Indirect Proofs

An Excellent Read: “Scott and Scurvy”

http://idlewords.com/2010/03/scott_and_scurvy.htm

Outline for Today

- ***Today will be pretty packed!***
- **Preliminaries**
  - Disproving statements
  - Mathematical implications
- **Proof by Contrapositive**
  - The basic method.
  - An interesting application.
- **Proof by Contradiction**
  - The basic method.
  - Contradictions and implication.

Disproving Statements

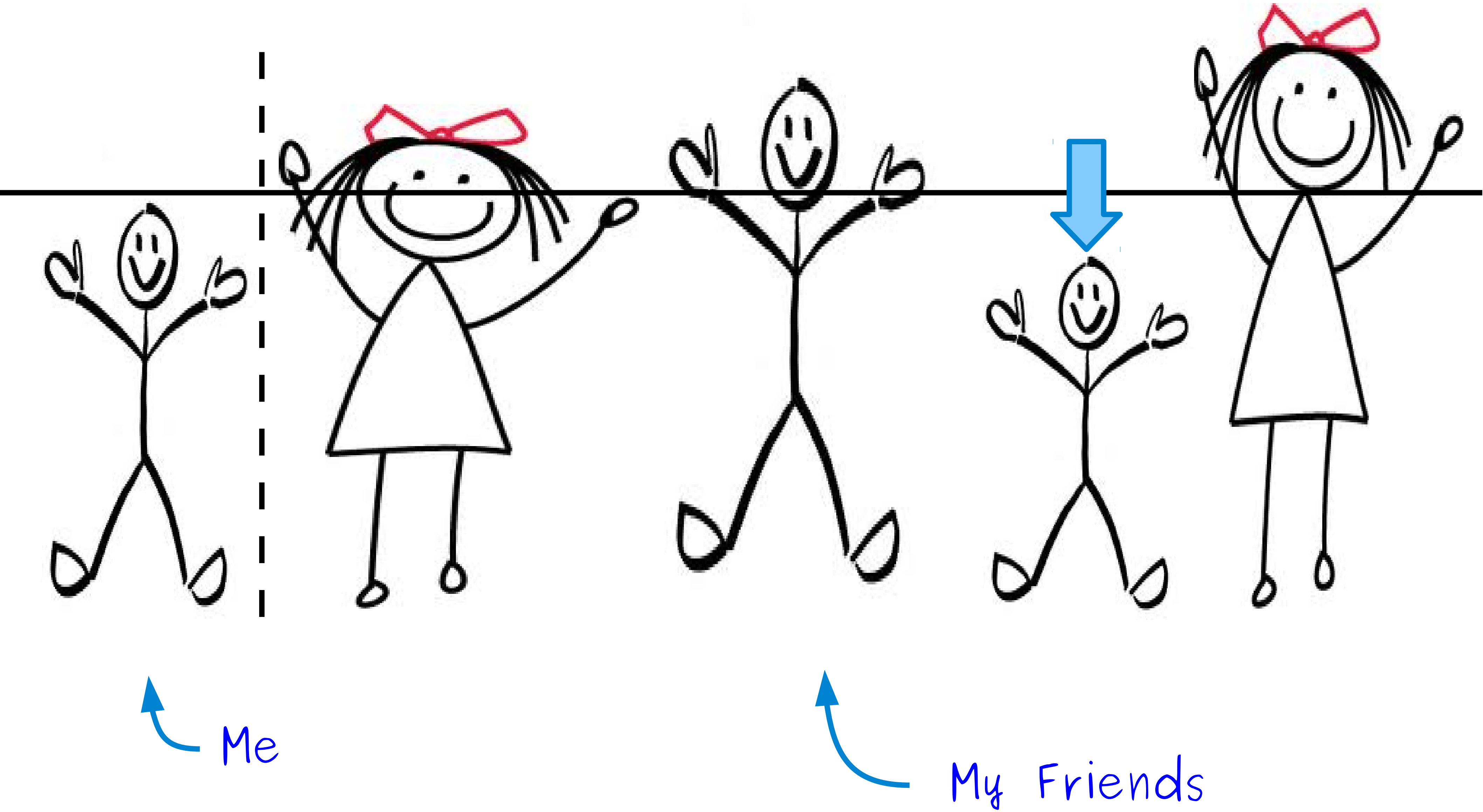
Proofs and Disproofs

- A ***proof*** is an argument establishing why a statement is true.
- A ***disproof*** is an argument establishing why a statement is *false*.
- Although proofs generally are more famous than disproofs, many important results in mathematics have been disproofs.
  - We'll see some later this quarter!

Writing a Disproof

- The easiest way to disprove a statement is to write a proof of the opposite of that statement.
  - The opposite of a statement X is called the ***negation*** of statement X.
- A typical disproof is structured as follows:
  - Start by stating that you're going to disprove some statement X.
  - Write out the negation of statement X.
  - Write a normal proof that statement X is false.

“All My Friends Are Taller Than Me”



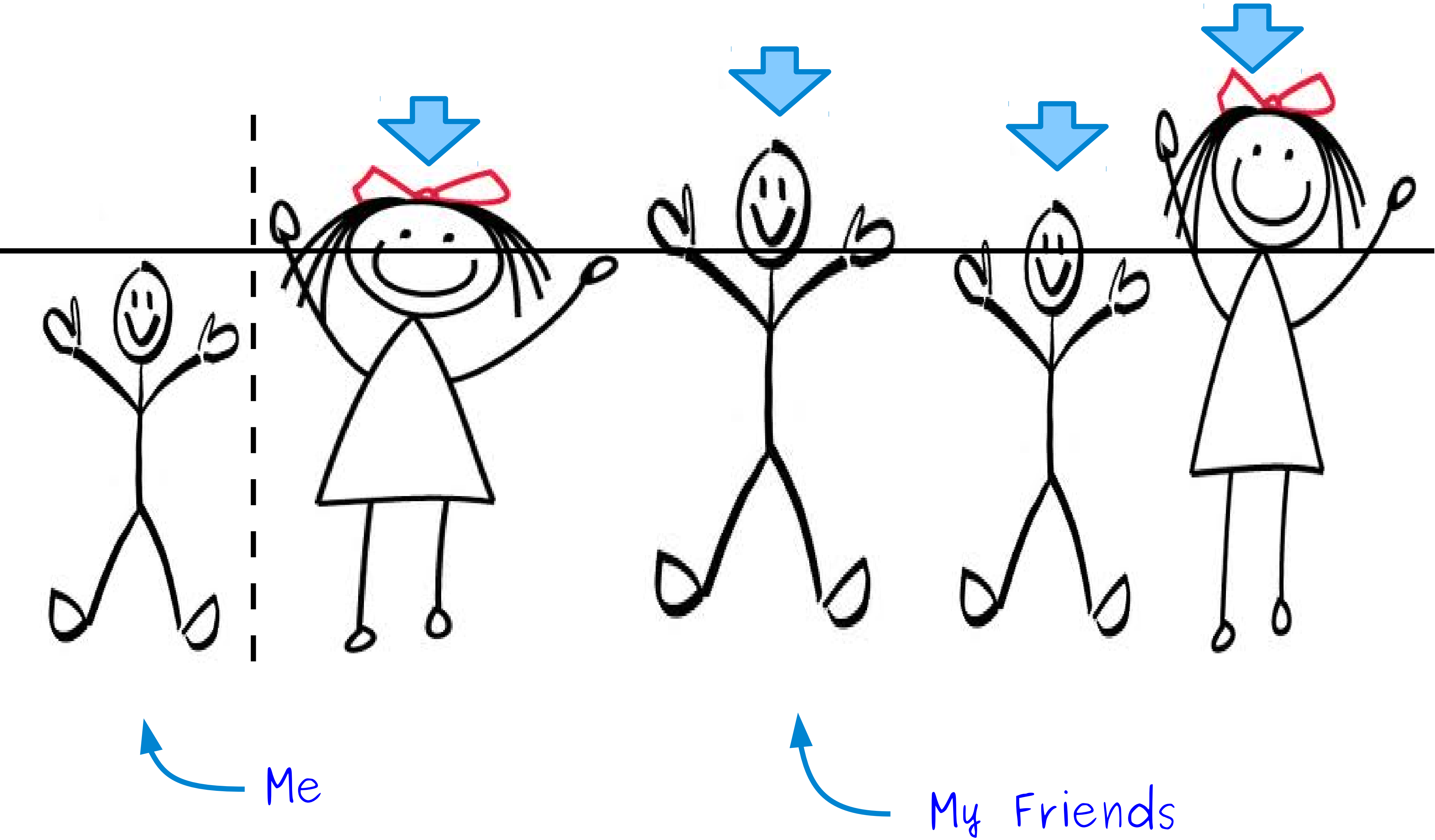
The negation of the *universal* statement

