ELCE 707: Advanced Topics in Control Systems

Classical Control Theory

A Review

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Part 1:
Introduction

Control: *what*, *why* and *how*?

Control

To exercise authoritative or dominating influence over; direct.
支配对…施加权威性的或优势性影响；指挥

控制, 调节, 操纵, 管理, 抑制, 监察, 监督, 监控, 支配, 掌握, 统制
To regulate, govern, adjust, direct, supervise, guide, manipulate, monitor, instruct, master, command, modulate, inspect, restrain, dominate, charge, rein, administrate, grip, dictate, … …

Control Systems

A control system is a device or set of devices to manage, command, direct or regulate the behavior of other devices or systems.

Control Engineering

Control engineering is the engineering discipline that focuses on the mathematical modeling systems of a diverse nature, analyzing their dynamic behavior, and using control theory to create a controller that will cause the systems to behave in a desired manner.

Control Theory (*Cybernetics*)

An interdisciplinary branch of engineering and mathematics that deals with the behavior of dynamical systems

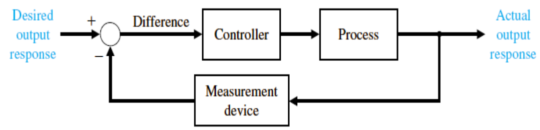
Feedback: the Foundation

Feedback

The return of a portion of the output of a process or system to the input, especially when used to maintain performance or to control a system or process.
反馈一个过程或系统的一部分由输出向输入的返回，尤指用来维持运转或控制一个系统或过程

The return of information about the result of a process or activity; an evaluative response
处理结果或活动信息的返回；一种有评估性的结果的返回

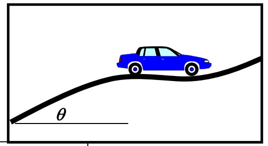
Closed-Loop System



Example

Automobile's Cruise Control

Vehicle speed control



1 MODEL:

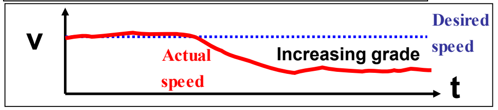
Inertia force / Traction: input, excitation / Friction: damping / Gravitation: disturbance

$$M\dot{v} = f_e(t) - Bv - Mg\sin\theta$$

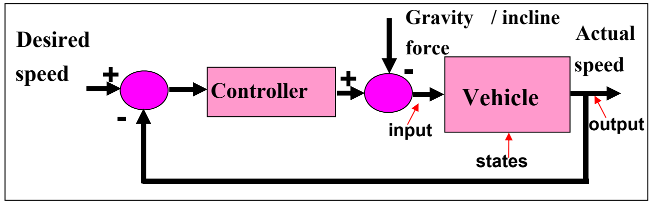
$$M\dot{v} + Bv \cong f_e(t) - Mg\theta$$

linearization

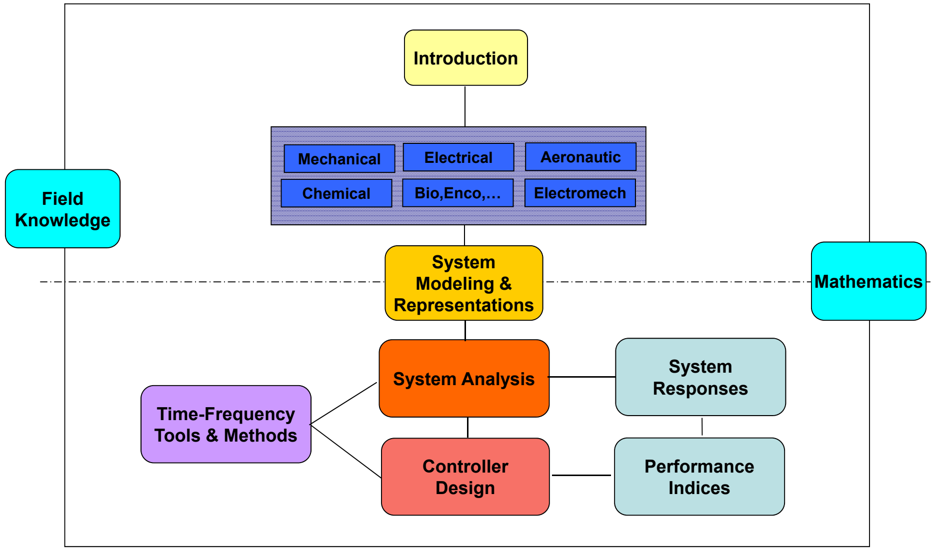
2 ANALYSIS:



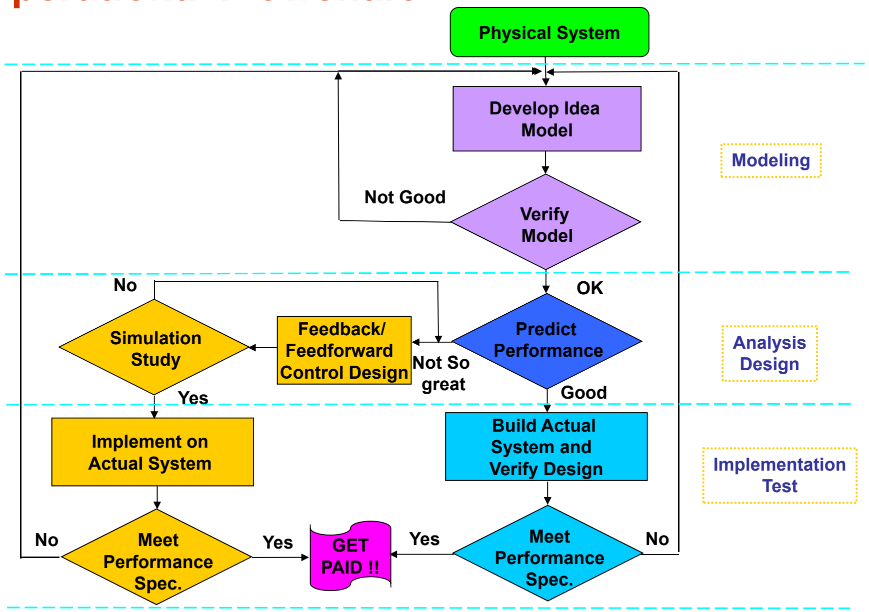
3 CONTROL:



Classical Control Theory: A Map



Operational Flowchart



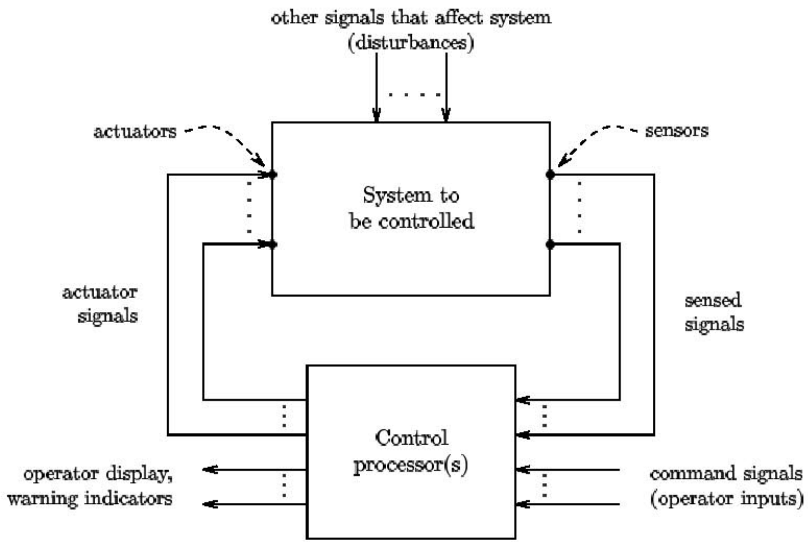


Figure 1.1 A schematic diagram of a general control system.

Challenges and Opportunities

Challenges

- Complicated systems
  - Large-scale, nonlinear, time-varying, distributed,…
- Higher quality requirements
  - Economic, environmental,…

Advances in Technologies

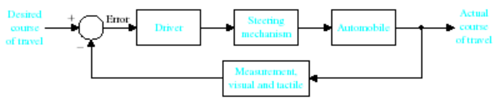
- Integrated/intelligent sensors
  - Cheaper, smaller, intelligent
- Improved actuator technology
- Powerful control processors

- Smart people, great ideas

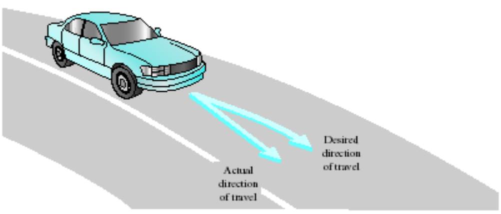
History: Feedback Control

- Preclassical period
  - Watt, Maxwell, …
- Classical period
  - Bode, Nyquist, Black, Evans, Wiener, …
- Modern period
  - Kalman,…
- Postmodern period
  - Robust control, H_∞ , intelligent control,…

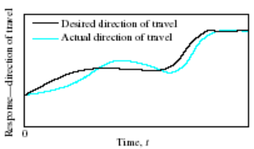
Examples of Modern Control Systems



(a)



(b)

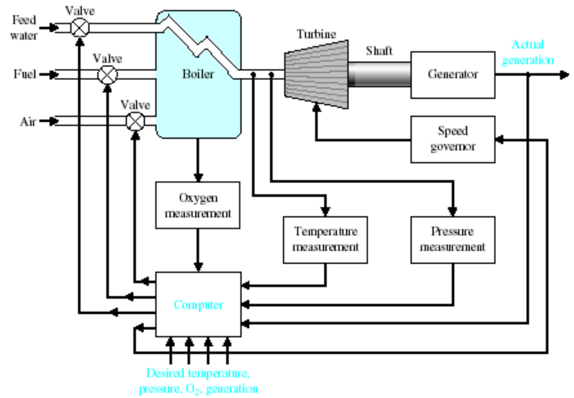


(c)

(a) Automobile automatic steering control system.

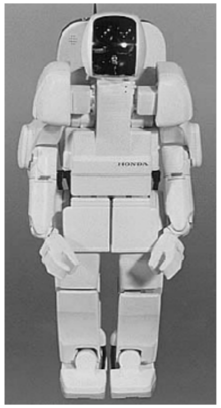
(b) The driver uses the difference between the actual and the desired direction of travel to generate a controlled adjustment of the steering wheel.

(c) Typical direction-of-travel response.

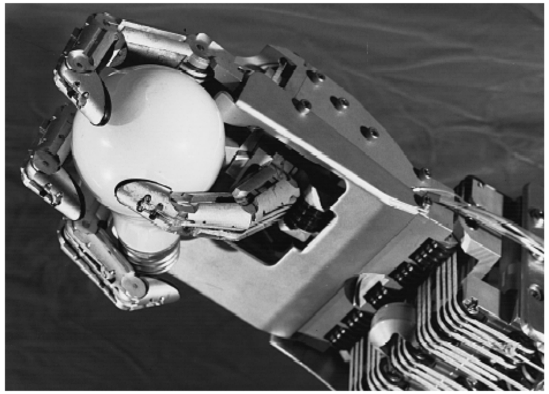


Coordinated control system for a boiler-generator.

The Future of Control Systems

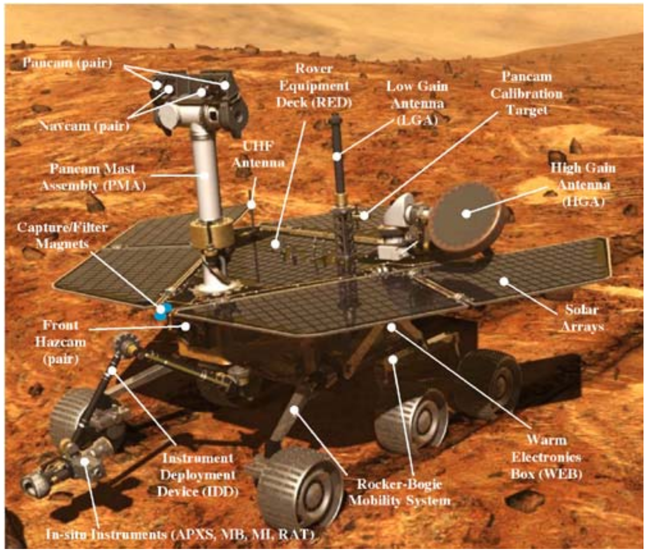


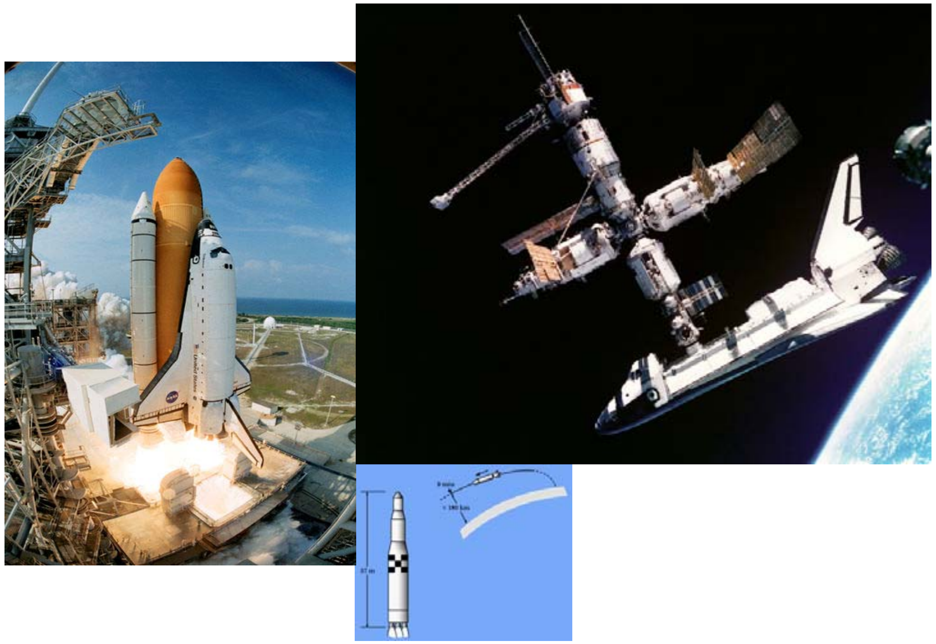
The Honda P3 humanoid robot. P3 walks, climbs stairs and turns corners.
Photo courtesy of American Honda Motor, Inc.



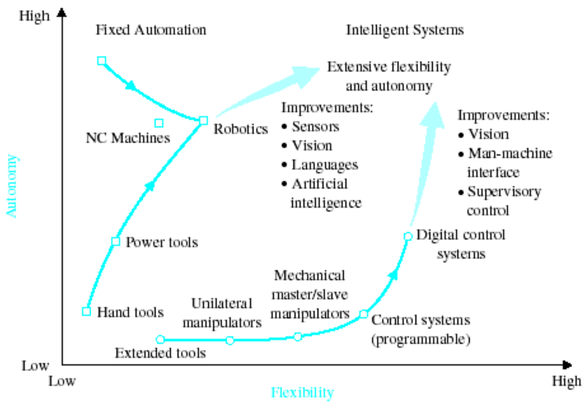
The Utah/MIT Dextrous Robotic Hand: A dextrous robotic hand having 18 degrees of freedom, developed as a research tool by the Center for Engineering Design at the University of Utah and the Artificial Intelligence Laboratory at MIT. It is controlled by five Motorola 68000 microprocessors and actuated by 36 high-performance electropneumatic actuators via high-strength polymeric tendons. The hand has three fingers and a thumb. It uses touch sensors and tendons for control.
(Photograph by Michael Milochik. Courtesy of University of Utah.)

Mars Rover





The Future of Control Systems



Future evolution of control systems and robotics.

Part 2:
System Modeling

Main Contents

- Introduction
- Mathematic modeling of common systems
- System identifications
- Linguistic model
- Model representations
- Linearization of nonlinear systems
- Matlab tools

Introduction

- Modeling: what and why
- Model types
- Modeling routes
- Modeling considerations

Modeling: What and Why

- When we interact with a system, we need some concepts of how its variables relate to each others.
- A model tells us the relationships among system variables.
- Modeling is one of the most important tasks in control system analysis and design.

Model Types

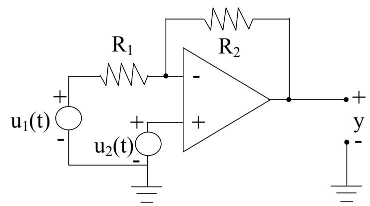
- Graphical models (numerical tables/plots)
  – Impulse/step responses, frequency plots, nonlinear characteristics …
- Mathematical models (analytical)
  – First-principle modeling method, system behaviors described by mathematical expressions (difference/differential equations)
- Software models (computer programs)
  – Built on interconnected subroutines and lookup tables
- Mental models (linguistic)
  – Fuzzy rule systems, expert systems, descriptions in daily language

Modeling Routes (1)

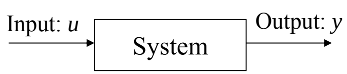
- First-principle modeling method (white-box)
  – Models constructed from physical laws governing the system
  – Split up the systems into subsystems that are well-understood from earlier empirical work/experience
  – Quite application dependent
- Empirical modeling method (black-box)
  – Models constructed from observed input-output data of the system
  – Selection of mathematical relationships that seem to fit observed input-output data
  – System identification/data analysis

Modeling Routes (2)

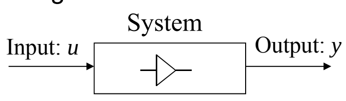
- Hybrid method (gray-box)
  – A combination of first-principle and empirical modeling methods



white-box modeling



black-box modeling

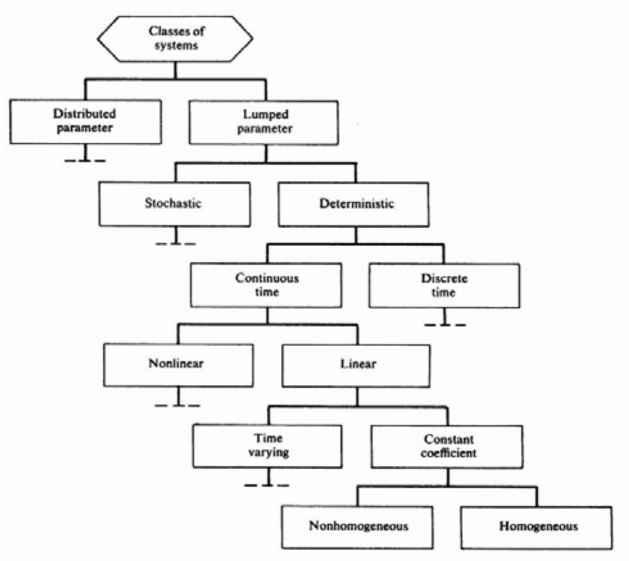


gray-box modeling

Modeling Considerations

- Simplicity *vs.* accuracy
  - Accuracy ↑, complexity ↑
  - Too complicated to analyze/use *vs.* too simple to describe the reality
  - Proper assumptions/approximations
- Fiction of true system
  - Usefulness rather than truth
- Model reduction
  - Reduce redundancy
- Model validation
  - Test whether the model is valid for its purpose

System Model Classifications (1)



System Model Classifications (2)

- ***Distributed System*** *A System with infinitely many state variables*
  - *Continuous elastic structures (beams, shells, and plates)*
  - *Fluid systems (ocean and atmosphere)*
  - *Can often be approximately described with lumped models (FEM, AMM)*
- ***Lumped System*** *A System with a finite number of state variables*
  - *Lumped parameter/ discrete system*
  - *Usually an artificial/modeling concept*
- ***Continuous-time System*** *All the signals are continuous in time*
  - *Everything is defined at each instant time*
  - *Also called Analog systems*
- ***Discrete-time Systems***
  - *Variables are only defined at discrete times*
  - *Also called sampled data systems*
- *Hybrid System*
  - *Continuous-time + discrete-time*

System Model Classifications (3)

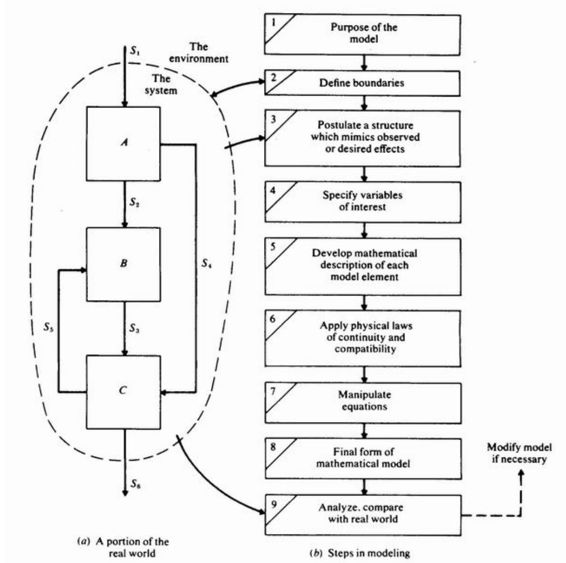
- ***Time-varying System*** *(in practice)*
  - *The characteristics of system changes with time going*
    - *time-varying parameters*
    - *time-varying dynamics*
- ***Time-invariant System*** *(ideal)*
  - *The features of system never ever changes*
  - *Usually a good approximation for most engineering application*
  - *A good starting point to obtain main features of system*
  - *Relatively easy to analyze*
- ***Linear System***
  - *Equations describing system are linear*
  - *Principle of superposition*
- ***Nonlinear System***
  - *Linearize it near a operating condition to obtain a linear approximation*

Mathematic Modeling

- Mathematic modeling: (analytical modeling)
  - First-principle modeling method
  - A systematic application of basic physical laws to system components and their interactions
  - System behaviors described by mathematical expressions (difference/differential equations)

Physical, chemical, biologic, social systems …

Mathematic Modeling Approach: Outline



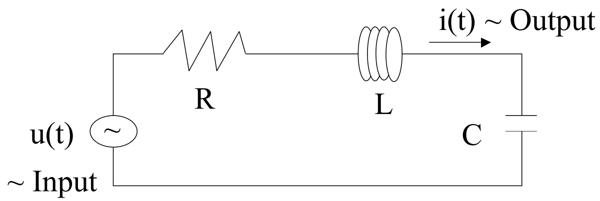
Mathematic Modeling of Electric Systems (1)

- Basic elements
  - Resistor $v(t)=R\cdot i(t)$
  - Capacitor $i(t)=C\cdot dv(t)/dt$
  - Inductor $v(t)=L\cdot di(t)/dt$
  - Voltage source $v(t)=v_S$
  - Currency source $i(t)=i_S$
  - Operational amplifier

Mathematic Modeling of Electric Systems (2)

- Basic laws
  - Kirchhoff's current law (KCL)
    - The algebraic sum of all currents entering and leaving a node is zero.
  - Kirchhoff's voltage law (KVL)
    - The algebraic sum of the voltages around any loop in an electrical circuit is zero.

Example: A Simple Electric Circuit (1)

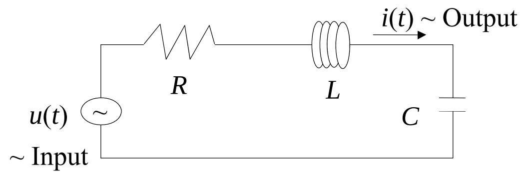


Use physical laws to model/describe system behavior:

What are the components? What properties do they have?

$$v_R = Ri_R, \quad v_L = L\frac{di_L}{dt}, \quad i_C = C\frac{dv_C}{dt}$$

- Relationship among the variables by physical law:
  - KCL: Current to a node = 0, $i_R = i_C = i_L = i$.
  - KVL: Voltage across a loop = 0.



$$KVL: \quad Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau + v_0 = u(t) \qquad L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{du(t)}{dt}$$

- An integral-differential or differential equation
- Input-output description or external description

- How to analyze the input-output relationship?
  - For example, find the output $i(t)$ given $u(t)$ and IC.
- We can use Laplace transform
  - Note: only effective for LTI systems

Laplace Transform: A Quick Review (1)

$$f(t) \Leftrightarrow F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

- Key Properties
  - Linearity:

    $$a_1 f_1(t) + a_2 f_2(t) \Leftrightarrow a_1 F_1(s) + a_2 F_2(s)$$

  - Derivative theorem:

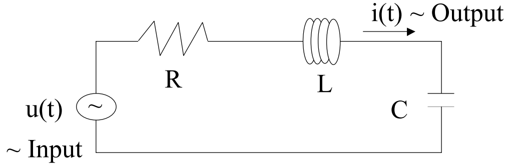
    $$\dot{f}(t) \Leftrightarrow sF(s) - f(0^-), \qquad \int f(\tau)d\tau \Leftrightarrow F(s)/s$$

➢ Converting linear constant coefficient differential equations into algebraic equations

Laplace Transform: A Quick Review (2)

- Other properties
  - Differentiation in the frequency domain: $t \cdot f(t) \leftrightarrow (-1)F'(s)$
  - Convolution: $h(t) * f(t) \leftrightarrow H(s) \cdot F(s)$
  - Time and frequency shifting: $f(t-t_0) \leftrightarrow e^{-st_0} F(s)$; $e^{s_0 t} f(t) \leftrightarrow F(s - s_0)$
  - Time and frequency scaling: $f(at) \leftrightarrow 1/a\, F(s/a)$ for $a > 0$
  - Initial Value Theorem: $f(0^+) = \lim_{s\to\infty} sF(s)$
  - Final Value Theorem: $f(\infty) = \lim_{s\to 0} sF(s)$ if all the poles of $sF(s)$ have strictly negative real parts

Example: A Simple Electric Circuit (2)



$$KVL:\quad Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau + v_0 = u(t)$$

$$RI(s) + L[sI(s) - i_0] + \frac{I(s)}{Cs} + \frac{v_0}{s} = U(s)$$

- An algebraic equation vs integral-differential equation. Solution:

$$\left(Ls + R + \frac{1}{cs}\right)I(s) = U(s) + \left[Li_0 - \frac{v_0}{s}\right]$$

$$I(s) = \frac{cs}{LCs^2 + RCs + 1}U(s) + \frac{LCsi_0 - cv_0}{LCs^2 + RCs + 1}$$

- Limitation of Laplace transform: not effective for time varying/nonlinear systems such as

$$\ddot{y}(t) + a_1(y,t)\dot{y}(t) + a_0(y,t)y(t) = b(y,t)u(t)$$

- The state space description to be studied later on will be able to handle more general systems

Mathematic Modeling of Mechanical Systems (1)

- Basic elements
  - Mass $\quad f(t)=M\cdot a(t)=M\cdot\dot{v}(t)= M\cdot\ddot{x}(t)$
    - Quantity of matter in a body, normally constant
    - a: acceleration; v: velocity; x: displacement
  - Spring $\quad f(t)=K\cdot x(t)$
    - Strentched or compressed develops a force
    - K: stiffness: x: total displacement from the equilibrium position
  - Damper $\quad f(t)=B\cdot v(t)=B\cdot\dot{x}(t)$
    - Reduce vibrations and tendency to hunt
    - Provide a resistance only to movement

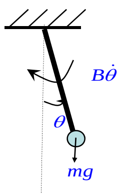
Mathematic Modeling of Mechanical Systems (2)

- Basic laws
  - Newton's law (2nd law of motion): $f(t)=Ma(t)$
- Translational motion
  - Applied forces=Mass $\times$ Resultant acceleration
  - $\Sigma f(t)=Ma(t)$
- Rotational motion
  - Applied torques=Inertia $\times$ Resultant angular acceleration
  - $\Sigma T(t)=J\alpha(t)$

Mathematic Modeling of Mechanical Systems (3)

- Forces
  - Contact/field force
  - Friction forces
    - Viscous/static/Coulomb friction forces
- Common subsystems
  - Spring-mass-damper systems
  - Inverted pendulum systems
  - Gear train systems
  - Translation-rotation conversion systems
  - Timing belts

Example (Simple Pendulum)



EOM: $I_o\ddot{\theta}=-mgl\sin\theta-B\dot{\theta}$

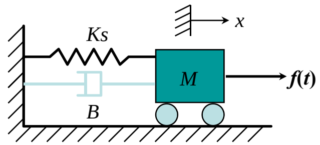
Equilibrium position: $\left.\begin{array}{l}\ddot{\theta}=0\\ \dot{\theta}=0\end{array}\right\}\Rightarrow\theta=0$

Assumption: θ is very small

Linearized EOM:
$$I_o\ddot{\theta}=-mgl\theta-B\dot{\theta}$$
$$I_o\ddot{\theta}+B\dot{\theta}+\underbrace{mgl}_{K}\theta=0$$

- **Mass-Spring-Damper System**

I/O Model:



$$M\ddot{x}+B\dot{x}+K_s x=f(t)$$

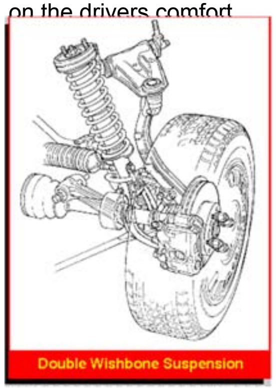
$$\boxed{\begin{aligned}G(s)&=\frac{1}{Ms^2+Bs+K_s}\\&=\frac{1/M}{s^2+\frac{B}{M}s+\frac{K_s}{M}}\\&=\frac{K\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}\end{aligned}}$$

$$\boxed{\begin{aligned}\omega_n&=\sqrt{\frac{K_s}{M}}\\ \varsigma&=\frac{B}{2\sqrt{MK_s}}\\ K&=\frac{1}{K_s}\end{aligned}}$$

Example -- SDOF Suspension

- **Suspension System**

Minimize the effect of the surface roughness of the road on the drivers comfort



– **Simplified Schematic (neglecting tire model)**

From the "absolute zero"

$$M\ddot{x}+B\left(\dot{x}-\dot{x}_p\right)+K\left(x-x_p\right)=-Mg$$

From the path

$$M\ddot{x}_{op}+B\dot{x}_{op}+Kx_{op}=-Mg-M\ddot{x}_p$$

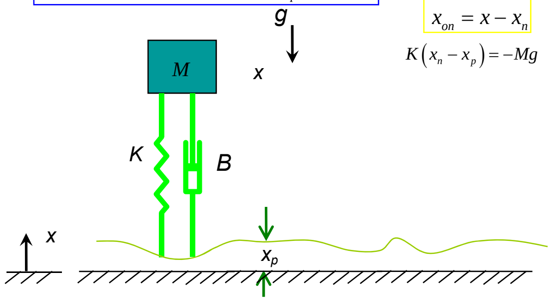
$$x_{op}=x-x_p$$

From nominal position

$$M\ddot{x}_{on}+B\dot{x}_{on}+Kx_{on}=-M\ddot{x}_p$$

$$x_{on}=x-x_n$$

$$K\left(x_n-x_p\right)=-Mg$$



Mathematic Modeling of Electromechanical Systems (1)

- Basic elements
  - Motor $f(t)=Bl\cdot i(t)$
    - Converts electric energy into mechanical energy
    - B: magnetic strength; i: current ; l: conductor length
  - Generator $e(t)= Bl\cdot v(t)$
    - Converts mechanical energy into electric energy
    - B: magnetic strength; v: velocity; l: conductor length

Mathematic Modeling of Electromechanical Systems (2)

- Sensors and encoders in control systems

  Important to performance monitoring and feedback
  - Potentiometer
    - Displacement measurement
    - Rotary Potentiometer
    - Linear motion Potentiometer
  - Tachometer
    - Angular velocity measurement
    - Voltage generator
  - Encoder
    - Convert linear/rotary displacement into digitally coded/pulse signals
    - Absolute/incremental encoders

Mathematic Modeling of Electromechanical Systems (3)

- DC motors in control systems

  widely used in the automation, machine-tool industries and computer peripheral equipments
  - Basic principles
    - $T_m(t)=K_m\Phi\cdot i_a(t)$
    - $e_b(t)=K_m\Phi\cdot\omega_m(t)$
    - T_m: motor torque; i_a: armature current
    - ω_m: shaft velocity; e_b: back emf
    - Φ: magnetic flux ; K_m: proportional constant
  - Classifications of permanent magnet DC motors

Mathematic Modeling of Fluid Systems

- Basic elements
  - Fluid resistor $q(t)=h(t)/R$
    - q: flow rate; h: pressure head; R: flow resistance
    - Pipe: conducts liquid
  - Fluid capacitor $q(t)=C\cdot\dot{h}(t)$
    - q: flow rate; h: pressure head; C: capacitance
    - Tank: stores fluid volume
- Common systems
    - Single-tank fluid-level system
    - Two-tank fluid-level system (Interacting/Noninteracting)
    - Drum-type boiler

Mathematic Modeling of Thermal Systems

- Basic elements
  - Thermal resistor $\dot{T}(t)=R\cdot q(t)$
    - T: temperature; q: heat transfer rate; R: thermal resistance
    - Heat flows from higher temperature to low temperature
  - Thermal capacitor $C=M\cdot c=\dot{T}$
    - C: capacitance; T: temperature; M: mass; c: specific heat
    - The capacity of an object to store heat
- Heating systems
  - An object heated by an external energy source (electric/steam/hot water)
  - Net heat rate gained = rate of heat addition - rate of heat loss
  - Lab. Project: thermal process control

Mathematic Modeling of Biologic Systems (1)

- Feedback is a central feature of life. The process of feedback governs how we grow, respond to stress and challenge, and regulate factors such as body temperature, blood pressure, and cholesterol level.

- It also makes it possible for us to stand upright. The mechanisms operate at every level, from the interaction of proteins in cells to the interaction of organisms in complex ecologies.

Mathematic Modeling of Biologic Systems (2)

- Predator-Prey model (Volterra)
- G(t): goats; W(t): wolves; t: time

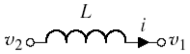
  G(t+1)=aG(t)-bG(t)W(t)

  W(t+1)=cW(t)+dG(t)W(t)

  Constants: a,b,c,d > 0; a>1, c<1
- Frequency G and W encounter is proportional, to the product
- System behavior
- Choice of a,b,c,d

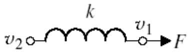
Analogy between Physical Systems

Electrical Inductance



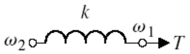
$$v_{21} = L \cdot \frac{d}{dt} i \qquad E = \frac{1}{2} \cdot L \cdot i^2$$

Translational Spring



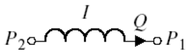
$$v_{21} = \frac{1}{k} \cdot \frac{d}{dt} F \qquad E = \frac{1}{2} \cdot \frac{F^2}{k}$$

Rotational Spring



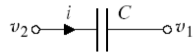
$$\omega_{21} = \frac{1}{k} \cdot \frac{d}{dt} T \qquad E = \frac{1}{2} \cdot \frac{T^2}{k}$$

Fluid Inertia



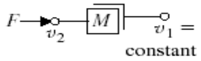
$$P_{21} = I \cdot \frac{d}{dt} Q \qquad E = \frac{1}{2} \cdot I \cdot Q^2$$

Electrical Capacitance



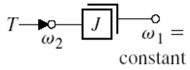
$$i = C \cdot \frac{d}{dt} v_{21} \qquad E = \frac{1}{2} \cdot M \cdot v_{21}^{\;2}$$

Translational Mass



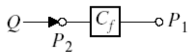
$$F = M \cdot \frac{d}{dt} v_2 \qquad E = \frac{1}{2} \cdot M \cdot v_2^{\;2}$$

Rotational Mass



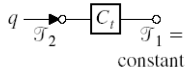
$$T = J \cdot \frac{d}{dt} \omega_2 \qquad E = \frac{1}{2} \cdot J \cdot \omega_2^{\;2}$$

Fluid Capacitance



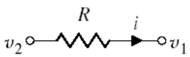
$$Q = C_f \cdot \frac{d}{dt} P_{21} \qquad E = \frac{1}{2} \cdot C_f \cdot P_{21}^{\;2}$$

Thermal Capacitance



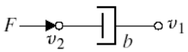
$$q = C_t \cdot \frac{d}{dt} T_2 \qquad E = C_t \cdot T_2$$

Electrical Resistance



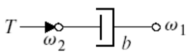
$$i = \frac{1}{R} \cdot v_{21} \qquad P = \frac{1}{R} \cdot v_{21}^{\;2}$$

Translational Damper



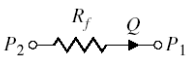
$$F = b \cdot v_{21} \qquad P = b \cdot v_{21}^{\;2}$$

Rotational Damper



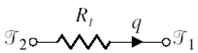
$$T = b \cdot \omega_{21} \qquad P = b \cdot \omega_{21}^{\;2}$$

Fluid Resistance



$$Q = \frac{1}{R_f} \cdot P_{21} \qquad P = \frac{1}{R_f} \cdot P_{21}^{\;2}$$

Thermal Resistance



$$q = \frac{1}{R_t} \cdot T_{21} \qquad P = \frac{1}{R_t} \cdot T_{21}$$

Main Contents

- Mathematic modeling of
  - Thermal systems
  - Fluid systems
  - Biologic/social systems
- System identifications
- Linguistic model
- Model representations
- Linearization of nonlinear systems
- Matlab tools

Model Representation

Model Representation Types

- Differential/difference equations
- Transfer functions
- Block diagrams
- Signal-flow graphs
- State-space representations

Differential/Difference Equations

- Differential/difference equations

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m\frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t)$$

- Given $u(t)$ for $t \geqq t_0$ and the IC of $y(t)$ and its derivatives at t_0, solve the equation for $y(t)$.

- The basic philosophy of linear control theory is that of developing analysis and design tools that would avoid the exact solution of the system differential equations.

Transfer Functions

- Transfer functions
  - TF of a linear system is the Laplace transform of the impulse response with zero initial conditions.

$$\frac{d^n y(t)}{dt^n}+a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}}+\cdots+a_1\frac{dy(t)}{dt}+a_0y(t)=b_m\frac{d^m u(t)}{dt^m}+b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}}+\cdots+b_1\frac{du(t)}{dt}+b_0u(t)$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_m s^m + b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n + a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$$

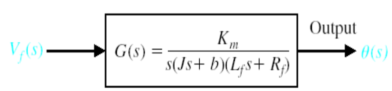
  - Characteristic equation

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

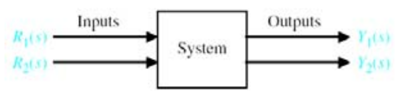
    - Stability of linear SISO systems is governed completely by the roots of the characteristic equation.
    - Valid for only LTI systems (Linear constant-coefficient differential/difference equations)
    - Difficult for MIMO cases

Block Diagram (1)

- Consist of unidirectional, operational blocks
- Blocks represent Transfer Functions
- Whole system divided into a set of subsystems/components
- Laplace Transform/mathematical simplification to get system I/O relationship --- complicated
- Block Diagram for system representation and I/O relationship --- simple



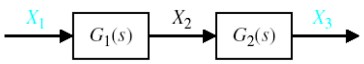
Block diagram of dc motor.



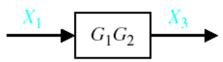
General block representation of two-input, two-output system.

Block Diagram (2)

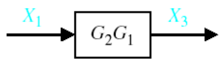
Original Diagram



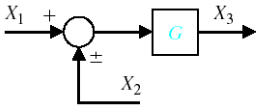
Equivalent Diagram



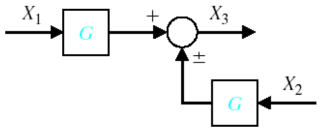
or



Original Diagram

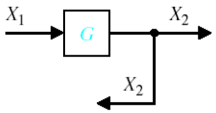


Equivalent Diagram

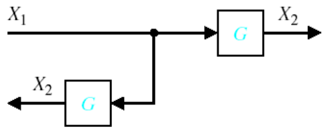


Block Diagram (3)

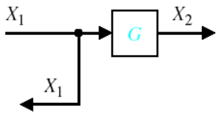
Original Diagram



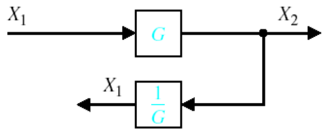
Equivalent Diagram



Original Diagram

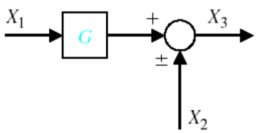


Equivalent Diagram

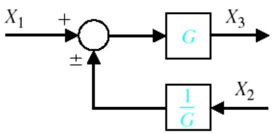


Block Diagram (6)

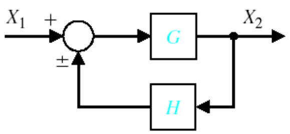
Original Diagram



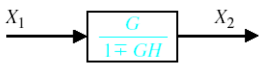
Equivalent Diagram



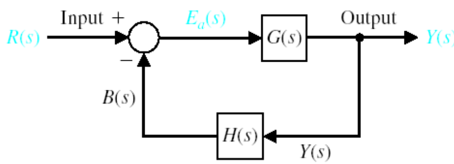
Original Diagram



Equivalent Diagram



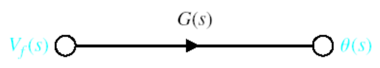
Block Diagram (7)



Negative feedback control system.

Signal-Flow Graph (1)

For complex systems, the block diagram method can become difficult to complete. By using the signal-flow graph model, the reduction procedure (used in the block diagram method) is not necessary to determine the relationship between system variables.



Signal-flow graph of the dc motor.

Basic Elements of SFG

- Node: input/output point
- Branch: unidirectional segment
- Path: a branch or a continuous sequence of branches, one signal (node) to another
- Forward Path: path from input to output
- Loop: a closed path (starts/ends at the same node), no node met twice (feedback loop!)
- Nontouching Loops: two loops with no common node
- Path/Loop Gain

Mason's Rule

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_k G_k \Delta_k}{\Delta}$$

- G_k: path gain of the *k*th forward path
- Δ: system determinant

  =1-Σ(all individual loop gains)+Σ(gain products of all possible two loops that do not touch)-Σ(gain products of all possible three loops that do not touch)+…
- Δ_k: *k*th forward path determinant

  =value of Δ for that part of SFG that doe not touch the *k*th forward path (remove the parts that touch path *k* from Δ)
- Δ_k=Δ-Σ(loop gain terms in Δ that touch the kth forward path) (or say, Δ_k is formed by eliminating from Δ those loop gains that touch path *k*).

State Space Representation

- Transfer function representation valid for only LTI systems and difficult for complex systems such as MIMO systems
- On the contrary, in state space framework linear and nonlinear systems, time-invariant and time-varying systems, single-variable and multivariable systems can all be represented in a unified manner.

Nonlinear Systems (1)

- Why nonlinear?
  - Most real systems are nonlinear.
    - Robotics: Geometric coordinate transformations (sin/cos)
  - Most components and actuators have nonlinear characteristics.
    - Backlash, hysteresis, deadzone, saturation,…
- Linear *vs*. Nonlinear
  - Nonlinear is general but difficult (complete treatment is impossible); Linear is comparatively easy but restricted (elegant analytical solution)
  - Linear models are descriptions of small perturbations away from a nominal operating point (approximation).

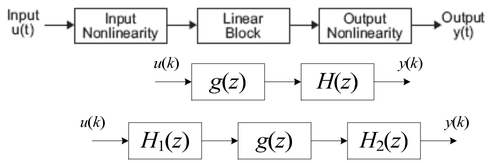
  - Superposition
  - Homogeneity (scaling)
  - Correlation between TF pole/zero and time response behavior
  - Analytical solutions
  - Stability (equilibrium points), controllability, observability (global consideration)

Nonlinear Systems (2)

Methods & Tools

- Linearization (linear approximate model around a known nominal solution or operating point.
  - Airplane & spacecraft
- Feedback linearization (state/output)
  - Robotics
- Describing function (approximately but useful)
- Phase-plane representations (useful but only for 2nd order systems)

- Special nonlinear systems
  - Bilinear
  - Affine (control-affine, output affine)
  - Finite Volterra model
  - Hammerstein model
  - Wiener model
  - Lur'e model
  - Sandwich model



Nonlinear Systems (3)

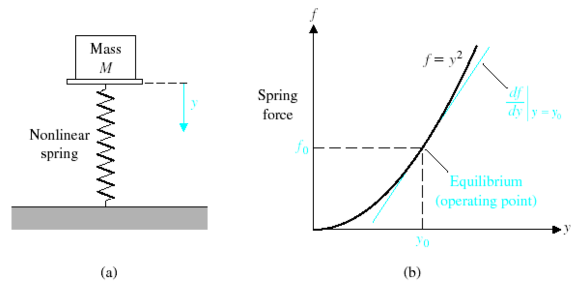
- Lyapunov stability theory
    - Stability is always a major concern
  - Very general method
  - Powerful for nonlinear analysis and controller design for certain types of nonlinear systems
  - Lyapunov function
  - Sufficient condition for global asymptotic stability

Linearization (1)

- Linearization (linear approximate model around a known nominal solution or operating point.
- Expand the nonlinear function into a Taylor series about the op. and retain only the linear term.

$$
\begin{aligned}
y &= f(x) \\
&= f(\bar{x}) + \frac{df}{dx}(x-\bar{x}) + \frac{1}{2!}\frac{d^2 f}{dx^2}(x-\bar{x})^2 + \cdots \\
&\approx f(\bar{x}) + K(x-\bar{x})
\end{aligned}
$$

$$K = \frac{df}{dx}\Big|_{\bar{x}}$$

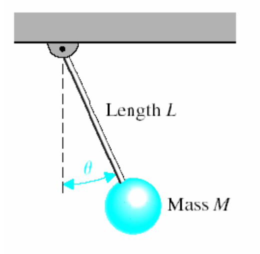


(a) A mass sitting on a nonlinear spring.
(b) The spring force versus y.

Linearization (2)

- Neglect higher-order terms by assuming they are small enough, thus variables should deviate only slightly from the op.
- Theoretic foundation: Lyapunov theory
  - A linearized approximation model is stable at an op., then in a vicinity of this op. the nonlinear system is stable.
- Controlled to this op. through feedback

Example



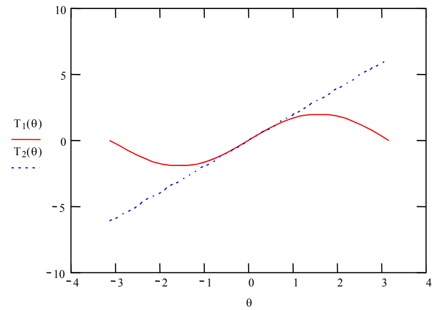
Pendulum oscillator.

$M := 200\ \text{gm}$ $\quad g := 9.8\,\frac{\text{m}}{\text{s}^2}$ $\quad L := 100\ \text{cm}$ $\quad \theta_0 := 0\ \text{rad}$ $\quad \theta := -\pi, \frac{-15\pi}{16} .. \pi$

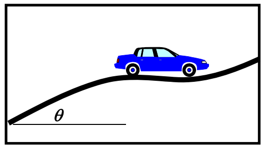
$T_0 := M \times g \times L \times \sin(\theta_0)$

$T_1(\theta) := M \times g \times L \times \sin(\theta)$

$T_2(\theta) := M \times g \times L \times \cos(\theta_0) \times (\theta - \theta_0) + T_0$



Students are encouraged to investigate linear approximation accuracy for different values of θ_0



1 MODEL:

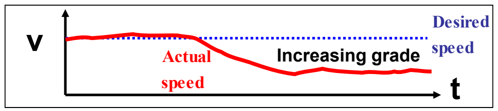
Inertia force / Traction: input, excitation / Friction: damping / Gravitation: disturbance

$$M\dot{v} = f_e(t) - Bv - Mg\sin\theta$$

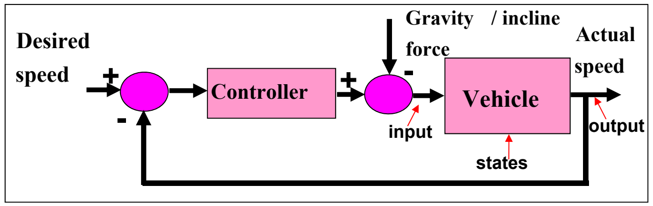
$$M\dot{v} + Bv \cong f_e(t) - Mg\theta$$

linearization

2 ANALYSIS:



3 CONTROL:



Part 3:

System Analysis

System Responses
&
Transfer Function Analysis

Dynamic Responses of LTI Systems

Ex: Let's look at a stable first order system:

$$\underset{\text{Time constant}}{\boxed{\tau}}\,\dot{y} + y = Ku$$

– Take LT of the I/O model and remember to keep tracks of the ICs:

$$\begin{aligned}
\mathcal{L}[\tau\,\dot{y} + y] &= [Ku] \\
\tau\,\mathcal{L}[\dot{y}] + [y] &= [u] \\
\tau\,(\; sY(s) - y(0)\;) + Y(s) &= K \cdot U(s)
\end{aligned}$$

– Rearrange terms s.t. the output $Y(s)$ terms are on one side and the input $U(s)$ and IC terms are on the other:

$$(\;\tau s + 1\;)\cdot Y(s) = K \cdot U(s) + \tau \cdot y(0)$$

– Solve for the output:

$$Y(s) = \underbrace{\left(\frac{K}{\tau s + 1}\right)\cdot U(s)}_{\text{Forced Response}} + \underbrace{\left(\frac{\tau}{\tau s + 1}\right)\cdot y(0)}_{\text{Free Response}}$$

Free & Forced Responses

- **Free Response** (**$u(t) = 0$ & nonzero ICs**)
  – The response of a system to *zero input* and *nonzero initial conditions*.
  – Can be obtained by
    - Let $u(t) = 0$ and use LT and ILT to solve for the free response.
- **Forced Response** (**zero ICs & nonzero $u(t)$**)
  – The response of a system to *nonzero input* and *zero initial conditions*.
  – Can be obtained by
    - Assume zero ICs and use LT and ILT to solve for the forced response (replace differentiation with s in the I/O ODE model).

In Class Exercise

Find the free and forced responses of the car suspension system without tire model:

$$\ddot{y} + 2\dot{y} + 4y = 2\dot{u} + 4u, \quad y = z, \quad u = x_r$$

– Take LT of the I/O model and remember to keep tracks of the ICs:

$$\begin{aligned}
\mathcal{L}[\ddot{y} + 2\dot{y} + 4] &= [2\dot{u} + 4u] \\
\mathcal{L}[\ddot{y}] + 2\;[\dot{y}] + 4\mathcal{L}[y] &= 2\;[\dot{u}] + 4\mathcal{L}[u] \\
\left[s^2Y(s) - sy(0) - \dot{y}(0)\right] + 2\left[sY(s) - y(0)\right] + 4Y(s) &= 2\left[sU(s) - u(0)\right] + 4U(s)
\end{aligned}$$

– Rearrange terms s.t. the output $Y(s)$ terms are on one side and the input $U(s)$ and IC terms are on the other:

$$\left[s^2 + 2s + 4\right]\cdot Y(s) = \left[2s + 4\right]\cdot U(s) + \left[s + 2\right]\cdot y(0) + \dot{y}(0)$$

– Solve for the output:

$$Y(s) = \underbrace{\left(\frac{2s + 4}{s^2 + 2s + 4}\right)\cdot U(s)}_{\text{Forced Response}} + \underbrace{\left(\frac{(s+2)\,y(0) + \dot{y}(0)}{s^2 + 2s + 4}\right)}_{\text{Free Response}}$$

Forced Response & Transfer Function

Given a general n-th order system model:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0 u$$

The **forced response** (zero ICs) of the system due to input $u(t)$ is:

– *Taking the LT of the ODE*:

$$\because\ \mathcal{L}\left[y^{(n)}\right] = s^n Y(s) \qquad \text{(WHY?)}$$

$$\therefore\ s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_1 sY(s) + a_0 Y(s) = b_m s^m U(s) + b_{m-1}s^{m-1}U(s) + \cdots + b_1 sU(s) + b_0 U(s)$$

$$\Rightarrow \underbrace{\left[s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right]}_{D(s)} \cdot Y(s) = \underbrace{\left[b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0\right]}_{N(s)} \cdot U(s)$$

$$\therefore\ Y(s) = \boxed{\underbrace{\frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}}_{G(s)}}\ U(s) = \frac{N(s)}{D(s)} \cdot U(s) = \boxed{G(s)}\ U(s)$$

Forced Response = **Transfer Function** **Inputs** = **Transfer Function** **Inputs**

Transfer Function

Given a general nth order system:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0 u$$

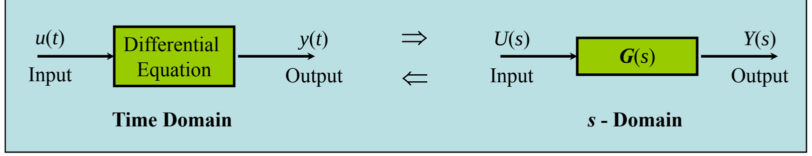
The transfer function of the system is:

$$G(s) \equiv \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Static gain

– The transfer function can be interpreted as:

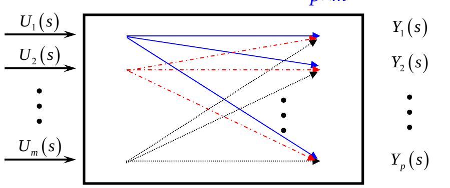
$$G(s) \equiv \left.\frac{Y(s)}{U(s)}\right|_{\text{zero I.C.s}}$$



Transfer Function Matrix

For Multiple-Input-Multiple-Output (MIMO) System with m inputs and p outputs:

Inputs $\quad G(s) \equiv \left\{G_{ij}(s)\right\}_{p\times m} \quad$ Outputs



$$G_{ij}(s) = \left.\frac{Y_i(s)}{U_j(s)}\right|_{\text{zero I.C.s}}$$

$$\underbrace{\begin{bmatrix} Y_1(s) \\ \vdots \\ Y_p(s) \end{bmatrix}}_{Y(s)} = \underbrace{\begin{bmatrix} G_{11}(s) & \cdots & G_{1m}(s) \\ \vdots & \vdots & \vdots \\ G_{p1}(s) & \cdots & G_{pm}(s) \end{bmatrix}}_{G(s)} \underbrace{\begin{bmatrix} U_1(s) \\ \vdots \\ U_m(s) \end{bmatrix}}_{U(s)}$$

$$Y_k(s) = \sum_{j=1}^{m} G_{kj}(s)U_j(s)$$

$$= G_{k1}(s)U_1(s) + G_{k2}(s)U_2(s) + \cdots + G_{km}(s)U_m(s) \qquad k = 1,\cdots,p$$

Poles and Zeros

Given a transfer function (TF) of a system:

$$G(s) \equiv \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

- **Poles**

  The roots of the denominator of the TF, i.e. the roots of the characteristic equation.

  $$D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = (s - p_1)(s - p_2)\cdots(s - p_n) = 0$$

  $$\boxed{\Rightarrow p_1, p_2, \cdots, p_n}$$

  ***n* poles of TF**

- **Zeros**

  The roots of the numerator of the TF.

  $$N(s) = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0 = b_m(s - z_1)(s - z_2)\cdots(s - z_m) = 0$$

  $$\boxed{\Rightarrow z_1, z_2, \cdots, z_m}$$

  ***m* zeros of TF**

$$G(s) \equiv \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)} = \frac{b_m(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

General Form of Free Response

Given a general nth order system model:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0 u$$

The **free response** (zero input) of the system due to ICs is:

– *Taking the LT of the model with zero input*

(*i.e.*, $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = 0$)

$$\underbrace{\left[s^nY(s) - s^{n-1}y(0) - \cdots - y^{(n-1)}(0)\right]}_{\mathcal{L}\left[y^{(n)}\right]} + a_{n-1}\underbrace{\left[s^{n-1}Y(s) - s^{n-2}y(0) - \cdots - y^{(n-2)}(0)\right]}_{\left[y^{(n-1)}\right]}$$

$$+ \cdots + a_1\underbrace{\left[sY(s) - y(0)\right]}_{\mathcal{L}[\dot{y}]} + a_0\underbrace{Y(s)}_{\mathcal{L}[y]} = 0$$

$$\Rightarrow$$

$$\left[s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right]\cdot Y_{Free}(s) = \underbrace{\left(s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1\right)y(0) + \cdots + y^{(n-1)}(0)}_{F(s)}$$

$$\Rightarrow \quad Y_{Free}(s) = \frac{F(s)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Free Response (Natural Response) = $\dfrac{\text{A Polynominal of } s \text{ that depends on ICs}}{\text{Same Denominator as TF } G(s)}$

Free Response (Examples)

Ex: Find the free response of the car suspension system without tire model (slinker toy):

$$\ddot{y} + 2\dot{y} + 4y = 2\dot{u} + 4u, \quad y = z, \quad u = x_r$$

$$Y(s) = \left(\frac{2s+4}{s^2+2s+4}\right)\cdot U(s) + \left(\frac{(s+2)y(0) + \dot{y}(0)}{s^2+2s+4}\right)$$

$$Y(s) = \frac{s+2}{s^2+2s+4}y(0) = \frac{s+2}{s^2+2s+4}$$

Ex: Perform partial fraction expansion (PFE) of the above free response when:

$\dot{y}(0) = 0$ and $y(0) = 1$ *(what does this set of ICs means physically)?*

$$Y(s) = \frac{s+2}{(s+1)^2 + \left(\sqrt{3}\right)^2} = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s+1)^2 + \left(\sqrt{3}\right)^2} + \frac{(s+1)}{(s+1)^2 + \left(\sqrt{3}\right)^2}$$

$$y(t) = e^{-t}\left[\frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t\right) + \cos\left(\sqrt{3}t\right)\right] = \frac{2}{\sqrt{3}}e^{-t}\sin\left(\sqrt{3}t + \tan^{-1}\left(\sqrt{3}\right)\right)$$

Decaying rate: damping, mass

Frequency: damping, spring, mass

phase: initial conditions

Q: Is the solution consistent with your physical intuition?

Free Response and Pole Locations

The free response of a system can be represented by:

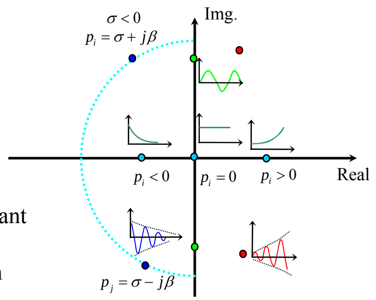
$$Y_{Free}(s) = \frac{F(s)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{F(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$= \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots + \frac{A_n}{s-p_n} \qquad \left(\begin{array}{c}\text{For simplicity, assume that} \\ p_1 \neq p_2 \neq \cdots \neq p_n\end{array}\right)$$

$$\Rightarrow \quad y_{Free}(t) = \mathcal{L}^{-1}\left[Y_{Free}(s)\right] = A_1 e^{p_1\cdot t} + A_2 e^{p_2\cdot t} + \cdots + A_n e^{p_n\cdot t}$$

p_i is real:
- $p_i < 0 \Rightarrow A_i e^{p_i t}$ exponential decrease
- $p_i = 0 \Rightarrow A_i$ constant
- $p_i > 0 \Rightarrow A_i e^{p_i t}$ exponential increase

$p_{i,j} = \sigma \pm j\beta$:
- $\sigma < 0 \Rightarrow$ decaying oscillation
- $\sigma = 0 \Rightarrow$ oscillation with constant magnitude
- $\sigma > 0 \Rightarrow$ increasing oscillation



$$A_i e^{p_i t} + \bar{A}_i e^{\bar{p}_i t} = 2\,\mathrm{Re}\left\{A_i e^{p_i t}\right\}$$
$$= 2\left|A_i\right|e^{\sigma_i t}\cos\left(\beta_i t + \varphi\right)$$

Complete Response

$U(s)$ Input → $G(s) \equiv \dfrac{N(s)}{D(s)}$ → $Y(s)$ Output

ICs: $y(0)$, $\dot{y}(0)$, $\cdots$, $y^{(n-1)}(0)$

- Complete Response

$$Y(s) = Y_{Forced}(s) + Y_{Free}(s)$$

$$= \frac{N(s)}{D(s)}\cdot U(s) + \frac{\text{A Polynomial of } s \text{ depending on I.C.s}}{D(s)}$$

Q: Which part of the system affects both the free and forced response ?

