Stochastic background searches using the optimal statistic

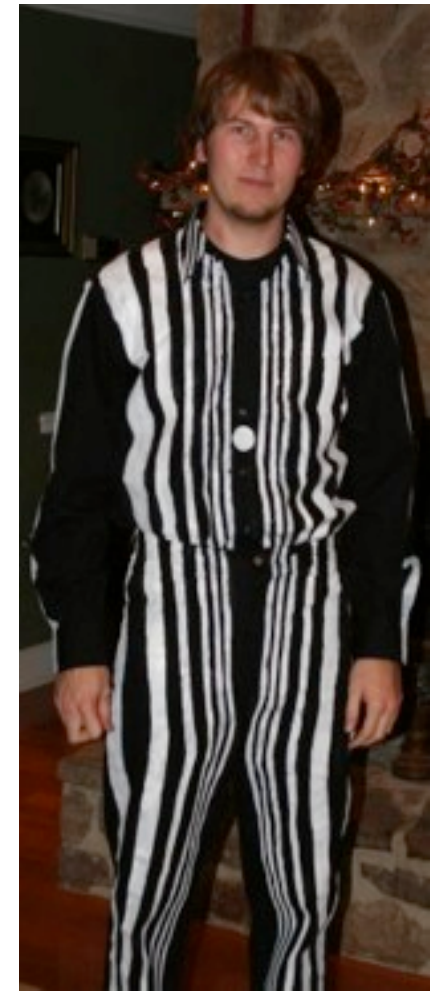
Xavi Siemens

Sydney Chamberlin, Jolien Creighton, Paul Demorest,
Justin Ellis, Larry Price, Joe Romano

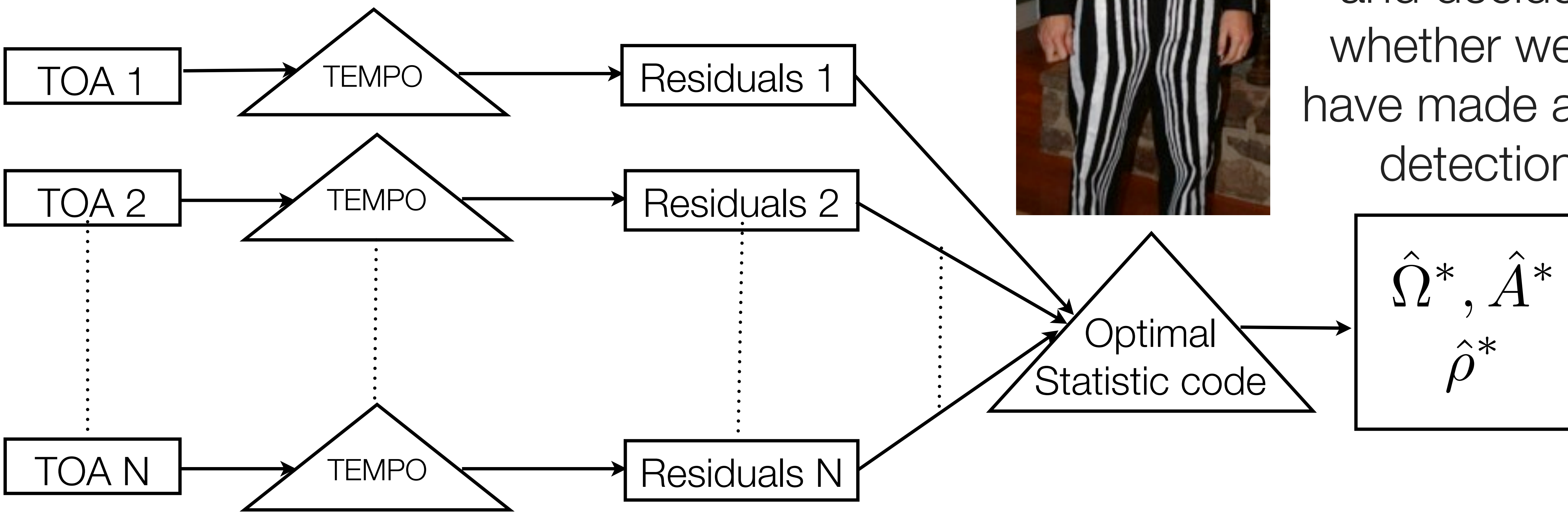
Anholm et al. 2009, van Haasteren et al. 2009, Demorest et al 2012

