Fluid Dynamics

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Fluid Mechanics vs. Classical Mechanics

Mean free path and Size of the system

Collisional, weakly collisional, collisionless

Multifluid (ions, dust, …): MHD and non-magnetized

Fluid Description based on the conservation laws

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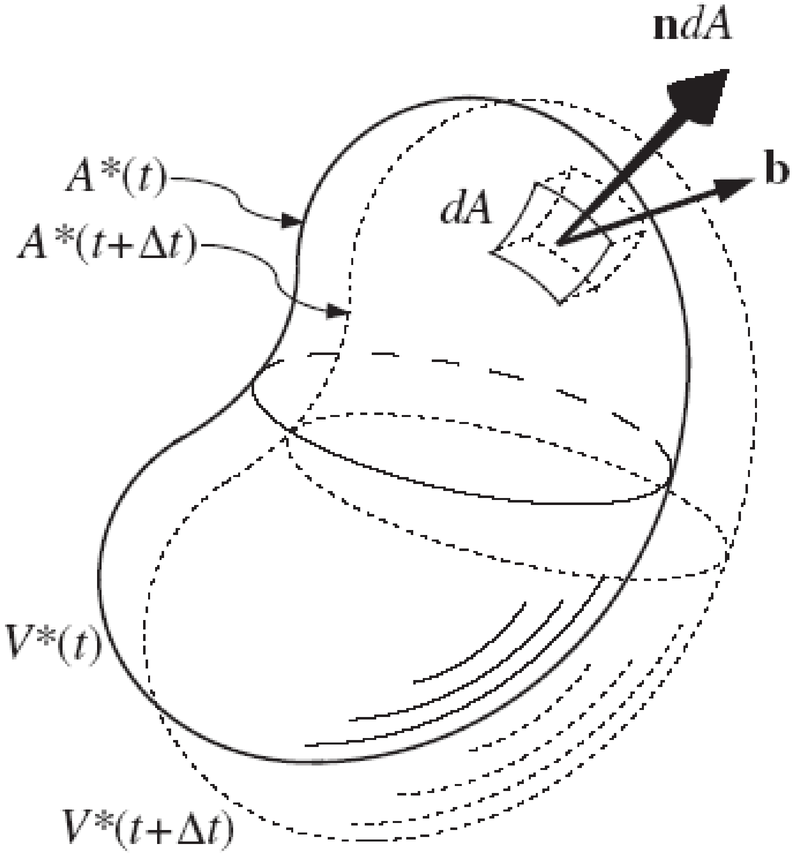
Conservation Laws

CONSERVATION OF MASS



$$\int_V \rho \, \mathrm{d}V$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \, \mathrm{d}V = -\int_S \rho \mathbf{u} \cdot \mathbf{dS}$$

$$\int_V \left\{ \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) \right\} \mathrm{d}V = 0$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = 0 \qquad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

