1 Notation and Definitions

All definitions are tailored to single-gameweek FPL optimization: maximizing expected points under budget (£100m), squad composition (15 players: 2 GK, 5 DEF, 5 MID, 3 FWD), club limits (≤ 3 per club), lineup formation, and bench rules (1 GK, 3 outfield).

1.1 Sets

- P: Set of all players.
- C: Set of clubs.
- $G \subset P$: Goalkeepers (GK).
- $D \subset P$: Defenders (DEF).
- $M \subset P$: Midfielders (MID).
- $F \subset P$: Forwards (FWD).
- $S \subset P$: Set of starting XI players.
- $B \subset P$: Set of bench players.
- $Q = S \oplus B \subset P$: Set of all squad players.

1.2 Parameters

- $q_i \in [0, 1]$: No-show probability for player $i \in P$.
- $p_i \in \mathbb{R}$: Cost of player $i \in P$.
- $u_i \in \{0, 1\}$: 1 if player $i \in P$ is unavailable for the upcoming gameweek, 0 if available.
- $y_i^0 \in \{0, 1\}$: 1 if player i is in current squad (known state, for Post-GW1).
- $B_{\text{bank}} \in \mathbb{R}^+ \cup \{0\}$: Cash in bank (known state, for Post-GW1).
- $p_{\text{buy},i}^0 \in \mathbb{R}^+ \cup \{0\}$: Purchase price paid for player $i \in P$
- $sp_i \in \mathbb{R}^+$: Selling price for player $i \in P$.
- $f \in \{0, 1, 2, 3, 4, 5\}$: Free transfers (1 + banked, ≤ 5).
- $\text{expected\_points}_i = \text{predicted\_points}_i \times (1 - q_i)$: Expected points for player $i \in P$ in upcoming gameweek.

1.3 Variables

- $x_i \in \{0, 1\}$: 1 if player $i \in P$ is in the starting XI .
- $c_i \in \{0, 1\}$: 1 if player $i \in P$ is captain .
- $y_i \in \{0, 1\}$: 1 if player $i \in P$ is in the squad .
- $t_i \in \{0, 1\}$: 1 if buy player i (transfer in).
- $s_i \in \{0, 1\}$: 1 if sell player i (transfer out).
- $e \in \{0, 1, 2, \ldots, 15\}$: Number of extra transfers beyond f (penalty).

2 Approach for pre-GW1 and post-GW1 Optimization

This section presents a solver-independent strategy for FPL optimization. This decomposition is independent of solver, making it a general algorithmic approach rather than a model-specific formulation.

2.1 Pre-GW1 Formulation (No Transfers)

2.1.1 Step 1: Optimize Starters and Captain

Decision variables: $y_i, x_i, c_i \in \{0, 1\}$ for all $i \in P$.

Solve the following optimization problem:

$$\max \quad \sum_{i \in P} \text{expected\_points}_i \cdot (x_i + c_i) \tag{1}$$

subject to constraints :

$$\begin{aligned}
\sum_{i \in P} x_i = 11, &\quad \text{(lineup size)} \\
\sum_{i \in G} x_i = 1, &\quad \text{(1 starting GK)} \\
3 \le \sum_{i \in D} x_i \le 5, &\quad \text{(3-5 starting DEF)} \\
2 \le \sum_{i \in M} x_i \le 5, &\quad \text{(2-5 starting MID)} \\
1 \le \sum_{i \in F} x_i \le 3, &\quad \text{(1-3 starting FWD)} \\
\sum_{i \in P} c_i = 1, &\quad \text{(exactly 1 captain)} \\
c_i \le x_i \quad \forall i \in P, &\quad \text{(captain must be starter)} \\
u_i + x_i \le 1 \quad \forall i \in P, &\quad \text{(unavailable players cannot be starter)} \\
\sum_{i \in P} y_i = 15, \quad \sum_{i \in G} y_i = 2, &\quad \sum_{i \in D} y_i = 5, \\
\sum_{i \in M} y_i = 5, \quad \sum_{i \in F} y_i = 3, &\quad \text{(squad quotas)} \\
\sum_{i \in P} p_i y_i \le 100, \quad \sum_{i \in P_c} y_i \le 3 \quad \forall c \in C, &\quad \text{(budget and club limits)} \\
x_i \le y_i \quad \forall i \in P &\quad \text{(starters must be in squad)}
\end{aligned}$$

Denote the optimal solution from Step 1 as x_i^*, y_i^*, c_i^*.