For all x, $P(x)$ is true.

is the *existential* statement

There exists an x where $P(x)$ is false.

"Some Friend Is Shorter Than Me"



The negation of the *existential* statement

There exists an x where $P(x)$ is true.

is the *universal* statement

For all x, $P(x)$ is false.

What would we have to show to disprove the following statement?

"Some set is the same size as its power set."

First, is this an existential statement or a universal statement?

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“Some set is the same size as its power set.”

First, is this an existential statement or a universal statement?

"There is a set S where S is the same size as its power set."

What happens when you negate an existential statement?

"There is a set S where S is the same size as its power set."

What happens when you negate an existential statement?

"For any set S, the set S is ***not*** the same size as its power set."

This is what we need to prove
to disprove the original statement.

"For any set S, the set S is ***not*** the same size as its power set."

Logical Implication

Implications

- An ***implication*** is a statement of the form

  If *P* is true, then *Q* is true.

- Some examples:
  - If n is an even integer, then n^2 is an even integer.
  - If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
  - If you liked it, then you should've put a ring on it.

Implications

- An ***implication*** is a statement of the form

  If *P* is true, then *Q* is true.

- In the above statement, the term "*P* is true" is called the ***antecedent*** and the term "*Q* is true" is called the ***consequent***.

What Implications Mean

- Consider the simple statement

If I put fire near cotton, it will burn.

- Some questions to consider:
  - Does this apply to all fire and all cotton, or just some types of fire and some types of cotton? *(Scope)*
  - Does the fire cause the cotton to burn, or does the cotton burn for another reason? *(Causality)*
- These are deeper questions than they might seem.
- To mathematically study implications, we need to formalize what implications really mean.

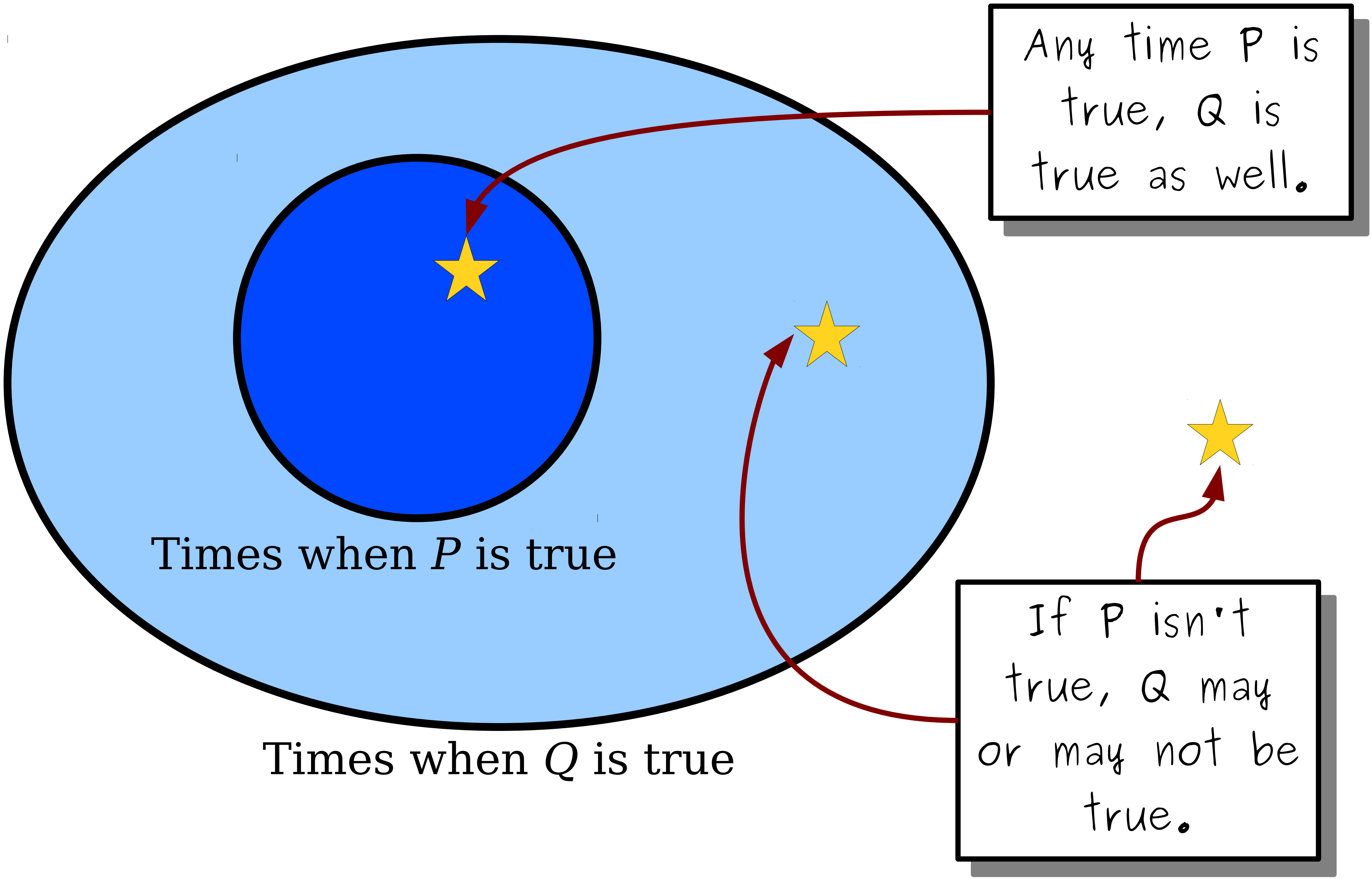
What Implications Mean

- In mathematics, the statement

  If *P* is true, then *Q* is true.

  means that any time *P* is true, *Q* will be true as well.
- There is no discussion of correlation or causation here. It simply means that if you find that *P* is true, you'll find that *Q* is true.

Implication, Diagrammatically



What Implication Doesn't Mean

“If there's a rainbow, it's raining somewhere”

- Implication is directional.
  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- Implication only cares about cases where the antecedent is true.
  - If there's no rainbow, it doesn't mean that there's no rain.
- Implication says nothing about causality.
  - Rainbows do not cause rain. ☺

Puppies Are Adorable

- Consider the statement

If *x* is a puppy, then I love *x*.

- Can you explain why the following statement is *not* the negation of the original statement?

If *x* is a puppy, then I don't love *x*.