Denominator *D*(*s*)

Q: When will free response converges to zero for all non-zero I.C.s ?

All the poles have negative real parts.

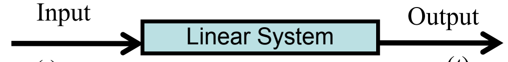
Forced Responses of LTI Systems

- **Forced Responses of LTI Systems**
  - Superposition Principle
  - Forced Responses to Specific Inputs
- **Forced Response of 1st Order Systems**
  - Transfer Function and Poles/Zeros
  - Forced Response of Stable 1st Order Systems
- **Forced Response of 2nd Order Systems**
  - Transfer Function and Poles/Zeros
  - Forced Response of Stable 2nd Order Systems

Forced Responses of LTI Systems

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_m u^{(m)} + \cdots + b_1\dot{u} + b_0 u$$

- **Superposition Principle**



$u_1(t)$ → $y_1(t)$

$u_2(t)$ → $y_2(t)$

$u(t)=k_1 u_1(t) + k_2 u_2(t)$ → $\boxed{y(t)=k_1 y_1(t) + k_2 y_2(t)}$

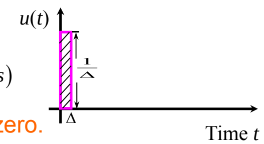
complicated input=$\sum$simple inputs $\sum$ forced responses of simple inputs =forced response of complicated input

The forced response of a linear system to a complicated input can be obtained by studying how the system responds to simple inputs, such as ***unit impulse*** *input*, ***unit step*** *input*, and ***sinusoidal*** *inputs with different input frequencies.*

Typical Forced Responses

- **Unit Impulse Response**
  - Forced response to unit impulse input

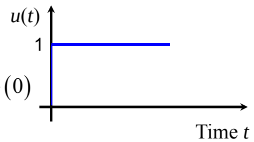
$$u(t)=\delta(t)=\begin{cases}+\infty, & t=0\\ 0, & t\neq 0\end{cases},\quad \int_{-\infty}^{+\infty}\delta(t)\,dt=1 \qquad Y(s)=G(s)U(s)=G(s)$$



If system is stable, then its SS impulse response is zero.

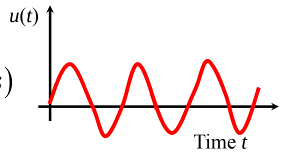
- **Unit Step Response**
  - Forced response to unit step input ($u(t) = 1$)

$$Y(s)=G(s)U(s)=\underbrace{\frac{1}{s}}_{U(s)}G(s) \qquad y(\infty)=\lim_{s\to 0} sY(s)=\lim_{s\to 0} s\frac{1}{s}G(s)=G(0)$$



- **Sinusoidal Response**
  - Forced response to sinusoidal inputs at different input frequencies
  - The SS (*steady state*) *response* of sinusoidal response is call the *Frequency Response*.

$$Y(s)=G(s)U(s)=G(s)\underbrace{\frac{\omega}{s^2+\omega^2}}_{U(s)} \qquad sY(s)=\frac{s\omega}{s^2+\omega^2}G(s)$$



Forced Response of 1st Order Systems

- **Standard Form of *Stable* 1st Order System**

$$\dot{y} + ay = bu \quad \Rightarrow \boxed{\tau\dot{y} + y = Ku}$$

$$\frac{1}{a}\dot{y} + y = \frac{b}{a}u$$

where

τ : Time Constant $\quad \tau = \dfrac{1}{a}$

K : Static (Steady State, DC) Gain $\quad K = \dfrac{b}{a}$

- **TF and Poles/Zeros**

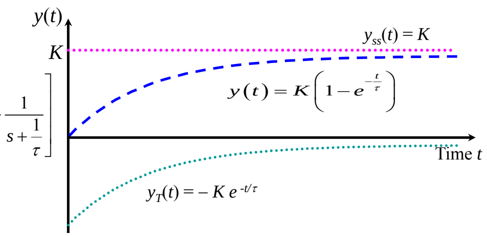
Stable system

$$G(s)=\frac{K}{\tau s+1} \qquad \text{pole: } p=-\frac{1}{\tau}<0 \qquad \text{zero: No zero}$$

- **Unit Step Response**
  - (u=1 and zero ICs)

$$Y(s)=G(s)U(s)=\frac{K}{(\tau s+1)s}=K\left[\frac{1}{s}-\frac{\tau}{\tau s+1}\right]=K\left[\frac{1}{s}-\frac{1}{s+\frac{1}{\tau}}\right]$$

$$y(t)=K\left(1-e^{-\frac{t}{\tau}}\right)$$



Normalized Unit Step Response

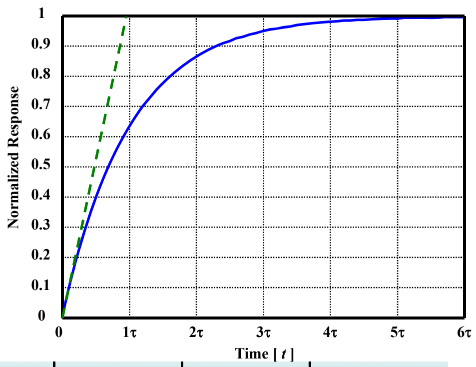
Normalized Unit Step Response (u = 1 & zero ICs)

$$\tau \dot{y} + y = Ku$$

$$\Rightarrow \; y(t) = K(1 - e^{-t/\tau})$$

Normalized
(such that as $t \to \infty$, $y_n \to 1$):

$$\Rightarrow \; y_n(t) = \frac{y(t)}{K} = 1 - e^{-t/\tau}$$



Time t	τ	2τ	3τ	4τ	5τ
$(1 - e^{-t/\tau})$	0.6321	0.8647	0.9502	0.9817	0.9933

Unit Step Response of Stable 1st Order System

Effect of Time Constant τ :

$$\tau \dot{y} + y = Ku$$

$$\Rightarrow \; y(t) = K(1 - e^{-t/\tau})$$

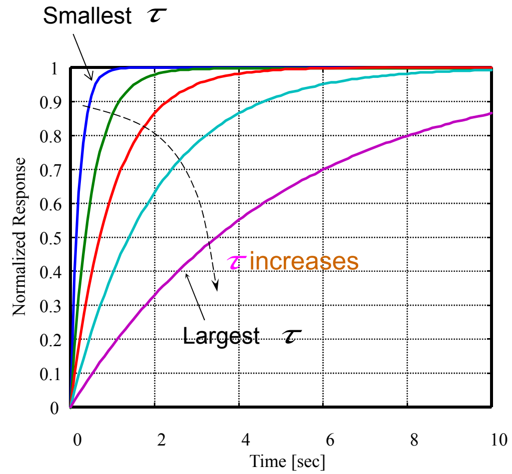
Normalized:

$$\Rightarrow \; y_n(t) = \frac{y(t)}{K} = (1 - e^{-t/\tau})$$

Initial Slope:

$$\Rightarrow \; \frac{d}{dt} y_n(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}$$

$$\Rightarrow \; \frac{d}{dt} y_n(0) = \frac{1}{\tau}$$

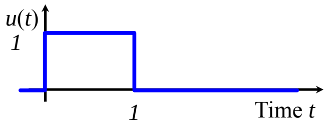


Q: What is your conclusion ?

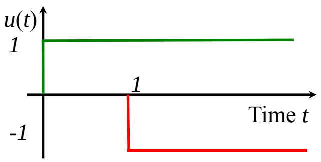
The smaller τ is,
the steeper the initial slope is, and
the faster the response approaches the steady state.

Forced Responses of Stable 1st Order System

Q: *How would you calculate the forced response of a 1st order system to a unit pulse (not unit impulse)?*



Q (Hint: superposition principle ?!)



$$u(t) = u_s(t) - u_s(t-1)$$

$$y(t) = y_s(t) - y_s(t-1)$$

Q: *How would you calculate the unit impulse response of a 1st order system?*

$$\delta(t) = \frac{d}{dt} u_s(t)$$

$$y(t) = \frac{d}{dt} y_s(t)$$

Q: *How would you calculate the sinusoidal response of a 1st order system?*

Standard Form of 2nd Order Systems

- I/O Model

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{u} + b_0 u$$

- TF and Pole/Zeros

$$G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \qquad \text{pole: } p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

$$\text{zero: } z = -\frac{b_0}{b_1}$$

- **Stability Condition**

$$a_1 > 0, \quad a_0 > 0$$

- **Standard Form of *Stable* 2nd Order Systems without Zeros**

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_0 u \quad \Rightarrow \quad \ddot{y} + \underbrace{2\zeta\omega_n}_{a_1} \dot{y} + \underbrace{\omega_n^2}_{a_0} y = \underbrace{K\omega_n^2}_{b_0} u$$

where

ω_n :	**Natural Frequency [rad/s]**	$\omega_n = \sqrt{a_0}$
ζ :	**Damping Ratio**	$\zeta = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$
K :	**Static (Steady State, DC) Gain**	$K = \frac{b_0}{\omega_n^2} = \frac{b_0}{a_0}$

Poles of Stable 2nd Order Systems

- **Stable 2nd Order Systems without Zeros**

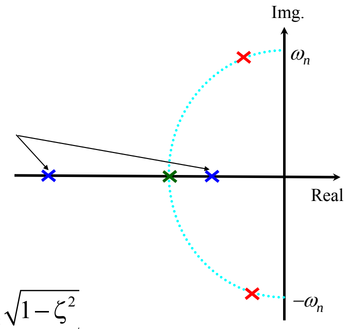
$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y \;=\; K\omega_n^2 u$$

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

- **Pole Locations**

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\Rightarrow \quad p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{(\zeta^2 - 1)}$$



- Over-damped ($\zeta > 1$)

  Two distinct real poles

- Critically damped ($\zeta = 1$)

  Two identical real poles at $p_{1,2} = -\zeta\omega_n$

- Under damped ($\zeta < 1$)

  Two complex poles at $p_{1,2} = -\underbrace{\zeta\omega_n}_{\sigma} \pm j\,\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$

Under-damped 2nd Order System

- **Unit Step Response** (u=1 and zero ICs)

$$Y(s) = G(s)\frac{1}{s} = \frac{K\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{K\omega_n^2}{s(s-p_1)(s-p_2)}, \quad p_{1,2} = -\sigma \pm j\omega_d$$

$$= \frac{K}{s} + \frac{A_1}{s+\sigma-j\omega_d} + \frac{\bar{A}_1}{s+\sigma+j\omega_d}, \qquad A_1 = -\frac{K}{2}\left(1 - j\frac{\sigma}{\omega_d}\right)$$

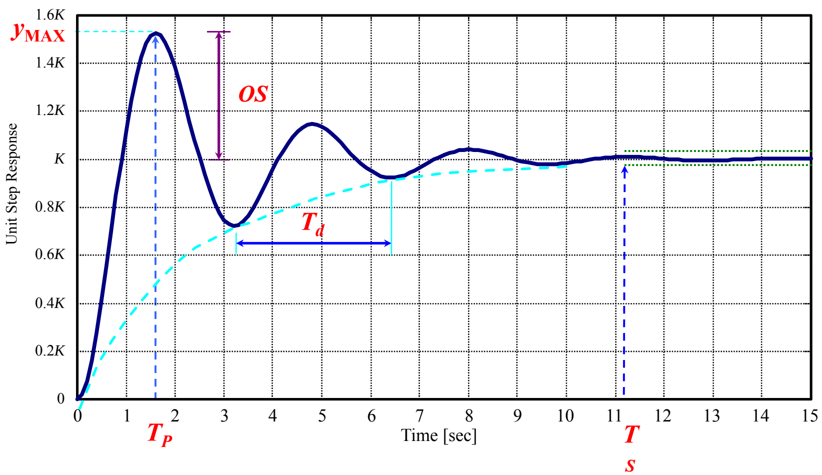
$\Rightarrow$

$$y(t) \;=\; K \;+\; \underbrace{A_1 e^{(-\sigma+j\omega_d)t} + \bar{A}_1 e^{(-\sigma-j\omega_d)t}}_{2\,\mathrm{Re}\left\{A_1 e^{(-\sigma+j\omega_d)t}\right\}}$$

$$= K\left[1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right)\right]$$

$$= K \;-\; \frac{K}{\sqrt{1-\zeta^2}}e^{-\sigma t}\sin(\omega_d t + \varphi), \qquad \varphi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Unit Step Response of 2nd Order Systems



Unit Step Response of 2nd Order System

- **Peak Time (T_P)**

Time when output $y(t)$ reaches its maximum value y_{MAX}.

$$y(t) = K\left[1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right)\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = Ke^{-\zeta\omega_n}\frac{\omega_n}{\sqrt{1-\zeta^2}}\sin(\omega_d t)$$

Find t_P such that $\dot{y}(t_P) = 0$

$$\omega_d t_P = \pi$$

$\Rightarrow$

$$t_P = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$T_d = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}$$

- **Percent Overshoot (%OS)**

At peak time t_P the maximum output

$$y_{MAX} = y(t_P) = K\left(1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}}\right)$$

The overshoot (OS) is:

$$OS = y_{MAX} - y_{SS} = Ke^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

The percent overshoot is:

$$\%OS = \left(\frac{OS}{y_{SS} - y(0)}\right)100\%$$

$$= e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%$$

Unit Step Response of 2nd Order System

- **Settling Time (t_s)**

  Time required for the response to be within a specific percent of the final (steady-state) value.

  Some typical specifications for settling time are: 5%, 2% and 1%.

  Look at the envelope of the response:

  ***x*% band settling time:**

$$t_s = -\tau \ln\left(x/100 \right) \approx$$
$$\tau = 1/\sigma$$

%	1%	2%	5%
t_s	4.6τ	3.9τ	3τ

Q: Which parameters of a 2nd order system affect the peak time?

Damping ratio and natural frequency

Q: Which parameters of a 2nd order system affect the % OS?

Damping ratio

Q: Which parameters of a 2nd order system affect the settling time?

Damping ratio and natural frequency

Q: Can you obtain the formula for a 3% settling time?

Transient and Steady State Responses

Transient and Steady State Response

Ex: Let's find the *total* response of a **stable** first order system:

$$\dot{y} + 5y = 10u$$

to a *ramp* input: $u(t) = 5t$ with I.C.: $y(0) = 2$

- total response

$$Y(s) = \underbrace{\underbrace{\frac{10}{s+5}}_{\text{Transfer Function}} \bullet \underbrace{\frac{5}{s^2}}_{U(s)}}_{\text{Forced Response}} + \underbrace{\frac{1}{s+5} \bullet \underbrace{2}_{y(0)}}_{\text{Free Response}}$$

- PFE

$$Y(s) = \boxed{\frac{a_1}{s} + \frac{a_2}{s^2}} + \boxed{\frac{a_3}{s+5} + \frac{2}{s+5}}$$

$$a_1 = \frac{1}{(2-1)!} \frac{d^{2-1}}{ds^{2-1}} \left\{ s^2 Y_{\text{forced}}(s) \right\} \Big|_{s=0} = \frac{d}{ds} \left\{ \frac{50}{s+5} \right\} \Big|_{s=0} = -\frac{50}{(s+5)^2} \Big|_{s=0} = -2$$

$$a_2 = \left[s^2 Y_{\text{forced}}(s) \right] \Big|_{s=0} = 10 \qquad a_3 = \left[(s+5) Y_{\text{forced}}(s) \right] \Big|_{s=-5} = 2$$

$$y(t) = \underbrace{-2 + 10t}_{\substack{\text{Steady state response} \\ \text{from Forced response}}} + \left(\underbrace{a_3 e^{-5t}}_{\substack{\text{Transient response} \\ \text{from Forced response}}} + \underbrace{2e^{-5t}}_{\substack{\text{Transient response} \\ \text{from free response}}} \right)$$

Transient and Steady State Response

In general, the *total response* of a **STABLE** LTI system

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \cdots + b_1 \dot{u} + b_0 u$$

to an input *u(t)* can be decomposed into two parts

$$y(t) = \underbrace{y_T(t)}_{\text{Transient Response}} + \underbrace{y_{SS}(t)}_{\text{Steady State Response}}$$

where

- Transient Response
  - contains the free response of the system plus a portion of forced response
  - will decay to zero at a rate that is determined by the characteristic roots (poles) of the system
- Steady State Response
  - will take the same (similar) form as the forcing input
  - Specifically, for a sinusoidal input, the steady response will be a sinusoidal signal with the **same frequency** as the input but with **different magnitude and phase**.

Transient and Steady State Response

Ex: Let's find the total response of a stable second order system:

$$\ddot{y}+4\dot{y}+3y=6u$$

to a sinusoidal input: $u(t)=5\sin(3t)$ with I.C.: $\dot{y}(0)=0, y(0)=2$

- Total response:

$$Y(s)=\underbrace{\frac{6}{s^2+4s+3}\bullet\frac{5\times 3}{s^2+3^2}}_{\text{Forced Response}}+\underbrace{\frac{2s+4\times 2}{s^2+4s+3}}_{\text{Free Response}}$$

- PFE

$$Y(s)=\left[\frac{a_1}{s+3}+\frac{a_2}{s+1}+\frac{a_3}{s+3j}+\frac{a_4}{s-3j}\right]+\left[\frac{b_1}{s+3}+\frac{b_2}{s+1}\right]$$

$$a_1=-\frac{5}{2}\quad a_2=\frac{9}{2}\qquad a_3=-1-\frac{1}{2}j\qquad a_4=-1+\frac{1}{2}j\qquad b_1=-1\qquad b_2=3$$

$$y(t)=\underbrace{2\,\mathrm{Re}\left\{a_3e^{-3jt}\right\}}_{\text{Steady state response}}+\underbrace{\left[(a_1+b_1)e^{-3t}+(a_2+b_2)e^{-t}\right]}_{\text{Transient response}}$$

$$=\sqrt{5}\sin\left(3t+\tan^{-1}(2)\right)+\left[-\frac{7}{2}e^{-3t}+\frac{15}{2}e^{-t}\right]$$

Steady State Response

- Final Value Theorem (FVT)

Given a signal's LT $F(s)$, if **all of the poles of $sF(s)$ lie in the LHP**, then $f(t)$ converges to a constant value as given in the following form

$$f(\infty)=\lim_{t\to\infty}f(t)=\lim_{s\to 0}sF(s)$$

Ex. A linear system is described by the following equation:

$$\ddot{y}+4\dot{y}+12y=4\dot{u}+3u$$

(1). If a constant input u=5 is applied to the system at time t=0, determine whether the output $y(t)$ will converge to a constant value?

(2). If the output converges, what will be its steady state value?

We did not consider the effects of IC since
- *it is a stable system*
- *we are only interested in steady state response*

$$Y(s)=\frac{4s+3}{s^2+4s+12}\bullet\frac{5}{s}\qquad y(\infty)=\lim_{t\to\infty}y(t)=\lim_{s\to 0}sY(s)=\frac{5}{4}$$

Steady State Response

Given a general n-th order **stable** system

$$y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1\dot{y}+a_0y=b_mu^{(m)}+b_{m-1}u^{(m-1)}+\cdots+b_1\dot{u}+b_0u$$

Transfer Function

$$G(s)=\frac{b_ms^m+b_{m-1}s^{m-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$$

Free Response

$$Y_{Free}(s)=\frac{F(s)}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$$

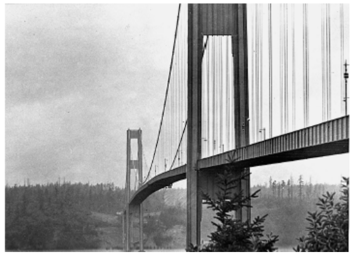
Steady State Value of Free Response (FVT)

$$y_{SS}=\lim_{s\to 0}sY_{Free}(s)=\lim_{s\to 0}\frac{sF(s)}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$$

$$=\lim_{s\to 0}\frac{0\times F(0)}{a_0}=0$$

In SS value of a stable LTI system, there is NO contribution from ICs.

System Stability



Stability

- **Stability Concept**

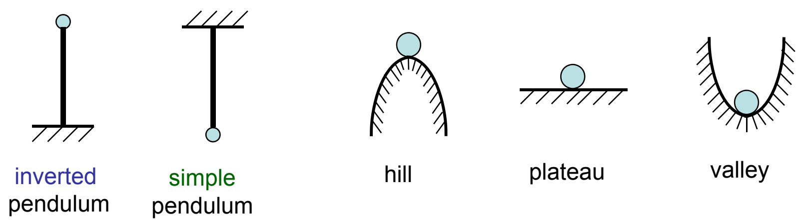
  Describes the ability of a system to stay at its equilibrium position in the absence of any inputs.

  where the derivatives of all states are zeros

  – **A linear time invariant (LTI) system is stable *if and only if* (iff) its free response converges to zero for all ICs.**

Ex: Pendulum

Ball on curved surface

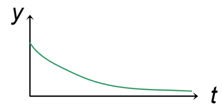


Examples (stable and unstable 1st order systems)

Q: free response of a 1st order system.

$$5\dot{y}+y=u(t) \qquad y(0)=y_0$$

$$\boxed{y(t)=y_0e^{-\frac{1}{5}t}}$$



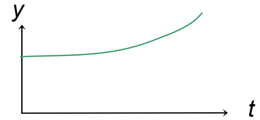
TF: $G=\frac{1}{5s+1}$

Pole: $p=-0.2$

Q: free response of a 1st order system.

$$-5\dot{y}+y=u(t) \qquad y(0)=y_0$$

$$\boxed{y(t)=y_0e^{\frac{1}{5}t}}$$



TF: $G=\frac{1}{-5s+1}$

Pole: $p=0.2$

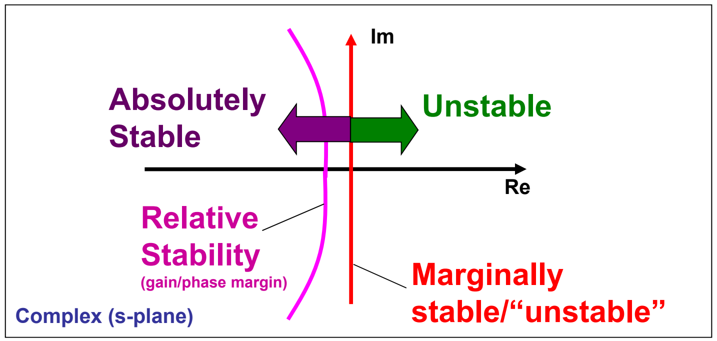
Stability of LTI Systems

- **Stability Criterion for LTI Systems**

$$y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1\dot{y}+a_0y=b_mu^{(m)}+b_{m-1}u^{(m-1)}+\cdots+b_1\dot{u}+b_0u$$

Stable $\Longleftrightarrow$ All poles lie in the left-half complex plane (LHP)

$\Longleftrightarrow$ All roots of $D(s)=\underbrace{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}_{\text{Characteristic Polynomial}}=0$ lie in the LHP



Stability of LTI Systems

- **Comments on LTI Stability**
  - Stability of an LTI system does not depend on the input (*why*?)
  - For 1st and 2nd order systems, stability is guaranteed if all the coefficients of the characteristic polynomial are positive (of same sign).

$$D(s)=s+a_0 \quad : \quad \text{Stable} \Leftrightarrow a_0>0$$
$$D(s)=s^2+a_1s+a_0 \quad : \quad \text{Stable} \Leftrightarrow a_1>0 \text{ and } a_2>0$$

  - Effect of Poles and Zeros on Stability
    - Stability of a system depends on its poles only.
    - Zeros do not affect system stability.
    - Zeros affect the specific dynamic response of the system.

System Stability (some empirical guidelines)

- **Passive systems are usually stable**
  - Any initial energy in the system is usually dissipated in real-world systems (poles in LHP);
  - If there is no dissipation mechanisms, then there will be poles on the imaginary axis
  - If any coefficients of the denominator polynomial of the TF are zero, there will be poles with zero RP
- **Active systems can be unstable**
  - Any initial energy in the system can be amplified by internal source of energy (feedback)
  - If all the coefficients of the denominator polynomial are NOT the **same sign**, system is unstable
  - Even if all the coefficients of the denominator polynomial are the same sign, instability can occur (Routh's stability criterion for continuous-time system)

In Class Exercises

(1) Obtain TF of the following system:

$$\ddot{y}+2\dot{y}+5y=\dot{u}-u$$

(2) Plot the poles and zeros of the system on the complex plane.

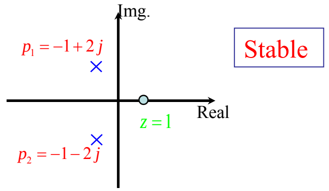
(3) Determine the system's stability.

$$\mathcal{L}[\circ]\quad s^2Y(s)+2sY(s)+5Y(s)=sU(s)-U(s)$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{s-1}{s^2+2s+5}$$

Poles: $s^2+2s+5=0 \Rightarrow p_{1,2}=\frac{-2\pm\sqrt{4-20}}{2}=-1\pm 2j$

Zero: $s-1=0 \quad \Rightarrow z=1$



Stable

(1) Obtain TF of the following system:

$$\dddot{y}+\ddot{y}+6\dot{y}=\ddot{u}-3\dot{u}+4u$$

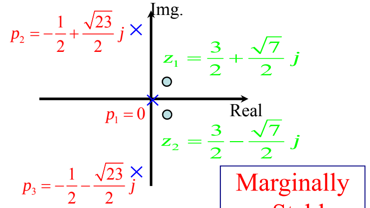
(2) Plot the poles and zeros of the system on the complex plane.

(3) Determine the system's stability.

TF: $G(s)=\frac{Y(s)}{U(s)}=\frac{s^2-3s+4}{s^3+s^2+6s}$

Poles: $s^3+s^2+6s=0 \Rightarrow p_1=0, p_{2,3}=\frac{-1\pm\sqrt{23}j}{2}$

Zeros: $s^2-3s+4=0 \Rightarrow z_{1,2}=\frac{3\pm\sqrt{7}j}{2}$

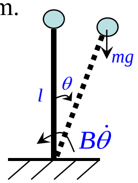


Marginally Stable

Example

Inverted Pendulum

(1) Derive a mathematical model for a pendulum.



(2) Find the equilibrium positions.

(3) Discuss the stability of the equilibrium positions.

EOM: $I_o\ddot{\theta}=mgl\sin\theta-B\dot{\theta}$

Equilibrium position: $\left.\begin{matrix}\ddot{\theta}=0\\ \dot{\theta}=0\end{matrix}\right\}\Rightarrow\theta=0$

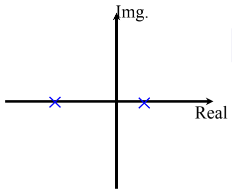
Assumption: θ is very small

Linearized EOM: $I_o\ddot{\theta}=mgl\theta-B\dot{\theta}$

$$I_o\ddot{\theta}+B\dot{\theta}-\underbrace{mgl}_{K}\theta=0$$

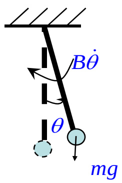
Characteristic equation: $s^2I_o+sB-K=0$

Poles: $p_{1,2}=\frac{-B\pm\sqrt{B^2+4KI_o}}{2I_o}$



Unstable

Example (Simple Pendulum)



EOM: $I_o\ddot{\theta}=-mgl\sin\theta-B\dot{\theta}$

Equilibrium position: $\left.\begin{matrix}\ddot{\theta}=0\\ \dot{\theta}=0\end{matrix}\right\}\Rightarrow\theta=0$

Assumption: θ is very small

Linearized EOM: $I_o\ddot{\theta}=-mgl\theta-B\dot{\theta}$

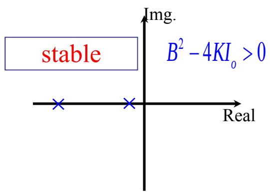
$$I_o\ddot{\theta}+B\dot{\theta}+\underbrace{mgl}_{K}\theta=0$$

How do the positions of poles change when K increases? (root locus)

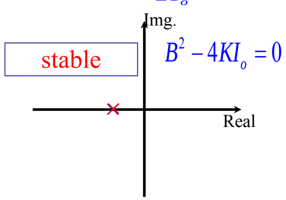
Characteristic equation: $s^2I_o+sB+K=0$

Poles: $p_{1,2}=\frac{-B\pm\sqrt{B^2-4KI_o}}{2I_o}$

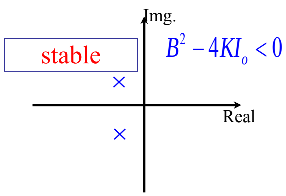
stable $B^2-4KI_o>0$



stable $B^2-4KI_o=0$



stable $B^2-4KI_o<0$



Stability

- System should be
  - Stable: BIBO stable if, for every bounded input, the output is bounded for all time
  - LTI system must have all poles in the left-half of the *s*-plane (negative real parts)
  - Transfer function poles are the same as roots of the characteristic polynomial and are the same as system eigenvalues
  - All eigenvalues must have negative real parts for BIBO stability (Poles on imaginary axis are not stable by this definition.)

$$T(s) = \frac{p(s)}{q(s)}$$

$$= \text{ratio of polynomials}$$

$T(s)$ has a characteristic equation:

$$\Delta(s) = q(s) = 0$$

Necessary condition

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

- If any coefficient a_i of $q(s)$ is zero or negative then not all roots lie in the left half of the s-plane
- Otherwise: set up *Routh Array* and use *Routh-Hurwitz Criterion*:

Routh Array

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$
\begin{array}{cccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

$$b_1 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_2 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_1 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}$$

$$c_2 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}$$

Routh-Hurwitz **Criterion**

Number of *sign changes* in the *first* column
= number of ***unstable*** poles.

Example:

$$q(s) = s^3 + s^2 + 2s + 8$$

$$
\begin{array}{ccc}
s^3 & 1 & 2 \\
s^2 & 1 & 8 \\
s^1 & -6 & \\
s^0 & 8 &
\end{array}
$$

– Number of sign changes in the first column = number of unstable poles.

$$q(s) = s^3 + s^2 + 2s + 8$$

$$
\begin{array}{ccc}
s^3 & 1 & 2 \\
s^2 & 1 & 8 \\
s^1 & -6 & \\
s^0 & 8 &
\end{array}
$$

Two sign Changes = two unstable poles

Routh Array

- Case 1: no row has a zero in the first column
- Case 2: a row has a zero in the first column, but at least one nonzero element elsewhere
- Case 3: a row has all zeros

Case 1: Example

$$q(s) = s^3 + 2s^2 + 3s + 4$$

$$\begin{array}{lll} s^3 & 1 & 3 \\ s^2 & 2 & 4 \\ s^1 & 1 & \\ s^0 & 4 & \end{array}$$

No sign changes, stable

Case 2: Example

$$q(s) = s^4 + 2s^3 + 2s^2 + 4s + 11$$

$$\begin{array}{llll} s^4 & 1 & 2 & 11 \\ s^3 & 2 & 4 & \\ s^2 & 0 & 11 & \\ & (\varepsilon > 0) \nearrow & & \\ s^1 & \dfrac{-22}{\varepsilon} & & \\ s^0 & 11 & & \end{array}$$

Two sign changes, unstable

Case 3: Example

$$q(s) = s^3 + 2s^2 + 3s + 6$$

$$\begin{array}{llll} s^3 & 1 & 3 & \\ s^2 & 2 & 6 & \Rightarrow (2s^2 + 6) \\ s^1 & 0 & 0 & \nearrow \\ s^0 & & & \end{array}$$

$$\begin{array}{llll} s^3 & 1 & 3 & \\ s^2 & 2 & 6 & \Rightarrow (2s^2 + 6) = f(s) \\ s^1 & 4 & & \leftarrow \left(= \dfrac{df(s)}{ds} = \dfrac{d(2s^2+6)}{ds}\right) \\ s^0 & 6 & & \end{array}$$

no sign changes, but:

$$2s^2 + 6 = 0 \quad s = \pm j\sqrt{3}$$

not stable

Comments

- Case 2 always works out to be unstable
- Case 3 occurs when there is an even polynomial factor such as s^2+a, s^4+b, or s^4+as^2+b
  - not stable by the necessary condition
  - Repeated roots on the imaginary axis *unstable*

$$q(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1$$

$$s^2 + 4 = 0 \quad s = \pm j2$$

$$s^4 + 16 = 0 \quad s^2 = \pm j4$$

$$s = \pm 1 \pm j1$$

$$s^4 + 7s^2 + 12 = 0$$

$$s^2 = -3, -4$$

$$s = \pm j\sqrt{3}, \pm j2$$

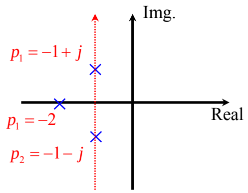
Relative Stability

- Degree of stability (*how much*)
- Characterized by the relative real part of each root/pair of roots

<u>Example 6.6</u>

$$q(s) = s^3 + 4s^2 + 6s + 4$$

s^3	1	6
s^2	4	4
s^1	5	
s^0	4	



$$p_1 = -2, \quad p_{2,3} = -1 \pm j$$

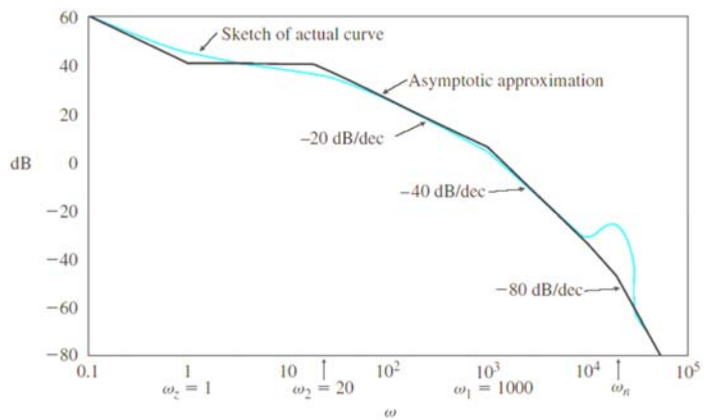
Shift imaginary axis to the left

with $s_n = s + 1$, $q(s_n) = s_n^3 + s_n^2 + s_n + 1$

s^3	1	1
s^2	1	1
s^1	0	0
s^0	1	0

- Response speed and overshoot

Frequency Response



Frequency Response

- **Forced Responses to Sinusoidal Inputs**
  - Transient and Steady-State Response
- **Frequency Response**
  - *Steady-State Response* to *sinusoidal inputs* at various input frequencies
- **Bode Plots**
  - A convenient *graphic display of frequency response* at *all input frequencies*

Forced Response to Sinusoidal Inputs

Ex: Let's find the forced response of a stable first order system:

$$\dot{y} + 5y = 10u$$

to a sinusoidal input: $u(t) = \sin(2t)$

- *Forced response in s-domain:*

$$Y(s) = G(s)\cdot U(s) = \frac{10}{s+5}\cdot\frac{2}{s^2+2^2} = \frac{20}{(s+5)(s-j2)(s+j2)}$$

- PFE:

$$Y(s) = \frac{A_1}{s+5} + \frac{B_1}{s-j2} + \frac{\overline{B}_1}{s+j2}$$

$$A_1 = (s+5)Y(s)\big|_{s=-5} = \frac{20}{29}$$

$$B_1 = (s-j2)Y(s)\big|_{s=j2} = -\frac{10}{29} - j\frac{25}{29}$$

- *Use ILT to obtain forced response in time-domain:*

$$y(t) = A_1e^{-5t} + B_1e^{j2t} + \overline{B}_1e^{-j2t}$$
$$= A_1e^{-5t} + 2\,\mathrm{Re}\left\{B_1e^{j2t}\right\}$$
$$= \frac{20}{29}e^{-5t} + 2\left[\underbrace{-\frac{10}{29}}_{B}\cos(2t) + \underbrace{\frac{25}{29}}_{A}\sin(2t)\right]$$
$$= \underbrace{\frac{20}{29}e^{-5t}}_{\text{transient response}} + \underbrace{\frac{10}{\sqrt{29}}\sin(2t - 21.8^o)}_{\text{steady-state response}}$$

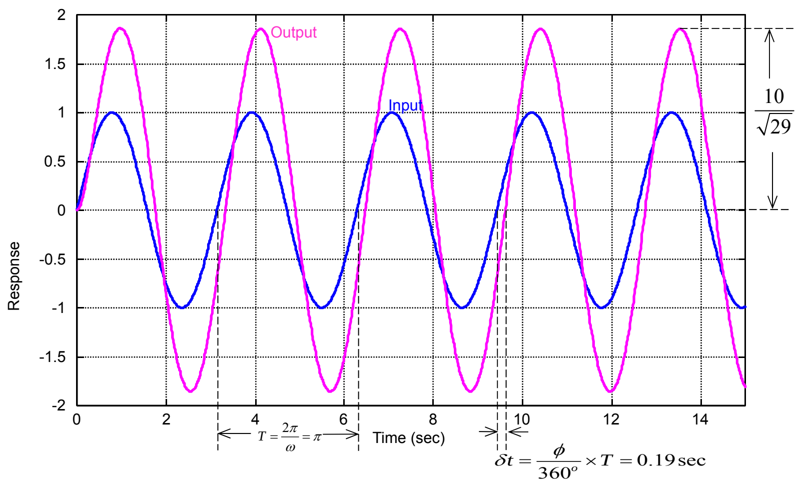
$$B_1e^{j2t} = \left(-\frac{10}{29} - j\frac{25}{29}\right)\left[\cos(2t) + j\sin(2t)\right]$$
$$= \underbrace{-\frac{10}{29}\cos(2t) + \frac{25}{29}\sin(2t)}_{\mathrm{Re}\{B_1e^{j2t}\}}$$
$$+ j\left[-\frac{25}{29}\cos(2t) - \frac{10}{29}\sin(2t)\right]$$

Useful Formula: $A\sin(\omega t) + B\cos(\omega t) = \sqrt{A^2+B^2}\,\sin(\omega t + \phi)$

Where $\phi = \mathrm{atan2}(B, A) = \angle(A + jB)$

Forced Response to Sinusoidal Inputs

Input is sin(2*t*)



Forced Response to Sinusoidal Inputs

In-class Ex: Given the same system as in the previous example, find the forced response to $u(t) = \sin(10\,t)$.

$$Y(s) = G(s)\cdot U(s),\quad G(s) = \frac{10}{s+5}\quad \text{and}\quad U(s) = \mathcal{L}\left[\sin(10t)\right] = \frac{10}{s^2+10^2}$$

$$Y(s) = G(s)\cdot U(s) = \frac{10}{s+5}\cdot\frac{10}{s^2+10^2} = \frac{100}{(s+5)(s-j10)(s+j10)}$$

$$Y(s) = \frac{A_1}{s+5} + \frac{B_1}{s-j10} + \frac{\overline{B}_1}{s+j10}$$

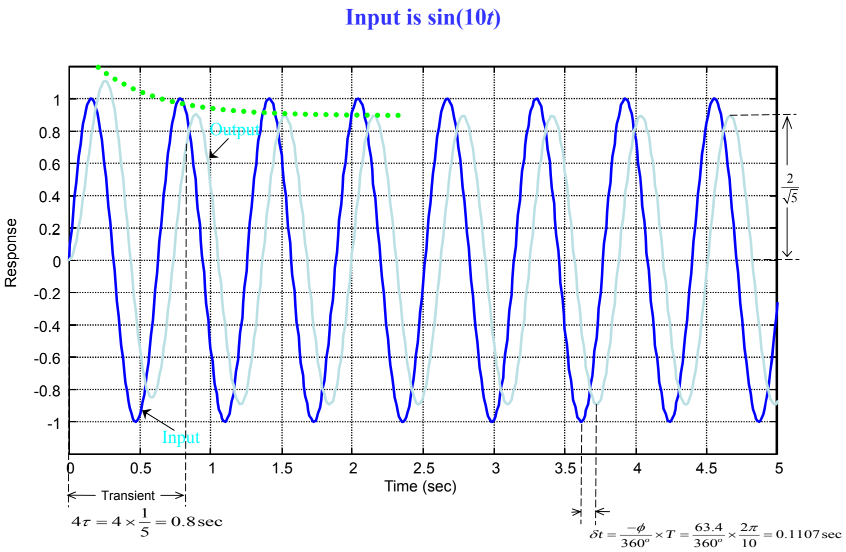
$$A_1 = (s+5)Y(s)\big|_{s=-5} = 0.8$$

$$B_1 = (s-j10)Y(s)\big|_{s=j10} = -0.4 - 0.2j$$

$$y(t) = A_1e^{-5t} + B_1e^{j10t} + \overline{B}_1e^{-j10t}$$
$$= A_1e^{-5t} + 2\,\mathrm{Re}\left\{B_1e^{j10t}\right\}$$
$$= \frac{4}{5}e^{-5t} + 2\left[\underbrace{-\frac{2}{5}}_{B}\cos(10t) + \underbrace{\frac{1}{5}}_{A}\sin(10t)\right]$$
$$= \underbrace{\frac{4}{5}e^{-5t}}_{\text{transient response}} + \underbrace{\frac{2}{\sqrt{5}}\sin(10t + \phi)}_{\text{steady-state response}}$$

$$\phi = \mathrm{atan2}\left(-\frac{2}{5}, \frac{1}{5}\right) = -1.107\,\mathrm{rad} = -63.435^o$$

Forced Response to Sinusoidal Inputs



Forced Response to Sinusoidal Inputs

Ex: Let's revisit the same example where

$$\dot{y} + 5y = 10u$$

and the input is a general sinusoidal input: sin(ωt).

$$Y(s) = G(s)\cdot U(s) = \frac{10}{s+5}\cdot\frac{\omega}{s^2+\omega^2} = \frac{10}{s+5}\cdot\frac{\omega}{(s-j\omega)(s+j\omega)}$$

$$Y(s) = \frac{A_1}{s+5} + \frac{A_2}{s-j\omega} + \frac{A_3}{s+j\omega}$$

– Use the *residue formula* to find A_i's:

$$A_1 = (s+5)Y(s)\Big|_{s=-5} = (s+5)\frac{10}{(s+5)}\frac{\omega}{s^2+\omega^2}\Big|_{s=-5} = \frac{10\omega}{\omega^2+25}$$

$$A_2 = (s-j\omega)Y(s)\Big|_{s=j\omega} = (s-j\omega)G(s)\frac{\omega}{s^2+\omega^2}\Big|_{s=j\omega} = \boxed{\frac{1}{2j}G(j\omega)} = \frac{10}{-2\omega+10j}$$

$$A_3 = \bar{A}_2 = \boxed{-\frac{1}{2j}G(-j\omega)} = \frac{10}{-2\omega-10j}$$

Forced Response to Sinusoidal Inputs

Ex: Forced response in time-domain:

$$Y(s) = G(s)\frac{\omega}{s^2+\omega^2} = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}\frac{\omega}{s^2+\omega^2}$$

$$= B_1\frac{1}{s+j\omega} + B_2\frac{1}{s-j\omega} + \sum_{i=1}^{n} A_i\frac{1}{s-p_i}$$

$$\boxed{y(t) = \underbrace{B_1 e^{j\omega t} + \bar{B}_1 e^{-j\omega t}}_{\text{steady state}} + \underbrace{\sum_{i=1}^{n} A_i e^{p_i t}}_{\text{transient}}}$$

$$\boxed{B_1 = (s-j\omega)G(s)\frac{\omega}{s^2+\omega^2}\Big|_{s=j\omega} = \frac{1}{2j}G(j\omega)}$$

The steady state sinusoidal response in time-domain:

$$y_{SS}(t) = B_1 e^{j\omega t} + \bar{B}_1 e^{-j\omega t} = 2\,\mathrm{Re}\{B_1 e^{j\omega t}\}$$

$$= 2\,\mathrm{Re}\left\{\frac{1}{2j}G(j\omega)e^{j\omega t}\right\} = \mathrm{Re}\left\{-j\underbrace{|G(j\omega)|e^{j\phi}}_{G(j\omega)}e^{j\omega t}\right\}$$

$$= \mathrm{Re}\left\{-j|G(j\omega)|e^{j(\omega t+\phi)}\right\}$$

$$= \mathrm{Re}\left\{|G(j\omega)|\sin(\omega t+\phi) - j|G(j\omega)|\cos(\omega t+\phi)\right\}$$

$$= |G(j\omega)|\sin(\omega t+\phi)$$

where $\phi = \angle G(j\omega)$

Sinusoidal input → Stable LTI System → Sinusoidal output

Phase shift
Changed Magnitude

What happens to frequency? No Change!

Forced Response to Sinusoidal Inputs

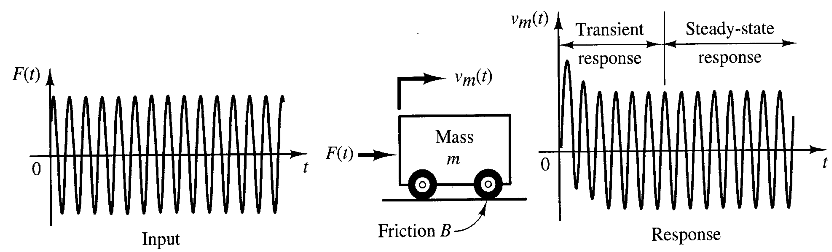


Figure 14.1: Response of a linear system to sinusoidal inputs.

Frequency Response

Frequency response is used to study *the steady state output $y_{SS}(t)$* of a ***stable LTI*** system to *sinusoidal inputs at different frequencies*.

In general, given a stable system:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0 u$$

$$G(s) \equiv \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)} = \frac{b_m(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

If the input is a sinusoidal signal with frequency ω, i.e.

$$u(t) = A_u \sin(\omega \cdot t)$$

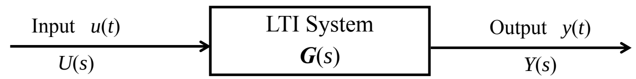
then the steady state output $y_{SS}(t)$ is also a sinusoidal signal with the same frequency as the input signal but with different magnitude and phase:

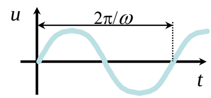
$$y_{SS}(t) = |G(j\omega)| \cdot A_u \sin(\omega \cdot t + \angle G(j\omega))$$

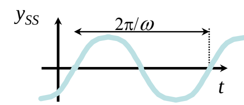
where $G(j\omega)$ is the complex number obtained by substitute $j\omega$ for s in $G(s)$, i.e.

$$G(j\omega) = G(s)\big|_{s=j\omega} \equiv \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \cdots + b_1(j\omega) + b_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1(j\omega) + a_0}$$

Frequency Response







$$u(t) = A_u \sin(\omega \cdot t) \quad \Rightarrow \quad y_{SS}(t) = |G(j\omega)| \cdot A_u \sin(\omega t + \angle G(j\omega))$$

$$u(t) = A_u \cos(\omega \cdot t) \quad \Rightarrow \quad y_{SS}(t) = |G(j\omega)| \cdot A_u \cos(\omega t + \angle G(j\omega))$$

$$u(t) = A_u \sin(\omega \cdot t + \phi) \quad \Rightarrow \quad y_{SS}(t) = |G(j\omega)| \cdot A_u \sin(\omega t + \phi + \angle G(j\omega))$$

– A different perspective of the role of the transfer function:

$$|G(j\omega)| = \frac{\text{Amplitude of the steady state sinusoidal output}}{\text{Amplitude of the sinusoidal input}}$$

$$\angle G(j\omega) = \text{ Phase difference (shift) between } y_{SS}(t) \text{ and the sinusoidal input}$$

Frequency Response

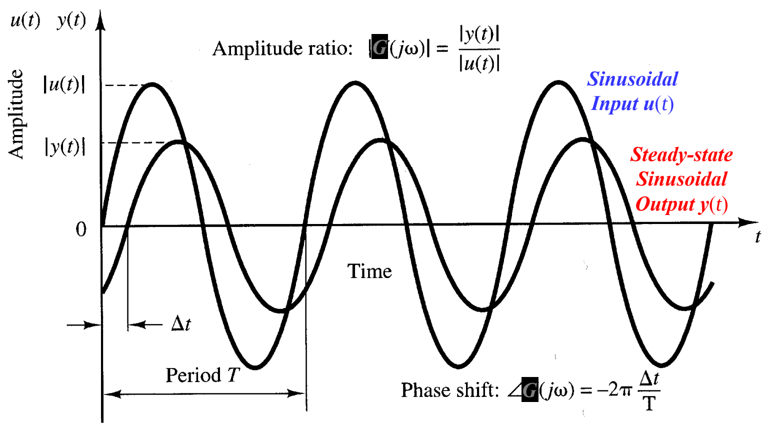


Figure 14.3: Steady-state response of a linear system with a sinusoidal forcing function.

Concepts

$$Y(s) = T(s)R(s)$$

$$R(s) = \frac{A\omega}{s^2 + \omega^2} \qquad T(s) = \frac{m(s)}{q(s)} = \frac{m(s)}{\prod_{i=1}^{n}(s + p_i)}$$

System response:

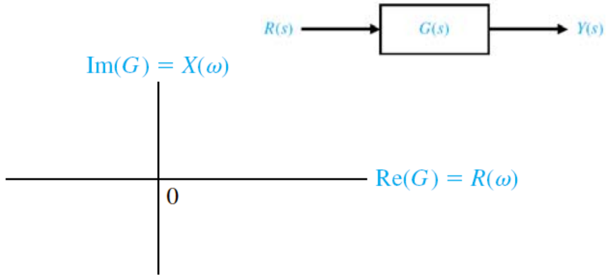
$$Y(s) = \frac{k_1}{s + p_1} + \cdots + \frac{k_n}{s + p_n} + \frac{\alpha s + \beta}{s^2 + \omega^2}$$

$$y(t) = k_1 e^{-p_1 t} + \cdots + k_n e^{-p_n t} + \mathscr{L}^{-1}\left\{\frac{\alpha s + \beta}{s^2 + \omega^2}\right\}$$

$$t \to \infty$$

$$\Rightarrow \quad y(t) = A\,|T(j\omega)|\,\sin(\omega t + \measuredangle T(j\omega))$$

Polar Plots



$$G(j\omega) = G(s)|_{s=j\omega} = R(\omega) + jX(\omega)$$

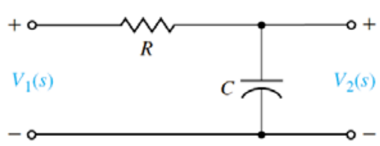
$$R(\omega) = \mathrm{Re}\left[G(j\omega)\right],\ X(\omega) = \mathrm{Im}\left[G(j\omega)\right]$$

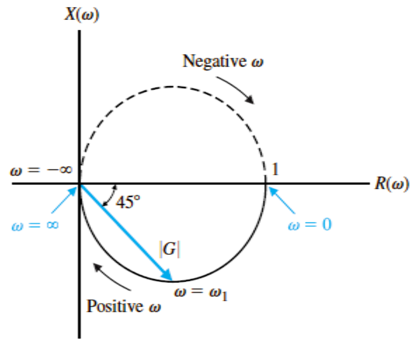
$$G(j\omega) = |G(j\omega)|\, e^{j\phi(j\omega)} = |G(\omega)| \measuredangle \phi(\omega)$$

$$\phi(\omega) = \tan^{-1}\frac{X(\omega)}{R(\omega)},\ |G(j\omega)|^2 = \left[R(\omega)\right]^2 + \left[X(\omega)\right]^2$$

Polar Plots

Example 8.1





$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs+1}$$

$$G(j\omega) = \frac{1}{j\omega RC + 1}$$

$$= \frac{1}{(\omega/\omega_1)^2 + 1} - \frac{j(\omega/\omega_1)}{(\omega/\omega_1)^2 + 1}$$

$$\omega_1 = \frac{1}{RC}$$

$$G(j\omega) = |G(\omega)| \measuredangle \phi(\omega)$$

$$|G(\omega)| = \frac{1}{\left[(\omega/\omega_1)^2 + 1\right]^{1/2}}$$

$$\phi(\omega) = -\tan^{-1}\frac{\omega}{\omega_1}$$

Polar Plots

Example 8.2

$$G(j\omega) = \frac{K}{j\omega(j\omega\tau + 1)}$$

$$|G(\omega)| = \frac{K}{(\omega^2 + \omega^4\tau^2)^{1/2}}$$

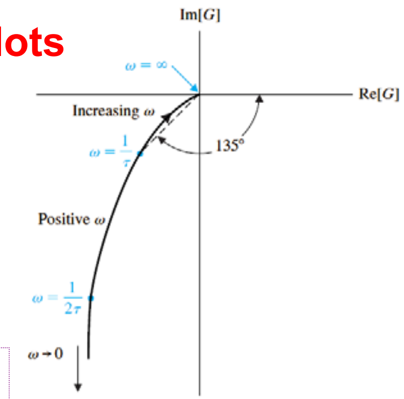
$$\phi(\omega) = -\tan^{-1}\left(\frac{1}{-\omega\tau}\right)$$

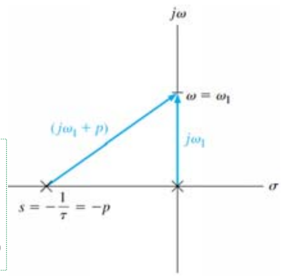
$$G(j\omega) = \frac{K}{j\omega - \omega^2\tau} = \frac{K(-j\omega - \omega^2\tau)}{\omega^2 + \omega^4\tau^2}$$

$$= \frac{-K\omega^2\tau}{\omega^2 + \omega^4\tau^2} + \frac{-j\omega K}{\omega^2 + \omega^4\tau^2}$$

$$|G(\omega_1)| = \frac{K/\tau}{|j\omega_1||j\omega_1 + p|}$$

$$\phi(\omega) = -\measuredangle(j\omega_1) - \measuredangle(j\omega_1 + p) = -90^\circ - \tan^{-1}(\omega_1/p)$$





Bode Plots

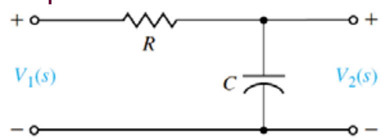
Logarithmic plots (H.W. Bode):

$$G(j\omega) = |G(\omega)|\, e^{j\phi(\omega)}$$

Two plots: log gain *vs* ω and angle *vs* ω:

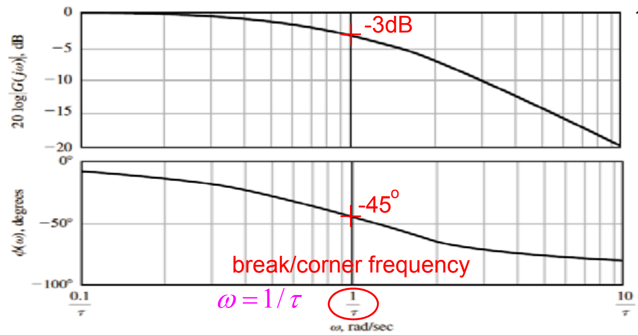
$$\text{log Gain} = 20\log_{10}|G(\omega)|$$

Example 8.3



$$G(j\omega) = \frac{1}{j\omega\tau + 1} \qquad \tau = RC$$

$$20\log_{10}|G(\omega)| = -10\log(1 + (\omega\tau)^2)$$



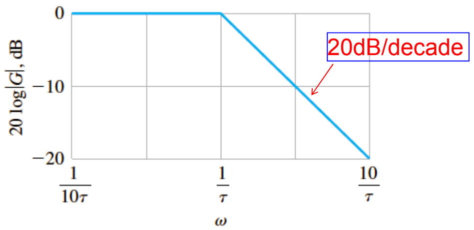
$$20\log_{10}|G(\omega)| = \begin{cases} 0\ \text{dB}, & \omega \ll 1/\tau \\ -3.01\ \text{dB}, & \omega = 1/\tau \\ -20\log(\omega\tau), & \omega \gg 1/\tau \end{cases}$$

$$\phi(\omega) = -\tan^{-1}(\omega\tau)$$

Bode Plots

$$G(j\omega)=\frac{1}{j\omega\tau+1} \qquad \tau=RC \qquad 20\log_{10}|G(\omega)|=-10\log(1+(\omega\tau)^2)$$

Asymptotic curve:



$$20\log|G|=-10\log(1+(\omega\tau)^2)=0 \text{ dB}, \ \omega \ll 1/\tau$$

$$20\log|G|=-3.01 \text{ dB}, \ \omega=1/\tau$$

$$20\log|G|=-20\log(\omega\tau), \ \omega \gg 1/\tau$$

Bode Plots

General TF:

$$G(j\omega)=\frac{K_b\prod_{i=1}^{Q}(1+j\omega\tau_i)}{(j\omega)^N\prod_{m=1}^{M}(1+j\omega\tau_m)\prod_{k=1}^{R}(1+(2\zeta_k/\omega_{n_k})j\omega+(j\omega/\omega_{n_k})^2)}$$

$$20\log|G|=20\log K_b+20\sum_{i=1}^{Q}\log|1+j\omega\tau_i|-20\log|(j\omega)^N|$$

$$-20\log\sum_{m=1}^{M}|(1+j\omega\tau_m)|-20\log\sum_{k=1}^{R}|(1+(2\zeta_k/\omega_{n_k})j\omega+(j\omega/\omega_{n_k})^2)|$$

$$\phi(\omega)=\sum_{i=1}^{Q}\tan^{-1}(\omega\tau_i)-N(90^\circ)-\sum_{M=1}^{M}\tan^{-1}(\omega\tau_m)-\sum_{k=1}^{R}\tan^{-1}\left(\frac{2\zeta_k\omega_{n_k}}{\omega_{n_k}^2}-\omega^2\right)$$

Four types of factors:

1. Constant gain; K_b
2. Poles/zeros at the origin; $(j\omega)$
3. Poles/zeros at the real axis; $(1+j\omega\tau)$
4. Complex conjugate poles/zeros. $(1+(2\zeta/\omega_n)j\omega+(j\omega/\omega_n)^2)$

Bode Plots

Four types of factors:

1. Constant gain: K_b

$20\log K_b=\text{constant in dB}$

$\Rightarrow$ horizontal line in BD

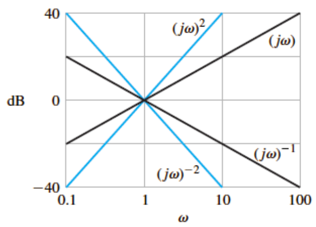
$$\phi(\omega)=\begin{cases}0^\circ, & \text{if } K_b\ge 0\\ -180^\circ, & \text{if } K_b<0\end{cases}$$

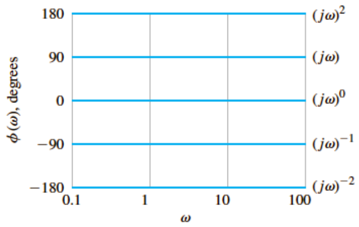
2. Poles/zeros at the origin: $(j\omega)^N$

zero: $20\log|j\omega|=20\log\omega \text{ dB}$ $\qquad \phi(\omega)=90^\circ$

pole: $20\log\left|\frac{1}{j\omega}\right|=-20\log\omega \text{ dB}$ $\qquad \phi(\omega)=-90^\circ$

general: $20\log|(j\omega)^N|=20N\log\omega \text{ dB}$ $\qquad \phi(\omega)=sign(N)N90^\circ$





Bode Plots

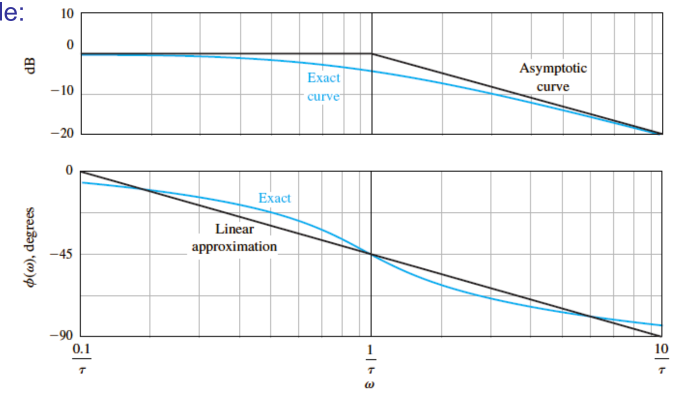
Four types of factors:

3. Poles/zeros at the real axis: $(1+j\omega\tau)^M$

$M=1$: zero: $20\log|1+j\omega|=10\log(1+\omega^2\tau^2) \text{ dB}$ $\qquad \phi(\omega)=\tan^{-1}(\omega\tau)$

$M=-1$: pole: $20\log\left|\frac{1}{1+j\omega}\right|=-10\log(1+\omega^2\tau^2) \text{ dB}$ $\qquad \phi(\omega)=-\tan^{-1}(\omega\tau)$

pole:



Bode Plots

Four types of factors:

4. Complex conjugate Poles/zeros: $(1+(2\zeta/\omega_n)j\omega+(j\omega/\omega_n)^2)^R$

$R=-1$: pole:

$$20\log|G| = 20\log\left|\frac{1}{1+(2\zeta/\omega_n)j\omega+(j\omega/\omega_n)^2}\right|$$

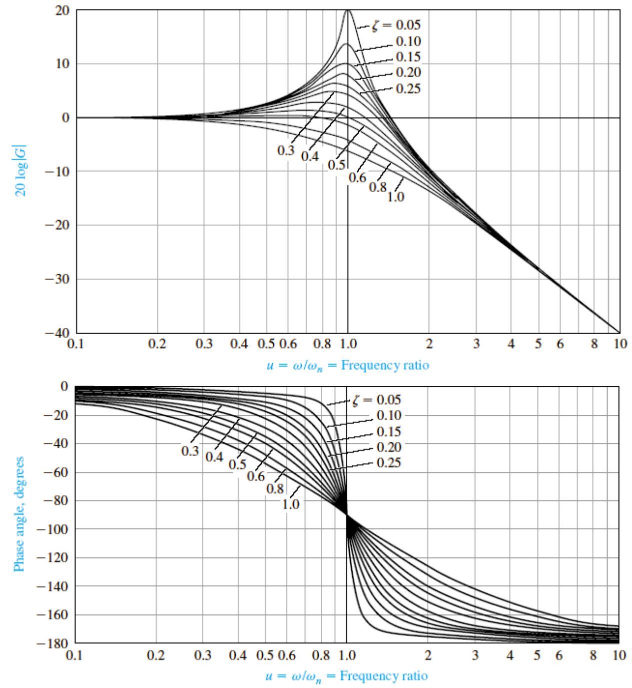
$$= -10\log((1+u^2)^2+4\zeta^2u^2) \qquad (u=\omega/\omega_n\ )$$

$$=\begin{cases} 0, \text{ if } u \ll 1 \\ -40\log u, \text{ if } u \gg 1 \\ f(\zeta), \text{ if } u=1 \end{cases}$$

Asymptotes meet at 0-dB line, but the difference must be counted when ζ <0.707

$$\phi(\omega) = -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right)$$

$$=\begin{cases} 0^\circ, \text{ if } u \ll 1 \\ -90^\circ, \text{ if } u=1 \\ -180^\circ, \text{ if } u \gg 1 \end{cases}$$

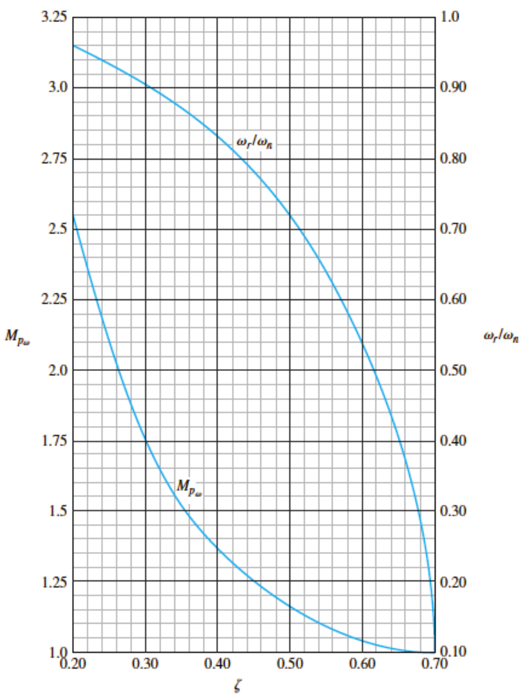


if $\zeta < 0.707$

$\omega_r = \omega_n\sqrt{1-2\zeta^2}$

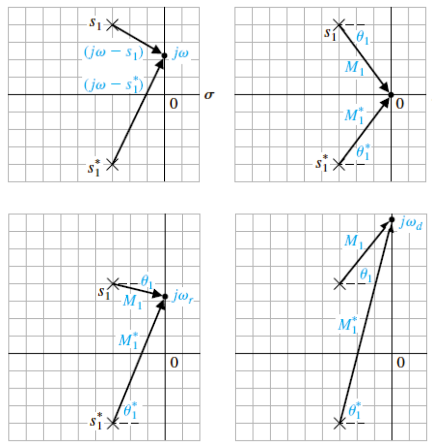
$M_{p_\omega} = |G(\omega_r)|$

$=(2\zeta\sqrt{1-\zeta^2})^{-1}$



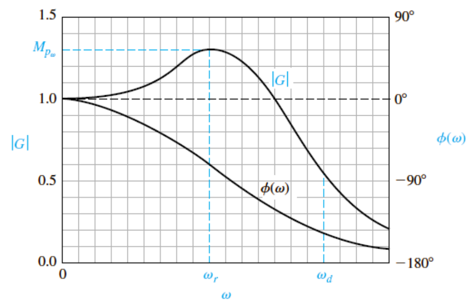
$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$G(j\omega) = \frac{\omega_n^2}{(s-s_1)(s-s_1^*)}\Big|_{s=j\omega} = \frac{\omega_n^2}{(j\omega-s_1)(j\omega-s_1^*)}$$



$$|G(\omega)| = \frac{\omega_n^2}{|j\omega-s_1||j\omega-s_1^*|}$$

$$\phi(\omega) = -\measuredangle(j\omega-s_1)-\measuredangle(j\omega-s_1^*)$$



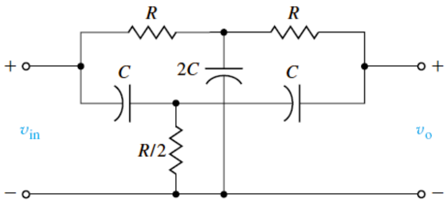
Bode Plots

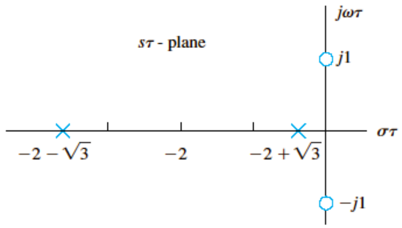
Example 8.4

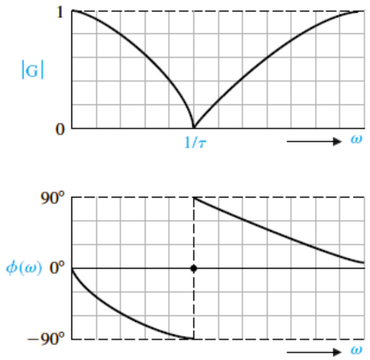
Twin-T network:

$$G(s)=\frac{V_o(s)}{V_{in}(s)}=\frac{(\tau s)^2+1}{(\tau s)^2+4\tau s+1}$$

$$\tau = RC$$







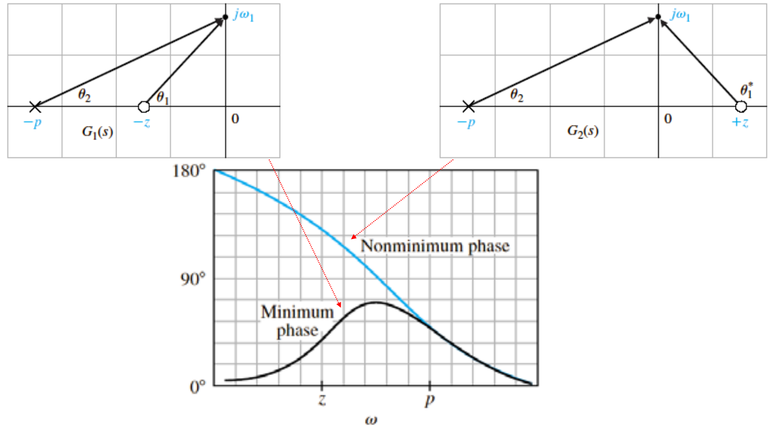
Minimum Phase *vs.* Nonminimum Phase

Minimum phase system:
(all zeros in LHP)

$$G(s)=\frac{s+z}{s+p}$$

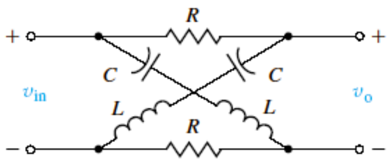
Nonminimum phase system:
(zero(s) in RHP)

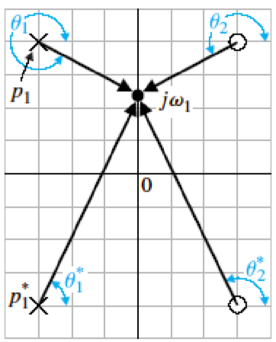
$$G(s)=\frac{s-z}{s+p}$$

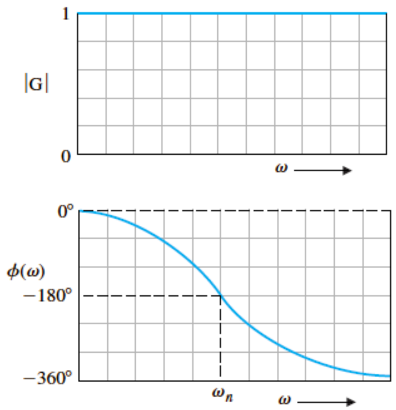


Minimum Phase *vs.* Nonminimum Phase

Example All-pass network:







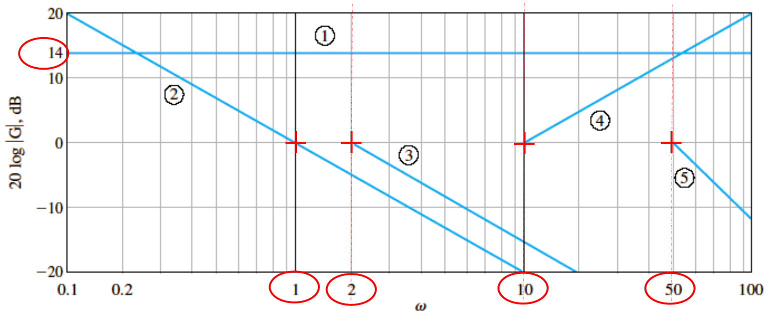
Bode Plots

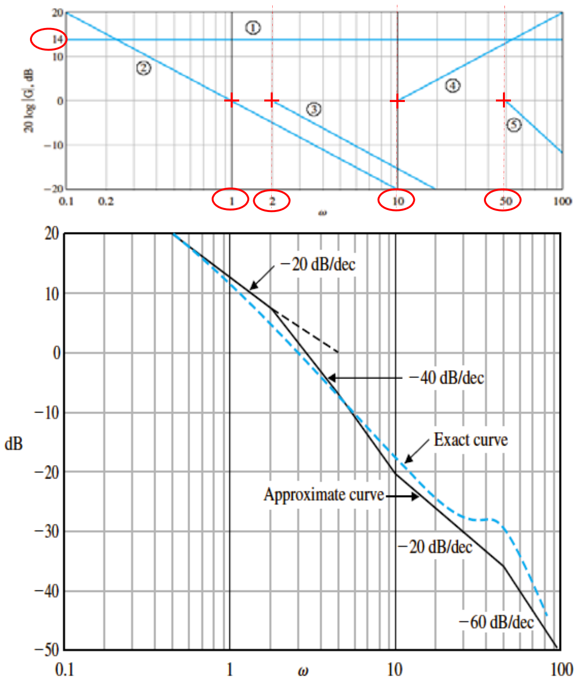
Example:

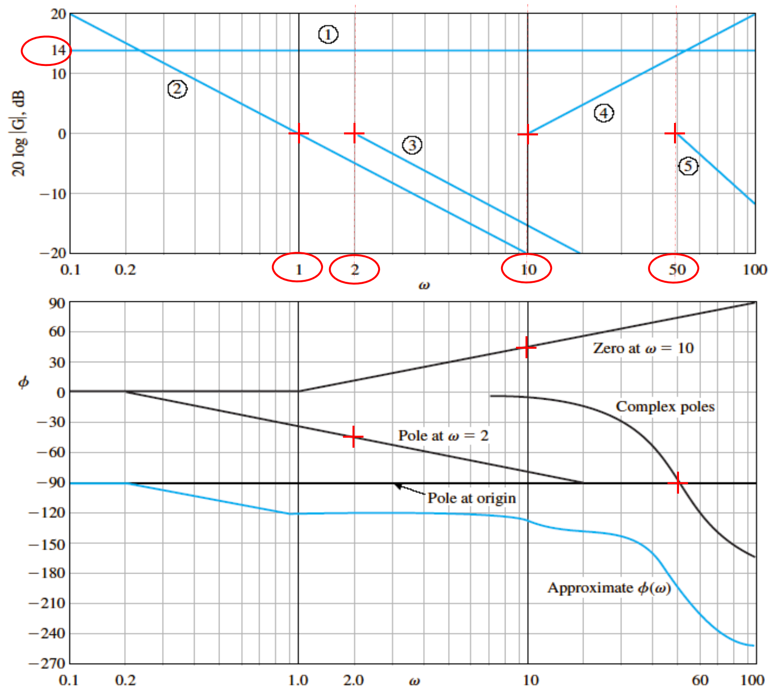
$$G(j\omega)=\frac{5(1+j0.1\omega)}{j\omega(1+j0.5\omega)(1+j0.6(\omega/50)+(j\omega/50)^2)}$$

Four types of factors:

1. Constant gain; K_b
2. Poles/zeros at the origin; $(j\omega)$
3. Poles/zeros at the real axis; $(1+j\omega\tau)$
4. Complex conjugate poles/zeros. $(1+(2\zeta/\omega_n)j\omega+(j\omega/\omega_n)^2)$







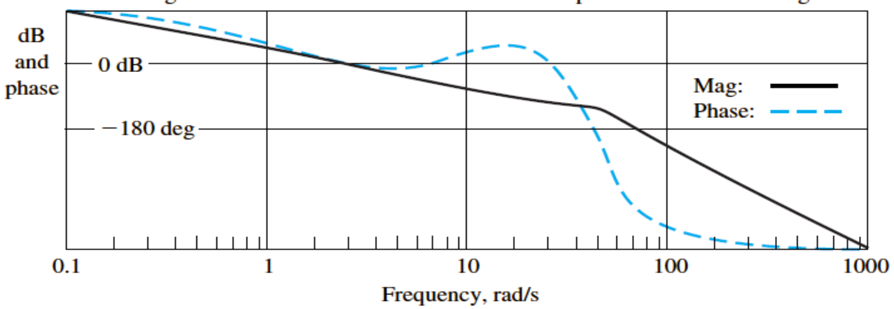
Bode Plots

Example:

$$G(j\omega)=\frac{5(1+j0.1\omega)}{j\omega(1+j0.5\omega)(1+j0.6(\omega/50)+(j\omega/50)^2)}$$

Max. mag = 33.96906 dB
Max. phase = −92.35844 deg
The gain is 2500

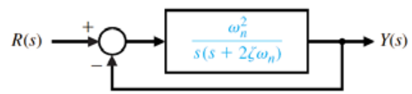
Min. mag = −112.0231 dB
Min. phase = −268.7353 deg



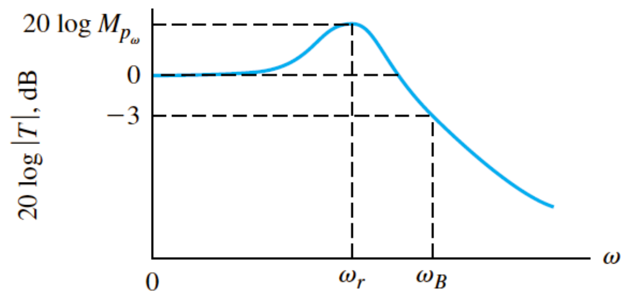
Matlab script:

```
kb=5; num=kb*[0.1 1]; d_1=[1 0]; d_2=[0.5 1]; d_3=[1/50^2 0.6/50 1];
bode(tf(num, conv(conv(d_1,d_2),d_3)))
```

Performance Specification in Frequency Domain



$$T(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$



1. Resonant Frequency: $\omega_r=\omega_n\sqrt{1-2\zeta^2}$ (if $\zeta<0.707$)
2. Maximum Magnitude: $M_{p_\omega}=(2\zeta\sqrt{1-\zeta^2})^{-1}$
3. Bandwidth (-3dB): $\omega_B=\omega_n\sqrt{1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}}$

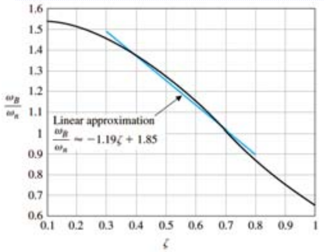
Performance Specification in Frequency Domain

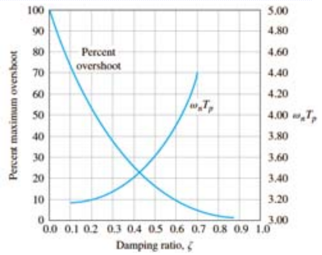
1. Resonant Frequency: $\omega_r = \omega_n \sqrt{1-2\zeta^2}$ (if $\zeta < 0.707$)
2. Maximum Magnitude: $M_{p_\omega} = (2\zeta\sqrt{1-\zeta^2})^{-1}$
3. Bandwidth (-3dB): $\omega_B = \omega_n \sqrt{1-2\zeta^2 + \sqrt{2-4\zeta^2+4\zeta^4}}$

Transient response

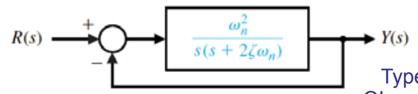
ω_r, ω_B → Rise time $\omega_r \uparrow$, $\omega_B \uparrow$ $\Rightarrow T_r \downarrow$

M_{p_ω} → Overshoot $M_{p_\omega} \uparrow$ $\Rightarrow O.S. \uparrow$





Performance Specification in Frequency Domain



Type 1 OL system

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Steady-State Error

$$\lim_{t\to\infty} e(t) = \frac{A}{K_v}$$ *for a ramp input with magnitude A*

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} = \frac{\omega_n}{2\zeta}$$

$$G(j\omega) = \frac{K\prod_{i=1}^{M}(j\omega\tau_i + 1)}{(j\omega)^N \prod_{k=1}^{Q}(j\omega\tau_k + 1)}$$

if N=0:

$$G(j\omega) = \frac{K}{(j\omega\tau_1+1)(j\omega\tau_2+1)}$$ position error constant

How to find in BD?

if N=1:

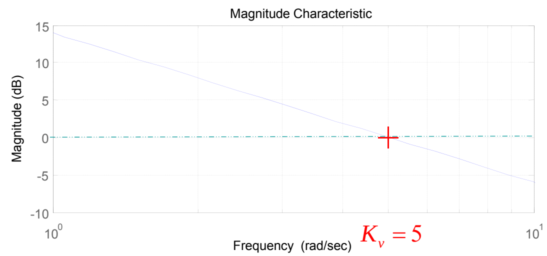
$$G(j\omega) = \frac{\omega_n/2\zeta}{j\omega(j\omega/2\zeta\omega_n + 1)} = \frac{K_v}{j\omega(j\omega\tau+1)}$$ velocity error constant

Example:

$$G(j\omega) = \frac{5(1+j0.1\omega)}{j\omega(1+j0.5\omega)(1+j0.6(\omega/50)+(j\omega/50)^2)}$$ Type 1 OL system

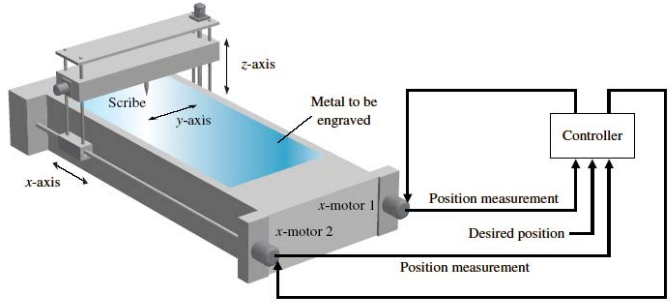
$K_v = 5$

$$G(j\omega) = \frac{5}{j\omega}$$ Type 1 OL system

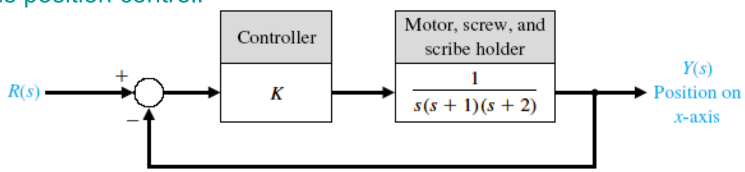


Design Example

Engraving Machine:



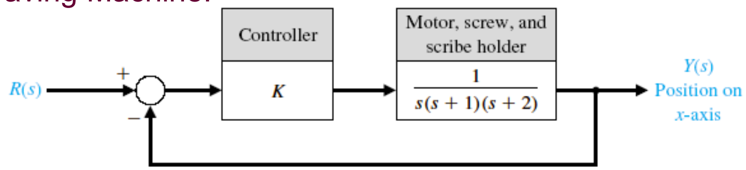
X-axis position control:



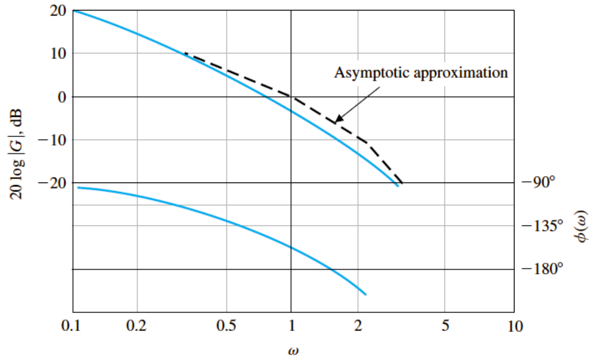
select gain K to achieve a satisfactory step response

Design Example

Engraving Machine:



BD of the open-loop system:

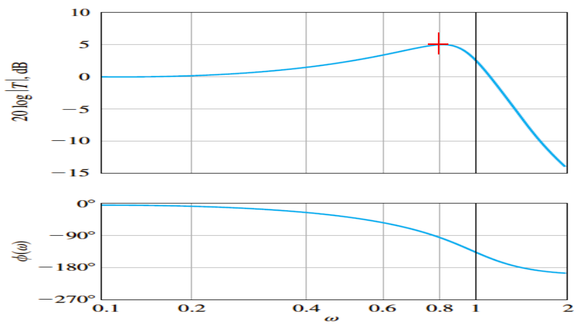


Design Example

TF of closed-loop system (K=2):

$$T(s)=\frac{2}{s^3+3s^2+2s+2} \overset{s=j\omega}{\Rightarrow} T(j\omega)=\frac{2}{(2-3\omega^2)+j\omega(2-\omega^2)}$$

BD of the closed-loop system (K=2):



Second-order dominant roots approximation:

$$20\log M_{p_\omega}=5 \qquad \omega_r=0.8$$

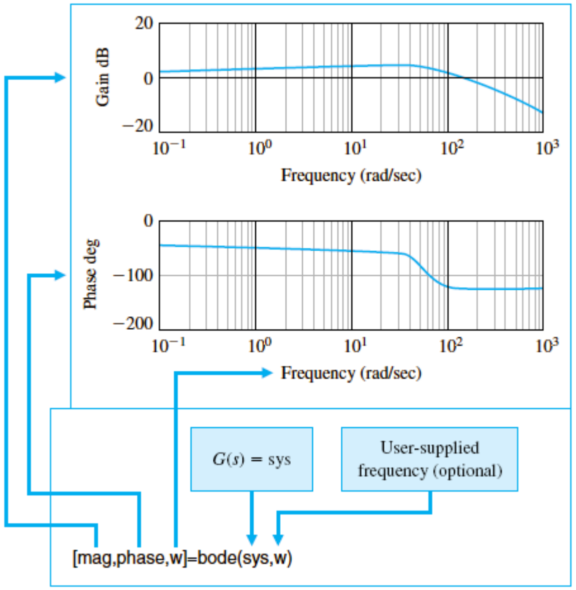
$$\zeta=0.29 \qquad \omega_n=0.88$$

$$T(s)=\frac{0.774}{s^2+0.51s+0.774}$$

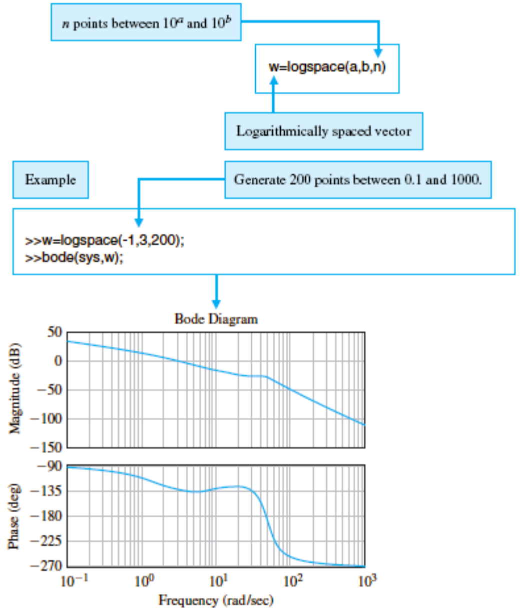
$$T_s=15.7 \quad (17)(\text{sec})$$

$$OS=37\% \quad (34\%)$$

Frequency Response *via* Matlab

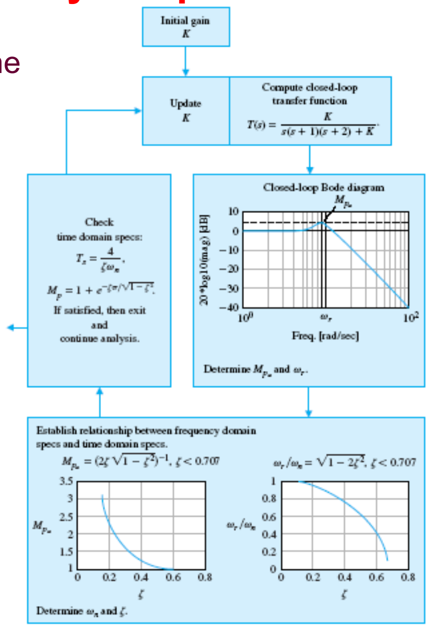


Frequency Response *via* Matlab



Frequency Response *via* Matlab

Example 8.2:
Engraving Machine



Frequency Response *via* Matlab

Example 8.2:
Engraving Machine

engrave.m

```
num=[K]; den=[1 3 2 K];
sys=tf(num,den);
w=logspace(-1,1,400);
[mag,phase,w]=bode(sys,w);
[mp,l]=max(mag);wr=w(l);

zeta=sqrt(0.5*(1-sqrt(1-1/mp^2)));
wn=wr/sqrt(1-2*zeta^2);

ts=4/zeta/wn
po=100*exp(-zeta*pi/sqrt(1-zeta^2))
```

Closed-loop transfer function.

Closed-loop Bode plot.

Solving Eqs. (8.36) and (8.37) for ζ and ω_n.

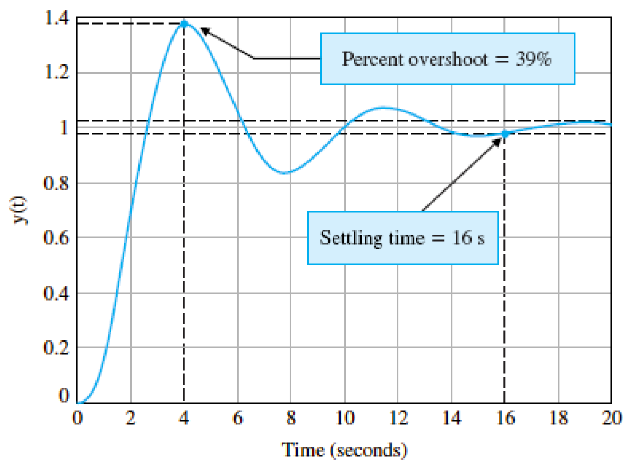
```
>> K=2; engrave

ts =
   15.7962
po =
   39.4570
```

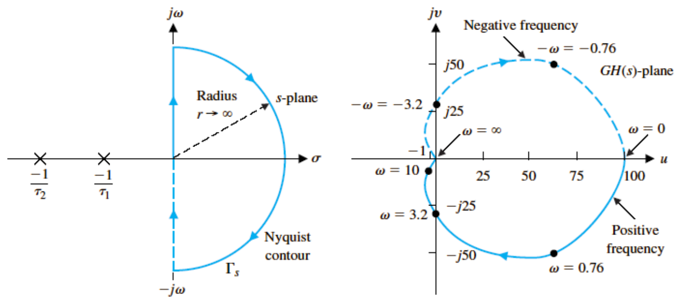
Check specifications and iterate, if necessary.

Frequency Response *via* Matlab

Example 8.2:
Engraving Machine



Stability in Frequency Domain

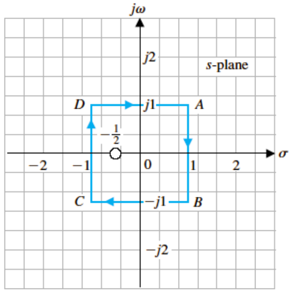


Contour Mapping

$$F(s):\ s \to F(s)$$

$$\boxed{\begin{aligned} F(j\omega) &= R(\omega) + j\cdot X(\omega) \\ &= u + jv \end{aligned}}$$

$$\begin{cases} F(s) = 2s+1 = 2(\sigma + j\omega)+1 \\ F(s) = u + jv \end{cases} \Rightarrow \begin{cases} u = 2\sigma + 1 \\ v = 2\omega \end{cases}$$

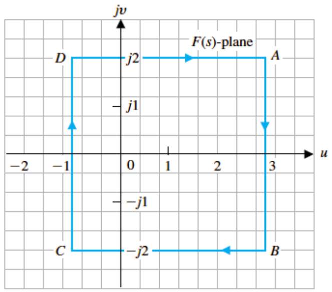


s-plane

$u=2\sigma+1$
$v=2\omega$

$\Rightarrow$

Conformal mapping



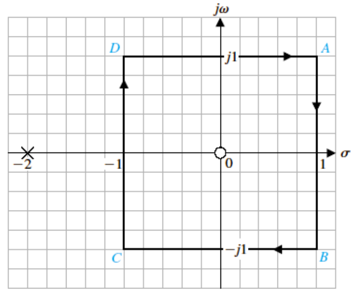
F(s)-plane

Clockwise traversal: positive direction

Contour enclosed area: right hand side area

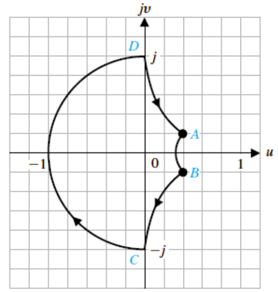
Contour Mapping

$$F(s) = \frac{s}{s+2}$$

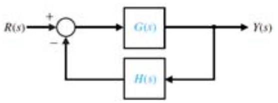


s-plane

$$F(s) = \frac{s}{s+2} \Rightarrow$$



F(s)-plane



Cauchy's Theorem

(Principle of the Argument)

CLCE: (single-loop system)

$$F(s) = 1 + L(s) = 1 + G(s)H(s) = 0$$

Poles of $F(s)$ are poles of $L(s)$,
Zeros of $F(s)$ are poles of the CL system.

Characteristic polynomial:

$$F(s) = \frac{K\prod_{i=1}^{n}(s+s_i)}{\prod_{k=1}^{m}(s+s_k)}$$

CLCE: (multi-loop system)

$$T(s) = \frac{\sum P_k \Delta_k}{\Delta(s)}$$

Characteristic polynomial:

$$F(s) = \Delta(s) = 1 - \sum L_n + \sum L_m L_q \cdots$$

Cauchy's Theorem (Principle of the Argument):

If a contour Γ_s in the *s*-plane encircles *Z* zeros and *P* poles of *F*(*s*) and does not pass through any poles or zeros of *F*(*s*) and the traversal is in the clockwise direction along the contour, the corresponding contour Γ_F in the *F*(*s*)-plane encircles the origin of the *F*(*s*)-plane N=Z-P times in the clockwise direction.

Cauchy's Theorem

(Principle of the Argument)

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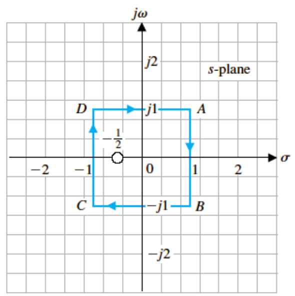
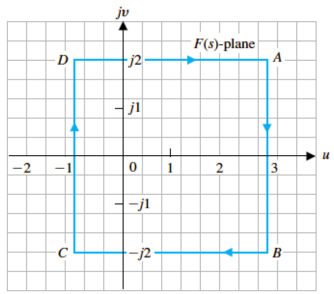
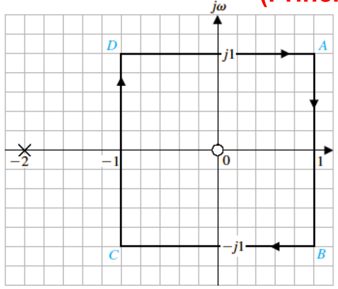


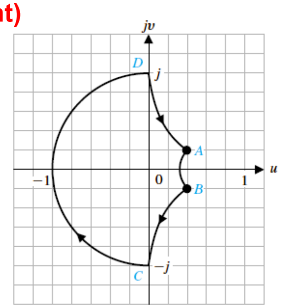
$$F(s) = 2s+1 \Rightarrow$$

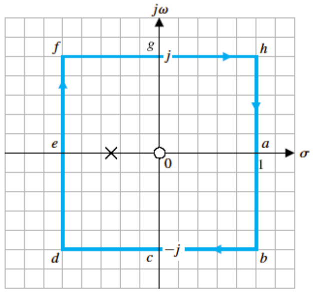


Cauchy's Theorem
(Principle of the Argument)

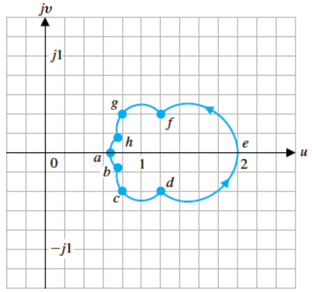


$$F(s)=\frac{s}{s+2}\quad\Rightarrow$$





$$F(s)=\frac{s}{s+0.5}\quad\Rightarrow$$

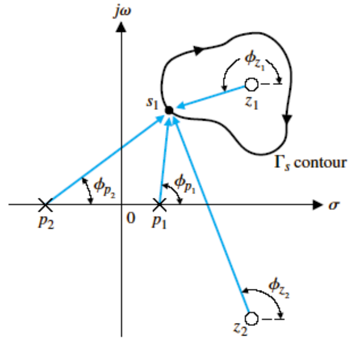


Cauchy's Theorem
(Principle of the Argument)

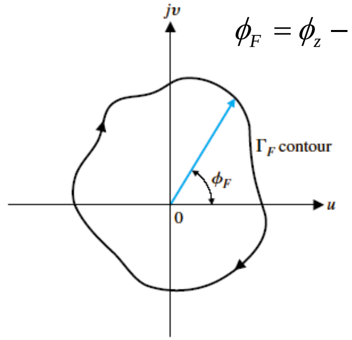
$$F(s)=\frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}=|F(s)|\measuredangle F(s)$$

$$=\frac{|s+z_1||s+z_2|}{|s+p_1||s+p_2|}(\measuredangle(s+z_1)+\measuredangle(s+z_1)-\measuredangle(s+p_1)-\measuredangle(s+p_2))$$

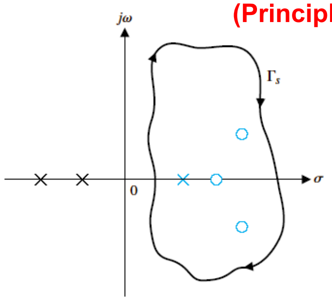
$$=|F(s)|(\phi_{z_1}+\phi_{z_2}-\phi_{p_1}-\phi_{p_2})$$



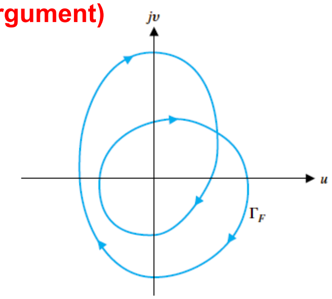
$$\phi_F=\phi_z-\phi_p$$

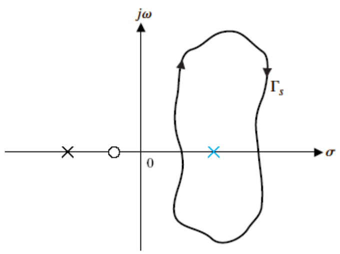


Cauchy's Theorem
(Principle of the Argument)

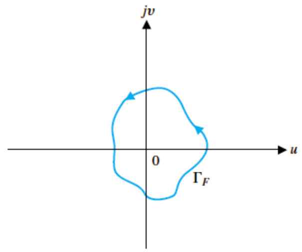


$$N=3-1$$





$$N=0-1$$

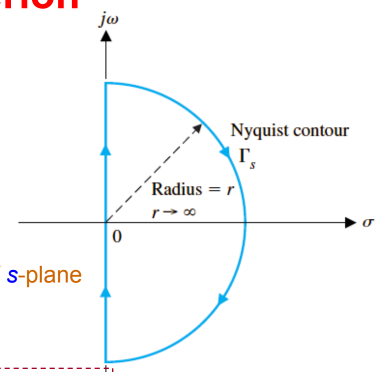


What could you imagine from this example? *For stability?*

Nyquist Criterion

Characteristic polynomial:

$$F(s)=\frac{K\prod_{i=1}^{n}(s+s_i)}{\prod_{k=1}^{m}(s+s_k)}$$



Zeros of *F*(s) are poles of the CL system.

System stable ⇔ All zeros of *F*(s) are in LHP of *s*-plane

⇔ All zeros of *F*(s) lie to the left of the *jω*-axis in the *s*-plane

Find the # of zeros of *F*(s) in RHP of the *s*-plane

⇒ Find the # of encirclements of the origin of F(s)-plane

$$Z = N + P$$

# of zeros of F(s) → Z

# of encirclements of the origin of F(s)-plane → N

# of poles of F(s) (Normally *P*=0) → P

If *P*=0, # of unstable CL poles = # of encirclements of the origin of F(s)-plane.

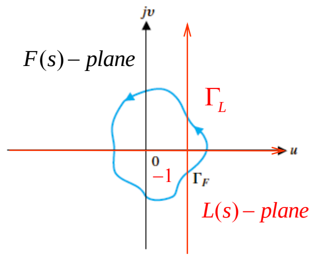
Nyquist Criterion

$F(s)=1+L(s)$

$$Z = N + P$$

- Z: # of zeros of F(s)
- N: # of encirclements of the origin of F(s)-plane
- P: # of poles of F(s) (Normally P=0)

If P=0, # of unstable CL poles = # of encirclements of the origin of F(s)-plane.



$L(s)=F(s)-1$

$$Z = N + P$$

- Z: # of zeros of F(s)
- N: # of encirclements of (-1,0) of L(s)-plane
- P: # of poles of L(s) (Normally P=0)

If P=0, # of unstable CL poles = # of encirclements of the (-1,0) point of L(s)-plane.

Nyquist Criterion

Nyquist stability criterion:

A feedback system is stable *iff* in the $L(s)$-plane, the contour Γ_L does not encircle the (-1,0) point when # of poles of $L(s)$ in the RHP(s) is 0.

A feedback system is stable *iff* in the $L(s)$-plane, the # of counterclockwise encirclements of the (-1,0) point by the contour Γ_L is equal to # of poles of $L(s)$ in the RHP(s).

A simple form:

If $L(s)$ has no RHP pole, then stable *iff* no encirclements of (-1,0).

If $L(s)$ has P RHP pole, then stable *iff* P counterclockwise encirclements of (-1,0).

Nyquist Criterion

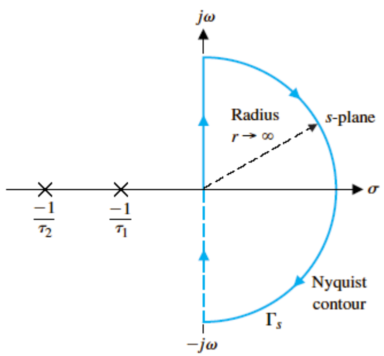
Example 9.1:

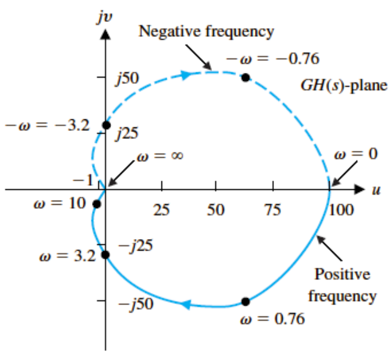
$$L(s)=GH(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}=\frac{100}{(s+1)(\frac{1}{10}s+1)}$$

$P = 0$ $\Rightarrow$ $Z = N = 0$

stable

Table 9.2: Magnitude and Phase of $L(j\omega)$





$\boxed{\text{what if } K \uparrow ?}$ $\Rightarrow$ *always stable*

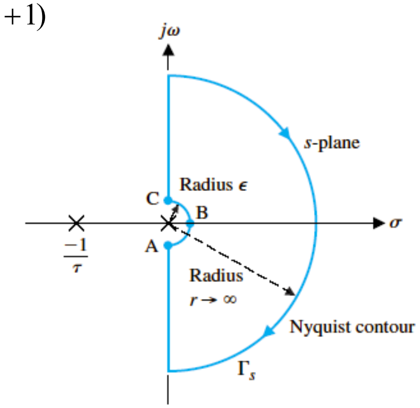
Nyquist Criterion

Example 9.2:

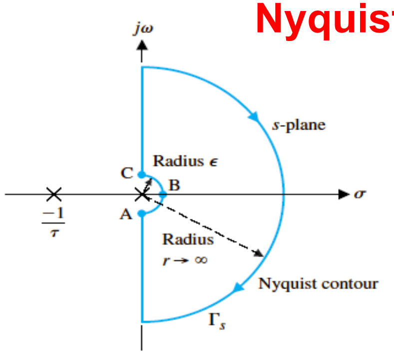
$$L(s)=GH(s)=\frac{K}{s(\tau s+1)}$$

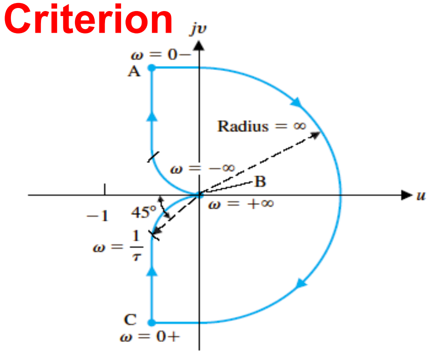
Cauchy' Theorem requires no pass through poles and zeros.

Infinitesimal detour around the pole at the origin.
(a small semicircle with radius ε→0)



Nyquist Criterion

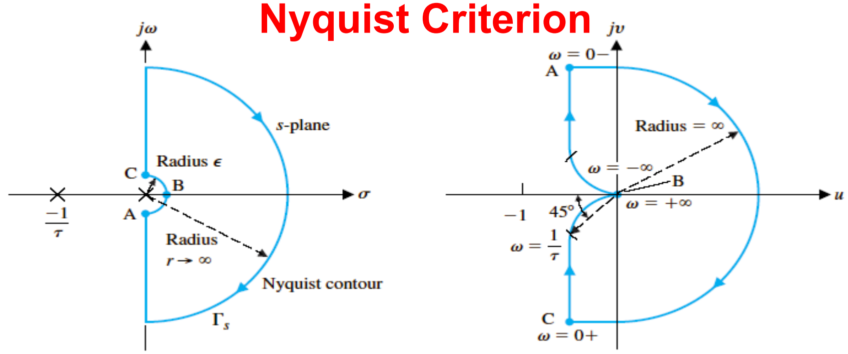




Four portions of the contour Γ_L :

1. ω: $0^-\to0^+$ $\quad s=\varepsilon e^{j\phi}$ $\quad \lim_{\varepsilon\to0} GH(s)=\lim_{\varepsilon\to0}\frac{K}{\varepsilon e^{j\phi}}=\lim_{\varepsilon\to0}\left(\frac{K}{\varepsilon}\right)e^{-j\phi}$
2. ω: $0^+\to+\infty$ $\quad \lim_{\omega\to\infty} GH(s)|_{s=j\omega}=\lim_{\omega\to\infty}\left(\frac{K}{j\omega(j\omega\tau+1)}\right)=\lim_{\omega\to\infty}|\frac{K}{\tau\omega^2}|\measuredangle(-\frac{\pi}{2}-\tan^{-1}\omega\tau)$
3. ω: $+\infty\to-\infty$ $\quad \lim_{\omega\to\infty} GH(s)|_{s=re^{j\phi}}=\lim_{\omega\to\infty}|\frac{K}{(\tau r)^2}|e^{-2j\phi}$
4. ω: $-\infty\to0^-$ $\quad$ symmetrical to Portion 2

Nyquist Criterion



s-plane → *L*(*s*)-plane

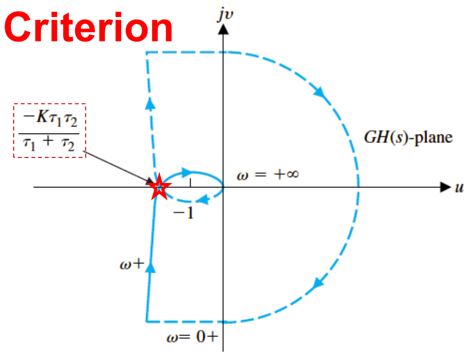
1. Origin → Semicircle of *inf* radius
2. Semicircle of infinite radius → Origin
3. ω: $0^+\to+\infty$: $\quad$ ω=0^+ → *inf* at ∠-90° $\quad$ ω=+∞ → 0 at ∠+90°

   (Passing *u*-axis at ω=0)

Nyquist Criterion

Example 9.3:

$$L(s)=GH(s)=\frac{K}{s(\tau_1 s+1)(\tau_2 s+1)}$$



s-plane → *L*(*s*)-plane

1. Origin → Semicircle of *inf* radius
2. Semicircle of infinite radius → Origin
3. ω: $0^+\to+\infty$: $\quad$ ω=0^+ → *inf* at ∠-90° $\quad$ ω=+∞ → 0 at ∠-270°

   (Passing *u*-axis, possibly encircle (-1,0))

$$GH(j\omega)=\frac{K}{[\omega^4(\tau_1+\tau_2)+\omega^2(1-\omega^2\tau_1\tau_2)^2]^{1/2}}\measuredangle(-\tan^{-1}\omega\tau_1-\tan^{-1}\omega\tau_2-\frac{\pi}{2})$$

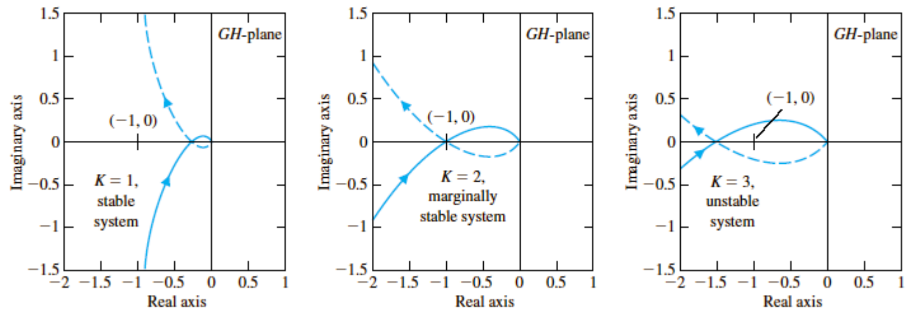
ω+→∞: $\quad \lim_{\varepsilon\to0} GH(s)=\lim_{\varepsilon\to0}\left(\frac{1}{\omega^3\tau_1\tau_2}\right)\measuredangle(-\frac{3\pi}{2})$

Nyquist Criterion

Intersection of *GH*(*jω*)-locus and *u*-axis:

$$v=\frac{-K(1/\omega)(1-\omega^2\tau_1\tau_2)}{[1+\omega^2(\tau_1^2+\tau_2^2)+\omega^4\tau_1^2\tau_2^2]}=0$$

$$u=\frac{-K(\tau_1+\tau_2)}{[1+\omega^2(\tau_1^2+\tau_2^2)+\omega^4\tau_1^2\tau_2^2]}\Big|_{\omega^2=1/\tau_1\tau_2}=\frac{-K\tau_1\tau_2}{(\tau_1+\tau_2)} \quad \geq -1 \quad ?$$



Stable if *K*≤2, when $\tau_1=\tau_2=1$

Nyquist Criterion

Example 9.4:

$$L(s) = GH(s) = \frac{K}{s^2(\tau s + 1)}$$

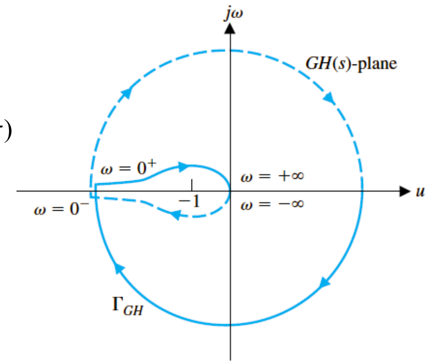
$$GH(j\omega) = \frac{K}{-\omega^2(j\omega\tau + 1)}$$

$$= \frac{K}{[\omega^4 + \tau^2\omega^6]^{1/2}} \measuredangle(-\pi - \tan^{-1}\omega\tau)$$

$$\lim_{\omega \to 0^+} GH(j\omega) = \lim_{\omega \to 0^+} \left(\frac{K}{\omega^2}\right) \measuredangle(-\pi)$$

$$\lim_{\omega \to +\infty} GH(j\omega) = \lim_{\omega \to +\infty} \left(\frac{K}{\omega^3}\right) \measuredangle(-\frac{3\pi}{2})$$

$$\lim_{\varepsilon \to 0} GH(s) = \lim_{\varepsilon \to 0} \left(\frac{K}{\varepsilon^2}\right) e^{-2j\phi}, \quad -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$$

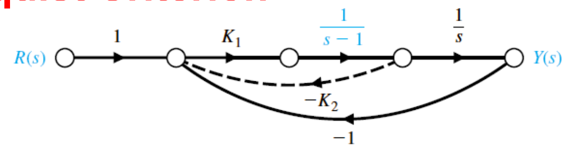


$Z = N + P = 2 + 0 = 2 \quad \Rightarrow$ unstable irrespective of K

Nyquist Criterion

Example 9.5:

K_2=0:



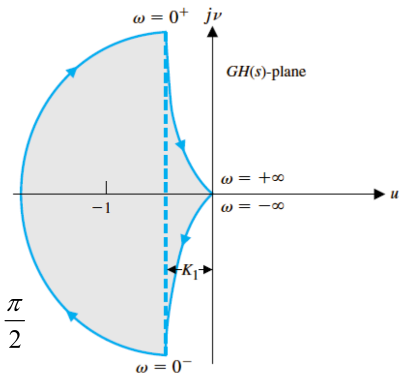
$$L(s) = GH(s) = \frac{K_1}{s(s-1)}$$

$$GH(j\omega) = \frac{K_1}{-j\omega(j\omega - 1)}$$

$$= \frac{K_1}{(\omega^2 + \omega^4)^{1/2}} \measuredangle(\frac{\pi}{2} + \tan^{-1}\omega)$$

$$\lim_{r \to \infty} GH(s)|_{s = re^{j\phi}} = \lim_{r \to \infty} \left(\frac{K_1}{r^2}\right) e^{-2j\phi}$$

$$\lim_{\varepsilon \to 0} GH(s) = \lim_{\varepsilon \to 0} \left(\frac{K}{\varepsilon}\right) \measuredangle(-\pi - \phi), \quad -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$$

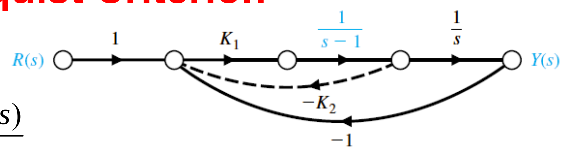


$Z = N + P = 1 + 1 = 2 \quad \Rightarrow$ unstable irrespective of K_1

Nyquist Criterion

Example 9.5:

K_2≠0:



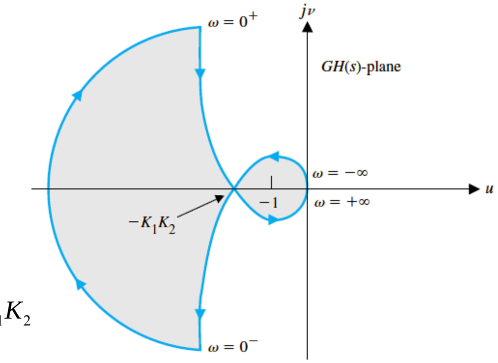
$$L(s) = GH(s) = \frac{K_1(1 + K_2 s)}{s(s-1)}$$

$$GH(j\omega) = \frac{K_1(1 + K_2 j\omega)}{-j\omega(j\omega - 1)}$$

Intersection of $GH(j\omega)$-locus and u-axis:

$$\omega^2 = 1/K_2$$

$$u|_{\omega^2 = 1/K_2} = \frac{-\omega^2 K_1(1 + K_2)}{(\omega^2 + \omega^4)}|_{\omega^2 = 1/K_2} = -K_1 K_2$$



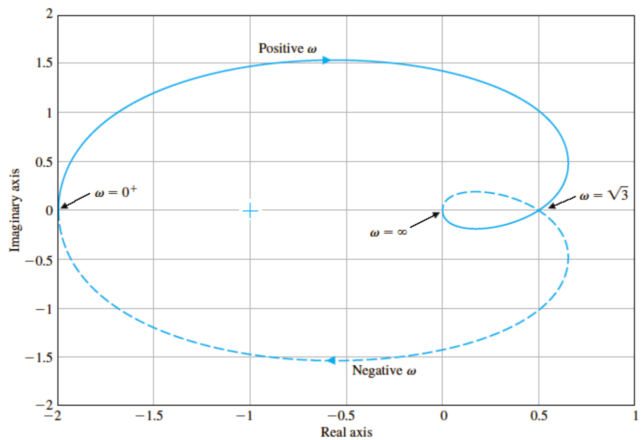
$Z = N + P = -1 + 1 = 0 \quad \Rightarrow$ stable when $K_1 K_2 > 1$

Nyquist Criterion

Example 9.6:

$$L(s) = GH(s) = \frac{K(s-2)}{(s+1)^2}$$

(Open-loop stable)



stable when $K>1/2$

Relative Stability: Gain Margin

$$GH(j\omega) = \frac{K}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)}$$

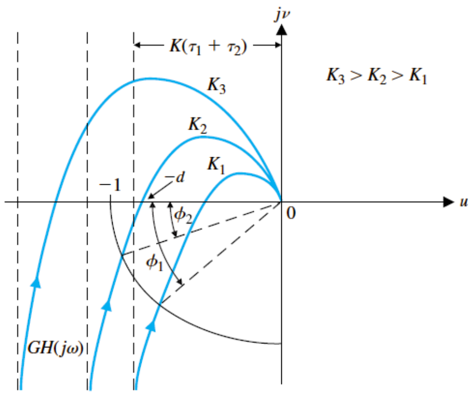
Intersection of $GH(j\omega)$-locus and u-axis:

$$v = 0$$

$$u = \frac{-K\tau_1\tau_2}{(\tau_1 + \tau_2)}$$

Marginally stable when

$$K = \frac{\tau_1 + \tau_2}{\tau_1\tau_2}$$



Gain Margin: the reciprocal of $|GH(j\omega)|$ at frequency at which phase angle **=-180°**.
GM is the increase in the system gain when phase angle **=-180°** (resulting in marginally stable system or intersection of (-1,0) on the Nyquist diagram).

$$\frac{1}{|GH(j\omega)|} = \left(\frac{-K_2\tau_1\tau_2}{\tau_1 + \tau_2}\right)^{-1} = \frac{1}{d} \qquad (\omega = 1/\sqrt{\tau_1\tau_2})$$

Relative Stability: Phase Margin

$$GH(j\omega) = \frac{K}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)}$$

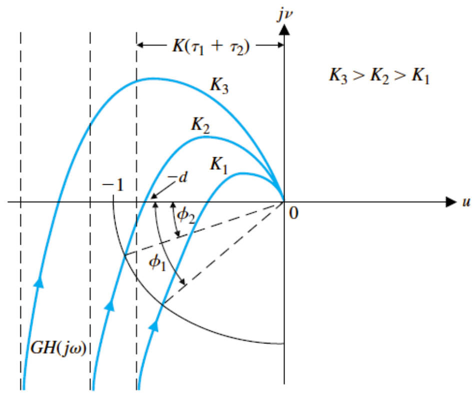
Intersection of $GH(j\omega)$-locus and u-axis:

$$v = 0$$

$$u = \frac{-K\tau_1\tau_2}{(\tau_1 + \tau_2)}$$

Marginally stable when

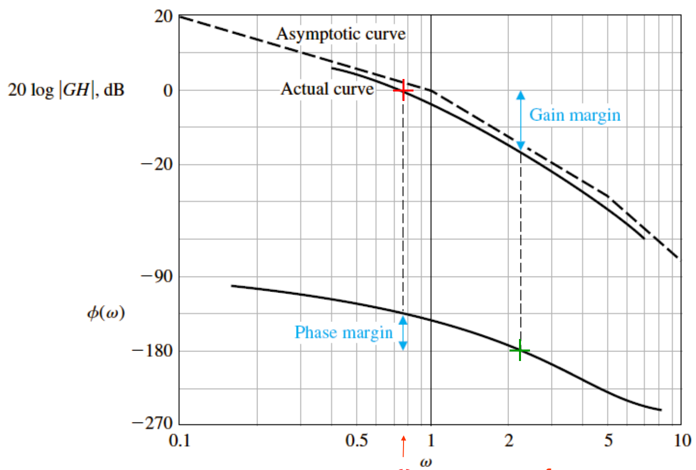
$$K = \frac{\tau_1 + \tau_2}{\tau_1\tau_2}$$



Phase Margin: the phase angel through which the $GH(j\omega)$ locus rotates to make the unity magnitude point $|GH(j\omega)|$ **=1** pass (-1,0).
PM is the amount of phase shift of $GH(j\omega)$ at **unity** magnitude (resulting in marginally stable system or intersection of (-1,0) on the Nyquist diagram).

