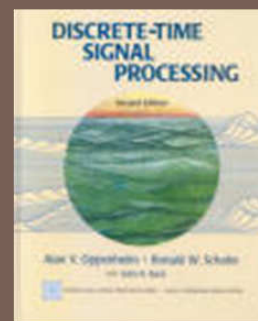
ELCE 705
DIGITAL SIGNAL PROCESSING

Sampling of
continuous-time signals
(4.1- 4.5)

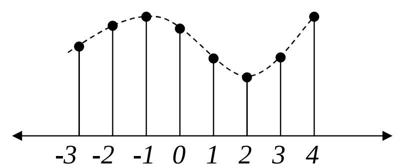


Contents

- Sampling Theorem
- Reconstruction
- Discrete-time Processing of CT Signals
- Continuous-time Processing of DT Signals

Periodic (Uniform) Sampling

- Sampling is a continuous to discrete-time conversion



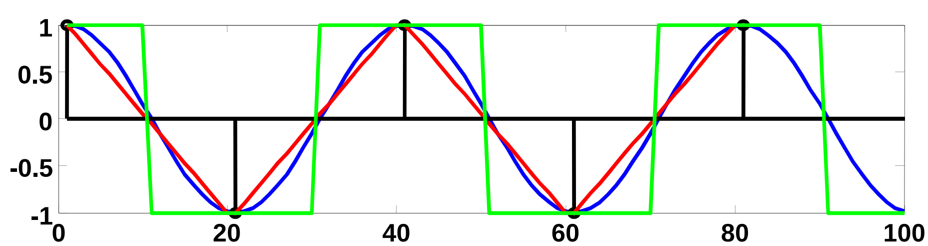
- Most common sampling is periodic

$$x[n] = x_c(nT) \quad -\infty < n < \infty$$

- T is the sampling period in second
- $f_s = 1/T$ is the sampling frequency in Hz
- Sampling frequency in radian-per-second $\Omega_s=2\pi f_s$ rad/sec
- This is the ideal case not the practical but close enough
  - In practice it is implemented with an analog-to-digital converters
  - Get digital signals that are quantized in amplitude and time

Periodic Sampling

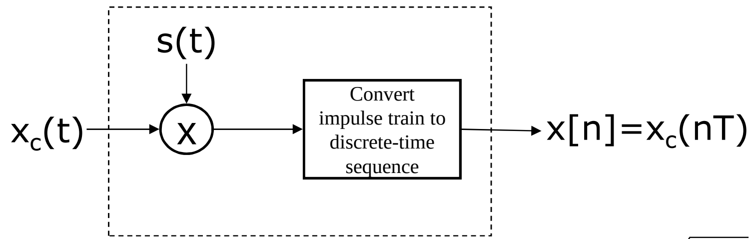
- Sampling is, in general, not reversible
- Given a sampled signal one could fit infinite continuous signals through the samples

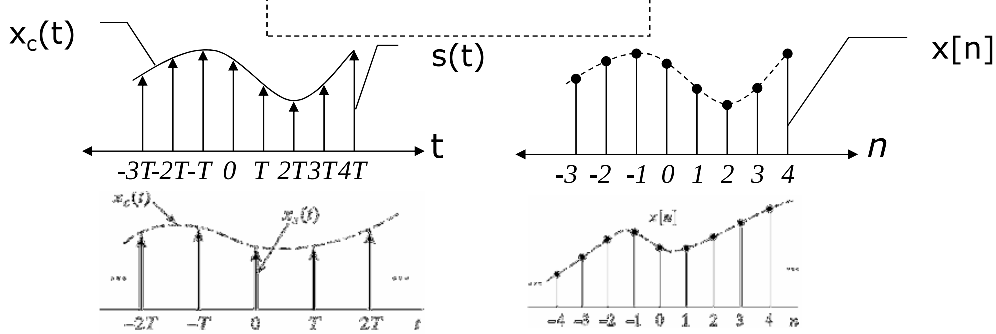


- Fundamental issue in digital signal processing
  - If we loss information during sampling we cannot recover it
- Under certain conditions an analog signal can be sampled without loss so that it can be reconstructed perfectly

