Variational Approaches for Auto-Encoding Generative Adversarial Networks

Mihaela Rosca, Balaji Lakshminarayanan
David Warde-Farley, Shakir Mohamed

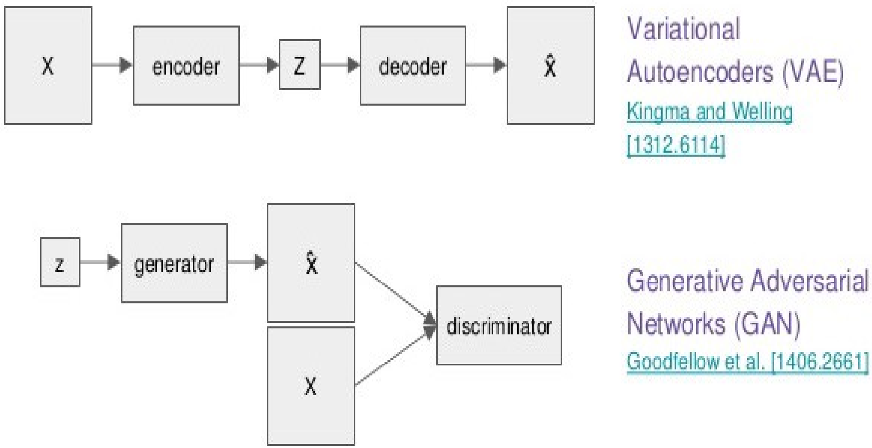
Presented by Shih-Ming Wang
ComputerVision Lab, UCSC

02-13-2019

Latent Variable Model I

- **Assumes** a generating process for real data $x \sim p^*(x)$
  - Unobserved quantity following prior distribution $z \sim p_\theta(z)$
  - Observation given z following likelihood $x|z \sim p_\theta(x|z)$
- Without losing generality, $z \sim \mathcal{N}(0, I)$ (dropping θ hereafter)
- Though $p_\theta(x, z) = p^*(x)p_\theta(z|x)$, it's not trivial to do inference (i.e. compute $p_\theta(z|x)$)
- Implicit Latent Variable Model, e.g. GAN
  - Learns a generator G_θ and makes likelihood implicit: $p_\theta(x|z) = \delta(x - G_\theta(z))$
- Prescribed Latent Variable Model, e.g. VAE
  - **Assumes** explicit (closed form) likelihood $p_\theta(x|z)$
  - **Assumes** explicit (closed form) posterior $p_\theta(z|x)$
  - Approximates posterior $p_\theta(z|x)$, hence able to do inference

Latent Variable Model II



Motivation

Generative Adversarial Network (GAN) Pros & Cons

- Excels at generating sharp samples
- Only make assumption on prior
- Optimization is hard (vanishing gradient) using its original loss
- The samples might lack diversity (mode-collapse)
- Can't do inference

Variational Auto Encoder (VAE) Pros & Cons

- Generates blurry images
- Requires assumption on likelihood posterior, limiting model power
- Pairwise reconstruction penalty discourages mode-collapse
- Can do inference

KL divergence

- KL divergence measures "distance" between distributions

$$KL(p\|q) = \mathbb{E}_{p(x)}\left[\ln \frac{p(x)}{q(x)}\right] = \int p(x) \ln \frac{p(x)}{q(x)} dx$$

- $KL(p\|q) \geq 0$, with equality hold when $p(x) = q(x), \forall x$
- Requirement to approximate $KL(p\|q)$
  - Able to sample from $p(x)$
  - $p(x), q(x)$ bare close form
- In some cases, for e.g. when p, q are both Gaussian, $KL(p\|q)$ bares close form and sampling is not required
- KL is asymmetric, but Jenson-Shanon divergence (JSD) is