The pipeline

Justin Ellis



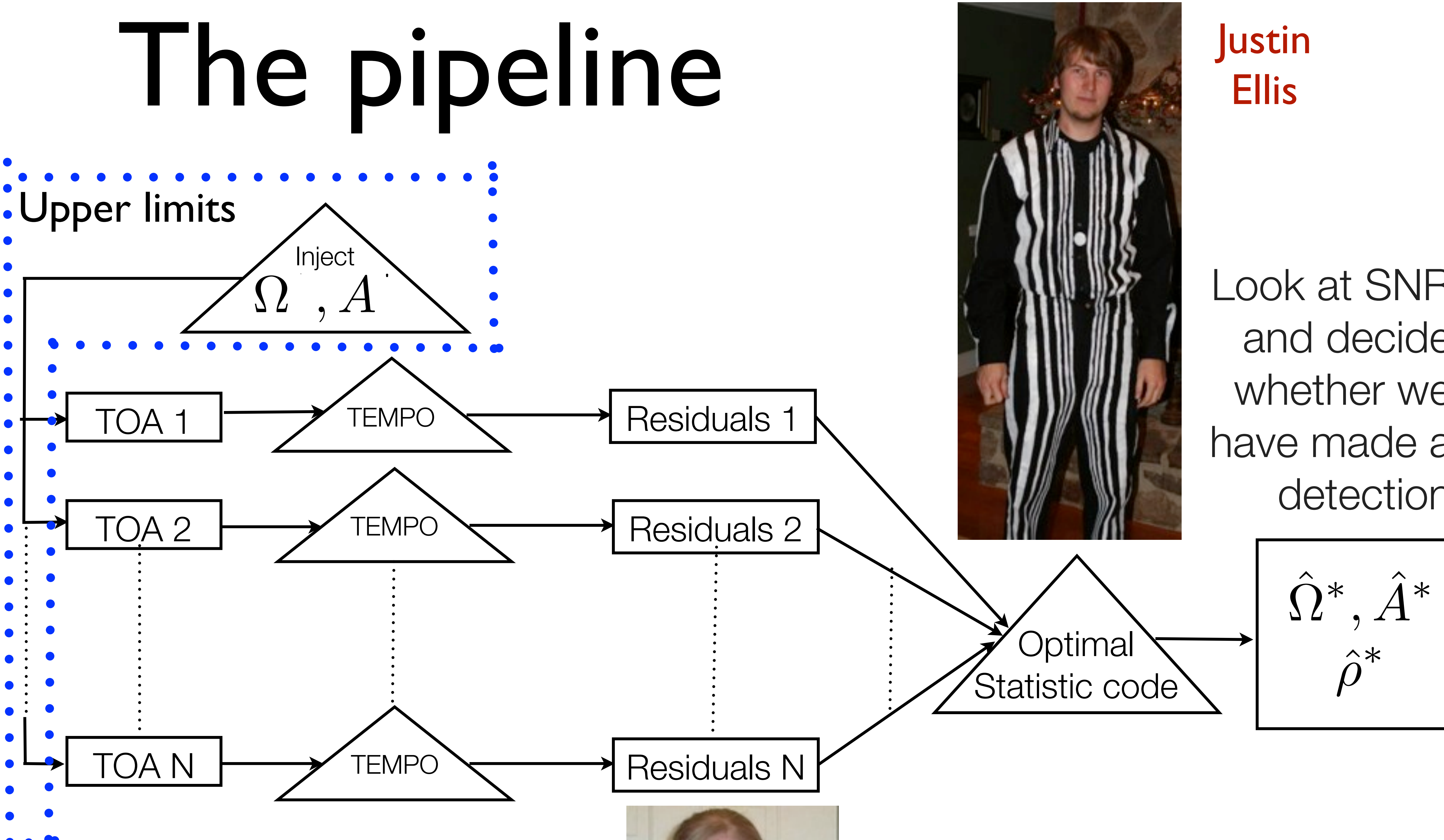
Look at SNR and decide whether we have made a detection



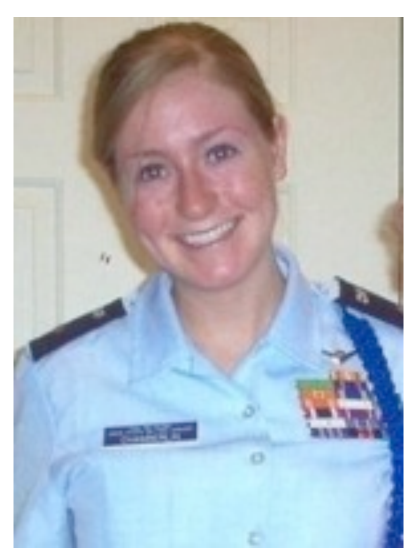
The pipeline

Justin Ellis

Look at SNR and decide whether we have made a detection



Find $\Omega_{95\%}, A_{95\%}$: Values of injected BG for which 95% of the time $\hat{\Omega} > \hat{\Omega}^*$



Sydney Chamberlin

Optimal statistic

Start from likelihood (standard Gaussian multivariate)

$$p(r|\Omega) = \frac{1}{\sqrt{\det(2\pi\Sigma_r)}} \exp(-\frac{1}{2} r^T \Sigma_r^{-1} r)$$

Residuals $\mathbf{r} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_l \end{bmatrix}$

Covariance matrix for residuals $\mathbf{\Sigma}_r = \langle \mathbf{r}\mathbf{r}^\mathbf{T} \rangle = \begin{bmatrix} \mathbf{P}_1 & \Omega\mathbf{S}_{12} & \cdots & \Omega\mathbf{S}_{1l} \\ \Omega\mathbf{S}_{21} & \mathbf{P}_2 & \cdots & \Omega\mathbf{S}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega\mathbf{S}_{l1} & \Omega\mathbf{S}_{l2} & \cdots & \mathbf{P}_l \end{bmatrix}$

$$\mathbf{\Sigma}_r = \begin{bmatrix} \mathbf{P}_1 & \Omega\mathbf{S}_{12} & \cdots & \Omega\mathbf{S}_{1l} \\ \Omega\mathbf{S}_{21} & \mathbf{P}_2 & \cdots & \Omega\mathbf{S}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega\mathbf{S}_{l1} & \Omega\mathbf{S}_{l2} & \cdots & \mathbf{P}_l \end{bmatrix}$$

covariance matrix

Auto-correlation matrices (diagonal elements of covariance matrix)

$$P_I = \langle r_I r_I^T \rangle_{ij} = R_I \left[\int_{-\infty}^{\infty} df e^{2\pi i f(t_i - t_j)} P_r(f) + \sigma_I^2 \delta_{ij} \right] R_I^T$$

Red noise power spectrum $P_r(f) = A f^{-\gamma}$

Red and white noise parameters modeled and measured from individual residual pulsar data (see poster by Justin Ellis)

