MATH 490 Final Project

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May 10, 2020

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1 Introduction

In this project, we will explore the Rubik's Cube puzzle with the help of SageMath. We will be discussing the Rubik's Cube Group and some of its interesting properties, various algorithms that can be used to solve a Rubik's cube, some of which utilize group theory, Sage's Rubik's cube group functions and finally talk about how SAT solvers could be used to solve Rubik's Cubes.

2 Notations and Terminologies

There are some terms that will be used to refer to the elements of a Rubik's Cube.

- Cube: a short-handed version of Rubik's Cube
- Cubie/piece: refers to the individual cuboid that makes up a cube. e.g. a Rubik's Cube has 9 cubies/pieces on a face.
- Face: refers to an entire face of the whole Rubik's Cube. e.g a Rubik's Cube has 6 faces.
- Facet: refers to a face of a cubie that is visible to the player. e.g an edge cubie has 2 facets, and a corner cubie has 3 facets.

In the following report, we will adapt the Singmaster Notation for Rubik's Cube, which uses a letter for each of the quarter (90°) clockwise face turns; i.e.

- U: up face turn
- D: down face turn
- L: left face turn
- R: right face turn
- F: front face turn
- B: back face turn

3 The Rubik's Cube Group

Using Group Theory we can analyze a Rubik's cube and study its various permutations. The Rubik's Cube Group, denoted by (G, $\cdot$), is a multiplicative group that includes all the possible permutations of a 3×3×3 Rubik's cube. The operations done to the group are the turns of faces of the Rubik's cube.

3.1 Order

The order of a group is defined to be the cardinality of the underlying set. We can calculate the order of the Rubik's Cube Group by looking at the possible permutations of the pieces. There are a total of 8 corner pieces, giving 8! possible permutations, and each of them can be oriented 3 ways, so we have $8! \cdot 3^8$ permutations. There are 12 edge pieces, each of which has 2 possible orientations, so we get $12! \cdot 2^12$.

However, we have to eliminate the permutations that are not solvable (imagine if you have a solved Rubik's cube, but take one corner piece and flip it, there is no way to solve the resulting cube). In fact, only $\frac{1}{3}$ of all the corner permutations are solvable; only $\frac{1}{2}$ of the permutations have the same edge-flipping orientation as the original cube, and only $\frac{1}{2}$ of these have the correct cubie-rearrangement parity.

Therefore, we get the order of the group by the multiplication principle and the division principle:

$$\frac{8! \cdot 3^8 \cdot 12! \cdot 2^{12}}{3 \cdot 2 \cdot 2} = 43,252,003,274,489,856,000 \approx 4.3252 \times 10^19 \approx 43 \text{ quintillion}$$

Suppose that we number each facet, then we get $6 \cdot 9 = 54$ facets. So clearly, the Rubik's Cube group is a subgroup of the permutation group S54. Furthermore, we have that the Rubik's Cube group is a proper subgroup.

3.2 Generation

Since we solve/scramble the Rubik's Cube by turning the six faces, we know that the Rubik's Cube Group can be generated by the six face turns {U, D, L, R, F, B}. However, despite a complex formula, it is possible to perform a face turn with only the other five face turns. Therefore, the Rubik's Cube groups can be generated by {U, D, L, R, F}

3.3 Parity

Theorem 1. *All permutations in the Rubik's Cube group are even. (Note: here the permutation refers to the permutations of the cubies, not the facets.)*

Proof. Since every permutation consists of a sequence of face turns, we can prove by showing that every face turn is even. A 90° turn rotates the 4 corner cubies as well as the 4 edge cubies, leaving the center one fixed, so we have 2 4-cycles, each of which can be written as a product of 3 2-cycles (proof can be found in basic group theory textbooks). So in the end we get an even permutation on the cubies. □

This theorem states an interesting property of the Rubik's cube group. We will use tools from Sage and Python to explore this theorem and further discuss special cases for Higher-Order Rubik's Cube.

3.4 Sage's Rubik's cube group functions

A Rubik's cube can be created in Sage either as a Permutation group or simply using the RubiksCube object that is provided. The RubiksCube object has various display methods and a solve method which has multiple algorithms implemented.

4 Solving a Rubik's Cube

As mentioned earlier, the order of the Rubik's Cube group is close to 43 quintillion, each of which can be achieved by a specific sequence of moves. Of all these permutations, one is what we called the "solved" Rubik's cube. So, a very naive algorithm would involve going through all of these possible permutations following a fixed sequence of moves to arrive at a solved Rubik's cube. This would not be an optimal solution but guarantees a solved cube.