Consider the steady form of the continuity

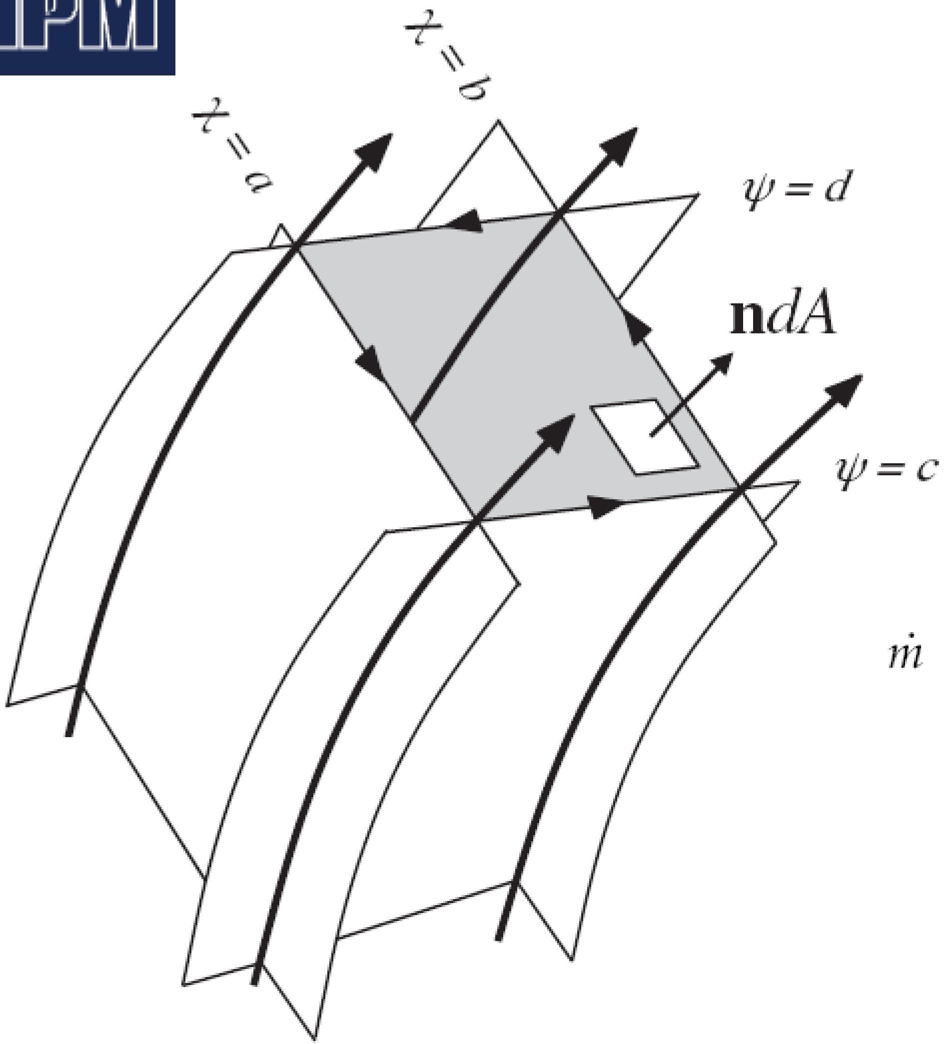
$$\nabla \cdot (\rho \mathbf{u}) = 0.$$

The divergence of the curl of any vector field is identically zero

$$\rho \mathbf{u} = \nabla \times \mathbf{\Psi}$$

vector potential $\mathbf{\Psi}$.

$$\mathbf{\Psi} = \chi \nabla \psi \Longrightarrow \rho \mathbf{u} = \nabla \chi \times \nabla \psi$$



$$\dot{m} = \int_A \rho \mathbf{u} \cdot \mathbf{n}\, dA = \int_A (\nabla \times \mathbf{\Psi}) \cdot \mathbf{n}\, dA = \int_C \mathbf{\Psi} \cdot d\mathbf{s} = \int_C \chi \nabla \psi \cdot d\mathbf{s} = \int_C \chi d\psi$$

$$= b(d-c) + a(c-d) = (b-a)(d-c).$$

$\nabla\chi$ is perpendicular to surfaces of constant χ.

CONSERVATION OF MOMENTUM

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathbf{u}\,\mathrm{d}V$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = f_i + \frac{\partial}{\partial x_j}[T_{ij}].$$

$$f_i = -\rho\frac{\partial\Phi}{\partial x_i} \qquad \nabla^2\Phi = 4\pi G\rho.$$

$$f_i = (\mathbf{j}\wedge\mathbf{B})_i$$

$$T_{ij} = -p\delta_{ij} + \tau_{ij}$$

Write out all the components of the stress tensor $\mathbf{T}$ in (x, y, z)-coordinates in terms of $\mathbf{u} = (u, v, w)$, and its derivatives.

$$\mathbf{T} = \begin{bmatrix} -p + 2\mu\dfrac{\partial u}{\partial x} + \left(\mu_v - \dfrac{2}{3}\mu\right)\nabla\cdot\mathbf{u} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & -p + 2\mu\dfrac{\partial v}{\partial y} + \left(\mu_v - \dfrac{2}{3}\mu\right)\nabla\cdot\mathbf{u} & \mu\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & -p + 2\mu\dfrac{\partial w}{\partial z} + \left(\mu_v - \dfrac{2}{3}\mu\right)\nabla\cdot\mathbf{u} \end{bmatrix}$$

The Lagrangian derivative

$\partial\rho/\partial t$ → It measures the way it changes with time at a fixed position

$D\rho/Dt$ → It measures the way it changes as moves with the fluid

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho.$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$$

CONSERVATION OF ENERGY

$$T\frac{DS}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

S is the entropy per unit mass

e is the internal energy

We can introduce n = N / V. Here, n is the particle number density. So,

$$P = \left(\frac{N}{V}\right) kT = nkT$$

QUESTION

HOW CAN WE DETERMINE THE PARTICLE NUMBER DENSITY n ?