Timing model $R_I = I - A_I (A_I^T C_I^{-1} A_I)^{-1} A_I^T C_I^{-1}$

$$\mathbf{\Sigma}_r = \begin{bmatrix} \mathbf{P}_1 & \Omega\mathbf{S}_{12} & \cdots & \Omega\mathbf{S}_{1l} \\ \Omega\mathbf{S}_{21} & \mathbf{P}_2 & \cdots & \Omega\mathbf{S}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega\mathbf{S}_{l1} & \Omega\mathbf{S}_{l2} & \cdots & \mathbf{P}_l \end{bmatrix}$$

covariance matrix

Cross-correlation (off-diagonal elements of covariance matrix)

$$\Omega_\beta S_{IJ} = \langle r_I r_J^T \rangle_{ij} = R_I \left[\frac{\chi_{IJ}}{2} \int_{-\infty}^{\infty} df e^{2\pi i f (t_i - t_j)} P_g(f) \right] R_J^T$$

Hellings-Downs curve χ_{IJ}

GW power spectrum $P_g(f) = \frac{H_0^2}{16\pi^4} \Omega_\beta f^{\beta-5} = \frac{A_g^2}{24\pi^2} \left(\frac{f}{f_{1\mathrm{yr}}} \right)^{2\alpha} f^{-3}$

Timing model $R_I = I - A_I (A_I^T C_I^{-1} A_I)^{-1} A_I^T C_I^{-1}$

Optimal statistic is obtained by maximizing the likelihood over GW amplitude (but fixed spectrum)

Pair-wise optimal statistic

$$\mathcal{Q}_{\mathrm{IJ}} = N_{IJ} r_I^T P_I^{-1} S_{IJ} P_J^{-1} r_J$$

Normalization

$$N_{\mathrm{IJ}} = \left(\mathrm{Tr}\left[P_I^{-1} S_{IJ} P_J^{-1} S_{IJ}^T\right]\right)^{-1}$$

Pairwise standard deviation

$$\sigma_{IJ} \approx \sqrt{N_{IJ}}$$

Normalization is chosen so that $\langle \mathcal{Q}_{\mathrm{IJ}} \rangle = \Omega_\beta$

PTA optimal statistic

$$\mathcal{Q}_{\mathrm{opt}} = \frac{\sum_{IJ} \mathcal{Q}_{IJ} \sigma_{IJ}^{-2}}{\sum_{IJ} \sigma_{IJ}^{-2}}$$

