Quantum Field Theory

Textbook: David Tong & Greiner Reinhardt & QFT by 刘川

18 weeks, 20% 是平时作业

- 场是更基本的量，粒子只是其激发态。
- QFT 理论是相当精确的理论。

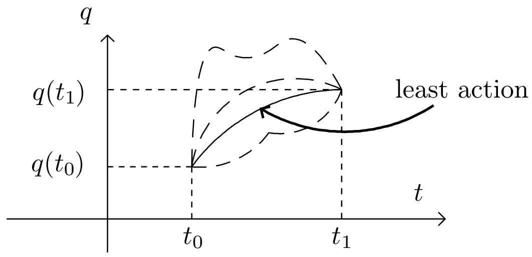
1 Classical Mechanics

Consider a point particle of mass m in one-dimentional under a time-independent potential $V(q)$.

Newton's equation of motion: $m\ddot{q} = -\frac{\partial V}{\partial q} = F$.

1.1 Lagrangian formalism

Define Lagrangian: $L(q, \dot{q}) = T - V = \frac{1}{2}m\dot{q}^2 - V(q)$, the action: $S = \int_{t_0}^{t_1} dt L(q, \dot{q})$.



Hamilton's *Principle of Least Action*: the classical equation of motion gives the least S.

$$\delta S = \int_{t_0}^{t_1} dt \delta L(q, \dot{q})$$

$$\delta L(q, \dot{q}) = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \quad \text{where } \delta\dot{q} = \frac{d}{dt}\delta q \text{ and } \delta L \text{ means function variation.}$$

Note: $\frac{\partial L}{\partial \dot{q}}\delta\dot{q} = \frac{\partial L}{\partial q}\frac{d}{dt}\delta q = \frac{d}{dt}\left(\frac{\partial L}{\partial q}\delta q\right) - \frac{d}{dt}\frac{\partial L}{\partial q}\delta q$, thus,

$$\delta S = \int_{t_0}^{t_1} dt \delta L = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q + \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial L}{\partial q}\delta q\right).$$

Since $\delta S = 0, \delta q(t_0) = 0 = \delta q(t_1)$ and $\int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial L}{\partial q}\delta q\right) = \frac{\partial L}{\partial q}\delta q|_{t_0}^{t_1}$, then we get

$$\frac{d}{dt}\left(\frac{\partial L}{d\dot{q}}\right) - \frac{\partial L}{\partial q} = 0,$$

i.e. *the Euler-Lagrange equation.*

For $L = \frac{1}{2}m\dot{q}^2 - V(q)$, we have $\frac{\partial L}{\partial \dot{q}} = m\dot{q} \Rightarrow m\ddot{q} = -\frac{\partial V}{\partial q}$ (EoM).

Question.

$L = T - V$ is not the total energy, so what's the meaning of the "$-$" in the least action?

1.2 Hamiltonian formulation

Define canonically conjugate momentum: $p = \frac{\partial L}{\partial \dot{q}}$.

Define Hamiltonian: $H(p, q) = p\,\dot{q}(p, q) - L(q, \dot{q}(p, q)) \leftarrow$ Legendre transformation.

For $L = \frac{1}{2}m\dot{q}^2 - V(q) \Rightarrow p = m\dot{q}$ and $H(p, q) = \frac{p^2}{2m} + V(q) = T + V$ (total energy).

Then, the classical equation of motion becomes *two coupled first-order differential equations*, i.e.

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} \\ \dot{q} &= \frac{\partial H}{\partial p} \end{aligned}$$

Exercise. Show this equivalent to the Newton's equation for $L = \frac{1}{2}mv^2 - V(q)$.

1.3 Poisson bracket

$A(p, q), B(p, q)$ are two dynamical variables (i.e. quantities which depend on p and q)

$$\{A, B\}_{\mathrm{PB}} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} \quad \text{(bilinear differenetial equation)}$$

Note $\{A, B\}_{\mathrm{PB}} = -\{B, A\}_{\mathrm{PB}} \Rightarrow \{A, A\}_{\mathrm{PB}} = 0$ and Jacobi identity, i.e.,

$$\{\{A, B\}_{\mathrm{PB}}, C\}_{\mathrm{PB}} + \{\{B, C\}_{\mathrm{PB}}, A\}_{\mathrm{PB}} + \{\{C, A\}_{\mathrm{PB}}, B\}_{\mathrm{PB}} = 0.$$

- 用场描述后，力的概念被抛弃；相互作用力可以被推导出来，无需事先给定。

For any physical quantity A,

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial t} + \frac{\partial A}{\partial p}\dot{p} + \frac{\partial A}{\partial q}\dot{q} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial p}\left(-\frac{\partial H}{dq}\right) + \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} \\ &= \frac{\partial A}{\partial t} + \{A, H\}_{\mathrm{PB}}. \end{aligned}$$

Thus A is conserved if A is not explicitly depending on t and $\{A, B\}_{\mathrm{PB}} = 0$.

