1. 插值：

① 线性插值：$L_1(x) = y_k \frac{x_{k+1}-x}{x_{k+1}-x_k} + y_{k+1}\frac{x-x_k}{x_{k+1}-x_k}$ $\left(f(x_0)\frac{x_1-x}{x_1-x_0} + f(x_1)\frac{x-x_0}{x_1-x_0} \right)$

② 二次插值：$L_2(x) = y_{k-1}\frac{(x-x_k)(x-x_{k+1})}{(x_{k-1}-x_k)(x_{k-1}-x_{k+1})} + y_k \frac{(x-x_{k-1})(x-x_{k+1})}{(x_k-x_{k-1})(x_k-x_{k+1})} + y_{k+1}\frac{(x-x_{k-1})(x-x_k)}{(x_{k+1}-x_{k-1})(x_{k+1}-x_k)}$

③ 多项式插值（拉格朗日）：$L_n(x_j)=\sum_{k=0}^n y_k l_k(x_j) = y_j,\ j=0,1,\dots,n$

$$l_k = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)},\ k=0,1,\dots,n$$

$|R_n(x)| \le \frac{M_{n+1}}{(n+1)!}|\omega_{n+1}(x)|$, $\omega_{n+1}=(x-x_0)(x-x_1)\cdots(x-x_n)$,

$M_{n+1} = \max_{a<x<b}|f^{(n+1)}(x)|$, $\sum_{j=0}^n x_j^k l_j(x) = x^k$, $\sum_{j=0}^n l_j(x)=1$

（多次）（高次插值）

④ 均差&牛顿插值：$f[x_0,x_1,\dots,x_n] = \frac{f^{(n)}(\xi)}{n!},\ \xi\in(a,b)$

任意节点：$P_n(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)+\dots+f[x_0,x_1,\dots,x_n](x-x_0)\cdots(x-x_{n-1})$

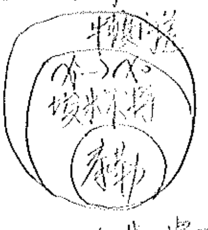
$R_n(x) = f(x)-P_n(x) = f[x,x_0,\dots,x_n]\omega_{n+1}(x)$

等距节点：$x=x_0+th$，牛顿前插公式：$P_n(x_0+th) = f_0 + t\Delta f_0 + \frac{t(t-1)}{2!}\Delta^2 f_0+\dots+\frac{t(t-1)\cdots(t-n+1)}{n!}\Delta^n f_0$

$R_n(x) = \frac{t(t-1)\cdots(t-n)}{(n+1)!}h^{n+1}f^{(n+1)}(\xi),\ \xi\in(x_0,x_n)$

⑤ 埃尔米特插值（导数值相等） 泰勒多项式：$P_n(x) = f(x_0)+f'(x_0)(x-x_0)+\dots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},\ \xi\in(a,b)$



① $H_{2n+1}(x) = \sum_{j=0}^n f(x_j)H_{n,j}(x) + \sum_{j=0}^n f'(x_j)\hat{H}_{n,j}(x)$ ④ $L_{1,0}(x) = \frac{x-x_1}{x_0-x_1}$, $L_{1,1}(x) = \frac{x-x_0}{x_1-x_0}$

② $H_{n,j}(x) = [1-2(x-x_j)L'_{n,j}(x_j)]L^2_{n,j}(x)$ ③ $\hat{H}_{n,j}(x) = (x-x_j)L^2_{n,j}(x)$

（分段光滑度）

⑥ 三次样条插值：① 一阶导数值（两端） ② 两端二阶导数 ③ x_n x_0 为周期性函数

$S(x) = \sum_{j=0}^n [y_j\alpha_j(x) + m_j\beta_j(x)]$, $\alpha_k(x) = \left(1+2\frac{x-x_k}{x_{k+1}-x_k}\right)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2$, $\beta_k(x) = (x-x_k)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2$

$S'(x_j)=m_j,\ j=0,1,\dots,n$ [分段三次埃尔米特]

