

# EFFICIENT MONOTONE SUBMODULAR MAXIMIZATION SUBJECT TO MATROID CONSTRAINTS IN SUBMODLIB

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## ABSTRACT

Maximization of monotone submodular functions subject to cardinality constraints has been a topic of interest with various greedy algorithms, with many of these algorithms achieving a  $(1 - \frac{1}{e})$  approximation on the optimal solution (OPT) at high computational speeds. However, these variants of classical greedy do not work well on matroid constraints, and only provide a  $\frac{1}{2}$  approximation of OPT. To overcome this, CONTINUOUS-GREEDY was introduced in Calinescu et al. (2011) which gives a  $(1 - \frac{1}{e} - \epsilon)$  approximation of OPT but it has a  $O(n^8)$  running time. An improvement to the algorithm is ACCELERATED-CONTINUOUS-GREEDY introduced in Badanidiyuru & Vondrák (2014) has  $\tilde{O}(n^2)$  running time and is much more efficient than the previous version. SUBMODLIB Kaushal et al. (2022) is an efficient and scalable Python library for submodular optimization in Python with a C++ optimization engine, and we plan to implement this algorithm along with corresponding submodular functions into SUBMODLIB.

## 1 INTRODUCTION

In the recent years, a lot of attention has been given to optimization of functions which give diminishing returns, termed as submodular functions. The reason mainly being - they arise naturally when dealing with certain kinds of problems. For example, the task of placing sensors in an area to maximize the amount of information gained from them has been shown to be submodular Guestrin et al. (2005). Similarly, influence maximization over social network graphs also has close relations to submodularity Kempe et al. (2003). Given a submodular function  $f$  and a constraint  $\mathcal{F}$ , we are interested in the optimization problem given as

$$\max_S f(S) \text{ over } S \in \mathcal{F} \quad (1)$$

The type of constraint usually depends on the problem we are tackling. In case of sensor placement or influence maximization, we have a cardinality constraint, i.e  $|S| \leq k$ . In case of welfare maximization, we have a matroid constraint, i.e  $S \in \mathcal{M}$  where  $\mathcal{M}$  is a matroid. It has been shown by Nemhauser et al. (1978) that for submodular functions which are monotone, a simple greedy algorithm works well under cardinality constraint and provides at least  $(1 - \frac{1}{e})$  approximation of the optimal solution. However, it has been shown by Nemhauser et al. (1978) that for the matroid constraints, i.e given  $\mathcal{M} = (E, \mathcal{I})$ , the problem

$$\max_{S \in \mathcal{I}} f(S) \quad (2)$$

the greedy solution provides a  $\frac{1}{2}$  approximation of the optimal solution. Hence for a complicated constraint, greedy algorithm doesn't work well anymore. The CONTINUOUS-GREEDY algorithm

proposed by Calinescu et al. (2011) provides a  $(1 - \frac{1}{e})$  solution to the above problem by converting the discrete optimization problem to a continuous one using multilinear relaxation and converting the solution obtained there back into a discrete one. The main benefit of multilinear relaxation is the usage of convexity and concavity, and the above problem is transformed to

$$\max\{F(y) : y \in P(\mathcal{M})\} \quad (3)$$

where  $P(\mathcal{M})$  is the matroid polytope of  $\mathcal{M}$ . Hence, it provides a two-step method to get the required approximation of OPT. One drawback of CONTINUOUS-GREEDY is that the running time of the algorithm is  $\tilde{O}(n^8)$  with  $\tilde{O}(n^7)$  oracle calls. An improvement to the above algorithm, called ACCELERATED-CONTINUOUS-GREEDY was proposed by Badanidiyuru & Vondrák (2014) which has a  $\tilde{O}(n^2)$  oracle calls and achieves a  $(1 - \frac{1}{e} - \epsilon)$  approximation. This variant is faster than the previous proposed speedup solutions which have  $\tilde{O}(n^4)$  oracle calls. This speedup can be related to how STOCHASTIC-GREEDY speeds up CLASSICAL-GREEDY Mirzasoleiman et al. (2014).

