int \frac{dp^3}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\big[a_{\vec{p}} + a^\dagger_{-\vec{p}}\big]e^{i\vec{p}\cdot\vec{x}},\\ \pi(\vec{y}) &= \int \frac{dp'^3}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{p}'}}{2}}\big[a_{\vec{p}'}e^{i\vec{p}'\cdot\vec{y}} - a^\dagger_{\vec{p}'}e^{-i\vec{p}'\cdot\vec{y}}\big] = \int \frac{dp'^3}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{p}'}}{2}}\big[a_{\vec{p}'} - a^\dagger_{-\vec{p}'}\big]e^{i\vec{p}'\cdot\vec{y}},\end{aligned}$$

where $\omega^2_{\vec{p}} = \vec{p}^2 + m^2$. Then the K-G solutions into (equal time) commutator, i.e., $[\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$, is given by

$$\begin{aligned}[\phi(\vec{x}), \pi(\vec{y})] &= \int \frac{dp^3dp'^3}{2(2\pi)^6}\sqrt{\frac{\omega_{\vec{p}'}}{\omega_{\vec{p}}}}(-i)\big[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}, a_{\vec{p}'}e^{i\vec{p}'\cdot\vec{y}} - a^\dagger_{\vec{p}'}e^{-i\vec{p}'\cdot\vec{y}}\big]\\ &= \int \frac{dp^3dp'^3}{2(2\pi)^6}\sqrt{\frac{\omega_{\vec{p}'}}{\omega_{\vec{p}}}}(-i)\big\{[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}},\ a_{\vec{p}'}e^{i\vec{p}'\cdot\vec{y}}] + \big[a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}},\ a_{\vec{p}'}e^{i\vec{p}'\cdot\vec{y}}\big] - \big[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}},\\ &\quad a^\dagger_{\vec{p}'}e^{-i\vec{p}'\cdot\vec{y}}\big] - \big[a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}, a^\dagger_{\vec{p}'}e^{-i\vec{p}'\cdot\vec{y}}\big]\big\}\\ &= \int \frac{dp^3dp'^3}{2(2\pi)^6}\sqrt{\frac{\omega_{\vec{p}'}}{\omega_{\vec{p}}}}(-i)\big\{[a_{\vec{p}},\ a_{\vec{p}'}]e^{i(\vec{p}\cdot\vec{x}+\vec{p}'\cdot\vec{y})} + \big[a^\dagger_{\vec{p}},\ a_{\vec{p}'}\big]e^{-i(\vec{p}\cdot\vec{x}-\vec{p}'\cdot\vec{y})} - \big[a_{\vec{p}},\\ &\quad a^\dagger_{\vec{p}'}\big]e^{i(\vec{p}\cdot\vec{x}-\vec{p}'\cdot\vec{y})} - \big[a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}\big]e^{-i(\vec{p}\cdot\vec{x}+\vec{p}'\cdot\vec{y})}\big\} \qquad (1)\\ &= i\delta^3(\vec{x}-\vec{y})\end{aligned}$$

Using the math identity for 3D Dirac delta function, which can be re-express as

$$\delta^3(\vec{x}-\vec{y}) = \int \frac{dp^3}{(2\pi)^3}e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} = \int \frac{dp^3}{2(2\pi)^3}\big[e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\big], \qquad (2)$$

on the RHS of the last row in (1), and matching terms, we can see that all terms where $\vec{p}' \neq \pm\vec{p}$ must equal to zero after the integration, since (2) has no terms in both $\vec{p}$ and $\vec{p}'$. Thus, These particular terms reduce to the following form, integrated over $\vec{p}$ and $\vec{p}'$.

- For terms where $\vec{p}' \neq \pm\vec{p}$:

$$\begin{aligned}&\Rightarrow \frac{1}{(2\pi)^3}\sqrt{\frac{\omega_{\vec{p}'}}{\omega_{\vec{p}}}}(-i)\big\{[a_{\vec{p}},\ a_{\vec{p}'}]e^{i(\vec{p}\cdot\vec{x}+\vec{p}'\cdot\vec{y})} + \big[a^\dagger_{\vec{p}},\ a_{\vec{p}'}\big]e^{-i(\vec{p}\cdot\vec{x}-\vec{p}'\cdot\vec{y})} - \big[a_{\vec{p}},\ a^\dagger_{\vec{p}'}\big]e^{i(\vec{p}\cdot\vec{x}-\vec{p}'\cdot\vec{y})} -\\ &\quad \big[a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}\big]e^{-i(\vec{p}\cdot\vec{x}+\vec{p}'\cdot\vec{y})}\big\} = 0\\ &\Rightarrow \frac{1}{(2\pi)^3}\sqrt{\frac{\omega_{\vec{p}'}}{\omega_{\vec{p}}}}(-i)\Big\{\underbrace{[a_{\vec{p}}, a_{\vec{p}'}]}e^{i(\vec{p}\cdot\vec{x}+\vec{p}'\cdot\vec{y})} + \underbrace{\big[a^\dagger_{\vec{p}}, a_{\vec{p}'}\big]}e^{-i(\vec{p}\cdot\vec{x}-\vec{p}'\cdot\vec{y})} - \underbrace{\big[a_{\vec{p}}, a^\dagger_{\vec{p}'}\big]}e^{i(\vec{p}\cdot\vec{x}-\vec{p}'\cdot\vec{y})} -\\ &\quad \underbrace{\big[a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}\big]}e^{-i(\vec{p}\cdot\vec{x}+\vec{p}'\cdot\vec{y})}\Big\} \qquad \big(\underbrace{\quad}\text{must} = 0\big)\end{aligned}$$

So, all possible coefficient commutators $[a_{\vec{p}}, a_{\vec{p}'}]$, $\big[a^\dagger_{\vec{p}}, a_{\vec{p}'}\big]$ and $\big[a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}\big]$ with $\vec{p}' \neq \vec{p}$ or $-\vec{p}$ vanish.

