IMSE 005
DIGITAL SIGNAL PROCESSING

Chapter3

Z-Transform

Contents

- Definition of z-transform
- Properties of ROC
- Inverse Z-transform
- Z-transform properties
- Analysis of LTI system using z-transform

Introduction

- The discrete-time Fourier transform (DTFT) was introduced.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

- *Z-transform* (ZT)
  - ZT can be thought of as a generalization of the DTFT
  - ZT is more complex than DTFT (both literally and figuratively), but provides a great deal of insight into system design and behavior
  - In going from the DTFT to the ZT we replace $e^{j\omega n}$ by z^n

Definition of the Z-transform

- Replace (generalize) the complex exponential building blocks $X(e^{j\omega})$ by z^n

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Two-sided
Bilateral z-transform

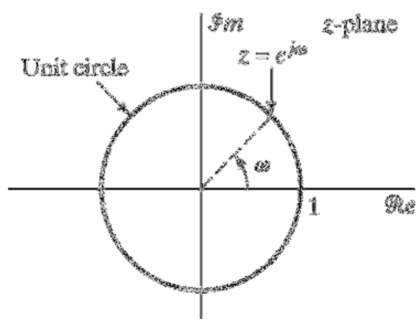
- For an arbitrary *z*, using polar notation : $z = re^{j\omega}$

  so $\boxed{z^n = r^n e^{j\omega n}}$
- If both *r* and *ω* are real, then z^n :
  - A complex exponential (*i.e.* sines and cosines)
  - With a real temporal envelope that can be either exponentially decaying or expanding

Generalizing the frequency variable

- For $z^n = r^n e^{j\omega n}$
- If r=1, |z|=1, DTFT corresponds to the particular case of z-transform

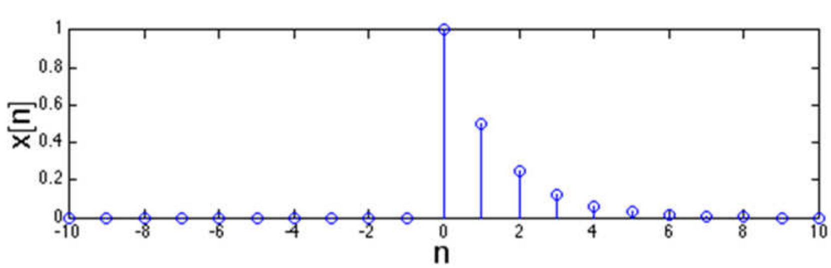
$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\left(re^{j\omega}\right)^{-n}$$
$$= \sum_{n=-\infty}^{\infty} \{x(n)r^{-n}\}e^{-j\omega n}$$
$$= \mathcal{F}\{x(n)r^{-n}\} \quad \Longrightarrow$$



$$X(z)\Big|_{z=e^{j\omega}} = X(\omega) = \mathcal{F}\{x(n)\}$$

Computing the Z-*transform: an example*

- Example 1: **Consider the time function** $x[n] = \alpha^n u[n]$

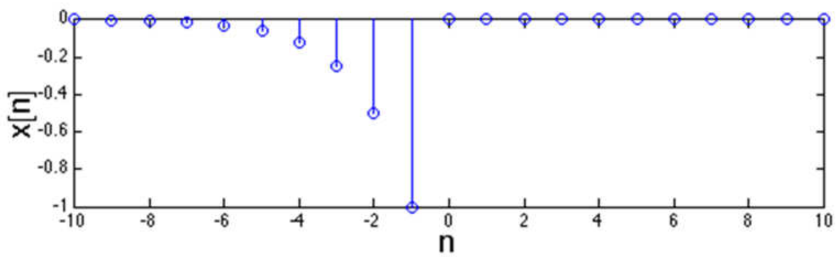


$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

$$= \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$$

Another example …

- Example 2: **Now consider the time function** $x[n] = -\alpha^n u[-n-1]$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = \sum_{n=-\infty}^{-1} -(\alpha z^{-1})^n$$

Let $l = -n; n = -\infty \Rightarrow l = \infty; n = -1 \Rightarrow l = 1$

Then, $\sum_{n=-\infty}^{-1} -(\alpha z^{-1})^n = \sum_{l=1}^{\infty} -(z\alpha^{-1})^l = 1 - \sum_{l=0}^{\infty} (z\alpha^{-1})^l = 1 - \frac{1}{1 - z\alpha^{-1}} = \frac{1}{1 - \alpha z^{-1}}$

Zeros and Poles

- The z-transform is most useful when the infinite sum can be expressed in closed form, i.e., rational function

X(z)= P(z) / Q(z)

Where P(z) and Q(z) are polynomials in z.