There are a variety of problems which are based around matroid-constraint-based optimization. One of them was stated in 2, and three related functions - Submodular Welfare Problem (SWP), Separable and Generalized Assignment Problem (SAP/GAP), have been described in Section 2.

SUBMODLIB Kaushal et al. (2022) is a submodular optimization library with a C++ optimization engine and has implemented many submodular functions, such as Facility Location, Graph Cut etc. along with constraint-based optimizers. However, currently there is no support for any matroid-based optimizers or matroid-constraint-based functions in the library. In this project, we introduce formally the problem at hand, and implement the functions and optimizers in the SUBMODLIB compatible format which can be merged with the main library.

## 2 PRELIMINARIES

### 2.1 SUBMODULAR FUNCTIONS

A set function  $f : 2^V \rightarrow \mathbb{R}$  is said to be *submodular*, if for all  $A, B \subseteq V$ ,

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

This can equivalently be stated as, for  $C \subseteq D \subseteq V$  and any  $i \in V \setminus D$ ,

$$f(C \cup \{i\}) - f(C) \geq f(D \cup \{i\}) - f(D) \equiv \Delta(i|C) \geq \Delta(i|D)$$

A set function  $f$  is said to be *monotone* if  $f(A) \geq f(B)$  whenever  $A \supseteq B$ . The set function  $f$  is said to be *non-negative* if the co-domain of  $f$  is  $\mathbb{R}_+$ . When considering the optimization problem at hand, we will consider non-negative monotone submodular functions.

The notion of submodular functions is extended with smoothness by Wolsey (1982). A function  $F : [0, 1]^V \rightarrow \mathbb{R}_+$  is *smooth submodular* if

$$F(x) + F(y) \geq F(x \vee y) + F(x \wedge y)$$

where  $(x \vee y) = \max\{x, y\}$  and  $(x \wedge y) = \min\{x, y\}$ , both in an coordinate-wise fashion.  $F$  is *monotone* if for  $x \leq y$  coordinate-wise,  $F(x) \leq F(y)$ .

The above can be stated together as follows,  $F : [0, 1]^V \rightarrow \mathbb{R}$  is smooth monotone submodular if

1. (Smoothness)  $F \in \mathcal{C}_2([0, 1]^V)$ , i.e the second-order partial derivatives exists everywhere
2. (Monotonicity) For each  $j \in V$ ,  $\frac{\partial F}{\partial y_j} \geq 0$  everywhere
3. (Submodularity) For any  $i, j \in V$ ,  $\frac{\partial^2 F}{\partial y_i \partial y_j} \leq 0$  everywhere

### 2.2 MATROIDS

A matroid  $\mathcal{M} = (E, \mathcal{I})$  is a structure with a finite ground set  $E$ , the universe and a collection of subsets of  $2^E$  (power set of  $E$ ),  $\mathcal{I}$  called independent sets, such that

1.  $\emptyset \in \mathcal{I}$

2.  $\forall I \in \mathcal{I}$  and  $J \subseteq I, J \in \mathcal{I}$
3. (Exchange axiom)  $\forall I, J \in \mathcal{I}$  and  $|I| < |J|$ , there exists  $x \in J \setminus I$  such that  $I \cup \{x\} \in \mathcal{I}$

A base  $B$  of a matroid  $\mathcal{M}$  is a maximal independent set, and the exchange axiom guarantees that all bases of  $\mathcal{M}$  have the same cardinality. Some examples of matroids are -

1. **k-Uniform Matroid:** A matroid  $\mathcal{M} = (E, \mathcal{I})$  is *k-Uniform* if
 
$$\mathcal{I} = \{X \subseteq E : |X| \leq k\}$$
2. **Partition Matroid:** A matroid  $\mathcal{M} = (E, \mathcal{I})$  is a partition matroid if  $E$  is partitioned into disjoint sets  $E_1, \dots, E_e$  and
 
$$\mathcal{I} = \{X \subseteq E : |E_i \cap X| \leq k_i \text{ for } i = 1, \dots, e\}$$