The remaining terms all have $\vec{p}' = \pm\vec{p}$, which $\omega_{\vec{p}'} = \omega_{\vec{p}}$.

- For terms where $\vec{p}' = -\vec{p}$:

$$\begin{aligned}&\Rightarrow \frac{1}{(2\pi)^3}\sqrt{\frac{\omega_{-\vec{p}'}}{\omega_{\vec{p}}}}(-i)\Big\{\underbrace{[a_{\vec{p}}, a_{-\vec{p}}]}e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + \underbrace{\big[a^\dagger_{\vec{p}}, a_{-\vec{p}}\big]}e^{-i\vec{p}\cdot(\vec{x}+\vec{y})} - \underbrace{\big[a_{\vec{p}}, a^\dagger_{-\vec{p}}\big]}e^{i\vec{p}\cdot(\vec{x}+\vec{y})} -\\ &\quad \underbrace{\big[a^\dagger_{\vec{p}}, a^\dagger_{-\vec{p}}\big]}e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big\} = i\big[e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\big] \qquad (3)\end{aligned}$$

For these terms, the coefficient commutators all must vanish, because:

1) All terms with $(\vec{x}+\vec{y})$, i.e., $\underbrace{[\bullet, \bullet]}$, must equal zero, as the RHS in (3) only has term in $(\vec{x}-\vec{y})$;

*. This document has been written using the GNU $\text{T}_\text{E}\text{X}_\text{MACS}$ text editor (see `www.texmacs.org`).

2) If we make the operator $\phi(x)=\phi(\vec{x},\,t)=e^{iHt}\phi(\vec{x})e^{-iHt}$ and $\pi(x)=\pi(\vec{x},\,t)$ time-dependent in the Heisenberg picture, All terms with $(\vec{x}-\vec{y})$, i.e., $\underbrace{[\bullet,\bullet]}$, must equal zero for the reason that a non-zero factor $e^{\pm i(\omega_{\vec{p}}+\omega_{\vec{p}'})t}$ would appear in the each term. (Note: $e^{iHt}a_{\vec{p}}e^{-iHt}=a_{\vec{p}}e^{-i\omega_{\vec{p}}t}$ and $e^{iHt}a_{\vec{p}}^{\dagger}e^{-iHt}=a_{\vec{p}}^{\dagger}e^{i\omega_{\vec{p}}t}$, thus $[a_{\vec{p}},\;a_{\vec{p}'}]e^{-i(\omega_{\vec{p}}+\omega_{\vec{p}'})t}$ and $\left[a_{\vec{p}}^{\dagger},\;a_{\vec{p}'}^{\dagger}\right]e^{i(\omega_{\vec{p}}+\omega_{\vec{p}'})t}$ in the Heisenberg picture)

- For terms where $\vec{p}'=\vec{p}$:

$$\Rightarrow\;\frac{1}{(2\pi)^3}\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{p}}}}(-i)\left\{[a_{\vec{p}},\;a_{\vec{p}}]e^{i\vec{p}\cdot(\vec{x}+\vec{y})}+\left[a_{\vec{p}}^{\dagger},\;a_{\vec{p}}\right]e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}-\left[a_{\vec{p}},\;a_{\vec{p}}^{\dagger}\right]e^{i\vec{p}\cdot(\vec{x}-\vec{y})}-\left[a_{\vec{p}}^{\dagger},\;a_{\vec{p}}^{\dagger}\right]e^{-i\vec{p}\cdot(\vec{x}+\vec{y})}\right\}$$

$$=\frac{1}{(2\pi)^3}\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{p}}}}(i)\left\{-\underbrace{[a_{\vec{p}},a_{\vec{p}}]}e^{i\vec{p}\cdot(\vec{x}+\vec{y})}+\underline{\left[a_{\vec{p}},a_{\vec{p}}^{\dagger}\right]}e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}+\underline{\left[a_{\vec{p}},a_{\vec{p}}^{\dagger}\right]}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}+\underbrace{\left[a_{\vec{p}}^{\dagger},a_{\vec{p}}^{\dagger}\right]}e^{-i\vec{p}\cdot(\vec{x}+\vec{y})}\right\}=i\left[e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}+e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\right]\tag{4}$$

Similar to the above analysis, all time independent terms in integration with $\vec{p}'=\vec{p}$ must equal RHS above.

1) All terms with $(\vec{x}+\vec{y})$, i.e., $\underbrace{[\bullet,\bullet]}$, must equal zero, as the RHS above only has term in $(\vec{x}-\vec{y})$; (Also can be vanish if we proform analysis in the Heisenberg picture)

2) The only way the LHS of (4) matches the RHS is if each coefficient commutator marked with $\underline{[\bullet,\bullet]}$ must equal $(2\pi)^3$, i.e., $\left[a_{\vec{p}},a_{\vec{p}}^{\dagger}\right]=(2\pi)^3$.

In summary, we can conclude the coefficient commutation realtions as following

$$[a_{\vec{p}},a_{\vec{q}}]=0,\;\;\left[a_{\vec{p}}^{\dagger},a_{\vec{q}}^{\dagger}\right]=0\;\;,\;\;\left[a_{\vec{p}},a_{\vec{q}}^{\dagger}\right]=(2\pi)^3\delta^3(\vec{p}-\vec{q})\,.$$