- The values of z for which X(z)=0 are called the zeros of X(z)
- The values of z for which X(z) is infinite are referred to as the poles of X(z). The poles of X(z) for finite values of z are the roots of the denominator polynomial.

Region of convergence (ROC)

- The *Z*-transforms were identical for Examples 1 and 2 even though the time functions were different?
  - Yes, indeed, very different time functions can have the same *Z*-transform!
  - What's missing in this characterization? The region of convergence (ROC).
- The ROC consists of all values of z, such that the following inequality holds. $$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$
- It is possible for the z-transform to converge even if the Fourier transform does not. (e.g. u[n])

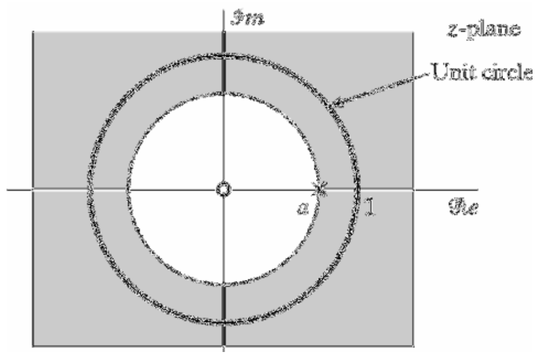
ROC

- So in general, a complete *Z*-transform includes
  - A time function
  - and its ROC
- If $z=z_1$, is in the ROC, then all values of z on the circle defined by $|z|=|z_1|$ will also be in the ROC.
- The ROC will consist of a ring in the z-plan centered about the origin.
  - Its outer boundary will be a circle (or may extend outward to infinity)
  - Its inner boundary will be a circle (or may extend inward to include the origin).

What shapes are ROCs for *Z*-transforms?

- In Example 1, $X(z)=\sum_{n=0}^{\infty}\alpha^n z^{-n}$
- the ROC was $|z|>|\alpha|$ We can represent this graphically as:

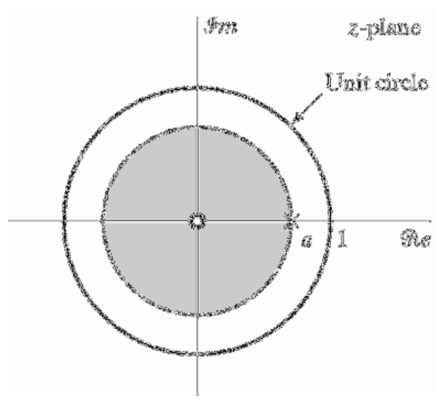
(ROC is Shaded area)



What shapes are ROCs for *Z*-transforms?

- In Example 2, $X(z)=\sum_{n=-\infty}^{-1}\alpha^n z^{-n}$
- the ROC was $|z|<|\alpha|$ We can represent this graphically as:

(ROC is shaded area)



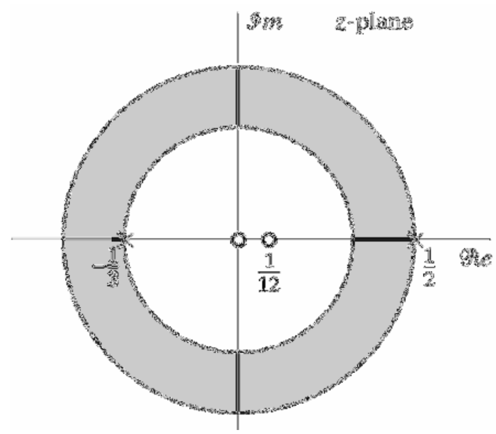
Sum of two exponentials

$$x[n] = \left(-\tfrac{1}{3}\right)^n u[n] - \left(\tfrac{1}{2}\right)^n u[-n-1].$$