The rank function of a matroid  $\mathcal{M} = (E, \mathcal{I})$  is a set function  $r_{\mathcal{M}} : 2^E \rightarrow \mathbb{N}$  such that for any  $S \subseteq E$ ,

$$r_{\mathcal{M}}(S) = \max\{|X| : X \subseteq S, X \in \mathcal{I}\}$$

As an analogy, it can be thought to be similar to the rank function of matrices, or dimension in vector spaces. For example, for the *k-Uniform* matroid defined above,  $r_{\mathcal{M}}(S) = \min\{|S|, k\}$  and the rank of the *partition* matroid is  $\sum_i k_i$ .

The matroid-rank theorem states that  $r : 2^E \rightarrow \mathbb{N}$  is a rank function for a matroid *if and only if*

1.  $r(\emptyset) = 0$  and  $r(A \cup \{e\}) - r(A) \in \{0, 1\} \forall A \subseteq E, e \in E$
2.  $r$  is submodular, i.e for any  $S, T \subseteq E$  we have  $r(S) + r(T) \geq r(S \cup T) + r(S \cap T)$

Having defined the matroid rank function, we can finally state what a matroid polytope is. Given a a matroid  $\mathcal{M} = (E, \mathcal{I})$ , the matroid polytope  $P(\mathcal{M})$  is defined as

$$P(\mathcal{M}) = \text{conv}(\{x_S \in \mathbb{R}^{|E|} : S \in \mathcal{I}\})$$

We can also write that

$$P(\mathcal{M}) = \{x \in \mathbb{R}^{|E|} : x(S) \leq r(S) \forall S \subseteq E, x_e \geq 0 \forall e \in E\}$$

where  $x(S) = \sum_{e \in S} x_e$ . Now, the base of a matroid can be defined as the set  $S \in \mathcal{I}$  such that  $r_{\mathcal{M}}(S) = r_{\mathcal{M}}(E)$  and the base polytope is defined as

$$B(\mathcal{M}) = \left\{y \in P(\mathcal{M}) : \sum_{i \in E} y_i = r_{\mathcal{M}}(E)\right\}$$

### 2.3 MULTILINEAR EXTENSION

For a monotone submodular set function  $f : 2^V \rightarrow \mathbb{R}_+$ , a canonical extension to a smooth monotone submodular function can be obtained as follows. For  $y \in [0, 1]^V$ , let  $\hat{y}$  denote a random vector in  $\{0, 1\}^V$  where each coordinate is independently rounded to 1 with probability  $y_j$ .  $\hat{y} \in \{0, 1\}^V$  can be identified with  $R \subseteq V$  with the indicator given as  $\hat{y} = \mathbf{1}_R$ . Then the multilinear function  $F$  (multilinear  $\implies \frac{\partial^2 F}{\partial y_j^2} = 0$ ) can be defined as

$$F(y) = \mathbb{E}[f(\hat{y})] = \sum_{R \subseteq V} f(R) \prod_{i \in R} y_i \prod_{j \notin R} (1 - y_j)$$

### 2.4 SUBMODLIB

As described in Section 1, SUBMODLIB is a python library for submodular optimization with a C++ optimization engine. The current optimizers in the library include CLASSICAL-GREEDY, LAZY-GREEDY and STOCHASTIC-GREEDY all of which provide a  $(1 - \frac{1}{e})$  approximation of OPT in the case of cardinality constraints, with a running time of atmost  $O(kn)$ . A list of common submodular, mutual information, conditonal gain and clustering functions have also been implemented which are based around cardinality constraints. The usage of the library can be seen on their GitHub page, but in short the functions and optimizers are exposed separately and for cardinality constraints are interchangeable. This is slightly in contrast with how matroid-constraint-based optimizers are formulated, and we describe that now.

