ELCE 705
DIGITAL SIGNAL PROCESSING

IIR Filter Design Techniques

7.0-7.4

Contents

- Introduction
- Design of IIR Filters
- Examples of IIR Filter Design
- Frequency Transformations of Lowpass IIR Filters

Introduction

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Definition: Digital filtering is just changing the frequency-domain characteristics of a given discrete-time signal

Filtering operations may include:

- noise suppression
- enhancement of selected frequency ranges or edges in images
- bandwidth limiting (to prevent aliasing of digital signals or to reduce interference of neighboring channels in wireless communications)
- removal or attenuation of specific frequencies
- special operations like integration, differentiation, etc.

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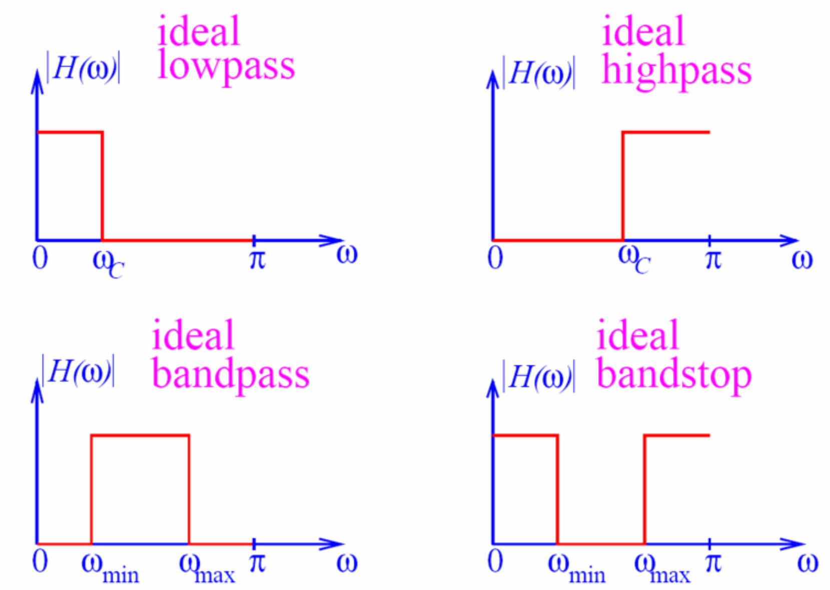
Basic filter types:

- lowpass filters (to pass low frequencies from zero to a certain cut-off frequency ω_C and to block higher frequencies)
- highpass filters (to pass high frequencies from a certain cut-off frequency ω_C to π and to block lower frequencies)
- bandpass filters (to pass a certain frequency range $[\omega_{\min}, \omega_{\max}]$, which does not include zero, and to block other frequencies)
- bandstop filters (to block a certain frequency range $[\omega_{\min}, \omega_{\max}]$, which does not include zero, and to pass other frequencies)

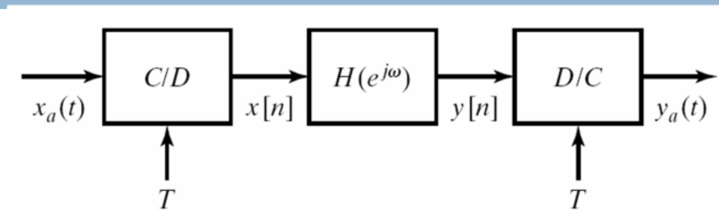
Basic filters



Design of filters

- Filter Design Steps
  - Specification
    - Problem or application specific
  - Approximation of specification with a discrete-time system
    - Our focus is to go from spec to discrete-time system
  - Implementation
    - Realization of discrete-time systems depends on target technology

Discrete-time filtering of continuous-time signals



Overall system behaves as a LTI continuous-time system if:

- LTI discrete-time system is used
- Input is bandlimited
- Sampling frequency is high enough

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}), |\Omega| < \pi/T \\ 0, \quad\quad |\Omega| \geq \pi/T \end{cases}$$

$$H(e^{j\omega}) = H_{eff}(j\omega/T), |\omega| < \pi$$

Filter Specifications

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- Specifications
  - Passband

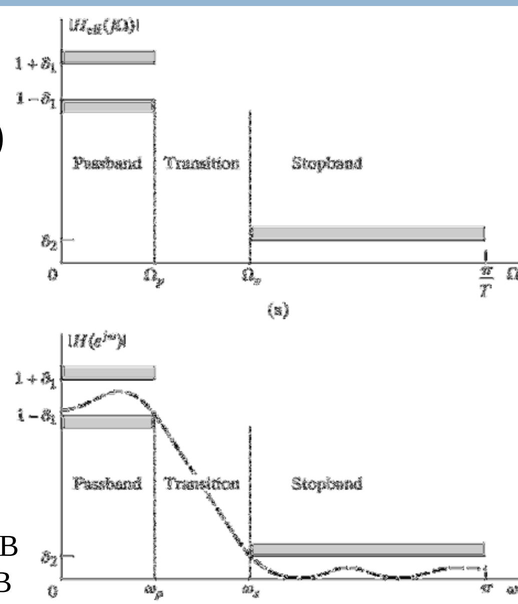
$$0.99 \leq |H_{eff}(j\Omega)| = 1.01 \quad 0 \leq \Omega \leq 2\pi(2000)$$