2.1.2 Post-hoc: Vice-Captain Assignment

Let $S = \{i \in P \mid x_i^* = 1\}$ denote the set of starting players.

After solving Step 1, assign the vice-captain as follows: From the starting lineup S, exclude the captain and select the player with the highest expected points among the remaining starters.

This ensures the vice-captain captures the best available backup bonus (e.g., for captain no-shows) without altering the optimization in Step 1.

2.1.3 Step 2: Optimize Bench Composition

Decision variables: $y_j \in \{0, 1\}$ for all $j \in P$.

State variable: $x_i^* \in \{0,1\}$ from Step 1 for all $i \in P$.
Solve the following optimization problem:

$$\max \quad \sum_{j \in P} \text{expected\_points}_j \cdot y_j \tag{2}$$

subject to :

$$\begin{aligned}
&\sum_{i \in P} y_i = 15 \quad \text{(total squad size)} \\
&\sum_{i \in G} y_i = 2, \quad \sum_{i \in D} y_i = 5, \quad \sum_{i \in M} y_i = 5, \quad \sum_{i \in F} y_i = 3 \quad \text{(position quotas)} \\
&\sum_{i \in P} p_i y_i \leq 100, \quad \sum_{i \in P_c} y_i \leq 3 \quad \forall c \in C \quad \text{(budget and club limits)} \\
&y_i = x_i^* = 1 \quad \forall i \in S \quad \text{(fix starters from Step 1)}
\end{aligned}$$

Denote the optimal solution from step 2 as y_i^{**}.

2.1.4 Post-hoc: Bench Position Assignment

Assign bench positions as follows to prioritize substitutions of high-expected-point players (bench order determines substitution priority):

1. Assign the backup goalkeeper: Place the single bench goalkeeper (the non-starting GK in the squad) into position 1, following the FPL rules.
2. Sort by expected points: Rank these three benched outfield players from highest to lowest expected points.
3. Assign to outfield slots: Place the highest-ranked player in position 2 (earliest substitute), the middle-ranked in position 3, and the lowest in position 4 (latest substitute). This ensures the best potential replacement enters first.

This ensures exactly 1 player per bench position, with GK in position 1 and outfield in positions 2-4.

2.1.5 Output for Pre GW1

The final output from the Pre-GW1 formulation consists of:

- Squad composition: $y_i^{**} \in \{0,1\}$ for all $i \in P$ (15 players).
- Starting lineup: $x_i^* \in \{0,1\}$ for all $i \in P$ (11 players).
- Captain: $c_i^* \in \{0,1\}$ for all $i \in P$ (1 player).
- Vice-captain: assigned post-hoc as per above.
- Bench positions: assigned post-hoc as per above.

2.1.6 Initial Condition for State Variables

After completing GW1 optimization, initialize the state variables for GW2:

$$y_i^0 \leftarrow y_i^{**} \quad \forall i \in P \quad \text{(GW1 squad becomes initial squad)} \tag{3}$$

$$B_{\text{bank}} = 100 - \sum_{i \in P} p_i y_i^{**} \quad \text{(remaining cash)} \tag{4}$$

$$f = 2 \quad \text{(1 free transfer per week, starts with 1 banked)} \tag{5}$$

$$p_{\text{buy},i}^0 = \begin{cases} p_i & \text{if } y_i^{**} = 1 \quad \text{(GW1 squad members)} \\ 0 & \text{if } y_i^{**} = 0 \quad \text{(all other players)} \end{cases} \quad \forall i \in P \tag{6}$$

These state variables form the complete initial state for the Post-GW1 formulation.

2.2 Post-GW1 Formulation (With Transfers)

For gameweeks after GW1, the current squad y_i^0 is known, and transfers are allowed subject to free transfer limit f and penalty -4 points per extra transfer.