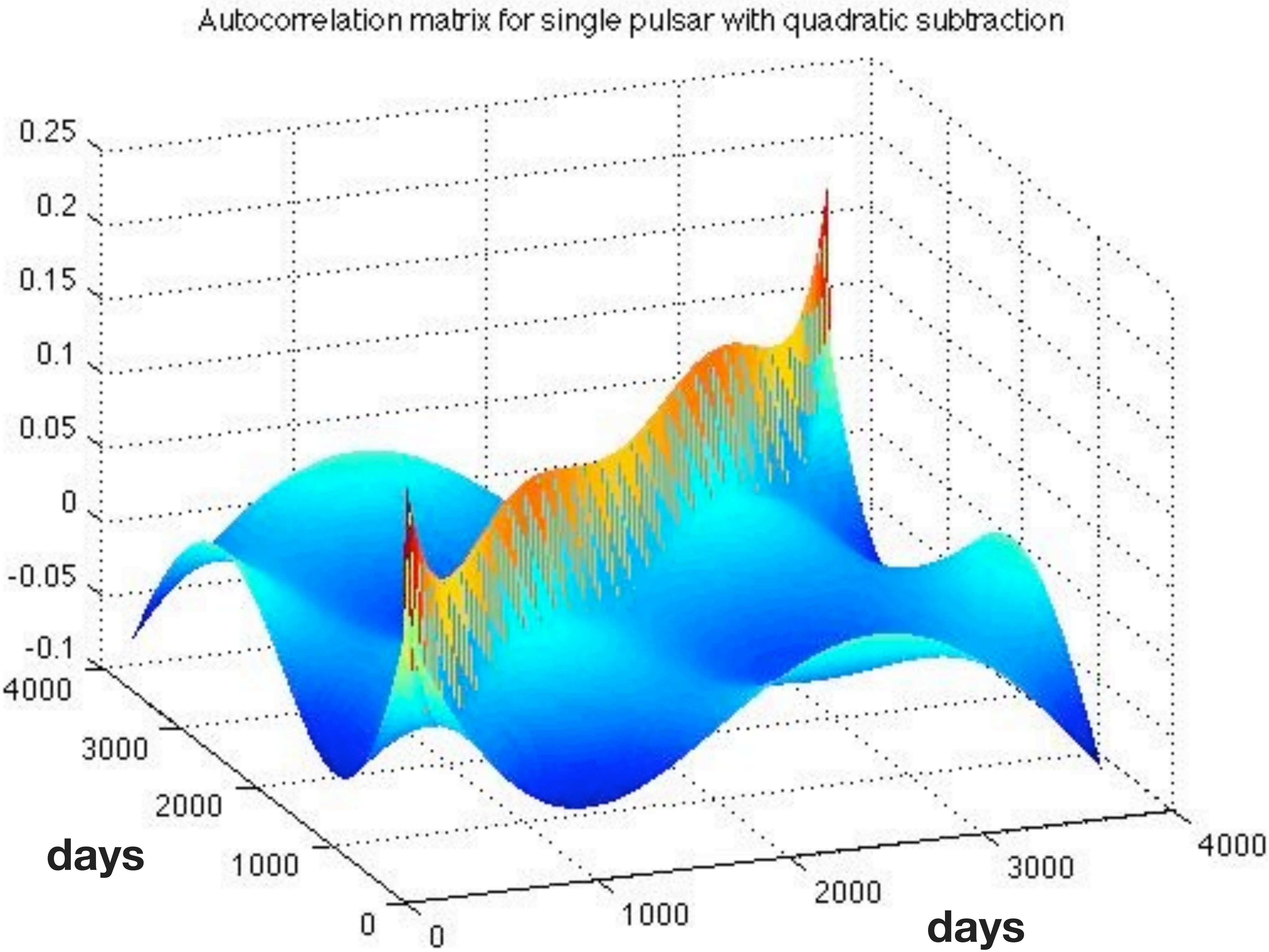
Pairwise standard deviation

$$\sigma_{IJ} \approx \sqrt{N_{IJ}}$$

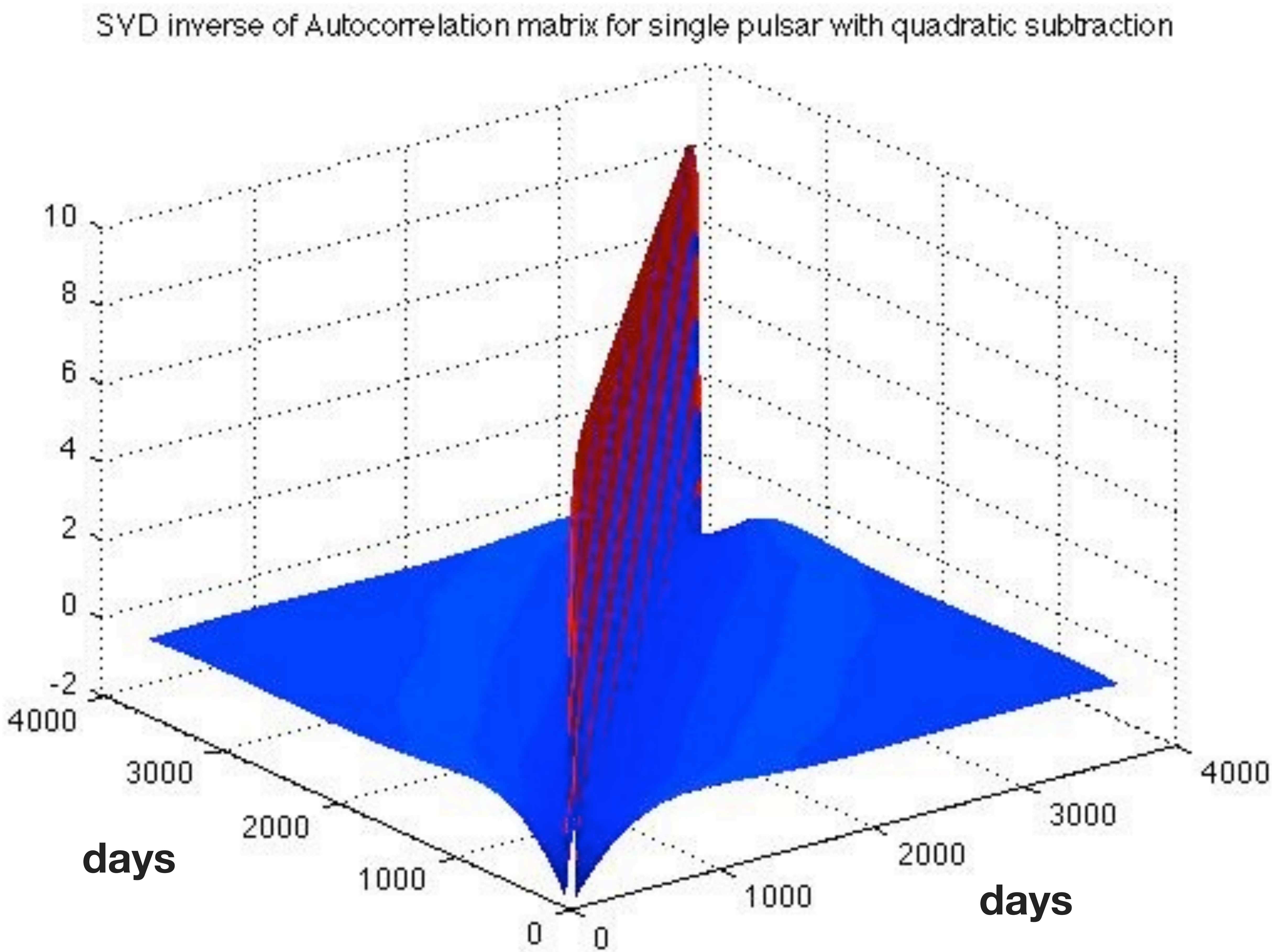
Normalization is chosen so that

$$\langle \mathcal{Q}_{\mathrm{opt}} \rangle = \Omega_\beta$$

$$P_I = \langle r_I r_I^T \rangle_{ij} = R_I \left[\int_{-\infty}^{\infty} df e^{2\pi i f (t_i - t_j)} P_r(f) + \sigma_I^2 \delta_{ij} \right] R_I^T$$