## 2.5 MATROID CONSTRAINTS

The constraint  $\mathcal{F}$  can be either simple, such as the cardinality constraint, or more complex such as a matroid or knapsack constraint. In particular, 2 is of importance and has been studied previously. In fact, there is a randomized algorithm which gives a  $(1 - \frac{1}{e})$  approximation to the problem  $\max\{f(S) : S \in \mathcal{I}\}$  where  $f : 2^E \rightarrow \mathbb{R}$  is a monotone submodular function given by a value oracle and  $\mathcal{M} = (E, \mathcal{I})$  is a matroid given by the membership oracle Calinescu et al. (2011). Apart from 2, there are some other matroid-constraint-based submodular problems of great interest as follows.

### 2.5.1 SUBMODULAR WELFARE PROBLEM

Firstly, the general framework for the welfare problem is called the Social Welfare Maximization Problem and is shown to come up often in combinatorial auctions Lehmann et al. (2002). Given a set  $X$  of  $m$  items, and  $n$  players each of which has a utility function  $w_i : 2^X \rightarrow \mathbb{R}_+$ , the goal is to partition  $X$  into disjoint subsets  $S_1, \dots, S_n$  in order to maximize the social welfare given as  $\sum_{i=1}^n w_i(S_i)$ . When the utility function  $w_i$  are submodular, the problem is of great interest, as it has been shown that CLASSICAL-GREEDY obtains a  $\frac{1}{2}$  approximation of OPT Nemhauser et al. (1978) and further has been shown that a  $(1 - \frac{1}{e} + \epsilon)$  approximation implies  $P = NP$  Khot et al. (2005). However, the problem can be reduced to a matroid-constraint-based problem with  $(1 - \frac{1}{e})$  approximation of OPT. The reduction following Calinescu et al. (2011) is as follows -

- (Step 1) Given the set of items  $A$  and  $n$  players, the ground set is defined as  $X = [n] \times A$ , i.e. one clone of each item for each player. For each player, define the mapping  $\pi_i : 2^X \rightarrow 2^A$  which takes all clones in  $S$  to corresponding player as

$$\pi_i(S) = \{j \in A : (i, j) \in S\}$$

- (Step 2) Given the utility functions  $w_1, \dots, w_n : 2^A \rightarrow \mathbb{R}_+$ , define  $f : 2^X \rightarrow \mathbb{R}_+$  as

$$f(S) = \sum_{i=1}^n w_i(\pi_i(S))$$

which is submodular if  $w_i$  are submodular.

- (Step 3) We need to partition  $A = S_1 \cup \dots \cup S_n$  such that  $\sum_i w_i(S_i)$  is maximized. This problem is equivalent to finding  $S = \bigcup_{i=1}^n (\{i\} \times S_i) \subseteq X$  containing at most one clone of each item, so that  $f(S)$  is maximized. Hence, we have the independence set as (for  $X_j = [n] \times \{j\}$ )

$$\mathcal{I} = \{S \subseteq X : \forall j; |S \cap X_j| \leq 1\}$$

Thus, we have reduced the SWP to the form of 2.

### 2.5.2 GENERALIZED AND SEPARABLE ASSIGNMENT PROBLEMS

The Separable Assignment Problem consists of  $m$  items and  $n$  bins. Each bin has a collection of feasible sets  $F_j$  satisfying down-closure (if  $A \in F_j$  and  $B \subseteq A$ , then  $B \in F_j$ ). Each item has a value  $v_i^j$  depending on the bin it is placed in. In SAP, we need to choose disjoint feasible sets  $S_j \in F_j$  to maximize  $\sum_j \sum_i v_i^j$ . The reduction to matroid constraint following Calinescu et al. (2011) is as follows -

- (Step 1) Define the ground set  $X = \{(j, S) : 1 \leq j \leq n, S \in F_j\}$ , i.e. for each bin, create a separate copy of the items which are packed only in  $j$

- (Step 2) Define the function  $f : 2^X \rightarrow \mathbb{R}_+$  which is monotone submodular as

$$f(S) = \sum_i \max_j \{v_i^j : \exists (j, S) \in S, i \in S\}$$

- (Step 3) The matroid constraint is  $\mathcal{M} = (X, \mathcal{I})$  where  $S \in \mathcal{I}$  if and only if  $S$  contains at most one pair  $(j, S_j)$  for each  $j$ .