Thus if H is not explicitly depending on t, then

$$\frac{\mathrm{dH}}{dt} = \frac{\partial H}{\partial t} + \{H, H\}_{\mathrm{PB}} = \frac{\partial H}{\partial t} = 0,$$

the Hamiltonian is conserved total energy.

. FYI: $\{p, p\}_{\mathrm{PB}} = \{q, q)_{\mathrm{PB}} = 0$ and $\{q, p\}_{\mathrm{PB}} = 1$.

1.4 Hamiltonian oscillator

For $F = -kq \Rightarrow V = \frac{1}{2}kq^2$, then $L = \frac{1}{2}mv^2 - \frac{1}{2}kq^2 = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$ and $\omega^2 = \frac{k}{m}$.

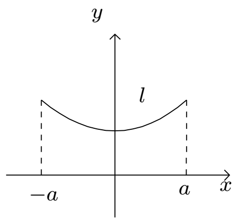
$$\begin{aligned} \dot{q} &= -\omega^2 q, \\ q &= a e^{-i\omega t} + a^* e^{i\omega t}, \end{aligned}$$

where a is a constant and a^* is the complex conjugate of a for real q.

$$\begin{aligned} p &= m\dot{q} = im\omega(-ae^{-i\omega t} + a^* e^{i\omega t}) \\ H &= \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2 = m\omega^2 |a|^2 \end{aligned}$$

Thus, total energy is conserved.

Exercise. Chain suspendid by two point.



Find out the shape $y(x)$.

1.5 Quantum Mechanics

1.5.1 Operator

$$\begin{aligned} \{A, B\}_{\mathrm{PB}} &\Rightarrow \frac{1}{i\hbar}\{\hat{A}, \hat{B}\} \\ A, B\colon \text{c-number} &\Rightarrow \hat{A}, \hat{B}\colon \text{operator} \end{aligned}$$

In particular, $\{q, p\}_{\mathrm{PB}} = 1 \Rightarrow [\hat{q}, \hat{p}] = i\hbar$. i.e., *Physical observables are promoted to become operators.*

A physical system is descibed by a state vector $|\psi>$ in Hilbert space.

$$\begin{aligned} \text{bra} <\psi| &\Leftarrow \text{dual vector space} \\ \text{ket} |\psi> &\Rightarrow \text{vector space} \end{aligned}$$

$$<\phi|\psi> \text{gives a c-number}, i.e., \text{complex number}$$

$$<\phi|\alpha\psi_1 + \beta\psi_2> = \alpha <\phi|\psi_1> + \beta <\phi|\psi_2>$$

$$\begin{aligned} <\phi|\psi>^* = <\psi|\phi> &\Rightarrow <\alpha\phi_1 + \beta\phi_2|\psi> = \alpha^* <\phi_1|\psi> + \beta <\phi_2|\psi> \\ <\phi|\phi> = 0 &\Rightarrow |\phi> = 0 \end{aligned}$$

A physical operator $\hat{O}$ is hermitian. $<\phi|\hat{O}^\dagger\psi> = <\hat{O}\phi|\psi>$

1.5.2 Expectation value

$$O \equiv <\psi|\hat{O}|\psi>$$

$$(\Delta O)^2 \equiv <\psi|(\hat{O} - O)^2|\psi> = <\psi|\hat{O}^2|\psi> - <\psi|\hat{O}|\psi>^2$$

If $\hat{O}|\psi> = O|\psi>$, $(\Delta O)^2 = 0$.

If $[\hat{A}, \hat{B}] \neq 0$, $|\psi>$ cannot be simultaneously eigenstate of both $\hat{A}$ and $\hat{B}$ $\Rightarrow$ Uncertainty priciple. i.e.,

$$\Delta A \Delta B \geqslant \frac{1}{2}|<\psi|[\hat{A}, \hat{B}]|\psi>|$$

For (p,q), $[\hat{q},\hat{p}]=i\hbar$, $\Delta p\Delta q\geqslant\frac{1}{2}\hbar$.

Note if $\left[\hat{A},\hat{B}\right]$ does not give rise to a "c", it still could be $\Delta A\Delta B\geqslant 0$, even if $\left[\hat{A},\hat{B}\right]\neq 0$.