- This second statement is too strong.
  - The initial statement means “I love all puppies.”
  - The second statement says “I don't love *any* puppies.”
- Here's the correct negation:

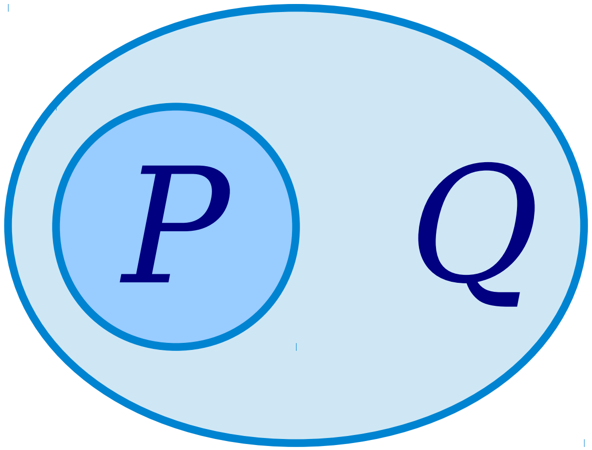
There is some puppy that I don't love.

The negation of the statement

“If *P* is true, then *Q* is true”

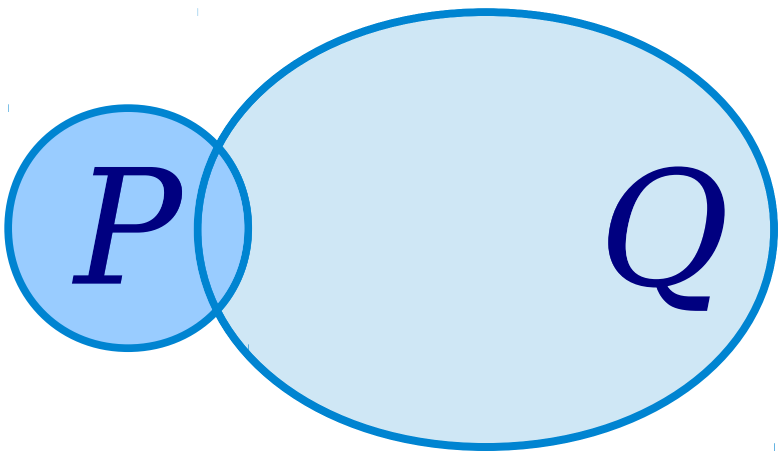
is the statement

“There is at least one case where *P* is true and *Q* is false.”



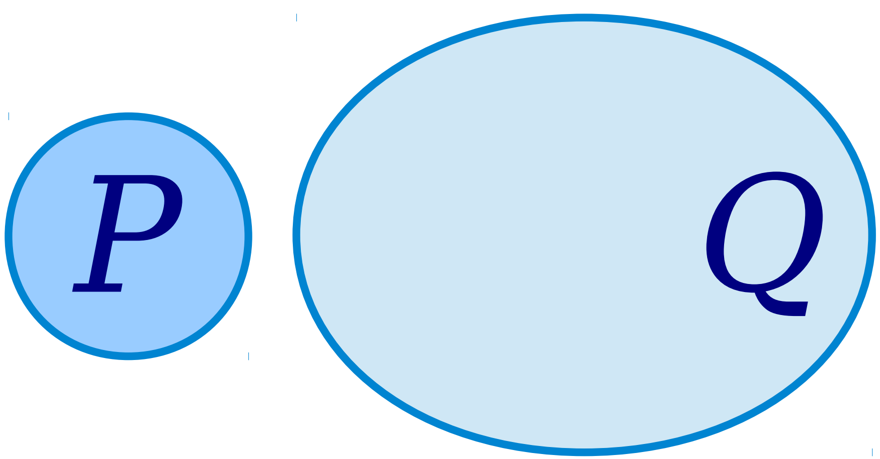
P implies Q

"If P is true, then Q is true."



P does not imply Q
-and-
P does not imply not Q

"Sometimes P is true and Q is true,
-and-
sometimes P is true and Q is false."



P implies not Q

If P is true, then Q is false

Proof by Contrapositive

The Contrapositive

- The ***contrapositive*** of the implication "If P, then Q" is the implication "If ***not*** Q, then ***not*** P."
- For example:
  - "If Harry had opened the right book, then Harry would have learned about Gillyweed."
  - Contrapositive: "If Harry didn't learn about Gillyweed, then Harry didn't open the right book."
- Another example:
  - "If I store the cat food inside, then wild raccoons will not steal my cat food."
  - Contrapositive: "If wild raccoons stole my cat food, then I didn't store it inside."

To prove the statement

If *P* is true, then *Q* is true

You may instead prove the statement

If *Q* is false, then *P* is false.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Proof: By contrapositive;

We're starting this proof by telling the reader that it's a proof by contrapositive. This helps cue the reader into what's about to come next.

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Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

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We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, there is some integer k such that $n = 2k + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, there is some integer k such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$n^2 = (2k + 1)^2$$

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, there is some integer k such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Since n is odd, there is some integer k such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

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Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, there is some integer k such that $n = 2$
and s

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The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

Biconditionals

- Combined with what we saw on Wednesday, we have proven that, if n is an integer:

If n is even, then n^2 is even.

If n^2 is even, then n is even.

- Therefore, if n is an integer:

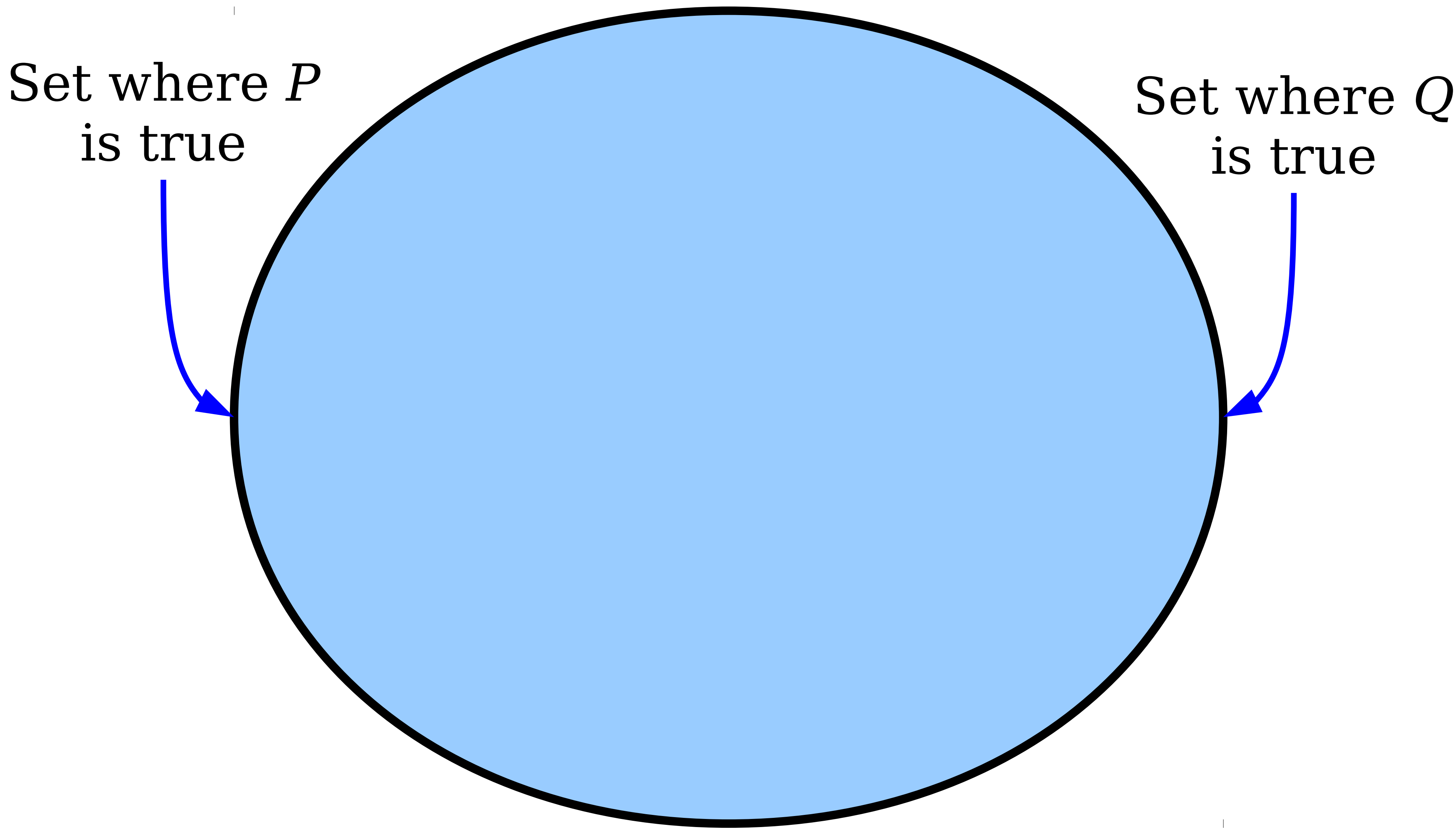
n is even if and only if n^2 is even.