$$X(z) = \underbrace{\frac{1}{1+\frac{1}{3}z^{-1}}}_{\frac{1}{3} < |z|} + \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z| < \frac{1}{2}},$$

$$= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

$$= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$



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- Z-transform properties
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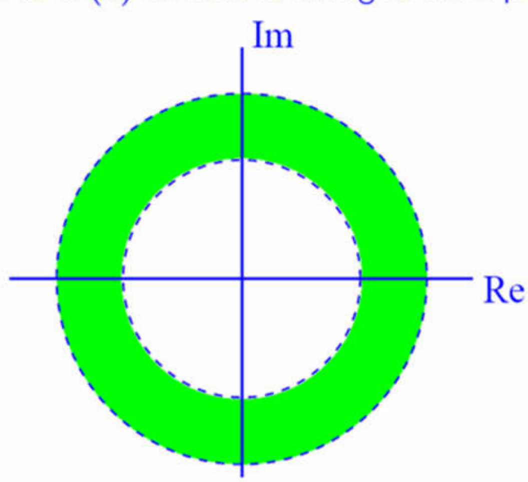
General form of ROCs

- There are four types of ROCs for Z-transforms, and they depend on the type of the corresponding time functions
- Four types of time functions:
  - Right-sided
  - Left-sided
  - "Both"-sided (infinite duration)
  - Finite duration

Properties of the ROC

Property 1: The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.



The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform includes the unit circle.

Properties of the ROC

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Property 2: The ROC does not contain any poles.

Property 3: If $x(n)$ is of finite duration, then the ROC is the entire z-plane except possibly $z = 0$ and/or $z = \infty$.

$$X(z) = \sum_{n=N_1}^{N_2} x(n) z^{-n} \qquad \text{finite duration signal}$$

Particular cases:

- if $N_1 < 0$ and $N_2 > 0$ then the ROC does not include $z = 0$ and $z = \infty$
- if $N_1 \geq 0$ then the ROC includes $z = \infty$, but does not include $z = 0$
- if $N_2 \leq 0$ then the ROC includes $z = 0$, but does not include $z = \infty$

Finite Length Sequence

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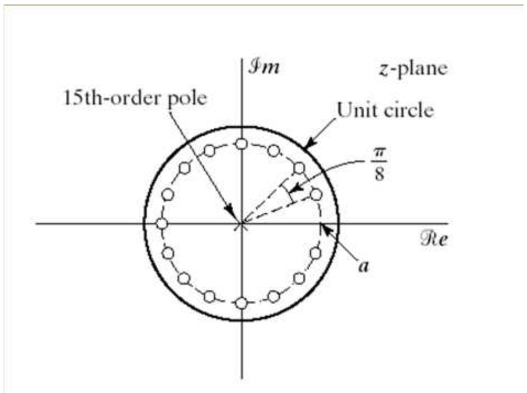
$$x[n] = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$



Properties of the ROC

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Property 4: If $x(n)$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.

$$X(z) = \sum_{n=N_1}^{\infty} x(n)z^{-n} \qquad \text{right} - \text{sided sequence}$$

Particular cases:

- if $N_1 < 0$ then the ROC does not include $z = \infty$
- if $N_1 \geq 0$ then the ROC includes $z = \infty$

Properties of the ROC

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Property 5: If $x(n)$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC.

$$X(z) = \sum_{n=-\infty}^{N_2} x(n)z^{-n} \qquad \text{left} - \text{sided sequence}$$

Particular cases:

- if $N_2 > 0$ then the ROC does not include $z = 0$
- if $N_2 \leq 0$ then the ROC includes $z = 0$

Properties of the ROC

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Property 6: If $x(n)$ is a two-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then the ROC will be a ring in the z-plane that includes the circle $|z| = r_0$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad \text{two} - \text{sided sequence}$$

Any two-sided sequence can be represented as a direct sum of a right-sided and left-sided sequences $\Longrightarrow$ the ROC of this composite signal will be the intersection of the ROC's of the components.