1.5.3 Position operator

$|q>$ $\quad\hat{q}|q>=q|q>$ $\quad<q'|q>=\delta(q'-q)$. They satisfy $1=\int\, dq\,|q><q|$.

Wave function: $\psi(q)=<q|\psi>$.

Momentum eigenstates $\hat{p}|p>=p|p>$ $\quad<p'|p>=2\pi\hbar\delta(p'-p)\Rightarrow\int\,\frac{dp}{2\pi\hbar}|p><p|=1$, where a plane wave $<q|p>=e^{ipq/\hbar}$

1.5.4 Schrodinger picture

- Dynamics: Schrodinger Heinsenberg and Dirac (interaction) picture

State vectors evolve in time, but operators are constant with respect to time.

$$\begin{aligned}|\psi(t)> &= U(t,t_0)|\psi(t_0)>\\ <\psi(t)| &= <\psi(t_0)|U(t,t_0)^\dagger\end{aligned}$$

$$<\psi(t)|\psi(t)>=<\psi(t_0)|U(t,t_0)^\dagger U(t,t_0)|\psi(t_0)>=<\psi(t)|I|\psi(t)>$$

where unitary evolution.

Schrodinger equation: （绝对的）线性!

$$i\hbar\frac{d}{dt}|\psi(t)>=H|\psi(t)>\Rightarrow i\hbar\frac{d}{dt}U(t)=HU(t)\Rightarrow U(t)=e^{-iHt/\hbar}$$

If H is time dependent, then $U(t)=T\exp\!\left(-\frac{i}{\hbar}\int_0^t\,H(t')dt'\right)$, where T is time ordering operator.

1.5.5 Heinsenberg picture

State vectors are constant with respect to time, while physical operators evolve in time.

Start with Schrodinger picture:

$$<A>_t=<\psi(t)|A|\psi(t)>=<\psi(0)\left|e^{iH(t)}Ae^{-iH(t)}\right|\psi(0)>=<\psi(0)|A(t)|\psi(0)>$$

$$\begin{aligned}\frac{dA(t)}{dt} &= \frac{i}{\hbar}He^{iHt/\hbar}Ae^{-iHt/\hbar}+e^{iHt/\hbar}\frac{\partial A}{\partial t}e^{-iHt/\hbar}+\frac{i}{\hbar}He^{iHt/\hbar}A(-H)e^{-iHt/\hbar}\\ &= \frac{i}{\hbar}He^{iHt/\hbar}(HA-AH)e^{-iHt/\hbar}+e^{iHt/\hbar}\frac{\partial A}{\partial t}e^{-iHt/\hbar}\end{aligned}$$

Thus,

$$\frac{dA(t)}{dt}=\frac{i}{\hbar}[H,A(t)]+e^{iHt/\hbar}\frac{\partial A}{\partial t}e^{-iHt/\hbar}.$$

This equation, of course, is solved by $A(t)=e^{iHt/\hbar}A(t)e^{-iHt/\hbar}$.

Compare this to Poisson bracket: $\frac{dA}{dt}=\frac{\partial A}{\partial t}+\{A,H\}_{\rm PB}$

$$\begin{aligned}\frac{\partial A}{\partial t}=0\ \Rightarrow\ \frac{\partial A}{\partial t}&=\{A,H\}_{\rm PB}\quad\text{Classical Mechanics}\\ i\hbar\frac{dA}{dt}&=[A,H]\quad\ \text{Quantum Mechanics}\end{aligned}$$

Again, $\{\,\}_{\rm PB} = \frac{1}{i\hbar}[\,]$.

1.5.6 Hamiltonian oscillator

$$
\begin{aligned}
\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2 \hat{q}^2 \qquad\qquad [\hat{q},\hat{p}] = i\hbar \\
&= \frac{1}{2m}(\hat{p}^2 + m^2\omega^2\hat{q}^2)
\end{aligned}
$$

Define:

$$
\begin{aligned}
\hat{a} &= \sqrt{\frac{\omega m}{2\hbar}}\left(\hat{q} + i\frac{\hat{p}}{\hbar\omega}\right), \\
\hat{a}^\dagger &= \sqrt{\frac{\omega m}{2\hbar}}\left(\hat{q} - i\frac{\hat{p}}{\hbar\omega}\right).
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \hat{q} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)\,, \\
\Rightarrow \hat{p} &= \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger)\,.
\end{aligned}
$$