$S(x) = M_j\frac{(x_{j+1}-x)^3}{6h_j} + M_{j+1}\frac{(x-x_j)^3}{6h_j} + \left(y_j - \frac{M_jh_j^2}{6}\right)\frac{x_{j+1}-x}{h_j} + \left(y_{j+1}-\frac{M_{j+1}h_j^2}{6}\right)\frac{x-x_j}{h_j}$ 通用形式

Cubic polynomial $x(t) = [2(x_0-x_1)+(\alpha_0+\alpha_1)]t^3 + [3(x_1-x_0)-(\alpha_1+2\alpha_0)]t^2 + \alpha_0 t + x_0$

$(x_0+\alpha_0, y_0+\beta_0)$ $y(t)$ ……

(x_0,y_0) $(x_1-\alpha_1, y_1-\beta_1)$ guide point construct cubic Hermite approximations

(x_1,y_1)

Numerical Differentiation: $f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(\xi)$.

3 point formula: $f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)] + \frac{h^2}{3} f^{(3)}(\xi_0)$ | $\xi_0 \in (x_0, x_0+2h)$

$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6} f^{3}(\xi_1)$ | $\xi_1 \in (x_0-h, x_0+h)$

5 point: $f'(x_0) = \frac{1}{12h}[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)] + \frac{h^4}{30} f^{5}(\xi)$ | (x_0-2h, x_0+2h)

End: $f'(x_0) = \frac{1}{12h}[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)] + \frac{h^4}{5} f^{5}(\xi)$ | $\xi \in (x_0, x_0+4h)$

Trapezoidal: $\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi)$

Mid-point: $\int_{x_{-1}}^{x_1} f(x)dx = 2hf(x_0) + \frac{h^3}{3} f''(\xi)$ $h = \frac{b-a}{2}$

Simpson: $\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{4}(\xi)$ $x_0 < \xi < x_2$

Simpson's three-eighths rule: $\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3h^5}{80} f^{4}(\xi)$

复化辛普森: $\int_a^b f(x)dx = \frac{h}{3}[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(b)] - \frac{b-a}{180} h^4 f^{4}(\mu)$

复化梯形: $\int_a^b f(x)dx = \frac{h}{2}[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)] - \frac{b-a}{12} h^2 f''(\mu)$

$h = \frac{b-a}{n}$, $x_j = a + jh$

Adaptive quadrature methods: $\int_a^b f(x)dx = S(a,b) - \frac{h^5}{90} f^{4}(\mu)$, $\frac{b-a}{4} = \frac{h}{2}$, $n=4$

$\int_a^b f(x)dx = S(a, \frac{a+b}{2}) + S(\frac{a+b}{2}, b) - \frac{1}{16}(\frac{h^5}{90}) f^{4}(\tilde{\mu})$.

$p(\lambda) = \det(A - \lambda I)$, λ eigenvalue $(A - \lambda I)x = 0$, x eigenvector.

Jacobi iterative technique: $x^k = D^{-1}(L+U)x^{k-1} + D^{-1}b$, $T_j = D^{-1}(L+U)$, $c_j = D^{-1}b$

Gauss-Seidel: $x^k = (D-L)^{-1}Ux^{k-1} + (D-L)^{-1}b$, $x^k = T_g x^{k-1} + c_g$

exact solution not known $\frac{\|x^{k+1} - x^k\|}{\|x^{k+1}\|}$ to measure $\|\delta^{k+1} - \delta\|$ $A^{-1} = A \backslash I$

拉普拉斯变换: $\cosh t = \frac{e^t + e^{-t}}{2}$, $\sinh t = \frac{e^t - e^{-t}}{2}$ $L[e^{at}] = \frac{1}{s-a}$

二倍角公式 $\sin 2\alpha = 2\cos\alpha\sin\alpha$, $\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \cos^2\alpha - \sin^2\alpha$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$,

$L[u(t-a)f(t-a)] = e^{-as}F(s)$, $F(s) = L[f(t)]$, $a > 0$

$L[tf(t)] = -F'(s)$, $L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$

$L[y''] = s^2 L[y] - sf(0) - f'(0)$, $L[y'] = sL[y] - f(0)$, $L[u(t-\pi)\sin t] = -\frac{e^{-\pi s}}{s^2+1}$