Representation of Sampling

- Mathematically convenient to represent in two stages
  - Impulse train modulator
  - Conversion of impulse train to a sequence





Frequency Domain Representation of Sampling

- Modulate (multiply) continuous-time signal with pulse train:

$$x_s(t) = x_c(t)s(t) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT) \qquad s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Take the Fourier Transform of $x_s(t)$ and $s(t)$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \qquad S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

  - Fourier transform of pulse train is again a pulse train
  - Note that multiplication in time is convolution in frequency
  - We represent frequency with $\Omega = 2\pi f$ hence $\Omega_s = 2\pi f_s$

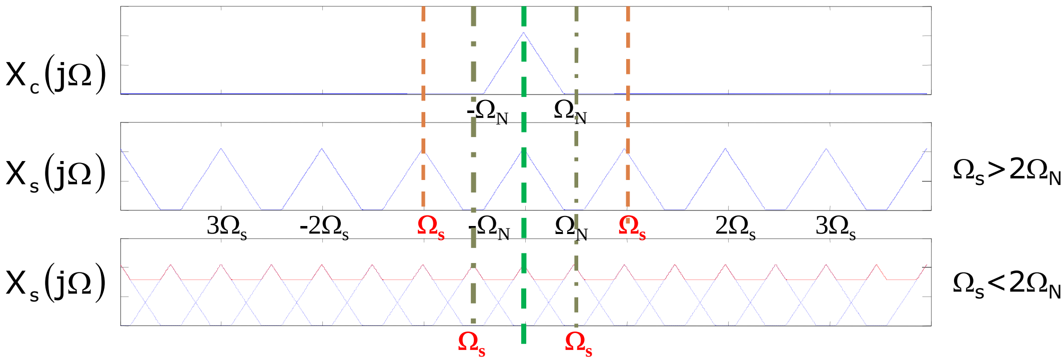
$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

Frequency Domain Representation of Sampling

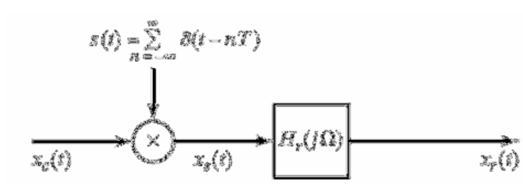
- Convolution with pulse creates replicas at pulse location:

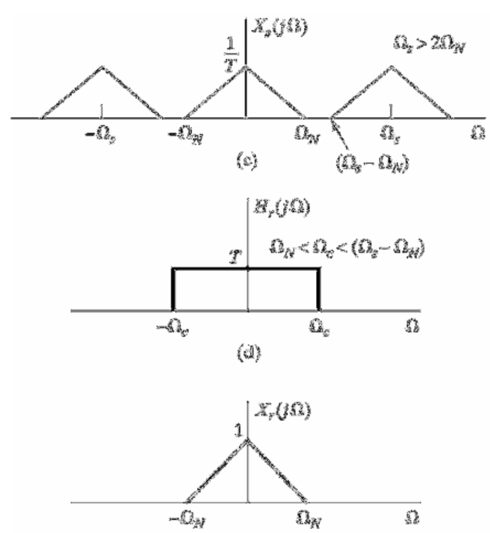
$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j(\Omega - k\Omega_s)\right)$$

  - Creates images of the Fourier transform of the input signal
  - Images are periodic with sampling frequency
  - If $\Omega_s < \Omega_N$ sampling maybe irreversible due to aliasing of images



