Stabilizer Formalism

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Groups

A group is a (non-empty) set G together with a function $\star : G \times G \to G$ such that

1. $a \star (b \star c) = (a \star b) \star c \quad \forall\ a, b, c \in G$
2. $\exists e \in G$ such that $e \star a = a \star e = a \quad \forall\ a \in G$
3. $\forall a \in G \;\; \exists a^{-1} \in G$ such that $a \star a^{-1} = a^{-1} \star a = e \quad \forall\ a \in G$

Examples:

1. $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ with addition.
2. $GL_{\mathbb{C}}(n) = \{n \times n$ invertible matrices$\}$ with matrix product.
3. $\mathcal{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$ with matrix product.
4. $\mathcal{P}_n = \{P_1 \otimes \cdots \otimes P_n : P_j \in \mathcal{P}_1\}$ with matrix product.

Groups

Let G be a group. A non-empty subset $S \subset G$ is a subgroup if

1. $x \in S \Rightarrow x^{-1} \in S$.
2. $x, y \in S \Rightarrow xy \in S$.

Let G be a group and fix $a \in G$. The subgroup of G generated by a is $< a >= \{a^n : n \in \mathbb{Z}\}$

Let $S \subset G$. A word on S is an element $w \in G$ of the form

$$w = x_1^{e_1} \cdots x_n^{e_n}$$

such that $x_i \in S,\ e_i \in \{+1, -1\},\ n \geq 1$

Theorem: Let $S \subset G$. If S is empty, then $< S >= \{e\}$. If S is not empty, then $< S >= \{\text{words on } S\}$

Groups

A group action $G \curvearrowright X$ of a group G on a non-empty set X is a function $G \times X \to X$ denoted by $(g, x) \mapsto g \cdot x$ such that

1. $e \cdot x = x \; \forall x \in X$
2. $g \cdot (h \cdot x) = (gh) \cdot x \; \forall x \in X$ and $\forall g, h \in G$

Example:

$$\mathcal{P}_n \curvearrowright \mathcal{H}^n$$

Let $G \curvearrowright X$ and fix $x \in X$. The stabilizer of x is defined as

$$G_x = Stab(x) = \{g \in G : g \cdot x = x\}$$

Example:

Consider the Bell pair $|\psi\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$. We have that $X_1 X_2, Z_1 Z_2 \in (\mathcal{P}_2)_{|\psi\rangle}$