- Equation of state

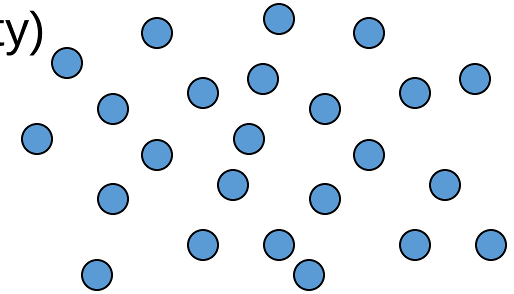
Let’s assume all particles of the gas are similar with mass m. Then,

$$n = \frac{\rho}{m}$$

Number of particles
per cm^3 (or number density)

Mass (gr)

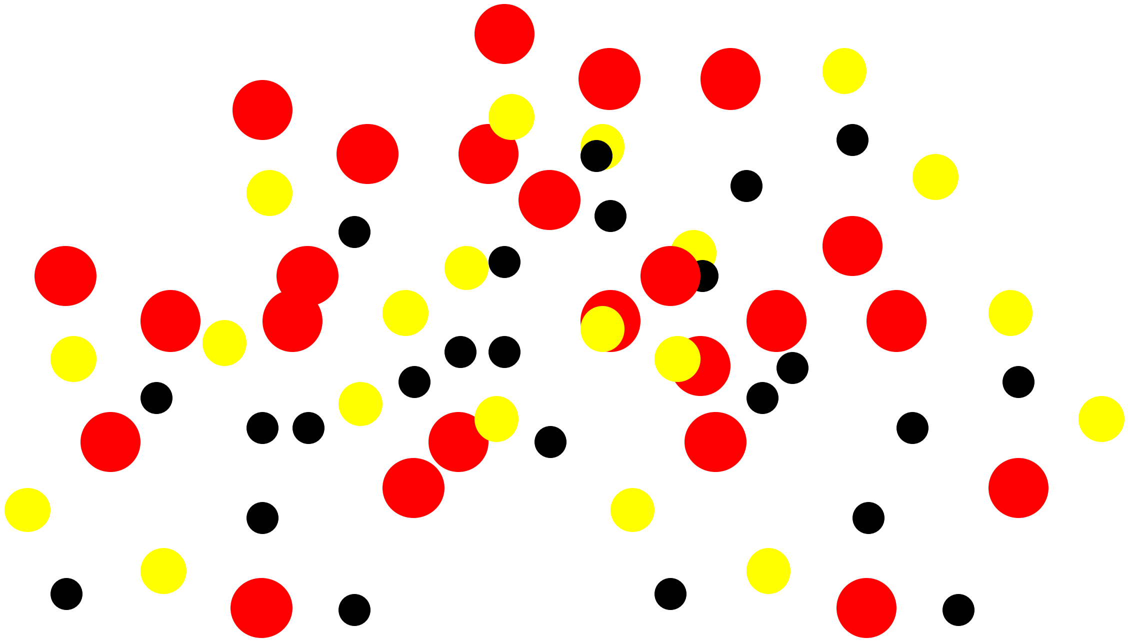
Density (gr/cm^3)