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Inverse z-transforms

- The following technique will be used for calculating inverse z-transform
  - Inspection Method
  - Partial fraction expansion
  - Long division
  - Taylor series

Partial fraction expansion

The partial fraction method of obtaining inverse z-transforms builds on the fact that we know that

$$a^n u[n] \Leftrightarrow \frac{z}{z-a} = \frac{1}{1-az^{-1}} \quad \text{for the ROC } |z| > |a| \text{ and that}$$

$$-a^n u[-n-1] \Leftrightarrow \frac{1}{1-az^{-1}} \quad \text{for the ROC } |z| < |a|$$

Let's consider another example ..... let

$$H(z) = \frac{z(3z-7)}{(z-2)(z-3)} = \frac{(3-7z^{-1})}{(1-2z^{-1})(1-3z^{-1})}$$

Note that this system has zeros at 0 and 7/3, and poles at 2 and 3.

Partial fraction expansion

The simplest case:

If

1. the order of the numerator of the polynomial in z^{-1} is less than the order of its denominator (as it is in this case), and
2. all the poles of the z-transform are at different locations in the z-plane (as they are in this case),

then we can write

$$H(z) = \frac{A}{(1-2z^{-1})} + \frac{B}{(1-3z^{-1})}$$

where the as-yet undetermined coefficients are referred to as the *residues* of the z-transform, following the term used in complex calculus.

Partial fraction expansion

The residues can be determined as follows:

$$A = H(z)(1-2z^{-1})\Big|_{z=2} = \left.\frac{3-7z^{-1}}{1-3z^{-1}}\right|_{z=2} = \frac{-\frac{1}{2}}{-\frac{1}{2}} = 1$$

$$B = H(z)(1-3z^{-1})\Big|_{z=3} = \left.\frac{3-7z^{-1}}{1-2z^{-1}}\right|_{z=2} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

Hence

$$H(z) = \frac{1}{1-2z^{-1}} + \frac{2}{(1-3z^{-1})}$$

and if we are told that the system is causal, then the corresponding inverse z-transform is

$$h[n] = [2^n + 2(3^n)]u[n]$$

General Consideration

- Obtain a partial fraction expansion and identify the sequences corresponding to the individual terms.

$$X(z)=\frac{b_0\prod_{k=1}^{M}(1-c_k z^{-1})}{a_0\prod_{k=1}^{N}(1-d_k z^{-1})} \Rightarrow \quad X(z)=\begin{cases}\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}} & M<N\\ \sum_{r=0}^{M-N}B_r Z^{-r}+\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}} & M\ge N\end{cases}$$

With B_r obtained by long division;

$$A_k=(1-d_k z^{-1})X(z)|_{z=d_k}$$

$$A_k/(1-d_k z^{-1})\leftrightarrow\begin{cases}(d_k)^n u[n]\\ -(d_k)^n u[-n-1]\end{cases}$$

ROC should be considered here.

Notes: Multi-order poles are not considered in our discussion

Partial fraction with numerator order greater than or equal to denominator order:

If the order of the numerator is too large, we can reduce it via long division. For example:

$$H(z)=\frac{3z^{-3}+2z^{-2}+z^{-1}+5}{1-3z^{-1}}$$

After we apply the long division, the result is that

$$H(z)=\frac{3z^{-3}+2z^{-2}+z^{-1}+5}{1-3z^{-1}}=-z^{-2}-z^{-1}-\frac{2}{3}+\frac{17/3}{1-3z^{-1}}$$

$$\boxed{-\delta[n-2]-\delta[n-1]}$$

Inverse z-transforms by long division

For example, consider the transform

$$H(z) = \frac{3 - 7z^{-1}}{1 - 5z^{-1} + 6z^{-2}}$$

Arranging the terms in order to increasing powers of z^{-1} and dividing.

```
                          3+ 8z^(-1)+22z^(-2)+ ...
1-5z^(-1)+6z^(-2) ) 3- 7z^(-1)+ 0z^(-2)+ 0z^(-3)+ 0z^(-4)+ ...
                    3-15z^(-1)+18z^(-2)
                       8z^(-1)-18z^(-2)+ 0z^(-3)
                       8z^(-1)-40z^(-2)+48z^(-3)
                                2z^(-2)-48z^(-3)+ 0z^(-4)
```