4.1 God's Number

God's Number refers to the maximum number of moves required to solve a Rubik's cube. In July 2020, this number was proved to be 26 or 20 (depending on how you define each turn) i.e. all of the 43 quintillion permutations of a Rubik's cube can be solved using at most 26 quarter turns or 20 half turns.

Mathematicians expected this number to be smaller than 20 because of the following two reasons:

- Out of the 43 quintillion permutations of the Rubik's cube, only 490 million require 20 moves to be solved.
- Out of the 43 quintillion permutations of the Rubik's cube, 1.5 quintillion require only 19 moves to be solved.

Although God's number may not give us an optimal solution or a fixed algorithm to solve all the possible permutations of a Rubik's cube, it could help us figure out the best algorithm to solve a particular permutation; an algorithm that requires at most 20 moves. If we count only the face moves referred to by the Singmaster notation, which is referred to as quarter turns, then God's number is 26.

4.2 The Super-Flip

The super-flip arrangement of a Rubik's cube is the most famous arrangement that requires 20 moves to solve. It was the first position that was found that could not be solved in less than 20 moves. It is interesting because every single corner cube is solved i.e. it is in its place, and every single edge is flipped in its place.

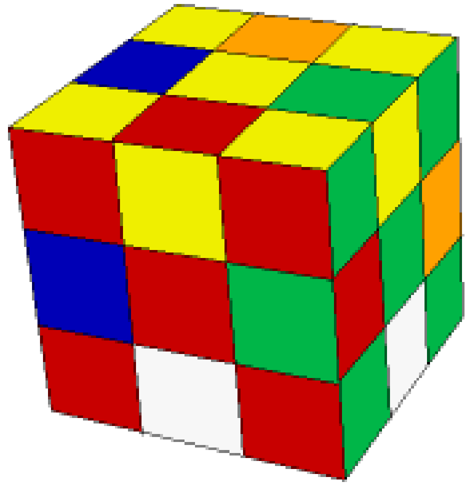

Figure 1: The Superflip Orientation

Using the method provided in Sage, we were able to get the center of the Rubik's Cube group; it turns out that the superflip orientation is, in fact, the only non-trivial element in the center.

There are several well-defined algorithms that can be used to solve a Rubik's cube. Some of these are:

4.3 Computer Algorithms

4.3.1 Thistlethwaite's Algorithm

Thistlethwaite's Algorithm utilizes group theory. It categorizes a permutation of a Rubik's cube into the five subgroups

- $G_0 = $ <L, R, F, B, U, D>
- $G_1 = $ <L, R, F, B, U2, D2>
- $G_2 = $ <L, R, F2, B2, U2, D2>
- $G_3 = $ <L2, R2, F2, B2, U2, D2>
- $G_4 = $ <1>

where G_0 is the entire group, and G4 only contains the solved state. The intermediate subgroups restrict some face turns to only 180° turns. The algorithm will permute the Rubik's Cube from the top subgroup to the bottom subgroup.

This algorithm is very useful, as many Rubik's Cube solvers as well as the optimal solution finders utilize it.

4.3.2 Kociemba's Algorithm

Kociemba's Algorithm is similar to Thistlethwaite's Algorithm, but with fewer intermediate steps.

- $G_0 = <\text{L, R, F, B, U, D}>$
- $G_1 = <\text{L2, R2, F2, B2, U, D}>$
- $G_2 = <1>$

4.3.3 Korf's Algorithm

Korf's algorithm utilizes a search algorithm named iterative-deepening-A* (IDA*) with a lower- bound heuristic function based on large memory-based lookup tables. These tables store the exact number of moves required to solve various subsets of the individual movable cubies.

4.4 Algorithms by Hand

4.4.1 CFOP

This is the most common method for a human to solve the cube. Each letter stands for a step in the algorithm. The four steps are: cross, first two layers (FLL), orienting the last layer (OLL), permuting the last layer (PLL), and they are illustrated below, where the uncolored cubes are the ones that need to be solved.