  - Stopband

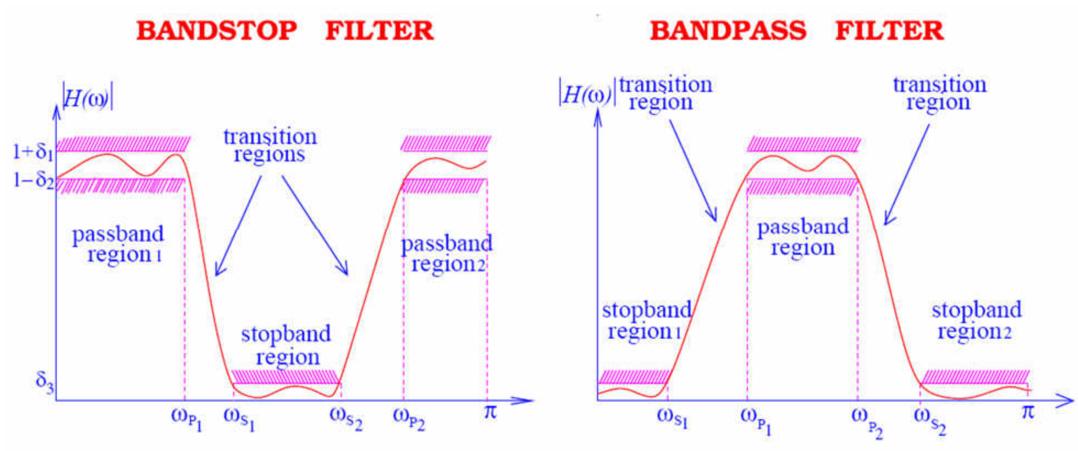
$$|H_{eff}(j\Omega)| \leq 0.001 \quad 2\pi(3000) \leq \Omega$$

- Parameters

$$\boxed{\begin{aligned} \delta_1 &= 0.01 \\ \delta_2 &= 0.001 \\ \Omega_p &= 2\pi(2000) \\ \Omega_s &= 2\pi(3000) \end{aligned}}$$

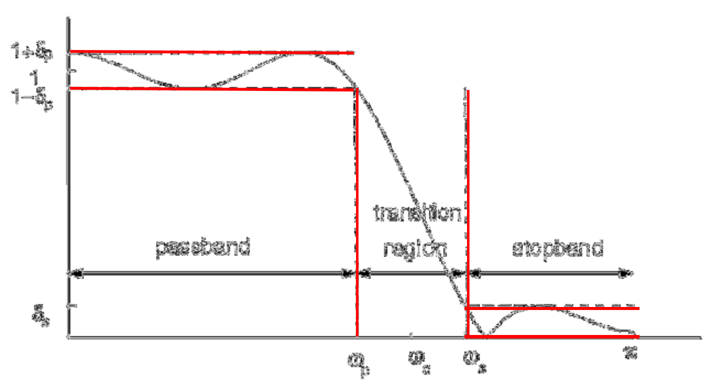


- Specs in dB
  - Ideal passband gain =20log(1) = 0 dB
  - Max passband gain = 20log(1.01) = 0.086dB
  - Max stopband gain = 20log(0.001) = -60 dB



Design discrete-time filters

- Determine the system function
  - frequency response falls within the prescribed tolerances.
- This is a problem in functional approximation.
  - Designing IIR filters implies approximation by a rational function of z
  - Designing FIR filters implies polynomial approximation.



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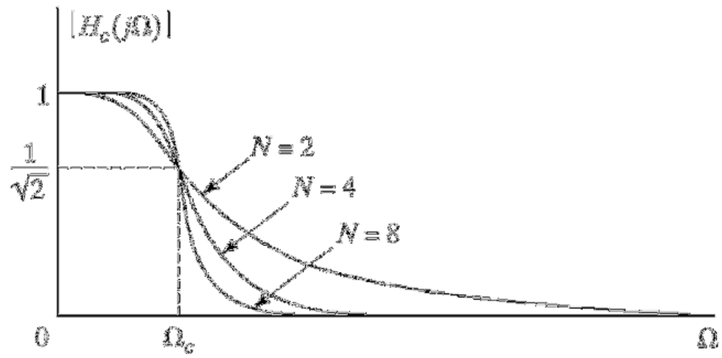
IIR Filter Design