- "If and only if" is often abbreviated ***iff***:

n is even iff n^2 is even.

- This is called a ***biconditional***.

P iff Q



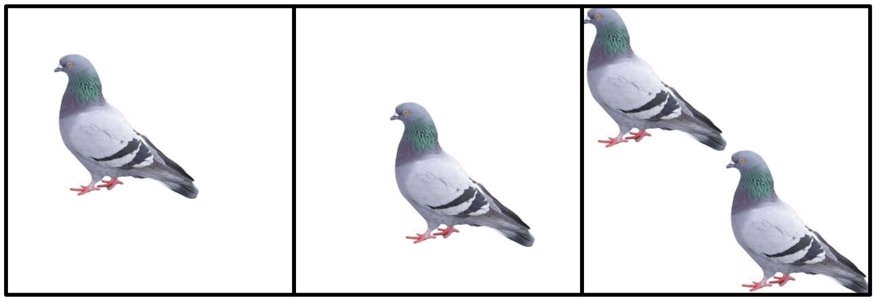
Proving Biconditionals

- To prove ***P* iff *Q***, you need to prove that P implies Q and that Q implies P.
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof and a proof by contrapositive.

The Pigeonhole Principle

The Pigeonhole Principle

- Suppose that you have n pigeonholes.
- Suppose that you have $m > n$ pigeons.
- If you put the pigeons into the pigeonholes, some pigeonhole will have more than one pigeon in it.



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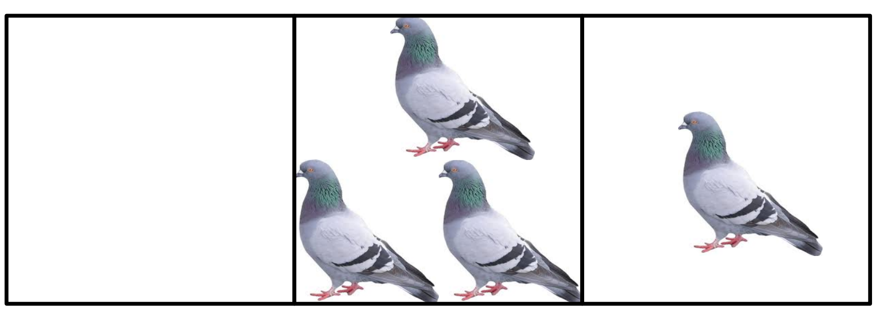

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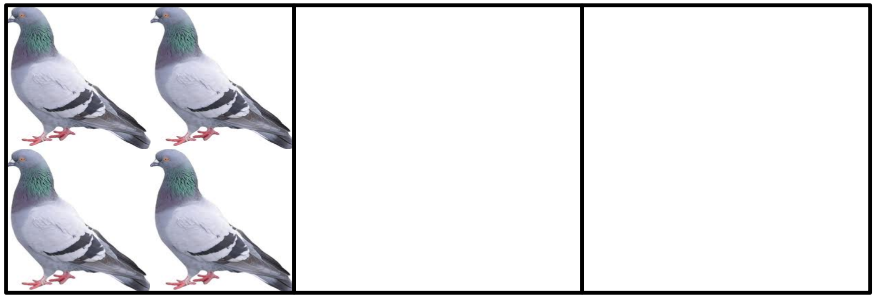

The Pigeonhole Principle

- Suppose that m objects are distributed into n bins.
- We want to prove the statement

  If $m > n$, then some bin contains at least two objects.

- What is the contrapositive of this statement?

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If "some bin contains at least two objects" is false, then "$m > n$" is false.

The Pigeonhole Principle

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Is this a universal statement or an existential statement?

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How do you take the negation of an existential statement?

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- What is the contrapositive of this statement?

  If every bin contains at most one object, then $m \leq n$.

- Look at the definitions of m and n. Does this make sense?

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i denote the number of objects in bin i. Since m is the number of total objects, we see that

$$m = x_1 + x_2 + \ldots + x_n.$$

We're assuming every bin has at most one object. In our notation, this means that $x_i \leq 1$ for all i. Using this inequality, we get the following:

$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &\leq 1 + 1 + \ldots + 1 \qquad (n \text{ times}) \\ &= n. \end{aligned}$$

So $m \leq n$, as required. ■

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes)
  - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.
- Each day, two people in New York City drink the same amount of water, to the thousandth of a fluid ounce.
  - No one can drink more than 50 gallons of water each day.
  - That's 6,400 fluid ounces. This gives 6,400,000 possible numbers of thousands of fluid ounces.
  - There are about 8,000,000 people in New York City proper.

Time-Out for Announcements!

Announcements

- Problem Set 1 out.
- **Checkpoint** due Monday, September 28.
  - Grade determined by attempt rather than accuracy. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
  - We will get feedback back to you with comments on your proof technique and style.
  - The more effort you put in, the more you'll get out.
- **Remaining problems** due Friday, October 2.
  - Feel free to email us with questions, stop by office hours, or ask questions on Piazza!

Submitting Assignments

- This quarter, we will be using GradeScope to handle assignment submissions. Visit www.gradescope.com and enter code **9JK26M**.
- Summary of the late policy:
  - Everyone has *three* 24-hour late days.
  - Late days can't be used on checkpoints.
  - Nothing may be submitted more than three days past the due date.
- Because this class is large, we rely on our tools to enforce deadlines. As a result, assignment due dates are tightly enforced.
- ***Very good idea:*** Leave at least two hours buffer time for your first assignment submission, just in case something goes wrong.

Working in Groups

- You can work on the problem sets individually, in a pair, or in a group of three.
- Each group should only submit a single problem set, and should submit it only once.
- Full details about the problem sets, collaboration policy, and Honor Code can be found in Handouts 05 and 06.

Please respect the Honor Code.

Office hours start tonight.

Schedule is available
on the course website.

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++ run a Fellowship to bring 30 high school students from across the country to Silicon Valley for a weekend
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APPLICATIONS DUE FRI OCT 2

StreetCode Academy

- StreetCode Academy (based in East Palo Alto) is looking for volunteers.
- Information session in **Old Union 219** on **Monday, September 28** at **7PM**.
- Interested? Fill out this form: https://docs.google.com/forms/d/1Hx4c8KG9PVyXMgmODbkXRYaDgtQOHqo954w2e_ylB-U/viewform?usp=send_form

CS+SOCIAL GOOD MIXER

Come to this year's first CS+Social Good Mixer to connect with incredible students, professors, and CS+Social Good industry partners!

