ELCE 705
Digital Signal Processing

Discrete-time signals and systems

Chapter 2

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Content

- Signals/Sequences
- Discrete-time Systems
- LTI system and Convolution
- Linear difference equations
- Frequency Response
- Review of CTFT
- DTFT

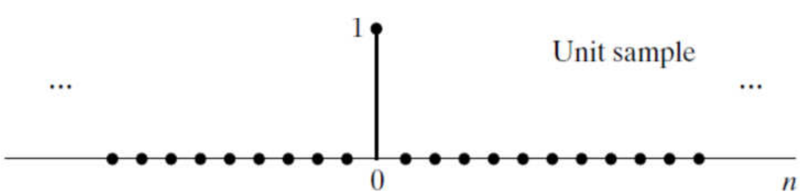
Signals and Sequences

- Consider a continuous-time signal x(t)
- This can be sampled every T units of time to create a discrete-time signal x(nT)
- T is usually implicit, so we write x(nT) as x[n]
- X[n] is a sequence of numbers defined on the domain of integers, i.e. for n=0, $\pm 1, \pm 2, \ldots$
- X[n] is not defined for non-integer values of n
- In general, we will denote continuous-time signals with parentheses and discrete-time signals with brackets
- Signals can be shifted in time by adding or subtracting from the indexing variable, e.g. for the case of n0>0
  - $y[n] = x[n - n_0]$: $y[n]$ is $x[n]$ delayed by n_0 samples
  - $z[n] = x[n + n_0]$: $z[n]$ is $x[n]$ advanced by n_0 samples

Expressions of discrete signals

- Discrete-time impulse (also known as the unit-sample function or delta function):

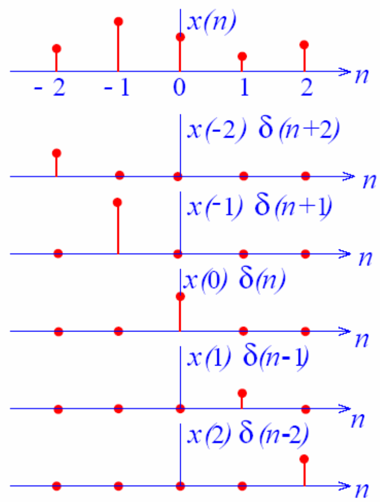
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



  - One useful property of the delta function is the *shifting property*:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Discrete-signal via impulse functions

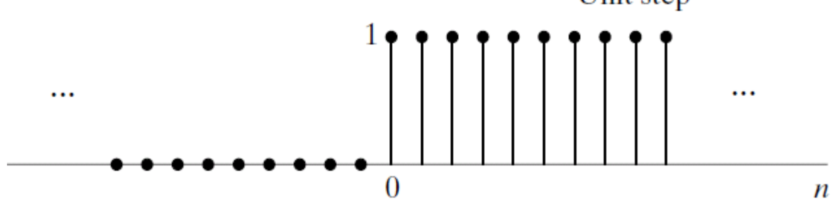


$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Unit step sequence

- Discrete-time step : $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

Unit step

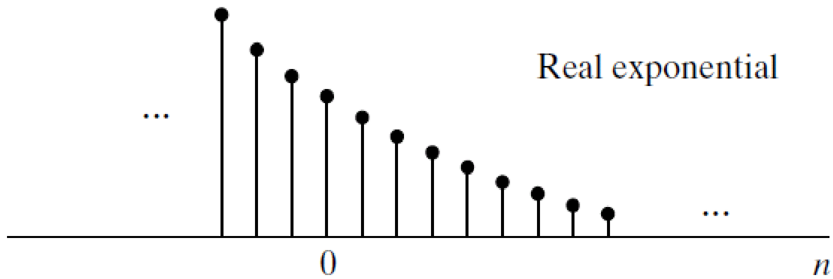


$$u[n] = \sum_{k=-\infty}^{n} \delta[k] \qquad u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

$$\delta[n] = u[n] - u[n-1]$$

Discrete-time exponential

$$a^n u[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Exponential and sinusoidal sequences are extremely important in representing and analyzing linear time-invariant discrete-time systems.

Continuous-time sinusoid

$$x_a(t) = A\cos(\Omega t + \theta),\ \Omega = 2\pi F$$

- For every fixed value of the frequency F, $x_a(t)$ is periodic.

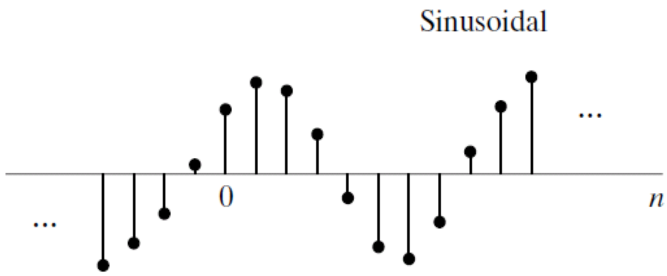
$$x_a(t+T_p) = x_a(t)$$

where $T_p=1/F$ is the fundamental period of the sinusoids signal

- Continuous-time sinusoidal signals with distinct frequencies are themselves distinct
- Increasing the frequency F results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in a given time interval.

Discrete-time sinusoid

$$A\cos(\omega_0 n + \phi)$$



$$\begin{aligned}
e^{j(\omega n+\phi)} &= \cos(\omega n+\phi) + j\sin(\omega n+\phi) \\
\cos(\omega n+\phi) &= \{e^{j(\omega n+\phi)} + e^{-j(\omega n+\phi)}\}/2 \\
\sin(\omega n+\phi) &= \{e^{j(\omega n+\phi)} - e^{-j(\omega n+\phi)}\}/2j
\end{aligned}$$

Periodicity

- Surprisingly, not all discrete-time sinusoids are periodic;
- A sequence x[n] is periodic if there exists an integer N such that x[n]=x[n+N] for all n;
- This means that cos (0.25πn) is periodic with N=8, but cos(n) is not periodic;
- In general, a sinusoid will only be periodic if there exists an N such that $\omega_0 N=2\pi k$, where k is an integer
- If expressed in frequency: $2\pi f_0 N=2\pi k \rightarrow f_0= k/N$
- A discrete-time sinusoid is period only if its frequency can be expressed as the ratio of two integers (i.e., f_0 is rational).

Discrete-time sinusoid (C)

- Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.

Let us consider the sinusoid $\cos(\omega_0 n+\theta)$, It follows that

$$\cos[(\omega_0+2\pi)n+\theta] = \cos(\omega_0 n+2\pi n+\theta) = \cos(\omega_0 n+\theta)$$

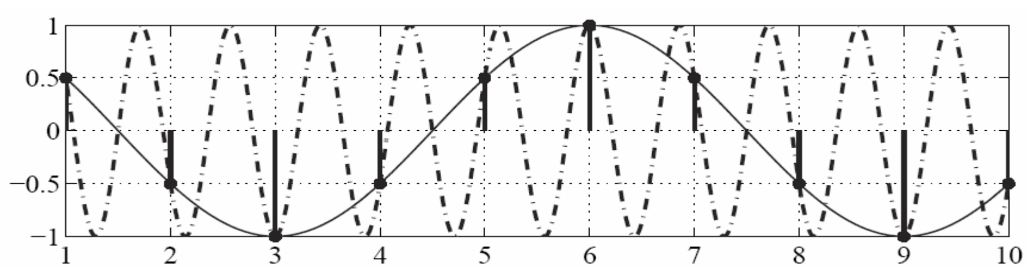
As a result, all sinusoidal sequences

$$x_k(n)=A\cos(\omega_k n+\theta), \qquad k=0,1,2,\ldots.$$

Where $\omega_k = \omega_0+2k\pi$, $-\pi\le\omega_0\le\pi$ are indistinguishable (identical)