Gain/Phase Margin in Bode Diagram

$$GH(j\omega) = \frac{1}{j\omega(j\omega + 1)(0.2j\omega + 1)}$$



ω_c : crossover frequency

GM=0-(-15dB)
=15dB

PM=180°-137°
=43°

Gain/Phase Margin in Bode Diagram

$$GH_1(j\omega) = \frac{1}{j\omega(j\omega + 1)(0.2j\omega + 1)}$$

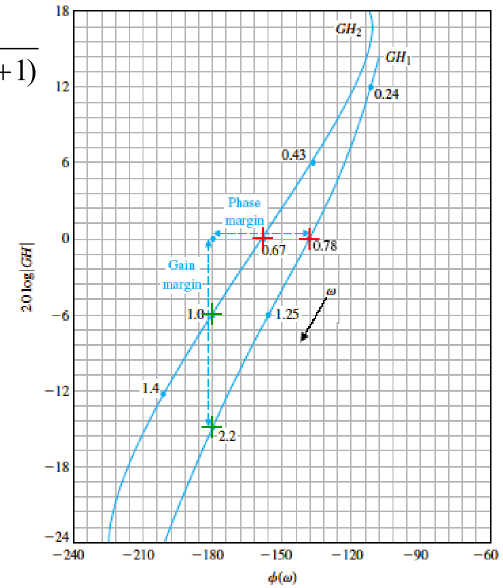
GM=15dB

PM=43°

$$GH_2(j\omega) = \frac{1}{j\omega(j\omega + 1)^2}$$

GM=5.7dB

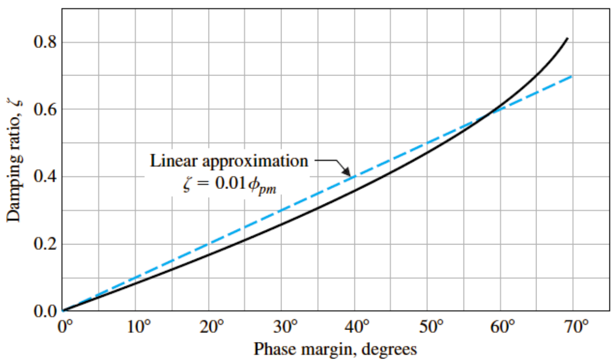
PM=20°



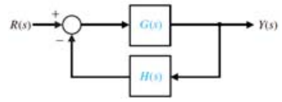
Gain/Phase Margin for 2nd Systems

$$GH(s)=\frac{\omega_n^2}{s(s+2\zeta\omega_n)}\qquad \phi_{pm}=\tan^{-1}\left(2\zeta\left[\frac{1}{(4\zeta^4+1)^{1/2}-2\zeta^2}\right]^{1/2}\right)$$

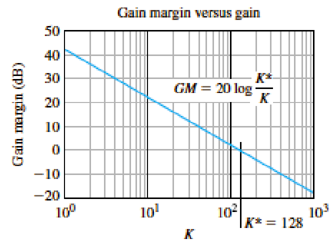
$$\zeta=0.01\phi_{pm}$$ (Linear approximation)

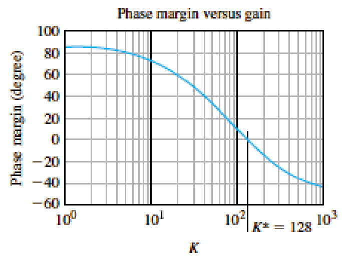


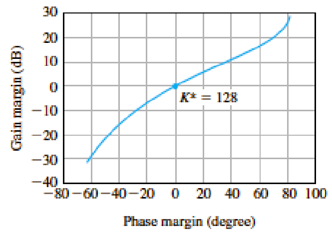
Gain/Phase Margin for 2nd Systems



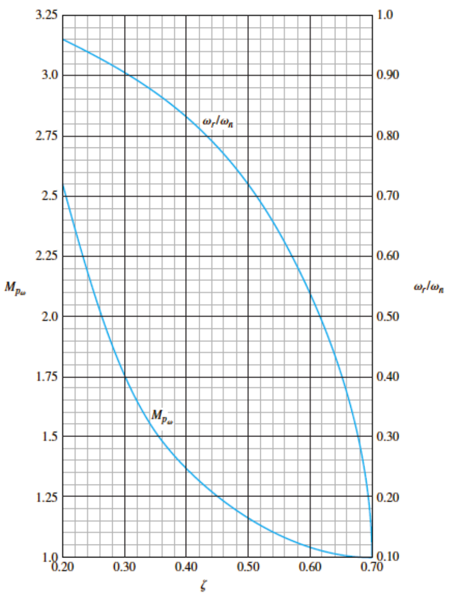
$$GH(s)=\frac{K}{s(s+4)^2}$$







Time-Domain Performance Criteria in Frequency Domain



$$T(j\omega)=\frac{1}{1+\frac{2\zeta j\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2}$$

if $\zeta<0.707$

$$\omega_r=\omega_n\sqrt{1\text{-}2\zeta^2}$$

$$M_{p_\omega}=|G(\omega_r)|=\left(2\zeta\sqrt{1\text{-}\zeta^2}\right)^{-1}$$

$$T(j\omega)\rightleftarrows GH(j\omega)$$

$$T(j\omega)=\frac{Y(j\omega)}{R(j\omega)}=\frac{G(j\omega)}{1+GH(j\omega)}\overset{H(j\omega)=1}{=}M(\omega)e^{j\phi(\omega)}=\frac{G(j\omega)}{1+G(j\omega)}$$

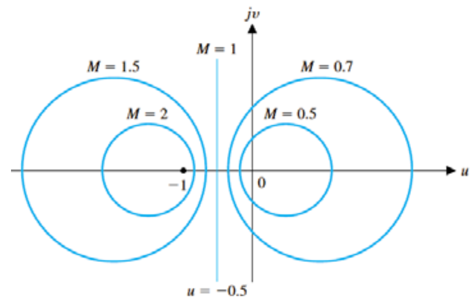
$$M(\omega)=\left|\frac{G(j\omega)}{1+G(j\omega)}\right|\overset{G(j\omega)=u+jv}{=}\left|\frac{u+jv}{1+u+jv}\right|=\frac{(u^2+v^2)^{1/2}}{[(1+u)^2+v^2]^{1/2}}$$

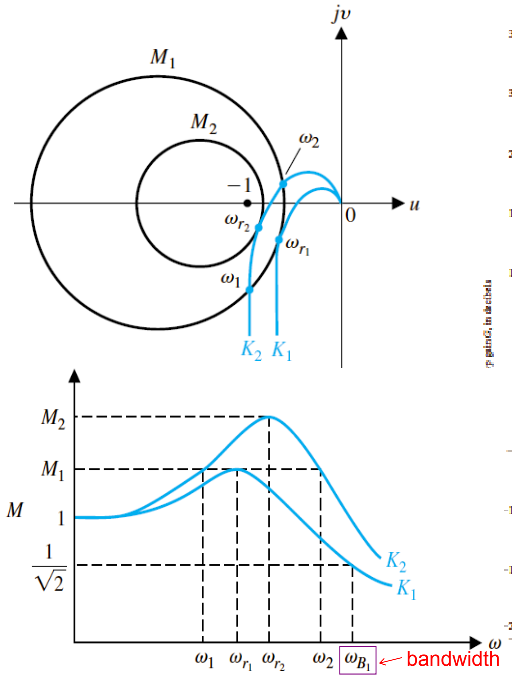
$$\phi(\omega)=\measuredangle T(j\omega)=\measuredangle\left|\frac{u+jv}{1+u+jv}\right|$$

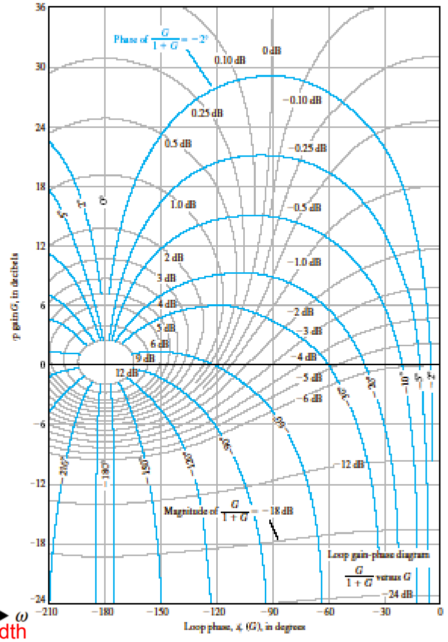
$$\left(u-\frac{M^2}{1-M^2}\right)^2+v^2=\left(\frac{M}{1-M^2}\right)^2$$

$$(u+0.5)^2+\left(v-\frac{1}{2N}\right)^2=\frac{1}{4}\left(1+\frac{1}{N^2}\right)$$

$$N=\tan\phi$$







Nichols Chart

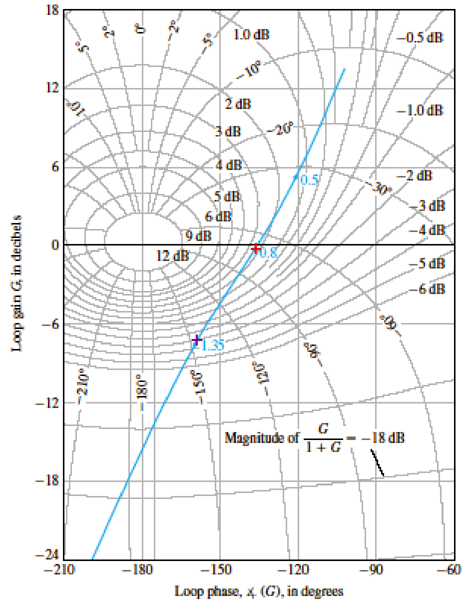
Example 9.7

$$G(j\omega)=\frac{1}{j\omega(j\omega+1)(0.2j\omega+1)}$$

M_{p_ω} =2.5dB, ω_r=0.8, **ψ**= -72°

M =-3dB, ω_B=1.33, **ψ**= -142°

$$\omega_B \approx 1.6\omega_c \qquad (0.2 \le \xi \le 0.8)$$



Nichols Chart

Example 9.8

$$G(j\omega)=\frac{0.64}{j\omega[(j\omega)^2+j\omega+1]}$$

M_{p_ω} =9dB, ω_r=0.88, **ψ**= -120°

PM=30° $\xRightarrow{\zeta=0.01\phi_{pm}}$ $\zeta=0.30$

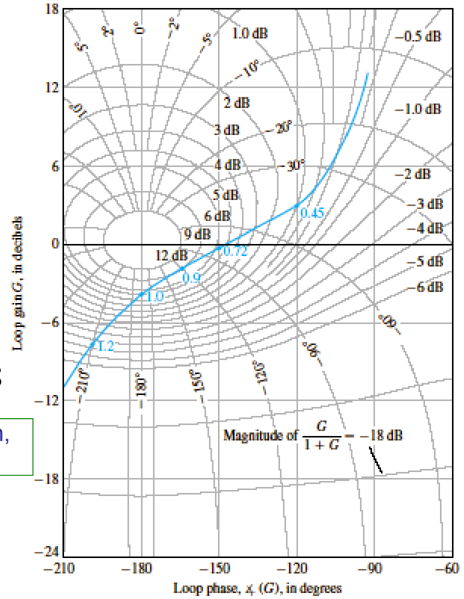
conflict

$M_{p_\omega}=(2\zeta\sqrt{1-\zeta^2})^{-1} \Rightarrow \zeta=0.18$

$\zeta=0.18$, not 0.30 by linear approximation, as the complex roots do not dominate.

To make $\zeta=0.30$, *how to select K?*

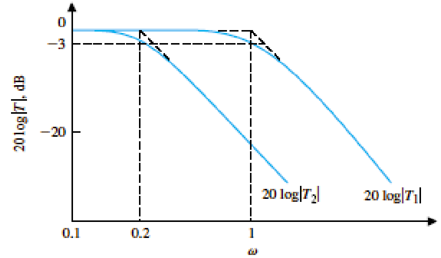
$$\zeta \to M_{p_\omega} \to K$$

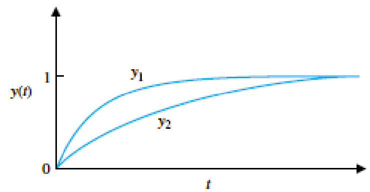


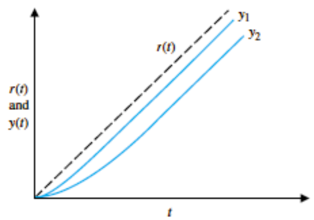
System Bandwidth

$$T_1(s)=\frac{1}{s+1}$$

$$T_2(s)=\frac{1}{5s+1}$$





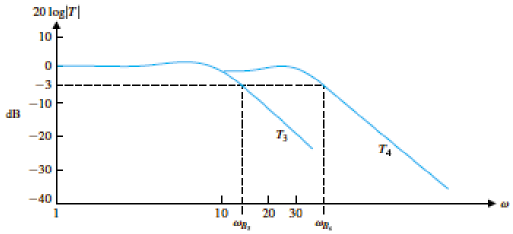


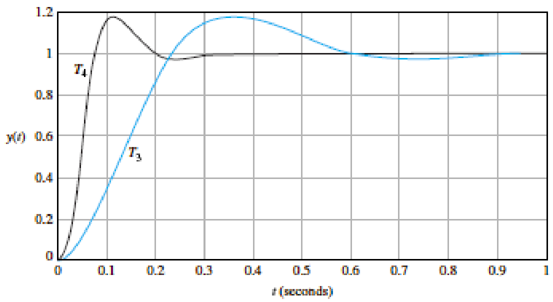
Larger bandwidth, faster response.

System Bandwidth

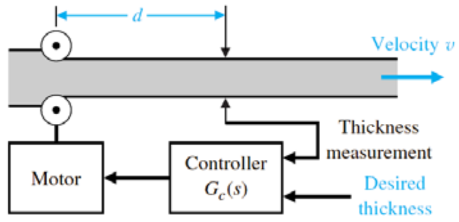
$$T_3(s) = \frac{100}{s^2 + 10s + 100}$$

$$T_4(s) = \frac{900}{s^2 + 30s + 900}$$





Time-Delay Systems

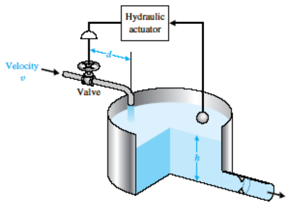


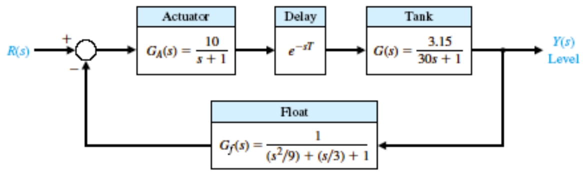
Delay TF: $G_d(s) = e^{-sT}$ $\qquad T = \dfrac{d}{v}$

Loop TF: $GG_c(j\omega) \cdot e^{-j\omega T}$ $\qquad \phi(\omega) = -\omega T$

Time-Delay Systems

Example 9.9



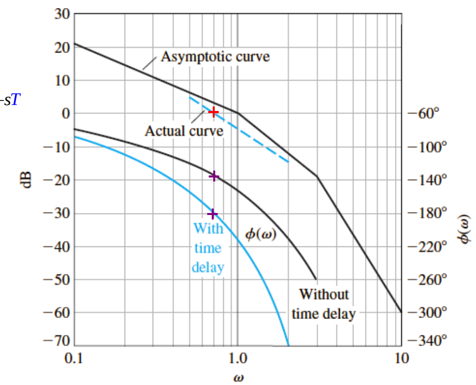


$$GH(s) = \frac{31.5}{(s+1)(30s+1)[s^2/9 + s/3 + 1]} e^{-sT}$$

Without delay: PM$=40^\circ$

With delay: PM$= -3^\circ$

To make PM$= 30^\circ$ *for the time delay system, how to select K?*



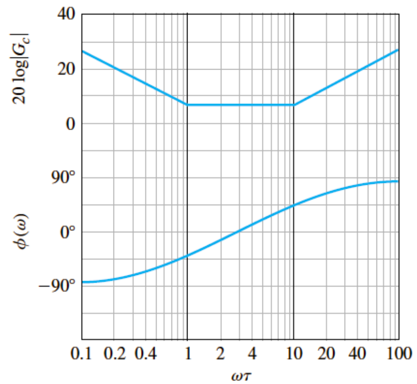
PID Controllers in Frequency Domain

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s = \frac{K_I(\tau s + 1)(\frac{\tau}{\alpha}s + 1)}{s}$$

$$\tau = \frac{K_D}{K_I} \qquad \frac{\tau}{\alpha} = \frac{K_P}{K_I}$$

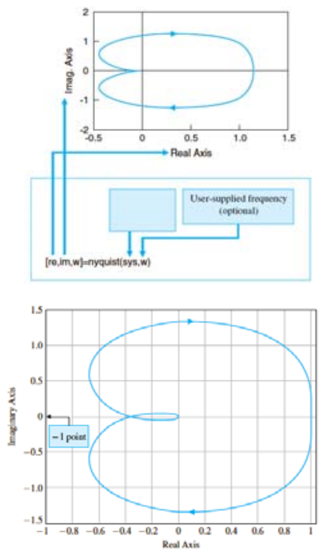
With two complex zeros:

$$G_c(s) = \frac{K_I[1 + (2\xi/\omega_n)j\omega - (\omega/\omega_n)^2]}{j\omega}$$



Useful Matlab Commands

nyquist(sys) bode(sys) margin(sys)




```
[mag,phase,w]=bode(sys);
[Gm,Pm,Wcg,Wcp]=margin(mag,phase,w);
```

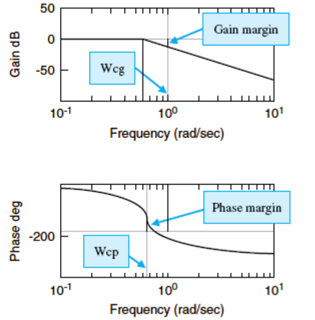
or

```
[Gm,Pm,Wcg,Wcp]=margin(sys);
```

Example

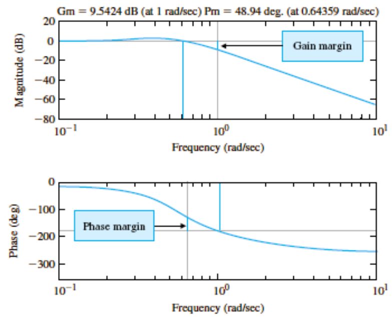
```
num=[0.5]; den=[1 2 1 0.5];
sys=tf(num,den);
margin(sys);
```

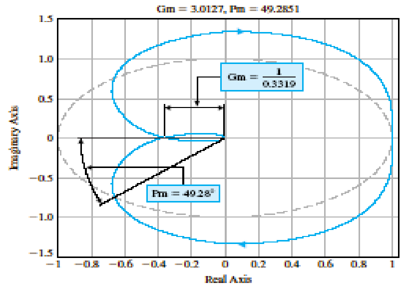
Gm = gain margin (dB)
Pm = phase margin (deg)
Wcg = freq. for phase = −180
Wcp = freq. for gain = 0 dB



Useful Matlab Commands

nyquist(sys) bode(sys) margin(sys)






```
num=[0.5];
den=[1 2 1 0.5 ];
sys=tf(num,den);
%
margin(sys)
```

Open-loop system

```
%   The Nyquist plot of
%
%                  0.5
%   G(s) = ---------------------------
%          s^3 + 2 s^2 + s + 0.5
%
%   with gain and phase margin calculation.
%
num=[0.5]; den=[1 2 1 0.5]; sys=tf(num,den);
%
[mag,phase,w]=bode(sys);
[Gm,Pm,Wcg,Wcp]=margin(mag,phase,w);
%
nyquist(sys);
title(['Gm = ',num2str(gm),' Pm = ',num2str(Pm)])
```

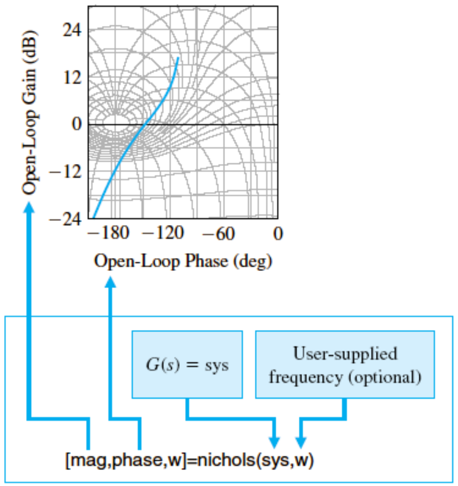
Compute gain and phase margins.

Nyquist plot

Label gain and phase margins on plot.

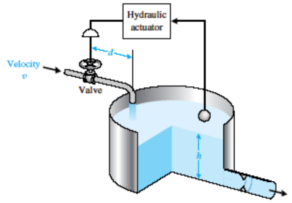
Useful Matlab Commands

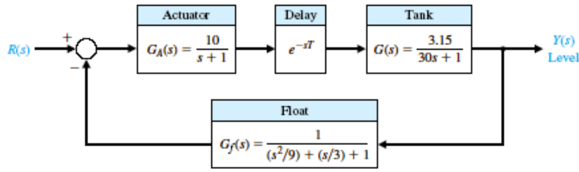
nichols(sys)



Useful Matlab Commands

Example 9.10

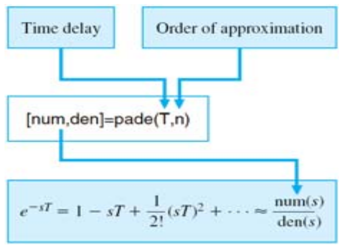




$$GH(s)=\frac{31.5}{(s+1)(30s+1)[s^2/9+s/3+1]}e^{-sT}$$

$$=\frac{31.5(s^2-6s+12)}{(s+1)(30s+1)(\frac{s^2}{9}+\frac{s}{3}+1)(s^2-6s+12)}$$

Pade approximation: $e^{-s}\approx\frac{s^2-6s+12}{s^2+6s+12}$ ($T=1$)




```
>>K=16; liquid
```

Command level input.

liquid.m

```
% Liquid Control System Analysis
%
[np,dp]=pade(1,2);
sysp=tf(np,dp);
num=K;
d1=[1 1]; d2=[30 1]; d3=[1/9 1/3 1];
den=conv(d1,conv(d2,d3));
sysg=tf(num,den);
sys=series(sysp,sysg);
%
margin(sys);
```

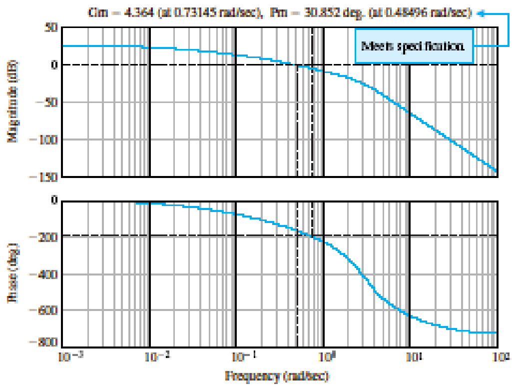
Compute GH(s).

Compute gain and phase margins.

Useful Matlab Commands

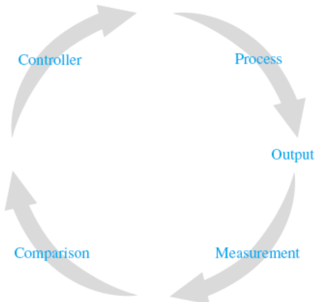
Example 9.10

$$GH(s) = \frac{31.5(s^2 - 6s + 12)}{(s+1)(30s+1)(\frac{s^2}{9} + \frac{s}{3} + 1)(s^2 - 6s + 12)}$$



Part 4:
Control

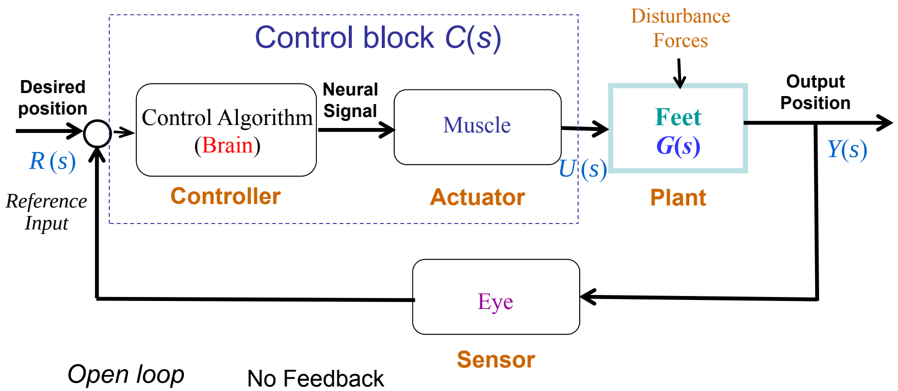
Introduction to Feedback Control



Introduction to Feedback Control

- **Automatic Control Systems**
  - Why Control?
  - Open-Loop (Feedforward) vs Closed-Loop (Feedback) Control
  - Fundamental Philosophies of Control Design
    - Roles of Feedforward and Feedback
- **Classical Feedback Control System Structure**
  - Elements of a Feedback Control System
  - Closed-Loop Transfer Functions (CLTF)
- **Typical Performance Specifications**
  - Steady State Performance Specifications
  - Transient (Dynamic) Performance Specifications

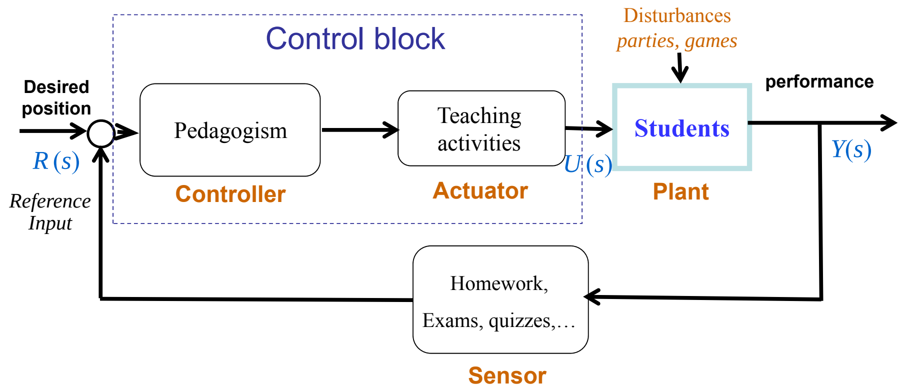
Example (*walking*)



Open loop No Feedback

Closed loop With Feedback

Another Example



Open loop No Feedback *Blindly teaching ?!*

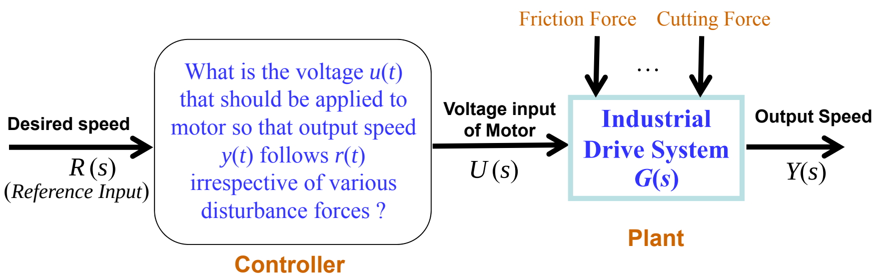
Closed loop With Feedback

Control System Design

- **Objectives of Control System Design**

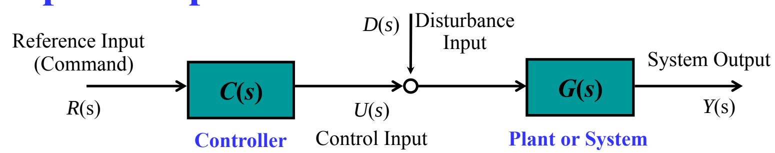
Output of a dynamic system (e.g., speed of a car) is normally affected by **various factors** that cannot be predicted in the design of a product (e.g., slope of road and wind forces) or may not possess the quality that one would like to have (chattering problem of fast car braking action),

the **objective** of a control system design is to **figure out a strategy** (e.g., the cruise control system) to **adjust certain inputs** to the system (e.g., the gas pedal position) so that the output of the system behaves in the way one would like to have (e.g., follow the desired reference output) in respect of various factors that cannot be predicted in the design of a product.



Open-Loop vs Closed-Loop

- **Open-Loop Control**



Q: *Ideally, if we want $Y(s)$ to follow $R(s)$ (i.e. want $Y(s) = R(s)$), how would you design the controller $C(s)$ for the above open-loop control system?*

$Y(s)=G(s)C(s)R(s)$ $G(s)C(s)=1$ $C(s)=1/G(s)$ System inversion!

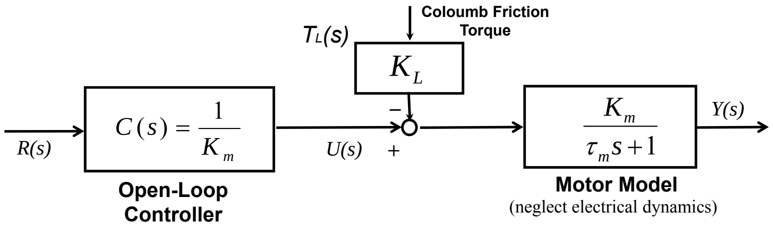
Q: *Can we attenuate the effect of disturbance $D(s)$ on system output $Y(s)$?* NO!

Q: *Can we attenuate the effect of variations of plant transfer function $G(s)$ on system output $Y(s)$?* NO!

The control input $u(t)$ (or $U(s)$) is synthesized based on the a priori knowledge of the system (plant) and the reference input $r(t)$ (or $R(s)$). The control system does not measure the output, and there is no comparison of the output to make it conform to the desired output (reference input).

Open-Loop Control Example

Speed Control of a DC Motor without Sensors



Block Diagram of Open-Loop Control System

Recall: Full model of DC Motor

$$\Omega(s) = \frac{K_T}{L_A J_A s^2 + (BL_A + R_A J_A)s + (R_A B + K_b K_T)} \cdot E_i(s) - \frac{L_A s + R_A}{L_A J_A s^2 + (BL_A + R_A J_A)s + (R_A B + K_b K_T)} \cdot T_L(s)$$

An **Implementable** Open-Loop Controller

$$C(s) = 1/G(s) = \frac{\tau_m}{K_m} s + \frac{1}{K_m} \qquad C(s) = \frac{1}{K_m}$$

Output Speed with Open-Loop Controller

$$Y(s) = \frac{K_m}{K_m(\tau_m s + 1)} R(s) + \left[-\frac{K_L K_m}{\tau_m s + 1}\right] T_L(s)$$

Open-Loop Control Example

Speed Control of a DC Motor without Sensors

Steady-State Output Speed for Constant Desired Speed Reference Inputs (no friction)

$$Y(s) = \frac{K_m}{K_m(\tau_m s + 1)} R(s) + \left[-\frac{K_L K_m}{\tau_m s + 1}\right] T_L(s)$$

$$T_L(s) = 0 \Rightarrow Y(s) = \frac{K_m}{K_m(\tau_m s + 1)} R(s)$$

$\dfrac{K_m}{\tau_m s+1}$

Time constant (no controller) τ_m

Time constant (with controller) τ_m

No Change

Steady state (no controller) $y_{ss}(t) = K_m r(t) \neq r(t)$

Steady state (with controller)

By FVT

$$y_{ss}(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{K_m}{K_m(\tau_m s + 1)} sR(s)$$

$$= \lim_{s\to 0} sR(s) = r$$

Open-Loop Control Example

Speed Control of a DC Motor without Sensors

Steady-State Output Speed for Constant Desired Speed Reference Inputs (with a constant friction)

$$Y(s) = \frac{K_m}{K_m(\tau_m s + 1)} R(s) + \left[-\frac{K_L K_m}{\tau_m s + 1}\right] T_L(s)$$

$$y_{ss}(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0}\left[\frac{K_m}{K_m(\tau_m s + 1)} sR(s) - \frac{K_m K_L}{\tau_m s + 1} sT_L(s)\right]$$

$$= r - K_m K_L T_L$$

There is no effort to attenuate the impact of friction !

Q: *In reality, the friction on a motor may change quite significantly. Will the customer be happy with such an open-loop controller ?*

NO!

Q: *What do we use in open-loop controller design?*

System model (TF of plant) only

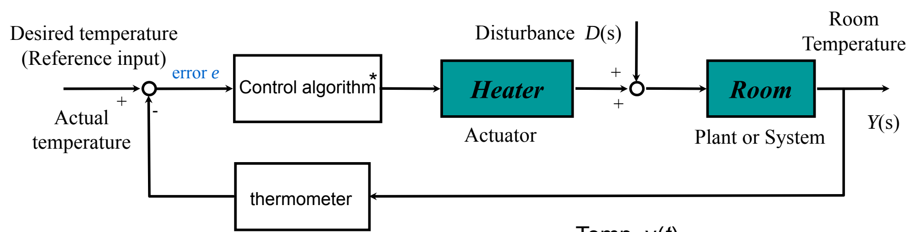
We did not use the information of system output.

Open-Loop *vs* Closed-Loop

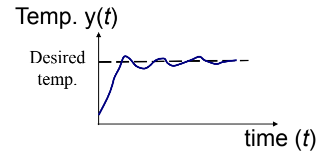
- **Closed-Loop (Feedback) Control**

  The control input $u(t)$ (or $U(s)$) is synthesized based on the a priori knowledge of the system (plant), the reference input $r(t)$ (or $R(s)$) and the measurement of the actual output $y(t)$ (or $Y(s)$).

For example the temperature control of this classroom:



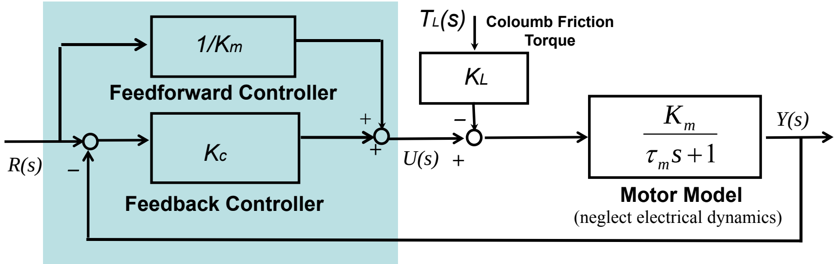
* on-off controller: $\begin{cases} e > 0, & \text{turn on heater} \\ e < 0, & \text{turn off heater} \end{cases}$



Closed-Loop Control Example

Speed Control of a DC Motor with Speed Sensors

Block Diagram of A Simple Closed-Loop Control System



Closed-loop Controller

A Simple Closed-Loop Controller

Feedback Controller $U_{fb}(s)=K_c(R(s)-Y(s))$

Feedforward Controller $U_{ff}(s)=R(s)/K_m$

$U(s)=U_{ff}(s)+U_{fb}(s)$

Closed-Loop Control Example

Speed Control of a DC Motor with Sensors

Output Speed with Closed-loop Controller

$$Y(s)=\underbrace{\frac{1}{\boxed{\frac{\tau_m}{1+K_mK_c}}s+1}}_{G_{YR}(s)}R(s)+\underbrace{\left[-\frac{\frac{K_LK_m}{1+K_mK_c}}{\frac{\tau_m}{1+K_mK_c}s+1}\right]}_{G_{YT}(s)}T_L(s)$$

Time constant

Steady-State Output Speed for Constant Desired Speed Reference Inputs (no friction)

$$T_L(s)=0 \quad y_{ss}=\lim_{s\to 0}G_{YR}(s)sR(s)=r$$

$\tau=\frac{\tau_m}{1+K_mK_c}$ decrease with increase of $K_c(K_m)$

Steady-State Output Speed for Constant Desired Speed Reference Inputs (with a constant friction T_c)

$$y_{ss}=\lim_{s\to 0}\left[G_{YR}(s)sR(s)+G_{YT}(s)sT_L(s)\right]=r-\frac{K_LK_m}{1+K_mK_c}T_c$$

$$e_{ss}=r-y_{ss}=\frac{K_LK_m}{1+K_mK_c}T_c$$

decrease with increase of $K_c(K_m)$

Closed-Loop Control Example

Speed Control of a DC Motor with Sensors

Q: When can we consistently have the desired steady-state speed regardless certain amount of Columb friction that may exist ?

High-gain feedback

Q: How is response speed of the closed-loop system compared with the response speed of original open-loop system ?

faster

compare time constants in different cases

Feedforward part depends on system model.

Recall $C_{feedforward}(s)\approx 1/G(s)$

TF of system

Feedback part does not strictly depend on system model.

In this example, we only need that $K_c>0$.

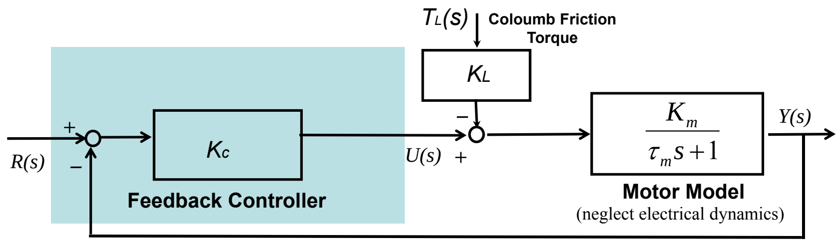
If system parameters vary, can we still use the same feedforward part? No!

What shall we do?

Closed-Loop Control Example

Speed Control of a DC Motor with Speed Sensors

Block Diagram of A Simple Closed-Loop Control System (No feedforward)



Closed-loop Controller

Steady-State Output Speed Error for Constant Desired Speed Reference Inputs (with a constant friction) *without Feedforward Control*

$$Y(s)=\underbrace{\frac{K_m}{\tau_m s+1}}_{G_p(s)}\left[U(s)-K_LT_L(s)\right]=G_p(s)\left[K_cR(s)-K_cY(s)-K_LT_L(s)\right]$$

$$\left(1+K_cG_p(s)\right)Y(s)=G_p(s)\left[K_cR(s)-K_LT_L(s)\right]$$

Feedforward *vs* Feedback Control

Speed Control of a DC Motor with Sensors and Feedback only

Steady-State Output Speed Error for Constant Desired Speed Reference Inputs (with a constant friction) *without Feedforward Control*

$$Y(s) = \frac{G_p(s)K_c}{1+K_cG_p(s)}R(s) + \left[\frac{-K_LG_p(s)}{1+K_cG_p(s)}T_L(s)\right]$$

$$= \frac{\frac{K_m}{\tau_m s+1}K_c}{1+K_c\frac{K_m}{\tau_m s+1}}R(s) - \frac{K_L\frac{K_m}{\tau_m s+1}}{1+K_c\frac{K_m}{\tau_m s+1}}T_L(s)$$

$$= \frac{\frac{K_mK_c}{1+K_mK_c}}{\frac{\tau_m}{1+K_mK_c}s+1}R(s) - \frac{\frac{K_mK_L}{1+K_mK_c}}{\frac{\tau_m}{1+K_mK_c}s+1}T_L(s)$$

Q: How is the performance of the closed-loop control with feedback control action only compared to that of the closed-loop control with both feedforward and feedback actions?

Feedforward *vs* Feedback Control

Q: How is the performance of the closed-loop control with feedback control action only compared to that of the closed-loop control with both feedforward and feedback actions?

Steady-State Output Speed Error for Constant Desired Speed Reference Inputs (with a constant friction) *with Feedback but without Feedforward Control*

$$Y(s) = \frac{\frac{K_mK_c}{1+K_mK_c}}{\frac{\tau_m}{1+K_mK_c}s+1}R(s) - \frac{\frac{K_mK_L}{1+K_mK_c}}{\frac{\tau_m}{1+K_mK_c}s+1}T_L(s)$$

Steady-State Output Speed Error for Constant Desired Speed Reference Inputs (with a constant friction) *with both Feedback and Feedforward Control*

$$Y(s) = \frac{1}{\frac{\tau_m}{1+K_mK_c}s+1}R(s) - \frac{\frac{K_LK_m}{1+K_mK_c}}{\frac{\tau_m}{1+K_mK_c}s+1}T_L(s)$$

Attenuation ability of the system to disturbances such as the Columb friction: Same

Response speed: Same

Steady-state output speed error: A little bit larger

$$\frac{K_mK_c}{1+K_mK_c} \approx 1$$
when K_c is very large.

Why Feedback ?

The previous speed control examples illustrate that, by using feedback, we can change the closed-loop system's dynamic behavior, as the Closed-Loop Transfer Function (CLTF) is different from the original system's (open-loop) transfer function. As such, through feedback, we have the ability to achieve the following objectives:

- **Stabilize Unstable Systems**

  For example, unstable plants such as ***inverted pendulum*** and the position control of DC motor can be stabilized using feedback. $G(s) = \frac{1}{I_o s^2 + Bs - K}$ try proportional controller

- **Improve System Performance** to meet stringent performance requirements
  - Steady State Performance -- for example, reduce steady state error due to disturbances ...
  - Transient Performance -- for example, reduce rise time and settling time to speed up system response , …

  Refer to Examples of DC Motor

- **Reduce (Attenuate) the effect of modeling uncertainty (error) and various disturbances *through High-Gain Feedback***
  - One of the key elements in all feedback control is to figure out how one can employ high gain feedback to have better disturbance and modeling error attenuation capability while without causing instability of closed-loop system in the presence of various physical constraints such as control input saturation and neglected high frequency dynamics.

Why Feedforwad (Model Compensation)?

The previous examples also illustrate that, feedforward control action makes the control input close to the desired control input that is needed to accomplish the task. As such, only small amount of control correction needs to be provided by feedback, which results in smaller tracking error (*keep in mind that feedback control needs tracking error to generate the control action*).

- Feedforward is very important for applications having *stringent* performance requirements such as precision electro-mechanical devices.
- Usefulness of feedforward heavily depends on the accuracy of models used for physical plants, which normally have quite large variations of system parameters. As such, *learning mechanisms* such as parameter adaptation in adaptive control are needed to build accurate model on-line based on various information obtained by sensors including stored past information.

General Control Design Principles

Control design is nothing but an *Inversion Process*. The inversion can be achieved by two key mechanisms:

(High-Gain) Feedback and (Model Compensation) Feedforward

$$\boxed{\frac{K}{1+KG} \overset{K \text{ is large}}{\approx} G^{-1}}$$

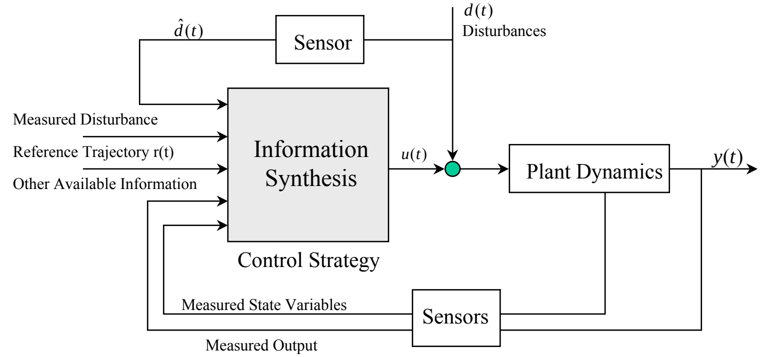
High-gain feedback **gives approximate inversion in the presence of modeling errors and disturbances, which is the essence of control. However, in practice, the choice of feedback gain is part of a complex web of design trade-offs; high-gain leads to high sensitivity to measurement noises and makes the stability of closed-loop system sensitive to control input saturation and neglected high-frequency dynamics. Understanding and balancing these trade-offs is the essence of feedback control system design.**

- Use nonlinear feedback instead of linear feedback to achieve a better trade-off !

On-line learning **is key to have a good model compensation or feedforward design.**

General Controller Structure

In general, a controller is nothing but a strategy to determine a control action based on all available information; information not only comes from the measured output but also from the measured internal state variables, measured disturbance, reference trajectory, and plant model structure. It can have any form and is illustrated below



So use your own imagination to come out new control schemes!

Feedback: Pros *vs* Cons

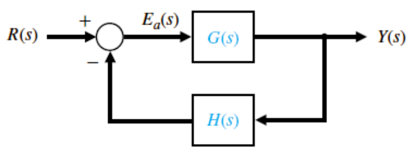
Pros:

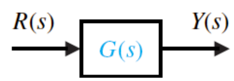
- Stability
- Transient Response
- Steady-State Error
- Disturbance
- Sensitivity

Cons:

- Complexity
- Gain loss
- Possibility of Instability

Open-Loop *vs.* Closed-Loop





$$Y(s) = \frac{G(s)}{1+G(s)H(s)}R(s)$$

$$Y(s) = \frac{G(s)}{1+G(s)}R(s), \text{ when } H(s)=1$$

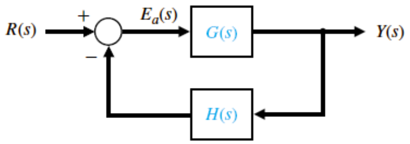
$$E(s) = R(s) - Y(s)$$

$$Y(s) = \frac{1}{H(s)}R(s), \text{ if } G(s)H(s) >> 1$$

To reduce the error, the magnitude of [1+G(s)H(s)] >> 1 (or the magnitude of [1+G(s)] >> 1) over the range of s under consideration.

Sensitivity to Parameter Variations

G(s) may vary due to the changing environment, aging and other factors
In such a case, OL produces inaccurate output,
while CL senses the variation and correct the output



$$G(s) \to G(s) + \Delta G(s)$$
$$\Downarrow$$
$$\Delta Y = \Delta G \cdot R$$

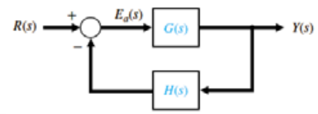
$$G(s) \to G(s) + \Delta G(s)$$
$$\Downarrow$$
$$\Delta Y = ?$$

$$\Rightarrow \quad \Delta Y = \frac{\Delta G}{(1+GH+\Delta GH)(1+GH)} R$$
$$\approx \frac{\Delta G}{(1+GH)^2} R, \quad \text{if } G >> \Delta G$$

Sensitivity to Parameter Variations

System Sensitivity: the ratio of the percentage change in the *system TF* to the percentage change in the *process TF (or parameter)*.

TF of the whole system: $T(s) = \dfrac{Y(s)}{R(s)}$



System sensitivity: $S = \dfrac{\Delta T / T}{\Delta G / G} = \dfrac{\partial T / T}{\partial G / G}$

CL:

$$S_G^T = \frac{\partial T/T}{\partial G/G} = \frac{1}{1+GH}$$

CL can reduce the system's sensitivity. Increasing *GH*(*s*) over the frequency range of interest can reduce S_G^T.

$$S_H^T = \frac{\partial T/T}{\partial H/H} = \frac{-GH}{1+GH}$$

Changes in *H*(*s*) directly affect the output.
(Choose appropriate feedback component.)

OL: $S_G^T = 1$

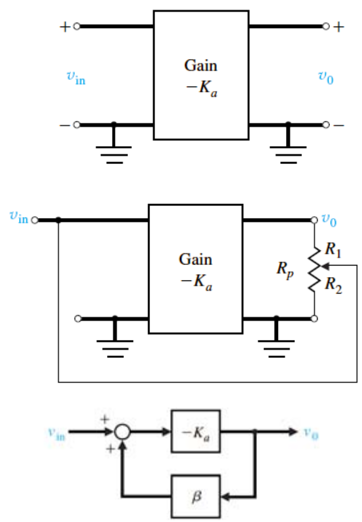
Sensitivity to Parameter Variations

If α is a parameter of G: $\quad S_\alpha^T = S_G^T S_\alpha^G$

If $\quad T(s,\alpha) = \dfrac{N(s,\alpha)}{D(s,\alpha)}$: $\quad S_\alpha^T = S_\alpha^N - S_\alpha^D$

Sensitivity to Parameter Variations

Example 4.1:

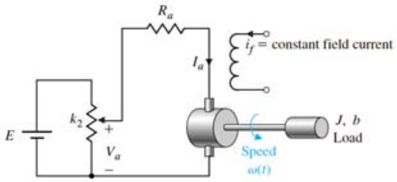


$$v_o = -K_a v_{in}$$
$$T = -K_a$$
$$S_{K_a}^T = 1$$

$$\beta = \frac{R_2}{R_p}$$
$$T = \frac{-K_a}{1+K_a\beta}$$
$$S_{K_a}^T = \frac{1}{1+K_a\beta}$$
$$= \frac{1}{1+10^3}$$
$$(K_a = 10^4, \beta = 0.1)$$

Control of the Transient Response

Example: OL control



TF:

$$G(s) = \frac{\omega(s)}{V_a(s)} = \frac{K_1}{\tau_1 s + 1}$$

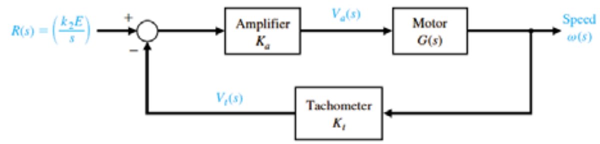
Response to a step command:

$$\omega(t) = K_1 K_2 E (1 - e^{-t/\tau_1})$$

$$K_1 = \frac{K_m}{(R_a b + K_b K_m)}, \quad \tau_1 = \frac{R_a J}{(R_a b + K_b K_m)}$$

Control of the Transient Response

Example: CL control



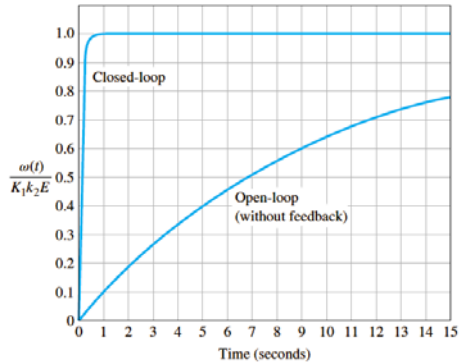
CLTF:

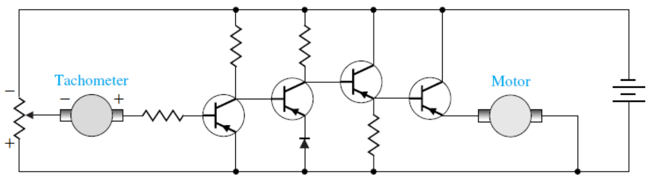
$$\frac{\omega(s)}{R(s)} = \frac{K_a K_1 / \tau_1}{s + (1 + K_a K_t K_1)/\tau_1}$$

Response to a step command:

$$\omega(t) = \frac{K_2 E}{K_t}(1 - e^{-(K_a K_t K_1)t/\tau_1})$$

Control of the Transient Response





Control of the Transient Response

Typically, OL pole: $1/\tau_1 = 0.10$

CL pole: $(K_a K_t K_1)/\tau_1 = 10$

⇒ Large K_a and thus higher-power motor is required. *disadvantage*

Sensitivity: (motor/potentiometer variation) $K_1 = \frac{K_m}{(R_a b + K_b K_m)}$

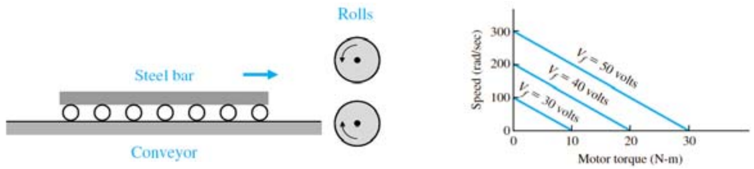
OL: $S^T_{K_m} \approx 1$

CL: $S^T_{K_m} = \frac{1}{1 + GH(s)} = \frac{s + 1/\tau_1}{s + (1 + K_a K_t K_1)/\tau_1}$

$$= \frac{(s + 0.10)}{(s + 10)}$$

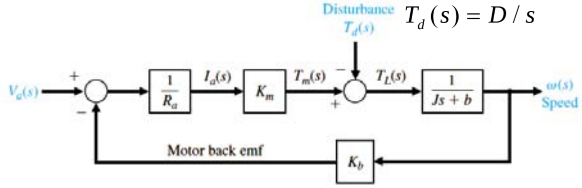
$$|S^T_{K_m}|_{s=j\omega} \approx 0.1$$

Control of Disturbance Effects



OL:

$T_d(s) = D/s$



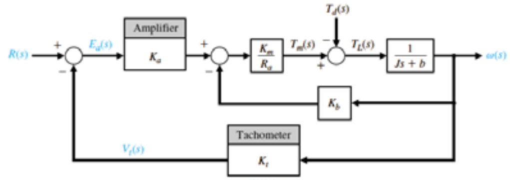
OLTF: $$E(s) = \frac{1}{Js + b + (K_m K_b / R_a)} T_d(s)$$

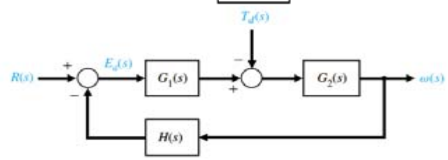
E_{ss}: $$\lim_{t\to\infty} E(t) = \frac{D}{b + (K_m K_b / R_a)}$$

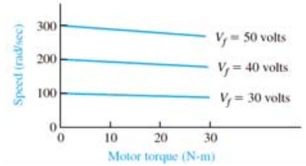
Control of Disturbance Effects

CL:

$T_d(s) = D/s$







$$G_1(s) = K_a K_m / R_a$$

$$G_2(s) = \frac{1}{Js + b}$$

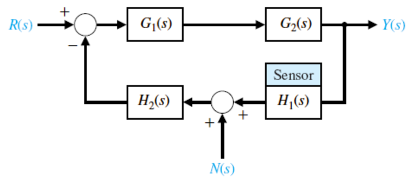
$$E(s) = \frac{1}{G_1(s) H(s)} T_d(s)$$

$$\lim_{t\to\infty} \omega(t) = \frac{-R_a D}{K_a K_m K_t}$$

Compare CL and OL:

$$\frac{\omega_c(\infty)}{\omega_o(\infty)} = \frac{R_a b + K_m K_b}{K_a K_m K_t}$$

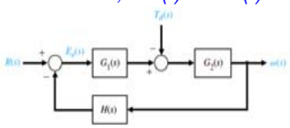
Control of Disturbance Effects



$$Y(s) = \frac{-G_1 G_2 H_2(s)}{1 + G_1 G_2 H_1 H_2(s)} N(s) \simeq \frac{-1}{H_1(s)} N(s) \qquad S_H^T \simeq 1$$

⇒ *H*(*s*) should have minimum noise, drift and parameter variation.
(This is usually possible as the feedback elements operate at low power levels and can be well designed at a reasonable cost.)