The Generalized Assignment Problem is a special case of SAP, where the feasible set is a knapsack constraint given as  $F_j = \{S : \sum_i s_i^j \leq 1\}$ . The rest of the analysis stands same as SAP.

### 3 METHODS AND ANALYSIS

#### 3.1 CONTINUOUS-GREEDY

As seen in Section 2, multilinear relaxation brings the discrete problem to a continuous space, and the optimization is performed there. There are two questions to answer now - how to approximate the function in the continuous space and how to bring it back to the discrete domain. To do this, we have two steps

- The *continuous greedy* process approximates  $\max\{F(y) : y \in P(\mathcal{M})\}$  within a  $(1 - \frac{1}{e})$  factor
- The *pipage-rounding* algorithm converts a fractional solution to a discrete solution satisfying  $f(S) \geq F(y) \geq (1 - \frac{1}{e})OPT$

The continuous greedy process functions with an overtime flow given as

$$\frac{dy}{dt} = v_{max}(y) \quad \text{where } v_{max} = \arg \max_{v \in P} (v \cdot \nabla F(y)) \quad (4)$$

It has been proved in Calinescu et al. (2011) that  $y(1) \in P$  and  $F(y(1)) \geq (1 - \frac{1}{e})OPT$ . In the earlier version of algorithms following the *continuous greedy* process,  $v_{max} = e_j$  and that produces a  $\frac{1}{2}$  approximation.

The *pipage rounding* technique was introduced in Ageev & Sviridenko (2004) and the goal is go from a solution  $y^* \in P(\mathcal{M})$  to the discrete solution  $S \in \mathcal{I}$ .

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**Algorithm 1:** Continuous Greedy

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```

1 function Euclid ( $a, b$ );
   Input : Two nonnegative integers  $a$  and  $b$ 
   Output:  $\gcd(a, b)$ 
2 if  $b = 0$  then
3   | return  $a$ ;
4 else
5   | return Euclid( $b, a \bmod b$ );
6 end
```

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#### 3.2 ACCELERATED-CONTINUOUS-GREEDY

Although CONTINUOUS-GREEDY works and provides the required approximation, the running time is  $\tilde{O}(n^8)$  due the repeated sampling and small time steps in the algorithm. Hence, we need a faster algorithm for any practical usage. The ACCELERATED-CONTINUOUS-GREEDY presented in Badanidiyuru & Vondrák (2014) interpolates the fast CLASSICAL-GREEDY with the slow CONTINUOUS-GREEDY with a  $\delta$  parameter such that  $\delta = 1$  corresponds to the former and  $\delta \in (0, 1)$  corresponds to discretezed version of the latter. The main reason why the acceleration is provided is because of updating partial derivatives after each increment, which gives a cleaner analysis and mimicks the discrete greedy algorithm. Along with that, due to the discrete greedy nature, we take larger steps and thus need fewer samples per iteration, and fewer iterations. Moreover, any  $\delta > 0$  gives a  $(1 - \frac{1}{e} - \mathcal{O}(\delta))$  approximation.

The implementation of ACCELERATED-CONTINUOUS-GREEDY currently utilizes the Swap-Rounding function, which we found to be inefficient. Along with that, the Pipage-Rounding function in CONTINUOUS-GREEDY can also have improvements. Hence, we enhance the working of the algorithm, and switch the Swap-Rounding procedure with our optimized Pipage-Rounding procedure.

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**Algorithm 2:** Accelerated Continuous Greedy

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```

1 function Euclid ( $a, b$ );
   Input : Two nonnegative integers  $a$  and  $b$ 
   Output:  $\gcd(a, b)$ 
2 if  $b = 0$  then
3 |   return  $a$ ;
4 else
5 |   return Euclid( $b, a \bmod b$ );
6 end

```

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## 4 IMPLEMENTATION AND CHANGES

## 5 FUTURE WORK

We have currently implemented the optimizers and functions in our fork of SUBMODLIB. The next step is allowing memoization to happen and merging with the main branch, after discussions with the SUBMODLIB team.

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