September 30, 2015
5:30pm to 7:30pm

The Gates Computer Science Building
5th floor Room 504
353 Serra Mall Stanford, CA 94305

The mixer will include food, fun activities, and swag!

RSVP at
https://docs.google.com/a/stanford.edu/forms/d/1bQm34MSQQmSkZUmaJZIefDApffdJVZMdmPIo5kWgk3I/viewform

Back to CS103!

Proof by Contradiction

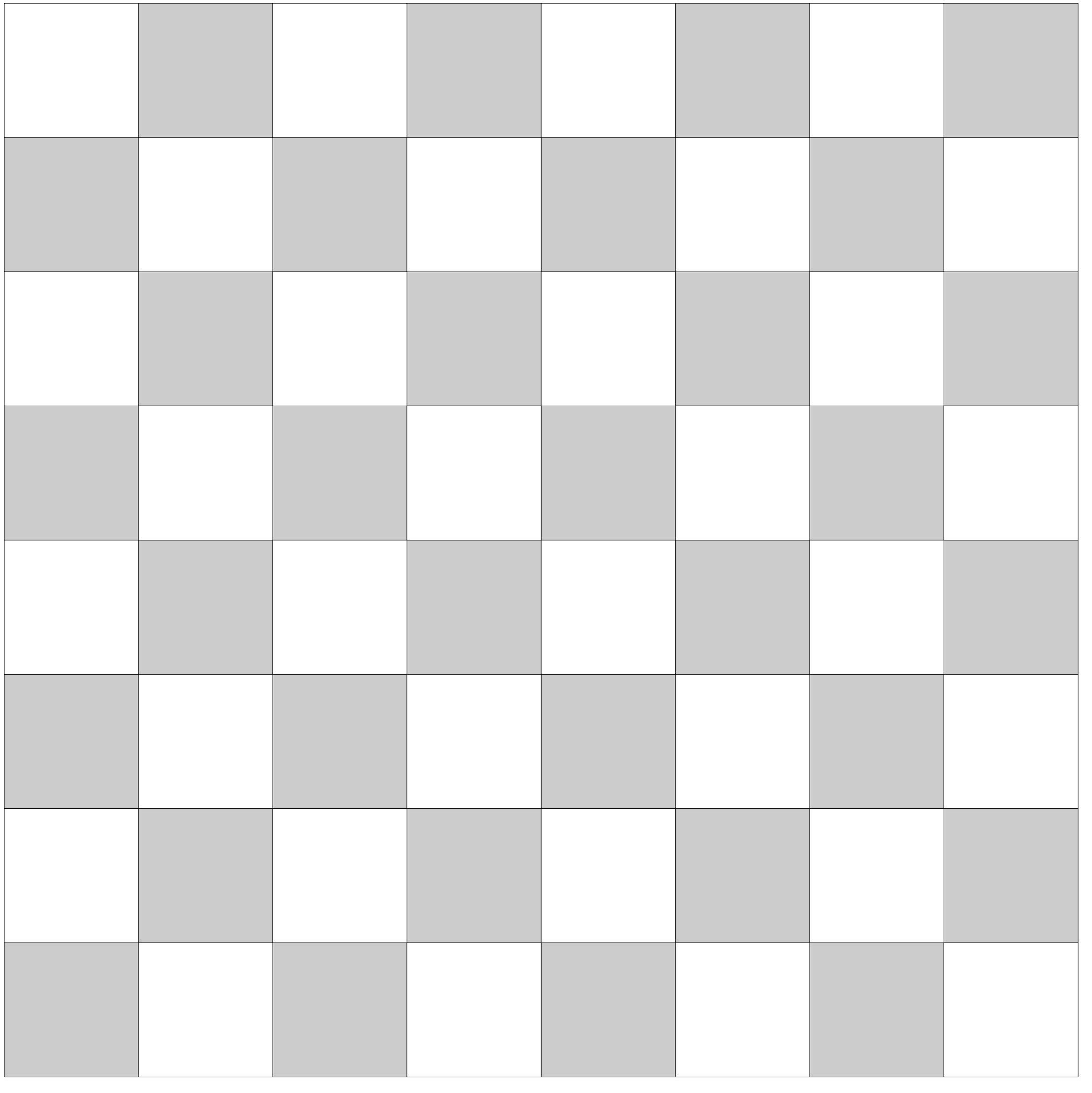
“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle*, The Adventure of the Blanched Soldier*

Proof by Contradiction

- A ***proof by contradiction*** is a proof that works as follows:
  - To prove that P is true, assume that P is *not* true.
  - Based on the assumption that P is not true, conclude something impossible.
  - Assuming the logic is sound, the only valid explanation is that the original assumption must have been wrong.
  - Therefore, P can't be false, so it must be true.

Tiling a Checkerboard



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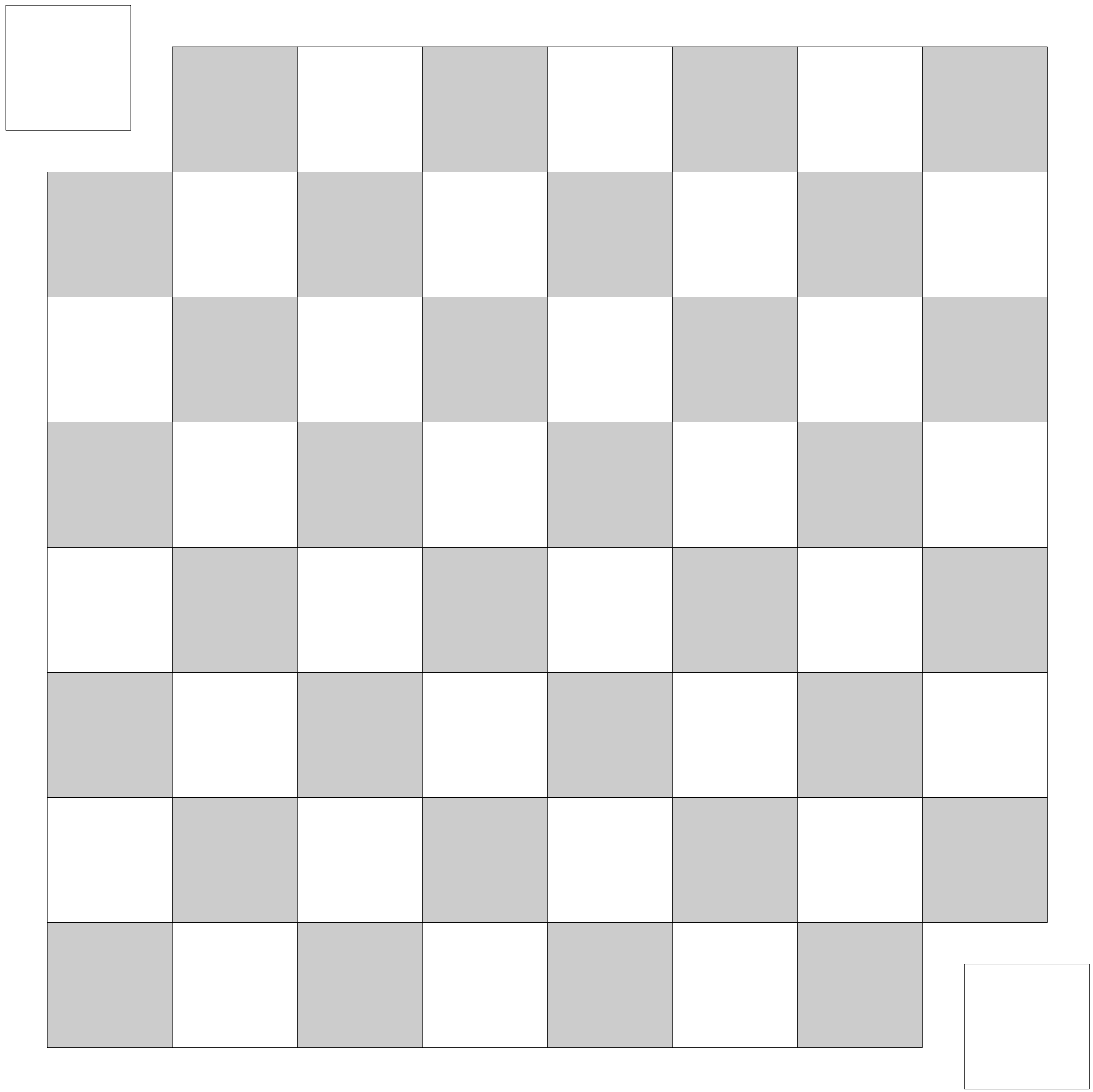