Noise at r(t) cannot be alleviated. (S/N not improved.)
But, if n(t) and r(t) are of different frequency characters, S/N can be maximized.



$$S_{G_2}^T = \frac{1}{1 + G_1 G_2 H(s)} \simeq \frac{1}{G_1 G_2 H(s)} \qquad \frac{Y(s)}{T_d(s)} = \frac{1}{1 + GH(s)}$$

Control of Steady-State Error

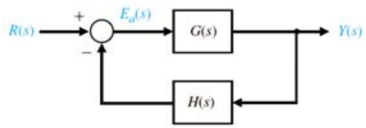
OL:

Unit step response:

$$E_o(s) = (1 - G(s))R(s)$$

$$e_o(\infty) = 1 - G(0)$$

CL:



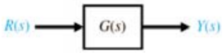
Unit step response, *H*(*s*)=1:

$$E_c(s) = \frac{1}{1 + G(s)} R(s)$$

$$e_c(\infty) = \frac{1}{1 + G(0)}$$

Control of Steady-State Error

OL:



$$G(s) = \frac{K}{\tau s + 1} \qquad \Delta K / K = 0.1$$

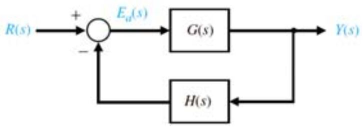
Unit step response:

$$E_o(s) = (1 - G(s))R(s)$$

$$e_o(t) = 1 - G(0) = 1 - K$$

$$\frac{\Delta e_o(\infty)}{|r(t)|} = \frac{0.1}{1}$$

CL:



$$G(s) = \frac{K}{\tau s + 1} \qquad H(s) = \frac{1}{\tau_1 s + 1}$$

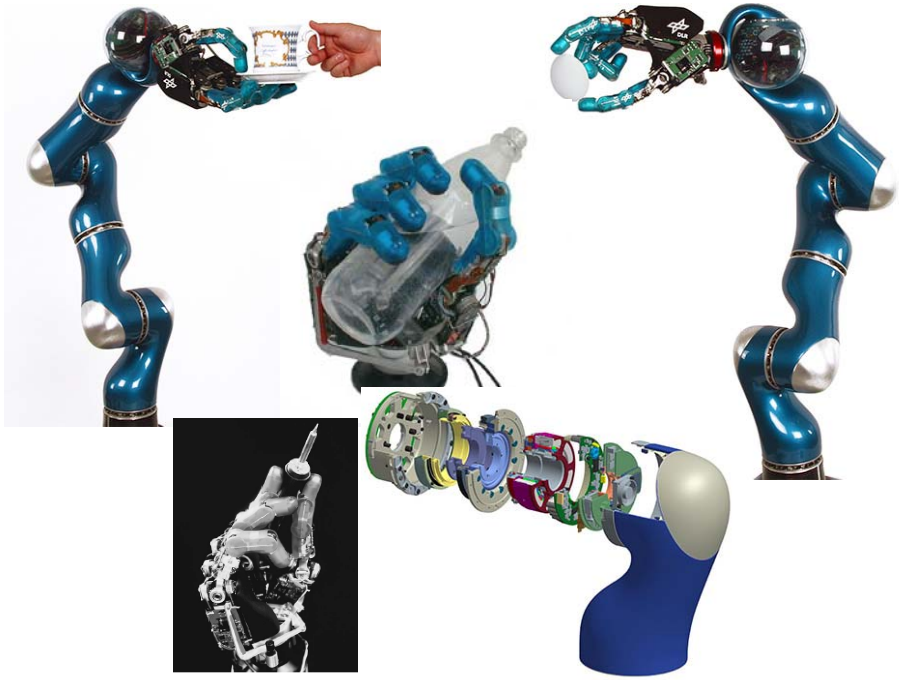
$$\Delta K / K = 0.1$$

Unit step response:

$$E_c(s) = \frac{1}{1 + G(s)} R(s)$$

$$e_c(t) = \frac{1}{1 + G(0)} = \frac{1}{1 + K}$$

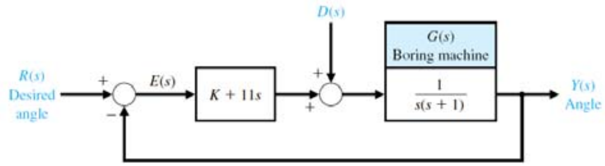
$$\frac{\Delta e_c(\infty)}{|r(t)|} = 0.0011$$



Purpose
Light, universal manipulator for space and service applications
Length
140 cm
Weight
Approximately 22 kg
Vision
Stereo vision
Sensors
Wrist force/torque, joint and motor position, joint torque
External Power
48 V DC, 20 kHz AC
KLOC
Hundreds Person-hours to develop software
Many, many...
Project Status
Complete - next generation hand soon operable, feels good, looks even better
Information Source
Max Fischer

Delicately handling a pretzel, the robotic hand developed at the Deutsches Zentrum fur Luft und Raumfahrt (German Aerospace Center), in the countryside outside Munich, demonstrates the power of a control technique called force-feedback. To pick up an object, Max Fischer, one of the hand's developers, uses the data-glove to transmit the motion of his hand to the robot. If he moves a finger, the robot moves the corresponding finger. Early work on remote-controlled robots foundered when the machines unwittingly crushed the objects they were manipulating. Researchers realized that they were trying to operate robots that didn't have any sense of the force they are exerting - feedback of the type ordinarily given by the nerves in the fingers. Now that the robot is equipped with sensors, it can feed back signals to the data-glove - giving Fischer the sensation of touching the object, and helping him handle it with appropriate delicacy.

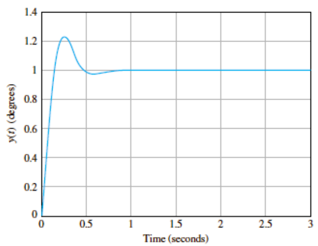
Example: English channel boring machines



Determine *K*, so that command response is desired and min. disturbance effect

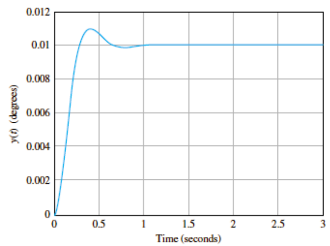
$$Y(s) = T(s)R(s) + T_d(s)D(s)$$

$$= \frac{K + 11s}{s^2 + 12s + K} R(s) + \frac{1}{s^2 + 12s + K} D(s)$$



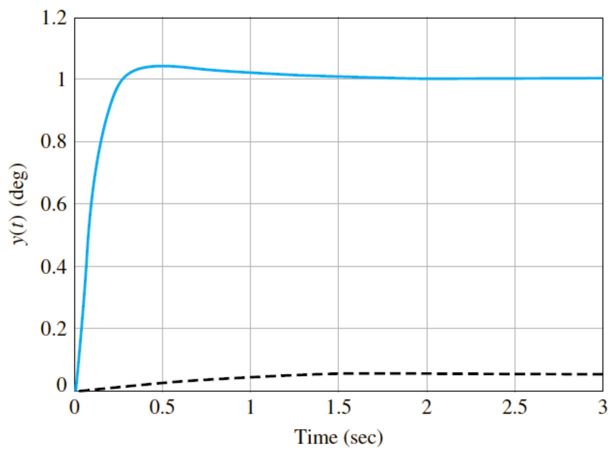
$$K = 100$$

$$d(t) = 0 \qquad r(t) = 0$$



Example: English channel boring machines

$K = 20$



Performance comparison for K=20 and K=100 is given in Table 4.1.

Example: English channel boring machines

Unit step command:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s \frac{1}{1+\dfrac{K+11s}{s(s+1)}} \frac{1}{s} = 0$$

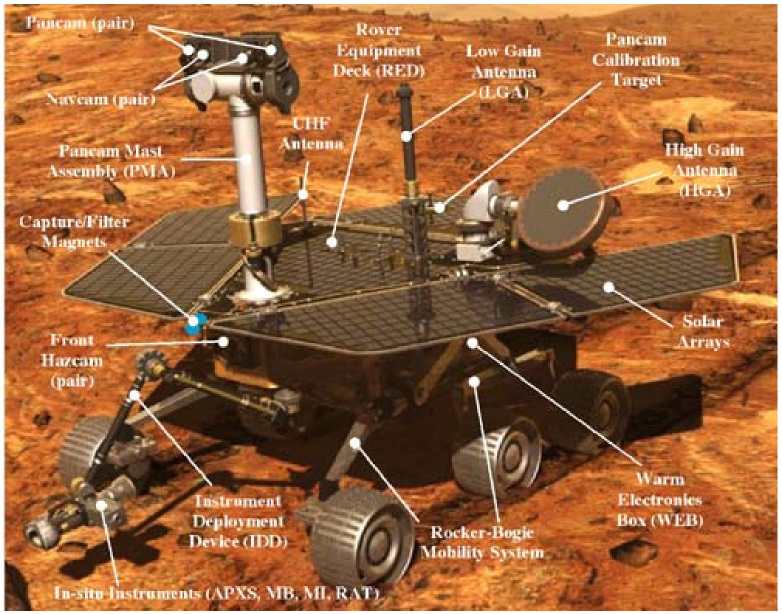
Unit step disturbance:

$$e_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0}\left[\frac{1}{s(s+12)+K}\right] = \frac{1}{K}$$

Sensitivity:

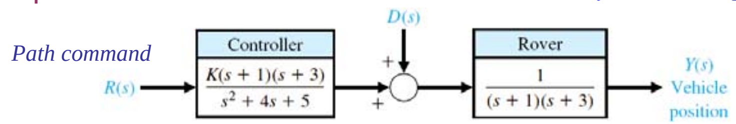
$$S_G^T = \frac{s(s+1)}{s(s+12)+K} \approx \frac{s}{K}$$

Example: Mars Rover Vehicle



Example: Mars Rover Vehicle

Disturbance, sensitivity → Find an appropriate K

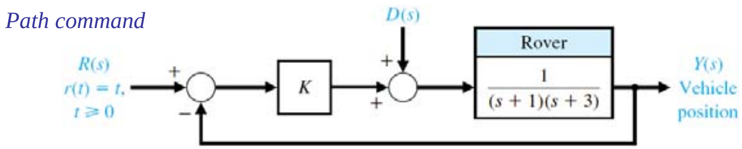


$K = 2$

$$T_o(s) = \frac{K}{s^2+4s+5} \overset{\Downarrow}{=} \frac{2}{s^2+4s+5}$$

Sensitivity:

$$S_K^{T_o} = \frac{dT_o}{dK}\frac{K}{T} = 1$$

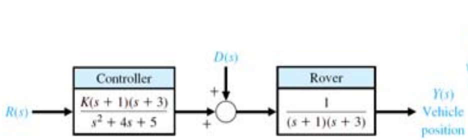


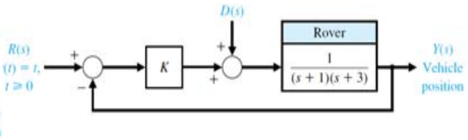
$K = 2$

$$T_c(s) = \frac{K}{s^2+4s+3+K} \overset{\Downarrow}{=} \frac{2}{s^2+4s+5}$$

Sensitivity:

$$S_K^{T_c} = \frac{s^2+4s+3}{s^2+4s+3+K} = \frac{(3-\omega^2)+j4\omega}{(3+K-\omega^2)+j4\omega}$$





$$T_o(s) = \frac{K}{s^2+4s+5} = \frac{2}{s^2+4s+5}$$

Sensitivity:

$$S_K^{T_o} = \frac{dT_o}{dK}\frac{K}{T} = 1$$

Unit disturbance effect ($r(t)$=0, $d(t)$=1):

$$y(\infty) = \lim_{s\to 0} s\left\{\frac{1}{(s+1)(s+3)}\right\}\frac{1}{s} = \frac{1}{3}$$

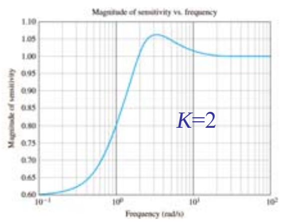
$$T_c(s) = \frac{K}{s^2+4s+3+K} = \frac{2}{s^2+4s+5}$$

Sensitivity:

$$S_K^{T_c} = \frac{s^2+4s+3}{s^2+4s+3+K} = \frac{(3-\omega^2)+j4\omega}{(3+K-\omega^2)+j4\omega}$$

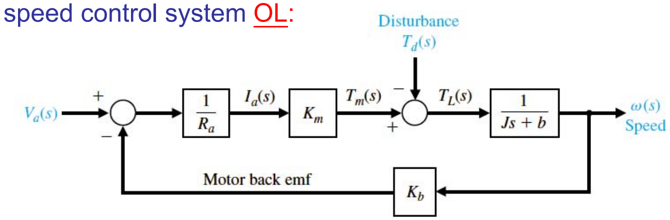
Unit disturbance effect ($r(t)$=0, $d(t)$=1):

$$y(\infty) = \lim_{s\to 0} s\left\{\frac{1}{(s^2+4s+3+K)}\right\}\frac{1}{s} = \frac{1}{3+K}$$



Control System Characteristics Using Matlab

Motor speed control system OL:



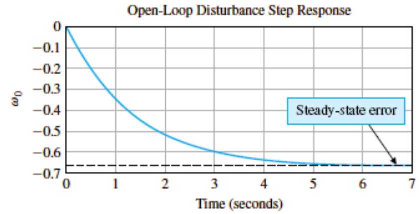
opentach.m

```
%Speed Tachometer Example
%
Ra=1; Km=10; J=2; f=0.5; Kb=0.1;
num1=[1]; den1=[J,b]; sys1=tf(num1,den1);
num2=[Km*Kb/Ra]; den2=[1]; sys2=tf(num2,den2);
sys_o=feedback(sys1,sys2);
%
sys_o=-sys_o
%
[yo,T]=step(sys_o);
plot(T,yo)
title('Open-Loop Disturbance Step Response')
xlabel('Time (seconds)'),ylabel('\omega_o'), grid
%
yo(length(T))
```

Change sign of transfer function since the disturbance has negative sign in the diagram.

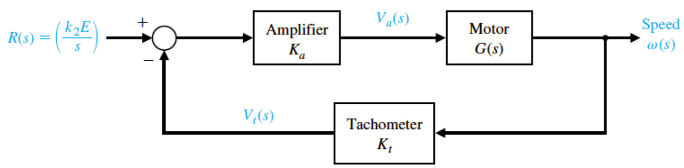
Compute response to step disturbance.

Steady-state error ⟶ last value of output yo.



Control system Characteristics using Matlab

Motor speed control system CL:



closedtach.m

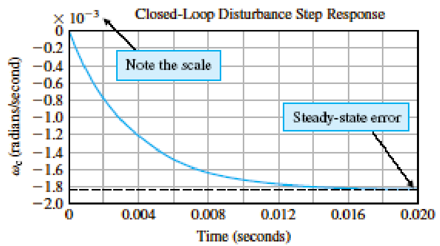
```
%Speed Tachometer Example
%
Ra=1;  Km=10; J=2; b=0.5; Kb=0.1; Ka=54; Kt=1;
num1=[1]; den1=[J,b]; sys1=tf(num1,den1);
num2=[Ka*Kt]; den2=[1]; sys2=tf(num2,den2);
num3=[Kb]; den3=[1]; sys3=tf(num3,den3);
num4=[Km/Ra]; den4=[1]; sys4=tf(num4,den4);
sysa=parallel(sys2,sys3);
sysb=series(sysa,sys4);
sys_c=feedback(sys1,sysb);
%
sys_c=-sys_c
%
[yc,T]=step(sys_c);
plot(T,yc)
title('Closed-Loop Disturbance Step Response')
xlabel('Time (seconds)'), ylabel('\omega_c (radians/second)'), grid
%
yc(length(T))
```

Block diagram reduction

Change sign of transfer function since the disturbance has negative sign in the diagram.

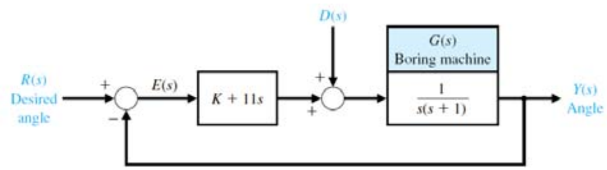
Compute response to step disturbance.

Steady-state error ⟶ last value of output yc.



Control system Characteristics using Matlab

Example: English channel boring machines



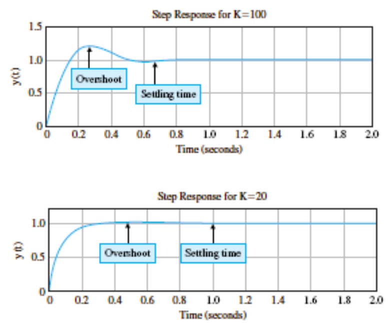
english1.m

```
% Response to a Unit Step Input R(s)=1/s for K=20 and K=100
%
numg=[1]; deng=[1 1 0]; sysg=tf(numg,deng);
K1=100; K2=20;
num1=[11 K1]; num2=[11 K2]; den=[0 1];
sys1=tf(num1,den);
sys2=tf(num2,den);
%
sysa=series(sys1,sysg); sysb=series(sys2,sysg);
sysc=feedback(sysa,[1]); sysd=feedback(sysb,[1]);
%
t=[0:0.01:2.0];
[y1,t]=step(sysc,t); [y2,t]=step(sysd,t);
subplot(211),plot(t,y1), title('Step Response for K=100')
xlabel('Time (seconds)'),ylabel('y(t)'), grid
subplot(212),plot(t,y2), title('Step Response for K=20')
xlabel('Time (seconds)'),ylabel('y(t)'), grid
```

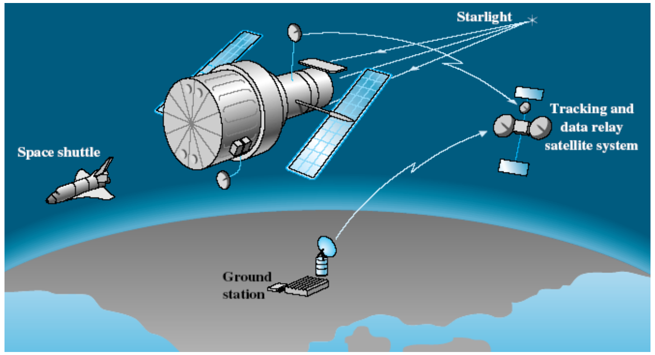
Closed-loop transfer functions.

Choose time interval.

Create subplots with x and y axis labels.



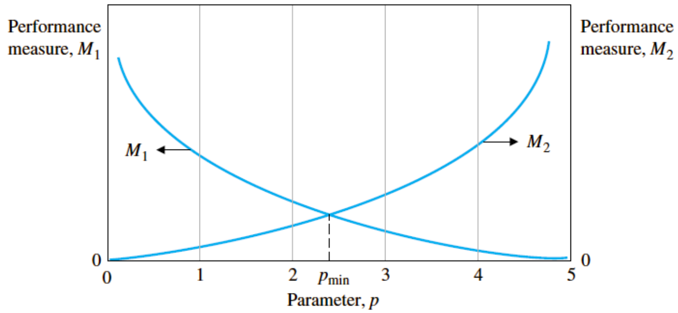
Performance of Feedback Control Systems



Introduction

Feedback control: the transient and steady-state performance can be adjusted. (*To what…?*)

Design Specifications (*include many but end with a compromise*)

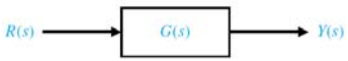


Performance Measures (time-domain performance)

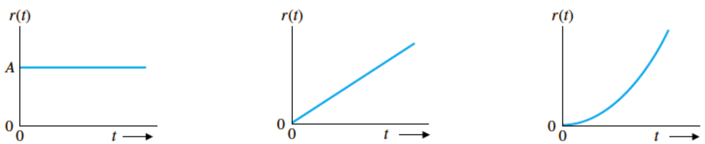
Test Input Signals

System responses to some specific input signals

(*Performance Measures*)



Standard Test Input Signals



Expressions in time-domain and s-domain

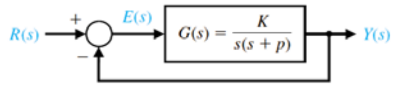
General form:

$$r(t) = t^n$$

$$R(s) = \frac{n!}{s^{n+1}}$$

2-Order Prototype System Performances

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



Damping ratio: ζ

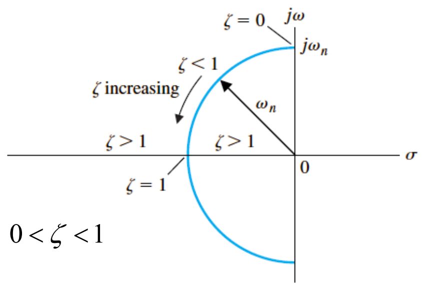
Natural frequency: ω_n

$$Y(s) = \frac{K}{s^2 + ps + K} R(s)$$

Zeros and Poles:

No zero

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

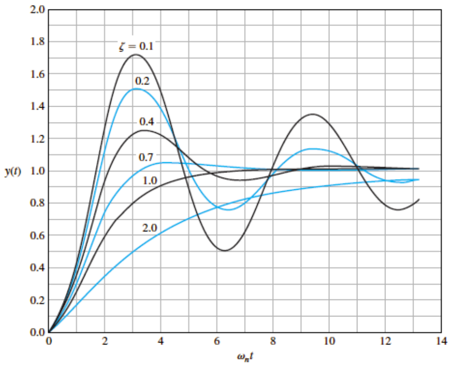


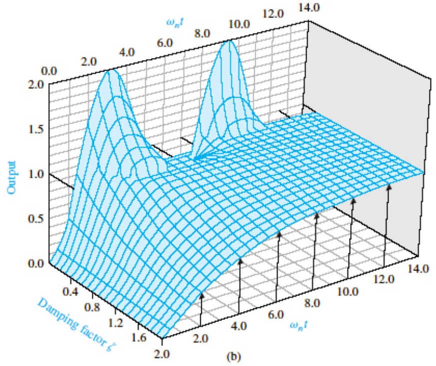
Unit Step Response:

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta), \quad 0 < \zeta < 1$$

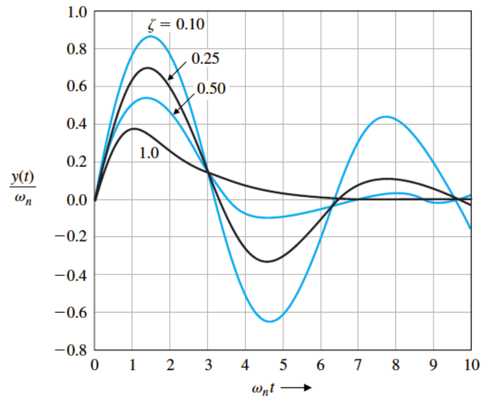
$$\beta = \sqrt{1-\zeta^2} \qquad \theta = \cos^{-1}\zeta$$

2-Order Prototype System Performances





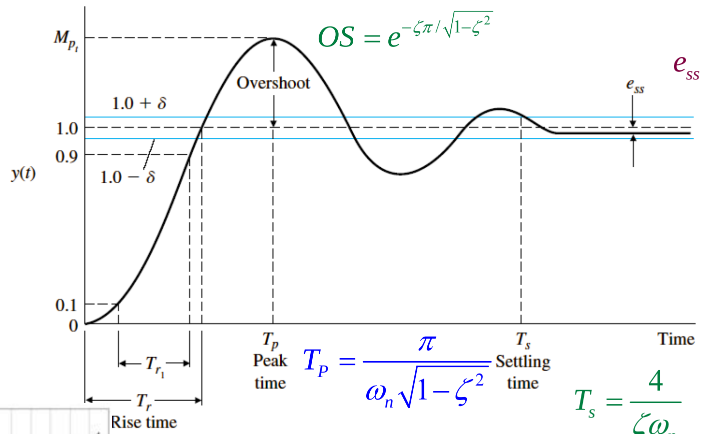
2-Order Prototype System Performances



$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (0 < \zeta < 1)$$

Impulse response: $y(t) = \dfrac{\omega_n}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t)$

2-Order Prototype System Performances



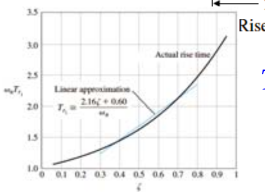
$$OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

e_{ss}

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$T_s = \frac{4}{\zeta\omega_n}$$

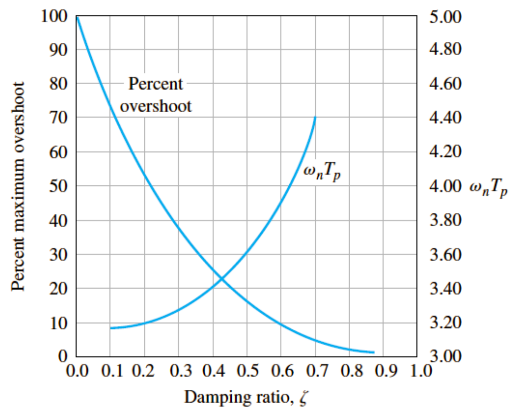
(δ = 2%)



$$T_{r_1} = \frac{2.16\zeta + 0.60}{\omega_n}$$

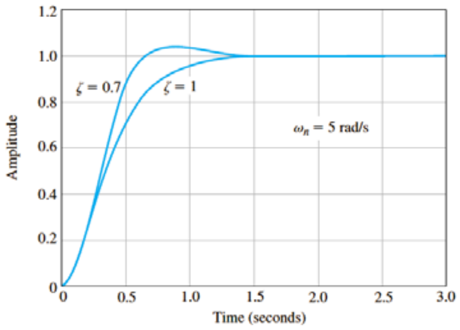
$(0.3 \le \zeta \le 0.8)$

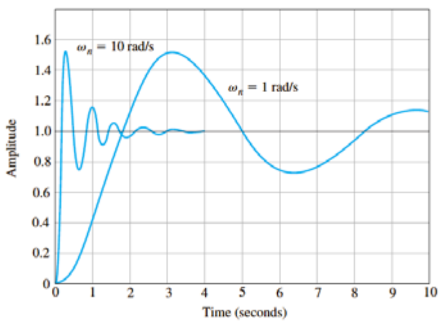
2-Order Prototype System Performances



$\zeta \uparrow \;\Rightarrow\; (\text{OS} \downarrow$ but $T_p, T_{r_1} \uparrow)$

2-Order Prototype System Performances





$\zeta \downarrow \;\Rightarrow\; (T_p, T_{r_1} \downarrow$ but OS $\uparrow)$

$(\omega_n$ fixed $)$

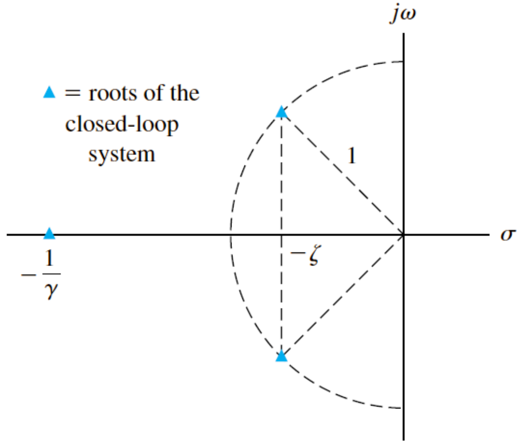
$\omega_n \uparrow \;\Rightarrow\; (T_p, T_{r_1} \downarrow)$

$(\zeta$ fixed $)$

OS is independent of ω_n

A Third Pole

$$T(s) = \frac{1}{(s^2 + 2\zeta s + 1)(\gamma s + 1)} \qquad (\omega_n = 1)$$



A Third Pole

Table 5.3

Table 5.3 Effect of a Third Pole (Equation 5.18) for $\zeta = 0.45$

γ	$\frac{1}{\gamma}$	Percent Overshoot	Settling Time*
2.25	0.444	0	9.63
1.5	0.666	3.9	6.3
0.9	1.111	12.3	8.81
0.4	2.50	18.6	8.67
0.05	20.0	20.5	8.37
0	∞	20.5	8.24

*Note: Settling time is normalized time, $\omega_n T_s$, and utilizes a 2% criterion.

If $\quad |1/\gamma| \geq 10 |\zeta\omega_n|$

Then

3-order system can be approximated by 2-order system with the dominant roots.

A Zero

$$T(s) = \frac{(\omega_n^2 / a)(s+a)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

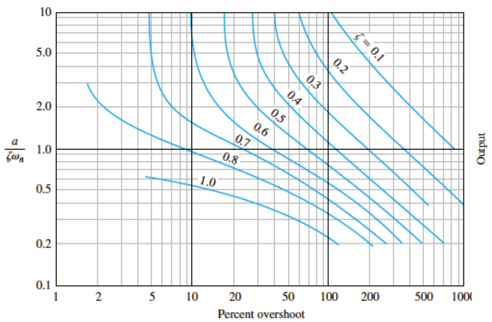
Table 5.4

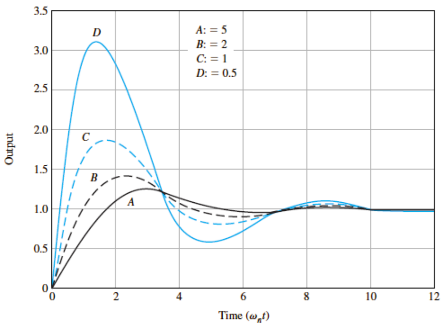
Table 5.4 The Response of a Second-Order System with a Zero and $\zeta = 0.45$

$a/\zeta\omega_n$	Percent Overshoot	Settling Time	Peak Time
5	23.1	8.0	3.0
2	39.7	7.6	2.2
1	89.9	10.1	1.8
0.5	210.0	10.3	1.5

If the zero is near the dominant poles, then it will *materially* affect system transient response.

A Zero

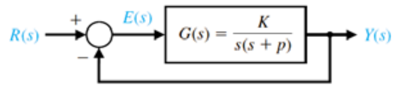




If the zero is near the dominant poles, then it will *materially* affect system transient response.

Transient Response and Zero/Pole Locations: Applications

Example 5.1



Select K and p, to satisfy time-domain specifications:

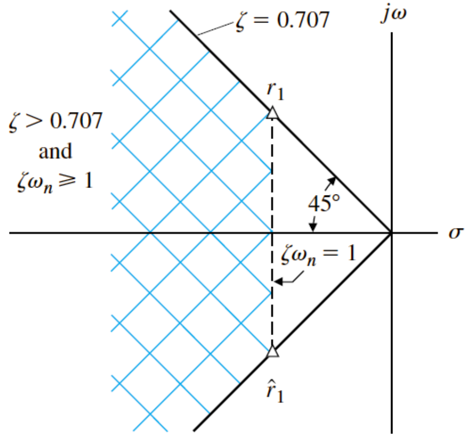
$$T_s = \frac{4}{\zeta\omega_n} \le 4(\text{sec}) \quad (\delta = 2\%)$$

$$OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \le 5\%$$

$$\Rightarrow \zeta > 0.707,\ \zeta\omega_n \ge 1$$

$$\Rightarrow r_{1,2} = -1 \pm j$$

$$\Rightarrow K = 2,\ p = 2$$



Transient Response and Zero/Pole Locations: Applications

$$T(s) = \frac{(\omega_n^2/a)(s+a)}{(s^2 + 2\zeta\omega_n s + \omega_n^2)(1+\tau s)}$$

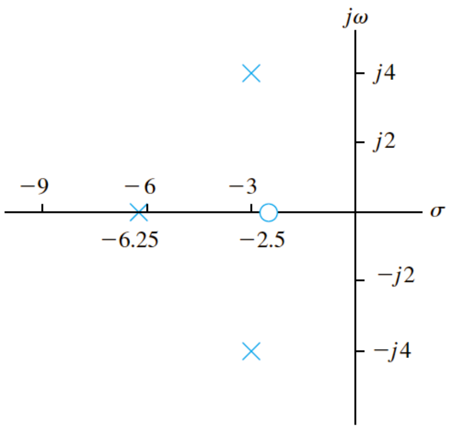
Example 5.2

$$T(s) = \frac{62.5(s+2.5)}{(s^2+6s+25)(s+6.25)}$$

$T_s = 1.6(\text{sec}),\ OS = 38\%$

$$T(s) = \frac{10(s+2.5)}{(s^2+6s+25)}$$

$T_s = 1.33(\text{sec}),\ OS = 55\%$



Estimation of ζ

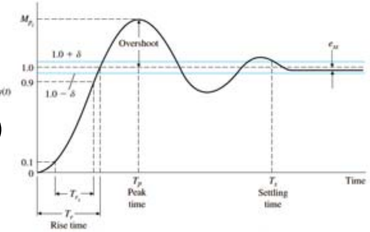
1. $\zeta = \dfrac{0.55}{\textit{cycles visible}}$ $\quad (0.2 \le \zeta \le 0.6)$

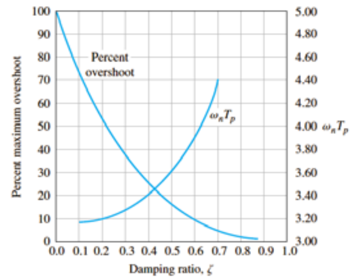
$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)$$

Cycles in τ **=** $\dfrac{\omega_n\beta}{2\pi\zeta\omega_n} = \dfrac{\beta}{2\pi\zeta}$

Cycle visible = $\dfrac{4\beta}{2\pi\zeta} = \dfrac{4(1-\zeta^2)^{1/2}}{2\pi\zeta}$

2. Use Figure 5.8





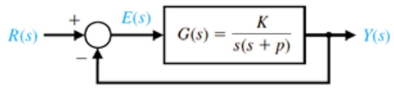
Transient Response and Zero/Pole Locations

$$T(s)=\frac{Y(s)}{R(s)}=\frac{\sum P_i(s)\Delta_i(s)}{\Delta(s)}$$

CL Poles:

Roots of $\Delta(s)=0$

Poles of $\sum P_i(s)\Delta_i(s)$



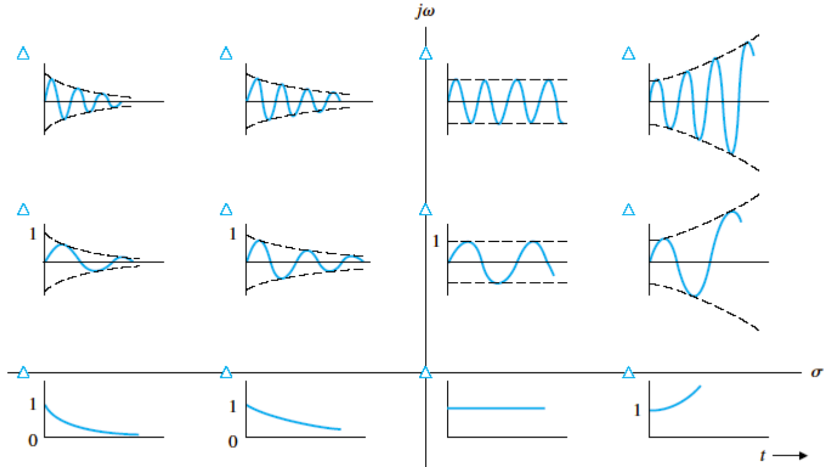
Unit step response (unity gain & no repeated roots)

$$Y(s)=\frac{1}{s}+\sum_{i=1}^{M}\frac{A_i}{s+\sigma_i}+\sum_{k=1}^{N}\frac{B_k s+C_k}{s^2+2\alpha_k s+(\alpha_k^2+\omega_k^2)}$$

$$y(t)=1+\sum_{i=1}^{M}A_i e^{-\sigma_i t}+\sum_{k=1}^{N}D_k e^{-\alpha_k t}\sin(\omega_k t+\theta_k)$$

constant **exponential terms** **damped sinusoidal terms**

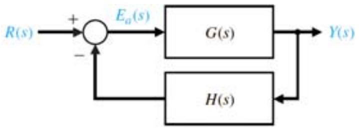
Transient Response and Zero/Pole Locations



Steady-State Error of Feedback Control Systems

$$E(s)=\frac{1}{1+G(s)}R(s)$$

$$e_{ss}=\lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$



$H(s)=1$

Unity feedback control system

Standard Test Input Signals (general form)

$$r(t)=t^n,\quad R(s)=n!/s^{n+1}$$

Loop Transfer Function

$$G(s)=\frac{K\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+p_k)}$$

Steady-State Error of Feedback Control Systems

$$e_{ss}=\lim_{s\to 0}\frac{n!}{s^n\left(1+\dfrac{K\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+p_k)}\right)}=\begin{cases}\dfrac{A}{1+K\prod_{i=1}^{M}z_i/\prod_{k=1}^{Q}p_k} & \textit{if } N=n\\ \infty & \textit{if } N=0\ \&\ n>0\\ \lim_{s\to 0}\dfrac{s^N}{s^n}=\begin{cases}0 & \textit{if } N>n\\ \infty & \textit{if } N<n\end{cases}\end{cases}$$

if N>0 & N≠n

System Type Number: *N*

(number of integrations)

Steady-State Error of Feedback Control Systems

$R(s) = A/s$

1. Step input

$$e_{ss} = \lim_{s\to 0} \frac{s(A/s)}{1+G(s)} = \frac{A}{1+G(0)} = \begin{cases} \dfrac{A}{1+K\prod_{i=1}^{M} z_i / \prod_{k=1}^{Q} p_k} & \textit{if } N=0 \text{ (Type 0)} \\ 0 & \textit{if } N>0 \text{ (Type 1,2,...)} \end{cases}$$

Position Error Constant: (if N=0)

$$K_p = \lim_{s\to 0} G(s) = G(0)$$

$$e_{ss} = \frac{A}{1+K_p}$$

Steady-State Error of Feedback Control Systems

$R(s) = A/s^2$

2. Ramp input

$$e_{ss} = \lim_{s\to 0} \frac{s(A/s^2)}{1+G(s)} = \lim_{s\to 0} \frac{A}{s+sG(s)} = \lim_{s\to 0} \frac{A}{sG(s)} = \begin{cases} \infty & \textit{if } N<1 \text{ (Type 0)} \\ \dfrac{A}{K_v} & \textit{if } N=1 \text{ (Type 1)} \\ 0 & \textit{if } N>1 \text{ (Type 2,3...)} \end{cases}$$

Velocity Error Constant: (if N=1)

$$K_v = \lim_{s\to 0} sG(s)$$

$$e_{ss} = \frac{A}{K_v}$$

Steady-State Error of Feedback Control Systems

$R(s) = A/s^3$

3. Acceleration input

$$e_{ss} = \lim_{s\to 0} \frac{s(A/s^3)}{1+G(s)} = \lim_{s\to 0} \frac{A}{s^2G(s)} = \begin{cases} \infty & \textit{if } N<2 \text{ (Type 0 \& 1)} \\ \dfrac{A}{K_a} & \textit{if } N=2 \text{ (Type 2)} \\ 0 & \textit{if } N>2 \text{ (Type 3,4,...)} \end{cases}$$

Acceleration Error Constant: (if N=2)

$$K_a = \lim_{s\to 0} s^2G(s)$$

$$e_{ss} = \frac{A}{K_a}$$

Steady-State Error of Feedback Control Systems

Table 5.5

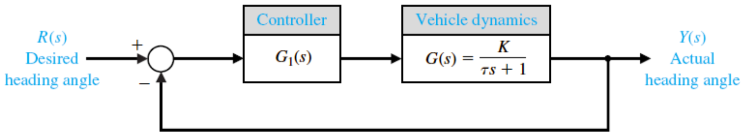
Table 5.5 Summary of Steady-State Errors

Number of Integrations in $G(s)$, Type Number	Input: Step, $r(t) = A$, $R(s) = A/s$	Ramp, At, A/s^2	Parabola, $At^2/2$, A/s^3
0	$e_{ss} = \dfrac{A}{1+K_p}$	Infinite	Infinite
1	$e_{ss} = 0$	$\dfrac{A}{K_v}$	Infinite
2	$e_{ss} = 0$	0	$\dfrac{A}{K_a}$

Error constants represent the ability of a system to reduce/eliminate e_{ss}.

Steady-State Error of Feedback Control Systems

Example 5.3



$$G_1(s) = K_1 + \frac{K_2}{s} \quad \text{(P+I controller)}$$

If $G_1(s) = K_1$ (P controller)

$\Rightarrow$ Type 0 CL system $\Rightarrow$ $e_{ss} = \dfrac{A}{1+K_p},\ K_p = KK_1$ for a step input

If $G_1(s) = K_1 + K_2 s$ (PI controller)

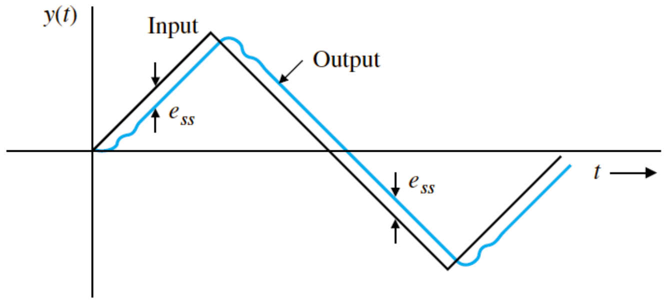
$\Rightarrow$ Type 1 CL system $\Rightarrow$ $e_{ss} = 0$ for a step input

Steady-State Error of Feedback Control Systems

Example 5.3

If $G_1(s) = K_1 + K_2 s$ (PI controller) for a ramp input

$\Rightarrow$ Type 1 CL system $\Rightarrow$ $e_{ss} = \dfrac{A}{K_v},\ K_v = K_2 K$

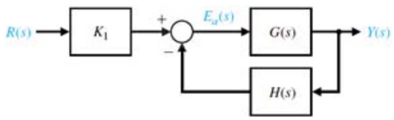


velocity meets the requirement

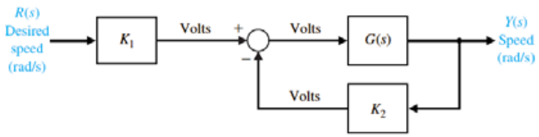
Nonunity Feedback Control Systems

How to calculate e_{ss} for nonunity feedback?

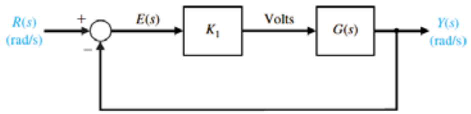
1. equivalent transformation



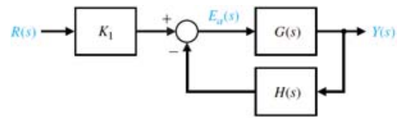
$\Downarrow$ $H(s) = K_2$



$\Downarrow$ $K_1 = K_2$

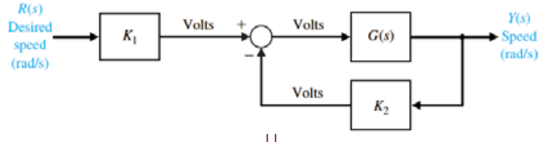


Nonunity Feedback Control Systems

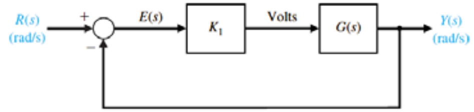


$\Downarrow$ $H(s) = \dfrac{K_2}{\tau s + 1},\ \lim_{s\to 0} H(s) = K_2$

(for DC gain calculation)



$\Downarrow$ $K_1 = K_2$



For a step input

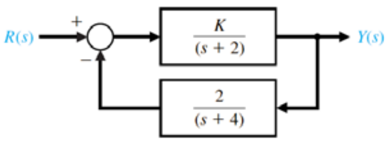
$$e_{ss} = \frac{1}{1 + K_1 G(0)}$$

Nonunity Feedback Control Systems

How to calculate e_{ss} for nonunity feedback?

2. Direct calculation

Example 5.5



$$K = ? \;\Rightarrow\; e_{ss} = 0$$

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K(s+4)}{(s+2)(s+4)+K}$$

For a step input $\quad e_{ss} = 1 - T(0)$

$$e_{ss} = 0 \;\Rightarrow\; T(0) = \frac{4K}{2K+8} = 1 \;\Rightarrow\; K = 4$$

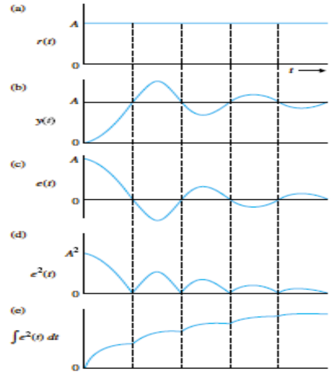
Performance Indices

Performance index:

A quantitative measure of the system performance.

Optimal Control:

Adjust system parameters to min (max) the performance index.



ISE $= \int_0^T e^2(t)dt$

IAE $= \int_0^T |e(t)|\,dt$

ITSE $= \int_0^T t e^2(t)dt$

ITAE $= \int_0^T t\,|e(t)|\,dt$

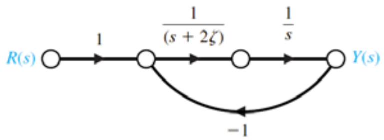
$$I = \int_0^T f(e(t), r(t), y(t), t)dt$$

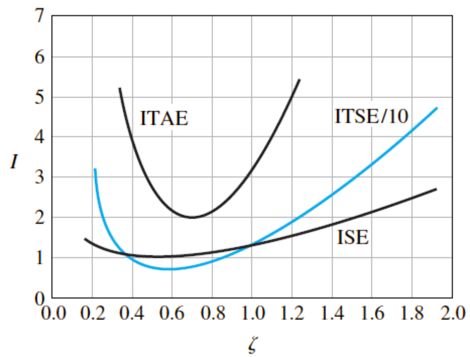
Performance Indices

Example 5.6

CLTF:

$$T(s) = \frac{1}{s^2 + 2\zeta s + 1}$$

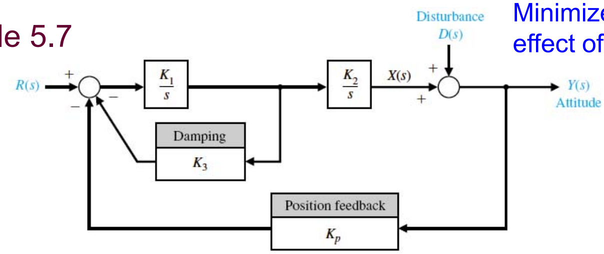




Performance Indices

Example 5.7

Minimize the effect of $D(s)$



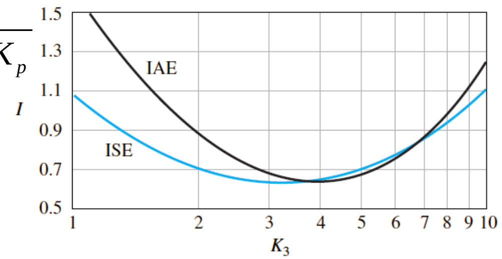
CLTF:

$$\frac{Y(s)}{D(s)} = \frac{s(s + K_1K_3)}{s^2 + K_1K_3 s + K_1K_2K_p}$$

Analytically calculate for

min(ISE): $\quad K_3 = 3.2$

min(IAE): $\quad K_3 = 4.2$



Performance Indices

$$T(s)=\frac{Y(s)}{R(s)}=\frac{b_0}{s^n+b_{n-1}s^{n-1}+\cdots+b_1 s+b_0}$$

Table 5.6

Table 5.6 The Optimum Coefficients of $T(s)$ Based on the ITAE Criterion for a Step Input

$s+\omega_n$
$s^2+1.4\omega_n s+\omega_n^2$
$s^3+1.75\omega_n s^2+2.15\omega_n^2 s+\omega_n^3$
$s^4+2.1\omega_n s^3+3.4\omega_n^2 s^2+2.7\omega_n^3 s+\omega_n^4$
$s^5+2.8\omega_n s^4+5.0\omega_n^2 s^3+5.5\omega_n^3 s^2+3.4\omega_n^4 s+\omega_n^5$
$s^6+3.25\omega_n s^5+6.60\omega_n^2 s^4+8.60\omega_n^3 s^3+7.45\omega_n^4 s^2+3.95\omega_n^5 s+\omega_n^6$

Table 5.7

Table 5.7 The Optimum Coefficients of $T(s)$ Based on the ITAE Criterion for a Ramp Input

$s^2+3.2\omega_n s+\omega_n^2$
$s^3+1.75\omega_n s^2+3.25\omega_n^2 s+\omega_n^3$
$s^4+2.41\omega_n s^3+4.93\omega_n^2 s^2+5.14\omega_n^3 s+\omega_n^4$
$s^5+2.19\omega_n s^4+6.50\omega_n^2 s^3+6.30\omega_n^3 s^2+5.24\omega_n^4 s+\omega_n^5$

System Simplification

Idea: to approximate a *high-order* system using a *lower-order* system, with a *minimun change* in the *system property of interest*.

1. Simple method: delete insignificant poles

$$G(s)=\frac{K}{s(s+2)(s+30)} \qquad G(s)=\frac{K/30}{s(s+2)}$$

2. Complicated & powerful method: to approximate a high-order system using a lower-order system, with frequency responses match as closely as possible.

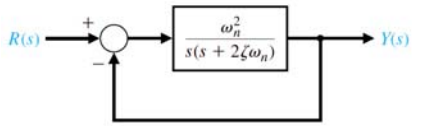
$$H(s)=K\frac{a_m s^m+a_{m-1}s^{m-1}+\cdots+a_1 s+1}{b_n s^n+b_{n-1}s^{n-1}+\cdots+b_1 s+1} \qquad \frac{H(s)}{L(s)}=\frac{M(s)}{\Delta(s)}\approx 1$$

Within the frequency range of interest

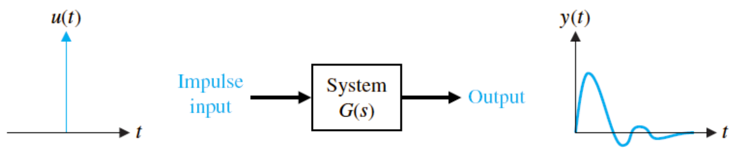
$$L(s)=K\frac{c_p s^p+c_{p-1}s^{p-1}+\cdots+c_1 s+1}{d_g s^g+d_{g-1}s^{g-1}+\cdots+d_1 s+1}$$

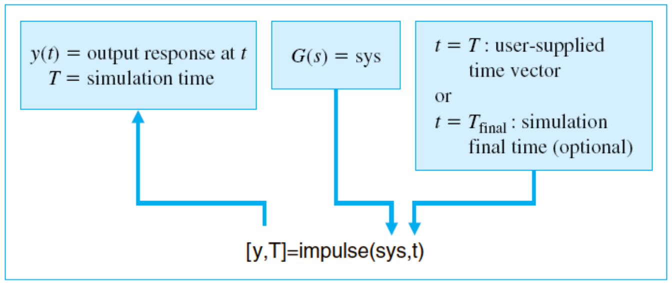
$$M_{2q}=\sum_{k=0}^{2q}\frac{(-1)^{k+q}M^{(k)}(0)M^{(2q-k)}(0)}{k!(2q-k)!}\underset{q=1,2,\ldots}{=}\Delta_{2q}=\sum_{k=0}^{2q}\frac{(-1)^{k+q}\Delta^{(k)}(0)\Delta^{(2q-k)}(0)}{k!(2q-k)!}$$

System Performance Using Matlab



$$Y(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}R(s)$$





System Performance Using Matlab

impresp.m

```
%Compute impulse response for a second-order system
%Duplicate Figure 5.6
%
t=[0:0.1:10]; num=[1];
zeta1=0.1; den1=[1 2*zeta1 1]; sys1=tf(num,den1);
zeta2=0.25; den2=[1 2*zeta2 1]; sys2=tf(num,den2);
zeta3=0.5; den3=[1 2*zeta3 1]; sys3=tf(num,den3);
zeta4=1.0; den4=[1 2*zeta4 1]; sys4=tf(num,den4);
%
[y1,T1]=impulse(sys1,t);
[y2,T2]=impulse(sys2,t);
[y3,T3]=impulse(sys3,t);
[y4,T4]=impulse(sys4,t);
%
plot(t,y1,t,y2,t,y3,t,y4)
xlabel(' \omega _nt'), ylabel('y(t)/\omega_n')
title('\zeta = 0.1, 0.25, 0.5, 1.0'), grid
```

Compute impulse response.

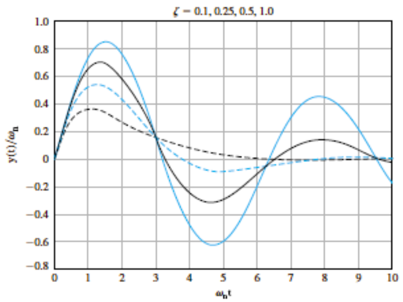
Generate plot and labels.

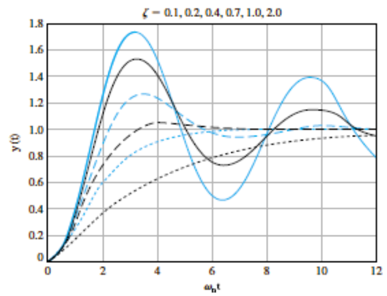
stepresp.m

```
%Compute step response for a second-order system
%Duplicate Figure 5.5 (a)
%
t=[0:0.1:12]; num=[1];
zeta1=0.1; den1=[1 2*zeta1 1]; sys1=tf(num,den1);
zeta2=0.2; den2=[1 2*zeta2 1]; sys2=tf(num,den2);
zeta3=0.4; den3=[1 2*zeta3 1]; sys3=tf(num,den3);
zeta4=0.7; den4=[1 2*zeta4 1]; sys4=tf(num,den4);
zeta5=1.0; den5=[1 2*zeta5 1]; sys5=tf(num,den5);
zeta6=2.0; den6=[1 2*zeta6 1]; sys6=tf(num,den6);
%
[y1,T1]=step(sys1,t); [y2,T2]=step(sys2,t);
[y3,T3]=step(sys3,t); [y4,T4]=step(sys4,t);
[y5,T5]=step(sys5,t); [y6,T6]=step(sys6,t);
%
plot(T1,y1,T2,y2,T3,y3,T4,y4,T5,y5,T6,y6)
xlabel(' \omega_n t'), ylabel('y(t)')
title('\zeta = 0.1, 0.2, 0.4, 0.7, 1.0, 2.0'), grid
```

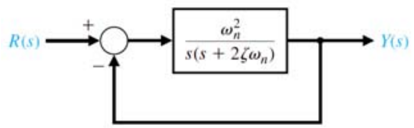
Compute step response.

Generate plot and labels.

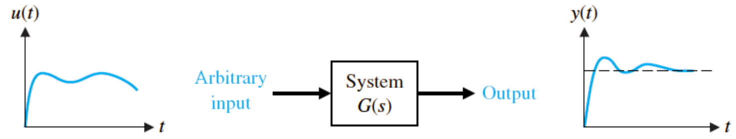


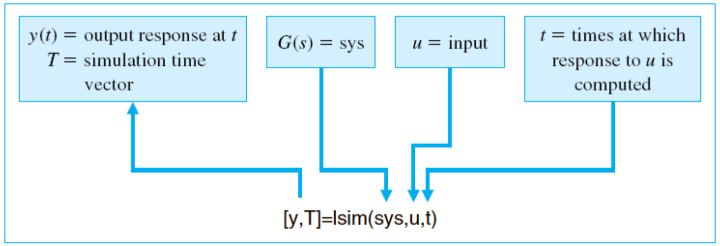


System Performance Using Matlab



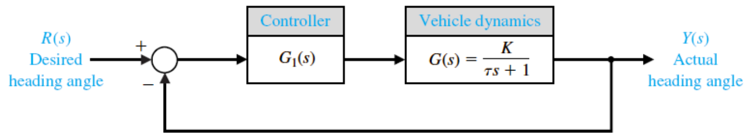
$$Y(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}R(s)$$





System Performance Using Matlab

Example 5.10 Robot steering control



$$G_1(s)=K_1+\frac{K_2}{s}$$ (P+I controller)

$$K_1=K=1, K_2=2, \tau=1/10$$

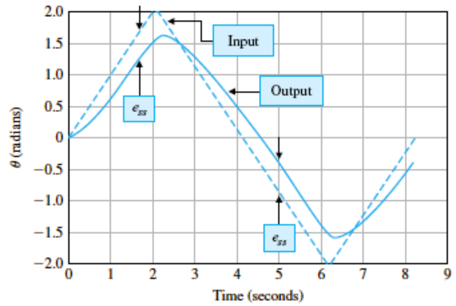
mobilerobot.m

```
%Compute the response of the Mobile Robot Control
%System to a triangular wave input
%
numg=[10 20]; deng=[1 10 0]; sysg=tf(numg,deng);
[sys]=feedback(sysg, [1]);
t=[0:0.1:8.2]';
v1=[0:0.1:2]';v2=[2:-0.1:-2]';v3=[-2:0.1:0]';
u=[v1;v2;v3];
[y,T]=lsim(sys,u,t);
plot(T,y,t,u,'--'),
xlabel('Time (seconds)'), ylabel('\theta (radians)'), grid
```

$GG_1(s)$

Compute triangular wave input.

Linear simulation.