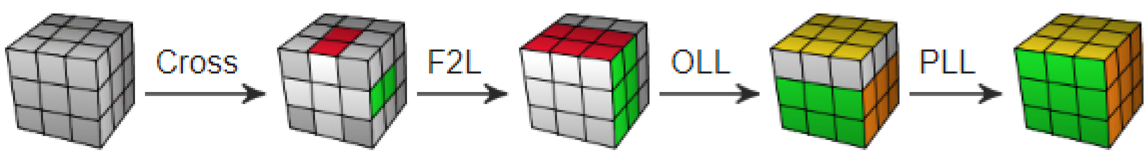

Figure 2: Illustration of CFOP Algorithm

Although there does not seem to be a direct contact of solving the cube, there are connections between this algorithm and group theory. As mentioned above, all permutations of the Rubik's Cube are even. As a consequence, in the last step of solving the cube (PLL), all the cases are either:

- Interchanging an even number of pairs (2-cycles)
- Any number of 3-cycles

The following picture shows some cases involved in PLL, which we could see fall into one of the categories above.

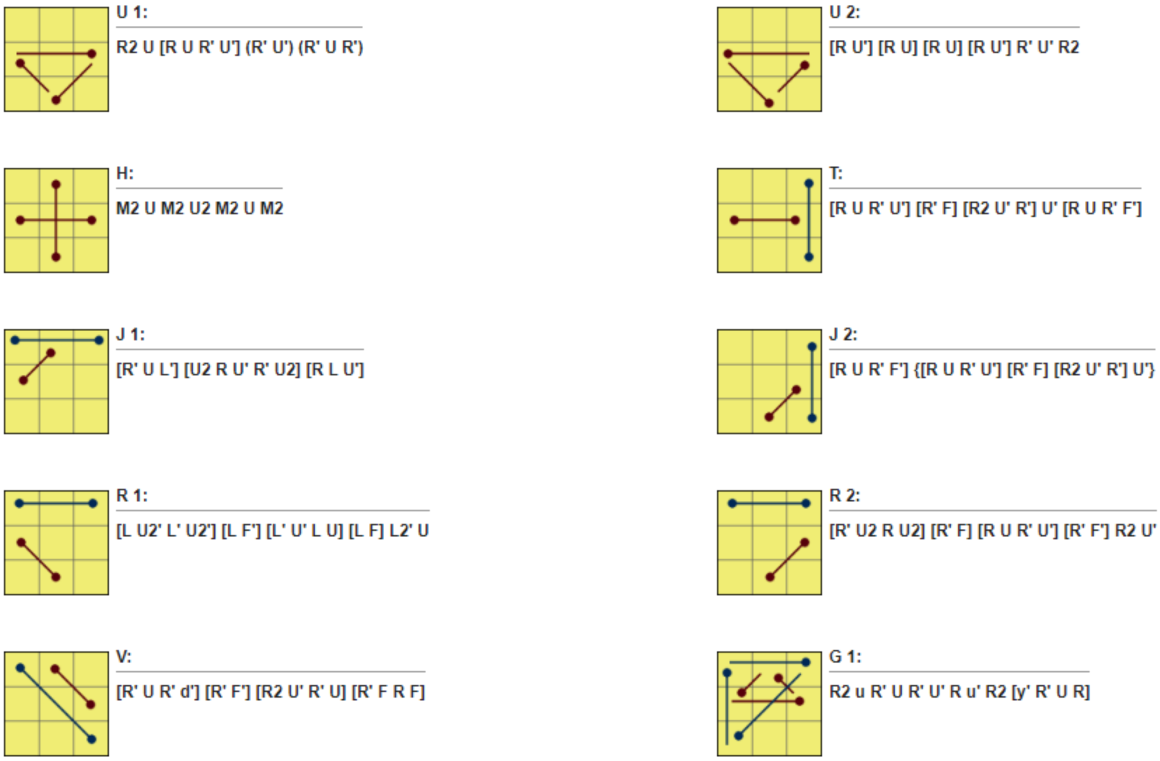

Figure 3: Formulas for Ten of The PLL Cases

4.4.2 Blindfolded Algorithm

A more challenging way to solve the cube is to solve it blindfolded. The algorithm to do so involves: orient the corner and edge pieces, permute the corner and edge pieces. The cube will not be solved step by step, but nevertheless the formulas in the CFOP algorithm could be useful.

Consider the case for rotating three edge pieces that are not in the same layer.

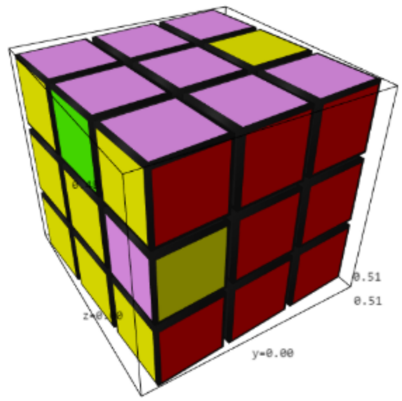

Figure 4: A 3-Edge-Cubie Rotation Not On The Same Layer

If we rotate the front face first, we get the three pieces on one layer; then the formula $U1$ or $U2$ above can be used. After that we could reverse the front face turn. So the algorithm looks like $F(U1)F^{-1}$, which is a conjugation operation of $U1$. From group theory we know that conjugation is an equivalence relationship, and in fact, the two conjugacy classes of $U1$ and $U2$ partition all the possible 3-edge rotations in the cube. So we could use only two formulas to solve most situations.