Thus,

$$[\hat{q},\hat{p}] = i\hbar \;\;\Rightarrow\;\; [\hat{a},\hat{a}^\dagger] = 1\,.$$

$$
\begin{aligned}
\hat{H} &= -\frac{i}{2m}\frac{\hbar m\omega}{2}(\hat{a} - \hat{a}^\dagger)^2 + \frac{m}{2}\omega^2\frac{\hbar}{2m\omega}(\hat{a} + \hat{a}^\dagger)^2 \\
&= \frac{1}{2}\hbar\omega(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) \\
&= \hbar\omega\left(\hat{a}\hat{a}^\dagger + \frac{1}{2}\right) = \hat{H}\,.
\end{aligned}
$$

1.5.7 Quantislization of energy

Consider $|\psi>$ is an eigenstate of $\hat{H}$,

$$
\begin{aligned}
H|\psi> &= E|\psi>\,, \\
H\hat{a}^\dagger|\psi> &= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hat{a}^\dagger|\psi> = (E + \hbar\omega)\hat{a}^\dagger|\psi>\,, \\
H\hat{a}|\psi> &= (E - \hbar\omega)\hat{a}|\psi>\,.
\end{aligned}
$$

Obsense of negative energy requires a ground state $|0>$ such that $\hat{a}|0> = 0$.

Note that $|0>$ cannot not be annihilated by both $\hat{a}$ and $\hat{a}^\dagger$, because $[\hat{a},\hat{a}^\dagger] = 1$, i.e., $\hat{a}^\dagger|0> = \hat{a}|0> = 0$ is impossible.

$$
\begin{aligned}
\hat{H}|0> &= \frac{1}{2}\hbar\omega|0> \qquad\qquad E_0 = \frac{1}{2}\hbar\omega \\
\hat{H}|n> &= E_n|n> \qquad\qquad E_n = \hbar\omega\left(n + \frac{1}{2}\right)
\end{aligned}
$$

where $|n> = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0>$.

Puzzle. What if start with a classical system $H(p,\,q) = H(a,\,a^\dagger) = \hbar\omega(a^* a)$, then quantice by write $H \to \hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a})$. In this formation, we do not have $E_0 = \frac{1}{2}\hbar\omega$.

Puzzle. Why "mass" have not been quantilize ?

1.5.8 Multiple particles

$(q, \dot{q}) \to (q_\alpha, \dot{q}_\alpha) \to (q_\alpha, p_\alpha), \alpha = 1, 2, \cdots, N.$

$$
\begin{aligned}
L(q, \dot{q}) &\to L(q_\alpha, \dot{q}_\alpha) \\
L(q, p) &\to H(q_\alpha, p_\alpha)
\end{aligned}
$$

In both classical and quantum mechanics particles are conserved. Naviely, they are related to conservations of energy and momentum. In fact, conservation of particles is neither necessary nor sufficient for energy conservation.

2 Classical Field Theory

A field is a quantity defined at every point of space and time $(\vec{x}, t)$. A general expression for a field is

$$
\begin{array}{ccl}
\text{field} & \text{coordinate} & \\
\phi_\alpha(\vec{x}, t) \longrightarrow & q_\alpha(t) & \alpha = 1, 2, \cdots, n \\
\swarrow\downarrow & \downarrow & \\
\text{label}(\alpha, \vec{x}) & \text{label: n degree of freedom} & \\
\nearrow & & \\
\end{array}
$$

infinite degrees of freedom at least one for each point $\vec{x}$ in space

Particle Mechanics: position, or coordinate is a dynnamical variable.

Field theory: coordinate is merely a label; they are not dynamical —— leads to a deeper geometric interpretations.

A familiar example is the electromagnetic field: $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$.

$$
\begin{aligned}
\vec{E} &= -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \\
\vec{B} &= \vec{\nabla} \times \vec{A}
\end{aligned}
$$

where $A^\mu(\vec{x}, t) = \left(\phi, \vec{A}\right)$ is a 4-vector.

2.1 The Lagrangian

The dynamics of the field is govened by a Lagrangian which is a function of $\phi_\alpha(\vec{x},\ t)$, $\dot{\phi}_\alpha(\vec{x},\ t)$ and $\vec{\nabla}\phi_\alpha(\vec{x}, t)$. In all the systems we study in this course, the Lagrangian is of the form:

$$
L(t) = \int d^3x \, \mathcal{L}(\phi_\alpha, \partial_\mu \phi_\alpha)\,,
$$

where the $\mathcal{L}$ is Lagrangian density or simply Lagrangian.

The action:

$$
S = \int_{t_1}^{t_2} dt \int d^3x \, \mathcal{L} = \int d^4x \, \mathcal{L}.
$$

In particle mechanics, $L = L(q, \dot{q})$ independent of $\ddot{q}$.