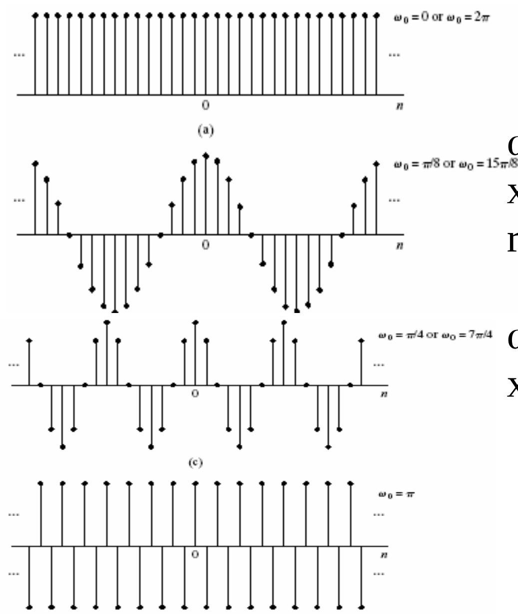
- Discrete-time sinusoid with frequencies $|\omega|\le\pi$ or $|f|\le 1/2$ are distinct.
- The sinusoids having the frequency $|\omega|>\pi$ is an alias of a corresponding sinusoid with $|\omega|\le\pi$. (different from continuous-time signal).

Frequency aliasing



- The continuous and dashed-dotted lines respectively show the real part of the analog complex exponential signals e^{jwt} and $e^{j(w+2\pi)t}$.
- Upon periodic sampling at integer values of t (i.e. using Ts = 1), the same sample values are obtained for both analog sinusoids.
- The two discrete signals are indistinguishable, even tough the original analog signals are different.
- This is the simplified illustration of the important phenomenon ---frequency aliasing.

Oscillation



ω increases from 0 toward π, x[n] oscillates more and more rapidly.

ω increases from π toward 2π, x[n] oscillates slower.

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Discrete-Time Systems

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- A discrete-time system is a device or algorithm that operates on a discrete-time signal.
- A mapping between the input sequence and the output sequence

  $x[n] \rightarrow \boxed{\text{DTS}} \rightarrow y[n]$ or $y[n]=T\{x[n]\}$
- Examples:

  delay $y[n] = x[n-n_d]$

  two-point average:

  $y[n] = (x[n]+x[n-1])/2$
- Classes of systems are defined by placing constraints on the properties of the transformation T{.}

Properties of Discrete-Time Systems

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- **Memoryless** (also called static)
  - A system is memoryless if the output is a function of the current input only and does not rely on past or future input values;
  - Example: $y[n]= x^2[n]$
  - Is the two-point average or the delay memoryless?

Properties of Discrete-Time Systems

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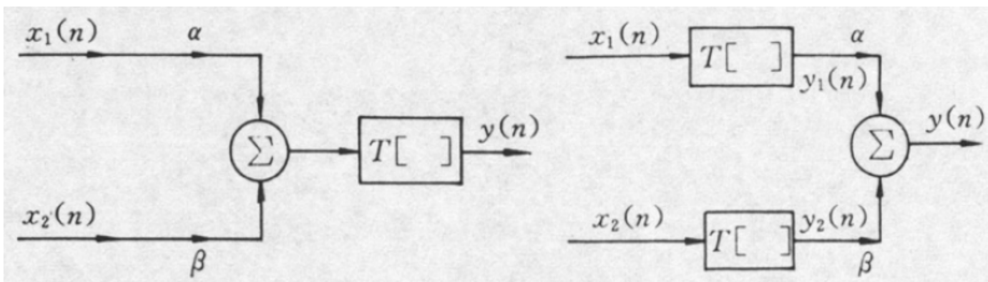
- **Linear**
  - Must satisfy the property $T\{ax_1[n]+bx_2[n]\}=aT\{x_1[n]\}+bT\{x_2[n]\}$
  - Homogeneity: $T\{ax[n]\} = a\,T\{x[n]\}$
  - Additivity: $T\{x_1[n]+x_2[n]\} = T\{x_1[n]\}+T\{x_2[n]\}$
  - Example: y[n]=x[n]z[n] (this could be a switch or a modulator)
  - Is $y[n]= x^2[n]$ linear?



Properties of Discrete-Time Systems

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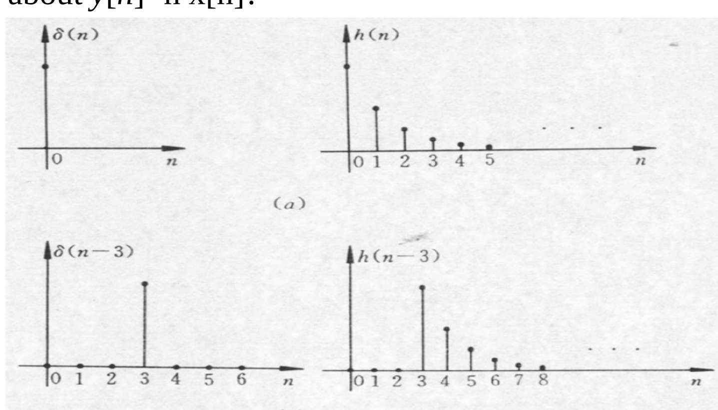
- **Time-invariant**
  - Must satisfy the property

    $y[n] = T\{x[n]\} \rightarrow y[n-k] = T\{x[n-k]\}$
  - Example: $y[n] = x[n-n_d]$
  - Is $y[n]$ = x(n)-x(n-1) time-invariant?
  - How about $y[n]$=n x[n]?



Properties of Discrete-Time Systems

- **Causal**
  - A system is causal if the output only depends on the current and previous input values and not on any future input values;
  - More detail description:
    - A system is causal if, for every choice of n_0, the output sequence value at the index $n=n_0$ depends only on the input sequence values for $n \le n_0$.
    - This implies if $x_1[n]=x_2[n]$ for $n \le n_0$, then $y_1[n]=y_2[n]$ for $n \le n_0$
  - Most discrete-time systems are causal (why?)
    - Future values of the signal can not be observed in real-time signal processing applications
    - A noncausal system is physically unrealizable (i.e., it cannot be implemented).

Stability of Discrete-time system

- A system is stable if every bounded input sequence produces a bounded output sequence; this is known as bounded-input bounded-output (BIBO) stability;
- A system is BIBO stable if $|x[n]| \le M1$ implies $|y[n]| \le M2$, for finite $M1$ and $M2$ and for all n $M1$ and $M2$ are fixed positive values
- Question: which of the following are BIBO stable: accumulator, finite average, shift, or amplifier with offset?

Examples of Discrete-Time Systems

- Accumulator:

$$y[n] = \sum_{k=0}^{n} x[k]$$

  - Causal (assuming n≥0)
  - Linear
  - Not time-invariant
  - unstable

- Finite averager:

$$y[n] = \sum_{k=n-N}^{n+N} x[k]$$

  - Not Causal (assuming N≠0)
  - Linear
  - time-invariant

Examples of Discrete-Time Systems

- Shifter:

$$y[n] = x[n - N]$$

  - Causal if N≥0: otherwise not causal
  - Linear
  - Time-invariant

- Amplifier with offset:

$$y[n] = ax[n] + b$$

  - Causal
  - Not linear (assuming b≠0)
  - Time-invariant

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LTI Systems

Discrete-time systems are systems that are both linear and time-invariant (LTI).