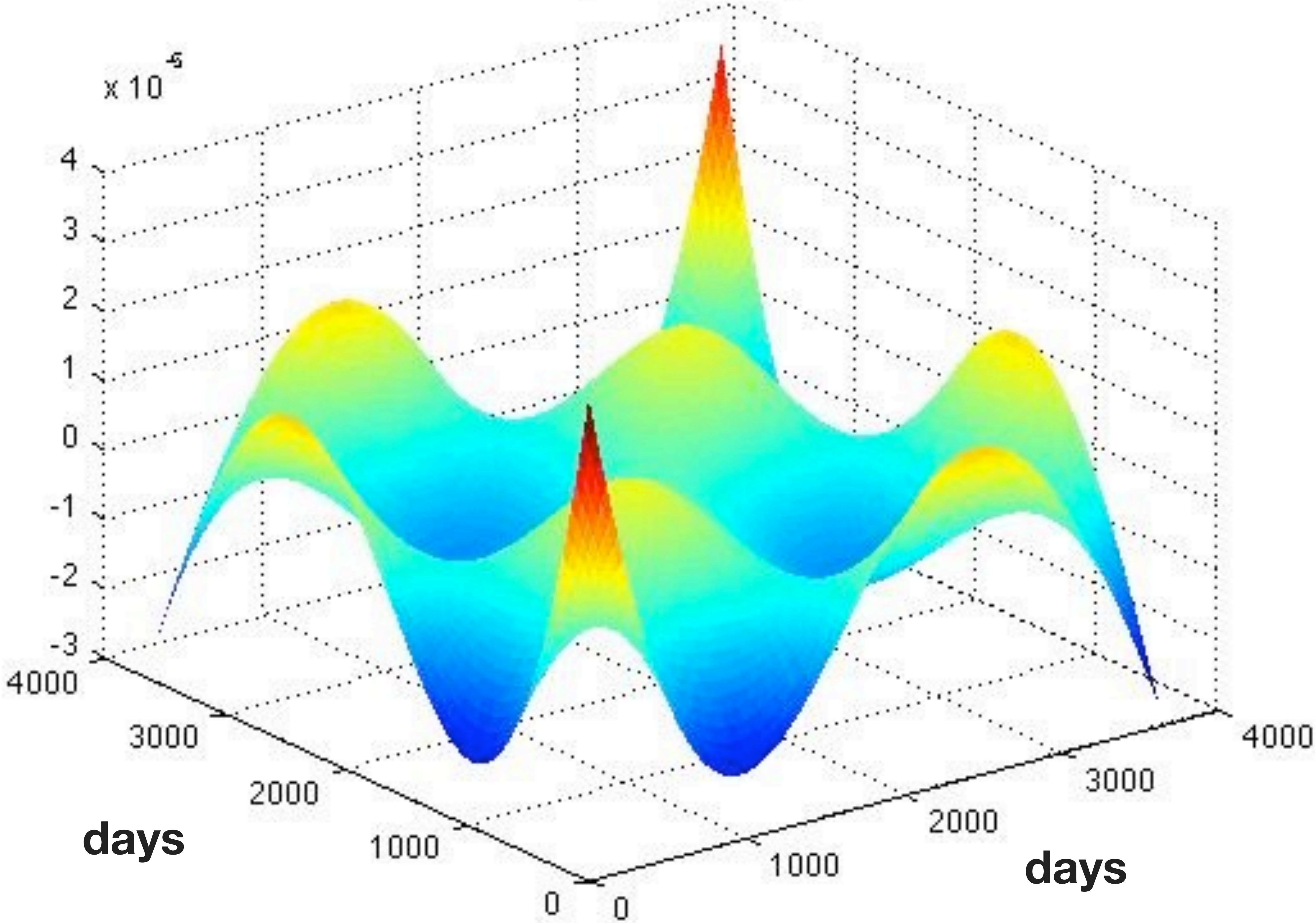
$$P_I^{-1}$$



SVD inverse of Autocorrelation matrix for single pulsar with quadratic subtraction
10
8
6
4
2
0
-2
4000
3000
2000
1000
0
days
0
1000
2000
3000
4000
days

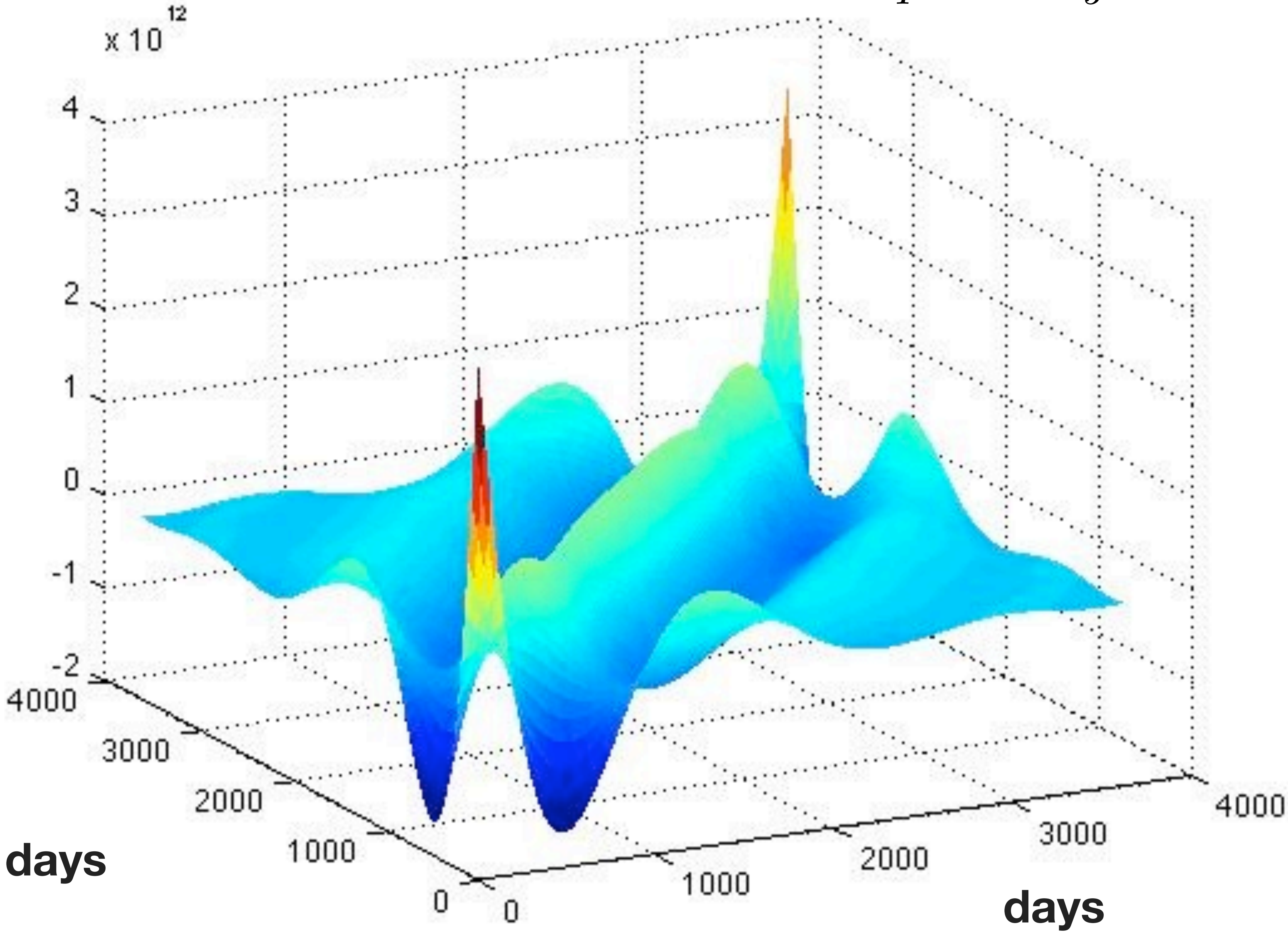
$$\Omega_\beta S_{IJ} = \langle r_I r_J^T \rangle_{ij} = R_I \left[\frac{\chi_{IJ}}{2} \int_{-\infty}^{\infty} df e^{2\pi i f (t_i - t_j)} P_g(f) \right] R_J^T$$