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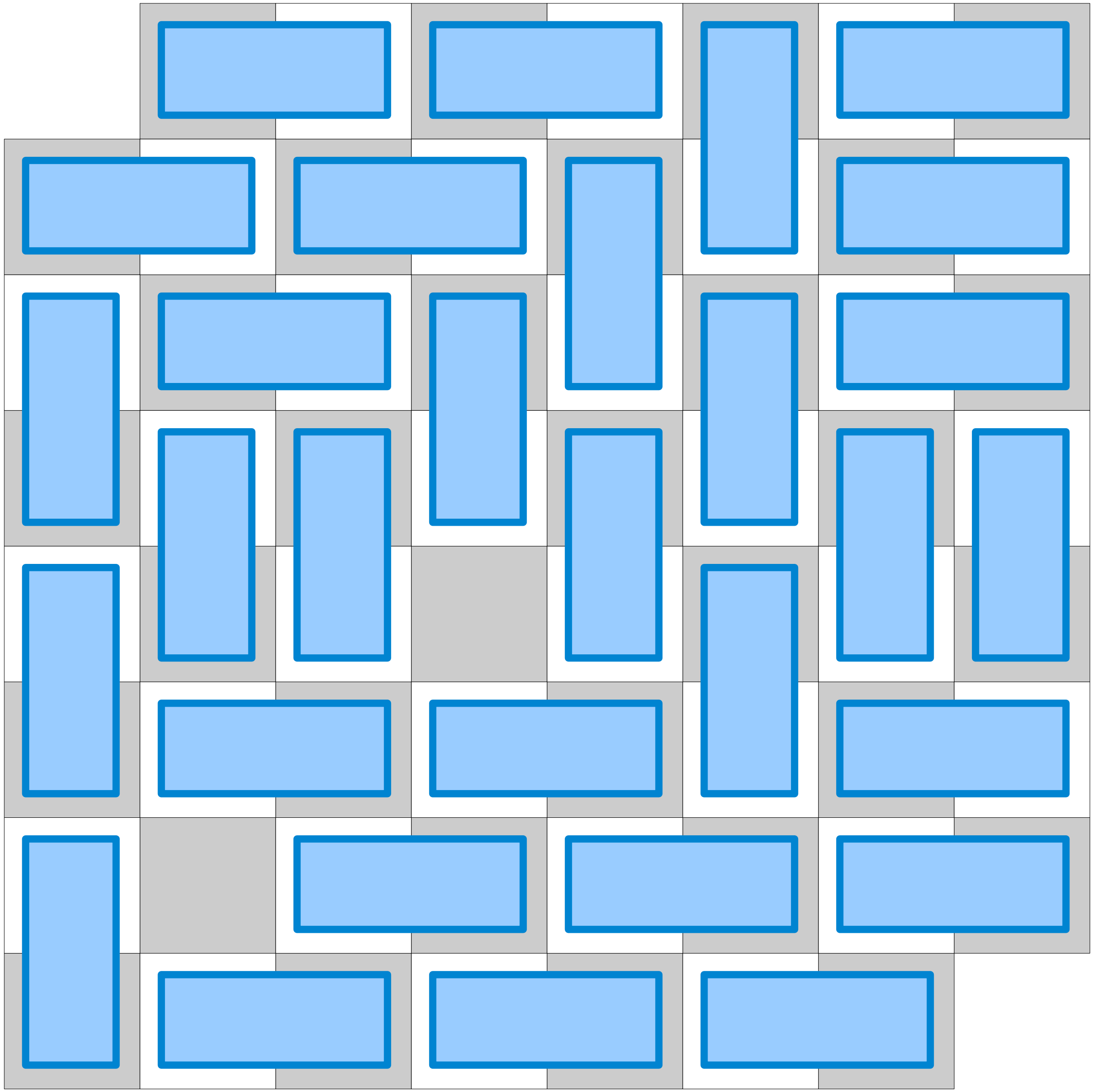