$$JS(p\|q) = 0.5KL(p\|(p+q)/2) + 0.5KL(q\|(p+q)/2)$$

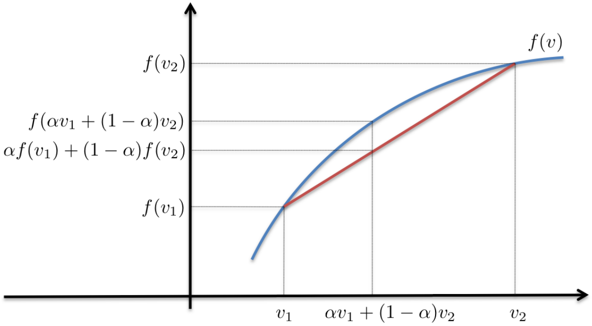
Jenson Inequality

- For concave function f and arbitrary distribution p

$$f\left(\mathbb{E}_{p(x)}[x]\right) \geq \mathbb{E}_{p(x)}[f(x)]$$

- Which implies for any function g

$$f\left(\mathbb{E}_{p(x)}[g(x)]\right) \geq \mathbb{E}_{p(x)}[f(g(x))]$$



Backpropagating through samples I

- Say we want to minimize $L(\theta, \eta) = \mathbb{E}_{p_\eta(x)}\left[f_\theta(x)\right]$
- Computing $\nabla_\theta L(\theta, \eta)$ is straightforward

$$\begin{aligned}\nabla_\theta L(\theta, \eta) &= \mathbb{E}_{p_\eta(x)}\left[\nabla_\theta L(\theta, \eta)\right] \\ &\approx \frac{1}{n}\sum_{i=1}^{n} \nabla_\theta f_\theta(x_i), \forall x_i \sim p_\eta(x)\end{aligned}$$

- However $\nabla_\eta L(\theta, \eta)$ is not in expectation form

$$\begin{aligned}\nabla_\eta L(\theta, \eta) &= \nabla_\eta \int p_\eta(x) f_\theta(x) dx \\ &= \int \nabla_\eta p_\eta(x) f_\theta(x) dx\end{aligned}$$

Backpropagating through samples II

Score function estimator (REINFORCE)

- use the identity $\nabla_\eta p_\eta(x) = p_\eta(x) \nabla_\eta \log p_\eta(x)$

$$
\begin{aligned}
\nabla_\eta L(\theta, \eta) &= \int \nabla_\eta p_\eta(x) f_\theta(x) dx \\
&= \int p_\eta(x) \nabla_\eta \log p_\eta(x) f_\theta(x) dx \\
&= \mathbb{E}_{p_\eta(x)} \left[f_\theta(x) \nabla_\eta \log p_\eta(x) \right] \\
&\approx \frac{1}{n} \sum_{i=1}^{n} f_\theta(x) \nabla_\eta \log p_\eta(x)
\end{aligned}
$$

- Such estimator of the gradient might have high variance

Backpropagating through samples III

Reparameterization Trick

- **Assume**
  - $f_\theta(x)$ is differentiable w.r.t. x
  - Exists g, $x = g(\eta, \epsilon)$, where $\epsilon \sim p(\epsilon)$.
  - For e.g. $x \sim \mathcal{N}(\mu(\eta) + \sigma^2(\eta))$ can be reparameterize as $x = \mu(\eta) + \sigma^2(\eta) * \epsilon, \epsilon \sim \mathcal{N}(0, 1)$
- Then we get an gradient estimator with low variance

$$\begin{aligned}
\nabla_\eta L(\theta, \eta) &= \nabla_\eta \mathbb{E}_{p_\eta(x)} [f_\theta(x)] \\
&= \nabla_\eta \mathbb{E}_{p(\epsilon)} [f_\theta(g(\eta, \epsilon))] \\
&= \mathbb{E}_{p(\epsilon)} [\nabla_\eta f_\theta(g(\eta, \epsilon))] \\
&= \mathbb{E}_{p(\epsilon)} \left[f'_\theta(g(\eta, \epsilon)) \nabla_\eta g(\eta, \epsilon)\right]
\end{aligned}$$

Maximum Log-Likelihood Principle (MLP)