S_{IJ} cross-correlation matrix



$$\mathcal{Q}_{\mathrm{IJ}} = N_{IJ} r_I^T P_I^{-1} S_{IJ} P_J^{-1} r_J$$

Optimal filter: $N_{IJ} P_I^{-1} S_{IJ} P_J^{-1}$



Open Mock Data Challenge

Dataset 1: 36 pulsars, 100ns white noise, SMBBH spectrum, $A = 5 \times 10^{-14}$

Dataset 2: 36 pulsars, different white noises for pulsars (given), SMBBH spectrum, $A = 5 \times 10^{-14}$

Dataset 3: 36 pulsars, different white noises for pulsars (given), SMBBH spectrum, $A = 10^{-14}$, and additional red noise

Open Mock Data Challenge

Dataset 1: 36 pulsars, 100ns white noise, SMBBH spectrum, $A = 5 \times 10^{-14}$

FOUND: $A = (4.9 \pm 0.19) \times 10^{-14}$

$\mathrm{SNR} = 13$

Dataset 2: 36 pulsars, different white noises for pulsars (given), SMBBH spectrum, $A = 5 \times 10^{-14}$

Dataset 3: 36 pulsars, different white noises for pulsars (given), SMBBH spectrum, $A = 10^{-14}$, and additional red noise

Open Mock Data Challenge

Dataset 1: 36 pulsars, 100ns white noise, SMBBH spectrum, $A = 5 \times 10^{-14}$

FOUND: $A = (4.9 \pm 0.19) \times 10^{-14}$

$$\mathrm{SNR} = 13$$

Dataset 2: 36 pulsars, different white noises for pulsars (given), SMBBH spectrum, $A = 5 \times 10^{-14}$

FOUND: $A = (4.7 \pm 0.27) \times 10^{-14}$

$$\mathrm{SNR} = 8.8$$

Dataset 3: 36 pulsars, different white noises for pulsars (given), SMBBH spectrum, $A = 10^{-14}$, and additional red noise

Open Mock Data Challenge

Dataset 1: 36 pulsars, 100ns white noise, SMBBH spectrum, $A = 5 \times 10^{-14}$

FOUND: $A = (4.9 \pm 0.19) \times 10^{-14}$

$\mathrm{SNR} = 13$

Dataset 2: 36 pulsars, different white noises for pulsars (given), SMBBH spectrum, $A = 5 \times 10^{-14}$