- Normally done by transforming a continuous-time filter into a discrete-time filter.
  - The art of continuous-time IIR filter design is highly advanced;
  - Simple closed-form design formulas are available;
  - Butterworth, Chebyshev, Elliptic. Etc.
- Essential properties of the continuous-time frequency response should be preserved in the fresulting discrete-time filter.
  - Imaginary axis of s-plane map onto the unit circle of z-plane
  - Stable continuous-time filter → stable discrete-time filter

Butterworth Lowpass Filters

- Passband is designed to be maximally flat
- The magnitude-squared function is of the form

$$|H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}} \qquad |H_c(s)|^2 = \frac{1}{1+(s/j\Omega_c)^{2N}}$$



$$s_k = (-1)^{1/2N}(j\Omega_c) = \Omega_c e^{(j\pi/2N)(2k+N-1)} \quad \text{for } k = 0,1,\ldots,2N-1$$

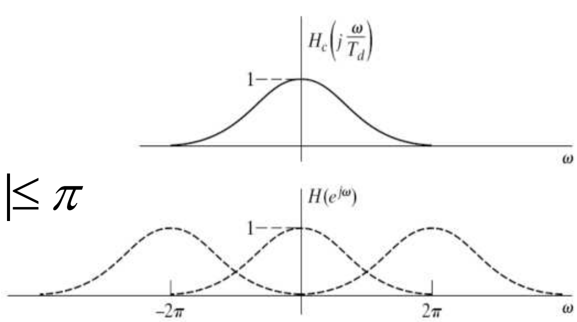
Impulse-Invariance Transformation

- Not widely used.
- Impulse response of discrete-time filter is obtained by sampling the impulse response of continuous-time filter
  - Impulse response is preserved by the mapping
  - Frequency response is not preserved due to aliasing

$$h[n] = T_d h_c(nT_d)$$

$$H(e^{j\omega}) = H_c(j\omega/T_d), |\omega| \leq \pi$$

If the continuous-time filter is bandlimited



Procedure-

Design IIR filter uses Impulse-invariance Transformation

- The discrete-time filter specifications are first transformed to continuous-time filter specifications.

$$\Omega=\omega/T_d$$

  - Aliasing involved in the transformation be negligible
- Obtain a suitable continuous-time filter $H_c(s)$
- Transform $H_c(s)$ to the desired discrete-time filter H(z).
  - To compensate for aliasing that might occur in the transformation, the continuous-time filter may be somewhat overdesigned.

$$H_c(s)=\sum_{k=1}^{N}\frac{A_k}{s-s_k} \qquad H(z)=\sum_{k=1}^{N}\frac{T_d A_k}{1-e^{s_k T_d}z^{-1}}$$

Bilinear Transformation

- Widely Used
- Avoids aliasing problem of impulse-invariance transformation

$$s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad H(z)=H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$

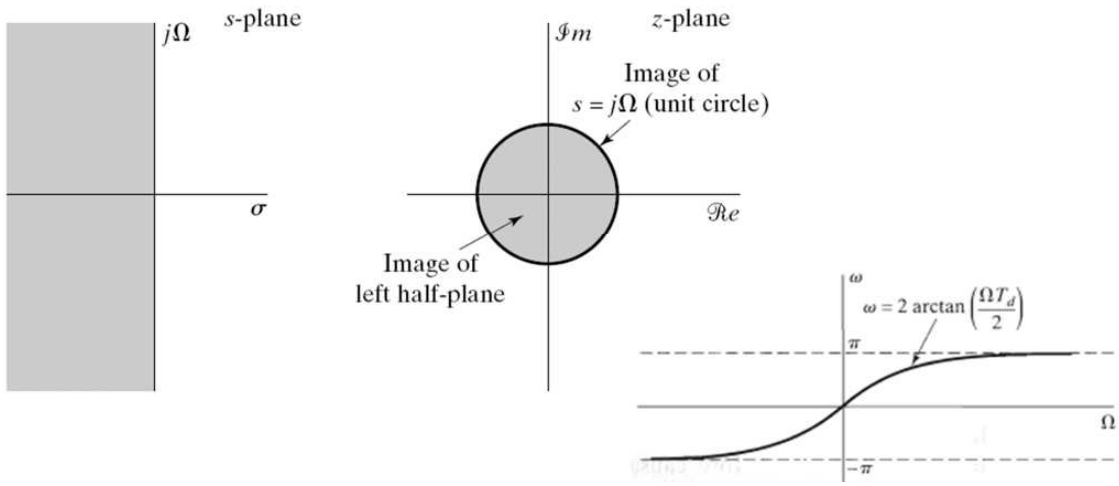
$$z=\frac{1+(T_d/2)s}{1-(T_d/2)s}=\frac{1+\sigma T_d/2+j\Omega T_d/2}{1-\sigma T_d/2-j\Omega T_d/2}$$

$$s=\sigma+j\Omega$$

$$\Omega=\frac{2}{T_d}\tan(\omega/2)$$

Mapping

- Map $j\Omega$ axis of the s-plane to the unit circle of the z-plane
- Map poles and zeros on the left half plane of s-plane to the inside of unit circle in z