4.4.3 Higher-Ordered Cubes Algorithm

Rubik's cubes come in many additional sizes. Sage does not have a built-in object for them, although one could use permutation groups for some purposes. A common size is the $4\times4\times4$ cube, called "Rubik's Revenge". In the following we would refer to it as a fourth-order Rubik's Cube.

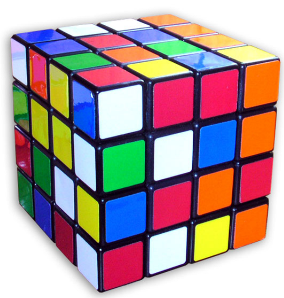

Figure 5: A Rubik's Revenge

Similar to solving some higher-ordered polynomials, the algorithm to solve a higher-ordered cube involves reducing its order, by ways of combining the same color on the edges and centers. For example, after combining the colors, the fourth-order Rubik's Cube looks like:

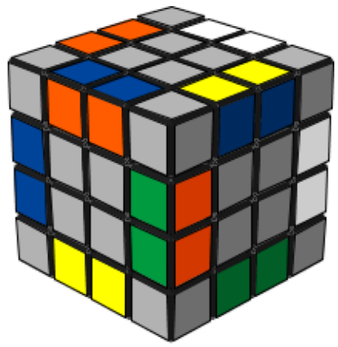

Figure 6: A Rubik's Revenge After Order Reduction

Now there is a homomorphism from the regular Rubik's Cube group to the 4×4×4 restricted to only moving the combined pieces together. However it is not an onto map, since there are special cases:

Consider a case in the last step (PLL) of solving a regular (3×3×3) Rubik's cube, where the three edge pieces need to be rotated.

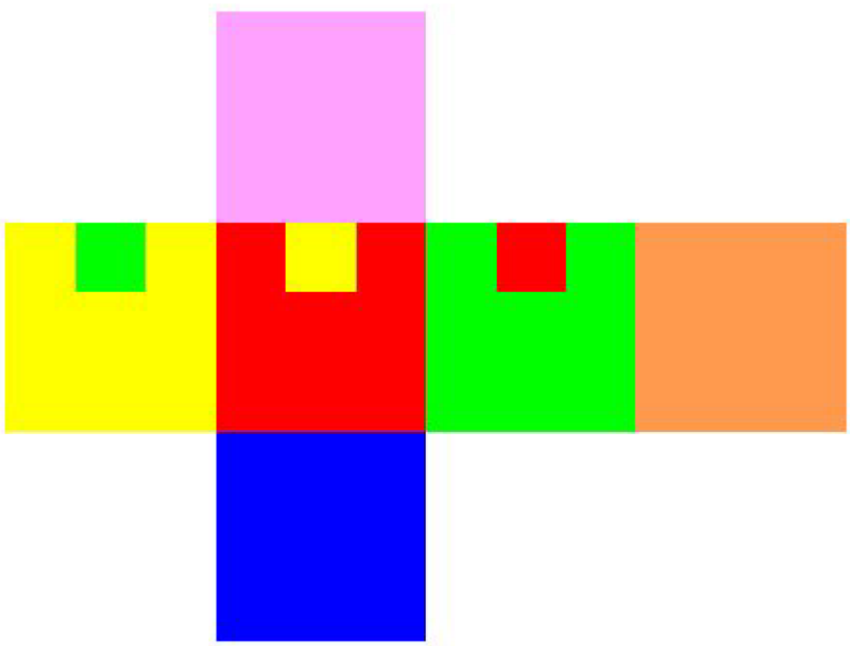

Figure 7: 3-Piece Rotation

As stated before, this is an even and valid permutation of the cube. Similar to a regular Rubik's Cube, a fourth order cube's permutations are all, in fact, even. We could show this using similar arguments as before. However, in the last step there are some possible cases which are not permitted for the regular cube, which are shown in the following picture