It plays a central role in digital signal processing:

- Many physical systems are either LTI or approximately so.
- Many efficient tools are available for the analysis and design of LTI systems. (e.g. Fourier analysis)

The input-output relation for the LTI system may be expressed as a convolution sum:

$$y[n]= x[n]*h[n]$$

Convolution

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$$
\begin{aligned}
y[n] &= T\{x[n]\} \\
&= T\left\{ \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right\} && \text{by the sifting property} \\
&= \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} && \text{by linearity} \\
&= \sum_{k=-\infty}^{\infty} x[k]h_k[n] && \text{by defining } h_k[n] \triangleq T\{\delta[n-k]\} \\
&= \sum_{k=-\infty}^{\infty} x[k]h_0[n-k] && \text{by time-invariance} \\
&= x[n] * h[n] && \text{note we defined } h[n] \triangleq h_0[n]
\end{aligned}
$$

- ***Convolution sum*** and is denoted by the symbol *

Graphic interpretation of convolution

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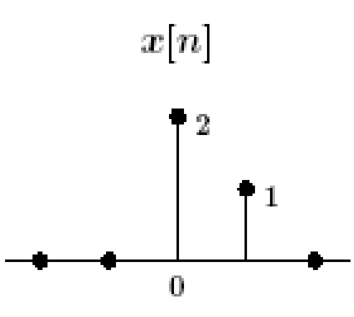
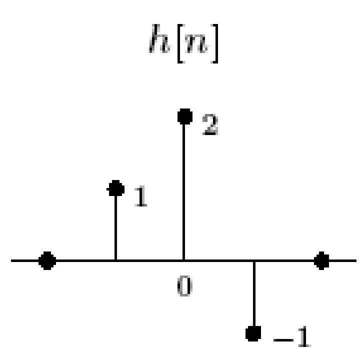
- Consider convolving the following two sequences:





First interpretation

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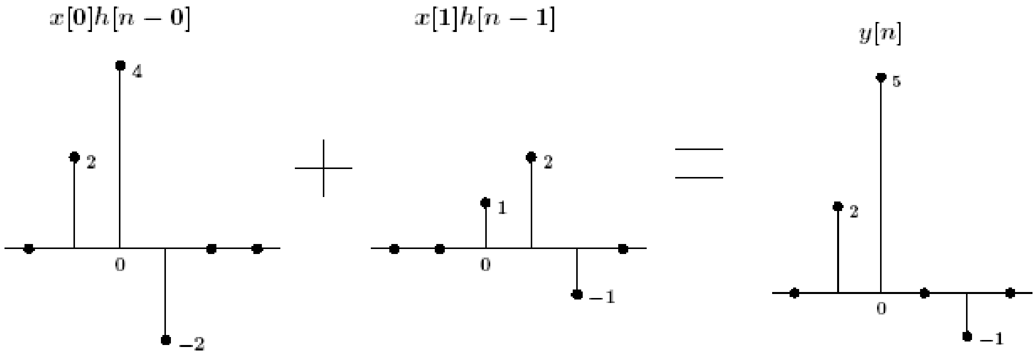
- Since x[n] is non-zero only for n=0 and n=1, convolution produces two terms: x[0]h[n-0] and x[1]h[n-1],
- Adding these two sequences result in y[n]



Flip-and-drag method

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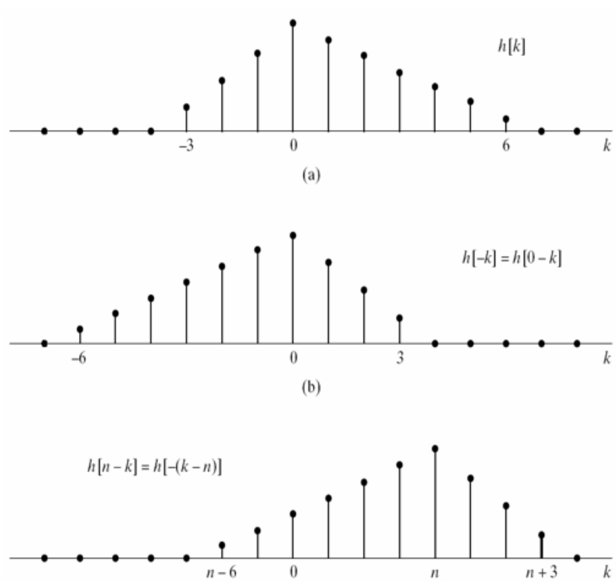
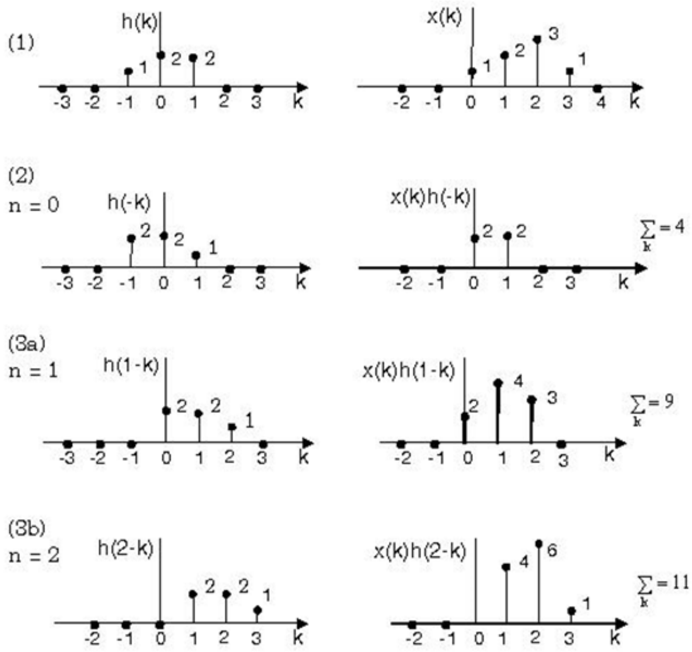
- For any fixed value of n, summing all the values of the products $x[k]h[n-k]$
- Key is to form ***h[n-k]***
  - Reflecting h[k] about the origin to obtain h[-k]
  - Shifting the origin of the reflected sequence to k=n



Example



$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k)$$

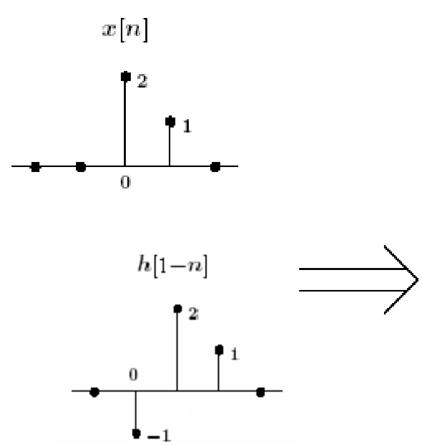
$$y(0) = 4$$

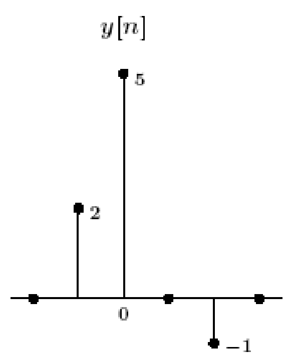
$$y(1) = 9$$

$$y(2) = 11$$

Second Interpretation

- Use the flip-and-drag method: fix one sequence, flip the other, then drag, multiply and accumulate:





$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k)$$

Analytical Convolution Example

- To convolve the sequences x[n]=u[n]-u[n-N] and h[n] = a^n u[n] ,
  - where N is a positive integer;
- Can do convolution graphically, do it analytically in this case:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} (u[k]-u[k-N])a^{n-k}u[n-k] \\ &= \sum_{k=0}^{N-1} a^{n-k}u[n-k] \\ &= \begin{cases} \sum_{k=0}^{n} a^{n-k}, & 0 \le n \le N-1 \\ \sum_{k=0}^{N-1} a^{n-k}, & n \ge N \end{cases} \\ &= \begin{cases} a^n \frac{1-a^{-(n+1)}}{1-a^{-1}}, & 0 \le n \le N-1 \\ a^n \frac{1-a^{-N}}{1-a^{-1}}, & n \ge N \end{cases} \end{aligned}$$

using $\sum_{k=M}^{N} a^k = \frac{a^M - a^{N+1}}{1-a}$

LTI System

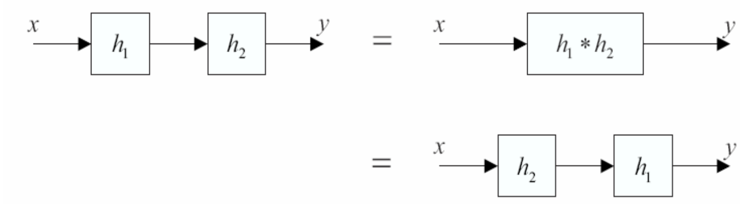
- A system is LTI if it obeys convolution
- LTI system is completely described by its impulse response

$$h[n] = T\{\delta[n]\}$$

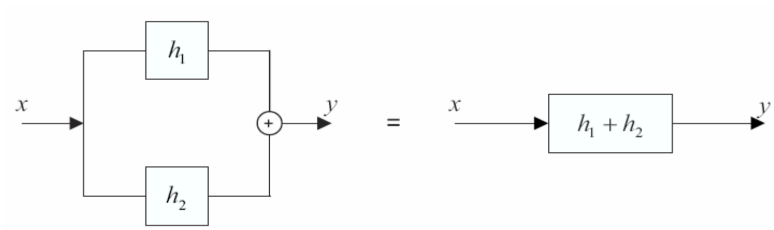
- Properties of convolution:
  - Commutative: $x_1[n] * x_2[n] = x_2[n] * x_1[n]$
  - Distributive: $x_1[n] * (x_2[n] + x_3[n]) = (x_1[n] * x_2[n]) + (x_1[n] * x_3[n])$
  - Associative $x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$
  - Identity $x[n] * \delta[n] = x[n]$
  - shift $x[n] * \delta[n-k] = x[n-k]$

Properties of LTI System

- If two LTI systems with impulse responses h1[n] and h2[n] are in cascade, the impulse response of the overall system is h1[n] * h2[n]



- If two LTI systems with impulse responses h1[n] and h2[n] are in parallel, the impulse response of the overall system is h1[n]+h2[n]



Stability of LTI System

- LTI system is stable if its impulse response is *absolutely summable*, i.e.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

- To prove this is sufficient, note that

$$\begin{aligned} |y[n]| &= \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \\ &\le \sum_{k=-\infty}^{\infty} |x[n-k]h[k]| \\ &= \sum_{k=-\infty}^{\infty} |x[n-k]| \cdot |h[k]| \\ &\le M_1 \sum_{k=-\infty}^{\infty} |h[k]| \end{aligned}$$

Stability of LTI System (c)

- To prove that is necessary, assume that $x[n]=\pm M_1$ for all n with the sign of x[n] chosen such that:

$$\operatorname{sgn}(x[n-k]) = \operatorname{sgn}(h[k]) \qquad \forall k$$

- Then, we have

$$\begin{aligned}|y[n]| &= \left|\sum_{k=-\infty}^{\infty} x[n-k]h[k]\right| \\ &= \sum_{k=-\infty}^{\infty} M_1|h[k]| \\ &= M_1 \sum_{k=-\infty}^{\infty} |h[k]|\end{aligned}$$

- And if the summation is not finite, then neither is |y[n]|

Stability of LTI System (c)

- Example:

Exponential impulse response $h[n]=a^n u[n]$ is stable if

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} |a|^k < \infty$$

- The above geometric series converges only for |a|<1, this system is stable only if -1<a<1

Causality of LTI system

- The class of causal systems was defined as: $y[n_0]$ depends only on the input $x[n]$, for $n \le n_0$.
- It implies the condition

  $h[n]=0, \quad n<0$

for a linear time-invariant system.

- A sequence that is zero for $n<0$ is referred to as a causal sequence, and it can be the impulse response of a causal LTI system.

Impulse response of LTI System

- Ideal Delay $$h[n]=\delta[n-n_d]$$
- Accumulator
$$h[n]=\sum_{k=-\infty}^{n}\delta[k]=\begin{cases}1, & n\ge 0\\ 0, & n<0\end{cases}=u[n]$$
- Forward Difference
$$h[n]=\delta[n+1]-\delta[n]$$
- Backward Difference
$$h[n]=\delta[n]-\delta[n-1]$$

Impulse response of LTI System

- An LTI system with a finite number impulse response coefficients is called a finite impulse response system, or FIR
- An LTI system with an infinite number of impulse response coefficients is called an infinite impulse response system, or IIR

For example, the LTI system with

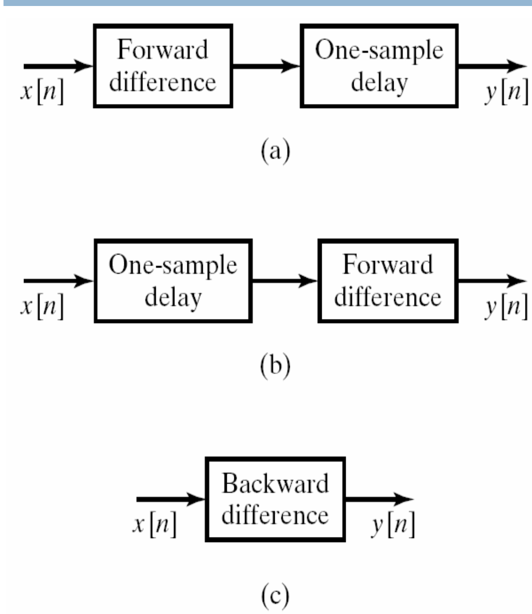
$$h[n] = u[n] - u[n-N] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

is FIR with $N_1 = 0$ and $N_2 = N - 1$. The LTI system with

$$h[n] = \alpha^n u[n] = \begin{cases} \alpha^n & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is IIR (cannot find an N_2...)

Example



- Commutative property is useful in simplifying the interconnections of the LTI system
- Inverse system

$h[n]*h_i[n]=h_i[n]*h[n]=\delta[n]$

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Linear difference equation systems

- A useful sub-class of discrete-time systems is described in general by N-th order linear difference equations:

$$a_0 y[n] + a_1 y[n-1] + \ldots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$a_0 = 1,\ a_N \neq 0$ and the a_k, b_k coefficients are constant (not time-varying)

- State of system at time n may be determined by the set of values
  - (y[n-1]……y[n-N]) – also called *initial conditions* at time n
- Given state at time n = n_0 and *input sequence x*; the unique output sequence y satisfying difference equation can be found.