Procedure

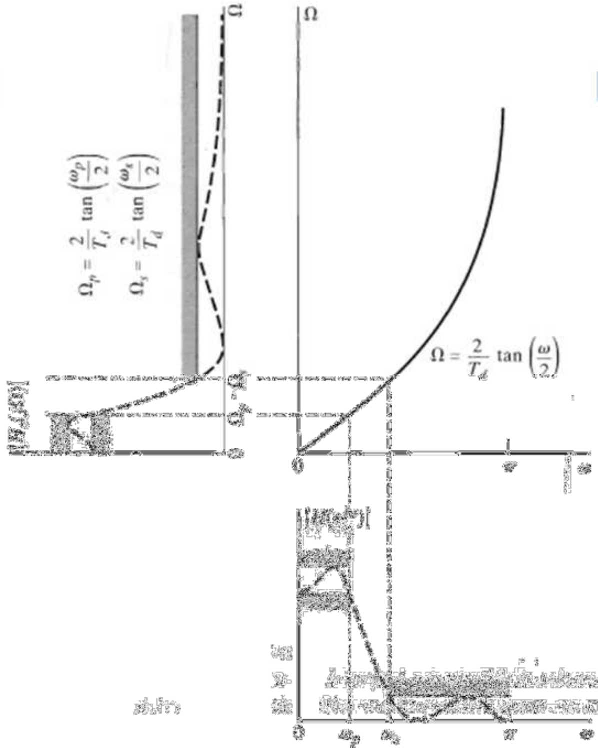
– Design IIR filter uses Bilinear Transformation

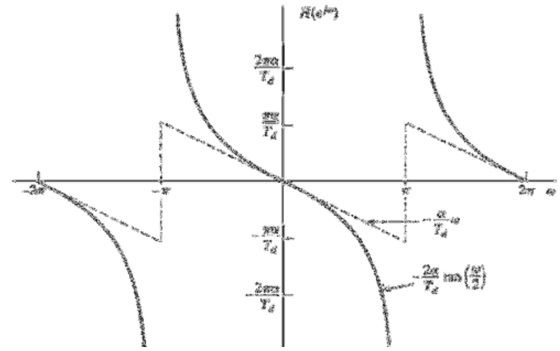
- The discrete-time filter specifications are first transformed to continuous-time filter specifications.

$$\Omega = \frac{2}{T_d}\tan(\omega/2)$$

- Obtain a suitable continuous-time filter $H_c(s)$
- Transform $H_c(s)$ to the desired discrete-time filter $H(z)$.

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$





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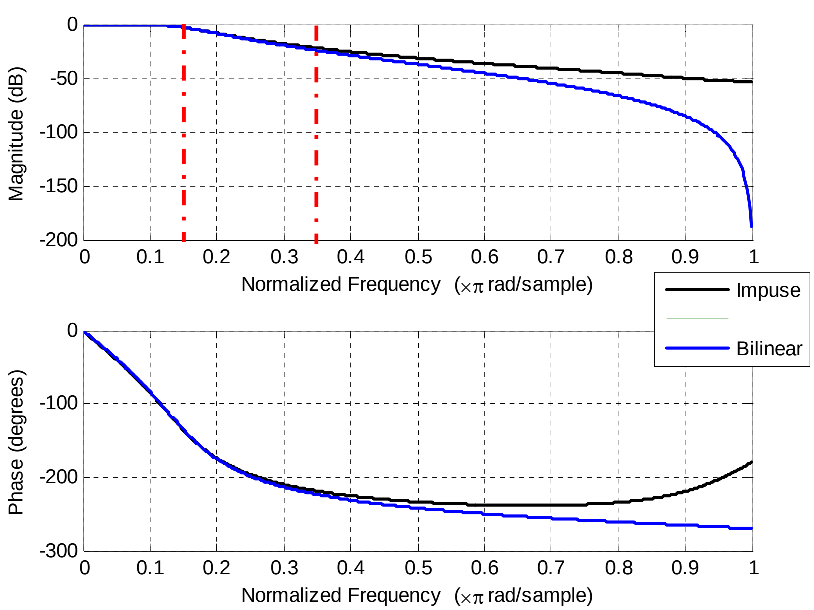
Example

- The specifications for the filter are

  - Passband cutoff frequency: $\omega_p = 0.15\pi$
  - Stopband cutoff frequency: $\omega_s = 0.35\pi$
  - Passband ripple[1]: $-3 \text{ dB} \le |H(e^{j\omega})| \le 0 \text{ dB}, |\omega| \le \omega_p$
  - Stopband attenuation: $|H(e^{j\omega})| \le -20 \text{ dB}, \omega_s \le |\omega| \le \pi$