FOUND: $A = (4.7 \pm 0.27) \times 10^{-14}$

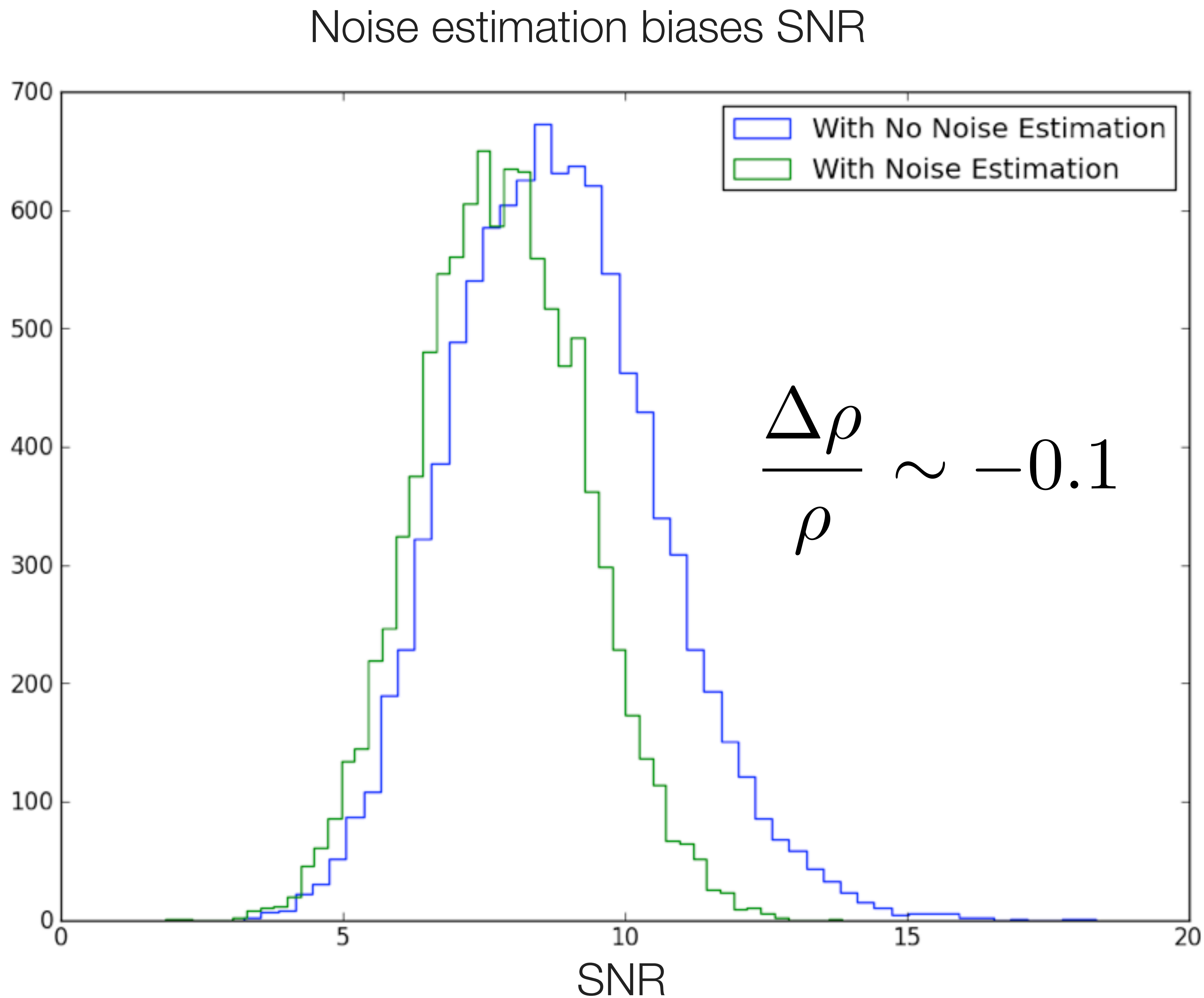
$\mathrm{SNR} = 8.8$

Dataset 3: 36 pulsars, different white noises for pulsars (given), SMBBH spectrum, $A = 10^{-14}$, and additional red noise