The first several terms of the quotient will be. $H(z) = 3 + 8z^{-1} + 22z^{-2} + \ldots$

If causal, the inverse transform is: $h[n] = 3\delta[n] + 8\delta[n-1] + 22\delta[n-2] + \ldots$

Inverse z-transforms by Taylor series expansion

Occasionally we are asked to obtain the inverse z-transform of a function that is not a ratio of polynomials in z or z^{-1}. We can use Taylor series expansion to accomplish this.

$$X(z) = \log(1 + az^{-1}) \qquad |z| > |a|$$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

$$x[n] = \frac{(-1)^{n+1} a^n}{n} u[n-1]$$

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Basic z-transform properties

- While the basic z-transform properties are very similar to those of the corresponding DTFTs, they are complicated a little by the fact that we now must also consider the region of convergence of the new transform as well.
- Not all the properties are reviewed, but the most important ones will be mentioned.
- The function x[n] has the z-transform X(z), with the corresponding ROC R_x. Following properties can be expressed.

Z-Transform Properties: Linearity

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- Notation $x[n] \overset{Z}{\longleftrightarrow} X(z) \qquad ROC = R_x$
- Linearity

$$ax_1[n] + bx_2[n] \overset{Z}{\longleftrightarrow} aX_1(z) + bX_2(z) \qquad ROC = R_{x_1} \cap R_{x_2}$$

  - Note that the ROC of combined sequence may be larger than either ROC
  - This would happen if some pole/zero cancellation occurs
  - Example: $x[n] = a^n u[n] - a^n u[n - N]$
    - Both sequences are right-sided
    - Both sequences have a pole z=a
    - Both have a ROC defined as |z|>|a|
    - In the combined sequence the pole at z=a cancels with a zero at z=a
    - The combined ROC is the entire z plane except z=0

Z-Transform Properties: Time Shifting

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$$x[n - n_o] \overset{Z}{\longleftrightarrow} z^{-n_o} X(z) \qquad ROC = R_x$$

- Here n_o is an integer
  - If positive the sequence is shifted right
  - If negative the sequence is shifted left
- The ROC can change the new term may
  - Add or remove poles at z=0 or z=∞
- Example

$$X(z) = z^{-1}\left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right) \qquad |z| > \frac{1}{4}$$

$$x[n] = \left(\frac{1}{4}\right)^{n-1} u[n - 1]$$

Z-Transform Properties: Multiplication by Exponential

$$z_o^n x[n] \xleftrightarrow{\;Z\;} X(z/z_o) \qquad \mathrm{ROC} = |z_o| R_x$$

- ROC is scaled by $|z_o|$
- All pole/zero locations are scaled
- If z_o is a positive real number: z-plane shrinks or expands
- If z_o is a complex number with unit magnitude it rotates
- Example: We know the z-transform pair

$$u[n] \xleftrightarrow{\;Z\;} \frac{1}{1 - z^{-1}} \qquad \mathrm{ROC}: |z| > 1$$

- Let's find the z-transform of

$$x[n] = r^n \cos(\omega_o n) u[n] = \frac{1}{2}\left(re^{j\omega_o}\right)^n u[n] + \frac{1}{2}\left(re^{-j\omega_o}\right)^n u[n]$$

$$X(z) = \frac{1/2}{1 - re^{j\omega_o} z^{-1}} + \frac{1/2}{1 - re^{-j\omega_o} z^{-1}} \qquad |z| > r$$

Z-Transform Properties: Differentiation

$$nx[n] \xleftrightarrow{\;Z\;} -z\frac{dX(z)}{dz} \qquad \mathrm{ROC} = R_x$$

- Example: We want the inverse z-transform of

$$X(z) = \log\left(1 + az^{-1}\right) \qquad |z| > |a|$$

- Let's differentiate to obtain rational expression

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}} \Rightarrow -z\frac{dX(z)}{dz} = az^{-1}\frac{1}{1 + az^{-1}}$$