So, $P = nkT = (\frac{\rho}{m})kT = \frac{\rho k T}{m}$

- Equation of state

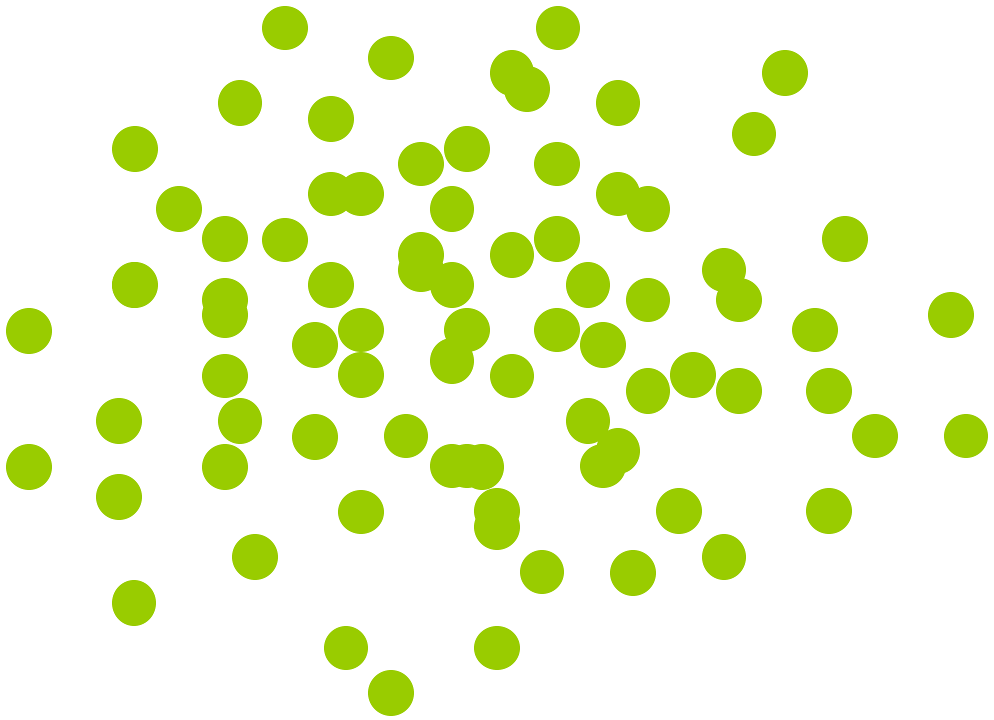
But, what about if we have a variety of particles of different mass?



- Equation of state

In that case, we can define the average mass of a gas particle as

$$\overline{m}$$



- Equation of state

Thus,

$$P = nkT = \left(\frac{\rho}{\overline{m}}\right)kT$$

We now define a new quantity, the **mean molecular weight**, as

$$\mu = \frac{\overline{m}}{m_H}$$

The mass of a hydrogen atom
1.67 * 10^(-24) gr

- Equation of state

$$\boxed{P = \frac{\rho k T}{\mu m_H}}$$

The mean molecular weight is dependent on the ***composition*** of the gas as well as the state of ***ionization*** of the system.

- Equation of state

Generally it is more useful to express the mean molecular weight in terms of mass ratios, know as mass fractions

$$X = \frac{total\ mass\ of\ \ hydrogen}{total\ mass\ of\ \ gas},$$

$$Y = \frac{total\ mass\ of\ \ helium}{total\ mass\ of\ \ gas},$$

$$Z = \frac{total\ mass\ of\ \ metals}{total\ mass\ of\ \ gas}$$

- Equation of state

Thus, X + Y + Z = 1

We consider a neutral gas

$$\bar{m} = \frac{(total\ \ number\ of\ \ H)\times m_H + (total\ number of\ \ He)\times m_{He} + (total number\ of\ \ metals)\times m_z}{Total\ numbers}$$

$$\bar{m} = \frac{N_H m_H + N_{He}\, m_{He} + N_z\, m_z}{N_H + N_{He} + N_z}$$

- Equation of state

$$\frac{1}{\bar{m}} = \frac{1}{\mu m_H} = \frac{N_H + N_{He} + N_z}{M},$$

$$M = N_H m_H + N_{He} m_{He} + N_z m_z$$

$$\frac{1}{\mu m_H} = \frac{N_H}{M} + \frac{N_{He}}{M} + \frac{N_z}{M}$$

$$= \frac{N_H}{N_H m_H} \times \underbrace{\frac{N_H m_H}{M}}_{X} + \frac{N_{He}}{N_{He} m_{He}} \times \underbrace{\frac{N_{He} m_{He}}{M}}_{Y} + \frac{N_z}{N_z m_z} \times \underbrace{\frac{N_z m_z}{M}}_{Z}$$