LCCDE

Consider the accumulator system:

$$x[n] \longrightarrow y[n] = \sum_{k=-\infty}^{n} x[k]$$

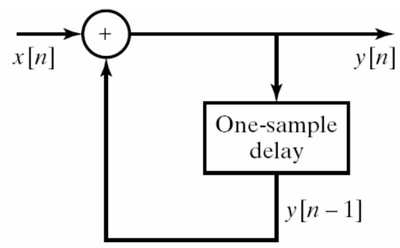
infinite number of adders

Observe that:

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n]$$
$$= y[n-1] + x[n]$$

one adder and one memory unit

This is an LCCDE of order $N = 1$ ($M = 0$, $a_0 = 1$, $a_1 = -1$, $b_0 = 1$)
LCCDE: Linear constant coefficient difference equations



Solution of LCCDE

- Given a particular input sequence $x[n]$, the problem of solving for an output sequence $y[n]$, is of particular interest in DSP. Only general properties of the solutions $y[n]$ are looked at here.
- Without additional constraints or information, a LCCDE does not provide a unique specification of the output.

The most general solution is expressed in the form

$$y[n]=y_p[n]+y_h[n]$$

- $y_p[n]$ is any particular solution of the LCCDE for a given input $x_p[n]$
- $y_h[n]$ is the general solution of the homogeneous equation for $x[n]=0$

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0$$

Proof of Result: Taking the sum of

$$\sum_{k=0}^{N} a(k) y_{\mathrm{P}}(n-k) = \sum_{k=0}^{M} b(k) x(n-k)$$
$$\sum_{k=0}^{N} a(k) y_{\mathrm{H}}(n-k) = 0$$

we obtain

$$\sum_{k=0}^{N} a(k) y(n-k) = \sum_{k=0}^{M} b(k) x(n-k)$$

with $y(n) = y_{\mathrm{P}}(n) + y_{\mathrm{H}}(n)$. Result is proven.

Recursive implementation

- LCCDE lead to efficient recursive implementation:
- Recursive because the computation of $y[n]$ makes use of past output signal values (e.g. $y[-1]$, y[-2],…y[-N], for calculating y[0]).
- These past output signal values contain all the necessary information about earlier states of the system.
- To generate values of y[n] for n<-N(again assuming that the values y[-1], y[-2],…, y[-N]) are given), back recursive can be used.

LCCD as recursive procedures

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If N sequential output values are provided.

•Later values of y[n] can be obtained by rearranging LCCDE equation as a recursive relation running forward in n

•Prior values of y[n] can be obtained by rearranging LCCDE equation as a recursive relation running backward in n

$$y(n) = \sum_{k=0}^{M} \frac{b(k)}{a(0)} x(n-k) - \sum_{k=1}^{N} \frac{a(k)}{a(0)} y(n-k) \qquad \text{forwards}$$

$$y(n-N) = \sum_{k=0}^{M} \frac{b(k)}{a(N)} x(n-k) - \sum_{k=0}^{N-1} \frac{a(k)}{a(N)} y(n-k) \qquad \text{backwards}$$

Example

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- Consider the difference equation in the form:

$$y(n) = -A_1 y(n-1) + B_0 x(n) + B_1 x(n-1)$$

- Find impulse response for this causal system.

$$h(0) = -A_1 h(-1) + B_0 x(0) + B_1 x(-1) = B_0$$

$$h(1) = -A_1 h(0) + B_0 x(1) + B_1 x(0) = -A_1 B_0 + B_1$$

$$h(2) = -A_1(-A_1 B_0 + B_1)$$

$$h(3) = A_1^2(-A_1 B_0 + B_1)$$

$$h(n) = (-A_1)^{n-1}(-A_1 B_0 + B_1)$$

$$h(n) = B_0 \delta(n) + (-A_1)^{n-1}(-A_1 B_0 + B_1) u(n-1)$$

General remarks

- The solution of an LCCDE is not unique; for an N th order LCCDE, uniqueness requires the specification of N initial conditions.
- LCCDE $\overset{?}{=}$ Causal LTI system.
- LCCDE corresponds to a unique causal LTI system if this system is *initially at rest*. That is:

  $x[n] = 0$ for $n < n_0 \rightarrow y[n] = 0$ for $n < n_0$

  - Equivalent to assuming zero initial conditions when solving LCCDE, that is:

  $\boxed{y[n_0 - l] = 0 \text{ for } l = 1, \ldots, N \text{ if } x[n]=0 \text{ for } n<n_0}$

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Frequency response of Discrete-time systems

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The response to a sinusoidal input is:

- Sinusoidal with the same frequency
- Amplitude and phase determined by the system.

Consider an input sequence $x(n) = e^{j\omega n}$

The corresponding output of a LTI system with impulse response h[n] is:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \longrightarrow y(n) = e^{j\omega n} H(e^{j\omega})$$

Frequency response

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Describes the change in complex amplitude of a complex exponential as a function of the frequency

$$\delta(n) \longrightarrow \boxed{\textbf{LTI system}} \longrightarrow h(n)$$

$$e^{j\omega n} \longrightarrow \boxed{\textbf{LTI system}} \longrightarrow e^{j\omega n}H(\omega)$$

Example: the delay system:

$$y(n) = x(n - n_d) \quad \text{with fixed integer } n_d$$

$$x(n) = e^{j\omega n} \implies y(n) = e^{j\omega(n-n_d)} \implies H(\omega) = e^{-j\omega n_d}$$

Since $|H(\omega)| = 1$, this system is frequency nonselective.

Properties of frequency response

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Periodic with 2π $$H(e^{j\omega}) = H(e^{j(\omega+2\pi)})$$

Expressed in terms of real and imaginary part

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

Or in terms of magnitude and phase as:

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\varphi(\omega)}$$

Even $$|H(e^{j\omega})| = [H_R{}^2(e^{j\omega}) + H_I{}^2(e^{j\omega})]^{\frac{1}{2}}$$

Odd $$\varphi(\omega) = tg^{-1} H_I(e^{j\omega}) / H_R(e^{j\omega})$$

Plot of Frequency response

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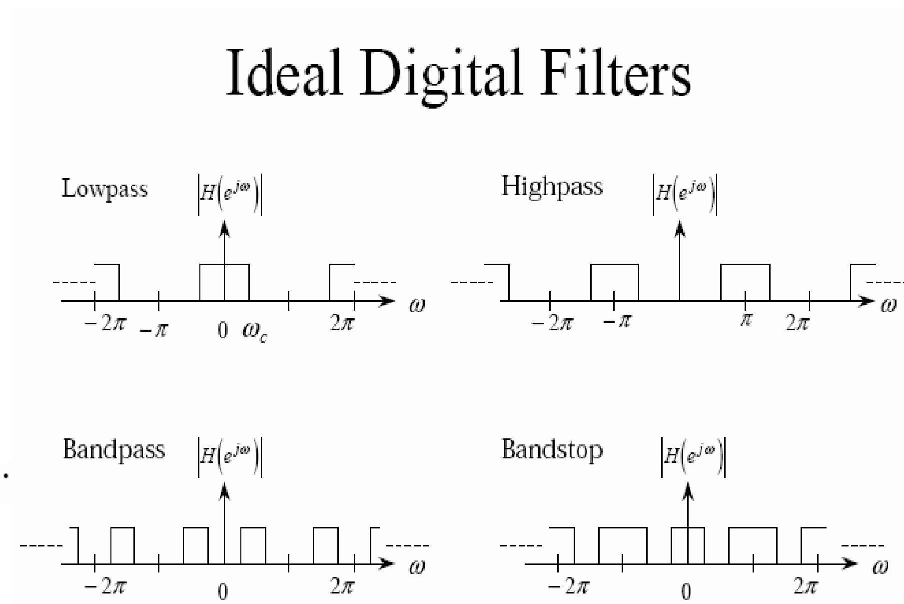