Filters obtained



Design IIR filter using matlab

- Choose the type of analog filter
- Estimate the order of the transfer function from the filter specifications

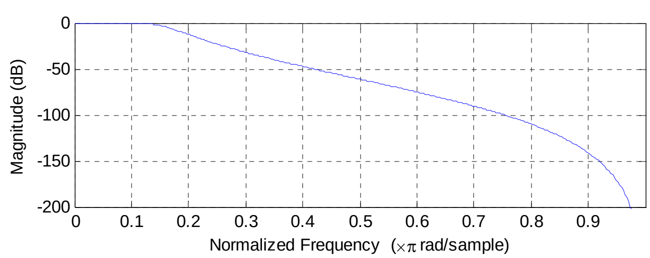
  [N, Wn]=buttord(Wp,Ws,Rp,Rs)

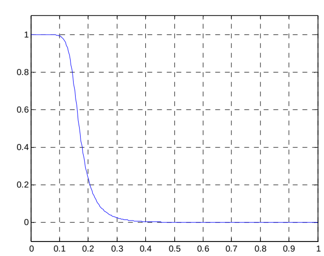
  - *cheb1ord, cheb2ord,ellipord*
- Determine the transfer function of the filter

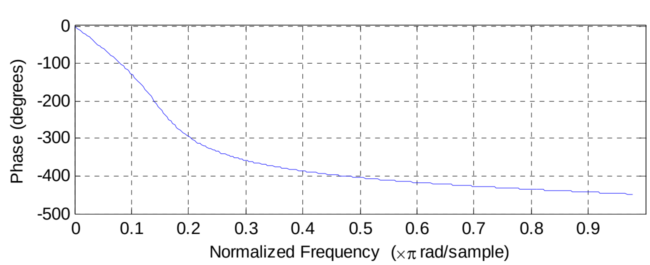
  [num,den]=butter(N,Wn,'filtertype')

  -*cheby1, cheby2,ellip*

Designed by Matlab





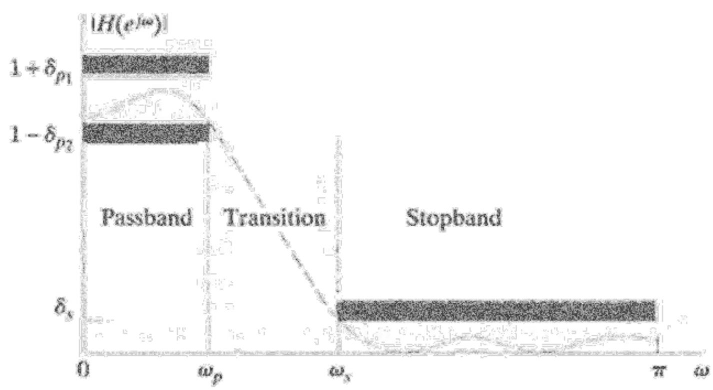


[N, Wn]=buttord(0.15,0.35,3,20);
[num,den]= butter(N, Wn);
w=0:0.01:pi;
h=freqz(num,den,w);
plot(w/pi,abs(h));

Analog transfer function

- Butterworth
  - Maximally flat magintude response at $\Omega = 0$
  - Monotonically decrease with increasing frequency
- Type 1 Chebyshev
  - Equiripple magnitude response in passband
  - Monotonically decrease outside passband
- Type 2 Chebyshev
  - Monotonically decrease in the passband
  - Equiripple magnitude response in stopband
- Elliptic
  - Equiripple magnitude response both in the passband and stopband