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- Equation of state

$$\frac{1}{\mu m_H} = \frac{1}{m_H} X + \frac{1}{m_{He}} Y + \frac{1}{m_z} Z$$

$$\frac{1}{\mu} = X + \frac{m_H}{m_{He}} Y + \frac{m_H}{m_z} Z$$

$$\frac{1}{4} \qquad < \frac{1}{A} >_n$$

Is a weight average of all elements
in the gas heavier than helium
For solar abundances 1/15.5

useful approximations

Incompressible approximation

The major difference between astrophysical fluids and those encountered in many terrestrial situations (including those encountered in many courses on fluid dynamics) is that astrophysical ones are highly compressible. However, in situations where fluid motions are slow compared with the sound speed, density gradients are quickly smoothed out and it is a useful approximation to treat the fluid as if it were incompressible. In physical terms this means that any particular element of the fluid does not change its density, which implies that

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0.$$

useful approximations

Adiabatic flow

If the flow occurs fast enough that no fluid element has time to exchange heat with its surroundings, and if energy generation within the fluid is negligible, the heat equation simplifies to

$$\frac{DS}{Dt} = 0.$$

useful approximations

Barotropic flow

We can avoid using the heat equation, and therefore simplify the analysis, by assuming that pressure is solely a function of density.

$$p = p(\rho).$$

Bernoulli equation for a non-magnetic barotropic fluid

For a barotropic fluid we have $p = p(\rho)$ and we can define the quantity $h = \int \mathrm{d}p/\rho$. Then, in a gravitational potential Φ, the momentum equation becomes

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \left(\frac{1}{2}u^2\right) - \operatorname{curl} \mathbf{u}.$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla h - \nabla \Phi \longrightarrow \frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \wedge \operatorname{curl} \mathbf{u} = -\nabla \left(\frac{1}{2}u^2 + h + \Phi\right)$$

useful approximations

Barotropic flow

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \wedge \operatorname{curl} \mathbf{u} = -\nabla \left(\frac{1}{2} u^2 + h + \Phi \right)$$

If the flow is steady, then taking the scalar product with **u** implies

$$\mathbf{u} \cdot \nabla \left(\frac{1}{2} u^2 + h + \Phi \right) = 0$$

and thus that the quantity $(\frac{1}{2}u^2 + h + \Phi)$ is constant on streamlines.

$$
\left\{
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \\
&\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \mathbf{f} \\
&P = \rho k T / \mu m_{\mathrm{H}} \\
&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \varepsilon\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho \varepsilon + P\right)\mathbf{v}\right] = \mathbf{f} \cdot \mathbf{v} - \nabla \cdot \mathbf{F}_{\mathrm{rad}} - \nabla \cdot \mathbf{q} \\
&\varepsilon = \frac{3}{2} k T / \mu m_{\mathrm{H}}
\end{aligned}
\right.
$$

$$\nabla \cdot (\rho \mathbf{v}) = 0,$$
$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \mathbf{f},$$
$$\nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho\varepsilon + P\right)\mathbf{v}\right] = \mathbf{f} \cdot \mathbf{v}.$$

Steady, spherically symmetric accretion

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \rho v\right) = 0 \quad \Rightarrow \quad 4\pi r^2 \rho(-v) = \dot{M}$$