System Performance Using Matlab

Example 5.11 A simplified model

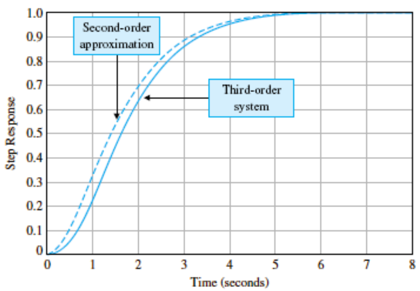
$$H(s)=\frac{6}{s^3+6s^2+11s+6} \qquad L(s)=\frac{1.60}{s^2+2.584s+1.60}$$

stepcomp.m

```
% Compare step response for second-order approximation
%
num1=[6]; den1=[1 6 11 6]; sys1=tf(num1,den1);
num2=[1.6]; den2=[1 2.584 1.6]; sys2=(tf(num2,den2);
t=[0:0.1:8];
[y1,T1]=step(sys1,t);
[y2,T2]=step(sys2,t);
plot(T1,y1,T2,y2,'--'), grid
xlabel('Time (seconds)'), ylabel('Step Response')
```

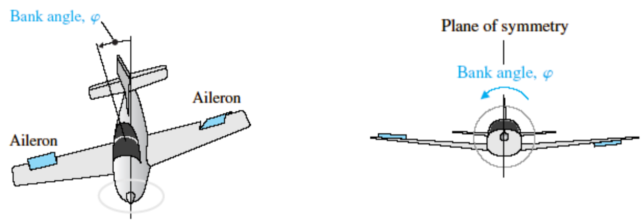
$H(s) = \frac{6}{s^3+6s^2+11s+6}$

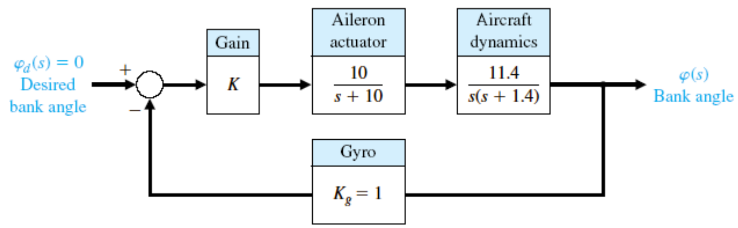
$L(s) = \frac{1.6}{s^2+2.584s+1.6}$

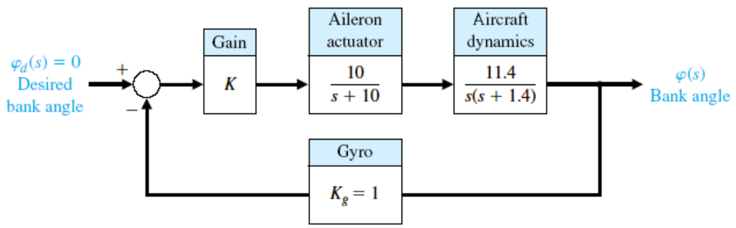


System Performance Using Simulink

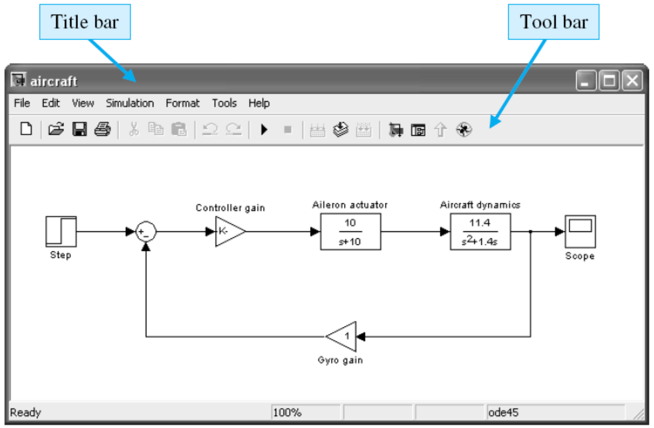
Example 5.12 Aircraft roll control



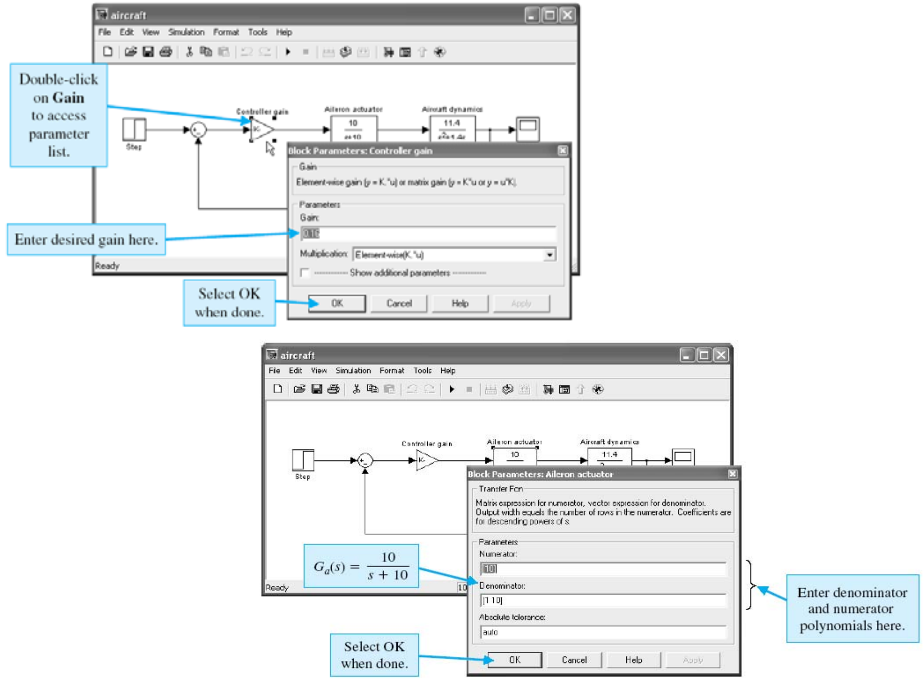




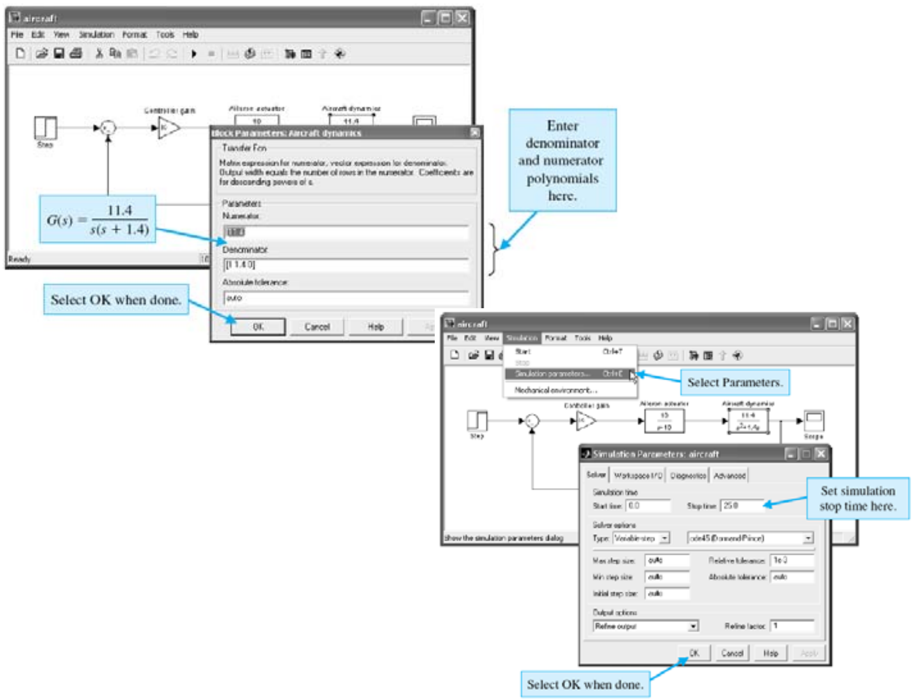
$\varphi_d(s) = 0$
Desired bank angle
Gain
K
Aileron actuator
$\frac{10}{s + 10}$
Aircraft dynamics
$\frac{11.4}{s(s + 1.4)}$
$\varphi(s)$
Bank angle
Gyro
$K_g = 1$



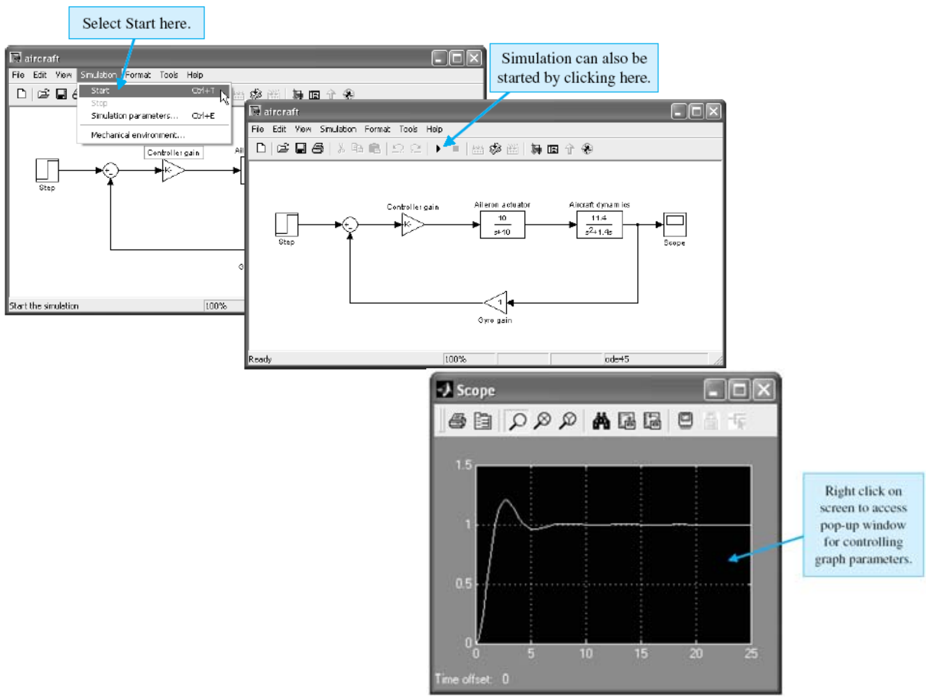
Title bar
Tool bar
aircraft
File Edit View Simulation Format Tools Help
Controller gain
Aileron actuator
Aircraft dynamics
Step
Scope
Gyro gain
Ready
100%
ode45



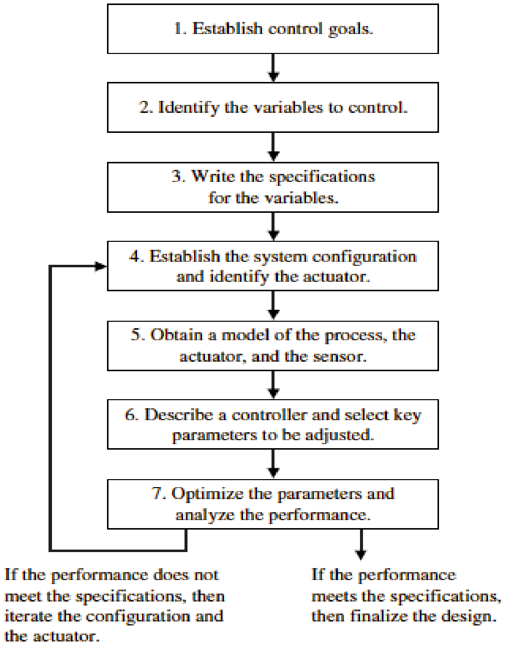
Double-click on Gain to access parameter list.
Enter desired gain here.
Select OK when done.
Block Parameters: Controller gain
$G_a(s) = \frac{10}{s + 10}$
Block Parameters: Aileron actuator
Enter denominator and numerator polynomials here.
Select OK when done.



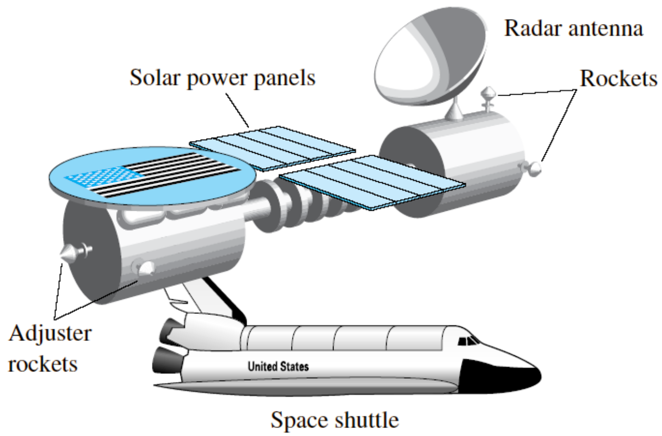
Block Parameters: Aircraft dynamics
Enter denominator and numerator polynomials here.
$G(s) = \frac{11.4}{s(s + 1.4)}$
Select OK when done.
Select Parameters.
Simulation Parameters: aircraft
Set simulation stop time here.
Select OK when done.



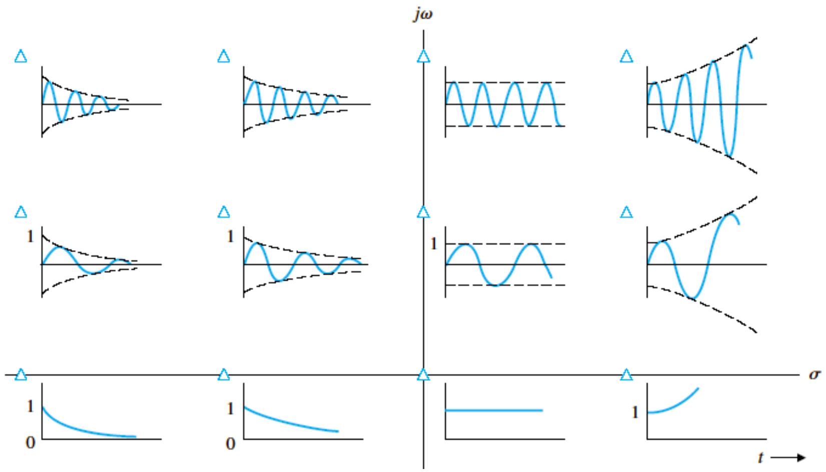
Select Start here.
Simulation can also be started by clicking here.
Scope
Right click on screen to access pop-up window for controlling graph parameters.
Time offset: 0



Root Locus

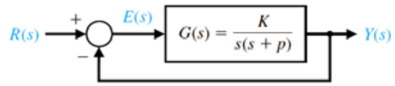


Transient Response and Zero/Pole Locations



Transient Response and Zero/Pole Locations: Applications

Example 5.1



Select K and p, to satisfy time-domain specifications:

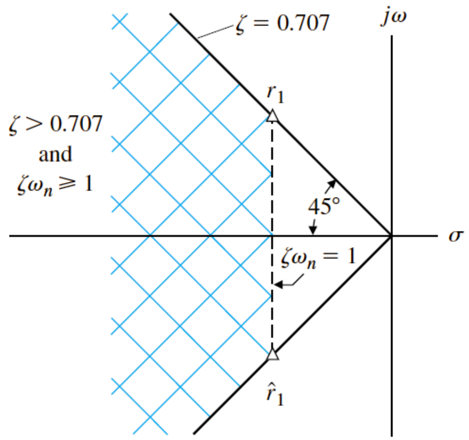
$$T_s = \frac{4}{\zeta\omega_n} \le 4(\text{sec}) \quad (\delta = 2\%)$$

$$OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \le 5\%$$

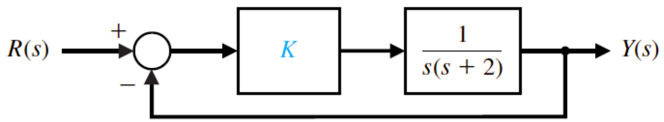
$$\Rightarrow \zeta > 0.707,\ \zeta\omega_n \ge 1$$

$$\Rightarrow r_{1,2} = -1 \pm j$$

$$\Rightarrow K = 2,\ p = 2$$



Root Locus Concepts: example



CLCE: $\Delta(s) = 1 + KG(s) = s^2 + 2s + K = 0$

Root Locus

- **Motivation**

  To satisfy transient performance requirements, it may be necessary to know how to choose certain controller parameters so that the resulting closed-loop poles are in the performance regions, which can be solved with Root Locus technique.

- **Definition**

  A graph displaying the roots of a polynomial equation when one of the parameters in the coefficients of the equation changes from 0 to ∞.

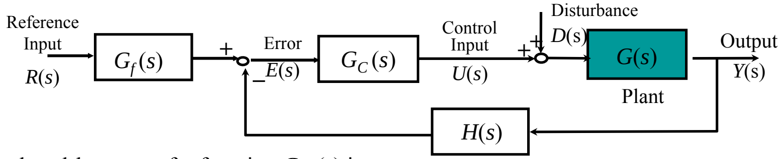
- **Rules for Sketching Root Locus**

- **Examples**

- **Controller Design Using Root Locus**

  Letting the CL characteristic equation (CLCE) be the polynomial equation, one can use the Root Locus technique to find how a positive controller design parameter affects the resulting CL poles, from which one can choose a right value for the controller parameter.

Closed-Loop Characteristic Equation (CLCE)



The closed-loop transfer function $G_{YR}(s)$ is:

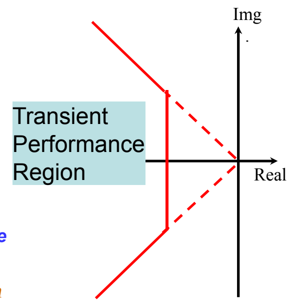
$$G_{YR}(s) = \frac{G(s)G_c(s)G_f(s)}{1 + G(s)G_c(s)H(s)}$$

The closed-loop characteristic equation (CLCE) is:

$$1 + G(s)G_c(s)H(s) = 0$$

For simplicity, assume a simple proportional feedback controller:

$$G_c(s) = K_p \Rightarrow 1 + K_p GH = 0$$



The transient performance specifications define a region on the complex plane where the closed-loop poles should be located.

Q: How should we choose K_P such that the CL poles are within the desired performance boundary?

Methods of Obtaining Root Locus

① Given a value of *K*, numerically solve the 1 + *K G(s)* = 0 equation to obtain all roots. Repeat this procedure for a set of *K* values that span from 0 to ∞ and plot the corresponding roots on the complex plane.

② In MATLAB, use the commands *rlocus* and *rlocfind*. A very efficient root locus design tool is the command *rltool*. You can use on-line help to find the usage for these commands.

$$1 + K_P \cdot 0.03 \cdot \frac{16}{s(0.0174s+1)} = 0 \quad \Rightarrow \quad 1 + K_P \cdot \frac{0.48}{0.0174s^2 + s} = 0$$

```
>> op_num=[0.48];
>> op_den=[0.0174 1 0];
>> rlocus(op_num,op_den);
>> [K, poles]=rlocfind(op_num,op_den);
```

No open-loop zeros

Two open-loop poles

③ Apply the following root locus sketching rules to obtain an *approximated* root locus plot.

Root Locus Sketching Rules

$$1+K\cdot\frac{N(s)}{D(s)}=0 \quad\Rightarrow\quad 1+K\cdot\frac{(s-z_1)(s-z_2)\cdots(s-z_{N_Z})}{(s-p_1)(s-p_2)\cdots(s-p_{N_P})}=0$$

Rule 1: The number of branches of the root locus is equal to the number of closed-loop poles (or roots of the characteristic equation). In other words, the number of branches is equal to the number of open-loop poles or open-loop zeros, whichever is greater. $D(s)+KN(s)=0$

Rule 2: Root locus starts at open-loop poles (*when* $K=0$) and ends at open-loop zeros (*when* $K=\infty$). If the number of open-loop poles is greater than the number of open-loop zeros, some branches starting from finite open-loop poles will terminate at zeros at infinity (i.e., go to infinity). If the reverse is true, some branches will start at poles at infinity and terminate at the finite open-loop zeros.

Rule 3: Root locus is symmetric about the real axis, which reflects the fact that closed-loop poles appear in complex conjugate pairs. $D(s)+KN(s)=0$ $\boxed{K=0}$ $\boxed{K=\infty?}$

Rule 4: Along the real axis, the root locus includes all segments that are to the left of an *odd* number of finite real open-loop poles and zeros.

$\boxed{\text{Check the phases}}$

$$\angle K\frac{N(s)}{D(s)}=\angle-1=\pi[\text{rad}]=180^\circ$$

Root Locus Sketching Rules

Rule 5: If number of poles N_P exceeds the number of zeros N_Z, then as $K\to\infty$, (N_P - N_Z) branches will become asymptotic to straight lines. These straight lines intersect the real axis with angles θ_k at σ_0.

$$\sigma_0=\frac{\sum p_i-\sum z_i}{N_P-N_Z}=\frac{\text{Sum of open-loop poles}-\text{Sum of open-loop zeros}}{\#\text{ of open-loop poles}-\#\text{ of open-loop zeros}}$$

$$\theta_k=(2k+1)\frac{\pi}{N_P-N_Z}\ [\text{rad}]=(2k+1)\frac{180^\circ}{N_P-N_Z}\ [\text{deg}],\quad k=0,1,2,\cdots$$

If N_Z exceeds N_P, then as $K\to 0$, (N_Z - N_P) branches behave as above.

Rule 6: Breakaway and/or break-in (arrival) points should be the solutions to the following equations:

$$\frac{d}{ds}\left(\frac{N(s)}{D(s)}\right)=0 \ \text{ or } \ \frac{d}{ds}\left(\frac{D(s)}{N(s)}\right)=0$$

Root Locus Sketching Rules

Rule 7: The departure angle for a pole p_i (the arrival angle for a zero z_i) can be calculated by slightly modifying the following equation:

Phase condition

$$\angle(s-z_1)+\angle(s-z_2)+\cdots+\angle(s-z_{N_Z})-\angle(s-p_1)-\angle(s-p_2)-\cdots-\angle(s-p_{N_P})=180^\circ$$

The departure angle q_j from the pole p_j can be calculated by replacing the term $\angle(s-p_j)$ with q_j and replacing all the s's with p_j in the other terms.

Rule 8: If the root locus passes through the imaginary axis (the stability boundary), the crossing point $j\omega$ and the corresponding gain K can be found as follows:

- Replace s in the left side of the closed-loop characteristic equation with $j\omega$ to obtain the real and imaginary parts of the resulting complex number.
- Set the real and imaginary parts to zero, and solve for ω and K. This will tell you at what values of K and at what points on the $j\omega$ axis the roots will cross.

magnitude condition

$$\boxed{K=\frac{|s-p_1||s-p_2|\cdots|s-p_{N_P}|}{|s-z_1||s-z_2|\cdots|s-z_{N_Z}|}}$$

Steps to Sketch Root Locus

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1+K\cdot\frac{N(s)}{D(s)}=0 \quad\text{or}\quad 1+K\cdot\frac{(s-z_1)(s-z_2)\cdots(s-z_{N_Z})}{(s-p_1)(s-p_2)\cdots(s-p_{N_P})}=0$$

Step 2: Find the open-loop zeros, z_i, and the open-loop poles, p_i. Mark the open-loop poles and zeros on the complex plane. Use x to represent open-loop poles and ○ to represent the open-loop zeros.

Step 3: Determine the real axis segments that are on the root locus by applying Rule 4.

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

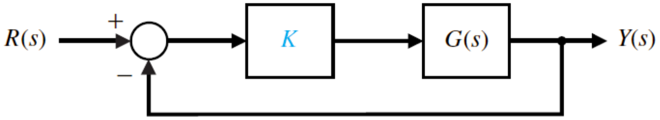
Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.

Root Locus Concepts

$$T(s)=\frac{Y(s)}{R(s)}=\frac{p(s)}{q(s)}$$



CLCE: $\quad 1+KG(s)=0$

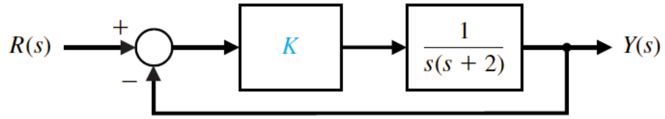
$$\Rightarrow \quad |KG(s)|\measuredangle KG(s)=-1+j0$$

Phase Condition

Magnitude Condition

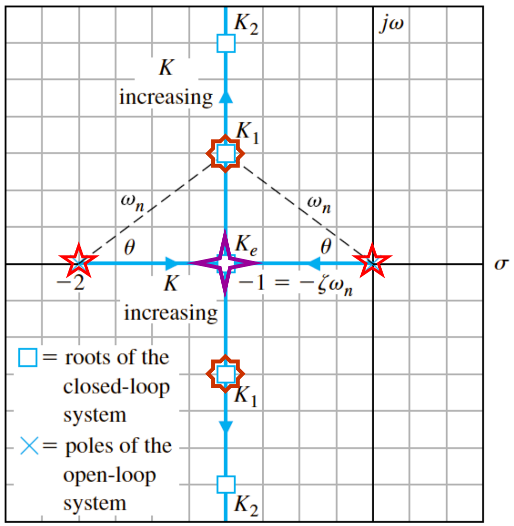
$$\Rightarrow \begin{cases} |KG(s)|=1 \\ \measuredangle KG(s)=180^\circ \pm k360^\circ \quad k=0,\pm1,\pm2,\cdots \end{cases}$$

Root Locus Concepts: example



CLCE: $\quad \Delta(s)=1+KG(s)=s^2+2s+K=0$

$$\begin{cases} |KG(s)|=1 \\ \measuredangle KG(s)=180^\circ \pm k360^\circ \\ \quad k=0,\pm1,\pm2,\cdots \end{cases} \Rightarrow \begin{cases} \left|\dfrac{K}{s(s+2)}\right|=1 \\ \measuredangle \dfrac{K}{s(s+2)}=180^\circ, 540^\circ, \cdots \end{cases}$$

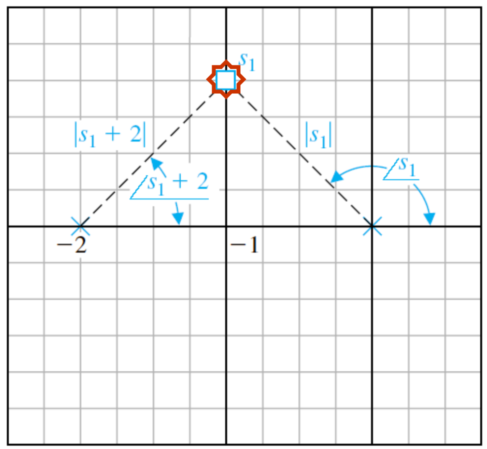


$$G(s)=\frac{1}{s(s+2)}$$

CLCE:

$$\Delta(s)=1+KG(s)=s^2+2s+K$$
$$=s^2+2\zeta\omega_n s+\omega_n^2=0$$

$$s_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$
$$\theta=\cos^{-1}\zeta, \quad \text{if } \zeta<1,$$



At s_1:
$$\begin{cases} \left|\dfrac{K}{s(s+2)}\right|_{s=s_1}=1 \quad \Rightarrow \quad K=|s_1||s_1+2| \\ \measuredangle \dfrac{K}{s(s+2)}\Big|_{s=s_1}=-\measuredangle s_1-\measuredangle(s_1+2)=-180^\circ \end{cases}$$

Root Locus Concepts

CLCE: $\Delta(s) = 1 + F(s) = 0$

$\Rightarrow \quad F(s) = -1 + j0$

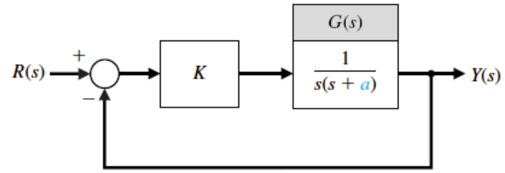
z_M : zeros of $F(s)$

p_n : poles of $F(s)$

$$F(s) = K\frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$\Rightarrow \begin{cases} |F(s)| = K\dfrac{|s+z_1||s+z_2|\cdots|s+z_M|}{|s+p_1||s+p_2|\cdots|s+p_n|} = 1 \\ \measuredangle F(s) = \measuredangle(s+z_1) + \measuredangle(s+z_2) + \cdots + \measuredangle(s+z_M) \\ \qquad - \measuredangle(s+p_1) - \measuredangle(s+p_2) - \cdots - \measuredangle(s+p_n) = 180^\circ \pm q360^\circ \end{cases}$$

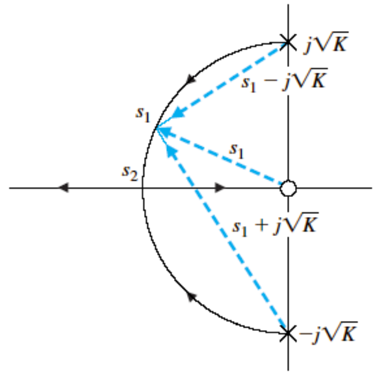
Root Locus Concepts



CLCE:

$$\Delta(s) = 1 + KG(s) = 1 + \frac{K}{s(s+a)} = 0$$

$$\Rightarrow \quad 1 + \frac{as}{s^2+K} = 0$$



Root Locus Procedure

Step 1: equation.

CE: $\Delta(s) = 1 + F(s) = 1 + KP(s) = 0, \quad 0 \le K < \infty$

$$1 + K\frac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0 \quad \Rightarrow \quad \begin{matrix} \prod_{j=1}^{n}(s+p_j) = 0, \text{ if } K = 0 \\ \prod_{i=1}^{M}(s+z_i) = 0, \text{ if } K \to \infty \end{matrix}$$

$$\Rightarrow \prod_{j=1}^{n}(s+p_j) + K\prod_{i=1}^{M}(s+z_i) = 0$$

Rule 1: Root locus starts at **open-loop poles** (*when K= 0*) and ends at **open-loop zeros** (*when K=∞*).

If the number of open-loop poles is greater than the number of open-loop zeros, some branches starting from finite open-loop poles will terminate at zeros at infinity (i.e., go to infinity). If the reverse is true, some branches will start at poles at infinity and terminate at the finite open-loop zeros.

Rule 2: The **number of branches** of the root locus is equal to the **number of closed-loop poles** (or roots of the characteristic equation). In other words, the number of branches is equal to the number of open-loop poles or open-loop zeros, whichever is greater.

Root Locus Procedure

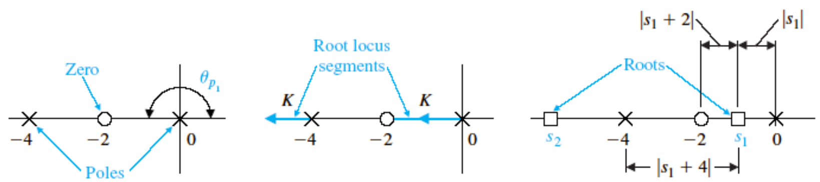
Step 2: loci segments.

Rule 3: Root locus is **symmetric** about the real axis, which reflects the fact that closed-loop poles appear in complex conjugate pairs.

Rule 4: Along the real axis, the root locus includes all segments that are to the **left** of an **odd** number of finite **real** open-loop poles and zeros.

Example 7.1

$$\Delta(s) = 1 + K\frac{2(s+2)}{s(s+4)} = 0$$



Root Locus Procedure

Step 3: asymptotes.

Rule 5: If number of poles n exceeds the number of zeros M, then as $K \to \infty$, $N=(n-M)$ branches will become **asymptotic** to straight lines. These straight lines intersect the real axis with angles ϕ_A at σ_A:

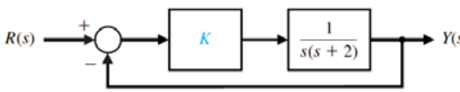
$$\sigma_A = \frac{\sum p_i - \sum z_i}{n-M} = \frac{\text{Sum of open-loop poles} - \text{Sum of open-loop zeros}}{\text{\# of open-loop poles} - \text{\# of open-loop zeros}}$$

$$\phi_A = \frac{(2q+1)\pi}{n-M}\ [\text{rad}] = \frac{(2q+1)180^\circ}{n-M}\ [\text{deg}],\quad q = 0, 1, 2, \cdots, (n-M-1)$$

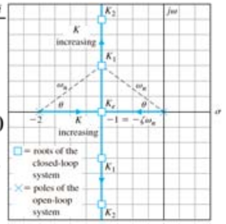
$$1 + K\frac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0 \quad \Rightarrow \quad 1 + K\frac{s^M}{s^n} = 0, \quad \text{for large } s$$

$$1 + \frac{K}{(s-\sigma_A)^{n-M}} = 0 \quad \Rightarrow \quad \sigma_A = \frac{\sum p_i - \sum z_i}{n-M}$$

Example:



$$\Rightarrow \quad \phi_A = \begin{cases} 90, & q=0 \\ 270, & q=1 \end{cases}$$

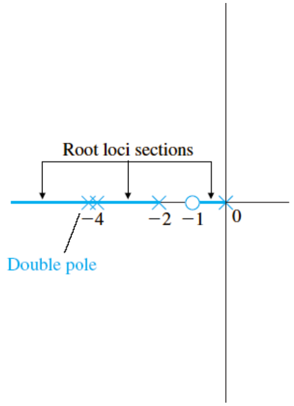


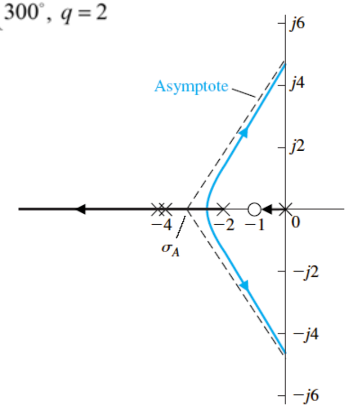
Root Locus Procedure

Example 7.2

CLCE: $$\Delta(s) = 1 + K\frac{(s+1)}{s(s+2)(s+4)^2} = 0$$

$$\sigma_A = \frac{\sum p_i - \sum z_i}{n-M} \qquad \phi_A = \begin{cases} 60^\circ, & q=0 \\ 180^\circ, & q=1 \\ 300^\circ, & q=2 \end{cases}$$





Root Locus Procedure

Step 4: cross points at imaginary axis (if any).

Rule 6: If the root locus passes through the imaginary axis (the stability boundary), the crossing point $j\omega$ and the corresponding gain K can be found or as follows:

- Replace s in the left side of the closed-loop characteristic equation with $j\omega$ to obtain the real and imaginary parts of the resulting complex number.
- Set the real and imaginary parts to zero, and solve for ω and K. This will tell you at what values of K and at what points on the $j\omega$ axis the roots will cross.

Example CLCE: $$\Delta(s) = 1 + K\frac{1}{s[(s+4)^2+16]} = 0$$

$$\Rightarrow \quad s^3 + 8s^2 + 32s + K = 0$$

s^3	1	32
s^2	8	K
s^1	$\frac{8\times 32 - K}{8}$	0
s^0	K	

$\Rightarrow \quad K = 256$

Critical points for marginally stable

$$(j\omega_0)^3 + 8(j\omega_0)^2 + 32(j\omega_0) + 256 = 0$$

$$\Rightarrow \begin{cases} -\omega_0^3 + 32j\omega_0 = 0 \\ -8j\omega_0^2 + 256 = 0 \end{cases}$$

$$\Rightarrow \quad \omega_0 = \pm\sqrt{32}$$

Root Locus Procedure

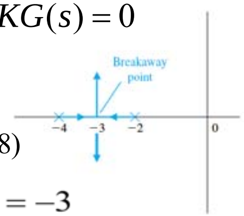
Step 5: breakaway points at real axis (if any).

Rule 7: If breakaway and/or break-in (arrival) points should be the solutions to the following equations:

$$\frac{dp(s)}{ds} = 0$$

$$p(s) = K \quad \Leftarrow \quad \Delta(s) = 1 + KG(s) = 0$$

Example: CLCE: $$\Delta(s) = 1 + K\frac{1}{(s+2)(s+4)} = 0$$



$$\Rightarrow \quad p(s) = K = -(s+2)(s+4) = -(s^2+6s+8)$$

$$\Rightarrow \quad \frac{dp(s)}{ds} = -(2s+6) = 0 \quad \Rightarrow \quad s = -3$$

Example 7.3:

CLCE:

$$\Delta(s) = 1 + K\frac{(s+1)}{s(s+2)(s+4)} = 0$$

$$\Rightarrow \quad p(s) = K = \frac{-s(s+2)(s+3)}{(s+1)}$$

$$\Rightarrow \quad \frac{dp(s)}{ds} = 2s^3 + 8s^2 + 10s + 6 = 0$$

$$\Rightarrow \quad s = -2.46, -0.77 \pm 0.79j$$

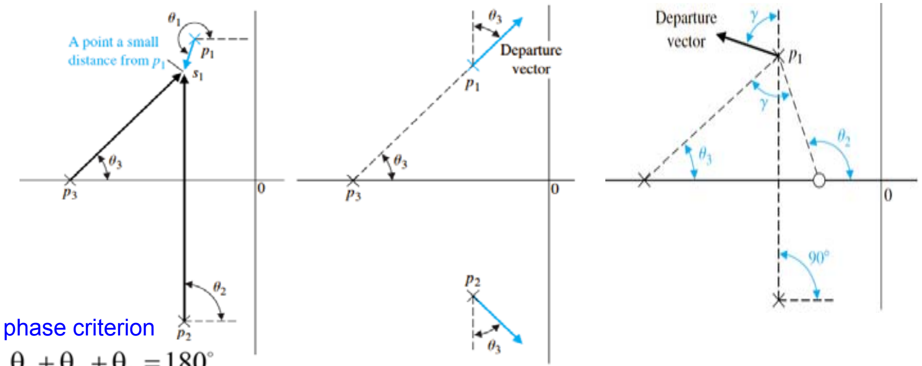
Root Locus Procedure

Step 6: departure/arrival angles.

Rule 8: The departure angle for a pole p_i (the arrival angle for a zero z_i) can be calculated by slightly modifying the following equation:

$$\angle(s-z_1)+\angle(s-z_2)+\cdots+\angle(s-z_M)-\angle(s-p_1)-\angle(s-p_2)-\cdots-\angle(s-p_n)=180^\circ$$

The departure angle q_j from the pole p_j can be calculated by replacing the term with $\angle(q_j-p_j)$ and replacing all the s's with p_j in the other terms.



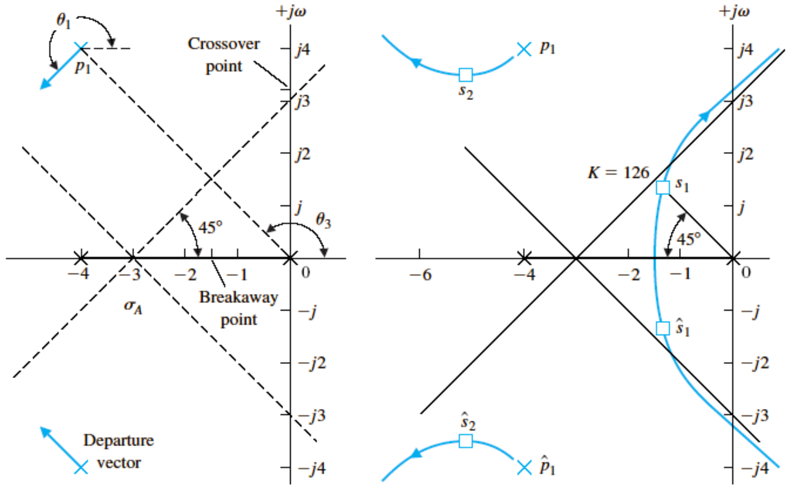
phase criterion

$$\theta_1+\theta_2+\theta_3=180^\circ$$

Examples

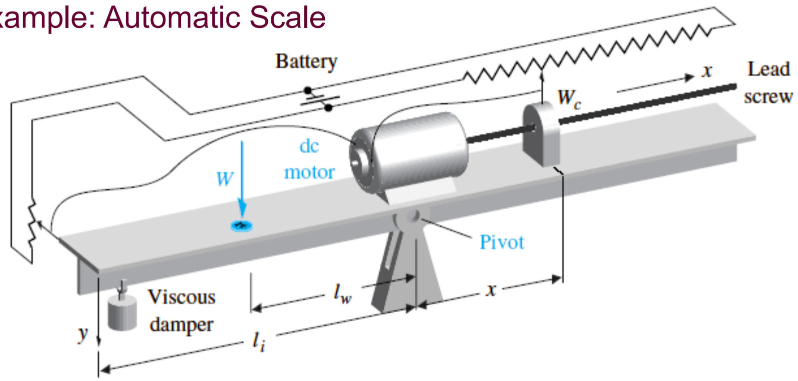
Example 7.4

$$1+K\frac{1}{s^4+12s^3+64s^2+128s}=0$$



Examples

Example: Automatic Scale



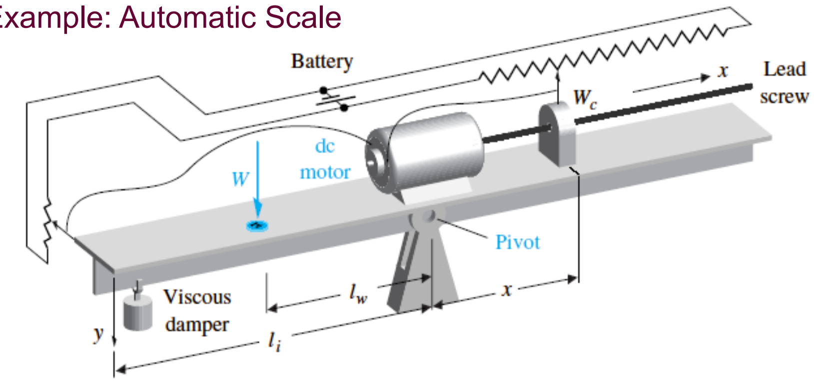
Tasks:

swiftness & closeness

1. Select the system parameters and the specifications;
2. Obtain a system model; $X(s)/W(s)$
3. Select the gain based on RL diagram; K_m : motor constant
4. Determine the dominant mode.

Examples

Example: Automatic Scale

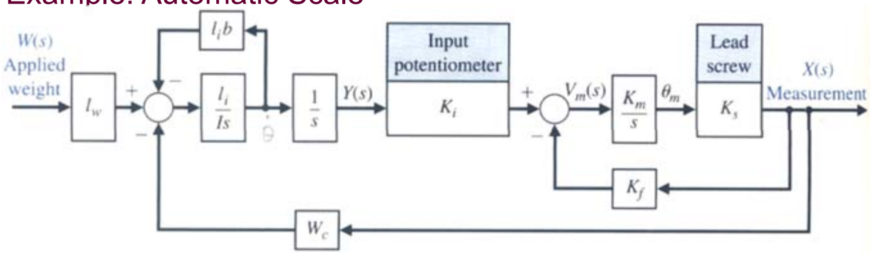


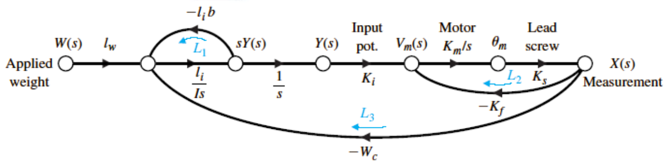
Specifications:

1. e_{ss}=0 for a step input;
2. ζ =0.5;
3. $T_s < 2$(sec) (δ = 2%)

Examples

Example: Automatic Scale





$$\frac{X(s)}{W(s)} = \frac{l_w l_i K_i K_m K_s / Is^3}{1 + (l_i^2 b / Is) + (K_m K_s K_f / s) + (l_i K_i K_m K_s W_c / Is^3) + (l_i^2 b K_m K_s K_f / Is^2)}$$

$$= \frac{l_w l_i K_i K_m K_s}{s(Is + l_i^2 b)(s + K_m K_s K_f) + l_i K_i K_m K_s W_c}$$

Examples

Example: Automatic Scale

CLTF: $\dfrac{X(s)}{W(s)} = \dfrac{l_w l_i K_i K_m K_s}{s(Is + l_i^2 b)(s + K_m K_s K_f) + l_i K_i K_m K_s W_c}$

DC gain: $\lim\limits_{s \to 0} \dfrac{X(s)}{W(s)} = \dfrac{l_w}{W_c} = 2.5 \text{ cm/kg}$

CLCE: $s(s + 8\sqrt{3})(s + \dfrac{K_m}{10\pi}) + \dfrac{96 K_m}{10\pi} = 0$

Standard form for RL:

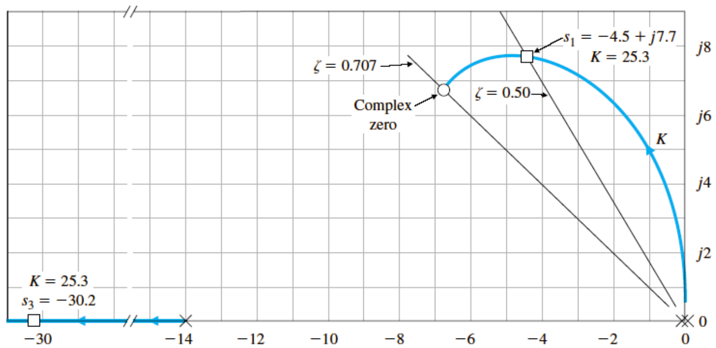
$$1 + KP(s) = 1 + \frac{K_m}{10\pi} \frac{(s + 6.93 + j6.93)(s + 6.93 - j6.93)}{s^2 (s + 8\sqrt{3})} = 0$$

Examples

Example: Automatic Scale

Standard form for RL:

$$1 + KP(s) = 1 + \frac{K_m}{10\pi} \frac{(s + 6.93 + j6.93)(s + 6.93 - j6.93)}{s^2 (s + 8\sqrt{3})} = 0$$



$$\zeta = 0.5 \;\Rightarrow\; K = 25.3 \;\Rightarrow\; K_m = 7600 \text{ rpm/v}$$

Parameter Design by RL Method

- select parameters by RL method

CE: $a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$

Standard form for RL: $1 + \dfrac{a_1 s}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_0} = 0$

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a s^{n-q} + \alpha s^{n-q} + \cdots + a_1 s + a_0 = 0, \quad a = (a_{n-q} - \alpha)$$

$$\Rightarrow \quad 1 + \frac{\alpha s^{n-q}}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a s^{n-q} + \cdots + a_1 s + a_0} = 0$$

Two parameters:

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a s^{n-q} + \alpha s^{n-q} + \cdots + b s^{n-r} + \beta s^{n-r} + \cdots + a_1 s + a_0 = 0$$

$$a = (a_{n-q} - \alpha), \quad b = (a_{n-r} - \beta)$$

Example:

$s^3 + s^2 + \beta s + \alpha = 0$ $\qquad$ $s^3 + s^2 + \alpha = 0$ with $\beta = 0$

$\Rightarrow \quad 1 + \dfrac{\beta s}{s^3 + s^2 + \alpha} = 0$ $\qquad$ $\Rightarrow \quad 1 + \dfrac{\alpha}{s^2(s+1)} = 0$

Two steps: (1) RL for α, (2) select a satisfactory α, RL for β.

Parameter Design by RL Method

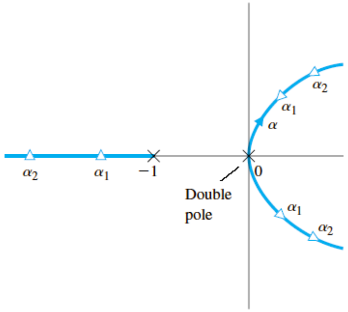
- select parameters by RL method

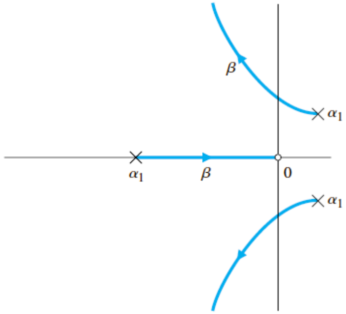
Example:

$$s^3+s^2+\beta s+\alpha=0 \qquad s^3+s^2+\alpha=0 \quad \text{with } \beta=0$$

$$\Rightarrow \quad 1+\frac{\beta s}{s^3+s^2+\alpha}=0 \qquad \Rightarrow \quad 1+\frac{\alpha}{s^2(s+1)}=0$$

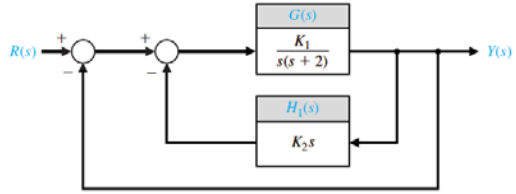
Two steps: (1) RL for α, (2) select a satisfactory α, RL for β.





Examples

Example: welding head control

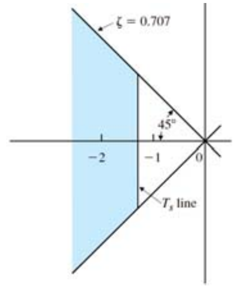


Specifications:

1. e_{ss} less than 35% of a ramp input slope;
2. ζ =0.707;
3. $T_s<3(\text{sec})$ $(\delta = 2\%)$ $\Rightarrow \ \sigma \geq \frac{4}{3}$

$$\Rightarrow \ \frac{e_{ss}}{R}=\frac{2+K_1K_2}{K_1}\leq 35\%$$

$\Rightarrow$ small K_2

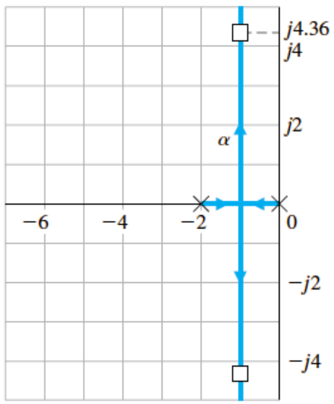


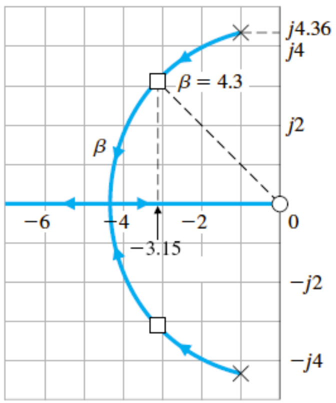
$$1+GH(s)=s^2+2s+\beta s+\alpha=0$$

$$\alpha=K_1, \quad \beta=K_1K_2$$

Examples

Example: welding head control

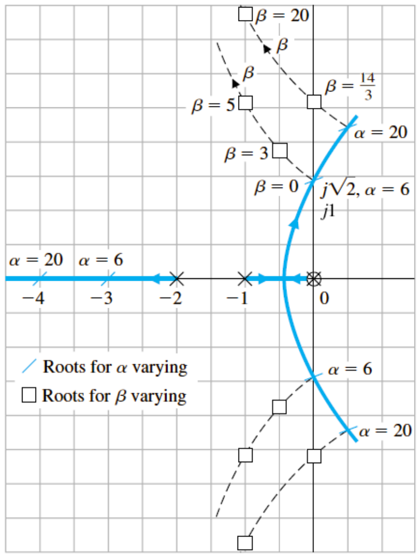




$$\alpha=K_1=20, \quad \beta=K_1K_2=20K_2=4.3 \quad (K_2=0.215)$$

Root Contours

CE: $\quad s^3+3s^2+2s+\beta s+\alpha=0$



PID Controller

Proportional-Integral-Derivative Controller:

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s$$

Proportional Controller:

$$G_c(s) = K_P$$

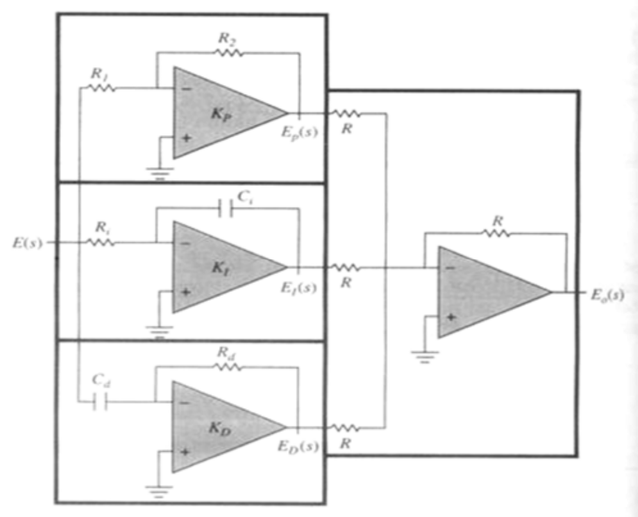
Proportional-Integral Controller:

$$G_c(s) = K_P + \frac{K_I}{s}$$

Proportional-Derivative Controller:

$$G_c(s) = K_P + K_D s$$

PID Controller: Implementation



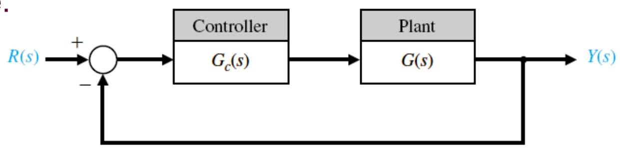
$$G(s) = \frac{R_2}{R_1} + \frac{1}{R_i C_i s} + R_d C_d s \qquad G(s) = K_p + \frac{K_I}{s} + K_D s$$

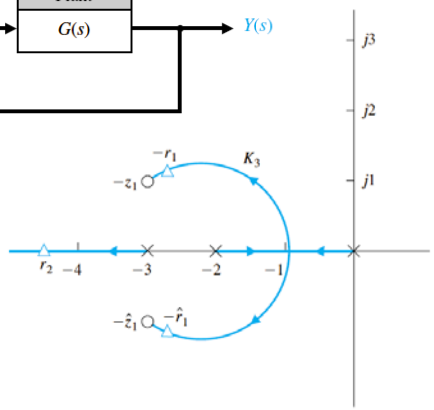
PID Controller: Application

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s = \frac{K_D\left(s + \frac{K_P}{K_D}s + \frac{K_I}{K_D}\right)}{s} = \frac{K_D(s+z_1)(s+z_2)}{s}$$

Three parameters free to choose: K_P, K_I, K_D (or K_D, z_1, z_2)

Example:



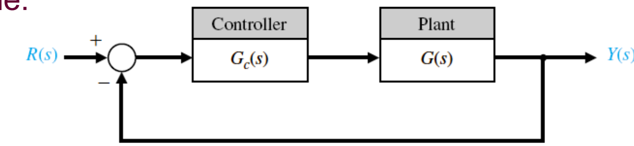


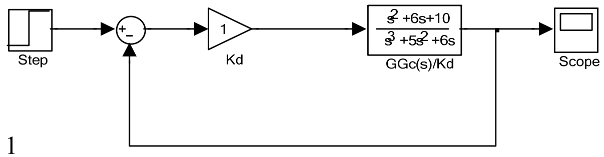
$$G(s) == \frac{1}{(s+2)(s+3)}$$

$$T(s) = \frac{GG_c(s)}{1+GG_c(s)} = \frac{K_D(s+z_1)(s+\hat{z}_1)}{(s+r_2)(s+r_1)(s+\hat{r}_1)}$$

PID Controller: Application

Example:

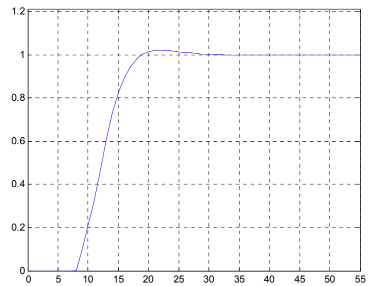




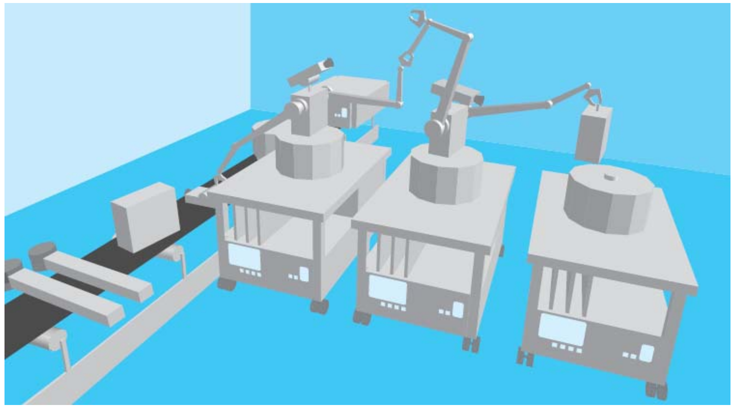
$$G(s) = \frac{1}{(s+2)(s+3)}$$

$$G(s) = \frac{GG_c(s)}{1+GG_c(s)}$$

$$= \frac{K_D(s+z_1)(s+\hat{z}_1)}{(s+r_2)(s+r_1)(s+\hat{r}_1)}$$



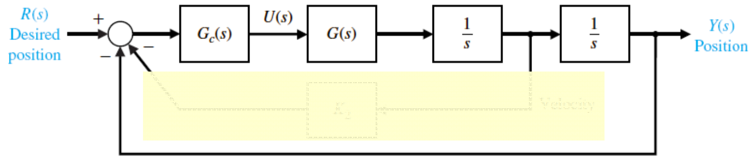
Robot Control System Design via RL Method



Specifications:

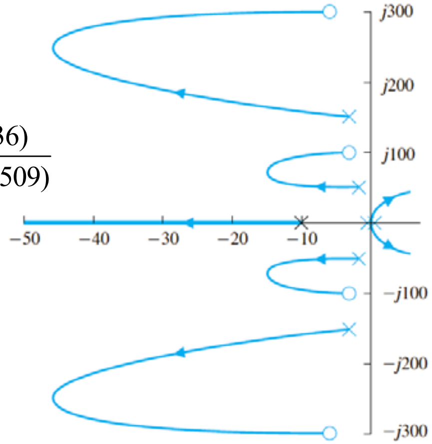
1. $e_{ss}=0$ for a step input;
2. O.S.<10%;
3. $T_s < 2\text{(sec)}$ $(\delta = 2\%)$

Robot Control System Design via RL Method

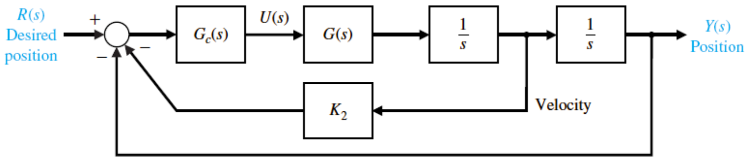


$$\frac{Y(s)}{U(s)} = (\frac{1}{s^2})G(s)$$

$$G(s) = \frac{(s^2+4s+10004)(s^2+12s+900036)}{(s+10)(s^2+2s+2501)(s^2+6s+22509)}$$



Robot Control System Design via RL Method



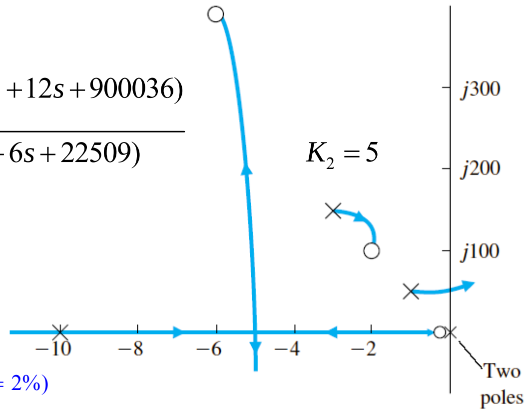
$$\frac{1}{s^2}G(s)H(s) =$$

$$\frac{K_1K_2(s+\frac{1}{K_2})(s^2+4s+10004)(s^2+12s+900036)}{s^2(s+10)(s^2+2s+2501)(s^2+6s+22509)}$$

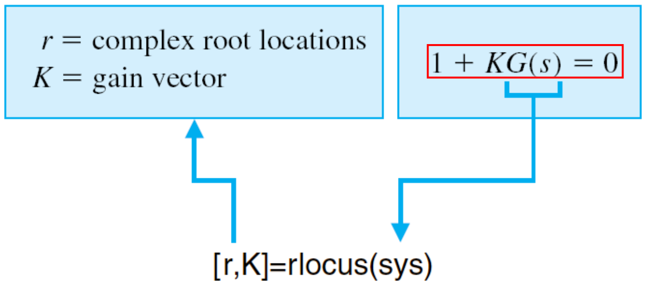
$K_2 = 5$

$$G_c(s) = K_1\frac{s+z}{s+p} = 5\frac{s+1}{s+5}$$

$\Rightarrow$ O.S.= 8%; $T_s = 1.6\text{(sec)}$ $(\delta = 2\%)$



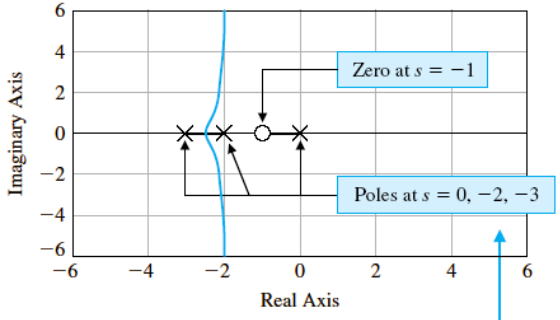
RL Method via Matlab



$$T(s) = \frac{Y(s)}{R(s)} = \frac{K(s+1)(s+3)}{s(s+2)(s+3)+K(s+1)}$$

$$= 1 + K\frac{(s+1)}{s(s+2)(s+3)}$$

RL Method via Matlab




```
>>p=[1 1]; q=[1 5 6 0]; sys=tf(p,q); rlocus(sys)
```

Generating a root locus plot.

```
>>p=[1 1]; q=[1 5 6 0]; sys=tf(p,q); [r,K]=rlocus(sys);
```

Obtaining root locations, r, associated with various values of the gain K.