Selling Price Calculation:

For each player $i \in P$ in the current squad ($y_i^0 = 1$), the selling price sp_i implements FPL's 50% profit lock mechanism:

$$sp_i = p^0_{\text{buy},i} + 0.5 \cdot \max(0, p_i - p^0_{\text{buy},i})$$

rounded up to nearest 0.1m if profit ($p_i > p^0_{\text{buy},i}$); otherwise $sp_i = p_i$ if loss ($p_i \leq p^0_{\text{buy},i}$).

This ensures:

- Players sold at a gain contribute only half their price increase (profit locked at 50%)
- Players sold at a loss contribute their full current price (losses fully realized)

Budget Dynamics:

Only new player purchases cost money; retained players are already owned and cost nothing. The budget constraint is derived from the requirement that cash must remain non-negative after transfers.

From the state update (Section 2.2.5):

$$B^{\text{new}}_{\text{bank}} = B_{\text{bank}} + \sum_{i \in P} sp_i s_i - \sum_{j \in P} p_j t_j$$

Requiring $B^{\text{new}}_{\text{bank}} \geq 0$ gives:

$$B_{\text{bank}} + \sum_{i \in P} sp_i s_i - \sum_{j \in P} p_j t_j \geq 0$$

Rearranging yields the budget constraint:

$$\sum_{j \in P} p_j t_j \leq B_{\text{bank}} + \sum_{i \in P} sp_i s_i$$

where p_j is the current market price for new purchases and sp_i is the locked selling price (with 50% profit lock) for any sold players.

2.2.1 Step 1: Optimize Starters, Captain, and Transfers

State Variables: The optimization uses four state variables carried over from the previous gameweek:

- $y_i^0 \in \{0, 1\}$: Current squad composition (15 players)
- $B_{\text{bank}} \in \mathbb{R}^+$: Cash in bank (remaining budget)
- $p^0_{\text{buy},i} \in \mathbb{R}^+ \cup \{0\}$: Purchase price for player $i \in P$ (0 if not in squad)
- $f \in \{0, 1, 2, 3, 4, 5\}$: Free transfers available (typically $f = 2$ for GW2)

Decision variables: $x_i, y_i, c_i, t_i, s_i \in \{0, 1\}$ for all $i \in P$, and $e \in \{0, 1, 2, \ldots, 15\}$.

Solve the following optimization problem:

$$\max \quad \sum_{i \in P} \text{expected\_points}_i \cdot (x_i + c_i) - 4e \tag{7}$$

subject to constraints:

$$
\begin{aligned}
\sum_{i\in P} x_i &= 11, \quad \text{(lineup size)} \\
\sum_{i\in G} x_i &= 1, \quad \text{(1 starting GK)} \\
3 \le \sum_{i\in D} x_i &\le 5, \quad \text{(3-5 starting DEF)} \\
2 \le \sum_{i\in M} x_i &\le 5, \quad \text{(2-5 starting MID)} \\
1 \le \sum_{i\in F} x_i &\le 3, \quad \text{(1-3 starting FWD)} \\
\sum_{i\in P} c_i &= 1, \quad \text{(exactly 1 captain)} \\
c_i &\le x_i \quad \forall i \in P, \quad \text{(captain must be starter)} \\
u_i + x_i &\le 1 \quad \forall i \in P, \quad \text{(unavailable players cannot start)} \\
\sum_{i\in P} y_i &= 15, \quad \sum_{i\in G} y_i = 2, \quad \sum_{i\in D} y_i = 5, \\
\sum_{i\in M} y_i &= 5, \quad \sum_{i\in F} y_i = 3, \quad \text{(squad quotas)} \\
\sum_{j\in P} p_j t_j &\le B_{\text{bank}} + \sum_{i\in P} sp_i s_i, \quad \sum_{i\in P_c} y_i \le 3 \quad \forall c \in C, \quad \text{(budget and club limits)} \\
x_i &\le y_i \quad \forall i \in P \quad \text{(starters must be in squad)} \\
y_i &= y_i^0 + t_i - s_i \quad \forall i \in P \quad \text{(squad update)} \\
t_i + s_i &\le 1 \quad \forall i \in P \quad \text{(no simultaneous buy/sell)} \\
t_i &\le 1 - y_i^0 \quad \forall i \in P \quad \text{(buy only non-squad)} \\
s_i &\le y_i^0 \quad \forall i \in P \quad \text{(sell only current squad)} \\
e &\ge \sum_{i\in P} t_i - f \quad \text{(extra transfers)} \\
\sum_{i\in P} t_i &\le 15 \quad \text{(logical upper bound on transfers)}
\end{aligned}
$$

Denote the optimal solution from Step 1 as $x_i^*, y_i^*, c_i^*, t_i^*, s_i^*, e^*$.