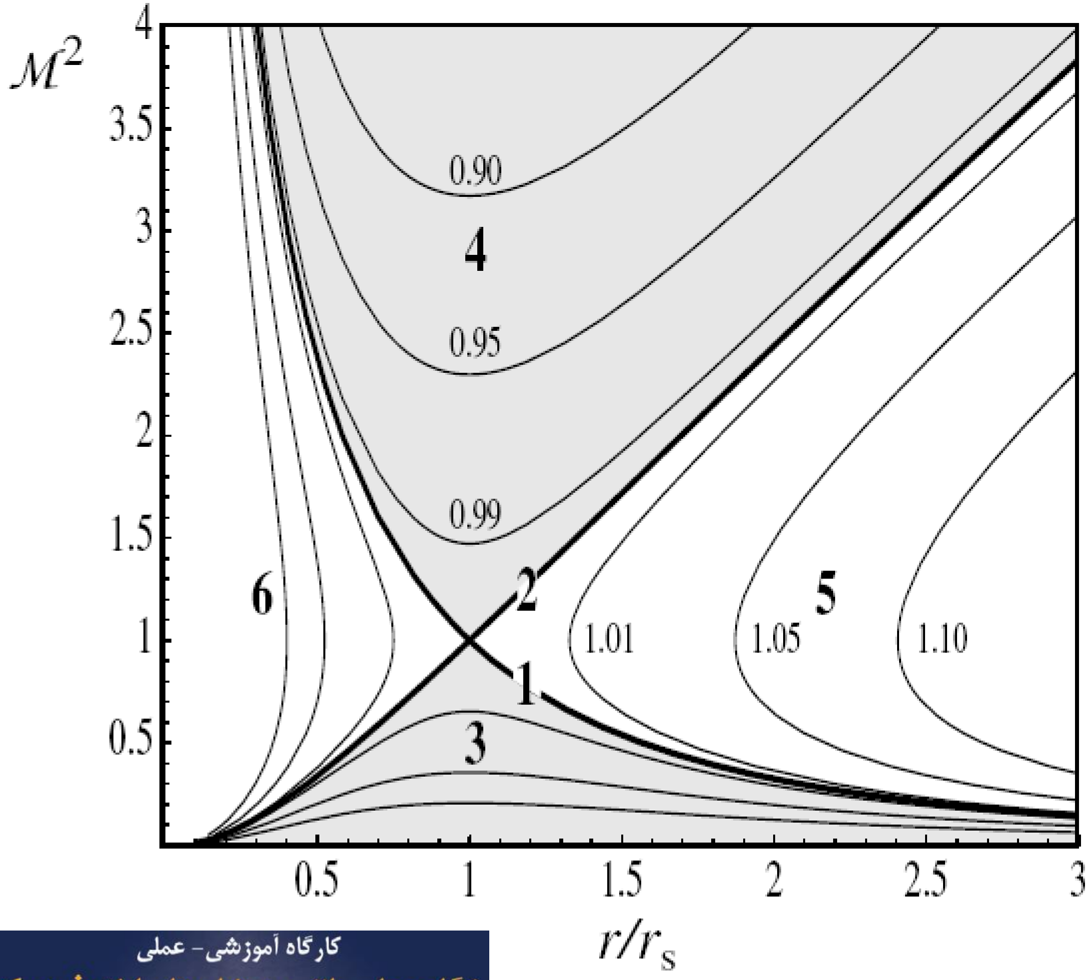
$$v\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2} = 0$$

$$P = K\rho^{\gamma},\ K = \text{constant}$$

$$T = \mu m_{\mathrm{H}} P/\rho k$$

$$\mathcal{M}^2 = v^2(r)/c_s^2(r)$$

$$\frac{1}{2}\left(1-\frac{c_s^2}{v^2}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(v^2\right) = -\frac{GM}{r^2}\left[1-\left(\frac{2c_s^2 r}{GM}\right)\right]$$



$$\dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{c_s^3(\infty)}\left[\frac{2}{5-3\gamma}\right]^{(5-3\gamma)/2(\gamma-1)}$$

The similarity (Taylor–Sedov) equations

Here we consider the effects of an explosion at a point in a uniform medium. The first studies of this problem concerned explosions of nuclear bombs in the Earth's atmosphere. In astronomy this analysis is used to model the effects of the early stages of the explosion of a supernova in the interstellar medium.

$$\frac{\partial \rho}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u).$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

For the energy conservation equation we assume that the shocked fluid does not cool, so that each fluid element conserves its entropy, i.e. $\mathrm{D}S/\mathrm{D}t = 0$. Then we have

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{1}{2}u^2\right)\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\rho u\left(e + \frac{p}{\rho} + \frac{1}{2}u^2\right)\right] = 0 \qquad e = p/(\gamma - 1)\rho$$