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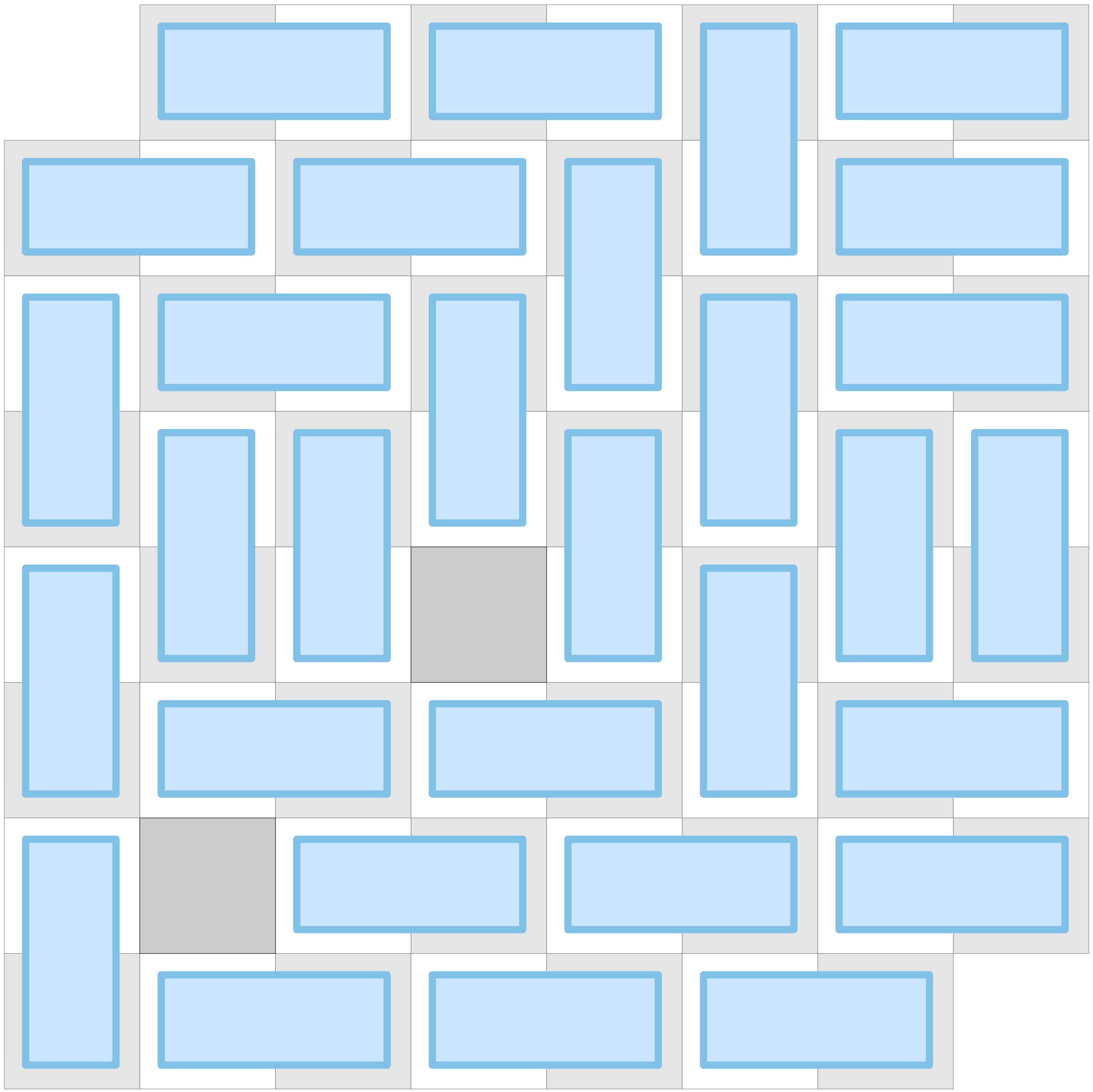
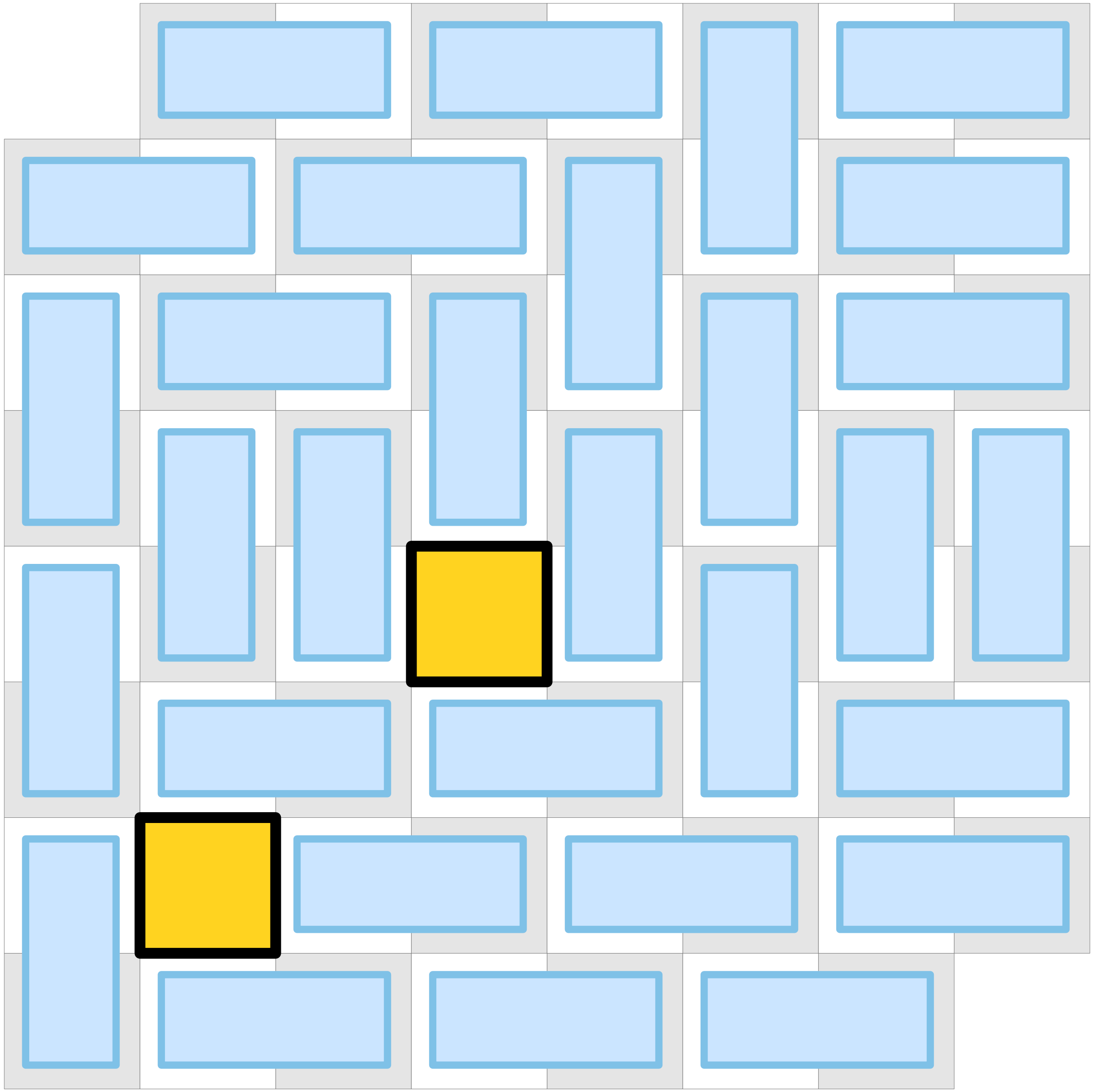
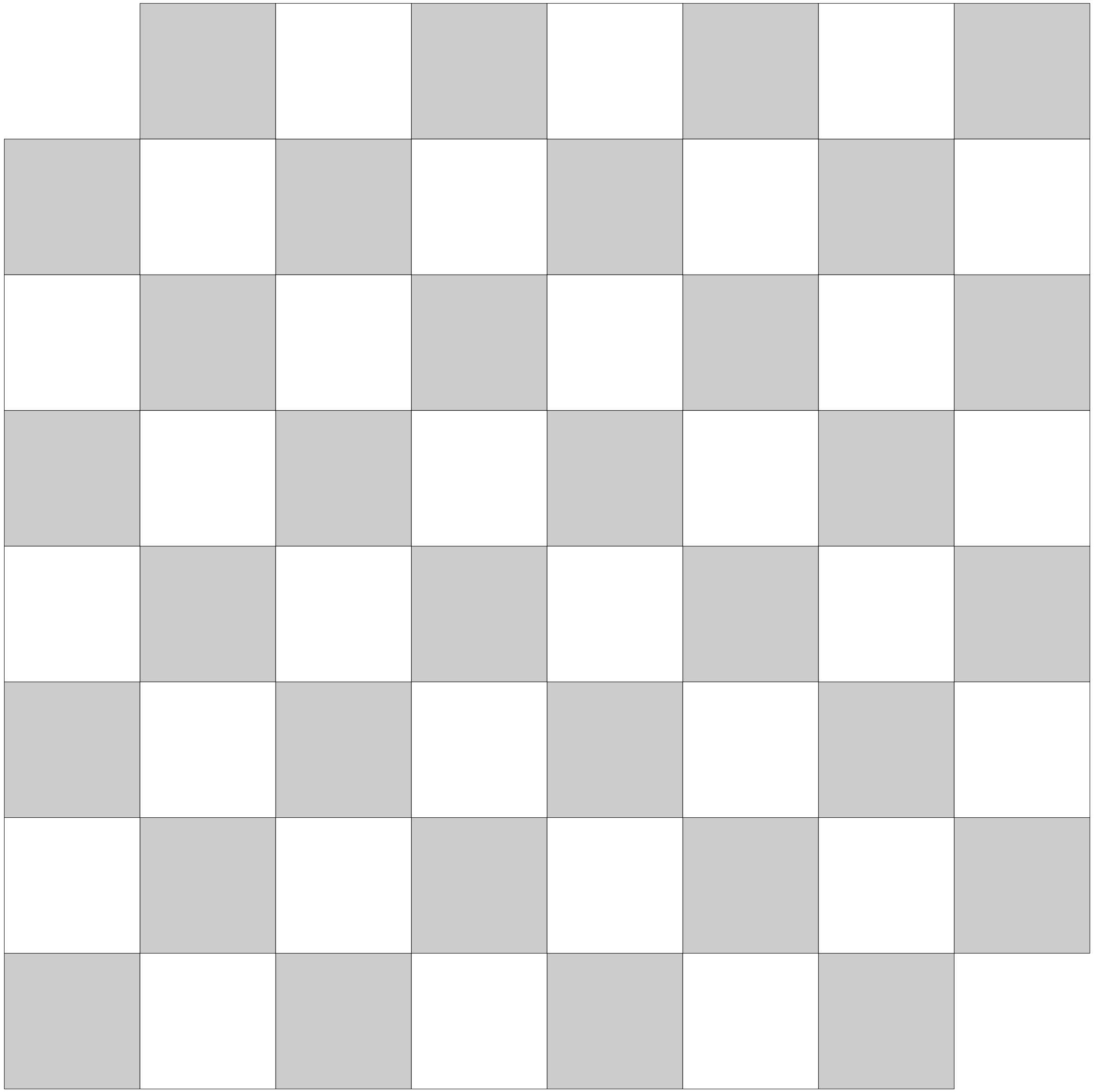