2.2.2 Post-hoc: Vice-Captain Assignment

Assign vice-captain as in Pre-GW1.

2.2.3 Post-hoc: Bench Position Assignment

Assign b_{ij} greedily as in Pre-GW1.

2.2.4 Output for Post GW1

The final output from the Post-GW1 formulation consists of:

- Squad composition: $y_i^* \in \{0, 1\}$ for all $i \in P$ (15 players, from Step 1).
- Starting lineup: $x_i^* \in \{0, 1\}$ for all $i \in P$ (11 players, from Step 1).
- Captain: $c_i^* \in \{0, 1\}$ for all $i \in P$ (1 player, from Step 1).

- Vice-captain: assigned post-hoc as per above.
- Bench positions: assigned post-hoc as per above.
- Transfers made: $t_i^*, s_i^* \in \{0,1\}$ for all $i \in P$ (from Step 1).
- Extra transfers penalty: e^* (from Step 1).

2.2.5 State Update for Next Gameweek

After completing Step 1, update the state variables to carry forward to the next gameweek using per-player cash calculations:

Cash Update:

Compute the new bank balance based on actual selling proceeds and buying costs:

$$B_{\text{bank}}^{\text{new}} = B_{\text{bank}} + \sum_{i \in P} sp_i s_i^* - \sum_{j \in P} p_j t_j^* \tag{8}$$

where:

- $\sum_{i \in P} sp_i s_i^*$: Cash from selling players (using locked selling prices)
- $\sum_{j \in P} p_j t_j^*$: Cost of buying players (at current market prices)

This formulation ensures:

- Players sold at a profit contribute $sp_i = p_{\text{buy},i}^0 + 0.5(p_i - p_{\text{buy},i}^0)$ (50% profit lock)
- Players sold at a loss contribute $sp_i = p_i$ (full current price, loss realized)
- The constraint $B_{\text{bank}}^{\text{new}} \geq 0$ is implicitly satisfied by the budget constraints in Step 1

Purchase Price State Update:

Update the purchase price history for all players:

$$\forall i \in P : \quad \begin{cases} p_{\text{buy},i}^0 \leftarrow 0 & \text{if } s_i^* = 1 \quad \text{(sold players - clear history)} \\ p_{\text{buy},i}^0 \leftarrow p_i & \text{if } t_i^* = 1 \quad \text{(bought players - record purchase)} \\ p_{\text{buy},i}^0 \text{ unchanged} & \text{otherwise} \quad \text{(retained/non-squad players)} \end{cases} \tag{9}$$

Squad State Update:

Update the state variables for the next gameweek:

- $y_i^0 \leftarrow y_i^*$ for all $i \in P$ (new squad becomes current squad)
- $B_{\text{bank}} \leftarrow B_{\text{bank}}^{\text{new}}$ (updated cash in bank from equation above)
- $p_{\text{buy},i}^0$ updated as specified (purchase price tracking)
- f: Free transfers for next gameweek (1 per week, bankable up to max 5)

The decomposition is solver-independent and serves as an algorithmic blueprint for efficient FPL optimization.

2.3 CP-Based Formulations with High-Level Set Abstraction

This subsection presents equivalent formulations using declarative set-based abstraction, as implemented in MiniZinc constraint programming models. The set-based abstraction maps directly to the binary formulations in Sections 2.1 and 2.2.

2.3.1 Sets and Parameters

Let $P = \{1, \dots, n\}$ be the set of all players.

Let $C = \{1, \dots, 20\}$ be the set of clubs.

Let $\mathcal{P} = \{\text{GK}, \text{DEF}, \text{MID}, \text{FWD}\}$ be the set of positions.

For each position $p \in \mathcal{P}$, let $P_p \subseteq P$ be the players available in position p.

For each club $c \in C$, let $P_c \subseteq P$ be the players in club c.