$$\xi = r\left(\frac{\rho_1}{Et^2}\right)^{1/5}$$

Similarity variables

$$\xi = r\left(\frac{\rho_1}{Et^2}\right)^{1/5}$$

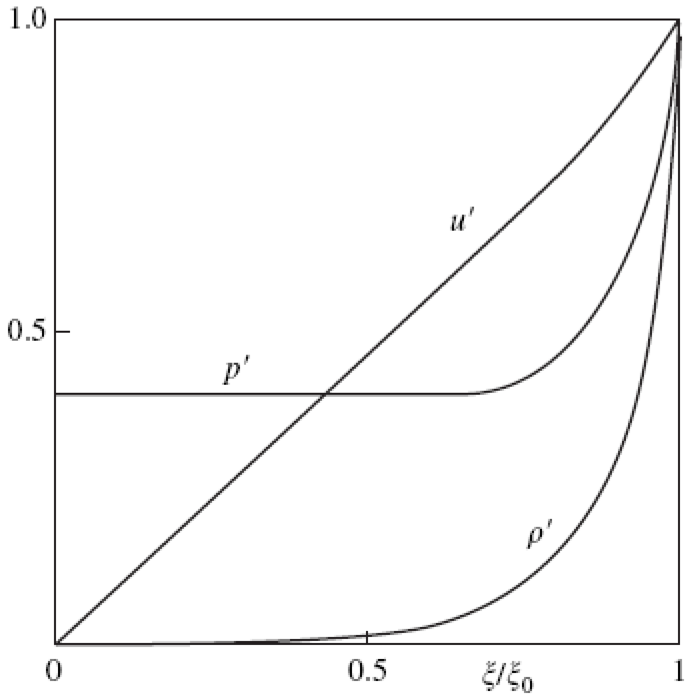
$$\rho(r,t) = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_1 A(\xi)$$

$$u(r,t) = \frac{4}{5(\gamma+1)}\left(\frac{r}{t}\right)V(\xi)$$

$$p(r,t) = \frac{8}{25(\gamma+1)}\rho_1\left(\frac{r}{t}\right)^2 B(\xi).$$

Taylor–Sedov
equations

$$
\left\{
\begin{aligned}
&-\xi\frac{\mathrm{d}A}{\mathrm{d}\xi}+\frac{2}{\gamma+1}\left(3AV+\xi\frac{\mathrm{d}}{\mathrm{d}\xi}(AV)\right)=0.\\
&-V-\frac{2}{5}\xi\frac{\mathrm{d}V}{\mathrm{d}\xi}+\frac{4}{5(\gamma+1)}\left(V^2+V\xi\frac{\mathrm{d}V}{\mathrm{d}\xi}\right)=-\frac{2}{5}\frac{\gamma-1}{\gamma+1}\frac{1}{A}\left(2B+\xi\frac{\mathrm{d}B}{\mathrm{d}\xi}\right)\\
&-2(B+AV^2)-\frac{2}{5}\xi\frac{\mathrm{d}}{\mathrm{d}\xi}(B+AV^2)\\
&\qquad+\frac{4}{5(\gamma+1)}\left(5V(\gamma B+AV^2)+\xi\frac{\mathrm{d}}{\mathrm{d}\xi}[V(\gamma B+AV^2)]\right)=0
\end{aligned}
\right.
$$



The Jeans instability

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} = -\frac{\nabla p'}{\rho} - \nabla\Phi' \\ \frac{\partial \rho'}{\partial t} + \rho \operatorname{div} \mathbf{u} = 0, \\ \nabla^2\Phi' = 4\pi G\rho' \\ p' = c_s^2 \rho' \end{cases}$$

the linearized equation of motion

linearized mass conservation

$\exp\{i(\omega t + \mathbf{k}\cdot\mathbf{r})\}$

$$\begin{cases} i\omega\mathbf{u} = -i\mathbf{k}c_s^2\frac{\rho'}{\rho} - i\mathbf{k}\Phi', \\ i\omega\frac{\rho'}{\rho} + i\mathbf{k}\cdot\mathbf{u} = 0 \\ -k^2\Phi' = 4\pi G\rho'. \end{cases}$$

$$\omega^2 = k^2c_s^2 - 4\pi G\rho$$

$$\lambda_J^2 = \frac{\pi c_s^2}{G\rho} \qquad M_J = \left(\frac{\pi}{G}\right)^{3/2}\frac{c_s^3}{\rho^{1/2}}$$

Axisymmetric perturbations – the Toomre criterion

The perturbed velocity field $\mathbf{u} = (u_R, R\Omega + u_\phi, 0).$ all linear quantities vary as $\propto \exp(\mathrm{i}\omega t)$

The linearized equations of motion

$$\mathrm{i}\omega u_R - 2\Omega u_\phi = -\frac{1}{\Sigma}\frac{\mathrm{d}P'}{\mathrm{d}R} - \frac{\mathrm{d}\Phi'}{\mathrm{d}R}$$

$$\mathrm{i}\omega u_\phi + \left[\Omega + \frac{\mathrm{d}}{\mathrm{d}R}(R\Omega)\right] u_R = 0,$$