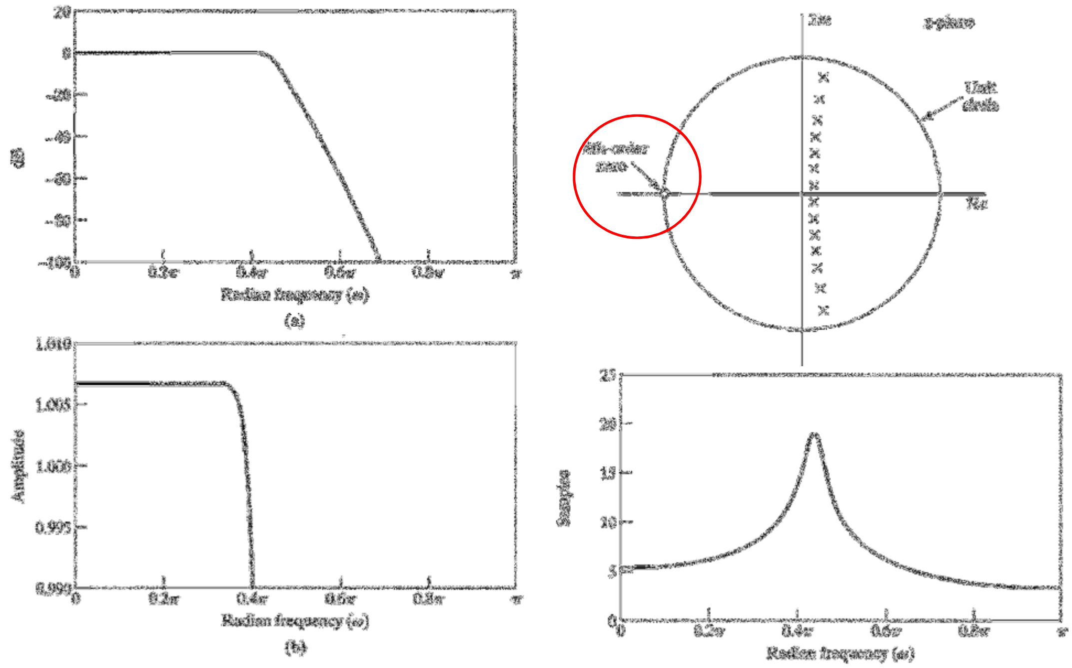
Example 7.5



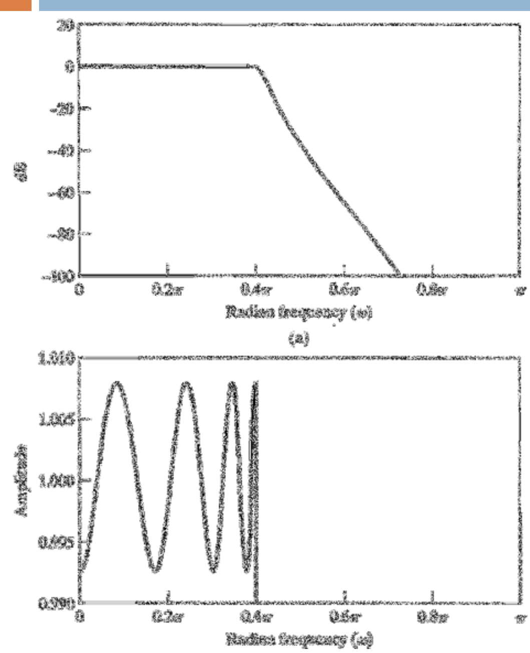
$$\delta_{p1} = \delta_{p2} = 0.01$$
$$\delta_s = 0.001$$
$$\omega_p = 0.4\pi \text{ radians}$$
$$\omega_s = 0.6\pi \text{ radians.}$$

$$\begin{aligned}
\text{ideal passband gain in decibels} &= 20\log_{10}(1) &&= 0\text{ dB}\\
\text{maximum passband gain in decibels} &= 20\log_{10}(1.01) &&= 0.0864\text{ dB}\\
\text{minimum passband gain at passband edge in decibels} &= 20\log_{10}(0.99) &&= -0.873\text{ dB}\\
\text{maximum stopband gain in decibels} &= 20\log_{10}(0.001) &&= -60\text{ dB}
\end{aligned}$$