Suddenly applied complex exponential inputs

- Causal LTI system

$x[n] = e^{j\omega n}\mathrm{u}[\mathrm{n}] \Rightarrow y[n] = \begin{cases} 0, & n < 0 \\ \left(\sum_{k=0}^{n} h[k]e^{-j\omega k}\right)e^{j\omega n} & n \geq 0 \end{cases}$

$$\mathrm{y}[n] = H\left(e^{j\omega}\right)e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

Steady-state response　　　Transient response

- FIR system $\mathrm{y}[n] = H\left(e^{j\omega}\right)e^{j\omega n}$ for n> M-1
- IIR system transient response to decay or system be stable when $\sum_{k=0}^{\infty} |h[k]| < \infty$

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Review of Fourier Analysis

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- The power of the Fourier transform in signal analysis and pattern recognition is its ability to reveals spectral structures that may be used to characterize a signal.
- Analysis of signals in terms of their components at different frequencies
  - Fourier analysis considers signals to be constructed from a sum of complex exponentials with appropriate frequencies, amplitude and phase
  - Frequency components are the complex exponentials (sines and cosines) which when added together, make up the signal.

Example of Fourier Analysis

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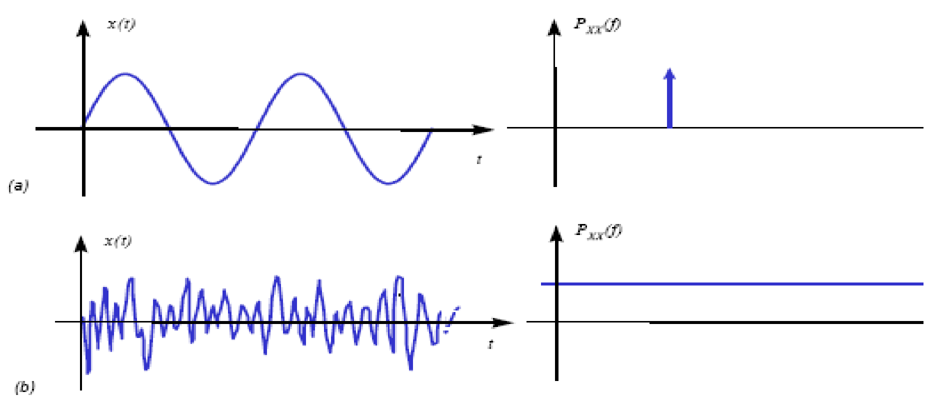
- For a periodic signal the power is concentrated in extremely narrow bands of frequencies indicating the existence of structure and the predictable character of the signal.
  - Pure sine wave: power is concentrated in one frequency.
  - Purely random signal: power is spread equally in the frequency domain.



Basic functions for the Fourier analysis

$$x_1(t) = \cos\omega_0 t \qquad x_3(t) = \cos\omega_0 t + j\sin\omega_0 t = e^{j\omega_0 t}$$

$$x_2(t) = \sin\omega_0 t$$

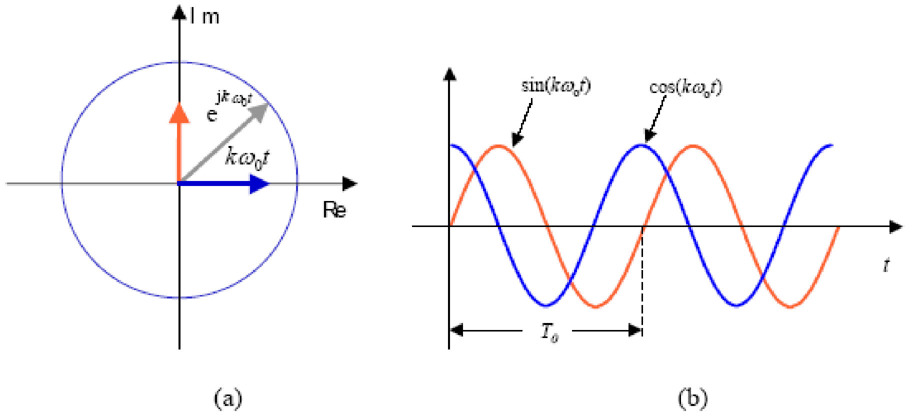


Figure - Fourier basis functions: (a) real and imaginary parts of a complex sinusoid, (b) vector representation of a complex exponential.

Fourier Series

- For a periodic continuous-time signal $x(t)$ with period T

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos k\Omega_0 t + \sum_{k=1}^{\infty} B_k \sin k\Omega_0 t$$

- This can be written in exponential form $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$

- The a_k are found from $a_k = \frac{1}{T}\int_T x(t) e^{-jk\Omega_0 t} dt$

- $x(t)$ is made from a d.c. term plus a weighted sum of sinusoids at integer multiples of the fundamental frequency $\Omega_0 = 2\pi/T$

Example

- Find the frequency spectrum of a 1 kHz sinewave:

  $x(t)=\sin(2000\pi t)$ $-\infty < t < \infty$

Solution A: The Fourier synthesis can be written as

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk2000\pi t} = \cdots + c_{-1}e^{-j2000\pi t} + c_0 + c_1 e^{j2000\pi t} + \cdots$$

The sine wave can be expressed as:

$$x(t)=\sin(2000\pi t)=\frac{1}{2j}e^{j2000\pi t}-\frac{1}{2j}e^{-j2000\pi t}$$

Equating the coefficients of the previous two Eq.

$$c_1=\frac{1}{2j},\ c_{-1}=-\frac{1}{2j} \text{ and } c_{k\neq\pm1}=0$$

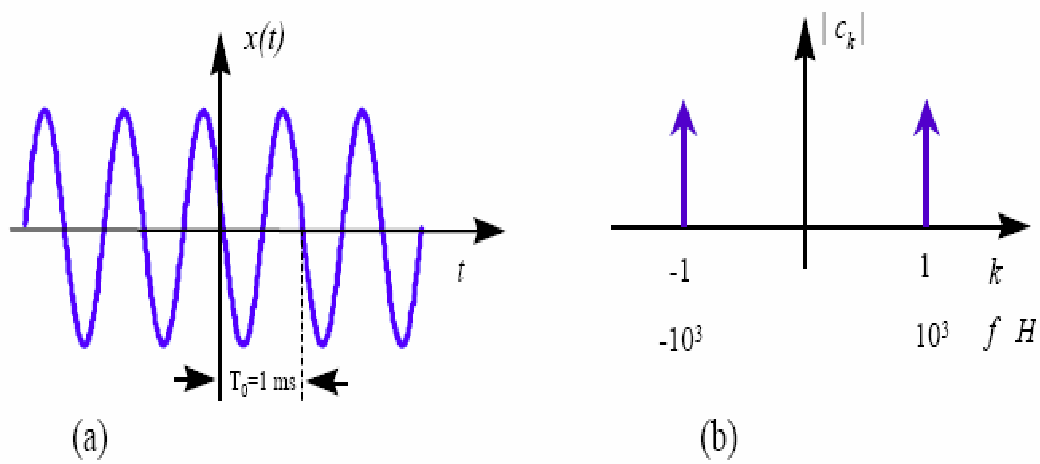


Figure 3.3 - A sinewave and its magnitude spectrum.