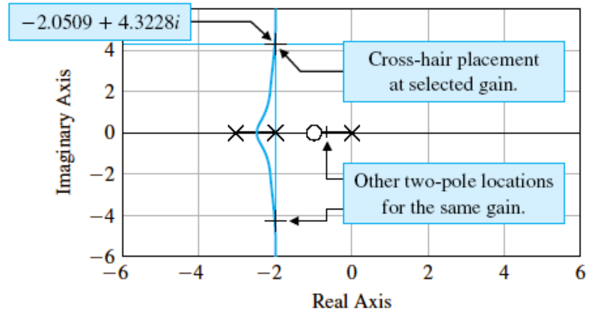
RL Method via Matlab

```
>>p=[1 1]; q=[1 5 6 0]; sys=tf(p,q); rlocus(sys)
>>rlocfind(sys)
```

rlocfind follows the rlocus function.

```
Select a point in the graphics window

 selected_point =
    -2.0509 + 4.3228i
 ans =
    20.5775
```

Value of K at selected point



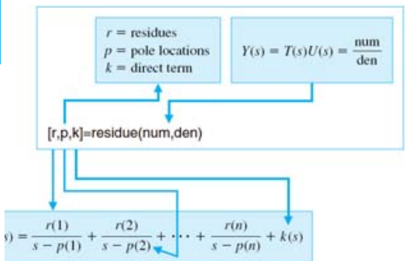
RL Method via Matlab

```
>>K=20.5775; num=K*[1 4 3]; den=[1 5  6+K  K 0];

>>[r,p,k]=residue(num,den)

 r =

-1.3786 + 1.7010i
-1.3786 - 1.7010i
-0.2429
 3.0000

 p=

-2.0505 - 4.3228i
-2.0505 + 4.3228i
-0.8989
      0

k =

 []
```

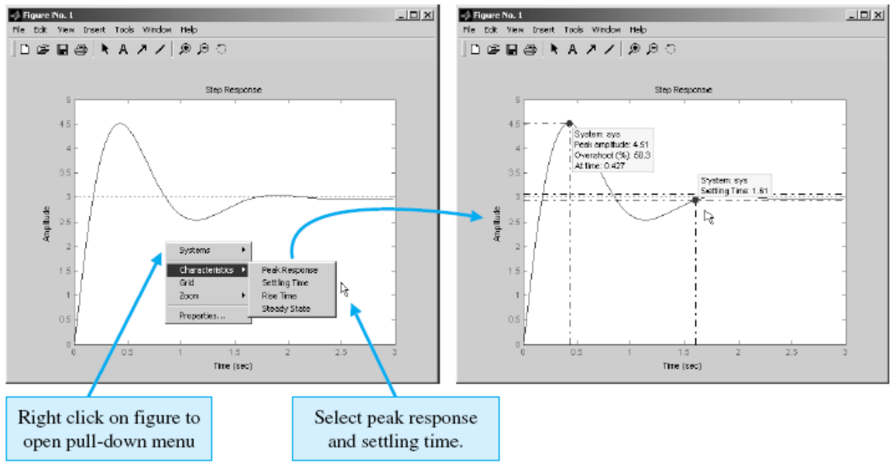
$Y(s) = \frac{\text{num}}{\text{den}}$

$$Y(s) = \frac{r(1)}{s-p(1)} + \frac{r(2)}{s-p(2)} + \frac{r(3)}{s-p(3)} + \frac{r(4)}{s-p(4)} + k(s)$$

r = residues
p = poles
k = direct term



RL Method via Matlab




```
>>K=20.5775;num=k"[1 4 3]; den=[1 5 6+K K]; sys=tf(num,den);
>>step(sys)
```

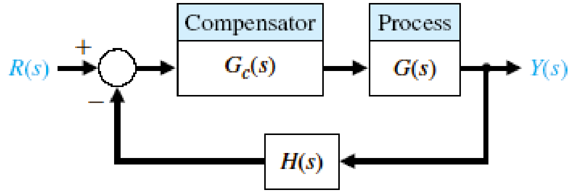
RL Method via Matlab

Root sensitivity: $S_K^{r_i} = \dfrac{\partial r_i}{\partial K / K}$ $\quad \left(\dfrac{\partial r_i / r_i}{\partial K / K}\right)$

```
% Compute the system sensitivity to a parameter
% variation
%
K=20.5775, den=[1 5 6+K K]; r1=roots(den);
%
dK=1.0289;          <-- 5% change in K
%
Km=K+dK; denm=[1 5 6+Km Km]; r2=roots(denm);
dr=r1-r2;           <-- Δr
%
S=dr/(dk/K);        <-- Sensitivity formula
```

$$S_K^{r_i} = 2.34\measuredangle 268.79^\circ$$

Feedback Control System Design



Introduction

Why Control?

– Unsatisfied system behaviors.

How to Control?

– Change the system: parameter, structure

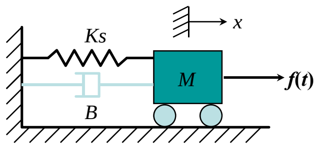
A best way to meet our requirements is to modify the system/process itself, if this is physically realizable and cost-efficient (usually *Not*, unfortunately).

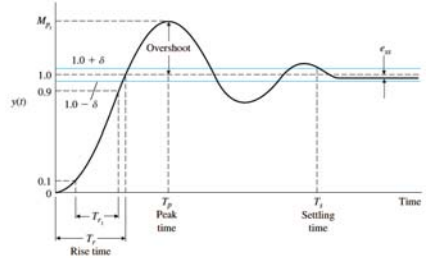
➢ Feedback: + sensor + controller (control algorithm)

Compensation:

adjustment to make up for performance deficiencies or inadequacies

Example





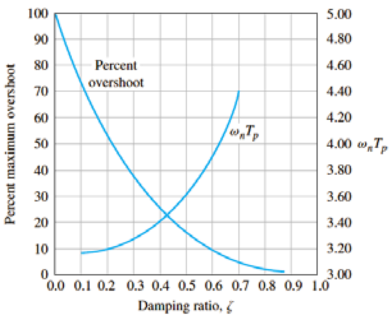
$$M\ddot{x} + B\dot{x} + K_s x = f(t)$$

$$G(s) = \frac{1}{Ms^2 + Bs + K_s} = \frac{1/M}{s^2 + \frac{B}{M}s + \frac{K_s}{M}} = \frac{K\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

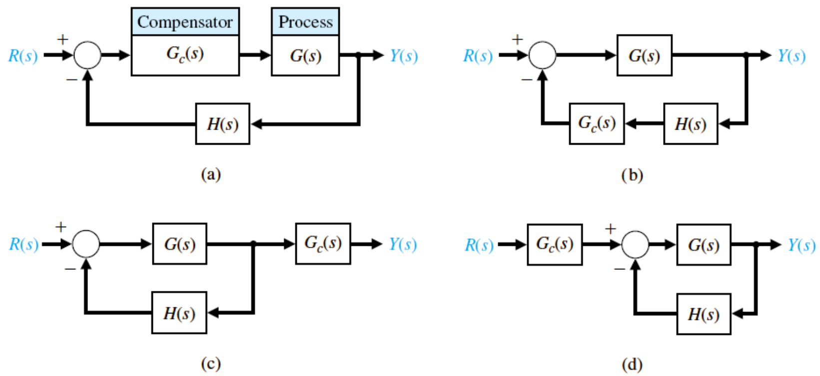
$$\omega_n = \sqrt{\frac{K_s}{M}}$$

$$\varsigma = \frac{B}{2\sqrt{MK_s}}$$

$$K = \frac{1}{K_s}$$



Types of Compensation



How to Design a Controller

Design of Control Systems:

Change system structure with appropriate components/parameters

– Control Object

Known by modeling, analysis, …

– Control Objectives (Performance Indices)

Time-domain: T_p, T_s, T_r, O.S., e_{ss}, …

Frequency domain: GM, PM, Mp_ω, ω_r, ω_B …

– Control Methods

Root locus: re-shape of root locus

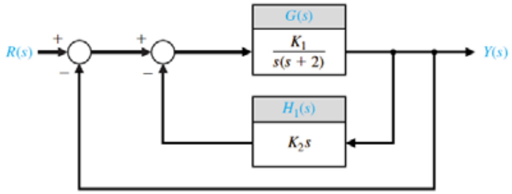
Frequency method: re-shape of frequency response

Modern control theory: pole placement

… …

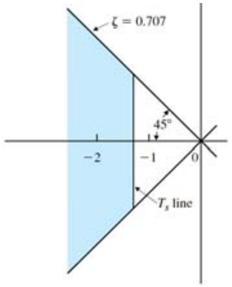
Examples (Section 9.4)

Example: welding head control



Specifications:

1. e_{ss} less than 35% of a ramp input slope; $\Rightarrow \frac{e_{ss}}{R} = \frac{2+K_1K_2}{K_1} \le 35\%$ $\Rightarrow$ small K_2
2. ζ =0.707;
3. $T_s < 3(\text{sec})$ $(\delta = 2\%)$ $\Rightarrow \sigma \ge \frac{4}{3}$

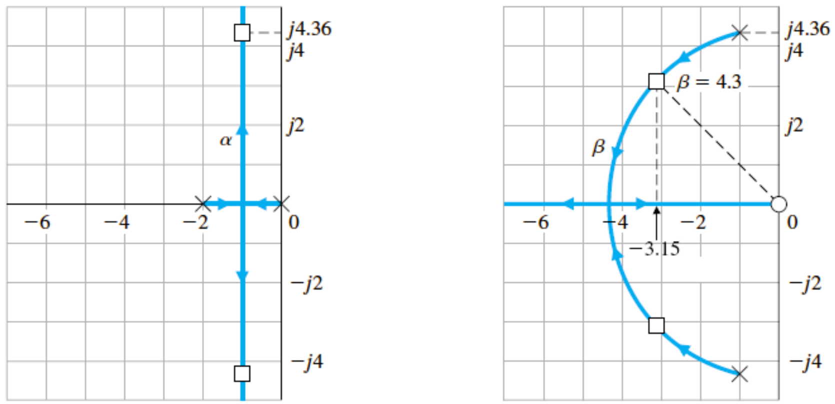


$$1 + GH(s) = s^2 + 2s + \beta s + \alpha = 0$$

$$\alpha = K_1, \quad \beta = K_1 K_2$$

Examples

Example: welding head control



$$\alpha = K_1 = 20, \quad \beta = K_1 K_2 = 20K_2 = 4.3 \quad (K_2 = 0.215)$$

Relative Stability: Gain Margin (Section 9.4)

$$GH(j\omega) = \frac{K}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)}$$

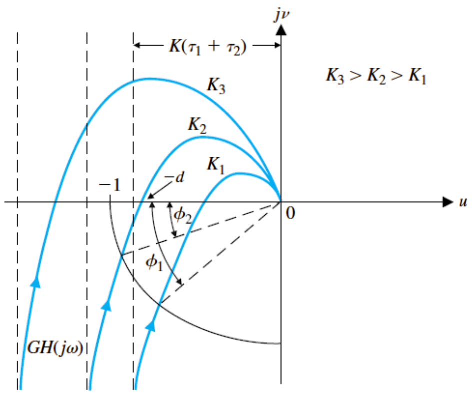
Intersection of $GH(j\omega)$-locus and u-axis:

$$v = 0$$

$$u = \frac{-K\tau_1\tau_2}{K(\tau_1 + \tau_2)}$$

Marginally stable when

$$K = \frac{\tau_1 + \tau_2}{\tau_1\tau_2}$$



Gain Margin: the reciprocal of $|GH(j\omega)|$ at frequency at which phase angle =-180°.
GM is the increase in the system gain when phase angle =-180° (resulting in marginally stable system or intersection of (-1,0) on the Nyquist diagram).

$$\frac{1}{|GH(j\omega)|} = \left(\frac{-K_2\tau_1\tau_2}{\tau_1 + \tau_2}\right)^{-1} = \frac{1}{d} \quad (\omega = 1/\sqrt{\tau_1\tau_2})$$

Relative Stability: Phase Margin

$$GH(j\omega) = \frac{K}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)}$$

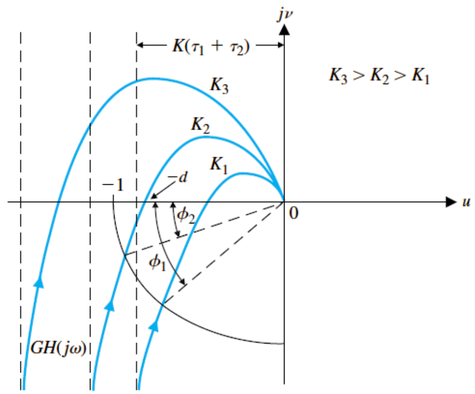
Intersection of $GH(j\omega)$-locus and u-axis:

$$v = 0$$

$$u = \frac{-\tau_1\tau_2}{K(\tau_1 + \tau_2)}$$

Marginally stable when

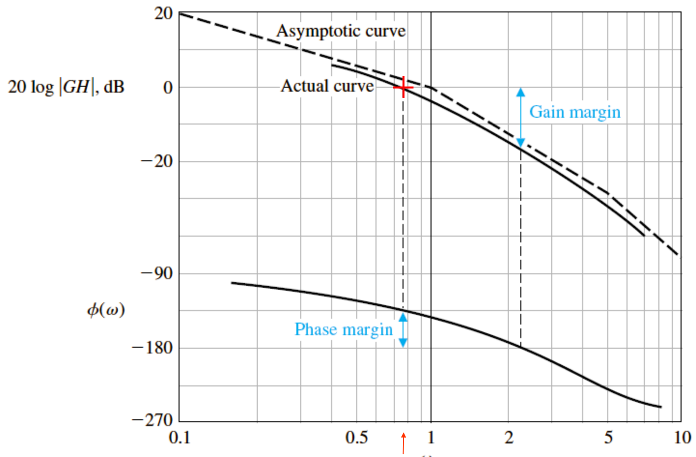
$$K = \frac{\tau_1 + \tau_2}{\tau_1\tau_2}$$



Phase Margin: the phase angel through which the $GH(j\omega)$ locus rotates to make the unity magnitude point $|GH(j\omega)| = 1$ pass (-1,0).
PM is the amount of phase shift of $GH(j\omega)$ at unity magnitude (resulting in marginally stable system or intersection of (-1,0) on the Nyquist diagram).

Gain/Phase Margin in Bode Diagram

$$GH(j\omega) = \frac{1}{j\omega(j\omega + 1)(0.2j\omega + 1)}$$



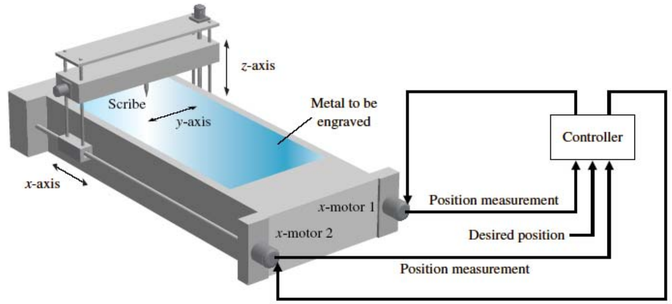
GM=0-(-15dB) =15dB

PM=180°-137° =43°

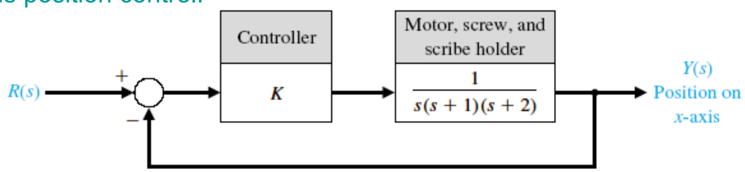
ω_c: crossover frequency

Design Example (Section 8.7)

Engraving Machine:



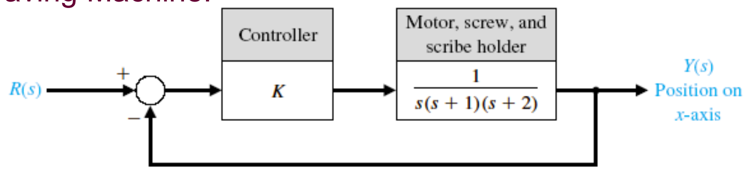
X-axis position control:



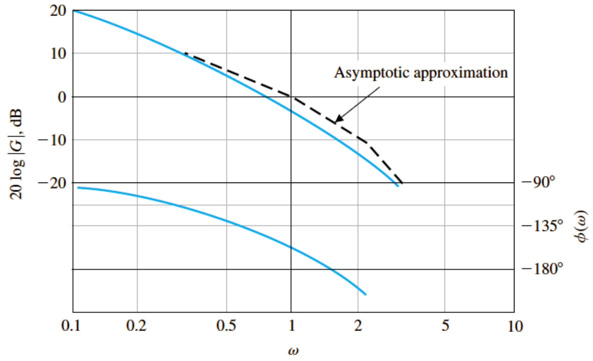
select gain K to achieve a satisfactory step response

Design Example

Engraving Machine:



BD *of the open-loop system:*

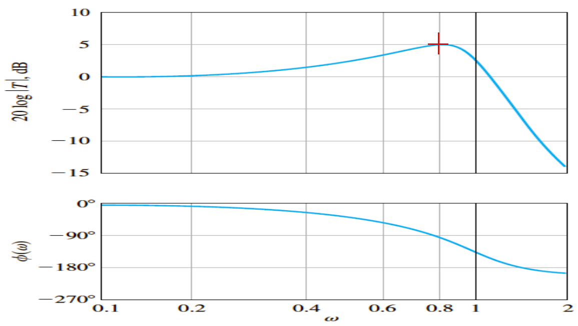


Design Example

TF *of closed-loop system* (K=2):

$$T(s)=\frac{2}{s^3+3s^2+2s+2}\quad \overset{s=j\omega}{\Rightarrow}\quad T(j\omega)=\frac{2}{(2-3\omega^2)+j\omega(2-\omega^2)}$$

BD *of the closed-loop system* (K=2):



Second-order dominant roots approximation:

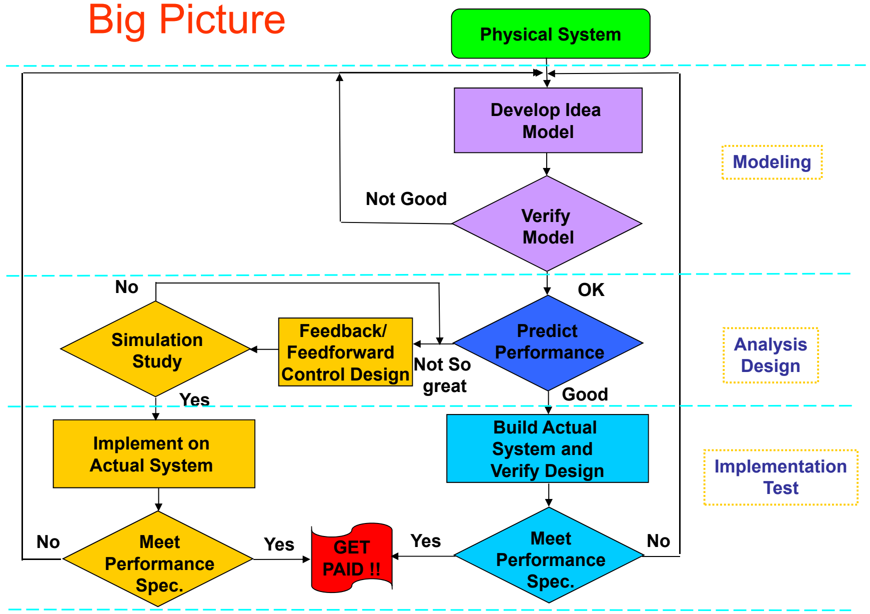
$$20\log M_{p_\omega}=5$$
$$\omega_r=0.8$$

$$\zeta=0.29$$
$$\omega_n=0.88$$

$$T(s)=\frac{0.774}{s^2+0.51s+0.774}$$

$$T_s=15.7\quad(17)(\text{sec})$$
$$OS=37\%\quad(34\%)$$

Big Picture



Control Objectives

Control Objectives:

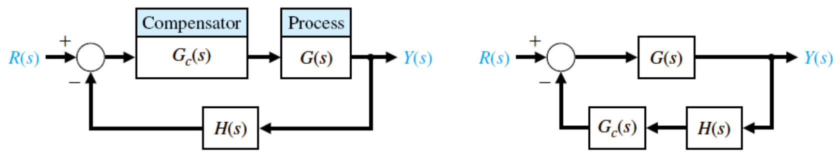
- Good stability
- Desired transient responses
- Minimum steady-state error
- Good regulation against disturbances
- Less sensitivity to system parameter changes

- Small critical signals
- Cheap implementation
- … …

Unfortunately, it is usually hard to find a controller to satisfy all the above conflicting and demanding specifications.

A compromise is more realizable in practice.

Cascade Compensators



$$G_c(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

Higher-order compensator can be constructed by cascading several first-order compensators.

First-order compensator:

$$G_c(s) = \frac{K(s+z)}{(s+p)}$$

Design of First-Order Compensator:

Select z, p and K to provide the overall system a satisfactory performance (transient response and steady-state error)

First-Order Compensator

Transfer Function:

$$G(s) = \frac{35s+35}{s+5}$$

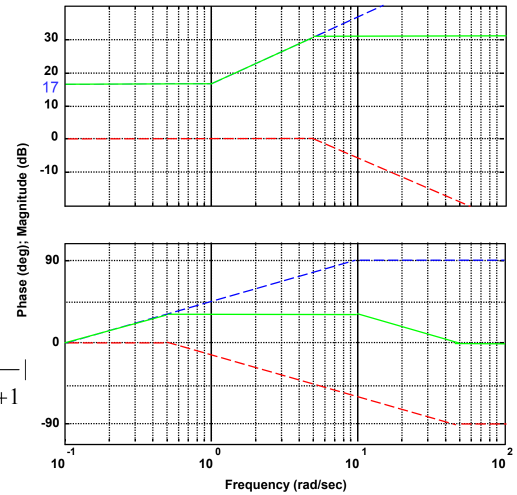
Plot the straight line approximation of $G(s)$'s Bode diagram:

$$G(j\omega) = \frac{35}{5}(j\omega+1)\bullet\frac{1}{\frac{1}{5}j\omega+1}$$

$$20\log_{10}|G(j\omega)| =$$

$$\underbrace{16.9}_{20\log_{10}7} + 20\log_{10}|j\omega+1| + 20\log_{10}\left|\frac{1}{\frac{1}{5}j\omega+1}\right|$$

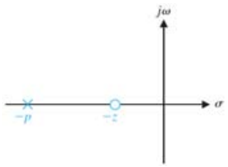
$$\angle G(j\omega) = \angle(j\omega+1) + \angle\left(\frac{1}{\frac{1}{5}j\omega+1}\right)$$



Phase-Lead Compensator

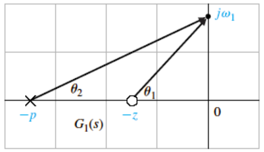
First-order compensator:

$$G_c(s) = \frac{K(s+z)}{(s+p)}$$



If $|z| < |p|$: phase-lead

$$G_c(j\omega) = \frac{K(j\omega+z)}{(j\omega+p)} = \frac{Kz}{p}\frac{[j(\omega/z)+1]}{[j(\omega/p)+1]} = \frac{K_1(1+j\omega\alpha\tau)}{(1+j\omega\tau)}$$

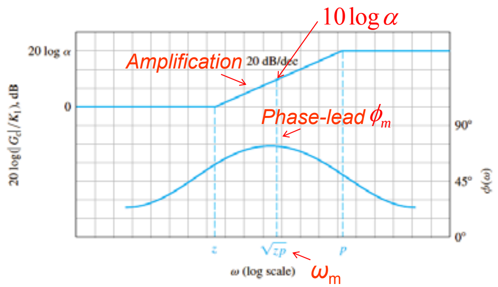


$\tau = 1/p,$

$p = \alpha z, \qquad \alpha > 1$

$K_1 = K/\alpha$

$$\phi(\omega) = \tan^{-1}\alpha\omega\tau - \tan^{-1}\omega\tau$$
$$= \tan^{-1}\frac{\alpha\omega\tau - \omega\tau}{1+(\omega\tau)^2\alpha}$$

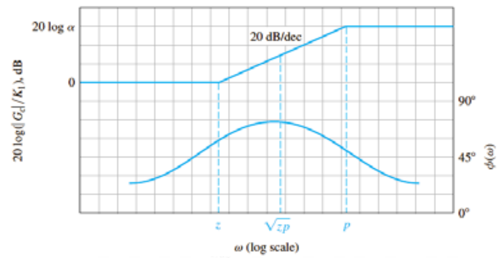


Purpose: increase PM (stabilize)

Phase-Lead Compensator

Phase-Lead compensator:

$$G_c(j\omega) = \frac{K(j\omega+z)}{(j\omega+p)} = \frac{K_1(1+j\omega\alpha\tau)}{(1+j\omega\tau)}$$



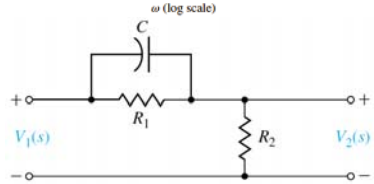
Circuit Implementation:

TF: $G_c(s) = \frac{V_2(s)}{V_1(s)}$

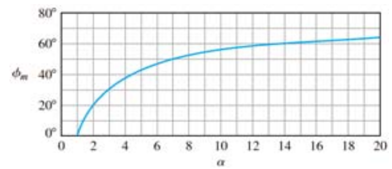
$$= \frac{R_2}{R_2 + \{R_1(1/Cs)/[R_1+(1/Cs)]\}}$$

$$= \left(\frac{R_2}{R_1+R_2}\right)\frac{1+R_1Cs}{\{1+[R_1R_2/(R_1+R_2)]Cs\}}$$

$$= \frac{(1+\alpha\tau s)}{\alpha(1+\tau s)} \qquad \tau = \frac{R_2}{R_1+R_2}C \qquad \alpha = \frac{R_1+R_2}{R_2}$$



$$\omega_m = \frac{1}{\tau\sqrt{\alpha}} \qquad \tan\phi_m = \frac{\alpha-1}{2\sqrt{\alpha}} \qquad \left(\sin\phi_m = \frac{\alpha-1}{\alpha+1}\right)$$

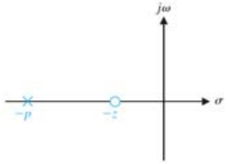


practically, $\phi_m < 70^\circ$ for a single first-order phase-lead compensator, cascade more for more phase-lead.

Differentiator

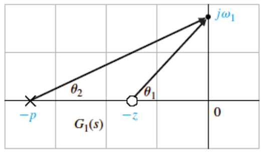
First-order compensator:

$$G_c(s) = \frac{K(s+z)}{(s+p)}$$



If $|z| < |p|$: phase-lead

$$G_c(j\omega) = \frac{K(j\omega+z)}{(j\omega+p)} = \frac{Kz}{p}\frac{[j(\omega/z)+1]}{[j(\omega/p)+1]} = \frac{K_1(1+j\omega\alpha\tau)}{(1+j\omega\tau)}$$

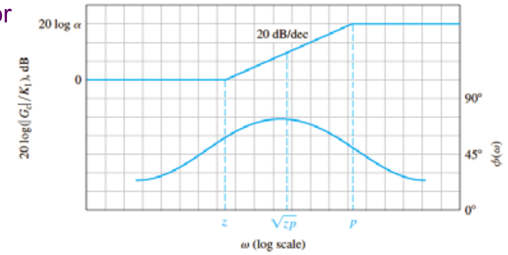


if $|z| << |p|$ and $z \approx 0$: differentiator

pole is negligible

$$G_c(s) \approx \left(\frac{K}{p}\right)s$$

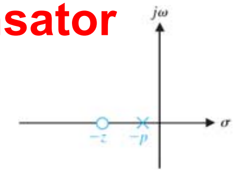
$$= \left(\frac{K}{p}\right)j\omega = \left(\frac{K}{p}\omega\right)e^{+j90^\circ}$$



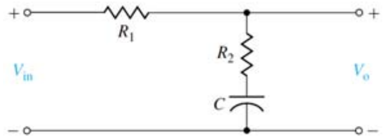
Phase-Lag Compensator

Phase-Lag compensator:

$$G_c(j\omega) = \frac{K(j\omega+z)}{(j\omega+p)} = \frac{K_1(1+j\omega\alpha\tau)}{(1+j\omega\tau)}$$



Circuit Implementation:



TF: $G_c(s) = \dfrac{V_o(s)}{V_i(s)}$

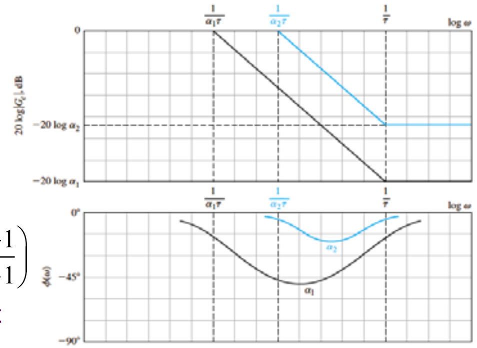
$$= \frac{R_2 + (1/Cs)}{R_1 + R_2 + (1/Cs)}$$

$$= \frac{1 + R_2 Cs}{1 + (R_1 + R_2)]Cs}$$

$$= \frac{1+\tau s}{1+\alpha\tau s} \qquad \tau = R_2 C \qquad \alpha = \frac{R_1 + R_2}{R_2}$$

$$\omega_m = \frac{1}{\tau\sqrt{\alpha}} \qquad \tan\phi_m = \frac{\alpha-1}{2\sqrt{\alpha}} \qquad \left(\sin\phi_m = \frac{\alpha-1}{\alpha+1}\right)$$

Purpose: increase ss error constant



Phase-Lead Design Using BD

Phase-Lead Design Procedure:

(add phase-lead to compensate the frequency response)

Step 1: Evaluate the uncompensated PM when the error constants are satisfied.

Step 2: Determine the required ϕ_m (with a small amount of safety)

Step 3: Calculate α $\left(\sin\phi_m = \dfrac{\alpha-1}{\alpha+1}\right)$

Step 4: Calculate $10\log\alpha$, determine the frequency at which the uncompensated magnitude = $-10\log\alpha$ dB (0-dB crossover frequency and also ω_m) $\omega_m = \omega_c$

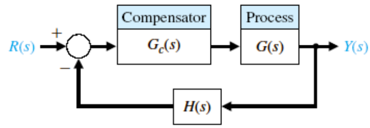
Step 5: Calculate $p = \omega_m\sqrt{\alpha}$, $z = p/\alpha$

Step 6: Draw the compensated frequency response, check the PM. Repeat the above steps if needed. Raise gain to make up for the attenuation $1/\alpha$.

Phase-Lead Design Using BD

Example 10.1: Lead Compensation for a Type-Two System

$$G(s) = \frac{K}{s^2}, \quad H(s) = 1$$



Without Compensation:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K}{s^2+K}$$

Specifications:

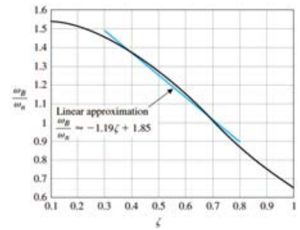
1. $T_s < 4(\text{sec})$ $(\delta = 2\%)$
2. $\zeta \geq 0.45$;

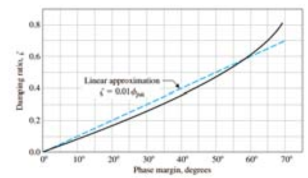
$\Rightarrow \omega_n = 2.22$

$$K = \omega_n^2 \approx 5$$

$$\Rightarrow \omega_B = 1.33\omega_n = 3.00 \quad (\zeta = 0.45)$$

$$\phi_{pm} = \frac{\zeta}{0.01} = 45^\circ$$



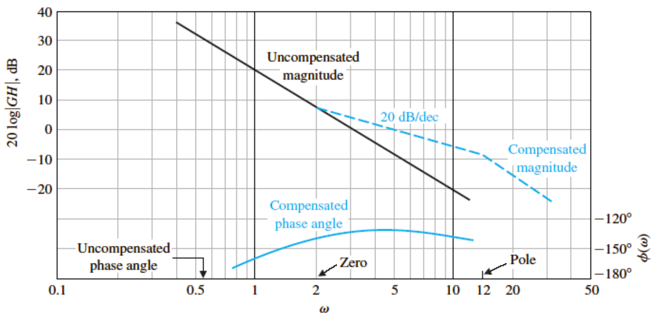


Phase-Lead Design Using BD

Example 10.1:

$$GH(j\omega)=\frac{K}{(j\omega)^2}$$

$$K=10$$



$$\phi_m=\sin^{-1}\left(\frac{\alpha-1}{\alpha+1}\right)=45^\circ$$

$$\Rightarrow\ \alpha=5.8\rightarrow 6$$

$$10\log\alpha=7.8\rightarrow\omega_m=4.95\qquad\Rightarrow\ p=12.0,z=2.0$$

$$G_c(j\omega)G(j\omega)H(j\omega)=\frac{10[(j\omega/2.0)+1]}{(j\omega)^2[(j\omega/12.0)+1]}\qquad G_c(s)=\boxed{\frac{1}{6}}\frac{(s/2.0)+1}{[(s/12.0)+1]}$$

$$T(s)=\frac{60(s+2)}{(s^2+6s+20.0)(s+6)}$$

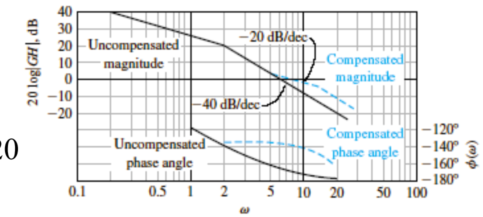
o.s.=34%, T_s=1.4s

Gain attenuation due to the passive RC network, an amplifier with gain=6 required

Phase-Lead Design Using BD

Example 10.2: Lead Compensation for a 2nd-Order System

$$GH(s)=\frac{K}{s(s+2)}$$



Specifications:

1. e_{ss}=5% (ramp input) $\Rightarrow K_v=20$
2. $\phi_{pm}\geq 45^\circ$

$$GH(j\omega)=\frac{K_v}{j\omega(0.5j\omega+1)}\leftarrow\frac{K}{2j\omega(0.5j\omega+1)}\qquad \measuredangle GH(j\omega)=-90^\circ-\tan^{-1}(0.5\omega)$$

$$\omega_c=6.2\text{ rad/s}\quad \phi(\omega)=-162^\circ\quad \phi_m=1.1\times(45^\circ-18^\circ)=30^\circ\quad \frac{\alpha-1}{\alpha+1}=\sin 30^\circ\Rightarrow\ \alpha=3$$

$$10\log\alpha=4.8\rightarrow\omega_m=8.4\qquad \omega_m=\omega_c=8.4\qquad\Rightarrow\ p=14.4,z=4.8$$

$$G_c(s)=\boxed{\frac{1}{3}}\frac{(s/4.8)+1}{[(s/14.4)+1]}\qquad G_cGH(s)=\frac{20[(s/4.8)+1]}{s(0.5s+1)[(s/14.4)+1]}$$

$$\phi(\omega_c)=-90^\circ-\tan^{-1}(0.5\omega_c)-\tan^{-1}\left(\frac{\omega_c}{14.4}\right)+\tan^{-1}\left(\frac{\omega_c}{4.8}\right)=-136.3^\circ\qquad \phi_{pm}=43.7^\circ$$

Phase-Lead Design Using RL

Phase-Lead Design Procedure:

(change the locations of the system dominant roots)

Step 1: Specifications → desired root locations for the dominant roots.

Step 2: Sketch the uncompensated RL, determine whether the desired root locations can be realized.

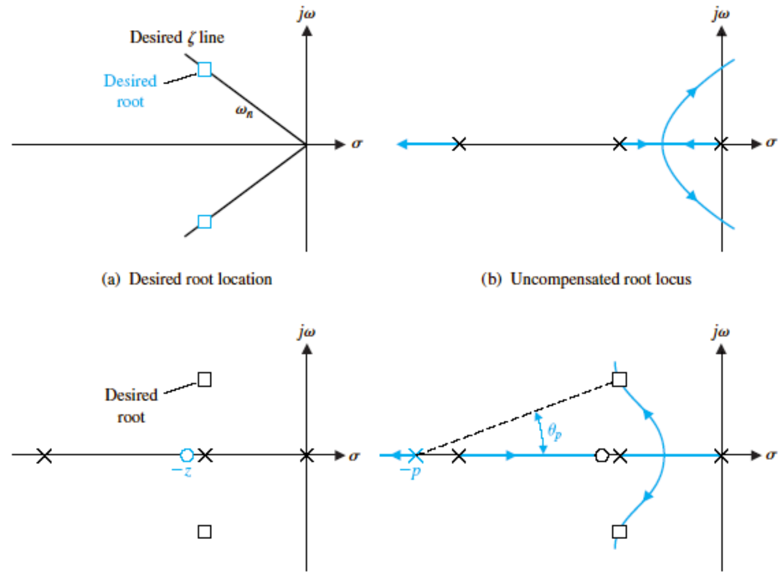
Step 3: If a compensator needed, place the compensator zero below the desired root location or to the left of the first two real poles.

Step 4: Determine the compensator pole → total angle at the desired root location is 180^0 (on the compensated RL).

Step 5: Evaluate the total gain at the desired root location, calculate the error constant.

Step 6: Repeat the above steps if the error constant is not satisfactory.

Phase-Lead Design Using RL



(a) Desired root location

(b) Uncompensated root locus

(c) Addition of zero

(d) Location of new pole

Phase-Lead Design Using RL

Example 10.3: Lead Compensation for a Type-Two System

$$G(s)=\frac{K}{s^2},\quad H(s)=1$$

Specifications:

1. $T_s < 4(\text{sec})$ $(\delta = 2\%)$
2. P.O. $\leqq 35\%$ $\Rightarrow \zeta \geq 0.32;$

$$r_1, \hat{r}_1 = -1 \pm j2 \quad (\zeta{=}0.45)$$

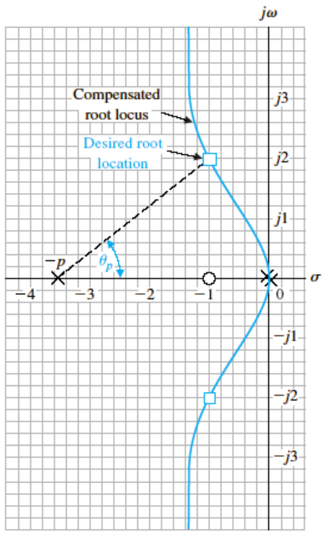
$$\phi_p = 180^\circ + (90^\circ - 116^\circ - 116^\circ) = 38^\circ$$

$$G_c(s)=\frac{s+1}{s+3.6} \qquad G_cGH(s)=\frac{K_1(s+1)}{s^2(s+3.6)}$$

$$K_1=\frac{(2.23)^2(3.25)}{2}=8.1$$

$$K_a = 8.1/3.6 = 2.25$$

$\Rightarrow$ P.O.=46% $\quad T_s = 3.8(\text{sec})$ $(\delta = 2\%)$



Phase-Lead Design Using RL

Example 10.4: Lead Compensation for a 2nd-Order System

$$GH(s)=\frac{K}{s(s+2)}$$

Specifications:

1. e_{ss}=5% (ramp input) $\Rightarrow K_v = 20$
2. $\zeta = 0.45$ $\qquad K = 40$

Without Compensation: Compensation needed

CLCP: $s^2+2s+40 \Rightarrow \zeta = \boxed{0.16}$

Desired system:

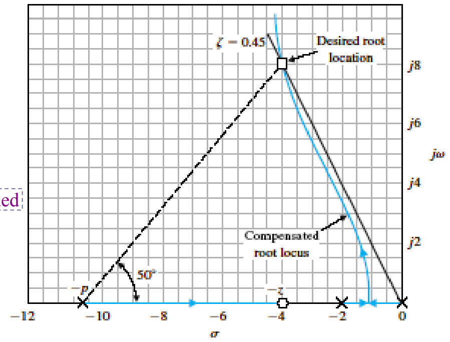
$$\zeta = 0.45, \omega_n = 9$$

Compensator:

$z=-4 \Rightarrow \phi = -116^\circ - 104^\circ + 90^\circ = -130^\circ \Rightarrow \theta_p = 50^\circ \Rightarrow p = 10.6$

$\Rightarrow K=\dfrac{9\times 8.25\times 10.4}{8}=96.5 \Rightarrow G_cGH(s)=\dfrac{96.5(s+4)}{s(s+2)(s+10.6)}$ $\quad \boxed{K_v = 18.2}$

Compensated system



Phase-Lead Design Using RL

Example 10.4: Lead Compensation for a 2nd-Order System

$$GH(s)=\frac{K}{s(s+2)}$$

Specifications:

1. e_{ss}=5% (ramp input) $\Rightarrow K_v = 20$
2. $\zeta = 0.45$ $\qquad K = 40$

Repeat:

Desired system:

$$\zeta = 0.45, \omega_n = 10$$

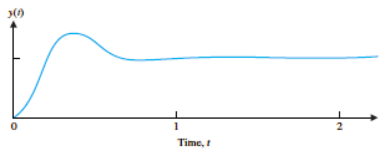
Compensator:

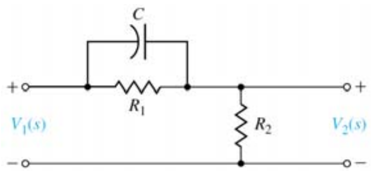
$z=-4.5 \Rightarrow p = 11.6 \Rightarrow K_v = 22.7$

$$G_c(s)=\frac{s+4.5}{s+11.6}$$

Compensated system:

O.S.=32%, T_s=0.8 *s*



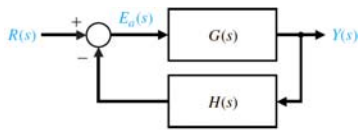


$$G_c(s)=\left(\frac{R_2}{R_1+R_2}\right)\frac{1+R_1Cs}{\{1+[R_1R_2/(R_1+R_2)]Cs\}}$$

Steady-State Error of Feedback Control Systems (Section 5.7)

$$E(s)=\frac{1}{1+G(s)}R(s)$$

$$e_{ss}=\lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$



$H(s)=1$

Unity feedback control system

Standard Test Input Signals (general form)

$$r(t)=t^n,\quad R(s)=n!/s^{n+1}$$

Loop Transfer Function

$$G(s)=\frac{K\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+p_k)}$$

Steady-State Error of Feedback Control Systems

$$e_{ss}=\lim_{s\to 0}\frac{n!}{s^{n}\left(1+\dfrac{K\prod_{i=1}^{M}(s+z_i)}{s^{N}\prod_{k=1}^{Q}(s+p_k)}\right)}=\begin{cases}\dfrac{A}{1+K\prod_{i=1}^{M}z_i/\prod_{k=1}^{Q}p_k} & \textit{if } N=n\\ \infty & \textit{if } N=0\ \&\ n>0\\ \lim\limits_{s\to 0}\dfrac{s^{N}}{s^{n}}=\begin{cases}0 & \textit{if } N>n\\ \infty & \textit{if } N<n\end{cases} & \end{cases}$$

if N>0 & N≠n

System Type Number: N

(number of integrations)

Steady-State Error of Feedback Control Systems

$R(s)=A/s$

1. Step input

$$e_{ss}=\lim_{s\to 0}\frac{s(A/s)}{1+G(s)}=\frac{A}{1+G(0)}=\begin{cases}\dfrac{A}{1+K\prod_{i=1}^{M}z_i/\prod_{k=1}^{Q}p_k} & \textit{if } N=0\ \text{(Type 0)}\\ 0 & \textit{if } N>0\ \text{(Type 1,2,…)}\end{cases}$$

Position Error Constant: (if $N=0$)

$$K_p=\lim_{s\to 0}G(s)=G(0)$$

$$e_{ss}=\frac{A}{1+K_p}$$

Steady-State Error of Feedback Control Systems

$R(s)=A/s^2$

2. Ramp input

$$e_{ss}=\lim_{s\to 0}\frac{s(A/s^2)}{1+G(s)}=\lim_{s\to 0}\frac{A}{s+sG(s)}=\lim_{s\to 0}\frac{A}{sG(s)}=\begin{cases}\infty & \textit{if } N<1\ \text{(Type 0)}\\ \dfrac{A}{K_v} & \textit{if } N=1\ \text{(Type 1)}\\ 0 & \textit{if } N>1\ \text{(Type 2,3…)}\end{cases}$$

Velocity Error Constant: (if $N=1$)

$$K_v=\lim_{s\to 0}sG(s)$$

$$e_{ss}=\frac{A}{K_v}$$

Steady-State Error of Feedback Control Systems

$R(s)=A/s^3$

3. Acceleration input

$$e_{ss}=\lim_{s\to 0}\frac{s(A/s^3)}{1+G(s)}=\lim_{s\to 0}\frac{A}{s^2G(s)}=\begin{cases}\infty & \textit{if } N<2\ \text{(Type 0 \& 1)}\\ \dfrac{A}{K_a} & \textit{if } N=2\ \text{(Type 2)}\\ 0 & \textit{if } N>2\ \text{(Type 3,4,…)}\end{cases}$$

Acceleration Error Constant: (if $N=2$)

$$K_a=\lim_{s\to 0}s^2G(s)$$

$$e_{ss}=\frac{A}{K_a}$$

Steady-State Error of Feedback Control Systems

Table 5.5

Table 5.5 Summary of Steady-State Errors

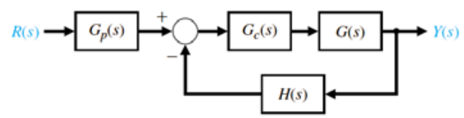
Number of Integrations in G(s), Type Number	Input: Step, $r(t) = A$, $R(s) = A/s$	Ramp, At, A/s^2	Parabola, $At^2/2$, A/s^3
0	$e_{ss} = \frac{A}{1+K_p}$	Infinite	Infinite
1	$e_{ss} = 0$	$\frac{A}{K_v}$	Infinite
2	$e_{ss} = 0$	0	$\frac{A}{K_a}$

Error constants represent the ability of a system to reduce/eliminate e_{ss}.

Integration Networks

Control Objectives:

1. Good Transient Responses
2. Minimum Steady-State Error
3. … …



? Increase the forward gain $\Leftrightarrow$ Increase the error constants

$\Rightarrow$ Bad/unacceptable/unstable transient response X

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\left[\frac{R(s)}{1+G_c(s)G(s)H(s)}\right]$$

System Type Number: N
(number of integrations)

Enlarge N

Integration Networks:

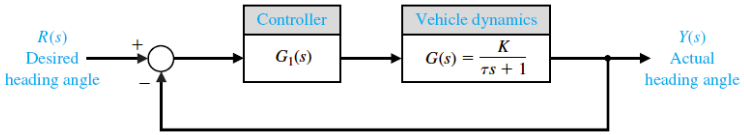
Purpose: increase ss error constant

PI Controller:

$$G_c(s) = K_P + \frac{K_I}{s}$$

Steady-State Error of Feedback Control Systems

Example 5.3



$$G_1(s) = K_1 + \frac{K_2}{s} \quad \text{(P+I controller)}$$

If $G_1(s) = K_1$ (P controller)

$\Rightarrow$ Type 0 CL system $\Rightarrow$ $e_{ss} = \frac{A}{1+K_p}$, $K_p = KK_1$ for a step input

If $G_1(s) = K_1 + K_2/s$ (PI controller)

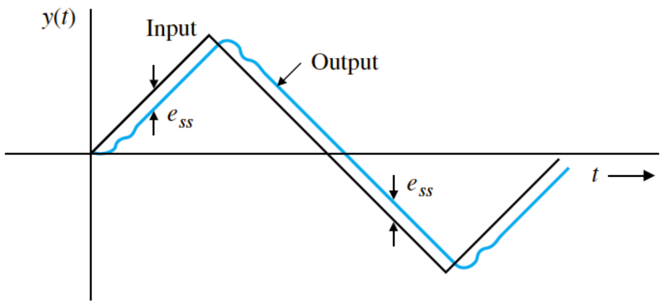
$\Rightarrow$ Type 1 CL system $\Rightarrow$ $e_{ss} = 0$ for a step input

Steady-State Error of Feedback Control Systems

Example 5.3

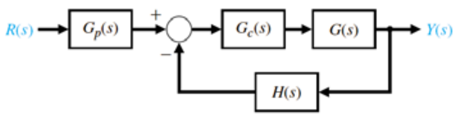
If $G_1(s) = K_1 + K_2/s$ (PI controller) for a ramp input

$\Rightarrow$ Type 1 CL system $\Rightarrow$ $e_{ss} = \frac{A}{K_v}$, $K_v = K_2 K$



velocity meets the requirement

Integration Networks



Heat Process:

$$G(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}$$

Uncompensated SS Error:

$$\lim_{t\to\infty} e(t)=\lim_{s\to 0} s\left[\frac{A/s}{1+G(s)H(s)}\right]=\frac{A}{1+K}$$

PI Controller:

$$G_c(s)=K_P+\frac{K_I}{s}$$

How about …

the transient response?

if a ramp input?

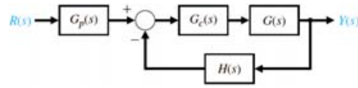
Compensated SS Error:

$$\lim_{t\to\infty} e(t)=\lim_{s\to 0} s\left[\frac{A/s}{1+G_c(s)GH(s)}\right]$$

$$=\lim_{s\to 0}\frac{A}{1+[(K_P s+K_I)/s]\{K/[(\tau_1 s+1)(\tau_2 s+1)]\}}=0$$

Integration Networks

Example 10.5: Temperature Control System

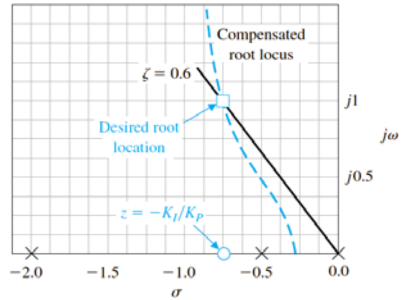


$$GH(s)=\frac{K_1}{(2s+1)(0.5s+1)}$$

Specifications:

1. O.S.≦10% (step input)
2. zero e_{ss}

$$\zeta=0.6,\quad \zeta\omega_n=0.75\ (T_s=5.33s)$$



PI Controller:

$$G_c(s)=K_P+\frac{K_I}{s}=K_P\left(\frac{s+K_I/K_P}{s}\right)$$

Determine the zero: $z=K_I/K_P$

Angle at the desired root = 180^0

$$\theta_z=-180^\circ+127^\circ+104^\circ+38^\circ=89^\circ$$

$$\Rightarrow\quad z=-0.75$$

$$K=K_1K_P=1.25\cdot 1.03\cdot 1.6/1.0=2.08$$

$$T(s)=\frac{2.08(s+0.75)}{(s+1)(s-0.75+j)(s-0.75-j)}$$

O.S.=17.6% (without G_p)

Prefilter:

$$G_P(s)=\frac{0.75}{s+0.75}$$

O.S.=2% (with G_p)

Phase-Lag Design Using RL

Uncompensated SS Error:

$$\lim_{t\to\infty} e(t)=\lim_{s\to 0} s\left[\frac{R(s)}{1+GH(s)}\right]\qquad K_v=\lim_{s\to 0} sGH(s)=\frac{K\prod_{i=1}^{M} z_i}{\prod_{j=1}^{n} p_j}$$

Phase-Lag compensator:

$$G_c(s)=\frac{1}{\alpha}\frac{(s+z)}{(s+p)}\qquad \alpha=\frac{z}{p}\gg 1$$

Type 1:

$$GH(s)=(K\prod_{i=1}^{M} z_i)/(s\prod_{j=1}^{n} p_j)$$

Compensated SS Error:

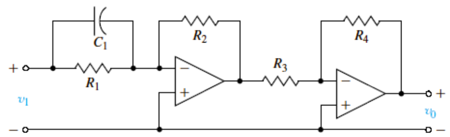
$$K_{v_{comp}}=\lim_{s\to 0} s\{G_c(s)GH(s)\}=\lim_{s\to 0} s\{G_c(s)\}K_{v_{uncomp}}$$

$$=\left(\frac{z}{p}\right)\left(\frac{1}{\alpha}\right)K_{v_{uncomp}}=\left(\frac{z}{p}\right)\left(\frac{K}{\alpha}\right)\left(\frac{\prod z_i}{\prod p_j}\right)$$

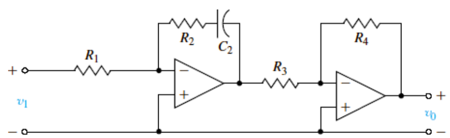
If the attenuation (due to the passive RC network) can be made up for (by raising the gain of the amplifier), then K_v will be increased by the ratio α=z/p.

If the zero-pole pairs appear relatively close together and near the origin, then their effect on the desired roots is negligible (no change on the *transient response*).

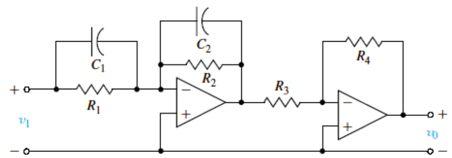
Operational Amplifier Circuit for Compensators



$$G_c(s)=\frac{R_4R_2}{R_3R_1}(R_1C_1s)+1$$



$$G_c(s)=\frac{R_4R_2(R_1C_1s+1)}{R_3R_1(R_2C_2s)}$$



$$G_c(s)=\frac{R_4R_2(R_1C_1s+1)}{R_3R_1(R_2C_2s+1)}$$

Phase-Lag Design Using RL

Phase-Lag Design Procedure:

(increase the error constant to reduce the steady-state error)

Step 1: Sketch RL of the uncompensated system.

Step 2: Transient performance specifications $\Rightarrow$ suitable dominant root locations on the uncompensated RL $\Rightarrow$ satisfy the specifications.

Step 3: Calculate the loop gain at the desired root locations $\Rightarrow$ error constant.

Step 4: Required α (error constant: uncompensated $\Rightarrow$ compensated)

Step 5: α $\Rightarrow$ pole-zero for the compensator $\Rightarrow$ the compensated RL passes through the desired root location.

Merge compensator pole and zero

$\Rightarrow$ At the root location the angles from the compensator pole and zero are essentially equal ($<2^0$)

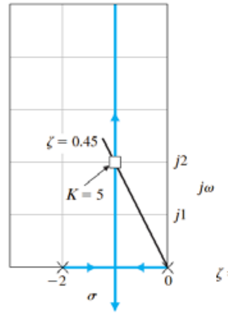
Phase-Lag Design Using RL

Example 10.6:

$$GH(s)=\frac{K}{s(s+2)}$$

Specifications:

1. e_{ss}=5% (ramp input) $\Rightarrow K_v=20$
2. $\zeta=0.45$



Desired roots:

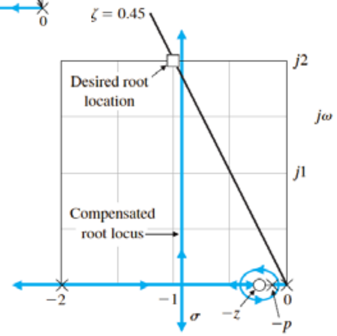
$s=-1\pm j2 \Rightarrow$ $K=2.24^2=5$, $K_v=K/2=2.5$

Compensator: $\frac{z}{p}=\frac{20}{2.5}=8$

Set: $z=-0.1 \Rightarrow p=-0.0125$

Compensated system:

$$G_cGH(s)=\frac{5(s+0.1)}{s(s+2)(s+0.0125)}$$



Phase-Lag Design Using RL

Example 10.7:

$$GH(s)=\frac{K}{s(s+10)^2}$$

Specifications:

Uncompensated:

1. $K_v=20$ $=K/10^2=20 \Rightarrow K=2000$
2. $\zeta=0.707$

Unstable!

Compensation:

Desired roots:

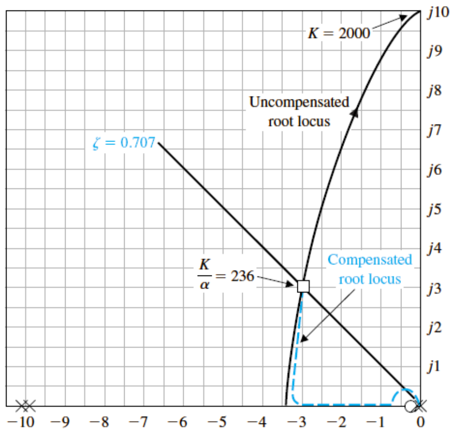
$s=-2.9\pm j2.9 \Rightarrow K=236$

Compensator: $\frac{z}{p}=\frac{2000}{236}=8.5$

Set: $z=-0.1 \Rightarrow p=-0.011$

Compensated system:

$$G_cGH(s)=\frac{236(s+0.1)}{s(s+10)^2(s+0.011)}$$



Phase-Lag Design Using BD

Phase-Lag Design Procedure:

$$G_c(j\omega)=\frac{1+j\omega\tau}{1+j\omega\alpha\tau}$$

(attenuation $\Rightarrow$ lower ω_c $\Rightarrow$ increase PM $\Rightarrow$ satisfy specifications)

Step 1: BD for the uncompensated system when the error constants are satisfied.

Step 2: Determine the PM of the uncompensated system.

Step 3: Determine $\omega_c^{'}$, the crossover frequency of the compensated system where PM requirement satisfied (assume 5º lag)

Step 4: Place the zero of the compensator a decade below $\omega_c^{'}$.

Step 5: Measure the required attenuation at $\omega_c^{'}$.

Step 6: Calculate α $\Rightarrow$ the attenuation at $\omega_c^{'}$ is $-20\log\alpha$.

Step 7: Calculate the ploe: $\omega_p=1/\alpha\tau=\omega_z/\alpha$.

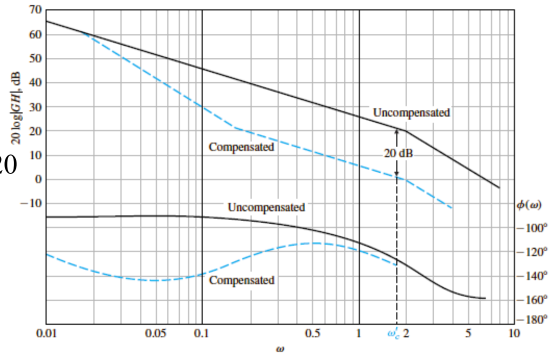
Phase-Lag Design Using BD

Example 10.8:

$$GH(j\omega)=\frac{K}{j\omega(j\omega+2)}$$

Specifications:

1. e_{ss}=5% (ramp input) $\Rightarrow K_v = 20$
2. $\zeta = 0.45 \quad \Rightarrow$ PM = 45°



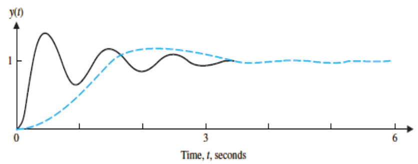
Uncompensated PM = 20°

$\Rightarrow 130^\circ = 180^\circ - 45^\circ - 5^\circ$

$\Rightarrow \omega_c' = 1.5$

$\Rightarrow$ Attenuation required = 20 dB

$20\log\alpha = 20 \text{ dB} \Rightarrow \alpha = 10$



Compensator zero: $\omega_z = \omega_c'/10 = 0.15$

Pole: $\omega_p = \omega_z/10 = 0.015$

Compensated system: $G_cGH(j\omega)=\dfrac{20(6.66j\omega+1)}{j\omega(0.5j\omega+1)(j66.6\omega+1)}$

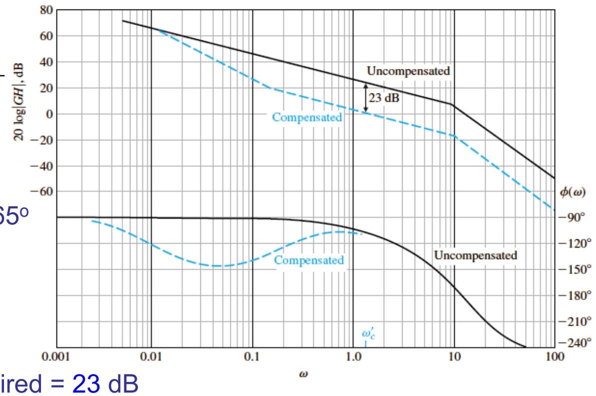
Phase-Lag Design Using BD

Example 10.9:

$$GH(j\omega)=\frac{K}{j\omega(j\omega+10)^2}$$

Specifications:

1. $K_v = 20$
2. $\zeta = 0.707 \quad \Rightarrow$ PM = 65°



Uncompensated PM = 0°

$\Rightarrow 110^\circ = 180^\circ - 65^\circ - 5^\circ$

$\Rightarrow \omega_c' = 1.5$

$\Rightarrow$ Attenuation required = 23 dB

$20\log\alpha = 23 \text{ dB} \Rightarrow \alpha = 14.2$

Compensator zero: $\omega_z = \omega_c'/10 = 0.15$ Pole: $\omega_p = \omega_z/14.2 = 0.0106$

Compensated system: $G_cGH(j\omega)=\dfrac{20(6.66j\omega+1)}{j\omega(0.5j\omega+1)(j66.6\omega+1)}$

$\Rightarrow \omega_c' = 1.5$ PM = 67°

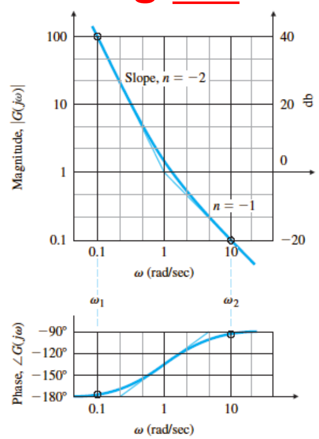
Phase-Lag Design Using BD

Remarks

1. Disadvantages:
   ω_c lowered $\Rightarrow$ CL Bandwidth reduced $\Rightarrow$ slow responses.
   phase-lag $\Rightarrow$ PM reduced
2. Implementation:
   Smaller ω'_c? $\Rightarrow$ Infeasible implementation
3. … … the system design is satisfactory when the asymptotic curve for the magnitude of the compensated system crosses the 0-dB line with a slope of -20dB/decade. ???
4. Lead-Lag Compensator?