- MLP optimizes θ so that $p_\theta(x) \to p^*(x)$

$$\theta = \underset{\theta}{argmax} \frac{1}{n} \sum_{i=1}^{n} \ln p_\theta(x), \forall x_i \sim p^*(x)$$

- MLP is equivalent to minimizing KL divergence

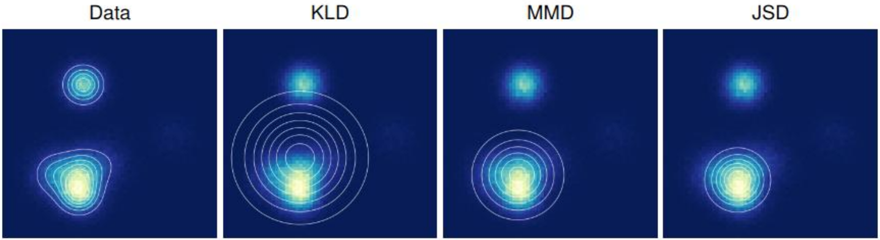
$$\begin{aligned}
\underset{\theta}{argmin}\, KL(p^*(x) \| p_\theta(x)) &= \underset{\theta}{argmin}\, E_{p^*(x)} \left[\ln \frac{p^*(x)}{p_\theta(x)} \right] \\
&= \underset{\theta}{argmin}\, E_{p^*(x)}[-\ln p_\theta(x)] + \cancel{\mathbb{E}_{p^*(x)} [\ln p^*(x)]} \\
&\sim \underset{\theta}{argmax} \frac{1}{n} \sum_{i=1}^{n} \ln p_\theta(x_i), \forall x_i \sim p^*(x)
\end{aligned}$$

- Marginal likelihood is intractable (no closed form) in Latent Variable Model

$$p_\theta(x) = \int p(z) p_\theta(x|z) dz$$

Choose of Loss

- When MLP is intractable, different training objective are adopted
- Most objectives are consistent with MLP given infinite data and model capacity
- With limited model capacity, different objective can lead to very different result
- GAN approximately minimizes JSD, leading to mode-collapse
- VAE approximates MLP (minimizes KLD), leading to unreal sample



GAN I

- GAN estimates $\frac{p^*(x)}{p_\theta(x)}$ with a discriminator D_ϕ
- The input to the discriminator follows $0.5p^*(x) + 0.5p_\theta(x)$

$$\frac{p^*(x)}{p_\theta(x)} = \frac{p_\phi(x|y=1)}{p_\phi(x|y=0)} = \frac{\cancel{p(x)}p_\phi(y=1|x)/\cancelto{0.5}{p(y=1)}}{\cancel{p(x)}p_\phi(y=0|x)/\cancelto{0.5}{p(y=0)}} = \frac{D_\phi(x)}{1-D_\phi(x)}$$

- The likelihood is $\prod_{i=1}^{n} D_\phi(x)^{y_i}(1-D_\phi(x))^{1-y_i}$, hence MLP for ϕ is

$$\phi = \underset{\phi}{argmax}\, \mathbb{E}_{p^*(x)}\left[\ln D_\phi(x)\right] + \mathbb{E}_{p_\theta(x)}\left[\ln(1-D_\phi(x))\right]$$

- Fixing D_ϕ, optimizes the following loss w.r.t. the generator G_θ

$$\theta = \underset{\theta}{argmin}\, \mathbb{E}_{p_\theta(x)}\left[\ln(1-D_\phi(x))\right]$$

GAN II

- This is the minmax game that GAN played

$$\begin{aligned} &\min_{\theta} \max_{\phi} \mathbb{E}_{p^*(x)} [\ln D_\phi(x)] + \mathbb{E}_{p_\theta(x)} [\ln(1 - D_\phi(x))] \\ \leftrightarrow &\min_{\theta} \max_{\phi} \mathbb{E}_{p^*(x)} [\ln D_\phi(x)] + \mathbb{E}_{p(z)} [\ln(1 - D_\phi(G_\theta(z)))] \end{aligned}$$