Recovery of a continuous-time signal





$$X_r(j\Omega) = H_r(j\Omega)X_s(j\Omega)$$

$$X_r(j\Omega) = X_c(j\Omega)$$

Nyquist Sampling Theorem

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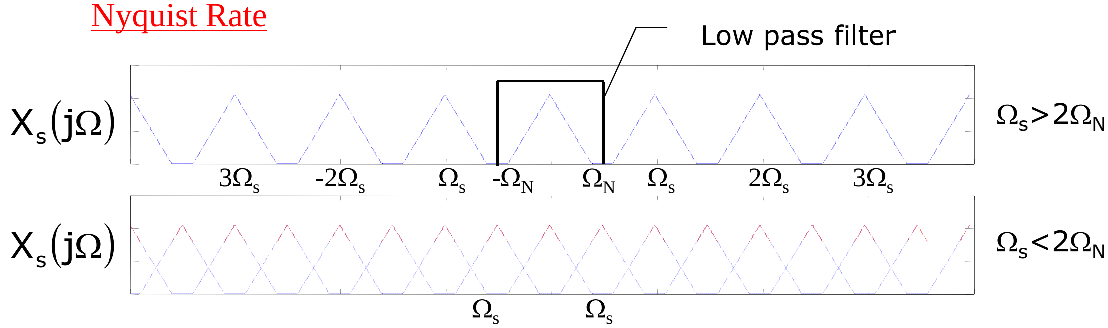
- Let $x_c(t)$ be a bandlimited signal with

$$X_c(j\Omega) = 0 \qquad \text{for } |\Omega| \geq \Omega_N$$

- Then $x_c(t)$ is uniquely determined by its samples $x[n]= x_c(nT)$ if

$$\Omega_s = \frac{2\pi}{T} = 2\pi f_s \geq 2\Omega_N$$

- Ω_N is generally known as the Nyquist Frequency
- The minimum sampling rate that must be exceeded is known as the Nyquist Rate



The expression of $X(e^{jw})$

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- Eventual objective is to express $X(e^{jw})$ for sampling sequence x[n].

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn}$$

$$\text{since} \quad x[n] = x_c(nT) \quad \text{and} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_c[n]e^{-j\omega n}$$

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T})$$

$$\text{have} \quad X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$\text{So} \quad X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - \frac{2\pi k}{T}))$$

- $X(e^{jw})$ is simply a frequency-scaled version of $X_s(j\Omega)$.
- A time normalization from $x_s(t)$ to x[n]

Contents

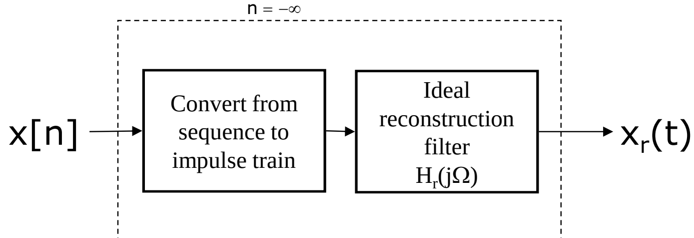
Reconstruction of Bandlimited Signal From Samples

- Sampling can be viewed as modulating with impulse train
- If Sampling Theorem is satisfied
  - The original continuous-time signal can be recovered by filtering sampled signal with an ideal low-pass filter (LPF)
- Impulse-train modulated signal

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

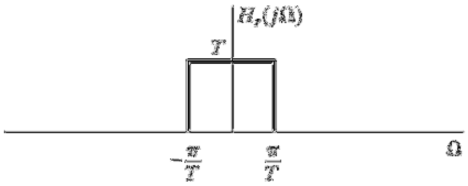
- Pass through LPF with impulse response $h_r(t)$ to reconstruct

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT)$$



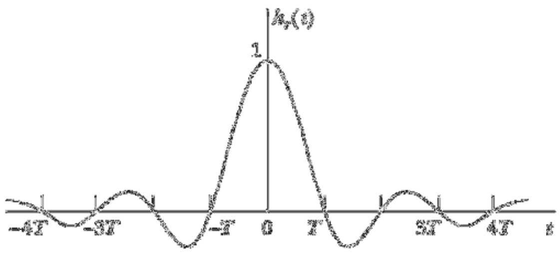
Ideal Reconstruction Filter

- Ideal LPC with cutoff frequency of $\Omega_c=\pi/T$ or $f_c=2/T$



(b)

$$X_r(j\Omega) = H_r(j\Omega) X_s(j\Omega)$$



(c)