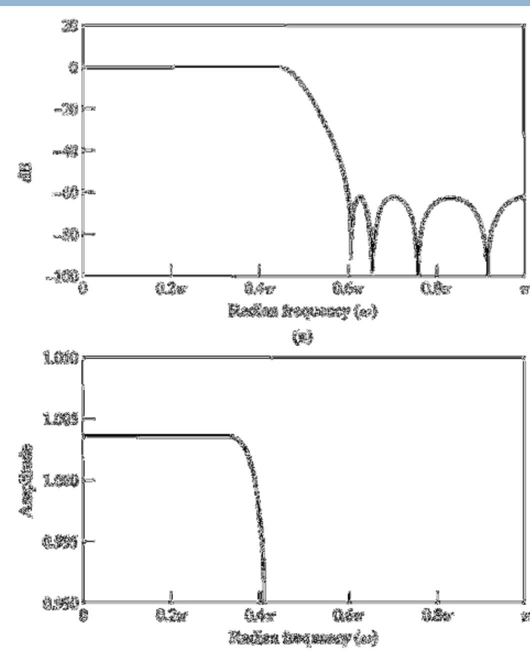
Butterworth



Chebyshev type I & II

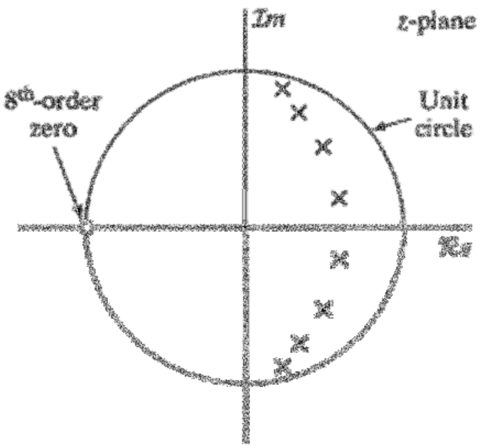


Type I

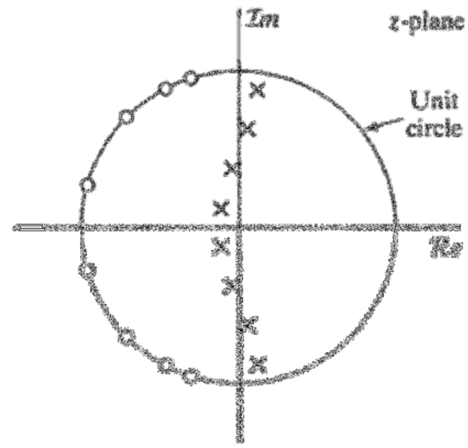


Type II

Chebyshev type I & II

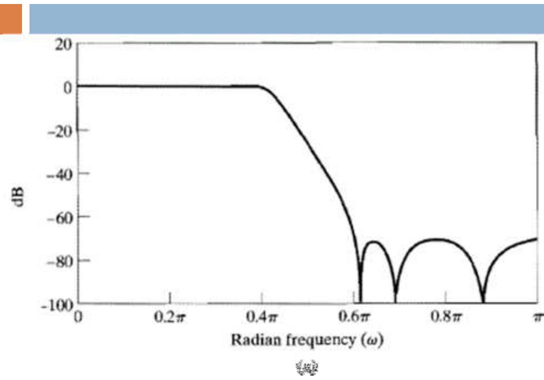


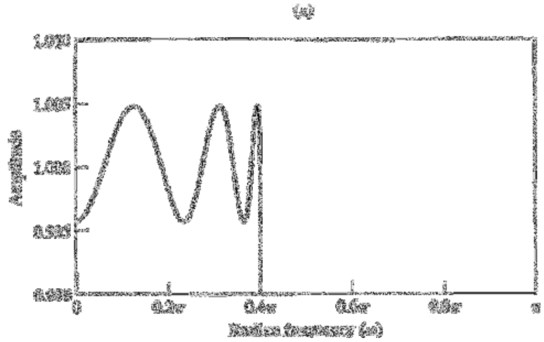
Type I

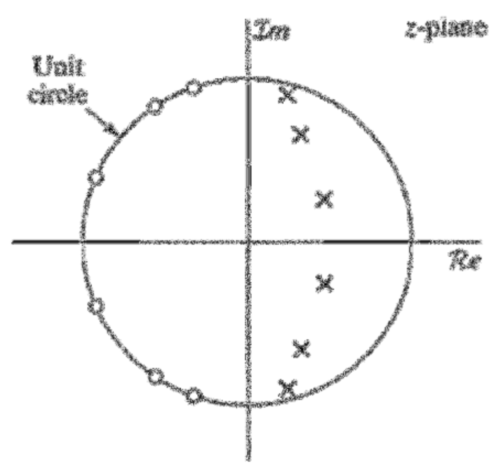


Type II

Elliptic Filter







Contents

Design other types of IIR filter

- Highpass, bandpass, bandstop, etc.
- Method 1
  - Design a continuous-time filter
    - Only work for bilinear transformation
  - Transform into discrete
- Method 2 (Preferred)
  - Design a discrete-time prototype lowpass filter
    - Impulse invariance or bilinear transformation
  - Perform an algebraic transformation to obtain other types of filter

Frequency Transformation

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- Begin with Hlp(z): rational function, stable and causal
  - Z with prototype lowpass filter
  - z with the transformed filter

$$H(z) = H_{lp}(Z)\big|_{Z^{-1}=G(z^{-1})}$$

- Target H(z): rational function, stable and causal $Z^{-1} = G(z^{-1})$
  - $G(z^{-1})$ be a rational function of z^{-1}
  - The inside of the unit circle of the Z-plane must map to the inside of the unit circle of the z-plane
  - The unit circle of the Z-plane must map onto the unit circle of the z-plane

Frequency Transformation

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- From condition 3 $e^{-j\theta} = |G(e^{-j\omega})|e^{j\angle G(e^{-j\omega})}$

$$|G(e^{-j\omega})| = 1 \qquad -\theta = \angle G(e^{-j\omega})$$

- General form of the function

$$Z^{-1} = G(z^{-1}) = \pm \prod_{k=1}^{N} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}}$$

- Simplest one: $Z^{-1} = G(z^{-1}) = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$

$$e^{-j\theta} = \frac{e^{-j\omega} - \alpha}{1 - \alpha e^{-j\omega}}$$

$$\omega = \arctan\left[\frac{(1-\alpha^2)\sin\theta}{2\alpha + (1+\alpha^2)\cos\theta}\right]$$