Fourier Transform

- For an aperiodic signal, we can say that it is equivalent to a periodic signal with infinite period
  - Hence the Fourier series becomes the Fourier Transform

$$T \Rightarrow \infty$$

$$\text{FT} \begin{cases} X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \\ x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{-j\Omega t}d\Omega \end{cases}$$

Existence of Fourier Transform

- A (Dirichlet Conditions) $\int_{-\infty}^{\infty}|x(t)|dt < \infty$
  - $x(t)$ is absolutely integrable
  - $x(t)$ has a finite number of maxima and minima in any finite interval
  - $x(t)$ has a finite number of discontinuities within any finite interval and all discontinuities are finite
- B
  - If $\int_{-\infty}^{\infty}|x(t)|^2 dt < \infty$ then

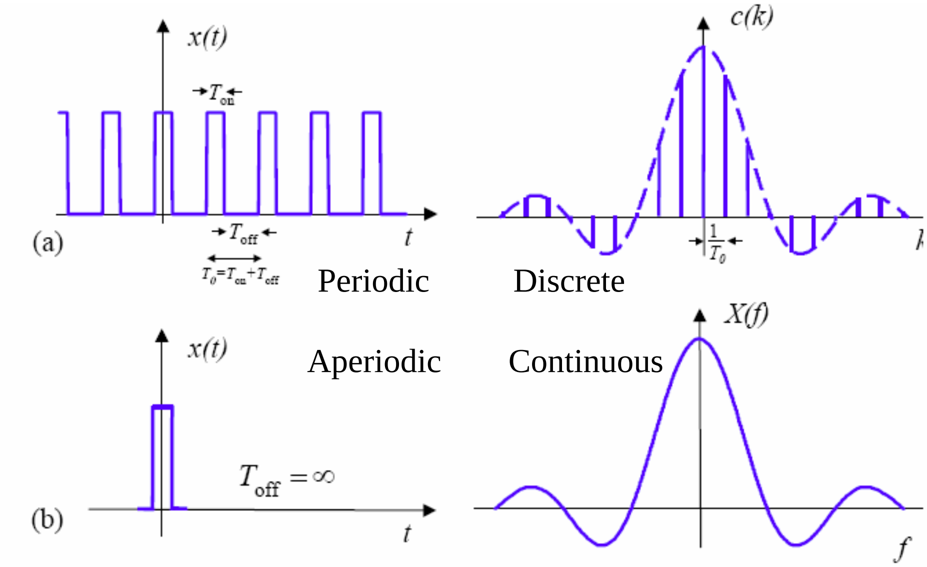
    1. $X(\Omega)$ is finite

    2. $\int_{-\infty}^{\infty}|e(t)|^2 dt = 0$

    for $e(t) = \hat{x}(t) - x(t)$

    where $\hat{x}(t) = IFT\{X(\omega)\}$

Comparison between FS and FT



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Discrete-Time Signals and the Fourier Transform

- CTFT has been reviewed.
- Now consider a discrete-time signal x[n]
  - X[n] is just a sequence of numbers
  - What does it mean for a sequence of numbers to have a spectrum?
- How can the spectrum of a discrete-time signal (sequence) be found using the Fourier transform when the Fourier transform is defined for continuous-time signals?

Discrete-Time Fourier Transform

The definition of DTFT:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad \text{DTFT}$$

The definition of inverse DTFT:

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Any interval of length 2π can be used.

DTFT derive from FT

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- Consider a discrete-time signal $x(n)$ and a continuous-time signal $x'(t)$ given by *

$$x'(t) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT)$$

- Now

$$X'(e^{j\omega}) = \int_{-\infty}^{\infty} x'(t)e^{-j\omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x(n)\int_{-\infty}^{\infty} e^{-j\omega t}\delta(t - nT)dt$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega T}$$

A sequence of numbers having a spectrum is OK if we use the trick of *

Continue

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- In DTFT, use the normalized frequency i.e.,

$$\omega \longrightarrow \omega \Delta t$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

- The DTFT $X(e^{j\omega})$ is a continuous function of frequency, and its period is 2π

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(e^{j\omega})$$

Example

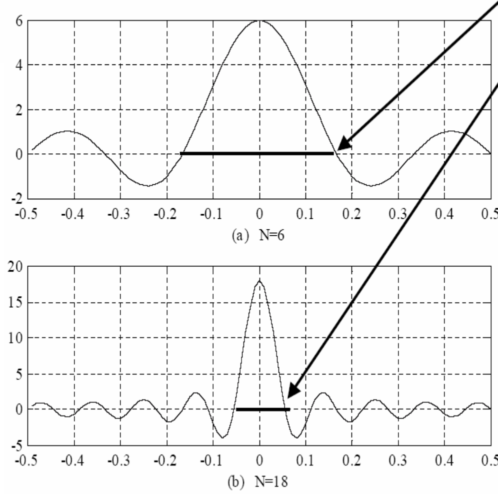
$$d(n)=\begin{cases}1 & n=0,1,\cdots,N-1\\ 0 & n\text{为其他值}\end{cases}$$

$$\frac{2\pi}{N}$$

$$D(e^{j\omega})=\sum_{n=0}^{N-1}e^{-j\omega n}=\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$=e^{-j(N-1)\omega/2}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$=e^{j\varphi(\omega)}D_g(\omega)$$



Zero point

$$\frac{\omega N}{2}=k\pi$$

$$\omega=\frac{2\pi}{N}k$$

$$D_g(e^{j\omega})=\frac{\sin(\omega N/2)}{\sin(\omega/2)}\qquad \sin c$$

Inverse DTFT

$$\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega m}d\omega=\int_{-\pi}^{\pi}[\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}]e^{j\omega m}d\omega$$

$$=\sum_{n=-\infty}^{\infty}x(n)\int_{-\pi}^{\pi}e^{-j\omega(n-m)}d\omega=2\pi x(m)$$

$$\int_{-\pi}^{\pi}e^{-j\omega(n-m)}d\omega=\begin{cases}2\pi & n=m\\ 0 & n\neq m\end{cases}$$

$$\therefore x(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$$

The Inverse DTFT is a synthesis formula. It represents x[n] as a superposition of infinite small complex sinusoids of the form $\frac{1}{2\pi}X(e^{j\omega})e^{j\omega n}d\omega$

with ω ranging over an interval of length 2π

Compare Fourier Series and DTFT

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$$x(t) = \sum_{n=-\infty}^{\infty} X_n \, e^{j\frac{2\pi n}{T}t}, \quad X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, e^{-j\frac{2\pi n}{T}t} \, dt \quad \textbf{FS}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \, e^{-j\omega n}, \quad x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega \quad \textbf{DTFT}$$

Observation: replacing in Fourier series

$$x(t) \longrightarrow X(\omega) \qquad X_n \longrightarrow x(n) \qquad t \longrightarrow -\omega \qquad T \longrightarrow 2\pi$$

we obtain DTFT!!!

Compare Fourier Series and DTFT (c)

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An important conclusion follows: DTFT is equivalent to Fourier series but applied to the "opposite" domain. In Fourier series, a periodic continuous signal is represented as a sum of exponentials weighted by discrete Fourier (spectral) coefficients. In DTFT, a periodic continuous spectrum is represented as a sum of exponentials weighted by discrete signal values.