$$h_r(t) = \frac{\sin(\pi t / T)}{\pi t / T}$$

Reconstructed Signal

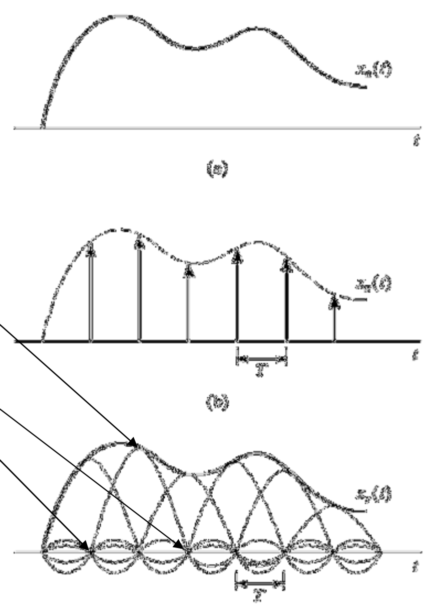
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

sinc function is 1 at t=0

sinc function is 0 at nT

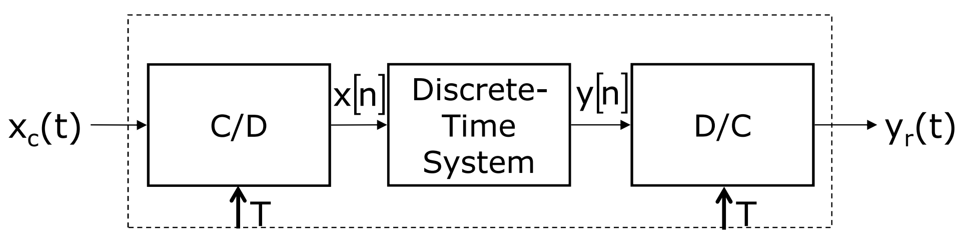
$$X_r(j\Omega) = X(e^{j\Omega T}) H_r(j\Omega)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$


(a)
(b)
(c)

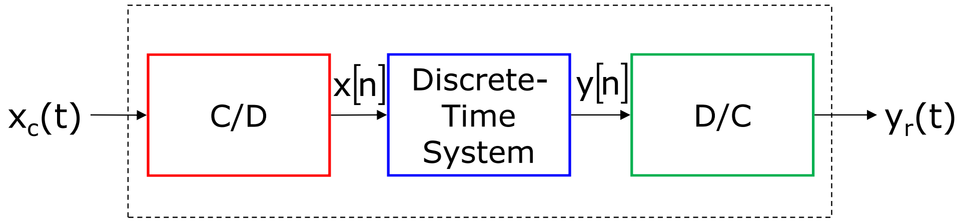
Contents

Discrete-time processing of continuous-time signals



- The overall system is equivalent to a continuous-time system.
- The properties of the overall system are dependent on the choice of the discrete-time system and the sampling rate.
  - The discrete-time system must be linear and time invariant.
  - The input signal must be bandlimited
  - The sampling rate must be high enough.

Discrete-time processing of continuous-time signals



$$x[n] = x_c(nT)$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

$$Y_r(j\Omega) = \begin{cases} TY(e^{j\Omega T}) & |\Omega| < \pi/T \\ 0 & \text{otherwise} \end{cases}$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{T}H(e^{j\omega})\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

LTI discrete-time system

- For the LTI system, we have:

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$
$$So \quad Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T})$$
$$\omega = \Omega T, \qquad X_c(j\Omega) = 0 \; for \; |\Omega| \geq \pi/T$$

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega), |\Omega| < \pi/T \\ 0, \qquad |\Omega| \geq \pi/T \end{cases}$$

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}), |\Omega| < \pi/T \\ 0, \qquad |\Omega| \geq \pi/T \end{cases}$$

Effect frequency response

Example

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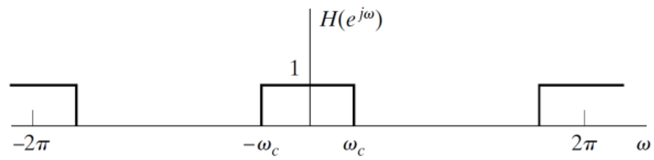
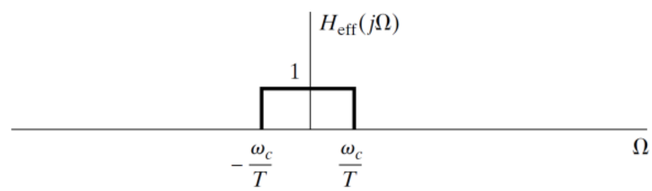
Given a fixed discrete-time lowpass filter

- Varying the sampling period T
- an equivalent continuous-time lowpass filter with a variable cutoff frequency can be implemented.