- Making use of z-transform properties and ROC

$$nx[n] = a(-a)^{n-1} u[n-1]$$

$$x[n] = (-1)^{n-1}\frac{a^n}{n} u[n-1]$$

Z-Transform Properties: Time Reversal

$$x[-n] \xleftrightarrow{Z} X(1/z) \qquad ROC = \frac{1}{R_x}$$

- ROC is inverted
- Example: $a^n u[n]$

$$x[n] = a^{-n} u[-n]$$

- Time reversed version of

$$X(z) = \frac{1}{1 - az} = \frac{-a^{-1}z^{-1}}{1 - a^{-1}z^{-1}} \qquad |z| < |a^{-1}|$$

Z-Transform Properties: Convolution

$$x_1[n] * x_2[n] \xleftrightarrow{Z} X_1(z)X_2(z) \qquad ROC : R_{x1} \cap R_{x_2}$$

- Convolution in time domain is multiplication in z-domain
- Example: calculate the convolution of

$$x_1[n] = a^n u[n] \text{ and } x_2[n] = u[n]$$

$$X_1(z) = \frac{1}{1 - az^{-1}} \quad ROC : |z| > |a| \qquad X_2(z) = \frac{1}{1 - z^{-1}} \quad ROC : |z| > 1$$

- Multiplications of z-transforms is $Y(z) = X_1(z)X_2(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}$
- ROC: if |a|<1 ROC is |z|>1 if |a|>1 ROC is |z|>|a|
- Partial fractional expansion of Y(z)

$$Y(z) = \frac{1}{1 - a}\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - az^{-1}}\right) \qquad \text{asume } ROC : |z| > 1$$

$$y[n] = \frac{1}{1 - a}\left(u[n] - a^{n+1}u[n]\right)$$

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Introduction

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

For LTI systems we can write

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Alternatively, this relationship can be expressed in the z-transform domain as

$$Y(z) = H(z)X(z),$$

where $H(z)$ is the **system function**, or the z-transform of the system impulse response.

System response for LCCD systems

- System described by a LCCD equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Applying z-transform at both sides

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z)$$

- System function is deduced:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

ROC of System Function

- *Zeros:* the roots of the numerator polynomial
- *Poles* : the roots of the denominator polynomial
- The *ROC* can't include the locations of the system's poles.
- *ROC is* bounded by circles that are centered at the origin of the z-plane, and that pass through the locations of the poles.

- A discrete-time system is causal if and only if the ROC of H(z) is the exterior of a circle including infinite.
- A discrete-time system is stable if and only if the ROC includes the unit circle.

Example

In case of the difference equation

$$y[n]-\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2] = 3x[n]-\frac{3}{4}x[n-1]$$

obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3-\frac{3}{4}z^{-1}}{1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

By multiplying numerator and denominator by z^2

$$H(z) = \frac{z^2\left(3-\frac{3}{4}z^{-1}\right)}{z^2\left(1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}\right)} = \frac{3z^2-\frac{3}{4}z}{z^2-\frac{1}{4}z-\frac{1}{8}} = \frac{3z\left(z-\frac{1}{4}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}$$

In this system, the zeros are at $z = 0$ and $z = \frac{1}{4}$ and the poles are at $z = \frac{1}{2}$ and $z = -\frac{1}{4}$

Example (continue)

- In this example, the potential boundaries of the ROCs are circles of radium ¼ and ½. This means that there are three possible ROCs for this system:

ROC	System	h [n]
$\|z\| < 1/4$	unstable	Left-sided
$1/4<\|z\| < 1/2$	unstable	Both-sided
$\|z\| > 1/2$	Stable & causal	Right-sided

Impulse response for rational system functions

- Infinite impulse response (IIR) systems
  - A rational transfer function, with at least one non-zero pole that is not cancelled by a zero
  - Impulse response h[n] is a infinite length sequence

- Finite impulse response (FIR) system
  - The system has no poles:
  - The system is determined to within a constant multiplier by its zeros,

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

THE END !