The linearized continuity equation

$$\mathrm{i}\omega\Sigma' + \Sigma\left(\frac{\mathrm{d}u_R}{\mathrm{d}R} + \frac{u_R}{R}\right) = 0,$$

$$\nabla^2\Phi' = 4\pi G\Sigma'\delta(z).$$

Linearization of Poisson's equation

$$-\omega^2 u_R + \left[4\Omega^2 + 2R\Omega \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right] u_R = -k^2 C_\mathrm{s}^2 u_R + 2\pi G |k| \Sigma u_R.$$

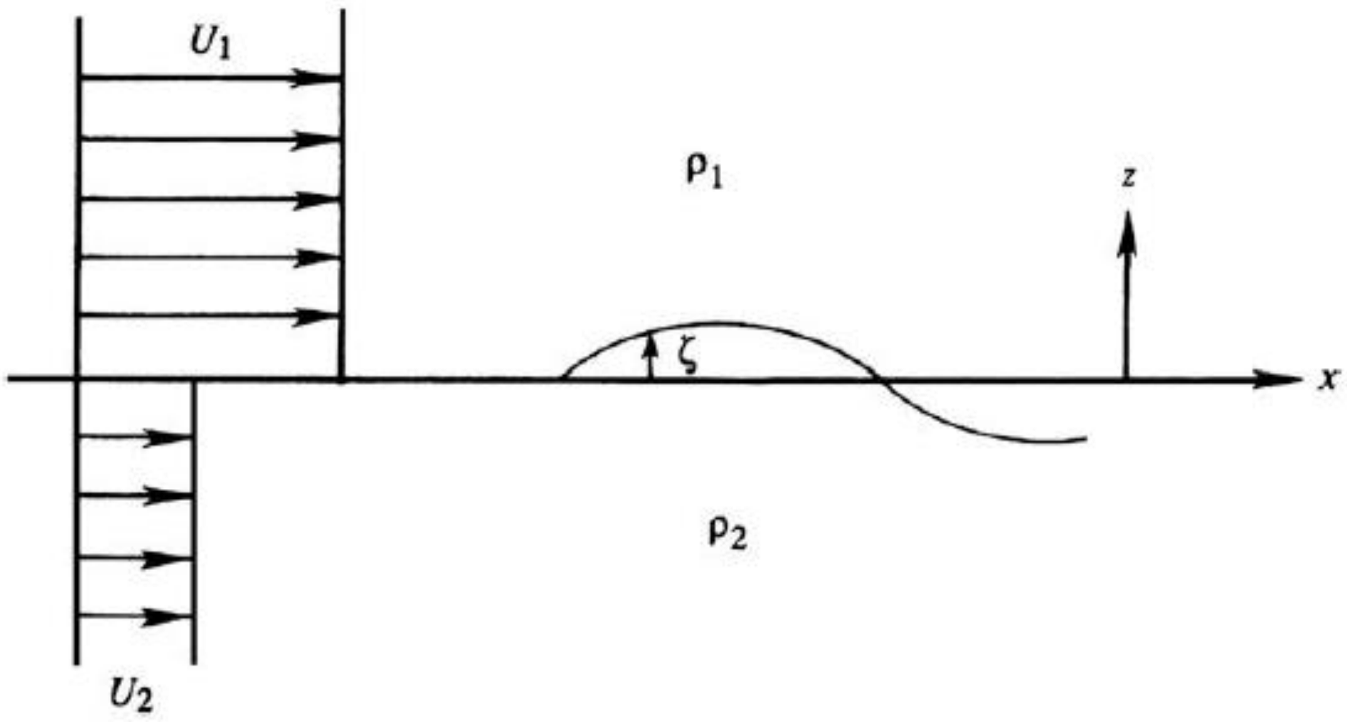
$$\Downarrow$$

$$\omega^2 = \kappa^2 - 2\pi G |k| \Sigma + k^2 C_\mathrm{s}^2.$$

epicyclic frequency

$$Q = \frac{\kappa C_\mathrm{s}}{\pi G \Sigma} < 1.$$

Role of cooling?



exponential growth

$$\left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right)\frac{g}{k} < \frac{\rho_2\rho_1}{(\rho_2+\rho_1)^2}(U_2-U_1)^2$$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