(a)



Example

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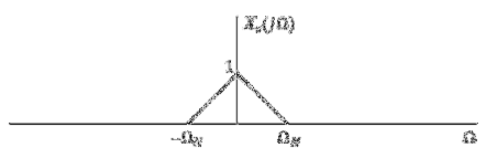
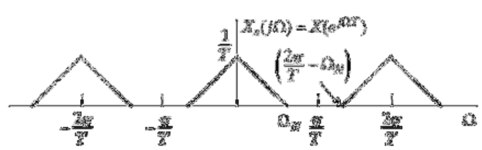
Continuous-time input signal



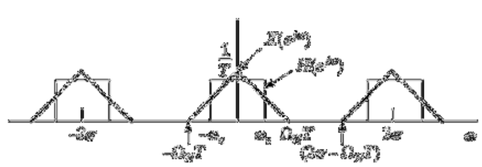
(a)

Sampled continuous-time input signal



(b)

Apply discrete-time LPF



(c)

Example

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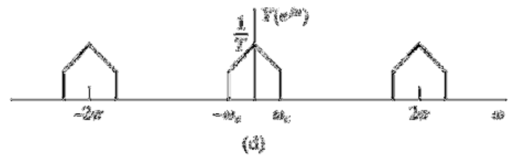
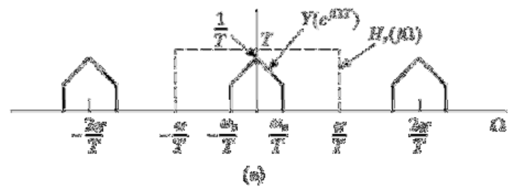
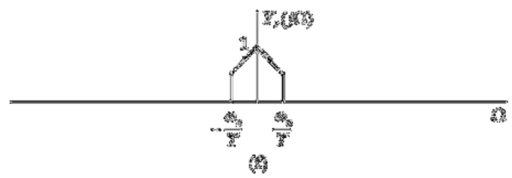
Signal after discrete-time LPF is applied



Application of reconstruction filter



Output continuous-time signal after reconstruction



Impulse Invariance

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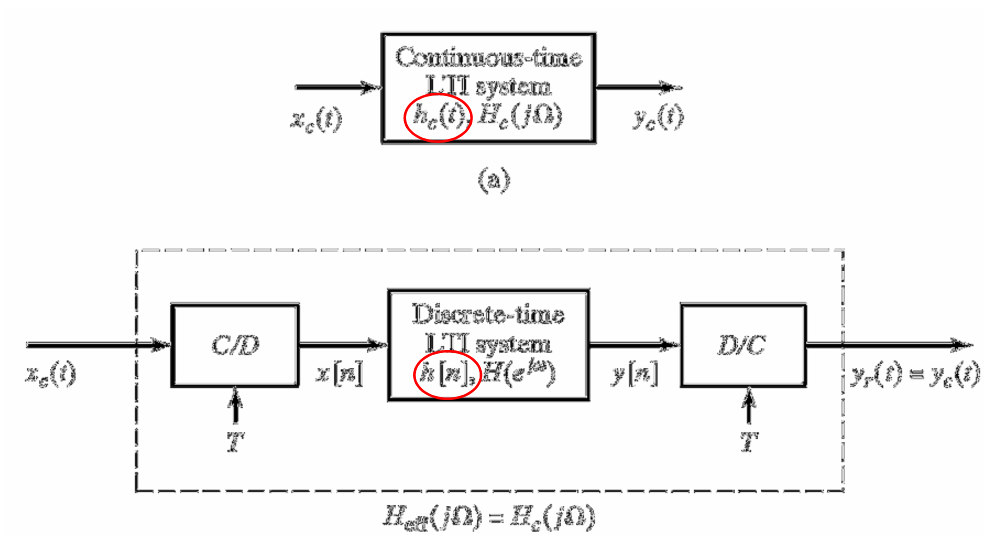