$L[u(t-\pi)\sin t] = -\frac{e^{-\pi s}}{s^2+1}$ $L[t^n] = \frac{n!}{s^{n+1}}$, $L^{-1}[\frac{1}{s^{n+1}}] = \frac{t^n}{n!}$

$$\frac{dy}{dx}+p(x)y=b(x) \Rightarrow y(x)=e^{-\int p(x)dx}\left(C+\int b(x)e^{\int p(x)dx}dx\right)$$

$$\int\frac{1}{\sqrt{1-y^2}}=\arcsin y$$

埃尔米特插值 ① 通过 $(x_0,f(x_0)),(x_1,f(x_1)),(x_2,f(x_2))$ 三点，$p'(x_1)=f'(x_1)$

$$p(x)=f(x_0)+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)+A(x-x_0)(x-x_1)(x-x_2)$$

$$A=\frac{f'(x_1)-f[x_0,x_1]-(x_1-x_0)f[x_0,x_1,x_2]}{(x_1-x_0)(x_1-x_2)},\quad R(x)=\frac{1}{4!}f^{4}(\xi)(x-x_0)(x-x_1)^2(x-x_2)$$

② 两个节点，四个函数. $H_3(x_k)=y_k,\ H_3(x_{k+1})=y_{k+1},\ H_3'(x_k)=m_k,\ H_3'(x_{k+1})=m_{k+1}$.

$$H_3(x)=\left(1+2\frac{x-x_k}{x_{k+1}-x_k}\right)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2y_k+\left(1+2\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2y_{k+1}$$

$$+(x-x_k)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2m_k+(x-x_{k+1})\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2m_{k+1}$$

$$R_3(x)=\frac{1}{4!}f^{4}(\xi)(x-x_k)^2(x-x_{k+1})^2,\quad \xi\in(x_k,x_{k+1})$$

分段线性插值：$I_h(x)=\frac{x-x_{k+1}}{x_k-x_{k+1}}f_k+\frac{x-x_k}{x_{k+1}-x_k}f_{k+1},\quad x_k\le x\le x_{k+1}$

$$\max|f(x)-I_h(x)|\le\frac{M_2}{2}\max_{x_k\le x\le x_{k+1}}|(x-x_k)(x-x_{k+1})|,\ \max|f(x)-I_h(x)|\le\frac{M_2}{8}h^2$$

$M_2=\max\limits_{a\le x\le b}|f''(x)|$

分段三次埃尔米特插值：

$$I_h(x)=\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2\left(1+2\frac{x-x_k}{x_{k+1}-x_k}\right)f_k+\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2\left(1+2\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)f_{k+1}$$

$$+\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2(x-x_k)f_k'+\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2(x-x_{k+1})f_{k+1}'$$

$$|f(x)-I_h(x)|\le\frac{1}{384}h_k^4\max$$

三次样条：

$$S'(x)=-M_j\frac{(x_{j+1}-x)^2}{2h_j}+M_{j+1}\frac{(x-x_j)^2}{2h_j}+\frac{y_{j+1}-y_j}{h_j}-\frac{M_{j+1}-M_j}{6}h_j$$

$$S''(x)=M_j\frac{x_{j+1}-x}{h_j}+M_{j+1}\frac{x-x_j}{h_j}$$

$$S(x)=M_j\frac{(x_{j+1}-x)^3}{6h_j}+M_{j+1}\frac{(x-x_j)^3}{6h_j}+\left(y_j-\frac{M_jh_j^2}{6}\right)\frac{x_{j+1}-x}{h_j}+$$

$$\left(y_{j+1}-\frac{M_{j+1}h_j^2}{6}\right)\frac{x-x_j}{h_j},\quad j=0,1,\dots,n-1$$

$$\mu_jM_{j-1}+2M_j+\lambda_jM_{j+1}=d_j,\quad \mu_j=\frac{h_{j-1}}{h_{j-1}+h_j},\quad \lambda_j=\frac{h_j}{h_{j-1}+h_j},\quad d_j=6f[x_{j-1},x_j,x_{j+1}]$$