FOUND: $A = (1.2 \pm 0.07) \times 10^{-14}$

$\mathrm{SNR} = 8.7$

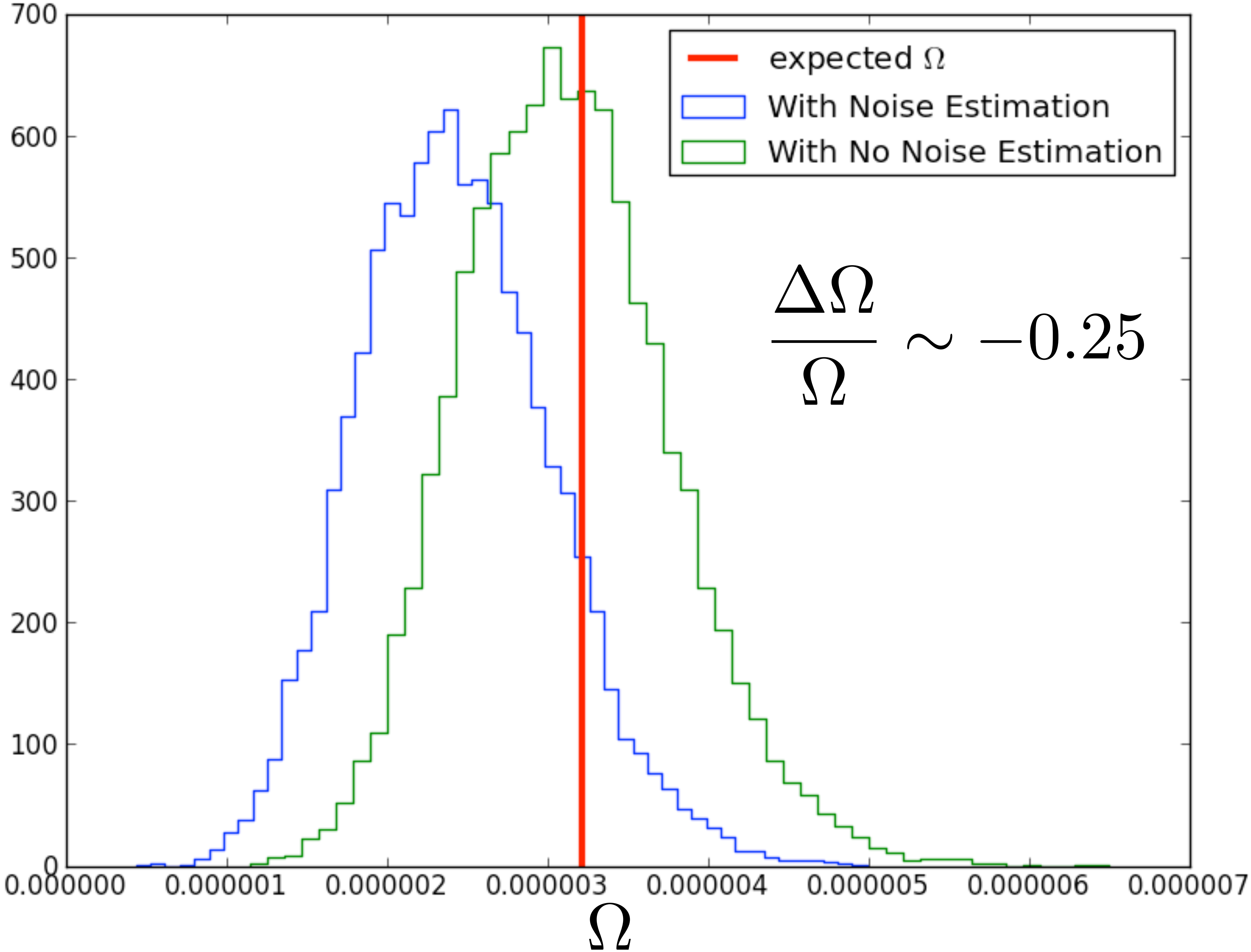
Noise estimation bias



$$\frac{\Delta\rho}{\rho} \sim -0.1$$

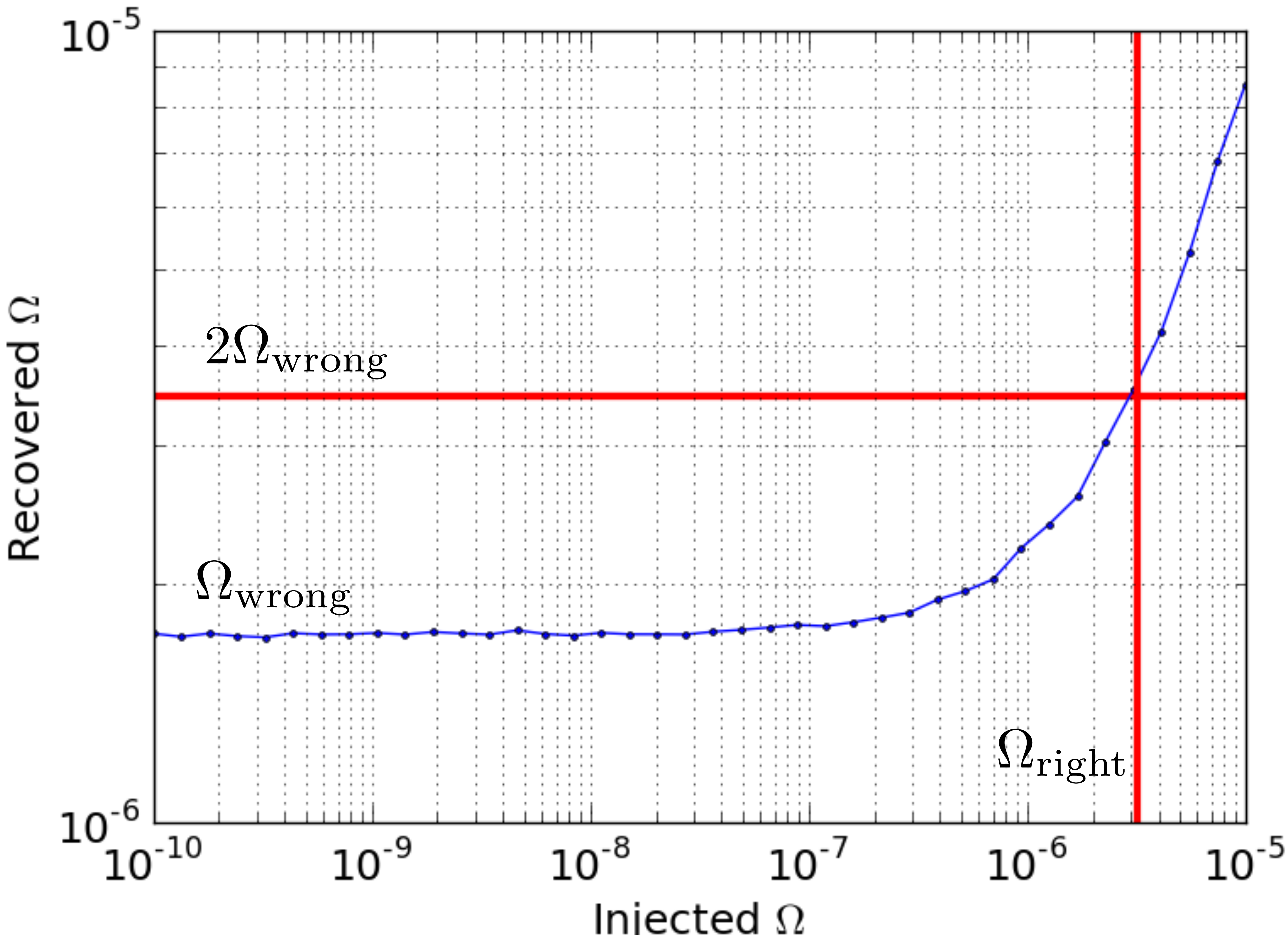
Noise estimation bias

Noise estimation biases amplitude estimation



Noise estimation bias

Two wrongs that make a right



Very preliminary, rough, upper limit estimates*

CAUTION: Based on variance of estimators, not injections

NANOGrav: $A_{95\%} = 6.4 \times 10^{-15}$ compare with 7.2×10^{-15}

Demorest et al. 2012

EPTA (5 pulsar dataset): $A_{95\%} = 6.5 \times 10^{-15}$ compare with 6×10^{-15}

van Haasteren et al. 2011

Agreement between three independent techniques!!

Working on combining both data sets

Working on PPTA data...

*Your mileage may vary

The end

How do we derive the likelihood for the residuals?

- Start from Gaussian likelihood for GWs

$$p(y|\Omega) = \frac{1}{\sqrt{\det(2\pi\Sigma_y)}} \exp(-\frac{1}{2} y^T \Sigma_y^{-1} y)$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_l \end{bmatrix} \quad \mathbf{\Sigma}_y = \begin{bmatrix} \mathbf{P}_1 & \Omega\mathbf{S}_{12} & \cdots & \Omega\mathbf{S}_{1l} \\ \Omega\mathbf{S}_{21} & \mathbf{P}_2 & \cdots & \Omega\mathbf{S}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega\mathbf{S}_{l1} & \Omega\mathbf{S}_{l2} & \cdots & \mathbf{P}_l \end{bmatrix}$$

$$\Omega S_{IJ} = \langle y_I y_J^T \rangle$$

$$P_I = \langle y_I y_I^T \rangle$$

$$y_I \rightarrow r_I = R_I y_I \qquad R_I = I - A_I (A_I^T C_I^{-1} A_I)^{-1} A_I^T C_I^{-1}$$

$$p(y|\Omega) \rightarrow p(r|\Omega) = \frac{1}{\sqrt{\det(2\pi\Sigma_r)}} \exp(-\frac{1}{2} r^T \Sigma_r^{-1} r)$$

$$\Sigma_r = R \Sigma_y R^T$$