Tiling a Checkerboard



Tiling a Checkerboard



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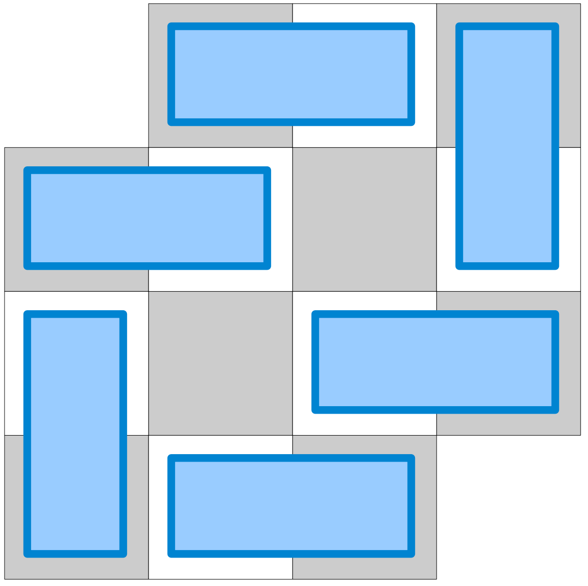

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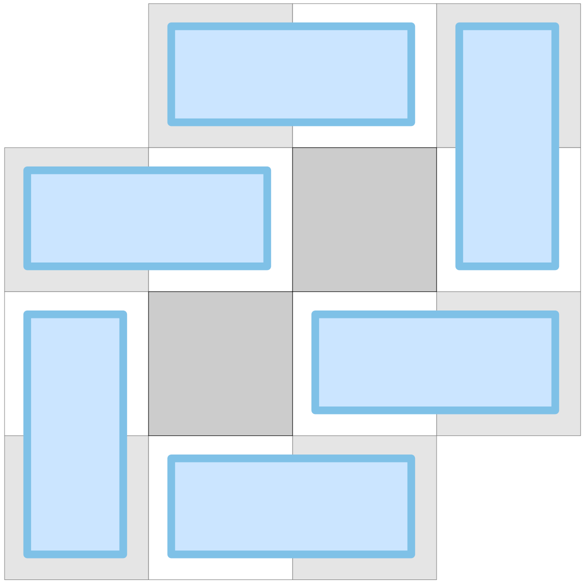

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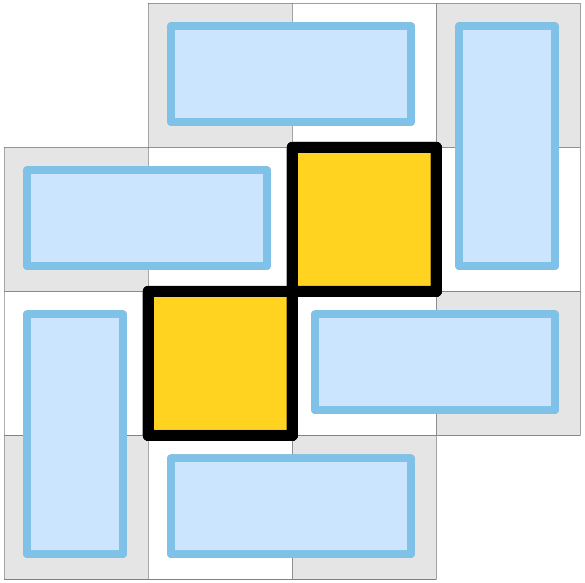
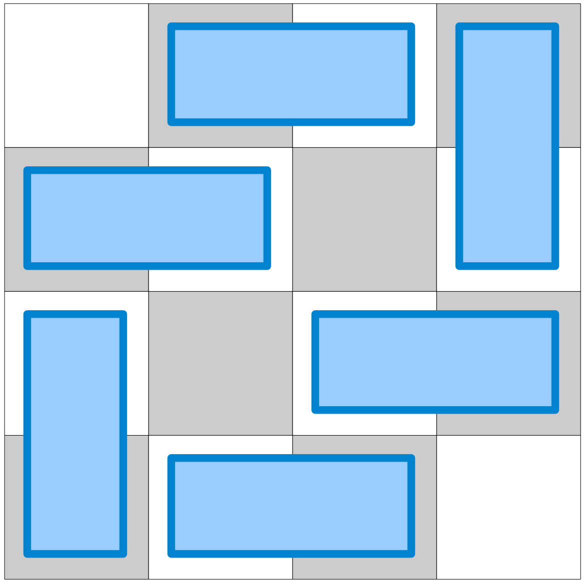


An Explanation



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Rational and Irrational Numbers

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- A number r is called a ***rational number*** if it can be written as

$$r = \frac{p}{q}$$

  where p and q are integers and $q \neq 0$.
- A number that is not rational is called ***irrational***.
- Useful theorem: If r is rational, r can be written as $p\,/\,q$ where $q \neq 0$ and where p and q have no common factors other than ± 1.

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Vi Hart on Pythagoras and
the Square Root of Two:

http://www.youtube.com/watch?v=X1E7I7_r3Cw

Proving Implications

- To prove the implication

  "If *P* is true, then *Q* is true."

- you can use these three techniques:
  - **Direct Proof.**
    - Assume P and prove Q.
  - **Proof by Contrapositive**
    - Assume not Q and prove not P.
  - **Proof by Contradiction**
    - … what does this look like?

Negating Implications

- To prove the statement

  "If *P* is true, then *Q* is true"

  by contradiction, we do the following:
  - Assume this statement is false.
  - Derive a contradiction.
  - Conclude that the statement is true.
- What is the negation of this statement?

  "*P* is true and *Q* is false"

Contradictions and Implications

- To prove the statement

  “If *P* is true, then *Q* is true”

  using a proof by contradiction, do the following:

  - Assume that P is true and that Q is false.
  - Derive a contradiction.
  - Conclude that if P is true, Q must be as well.

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Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned} \tag{2}$$

Theorem: If n is an integer and n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

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The three key pieces:

1. State that the proof is by contradiction.
2. State what the negation of the original statement is.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

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Skills from Today

- Disproving statements
- Negating universal and existential statements.
- Negating implications.
- Determining the contrapositive of a statement.
- Writing a proof by contrapositive.
- Writing a proof by contradiction.

Next Time

- **Mathematical Logic**
  - How do we formalize the reasoning from our proofs?
- **Propositional Logic**
  - Reasoning about simple statements.
- **Propositional Equivalences**
  - Simplifying complex statements.

Appendix: Helpful References

Negating Implications

“If *P*, then *Q*”

becomes

“*P* but not *Q*”

Negating Universal Statements

“For all *x*, P(*x*) is true”

becomes

“There is an *x* where P(*x*) is false.”

Negating Existential Statements

“There exists an *x* where P(*x*) is true”

becomes

“For all *x*, P(*x*) is false.”