Under the constraints:

$$H(e^{j\omega}) = H_c(j\omega/T), |\omega| < \pi$$

$$H_c(j\Omega) = 0, |\Omega| \geq \pi/T$$

The relationship between the continuous-time impulse response ***$h_c(t)$*** and the discrete-time impulse response ***$h[n]$***

$$h[n] = Th_c(nT)$$

The impulse response of the discrete-time system is a scaled, sampled version of $h_c(t)$.

Example: Impulse Invariance

- Ideal low-pass discrete-time filter by impulse invariance

$$H_c(j\Omega) = \begin{cases} 1 & |\Omega| < \Omega_c \\ 0 & \text{else} \end{cases}$$

- The impulse response of continuous-time system is

$$h_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

- Obtain discrete-time impulse response via impulse invariance

$$h[n] = Th_c(nT) = T\frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n}$$

- The frequency response of the discrete-time system is

$$H_c(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

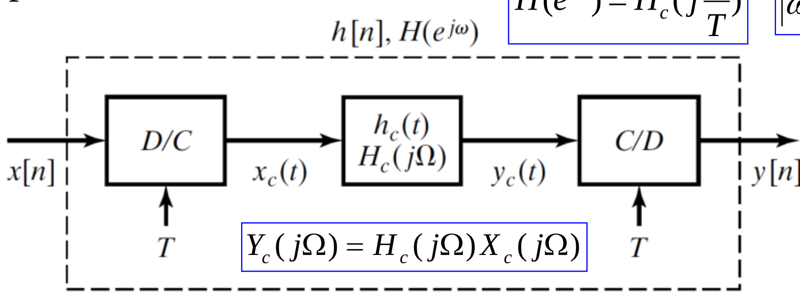
Contents

- Sampling Theorem
- Reconstruction
- Discrete-time Processing of CT Signals
- Continuous-time Processing of DT signals

CT Processing of DT Signals

- Provide interpretation of certain DTS, that have no simple interpretation in discrete domain.

$$\boxed{H(e^{j\omega}) = H_c(j\frac{\omega}{T})} \quad \boxed{|\omega| < \pi}$$



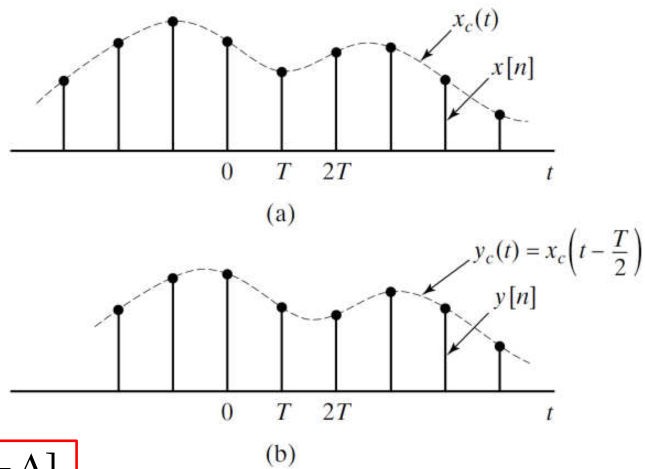
$$x[n] = x_c(nT)$$
$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$
$$X_c(j\Omega) = TX(e^{j\Omega T})$$

$$y[n] = y_c(nT)$$
$$y_c(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$
$$Y(e^{j\omega}) = \frac{1}{T} Y_c(j\frac{\omega}{T})$$

Example- Noninteger Delay

- A discrete-time system

$$H(e^{i\omega}) = e^{-j\omega\Delta} \qquad |\omega| < \pi$$



- Δ is not an integer

$$H_C(j\Omega) = e^{-j\Omega T\Delta}$$

$$y_C(t) = x_C(t - T\Delta)$$

$$y[n] = x_C(nT - T\Delta) = \boxed{x[n-\Delta]}$$

no formal meaning

THE END