In field theory, $\mathcal{L}$ can be depending on $\nabla\phi$, $\nabla^2\phi$, $\nabla^3\phi$ etc.

For current purpose, we shall consider only dependence on $\nabla\phi$. Equantions of motion are determined by the principle of least action. We vary the path, keeping in the end points fixed and require $\delta S=0$.

$$\begin{aligned}\delta S &= \int d^4x\Big[\frac{\partial\mathcal{L}}{\partial\phi_\alpha}\delta\phi_\alpha+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\delta(\partial_\mu\phi_\alpha)\Big]\\ &= \int d^4x\Big\{\Big[\frac{\partial\mathcal{L}}{\partial\phi_\alpha}-\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big)\Big]\delta\phi_\alpha+\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\delta\phi_\alpha\Big)\Big\}\end{aligned}$$

where $\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\delta\phi_\alpha\Big)$ is surface term and we have $\delta\phi_\alpha(\vec{x},t_0)=\delta\phi_\alpha(\vec{x},t)=0$.

$\delta S=0$ for all such paths $\Rightarrow$ Enler-Lagrange equations of motion for field ϕ_α :

$$\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\Big)-\frac{\partial\mathcal{L}}{\partial\phi_\alpha}=0.$$

Example: The Klein-Gordon Equantion:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}m^2\phi^2\\ &= \frac{1}{2}\dot{\phi}^2-\frac{1}{2}(\nabla\phi)^2-\frac{1}{2}m^2\phi^2\end{aligned}$$

Note that $\eta^{\mu\nu}=\eta_{\mu\nu}=\mathrm{diag}(+,-,-,-)$ and $\nabla\phi=(\partial_x\phi,\partial_y\phi,\partial_z\phi)$; and 号差-2的好处：动能项 $\frac{1}{2}\dot{\phi}^2$ 的符号是正的。

$$\begin{aligned}L=T-V\Rightarrow T &= \int d^3x\,\frac{1}{2}\dot{\phi}^2\\ V &= \int d^3x\left(\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m^2\phi^2\right)\end{aligned}$$

where $\frac{1}{2}(\nabla\phi)^2$ and $\frac{1}{2}m^2\phi^2$ term describe gradient energy and potential energy respectly.

With $\frac{\delta\mathcal{L}}{\delta\phi}=-m^2\phi$, and $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}=\partial^\mu\phi\equiv\big(\dot{\phi},-\nabla\phi\big)$,

The Enler-Lagrangian equation is then

$$\ddot{\phi}-\nabla^2\phi+m^2\phi=0$$

or in relativistic form $\partial_\mu\partial^\mu\phi+m^2\phi=0$, i.e., Klein-Gordon equation.

With $\partial_\mu\partial^\mu\equiv\Box$, the Lapacian in Minkowski spacetime,

$$(\Box+m^2)\phi=0.$$

A more general scalar system:

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)\Rightarrow\partial_\mu\partial^\mu\phi+\frac{\partial V}{\partial\phi}=0.$$

Another example:

Consider a complex scalar ψ and the complex conjugate ψ^*,

$$\mathcal{L}=\frac{1}{2}\big(\psi^*\dot{\psi}-\dot{\psi}^*\psi\big)-\nabla\psi^*\cdot\nabla\psi-m\psi^*\psi.$$

With $\frac{\partial\mathcal{L}}{\partial\psi^*} = \frac{i}{2}\dot{\psi} - m\psi$, $\frac{\partial\mathcal{L}}{\partial\psi} = -\frac{i}{2}\psi$ and $\frac{\partial\mathcal{L}}{\partial(\nabla\psi^*)} = -\nabla\psi$, we have

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\psi^*} - \nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla\psi^*)} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} &= 0 \\ i\psi^* - m\psi + \nabla^2\psi + \frac{i}{2}\dot{\psi} &= 0\end{aligned}$$

If we vary respect to ψ and ψ^*, respectly,

$$\begin{aligned}i\dot{\psi} &= -\nabla^2\psi + m\psi\,, \\ -i\dot{\psi}^* &= -\nabla^2\psi^* + m\psi^*\,.\end{aligned}$$

This looks remarkable like the Schrodinger equation indeed, if we replace on by $V(x,t)$, we obtain the Schrodinger equation. However, the interpretation is very different. In Schrodinger equation, ψ is the wave function, but here ψ is a scalar field.

3 Flat Spacetime Symmetry and Maxwell Theory

3.1 Galilean group