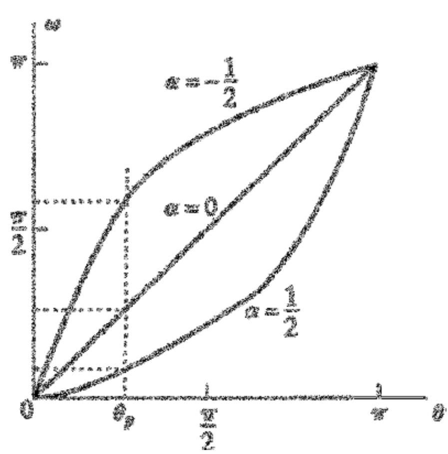


TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\boxed{\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\frac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

Example 7.6

Consider a Type I Chebyshev lowpass filter with system function

$$H_{\text{lp}}(Z) = \frac{0.001836(1 + Z^{-1})^4}{(1 - 1.5548Z^{-1} + 0.6493Z^{-2})(1 - 1.4996Z^{-1} + 0.8482Z^{-2})}. \tag{7.49}$$

This 4th-order system was designed to meet the specifications

$$0.89125 \le |H_{\text{lp}}(e^{j\theta})| \le 1, \quad 0 \le \theta \le 0.2\pi, \tag{7.50a}$$

$$|H_{\text{lp}}(e^{j\theta})| \le 0.17783, \quad 0.3\pi \le \theta \le \pi. \tag{7.50b}$$

The frequency response of this filter is shown in Figure 7.25.

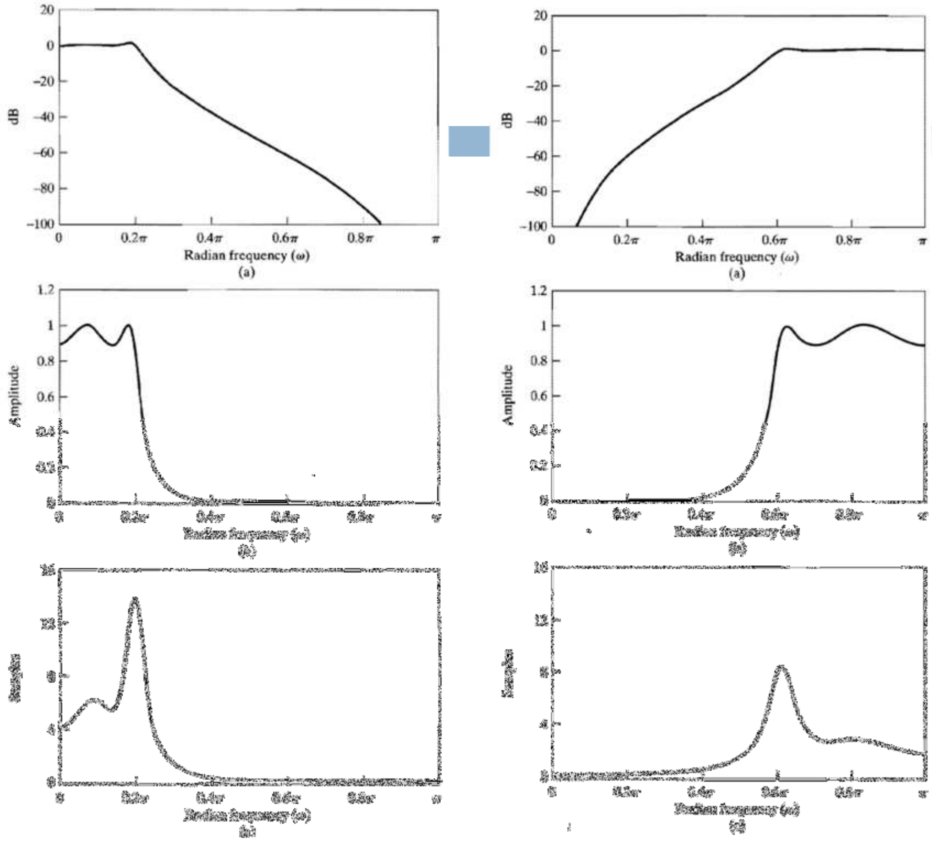
To transform this filter to a highpass filter with passband cutoff frequency $\omega_p = 0.6\pi$, we obtain from Table 7.1

$$\alpha = -\frac{\cos[(0.2\pi + 0.6\pi)/2]}{\cos[(0.2\pi - 0.6\pi)/2]} = -0.38197. \tag{7.51}$$

Thus, using the lowpass-highpass transformation indicated in Table 7.1, we obtain

$$H(z) = H_{\text{lp}}(Z)\Big|_{Z^{-1} = -[(z^{-1} - 0.38197)/(1 - 0.38197z^{-1})]}$$

$$= \frac{0.02426(1 - z^{-1})^4}{(1 + 1.0416z^{-1} + 0.4019z^{-2})(1 + 0.5661z^{-1} + 0.7657z^{-2})}. \tag{7.52}$$



20
0
−20
−40
−60
−80
−100
dB
0
0.2π
0.4π
0.6π
0.8π
π
Radian frequency (ω)
(a)
1.2
1
0.8
0.6
Amplitude

THE END