Let $e_i \in \mathbb{R}_{\geq 0}$ be the expected points for player $i \in P$.

Let $k_i \in \mathbb{R}_{\geq 0}$ be the cost for player $i \in P$.

Let $u_i \in \{0, 1\}$ indicate if player $i \in P$ is unavailable.

Let $U \in \mathbb{R}_{\geq 0}$ be an logical upper bound on the objective.

Let $L \in \mathbb{R}_{\leq 0}$ be an logical lower bound on the objective.

2.3.2 Decision Variables

The following set-based decision variables are common to both Pre-GW1 and Post-GW1 formulations:

Let $Q \subseteq P$ be the squad set, with $|Q| = 15$.

Let $S \subseteq Q$ be the starters set, with $|S| = 11$.

Let $K \subseteq S$ be the captain set, with $|K| = 1$.

For each position $p \in \mathcal{P}$, let $Q_p \subseteq P_p$ be the squad partition for p.

For each position $p \in \mathcal{P}$, let $S_p \subseteq Q_p$ be the starters partition for p.

2.3.3 Pre-GW1 Optimization

This formulation corresponds to the Pre-GW1 problem in Section 2.1.

Step 1: Optimize Starters and Captain

Solve the following optimization problem:

$$\max_{Q,S,K,\{Q_p,S_p\}_{p\in\mathcal{P}}} \quad z = \sum_{i\in S} e_i + \sum_{i\in K} e_i \tag{10}$$

subject to:

$$0 \leq z \leq U,$$

Cardinalities:

$$\begin{aligned}
&|Q| = 15, \quad |S| = 11, \quad |K| = 1,\\
&|Q_{\text{GK}}| = 2, \quad |Q_{\text{DEF}}| = 5, \quad |Q_{\text{MID}}| = 5, \quad |Q_{\text{FWD}}| = 3,\\
&|S_{\text{GK}}| = 1, \quad 3 \leq |S_{\text{DEF}}| \leq 5, \quad 2 \leq |S_{\text{MID}}| \leq 5, \quad 1 \leq |S_{\text{FWD}}| \leq 3,
\end{aligned}$$

Partitions:

$$\begin{aligned}
&Q = \bigcup_{p\in\mathcal{P}} Q_p, \quad \forall p_1 \neq p_2 \in \mathcal{P} : \, Q_{p_1} \cap Q_{p_2} = \emptyset,\\
&S = \bigcup_{p\in\mathcal{P}} S_p, \quad \forall p_1 \neq p_2 \in \mathcal{P} : \, S_{p_1} \cap S_{p_2} = \emptyset,\\
&\forall p \in \mathcal{P} : \, Q_p \subseteq P_p, \, S_p \subseteq Q_p,
\end{aligned}$$

Budget:

$$\sum_{i \in Q} k_i \leq 100,$$

Club Limits:

$$\forall c \in C : |Q \cap P_c| \leq 3,$$

Availability:

$$\forall i \in P : (u_i = 1) \implies (i \notin S),$$

Subsets:

$$S \subseteq Q, \quad K \subseteq S.$$

Denote the optimal solution from Step 1 as $Q^*, S^*, K^*, \{Q_p^*, S_p^*\}_{p \in \mathcal{P}}$.

Step 2: Optimize Bench

Fix S^* and $\{S_p^* \mid p \in \mathcal{P}\}$ from Step 1's optimal solution. Then solve:

$$\max_{Q, \{Q_p\}_{p \in \mathcal{P}}} \quad z = \sum_{i \in Q} e_i \tag{11}$$

subject to:

$$0 \leq z \leq U,$$
$$S = S^*, \quad \forall p \in \mathcal{P} : S_p = S_p^*,$$

and all other constraints from Step 1.

Denote the optimal solution from Step 2 as $Q^{**}, \{Q_p^{**}\}_{p \in \mathcal{P}}$.

Post-hoc Assignments:

After solving steps 1 and 2, perform the same post-hoc assignments as in Section 2.1 (vice-captain and bench position assignments).

2.3.4 Post-GW1 Optimization with Transfers

This formulation corresponds to the Post-GW1 problem in Section 2.2.