- We need to be able to
  - sample from implicit likelihood $p_\theta(x|z)$
- We can do
  - sample from model distribution $p_\theta(x) = p(z)p_\theta(x|z)$
- We can't do
  - compute $p_\theta(x)$ given x
  - compute $p_\theta(z|x)$ or sample $z \sim p_\theta(z|x)$ given x

VAE I

- Variational Inference (VI) bounds $p_\theta(x)$ with evidence lower bound (ELBO) and follows MLP to maximize the ELBO:

$$
\begin{aligned}
\ln p_\theta(x) &= \ln \int p(z) p_\theta(x|z) dz \\
&= \ln \int q_\eta(z) \frac{p(z)}{q_\eta(z)} p_\theta(x|z) dz \\
&= \ln \mathbb{E}_{q_\eta(z)} \left[\frac{p(z)}{q_\eta(z)} p_\theta(x|z) \right] \\
&\geq \mathbb{E}_{q_\eta(z)} \left[\ln \left(\frac{p(z)}{q_\eta(z)} p_\theta(x|z) \right) \right] && \text{(Jenson Inequality)} \\
&= \mathbb{E}_{q_\eta(z)} \left[\ln \left(\frac{p_\theta(x,z)}{q_\eta(z)} \right) \right] && (ELBO_1) \\
&= \mathbb{E}_{q_\eta(z)} \left[\ln p_\theta(x|z) \right] + KL(q_\eta(z) \| p(z)) && (ELBO_2)
\end{aligned}
$$

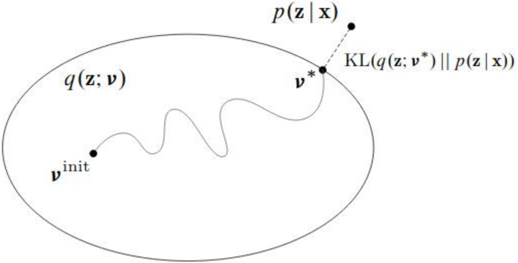
VAE II

- $q_\eta(z)$ (variational distribution) serves as approximate posterior
- EM algorithm alternates between the two steps
  - M-step: Fixing $q_\eta(z)$, optimize ELBO w.r.t. θ
  - E-step: Fixing p_θ, optimize ELBO w.r.t. $q_\eta(z)$

$$
\begin{aligned}
q_\eta(z) &= \underset{q_\eta(z)}{argmax}\, \mathbb{E}_{q_\eta(z)}\left[\ln\left(\frac{p_\theta(x,z)}{q_\eta(z)}\right)\right] \qquad (ELBO_1) \\
&= \underset{q_\eta(z)}{argmax}\, \mathbb{E}_{q_\eta(z)}\left[\ln\left(\frac{p(x)p_\theta(z|x)}{q_\eta(z)}\right)\right] \\
&= \underset{q_\eta(z)}{argmax}\, \ln p(x) + \mathbb{E}_{q_\eta(z)}\left[\ln\left(\frac{p_\theta(z|x)}{q_\eta(z)}\right)\right] \\
&= \underset{q_\eta(z)}{argmax}\, \cancel{\ln p(x)} - KL(q_\eta(z)\|p_\theta(z|x)) \\
&= p_\theta(z|x)
\end{aligned}
$$

VAE III

- $p_\theta(x)$ in $p_\theta(z|x) = \frac{p(z)p_\theta(x|z)}{p(x)}$ is usually intractable
- Usually assume parametric form $q_\eta(z)$ and optimize η so that $q_\eta(z) \to p(z|x)$
- $q_\eta(z)$ usually has local parameters for each sample x. Alternatively, we can "armortize" it by modeling $q_\eta(z|x)$ with global parameters