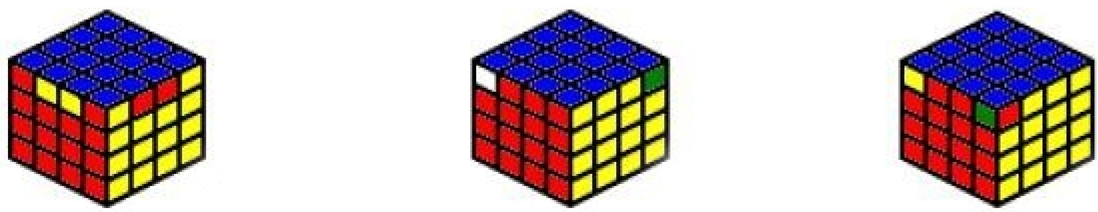

Figure 8: Special Cases in PLL for Rubik's Revenge

In the left-most case, we are switching two edges, which would not happen in the regular case. In a regular cube's case, switching two edges only involves switching 2 cubies, making such permutation odd. However, here there are two pairs of cubies being interchanged, so the permutation is even and permitted.
In the other two cases, it seems like that there are only two pieces being interchanged, making such permutation prohibited. This is due to the fact that for the fourth order cube, there is no one single cubie that stays right at the center of each face and does not move the entire time, instead, there are four center pieces which are indistinguishable and allowed to move around. The algorithm for solving these two special cases also involves interchanging a pair of cubies at the center, making it even.

To generalize, these special cases will occur if the order of the Rubik's Cube is even. They will not occur of Rubik's Cube of odd orders, since they all have fixed center cubies and switching an odd number of pairs of cubies is not allowed.

4.5 SAT Solvers

SAT stands for satisfiability. An SAT solver is used to solve decision problems i.e. problems that require answers in the form of boolean values. Such a problem has the following elements: **Variables**, **Literals** (which are variables and their negations), **Clauses** (which are sets of literals), and **Assignments** (which can be seen as a dictionary in which the keys are variables and the values are boolean values). How a clause is **satisfied** depends on the problem itself. By considering all possible clauses or conditions we do want, an SAT Solver can be used to output a clause that would satisfy all the given clauses.

Consider a 2×2×2 Rubik's cube. It has 24 squares each having an individual color and 6 4-squared faces. We label the faces using the dictionary {0, 1, 2, 3, 4, 5} and label the 6 colors using the dictionary {A, B, C, D, E, F}.

The idea is to run an SAT solver on each of the 6 faces such that if all colors on that face are different (say A, B, C, and D), the variables of the SAT Solver are initially set to be {A, B, C, D}. The program randomly picks two colors, say A and B, and replaces all colors in the square that aren't A with B. Now the SAT Solver's variables are updated to be {A, B}.

By doing so, we are picking A to be the color that we want this face to be. Now, we have our first clauses {A, A, A, A} and {∼B, ∼B, ∼B, ∼B}. The remaining clauses would be dependent on the structure of the Rubik's cube i.e. how it can be rotated to get particular colors at the four positions on this specific face. One way of understanding this is by looking at how it can be rotated. From the diagram we can see the directions it can be rotated in and each unit refers to a shift of one square

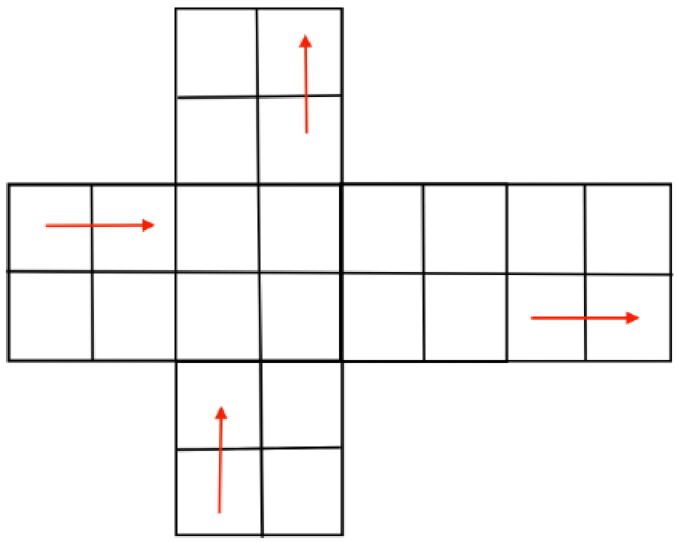

Figure 9: SAT Solver Implemented on A 2×2×2 Rubik's Cube

The purpose of having such an SAT Solver is to find a way of solving the Rubik's Cube in less time, so this idea can be extended to Cubes that do not have Sage functions to solve, such as 4x4x4 cubes.

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