clamped: $2M_0+M_1=\frac{6}{h_0}(f[x_0,x_1]-f_0'),\quad M_{n-1}+2M_n=\frac{6}{h_{n-1}}(f_n'-f[x_{n-1},x_n])$

if $\lambda_0=1$, $d_0=\frac{6}{h_0}(f[x_0,x_1]-f_0')$, $\mu_n=1$, $d_n=\frac{6}{h_{n-1}}(f_n'-f[x_{n-1},x_n])$

$$\begin{bmatrix} 2 & \lambda_0 & & & \\ \mu_1 & 2 & \lambda_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-1} & 2 & \lambda_{n-1} \\ & & & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

第二种边界：$M_0=f_0''$, $M_n=f_n''$

natural boundary

if $\lambda_0=\mu_n=0$, $d_0=2f_0''$, $d_n=2f_n''$

periodic: $M_0=M_n$, $\lambda_n M_1+\mu_n M_{n-1}+2M_n=d_n$, $\lambda_n=\frac{h_0}{h_{n-1}+h_0}$, $\mu_n=1-\lambda_n=\frac{h_{n-1}}{h_{n-1}+h_0}$

$d_n=6\,\frac{f[x_0,x_1]-f[x_{n-1},x_n]}{h_0+h_{n-1}}$

$$\begin{bmatrix} 2 & \lambda_1 & & & \mu_1 \\ \mu_2 & 2 & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-1} & 2 & \lambda_{n-1} \\ \lambda_n & & & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

二分法误差 $\frac{b-a}{2^{k+1}}$ $\quad k+1\geq\log_2(b-a)\times 10^8$

Jacobi: $a_{ii}x_i^{k+1}=-\sum_{j=1}^{i-1}a_{ij}x_j^{(k)}-\sum_{j=i+1}^{n}a_{ij}x_j^{(k)}+b_i$

$$\begin{cases} x^0=(x_1^0,x_2^0,\dots,x_n^0)^T \\ x_i^{k+1}=\left(b_i-\sum_{j=1,\,j\neq i}^{n}a_{ij}x_j^k\right)/a_{ii} \\ i=1,2,\dots,n;\quad k=0,1,\dots \text{ 迭代次数} \end{cases}$$

Gauss: $a_{ii}x_i^{k+1}=b_i-\sum_{j=1}^{i-1}a_{ij}x_j^{k+1}-\sum_{j=i+1}^{n}a_{ij}x_j^k$, $i=1,2,\dots,n$

$$\begin{cases} x^0=(x_1^0,\dots,x_n^0)^T \text{ 初值} \\ x_i^{k+1}=\left(b_i-\sum_{j=1}^{i-1}a_{ij}x_j^{k+1}-\sum_{j=i+1}^{n}a_{ij}x_j^k\right)/a_{ii} \\ i=1,2,\dots,n;\quad k=0,1,\dots \end{cases}$$

$$\Rightarrow\begin{cases} x^0=(x_1^0,\dots,x_n^0)^T \\ x_i^{k+1}=x_i^k+\Delta x_i \\ \Delta x_i=\left(b_i-\sum_{j=1}^{i-1}a_{ij}x_j^{k+1}-\sum_{j=i}^{n}a_{ij}x_j^k\right)/a_{ii} \\ i=1,2,\dots,n;\quad k=0,1,\dots \end{cases}$$

变分: $\varphi(x)=\frac{1}{2}(Ax,x)-(b,x)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j-\sum_{j=1}^{n}b_jx_j$

$Ax=b$, $\varphi(x)-\varphi(x^*)=\frac{1}{2}(A(x-x^*),x-x^*)$

最速: $\alpha_k=-\frac{(Ax^k-b,p^k)}{(Ap^k,p^k)}$, $\quad \|x^k-x^*\|_A\leq\left(\frac{\lambda_1-\lambda_n}{\lambda_1+\lambda_n}\right)^k\|x^0-x^*\|_A$

$\varphi(x^k+\alpha_kp^k)\leq\varphi(x^k+\alpha p^k)$, $\forall\alpha\in R$