VAE IV

- VAE models $q_\eta(z|x)$ with encoder E_η and $p_\theta(x|z)$ with decoder G_θ
- **Assumes** posterior $q_\eta(z|x_n) \sim \mathcal{N}(E_\eta(x_n), E_\eta(x_n)\mathbb{I})$
- **Assumes** likelihood $p_\theta(x_n|z) \sim \mathcal{N}(G_\theta(x_n), \mathbb{I})$
- Train η, ϕ simultaneously to minimizes $ELBO_2$

$$\eta, \phi = \underset{\eta,\phi}{argmax}\, \mathbb{E}_{q_\eta(z|x)}\left[\ln p_\theta(x|z)\right] + KL(q_\eta(z|x)\|p(z))$$

- VAE use reparameterization trick to estimate $\nabla_\eta \mathbb{E}_{q_\eta(z|x)}\left[\ln p_\theta(x|z)\right]$
- The first term is the reconstruction loss (auto-encoder)
- The second term serves a regularization, leading to higher reconstruction error

VAE V

- This is the game that VAE played

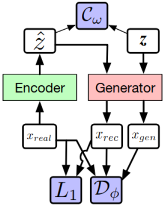
$$\eta, \phi = \underset{\eta,\phi}{argmax}\, \mathbb{E}_{q_\eta(z|x)} [\ln p_\theta(x|z)] + KL(q_\eta(z|x) \| p(z))$$

- We need to be able to
  - compute likelihood $p_\theta(x|z)$
  - compute posterior approximation $q_\eta(z|x)$
- We can do
  - sample from model distribution $p_\theta(x) = p(z)p_\theta(x|z)$
  - compute $q_\theta(z|x)$ or sample $z \sim q_\theta(z|x)$ given x
- We can't do
  - compute $p_\theta(x)$ given x

Fusion of VAE & GAN

Key Idea

- Starts with VAE architecture
- Remove the close form assumption for $q_\eta(z|x)$ and $p_\theta(x|z)$
- Use implicit distribution for
  - posterior: $q_\eta(z|x) = \delta(z - E_\eta(x))$
  - likelihood: $p_\theta(x|z) = \delta(x - G_\theta(z))$
- Use the GAN trick to learn two discriminators C_ω and D_ϕ to estimate the two terms in ELBO



Implicit Variational Decomposition

- Instead of Gaussian, let $q_\theta(z|x) = \delta(z - E_\eta(x))$
- The KL term in ELBO can be estimated by

$$KL(q_\eta(z|x)||p(z)) = \mathbb{E}_{q_\eta(z|x)}\left[\ln \frac{q_\eta(z|x)}{p(z)}\right] \approx \mathbb{E}_{q_\eta(z|x)}\left[\ln \frac{C_\omega(z)}{1 - C_\omega(z)}\right]$$

- C_ω is a discriminator trained to tell samples drawn from $q_\eta(z|x)$ or $p(z)$

Hybrid Likelihood

- Let $p_\theta(x|z) = \delta(x - G_\theta(z))$
- The reconstruction term in ELBO can be estimated by

$$\begin{aligned}
\mathbb{E}_{q_\eta(z|x)}\left[\ln p_\theta(x|z)\right] &= \mathbb{E}_{q_\eta(z|x)}\left[\ln\left(\frac{p_\theta(x|z)}{p^*(x)}p^*(x)\right)\right] \\
&= \mathbb{E}_{q_\eta(z|x)}\left[\ln\frac{p_\theta(x|z)}{p^*(x)}\right] + \cancel{\ln p^*(x)} \\
&\approx \mathbb{E}_{q_\eta(z|x)}\left[\ln\frac{D_\phi(G_\theta(z))}{1 - D_\phi(G_\theta(z))}\right]
\end{aligned}$$

- D_ϕ is a discriminator trained to tell samples from $p_\theta(x|z)$ or $p^*(x)$
- We can hybrid the implicit likelihood with an explicit likelihood
- For e.g. Laplace: $p_\theta(x|z) \propto \exp(-\lambda\|x - G_\theta(z)\|_1)$, which leads to the L_1 reconstruction loss $\mathbb{E}_{q_\eta(z|x)}\left[-\lambda\|x - G_\theta(z)\|_1\right]$