Phase-Lag Design Using BD

When the slope of |G(jω)| versus ω on a log–log scale persists at a constant value for approximately a decade of frequency, the relationship is particularly simple and is given by $\angle G(j\omega) = n \times 90^\circ$.



Bode's Gain-Phase Relationship:

For any stable minimum-phase system (that is, one with no RHP zeros or poles), the phase of G(jω) is uniquely related to the magnitude of G(jω).

$$\angle G(j\omega_o)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{dM}{du}\right)W(u)\,du \quad \text{(in radians)}$$

Phase-Lag Design Using BD

A Very Simple Design Criterion:

Adjust the slope of the magnitude curve $|KG(j\omega)|$ so that it crosses over magnitude 1 with a slope of −1 for a decade around ω_c .

Lead-Lag Compensator

Lead-Lag Compensator:

Lag part for attenuation $\Rightarrow$ low-frequency portion $\Rightarrow$ raise the error consant

+

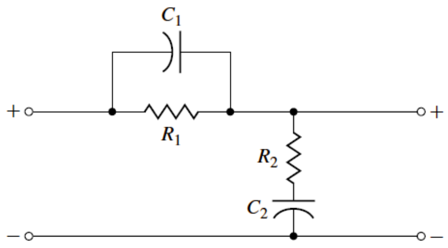
Lead part for phase-lead $\Rightarrow$ around ω_c $\Rightarrow$ increase PM

$$G_c(s) = \frac{V_2(s)}{V_1(s)} = \frac{(1+\alpha\tau_1 s)}{(1+\tau_1 s)}\frac{(1+\beta\tau_2 s)}{(1+\tau_2 s)}$$

$$\alpha > 1 \quad \beta < 1 \quad \alpha\beta = 1$$

$$= \frac{(R_1C_1s+1)(R_2C_2s+1)}{R_1R_2C_1C_2s^2+(R_1C_1+R_1C_2+R_2C_2)s+1}$$

$$\alpha\tau_1 = R_1C_1,\ \beta\tau_2 = R_2C_2,\ \tau_1\tau_2 = R_1C_1R_2C_2$$



Design on BD Analytically

Previous Methods: trial-and-error

Computational Methods: analytical using computers

Compensator: $G_c(s) = \dfrac{1+\alpha\tau s}{1+\tau s}$ $\quad \alpha > 1$ Lead compensator $\quad \alpha < 1$ Lag compensator

Phase added at ω_c:

$$p = \tan\phi = \frac{\alpha\omega_c\tau - \omega_c\tau}{1+(\omega_c\tau)^2\alpha}$$

Magnitude added at ω_c:

$$c = 10^{M/10} = \frac{1+(\omega_c\alpha\tau)^2}{1+(\omega_c\tau)^2}$$

Equation for α:

$$(p^2-c+1)\alpha^2 + 2p^2c\alpha + p^2c^2 + c^2 - c = 0$$

for a single-stage compensator

$$c > p^2+1$$

$$\tau = \frac{1}{\omega_c}\sqrt{\frac{1-c}{c-\alpha^2}}$$

Design on BD Analytically

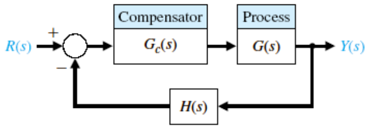
Phase-Lead Compensator Design on BD Analytically:

1. Select the desired ω_c
2. Determine the desired PM $\Rightarrow$ the required phase ϕ
3. Verify if the phase is applicable: $\phi > 0$ and $M > 0$
4. Determine whether a single stage is sufficient when $c > p^2+1$
5. Calculate α $\Leftarrow$ $(p^2-c+1)\alpha^2 + 2p^2c\alpha + p^2c^2 + c^2 - c = 0$
6. Calculate τ $\Leftarrow$ $\tau = \dfrac{1}{\omega_c}\sqrt{\dfrac{1-c}{c-\alpha^2}}$

Design on <u>BD</u> Analytically

Example 10.10: Lead Compensation for a Type-Two System

$$G(s)=\frac{K}{s^2},\quad H(s)=1$$



Without Compensation:

$$T(s)=\frac{Y(s)}{R(s)}=\frac{K}{s^2+K}$$

Specifications:

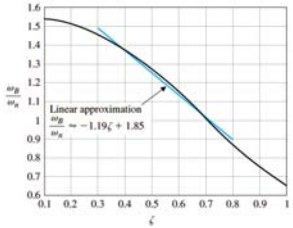
1. $T_s < 4(\text{sec})$ $(\delta = 2\%)$
2. $\zeta \geq 0.45$;

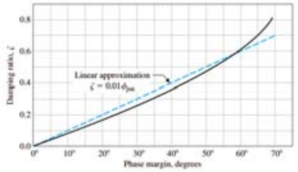
$$\Rightarrow \omega_n = 2.22$$

$$K=\omega_n^2 \approx 5$$

$$\Rightarrow \quad \omega_B = 1.33\omega_n = 3.00 \quad (\zeta{=}0.45)$$

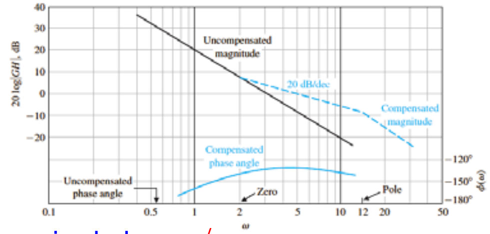
$$\phi_{pm}=\frac{\zeta}{0.01}=45^\circ$$





Design on <u>BD</u> Analytically

Example 10.10:



1. Select the desired ω_c

   $\omega_c = 5$

2. Determine the desired <u>PM</u> / required phase ϕ

   $p=\tan 45^\circ = 1$ $\qquad c = 10^{8/10}=6.31$

3. Verify if the phase is applicable: $\phi > 0$ and $M > 0$

   Yes

4. Determine whether a single stage is sufficient when $c > p^2+1$

   Yes

5. Calculate α

   $\alpha = 5.84 \quad \Leftarrow \quad -4.31\alpha^2+12.62\alpha+73.32=0$

6. Calculate τ

   $\tau = 0.087$

$$G_c(s)=\frac{1+0.515s}{1+0.087s}=5.92\frac{s+1.94}{s+11.5}$$

A Zero

$$T(s)=\frac{(\omega_n^2/a)(s+a)}{s^2+2\zeta\omega_n s+\omega_n^2}$$

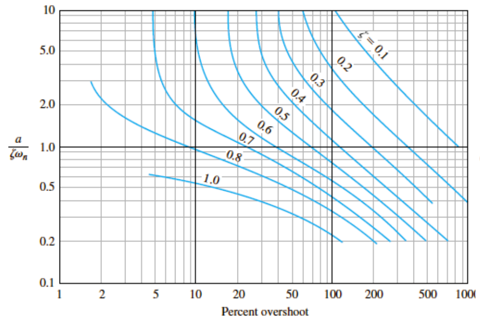
Table 5.4

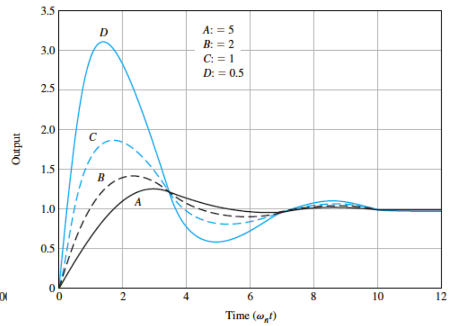
Table 5.4 The Response of a Second-Order System with a Zero and $\zeta = 0.45$

$a/\zeta\omega_n$	Percent Overshoot	Settling Time	Peak Time
5	23.1	8.0	3.0
2	39.7	7.6	2.2
1	89.9	10.1	1.8
0.5	210.0	10.3	1.5

If the zero is near the dominant poles, then it will *materially* affect system transient response.

A Zero



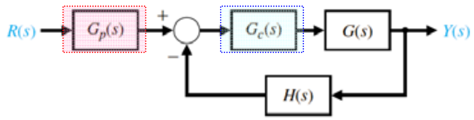


If the zero is near the dominant poles, then it will *materially* affect system transient response.

Prefilter

Compensator:

$$G_c(s)=\frac{s+z}{s+p}$$



CLTF (compensated):

$$T(s)=\frac{\boxed{G_pG_cG(s)}}{1+G_cG(s)}$$

contains zero of G_c

Example:

$$G(s)=\frac{1}{s}$$ Specifications: 1. T_s = 0.5s (2%) 2. O.S. = 4%

$$G_c(s)=K_P+\frac{K_I}{s}=\frac{(K_Ps+K_I)}{s}\qquad T(s)=\frac{(K_Ps+K_I)G_p(s)}{s^2+K_Ps+K_I}$$

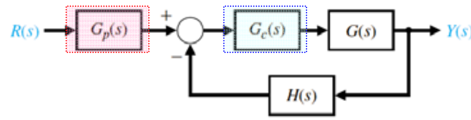
Spec: $\Rightarrow\ \zeta=1/\sqrt{2},\ \ \omega_n=8\sqrt{2}\ \Rightarrow\ K_P=16,\ \ K_I=128$

$$T(s)=\frac{16(s+8)G_p(s)}{s^2+16s+128}\qquad a/\zeta\omega_n=1\ \Rightarrow\ \text{O.S.}=21\%\quad \text{X}$$

Prefilter

Prefilter:

$$G_p(s)=\frac{8}{s+8}$$



CLTF (compensated & prefiltered):

$$T(s)=\frac{128}{s^2+16s+128}\quad\Rightarrow\quad \text{O.S.}=4.5\%$$

The zero will have significant effect if $a/\zeta\omega_n<5$.

Example 10.3:

$$T(s)=\frac{8.1(s+1)G_p(s)}{(s+1+j2)(s+1-j2)(s+1.62)}$$

$G_p(s)=\frac{1}{s+1}\ \Rightarrow$ O.S. = 6.7%, T_s=3.8s

$G_p(s)=1\ \Rightarrow$ O.S. = 46.6%, T_s=3.8s

Prefilter helps systems with *lead network* or *PI compensators*, but is not useful for systems with *lag compensators*.

zero effect insignificant

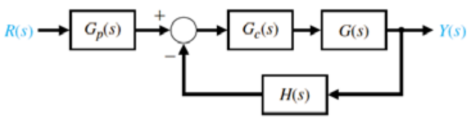
Example 10.6:

$$G_cGH(s)=\frac{5\,\boxed{(s+0.1)}}{s(s+2)\,\boxed{(s+0.0125)}}\qquad T(s)=\frac{5(s+0.1)}{(s^2+1.98s+5.1)(s+0.095)}\approx\frac{5}{(s^2+1.98s+5.1)}$$

Prefilter

Example 10.11: A Third-Order System

$$G(s)=\frac{1}{s(s+1)(s+5)}$$



Specifications:

1. O.S. ≦ 2% (step input)
2. Ts=3s

Lead Compensator:

$$G_c(s)=\frac{K(s+1.2)}{s+10}\quad\Rightarrow\ \zeta=0.1/\sqrt{2}\quad\Rightarrow\ K=78.7$$

$$T(s)=\frac{78.7(s+1.2)G_p(s)}{(s+1.71+j1.71)(s+1.71-j1.71)(s+1.45)(s+11.1)}$$

$$\approx\frac{7.1(s+1.2)G_p(s)}{(s^2+3.45s+5.85)(s+1.45)}$$

	O.S.	Tr	Ts
G_p=1:	9.9%	1.05	2.9
p=1.2:	0%	2.30	3.0
p=2.4:	4.8%	1.60	3.2

Prefilter:

$$G_p(s)=\frac{p}{s+p}\qquad T(s)\approx\frac{7.1p(s+1.2)}{(s^2+3.45s+5.85)(s+1.45)(s+p)}$$

Deadbeat Control

Deadbeat:

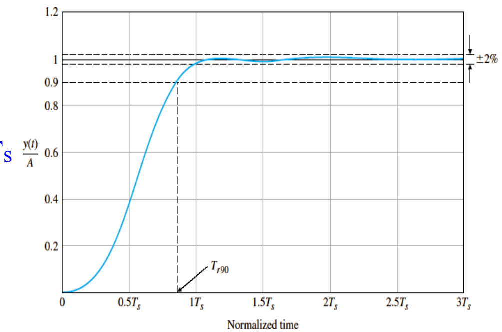
having an indicator that stops without oscillation (*American Heritage Dictionary*)

Deadbeat Response:

a response that proceeds rapidly to the desired level and holds at that level with minimal overshoot.

Characteristics:

1. $e_{ss}=0$
2. Fast response: min. T_r and T_s
3. 0.1 % < Overshoot < 2 %
4. Undershoot < 2 %



Deadbeat Control

CLTF: (3^{rd} order) $\quad \boxed{\bar{s} = s/\omega_n}$

$$T(s) = \frac{\omega_n^3}{s^3 + \alpha\omega_n s^2 + \beta\omega_n^2 s + \omega_n^3} \overset{\Downarrow}{=} \frac{1}{\bar{s}^3 + \alpha\bar{s}^2 + \beta\bar{s} + 1} \quad (normalized)$$

Coefficients for Deadbeat Response:

Table 10.2 Coefficients and Response Measures of a Deadbeat System

System Order	Coefficients α	β	γ	δ	ϵ	Percent Overshoot, P.O.	Percent Undershoot, P.U.	90% Rise Time, $T_{r_{90}}$	100% Rise Time, T_r	Settling Time, T_s
2nd	1.82					0.10%	0.00%	3.47	6.58	4.82
3rd	1.90	2.20				1.65%	1.36%	3.48	4.32	4.04
4th	2.20	3.50	2.80			0.89%	0.95%	4.16	5.29	4.81
5th	2.70	4.90	5.40	3.40		1.29%	0.37%	4.84	5.73	5.43
6th	3.15	6.50	8.70	7.55	4.05	1.63%	0.94%	5.49	6.31	6.04

Note: All time is normalized.

Deadbeat Control System Design:

1. Select coefficients for deadbeat response
2. Design the compensator, such that the compensated system CLTF has the the coefficients (*let two CLTFs equal*)

Deadbeat Control

CLTF: (3^{rd} order) $\quad \boxed{\bar{s} = s/\omega_n}$

$$T(s) = \frac{\omega_n^3}{s^3 + \alpha\omega_n s^2 + \beta\omega_n^2 s + \omega_n^3} \overset{\Downarrow}{=} \frac{1}{\bar{s}^3 + \alpha\bar{s}^2 + \beta\bar{s} + 1} \quad (normalized)$$

Coefficients for Deadbeat Response:

Table 10.2 Coefficients and Response Measures of a Deadbeat System

System Order	Coefficients α	β	γ	δ	ϵ	Percent Overshoot, P.O.	Percent Undershoot, P.U.	90% Rise Time, $T_{r_{90}}$	100% Rise Time, T_r	Settling Time, T_s
2nd	1.82					0.10%	0.00%	3.47	6.58	4.82
3rd	1.90	2.20				1.65%	1.36%	3.48	4.32	4.04
4th	2.20	3.50	2.80			0.89%	0.95%	4.16	5.29	4.81
5th	2.70	4.90	5.40	3.40		1.29%	0.37%	4.84	5.73	5.43
6th	3.15	6.50	8.70	7.55	4.05	1.63%	0.94%	5.49	6.31	6.04

Note: All time is normalized.

Example: A Third-Order System $\quad$ *Spec.*: $T_s = 1.2\ s$

Table $\Rightarrow \omega_n T_s = 4.04 \Rightarrow \omega_n = 4.04/1.2 = 3.37$

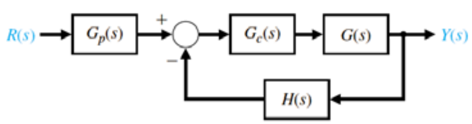
$\Downarrow$

$$\alpha = 1.90,\ \beta = 2.20 \qquad T(s) = \frac{\omega_n^3}{s^3 + \alpha\omega_n s^2 + \beta\omega_n^2 s + \omega_n^3} = \frac{38.3}{s^3 + 6.4s^2 + 25s + 38.3}$$

Deadbeat Control

Example 10.12:

Plant: $\quad G(s) = \dfrac{K}{s(s+1)}$



Lead Compensator: $\quad G_c(s) = \dfrac{s+z}{s+p}$

Prefilter: $\quad G_p(s) = \dfrac{p}{s+p}$

CLTF: $\quad T(s) = \dfrac{Kz}{s^3 + (1+p)s^2 + (K+p)s + Kz}$

Table $\Rightarrow \omega_n T_s = 4.04 \Rightarrow \omega_n = 4.04/2 = 2.02$

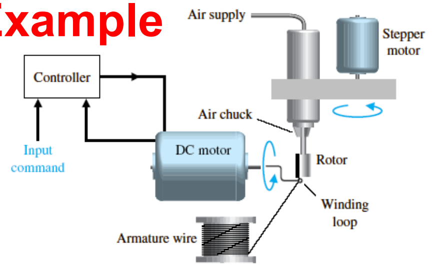
$\Downarrow$

$$\alpha = 1.90,\ \beta = 2.20 \quad T(s) = \frac{\omega_n^3}{s^3 + \alpha\omega_n s^2 + \beta\omega_n^2 s + \omega_n^3} = \frac{8.24}{s^3 + 3.84s^2 + 4.44s + 8.24}$$

$\Rightarrow p = 2.84,\ z = 1.34,\ K = 6.14$

$\Rightarrow T_s = 2\ s,\ T_r = 2.14\ s,\ T_{r_{90}} = 1.72\ s$

Design Example



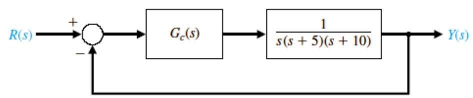
Rotor Winder Control System:

Plant: $\quad G(s) = \dfrac{1}{s(s+5)(s+10)}$

Step input: $\quad e_{ss} = 0$

Ramp input: $\quad e_{ss} = A/K_v \quad K_v = \lim_{s\to 0} \dfrac{G_c(s)}{50}$

Compensator:



P controller: $\quad G_c(s) = K \Rightarrow K_v = K/50$

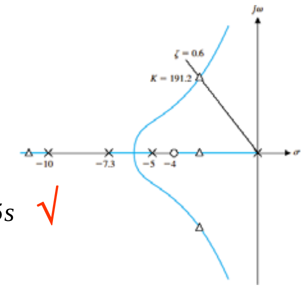
$K_v = 10 \Rightarrow K = 500$

$\Rightarrow OS_{step} = 70\%,\ T_s = 8s$ **X**

Lead Compensator: $\quad G_c(s) = K\dfrac{s+z_1}{s+p_1}$

$\zeta = 0.6 \Rightarrow z_1 = 4,\ p_1 = 7.3$

$\Rightarrow G_c(s) = 191.2\dfrac{s+4}{s+7.3} \Rightarrow$ $OS_{step} = 3\%,\ T_s = 1.5s$ √ $\quad K_v = 2.1$ **X**



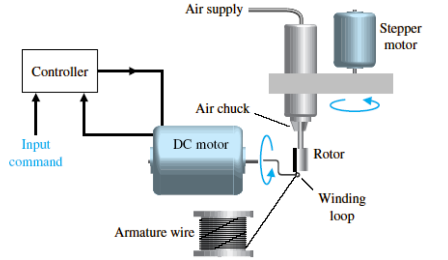
Design Example

Rotor Winder Control System:

Plant: $G(s)=\dfrac{1}{s(s+5)(s+10)}$

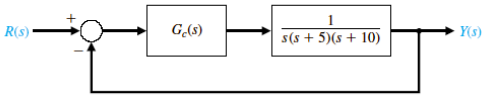
Step input: $e_{ss}=0$

Ramp input: $e_{ss}=A/K_v \quad K_v=\lim_{s\to 0}\dfrac{G_c(s)}{50}$



Compensator:

Lag compensator: $G_c(s)=K\dfrac{s+z_2}{s+p_2}$



$K_v=38=\dfrac{Kz_2}{50p_2} \qquad K=105$

$\Rightarrow \alpha=18.1 \Rightarrow z_2=0.1, p_2=0.0055 \Rightarrow OS_{step}=12\%, T_s=2.5s$

Table 10.3 Design Example Results

Controller	Gain, K	Lead Network	Lag Network	Lead-Lag Network
Step overshoot	70%	3%	12%	5%
Settling time (seconds)	8	1.5	2.5	2.0
Steady-state error for ramp	10%	48%	2.6%	4.8%
K_v	10	2.1	38	21

Design Example

Rotor Winder Control System:

Compensator:

Lead-lag compensator:

$$G_c(s)=K\frac{(s+z_1)}{(s+p_1)}\frac{(s+z_2)}{(s+p_2)}$$

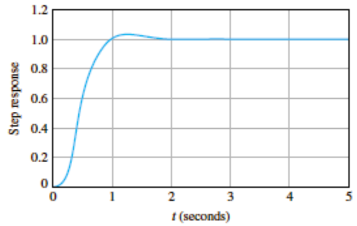
Lead Part: $G_{c,lead}(s)=191.2\dfrac{s+4}{s+7.3}$

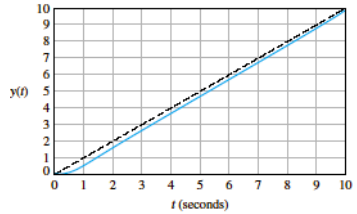
$\Rightarrow K_{v,lead}=2.1$

$K_{v,lead-lag}=21 \Rightarrow \alpha=10$

$\Rightarrow z_2=0.1, p_2=0.01$

$$G_cG(s)=\frac{191.2(s+4)(s+0.1)}{s(s+5)(s+10)(s+7.28)(s+0.01)}$$





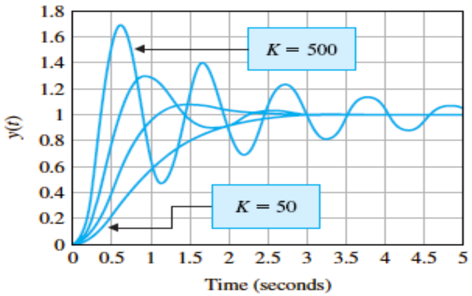
Design Example

```
K=[50 100 200 500];
%
numg=[1]; deng=[1 15 50 0];
sysg=tf(numg,deng);
t=[0:0.1:5];
%
for i=1:4
sys=feedback(K(i)*sysg,[1]);
y=step(sys,t);
Ys(:,i)=y;
end
%
plot(t,Ys(:,1),t,Ys(:,2),t,Ys(:,3),t,Ys(:,4))
xlabel('Time (seconds)'), ylabel('y(t)')
```

Compute response for four gains.

Closed-loop transfer function.

Store response for ith gain in Ys.



Design Example

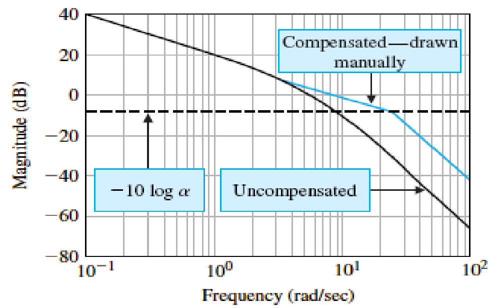
```
K=500;
numg=[1]; deng=[1 15 50 0]; sysg=tf(numg,deng);
sys=K*sysg;
%
[Gm,Pm,Wcg,Wcp]=margin(sys);
%
Phi=(60-Pm)*pi/180;
alpha=(1+sin(Phi))/(1-sin(Phi))
[mag,phase,w]=bode(sys);
mag_save(1,:)=mag(:,1,:);
%
M=-10*log10(alpha)*ones(length(w),1);
%
semilogx(w,20*log10(mag_save),w,M), grid
xlabel('Frequency (rad/sec)'), ylabel('Magnitude (dB)')
```

Compute phase margin.

Additional phase lead.

Compute α.

Plot $-10\log(\alpha)$ line to aid in locating ω_m.

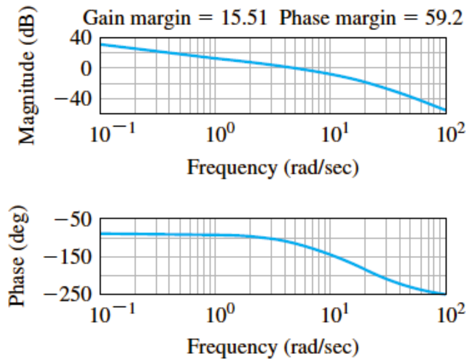


Design Example

```
K=1800;
numg=[1]; deng=[1 15 50 0];
numgc=K*[1 3.5]; dengc=[1 25];
sysg=tf(numg,deng);
sysgc=tf(numgc,dengc);
sys=series(sysgc,sysg);
margin(sys)
```

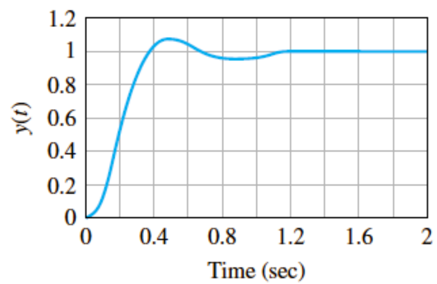
Increase K to account for attenuation of $1/\alpha$.

Lead compensator.



Design Example

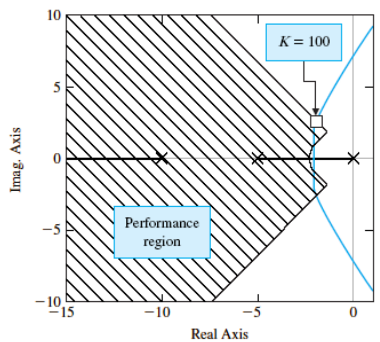
```
K=1800;
%
numg=[1]; deng=[1 15 50 0]; sysg=tf(numg,deng);
numgc=K*[1 3.5]; dengc=[1 25]; sysgc=tf(numgc,dengc);
%
syso=series(sysgc,sysg);
sys=feedback(syso,[1]);
%
t=[0:0.01:2];
step(sys,t)
ylabel ('y(t)')
```



Design Example

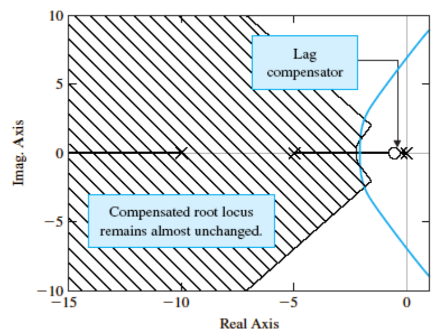
```
numg=[1]; deng=[1 15 50 0];
sysg=tf(numg,deng);
clf; rlocus(sysg); hold on
%
zeta=0.5912; wn=2.2555;
%
x=[-10:0.1:-zeta*wn]; y=-(sqrt(1-zeta^2)/zeta)*x;
xc=[-10:0.1:-zeta*wn]; c=sqrt(wn^2-xc.^2);
%
plot(x,y,':',x,-y,':',xc,c,':',xc,-c,':')
axis([-15,1,-10,10]);
```

Plot performance regions on locus.



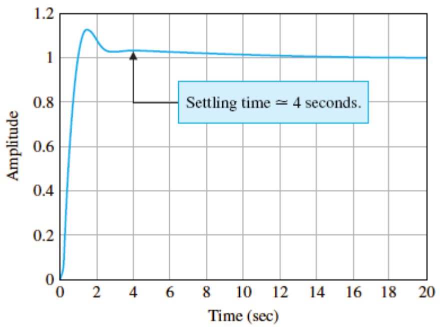
Design Example

```
numg=[1]; deng=[1 15 50 0]; sysg=tf(numg,deng);
numgc=[1 0.1]; dengc=[1 0.01]; sysgc=tf(numgc,dengc);
sys=series(sysgc,sysg);
clf; rlocus(sys); hold on
%
zeta=0.5912; wn=2.2555;
x=[-10:0.1:-zeta*wn]; y=-(sqrt(1-zeta^2)/zeta)*x;
xc=[-10:0.1:-zeta*wn];c=sqrt(wn^2-xc.^2);
plot(x,y,':',x,-y,':',xc,c,':',xc,-c,':')
axis([-15,1,-10,10]);
```



Design Example

```
K=100;
%
numg=[1]; deng=[1 15 50 0]; sysg=tf(numg,deng);
numgc=K*[1 0.1]; dengc=[1 0.01]; sysgc=tf(numgc,dengc);
%
syso=series(sysgc,sysg);
sys=feedback(syso,[1]);
%
step(sys)
```

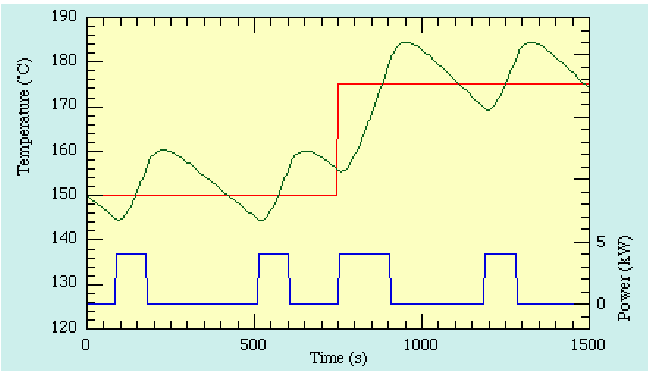


P/PI/PID Controllers

Different Types of Feedback Control

On-Off Control

This is the simplest form of control.

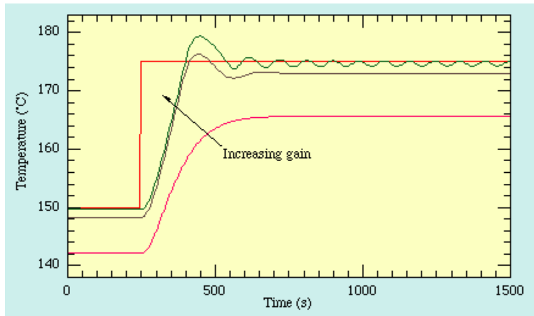


Proportional Control

A proportional controller attempts to perform better than the on-off type by applying power in proportion to the difference in temperature between the measured and the set-point.

As the gain is increased the system responds faster to changes in set-point but becomes progressively underdamped and eventually unstable.

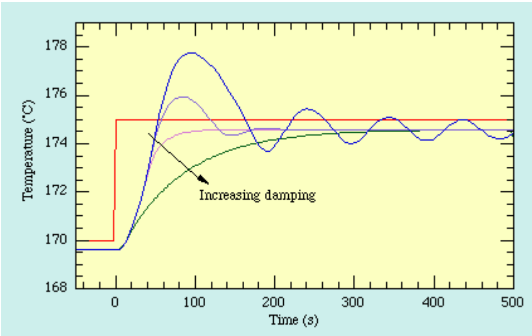
The final temperature lies below the set-point for this system because some difference is required to keep the heater supplying power.



Proportional + Derivative Control

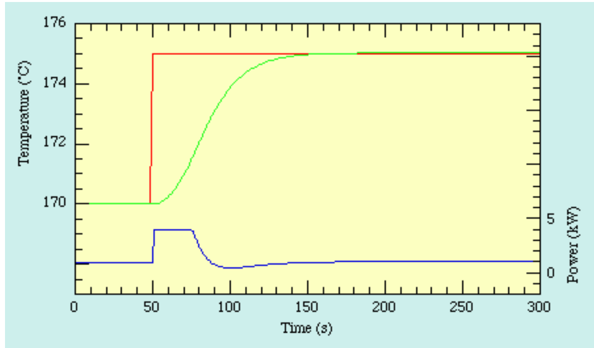
The stability and overshoot problems that arise when a proportional controller is used at high gain can be mitigated by adding a term proportional to the time-derivative of the error signal.

The value of the damping can be adjusted to achieve a critically damped response.



Proportional+Integral+Derivative Control

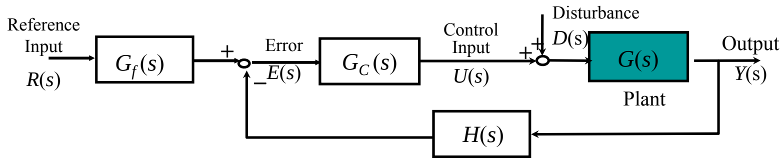
Although PD control deals neatly with the overshoot and ringing problems associated with proportional control it does not cure the problem with the steady-state error. Fortunately it is possible to eliminate this while using relatively low gain by adding an integral term to the control function which becomes.



PID Controller

- **Structure of Controller**
- **Effects of Proportional, Integral and Derivative Actions**
- **Design of PID Controllers**

PID Controller



Proportional plus Integral plus Derivative (PID) Control

In time domain

$$u(t) = K_{CP} \bullet e(t) + K_{CI} \bullet \int_0^t e(\tau)d\tau + K_{CD} \bullet \dot{e}(t)$$

current information | passed information | (prediction of) future information

In *s*-domain

$$U(s) = \left(K_{CP} + K_{CI} \bullet \frac{1}{s} + K_{CD} \bullet s\right) E(s)$$

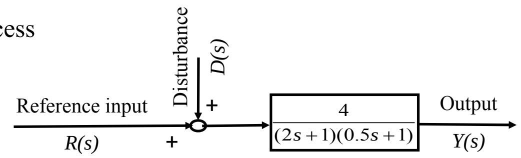
$$G_c(s) = \frac{U(s)}{E(s)} = \frac{K_{CD}s^2 + K_{CP}s + K_{CI}}{s} = \frac{K_{CD}(s - z_1)(s - z_2)}{s} \qquad \begin{cases} K_{CP} = -K_{CD}(z_1 + z_2) \\ K_{CI} = K_{CD} z_1 z_2 \end{cases}$$

Effect of P-I-D Actions

- **Derivative Action ($K_{CD}s$):**
  Provides added damping to the closed-loop system; reduces overshoot and oscillation in step response; tends to slow down the closed-loop system response. Dominant during initial transient, due to the effect of differentiation.
- **Integral Action (K_{CI}/s):**
  Eliminates steady-state error to step inputs; tends to destabilize the closed-loop system; has averaging effect. Dominant during steady-state by producing an accumulation of steady-state error to increase control effort.
- **Proportional Action (K_{CP}):**
  Introduces immediate action due to error; improves system response time. Has similar control authority for both transient and steady-state.

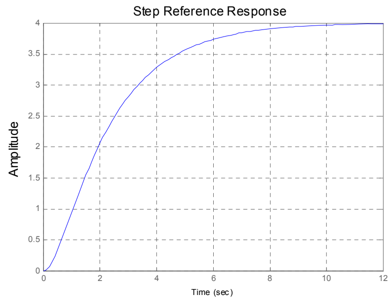
Effect of PID Actions

•Ex: For a given process



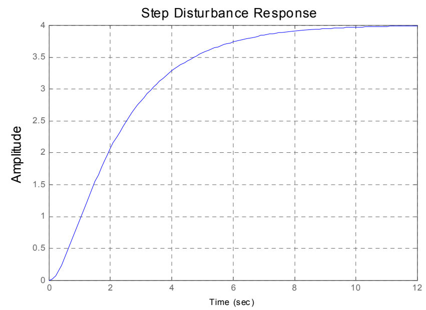
•Unit step reference response

$$Y(s)=\frac{4}{(2s+1)(0.5s+1)}\underbrace{\frac{1}{s}}_{R(s)}$$



•Unit step disturbance response

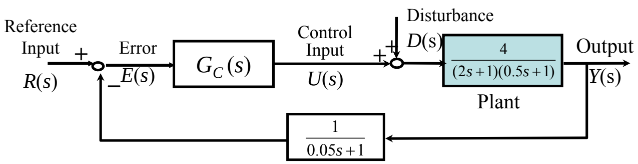
$$Y(s)=\frac{4}{(2s+1)(0.5s+1)}\underbrace{\frac{1}{s}}_{D(s)}$$



Effect of P Action

•Objective:

Design a system that has zero steady state error for step inputs with %OS<10% and *Ts* (2%)<6 [sec]



(A.) Let's try Proportional control: $G_c(s)=K_{CP}$

CLTF:
$$G_{YR}(s)=\frac{G_C(s)G_P(s)}{1+G_C(s)G_P(s)H(s)}=\frac{K_{CP}\dfrac{4}{(2s+1)(0.5s+1)}}{1+K_{CP}\dfrac{4}{(2s+1)(0.5s+1)}\dfrac{1}{0.05s+1}}$$

$$G_{YD}(s)=\frac{G_P(s)}{1+G_C(s)G_P(s)H(s)}=\frac{\dfrac{4}{(2s+1)(0.5s+1)}}{1+K_{CP}\dfrac{4}{(2s+1)(0.5s+1)}\dfrac{1}{0.05s+1}}$$

$$=\frac{4(0.05s+1)}{(2s+1)(0.5s+1)(0.05s+1)+4K_{CP}}$$

Effect of P Action

(A.1) Design P Controller using Root Locus:

Stability:
$0<K_{CP}<14.1$

CLCE:
$$1+K_{CP}\frac{4}{(2s+1)(0.5s+1)}\frac{1}{0.05s+1}$$
$$=1+K_{CP}\frac{4}{0.05}\frac{\overbrace{1}^{N(s)}}{\underbrace{(s+0.5)(s+2)(s+20)}_{D(s)}}$$

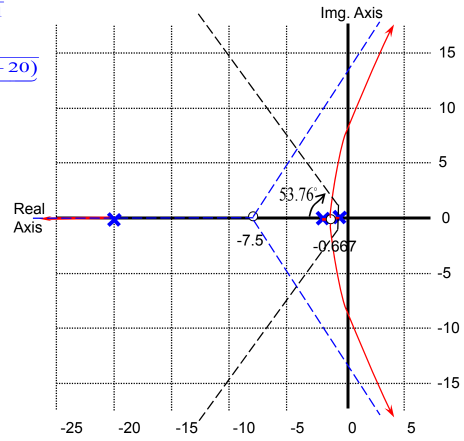
$$\sigma_0=\frac{\sum p_i-\sum z_i}{N_P-N_Z}=-7.5$$
$$\theta_k=(2k+1)\frac{\pi}{N_P-N_Z}\ [\text{rad}]=\begin{cases}\dfrac{\pi}{3}\\ \pi\\ \dfrac{5\pi}{3}\end{cases}$$

$$\frac{d}{ds}\left(\frac{D(s)}{N(s)}\right)=0,\Rightarrow 3s^2+45s+51=0$$
$$s_1=-1.235, s_2=-13.765$$

$$\left(4K_{CP}+1-1.125\omega^2\right)+\left(2.55-0.05\omega^2\right)\omega j=0$$

$$\Rightarrow\begin{cases}\omega_1=7.1414\\ K_{CP_1}=14.1\end{cases},\ \boxed{\begin{cases}\omega_2=0\\ K_{CP_2}=-0.25\end{cases}}$$

Not valid



Effect of P Action

(A.2) Check for Steady State Error:

Unit Step Reference Response

$$e_{ss} = r - y_{ss} = \left(1 - \frac{4K_{CP}}{1+4K_{CP}}\right) \bullet \underbrace{1}_{r}$$

Do NOT forget

Stability:
$0<K_{CP}<14.1$

Larger gain results in smaller steady-state error.

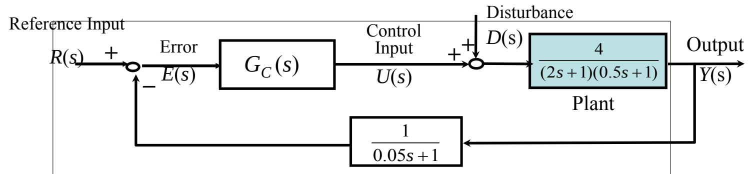
Unit Step Disturbance Response

$$y_{ss} = \lim_{s\to 0} sG_{YD}(s)\frac{1}{s} = G_{YD}(0) = \frac{4}{1+4K_{CP}}$$

Larger gain results in stronger attenuation of disturbance.

Think about the change of overshoot when gain increases

Effect of PI Actions



(B.) Add integral action (PI control):

CLTF:

$$G_{YR}(s) = \frac{G_C(s)G_P(s)}{1+G_C(s)G_P(s)H(s)} = \frac{K_{CP}\frac{s-z_{PI}}{s}\frac{4}{(2s+1)(0.5s+1)}}{1+K_{CP}\frac{s-z_{PI}}{s}\frac{4}{(2s+1)(0.5s+1)}\frac{1}{0.05s+1}}$$

$$G_{YD}(s) = \frac{G_P(s)}{1+G_C(s)G_P(s)H(s)} = \frac{\frac{4}{(2s+1)(0.5s+1)}}{1+K_{CP}\frac{s-z_{PI}}{s}\frac{4}{(2s+1)(0.5s+1)}\frac{1}{0.05s+1}}$$

$$= \frac{4s(0.05s+1)}{s(2s+1)(0.5s+1)(0.05s+1)+4K_{CP}(s-z_{PI})}$$

Effect of PI Actions

(B.1) Design PI Controller using Root Locus:

Stability:
$0<K_{CP}<11$

CLCE: $1+K_{CP}\frac{4}{0.05}\frac{\overbrace{(s-z_{PI})}^{N(s)}}{\underbrace{s(s+0.5)(s+2)(s+20)}_{D(s)}}=0$

Choose $z_{PI} = -0.5$:

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = -7.3$$

$$\theta_k = (2k+1)\frac{\pi}{N_P - N_Z}\ [\text{rad}] = \begin{cases}\frac{\pi}{3}\\ \pi\\ \frac{5\pi}{3}\end{cases}$$

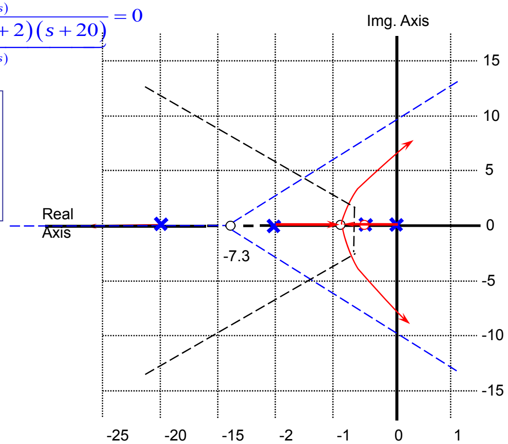
$$\frac{d}{ds}\left(\frac{D(s)}{N(s)}\right) = 0, \Rightarrow 3s^2+44s+40=0$$

$s_1 = -0.9737, s_2 = -13.6929$

$$(2K_{CP}-0.55\omega^2)+(1-0.025\omega^2)\omega j = 0$$

$$\Rightarrow \begin{cases}\omega_1 = 6.3246\\ K_{CP_1} = 11\end{cases}, \boxed{\begin{cases}\omega_2 = 0\\ K_{CP_2} = 0\end{cases}}$$

Not valid



Effect of PI Actions

(B.2) Check for Steady State Error:

Unit Step Reference Response

$$e_{ss} = r - y_{ss} = (1 - G_{YR}(0)) \bullet \underbrace{1}_{r} = 0$$

By using Integral action. steady state error is eliminated.

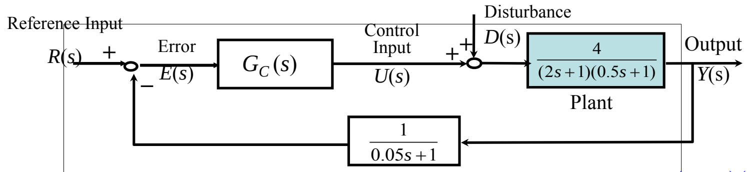
Unit Step Disturbance Response

$$y_{ss} = \lim_{s\to 0} sG_{YD}(s)\frac{1}{s} = G_{YD}(0) = 0$$

By using Integral action, the effect of a constant disturbance can also be eliminated.

Has transient performance been improved? Not much

Effect of PID Actions



(C.) Add derivative action (PID control): $G_C(s)=K_{CP}+K_{CI}\frac{1}{s}+K_{CD}s=K_{CD}\frac{(s-z_1)(s-z_2)}{s}$

CLTF:

$$G_{YR}(s)=\frac{G_C(s)G_P(s)}{1+G_C(s)G_P(s)H(s)}=\frac{K_{CD}\frac{(s-z_1)(s-z_2)}{s}\frac{4}{(2s+1)(0.5s+1)}}{1+K_{CD}\frac{(s-z_1)(s-z_2)}{s}\frac{4}{(2s+1)(0.5s+1)}\frac{1}{0.05s+1}}$$

$$G_{YD}(s)=\frac{G_P(s)}{1+G_C(s)G_P(s)H(s)}=\frac{\frac{4}{(2s+1)(0.5s+1)}}{1+K_{CD}\frac{(s-z_1)(s-z_2)}{s}\frac{4}{(2s+1)(0.5s+1)}\frac{1}{0.05s+1}}$$

$$=\frac{4s(0.05s+1)}{s(2s+1)(0.5s+1)(0.05s+1)+4K_{CD}(s-z_1)(s-z_2)}$$

Effect of PID Actions

Stability:
$0<K_{CD}$

(C.1) Design PID Controller using Root Locus:

CLCE: $1+K_{CD}\frac{4}{0.05}\frac{\overbrace{(s-z_1)(s-z_2)}^{N(s)}}{\underbrace{s(s+0.5)(s+2)(s+20)}_{D(s)}}=0$

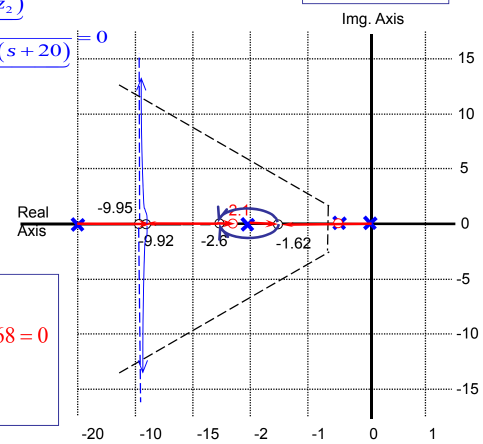
Choose $z_1=-0.5$, $z_2=-2.1$

$$\sigma_o=\frac{\sum p_i-\sum z_i}{N_P-N_Z}=-9.95$$

$$\theta_k=(2k+1)\frac{\pi}{N_P-N_Z}\ [\text{rad}]=\begin{cases}\frac{\pi}{2}\\ \frac{3\pi}{2}\end{cases}$$

$$\frac{d}{ds}\left(\frac{D(s)}{N(s)}\right)=0,\Rightarrow 4s^3+56.6s^2+184.8s+168=0$$

$$s_1=-9.92,\ s_2=-2.60,\ s_3=-1.62$$



Effect of PID Actions

(C.2) Check for Steady State Error:

Unit Step Reference Response

$$e_{ss}=r-y_{ss}=\left(1-G_{YR}(0)\right)\bullet\underbrace{1}_{r}=0$$

By using Integral action. steady state error is eliminated.

Unit Step Disturbance Response

$$y_{ss}=\lim_{s\to 0}sG_{YD}(s)\frac{1}{s}=G_{YD}(0)=0$$

By using Integral action, the effect of a constant disturbance can also be eliminated.

Has transient performance been improved? Yes

PID Controller Design via Root Locus

Pole-Zero structure of PID Controller:

$$G_C(s)=K_C\frac{N(s)}{D(s)}=\frac{K_{CD}s^2+K_{CP}s+K_{CI}}{s}=\frac{K_{CD}(s-z_1)(s-z_2)}{s},\begin{cases}K_{CD}=K_C\\ K_{CP}=-K_C(z_1+z_2)\\ K_{CI}=K_Cz_1z_2\end{cases}$$

PID Controller adds one open-loop pole at origin and two open-loop zeros, z_1 and z_2. These two open-loop zeros could be either real or complex conjugate pair.

Design Procedure

Step1: Select the position of the two zeros such that the root locus will intersect with the desired performance region.
Step 2: Pick the controller gain K_C such that CL poles are in the performance region
Step 3: Find the corresponding PID gain using the above formula.

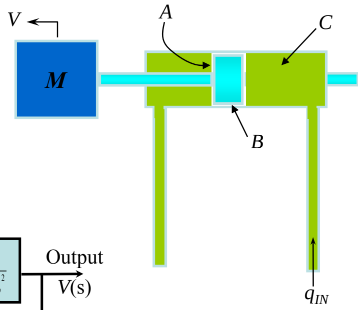
Design of PID Controller

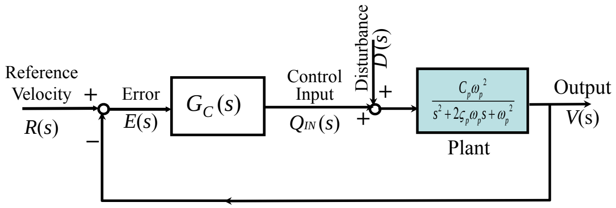
Ex: (*Motion Control of Hydraulic Cylinders*)

Recall the example of the flow control of a hydraulic cylinder that takes into account the capacitance effect of the pressure chamber. The plant transfer function is:

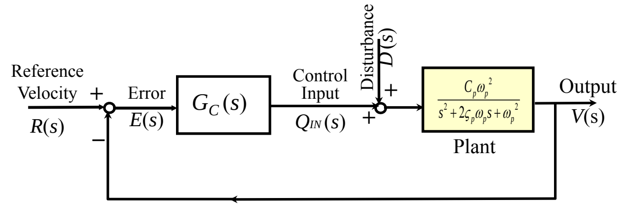
$$G(s)=\frac{V(s)}{Q_{IN}(s)}=\frac{C_p\omega_p^2}{s^2+2\varsigma_p\omega_p s+\omega_p^2}$$

where $\omega_p=6\pi$ rad/sec, $\varsigma_p=0.1$, $C_p=0.2$.





Design of PID Controller



$$G(s)=\frac{V(s)}{Q_{IN}(s)}=\frac{C_p\omega_p^2}{s^2+2\varsigma_p\omega_p s+\omega_p^2}=\frac{0.2(6\pi)^2}{s^2+1.2\pi s+(6\pi)^2}=\frac{0.2(6\pi)^2}{\Big(s-\underbrace{(-1.885+18.755j)}_{p_1}\Big)\Big(s-\underbrace{(-1.885-18.755j)}_{p_2}\Big)}$$

We would like to design a controller such that the closed loop system is better damped (smaller OS%)

CLTF: $$G_{CL}(s)=\frac{V(s)}{R(s)}=\frac{K_C N_P(s)N_C(s)}{D_P(s)D_C(s)+K_C N_P(s)N_C(s)},$$

with $G_C(s)=\frac{N_C(s)}{D_C(s)}$, and $G(s)=\frac{N_P(s)}{D_P(s)}$

Design of PID Controller

PID controller design $$G_C(s)=K_C\frac{N(s)}{D(s)}=\frac{K_{CD}s^2+K_{CP}s+K_{CI}}{s}=\frac{K_{CD}(s-z_1)(s-z_2)}{s},\begin{cases}K_{CD}=K_C\\K_{CP}=-K_C(z_1+z_2)\\K_{CI}=K_C z_1 z_2\end{cases}$$

Closed-loop Characteristic Equation

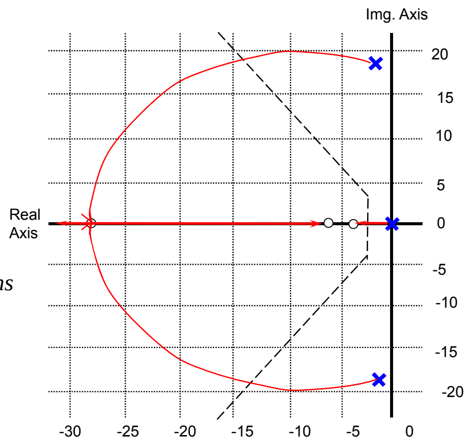
$$1+K_C\frac{0.2(6\pi)^2(s-z_1)(s-z_2)}{s(s-p_1)(s-p_2)}=0$$

Transient performance region

$$t_s(2\%)=\frac{4}{\sigma}\le T_s\ \Rightarrow\ \sigma\ge\frac{4}{T_s}$$

$$\varphi=\arctan\left(-\frac{\pi}{\ln(X\%)}\right)$$

Can a PID controller be designed to satisfy the transient design specifications (smaller overshoot and faster settling time) ?



Characteristics of P/PI/PID Controllers

- A proportional controller (Kp) will have the effect of reducing the rise time and will reduce, but never eliminate, the steady-state error.
- An integral control (Ki) will have the effect of eliminating the steady-state error, but it may make the transient response worse.
- A derivative control (Kd) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response.

Proportional Control

By only employing proportional control, a steady state error occurs.

Proportional and Integral Control

The response becomes more oscillatory and needs longer to settle, the error disappears.

Proportional, Integral and Derivative Control

All design specifications can be reached.

The Characteristics of P, I, and D controllers

CL RESPONSE	RISE TIME	OVERSHOOT	SETTLING TIME	S-S ERROR
Kp $\uparrow$	Decrease	Increase	Small Change	Decrease
Ki $\uparrow$	Decrease	Increase	Increase	Eliminate
Kd $\uparrow$	Small Change	Decrease	Decrease	Small Change

Tips for Designing a PID Controller

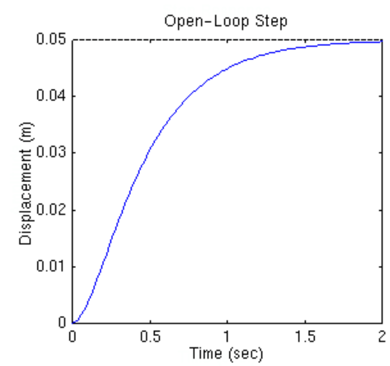
1. Obtain an open-loop response and determine what needs to be improved
2. Add a proportional control to improve the rise time
3. Add a derivative control to improve the overshoot
4. Add an integral control to eliminate the steady-state error
5. Adjust each of Kp, Ki, and Kd until you obtain a desired overall response.

Lastly, please keep in mind that you do not need to implement all three controllers (proportional, derivative, and integral) into a single system, if not necessary. For example, if a PI controller gives a good enough response (like the above example), then you don't need to implement derivative controller to the system. Keep the controller as simple as possible.

Open-Loop Control - Example

$$G(s) = \frac{1}{s^2 + 10s + 20}$$

```
num=1;
den=[1 10 20];
step(num,den)
```



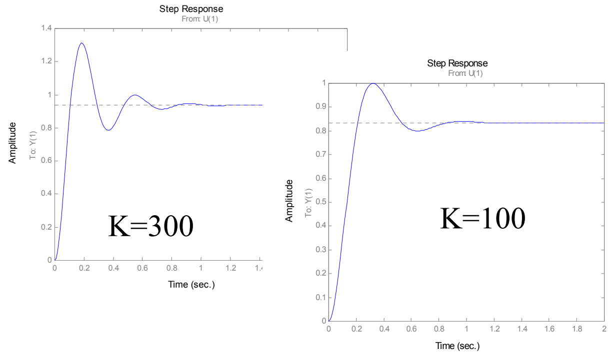
Proportional Control - Example

The proportional controller (Kp) reduces the rise time, increases the overshoot, and reduces the steady-state error.

MATLAB Example

$$T(s) = \frac{Kp}{s^2 + 10 \cdot s + (20 + Kp)}$$

```
Kp=300;
num=[Kp];
den=[1 10 20+Kp];
t=0:0.01:2;
step(num,den,t)
```



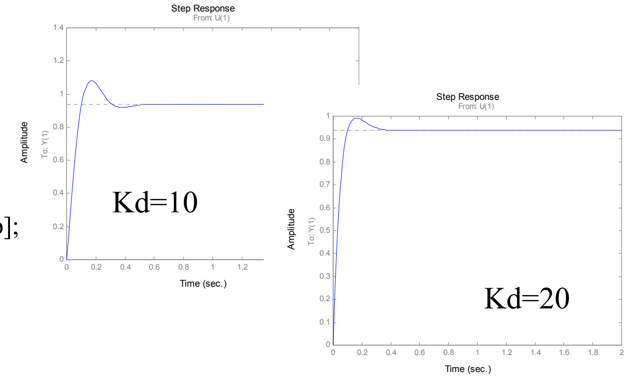
Proportional - Derivative - Example

The derivative controller (Kd) reduces both the overshoot and the settling time.

MATLAB Example

$$T(s) = \frac{Kd \cdot s + Kp}{s^2 + (10 + Kd) \cdot s + (20 + Kp)}$$

```
Kp=300;
Kd=10;
num=[Kd Kp];
den=[1 10+Kd 20+Kp];
t=0:0.01:2;
step(num,den,t)
```



Proportional - Integral - Example

The integral controller (Ki) decreases the rise time, increases both the overshoot and the settling time, and eliminates the steady-state error

$$T(s) = \frac{Kp \cdot s + Ki}{s^3 + 10 \cdot s^2 + (20 + Kp) \cdot s + Ki}$$

MATLAB Example

```
Kp=30;
Ki=70;
num=[Kp Ki];
den=[1 10 20+Kp Ki];
t=0:0.01:2;
step(num,den,t)
```