The key design choice is using explicit binary variables t_i, s_i for transfers (rather than set differences) to avoid propagation issues, while maintaining high-level set abstractions for squad/starters.

Additional State Variables:

For Post-GW1, the following state variables from the previous gameweek are required:

Let $y_i^0 \in \{0, 1\}$ indicate if player $i \in P$ is in the current squad.

Let $sp_i \in \mathbb{R}_{\geq 0}$ be the selling price for player $i \in P$ (scaled by 10, computed via 50% profit lock).

Let $B_{\text{bank}} \in \mathbb{R}_{\geq 0}$ be the cash in bank (scaled by 10).

Let $f \in \{0, 1, 2, 3, 4, 5\}$ be the number of free transfers available.

Additional Decision Variables:

For each player $i \in P$, let $t_i \in \{0, 1\}$ indicate transfer in (buy player i).

For each player $i \in P$, let $s_i \in \{0, 1\}$ indicate transfer out (sell player i).

Let $e \in \{0, 1, \ldots, 15\}$ be the number of extra transfers beyond f.

For each player $i \in P$, let $y_i \in \{0, 1\}$ indicate if player i is in the new squad (channeled with set Q).

Optimization Formulation:

Solve the following optimization problem:

$$\max_{Q,S,K,\{Q_p,S_p\}_{p\in\mathcal{P}},\{y_i,t_i,s_i\}_{i\in P},e} \quad z = \sum_{i\in S} e_i + \sum_{i\in K} e_i - 40 \cdot e \tag{12}$$

subject to:

$$L \le z \le U \quad \text{(precalculated logical bound)},$$

Transfer Logic:

$$\begin{aligned}
&\forall i \in P : \; y_i = y_i^0 + t_i - s_i \quad \text{(squad update)},\\
&\forall i \in P : \; t_i + s_i \le 1 \quad \text{(no buy and sell same player)},\\
&\forall i \in P : \; y_i^0 + t_i \le 1 \quad \text{(no buy if owned)},\\
&\forall i \in P : \; s_i \le y_i^0 \quad \text{(only sell owned)},\\
&e \ge \sum_{i\in P} t_i - f \quad \text{(extra transfers penalty)},\\
&\sum_{i\in P} t_i \le 15,
\end{aligned}$$

Budget with Transfers:

$$\sum_{i\in P} k_i \cdot t_i \le B_{\text{bank}} + \sum_{i\in P} sp_i \cdot s_i,$$

Channeling and Subsets:

$$\begin{aligned}
\forall i \in P : \; (y_i = 1) \iff (i \in Q),\\
S \subseteq Q, \quad K \subseteq S,
\end{aligned}$$

Cardinalities:

$$\begin{aligned}
&\sum_{i\in P} y_i = 15, \quad |S| = 11, \quad |K| = 1,\\
&|Q_{\text{GK}}| = 2, \quad |Q_{\text{DEF}}| = 5, \quad |Q_{\text{MID}}| = 5, \quad |Q_{\text{FWD}}| = 3,\\
&|S_{\text{GK}}| = 1, \quad 3 \le |S_{\text{DEF}}| \le 5, \quad 2 \le |S_{\text{MID}}| \le 5, \quad 1 \le |S_{\text{FWD}}| \le 3,
\end{aligned}$$

Partitions:

$$\begin{aligned}
&Q = \bigcup_{p\in\mathcal{P}} Q_p, \quad \forall p_1 \ne p_2 \in \mathcal{P} : \; Q_{p_1} \cap Q_{p_2} = \emptyset,\\
&S = \bigcup_{p\in\mathcal{P}} S_p, \quad \forall p_1 \ne p_2 \in \mathcal{P} : \; S_{p_1} \cap S_{p_2} = \emptyset,\\
&\forall p \in \mathcal{P} : \; Q_p \subseteq P_p, \, S_p \subseteq Q_p,
\end{aligned}$$

Club Limits and Availability:

$$\begin{aligned}
&\forall c \in C : \; |Q \cap P_c| \le 3,\\
&\forall i \in P : \; u_i = 1 \implies i \notin S.
\end{aligned}$$

Post-hoc Assignments:
After solving the optimization, perform the same post-hoc assignments as in Section 2.2 (vice-captain and bench position assignments). State updates follow Section 2.2.5.