α-GAN I

- This is the total loss of $\alpha - GAN$

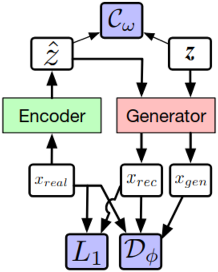
$$\mathcal{L}(\theta, \eta) = \mathbb{E}_{q_\eta(z|x)}\left[-\lambda\|x - G_\theta(z)\|_1 + \ln\frac{D_\phi(G_\theta(z))}{1 - D_\phi(G_\theta(z))} + \ln\frac{C_\omega(z)}{1 - C_\omega(z)}\right]$$

- The loss for D_ϕ and C_ω are not shown
- We need to be able to
  - train encoder E_η, decoder G_θ, and discriminators D_ϕ, C_ω
  - sample from implicit posterior $z \sim q_\eta(z|x)$
  - sample from implicit likelihood $x \sim p_\theta(x|z)$
  - compute explicit likelihood $p_\theta(x|z)$
- We can do
  - sample from model distribution $p_\theta(x) = p(z)p_\theta(x|z)$
  - sample from implicit posterior $q_\theta(z|x)$ given x
- We can't do
  - compute $p_\theta(x)$ given x
  - compute $q_\theta(z|x)$ given x

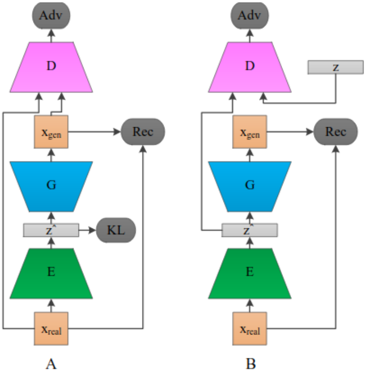
α-GAN II

Tricks to improve

- For the two discriminator, use a different loss to avoid vanishing gradient
- Pass reconstruct sample $x_{rec} \sim G_\theta(E_\eta(x))$ as well as sample from noise $x_{gen} \sim\sim G_\theta(z)$ to train D_ϕ

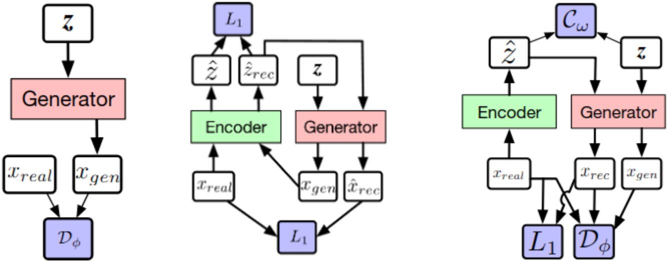


Related Work I



- (A) VAEGAN imposes a discriminator (D_ϕ) on the data space
- (B) AAE imposes a discriminator (C_ω) on the latent space
- (C) ALI & BiGan discriminate **jointly** in the data and latent space

Related Work II



(a) DCGAN (b) AGE (c) α-GAN

- DCGAN is the normal GAN with some architecture tweaks
- AGE adds a **loop** from Generator back to Encoder, which is used as discriminator
- $\alpha - GAN$ imposes two discriminator on the data space and the latent space

Evaluation

- Inception Score: samples should cluster compactly

$$\mathbb{E}_x \left[KL(p(y|x)) || p(y) \right]$$

- (1 - avg pairwise MS-SSIM): test in-class mode collapse (CelebA only)
- Wasserstein critic: discriminator to tell samples from train set or val set. Capture memorization and mode-collapse

Experiments

- Compared with DC-GAN, WGAN-GP, AGE
- Using ColorMnist, CelebA, CIFAR10 dataset
- Obserbations:
  - WGAN-GP, having no inference power, wins $\alpha - GAN$ most of the time
  - $\alpha - GAN$ wins AGE, having inference power,most of the time
  - By visual inspection, WGAN-GP still generates more realistic samples. It's hard to tell $\alpha - GAN$ generates better sample than AGE