Remarks:

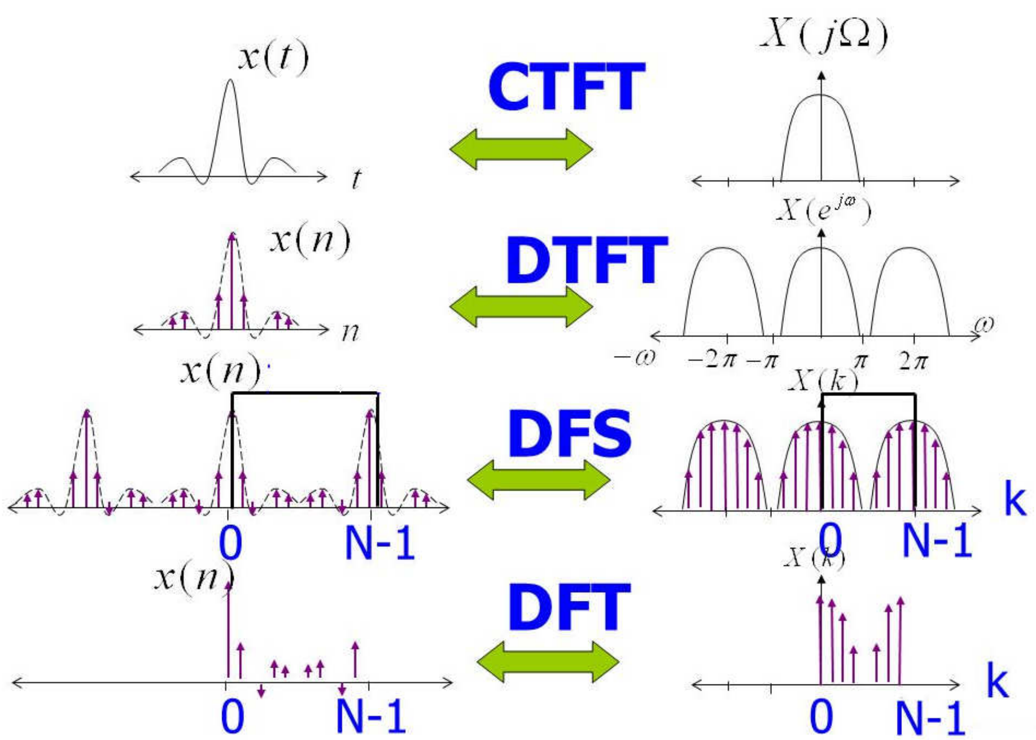
DTFT can be derived directly from the Fourier series

all developments for Fourier series can be applied to DTFT

the relationship between Fourier series and DTFT illustrates the duality between time and frequency domains

Four Transform

	Time Domain		Frequency Domain	
1.	Continuous Aperiodic	⟺	Continuous Aperiodic (Ω)	FT
2.	Continuous Periodic	⟺	Discrete Aperiodic (Ω)	FS
3.	Discrete Aperiodic	⟺	Continuous Periodic (ω)	DTFT
4.	Discrete Periodic		Discrete Periodic	DFS



Convergence of the infinite sum

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$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad \text{DTFT}$$

- X[n] is *absolutely summable* (Eg. 2.17)
  - Impulse response of stable system is absolutely summable
  - Any FIR system is absolutely summable
  - For IIR system, need to consider the convergence
- X[n] is square summable (Eg. 2.18)

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

- Fourier Transform of a constant sequence

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

Eg. 2.18

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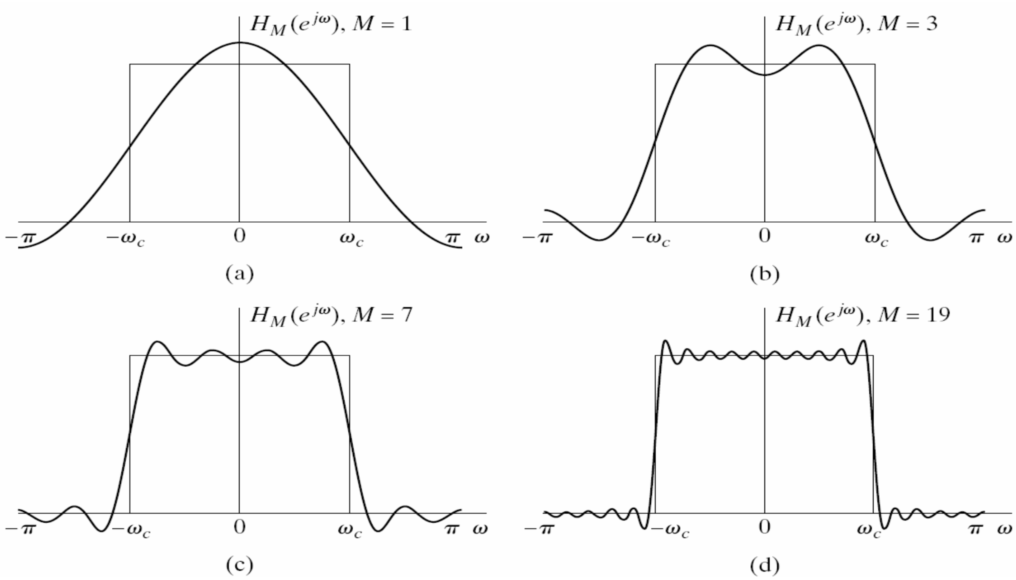

Symmetrical Properties

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- For arbitrary real sequence x[n]

$$\left|X(e^{j\omega})\right| = \left|X(e^{-j\omega})\right|$$ Even Function

$$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$$ Odd Function

Properties of DTFT

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- Linearity:

$$F[ax_1(n) + bx_2(n)] = aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

- Time-shifting:

$$\text{if} \quad x(n) = \mathcal{F}^{-1}\{X(\omega)\} \ \text{then} \ x(n-m) = \mathcal{F}^{-1}\{X(\omega)e^{-j\omega m}\}$$

Proof:

$$\begin{aligned} \mathcal{F}\{x(n-m)\} &= \sum_{n=-\infty}^{\infty} x(\underbrace{n-m}_{k})\, e^{-j\omega n} = \sum_{k=-\infty}^{\infty} x(k) e^{-j\omega(m+k)} \\ &= e^{-j\omega m} \sum_{k=-\infty}^{\infty} x(k) e^{-j\omega k} = X(\omega) e^{-j\omega m} \\ \mathcal{F}^{-1}\{X(\omega)e^{-j\omega m}\} &= \mathcal{F}^{-1}\{\mathcal{F}\{\, x(n-m)\}\} = x(n-m) \end{aligned}$$

Properties of DTFT (c)

Frequency shifting:

If $X(\omega) = \mathcal{F}\{x(n)\}$ then $X(\omega - \nu) = \mathcal{F}\{x(n)e^{j\nu n}\}$

Also if $x(n) = \mathcal{F}^{-1}\{X(\omega)\}$ then $x(n)e^{j\nu n} = \mathcal{F}^{-1}\{X(\omega - \nu)\}$

Proof:

$$\begin{aligned}
\mathcal{F}\{x(n)e^{j\nu n}\} &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega-\nu)n} = X(\omega - \nu) \\
\mathcal{F}^{-1}\{X(\omega - \nu)\} &= \mathcal{F}^{-1}\{\mathcal{F}\{\, x(n)e^{j\nu n}\}\} = x(n)e^{j\nu n}
\end{aligned}$$

Properties of DTFT (c)

Time reversal:

If $X(\omega) = \mathcal{F}\{x(n)\}$ then $X(-\omega) = \mathcal{F}\{x(-n)\}$

Also if $x(n) = \mathcal{F}^{-1}\{X(\omega)\}$ then $x(-n) = \mathcal{F}^{-1}\{X(-\omega)\}$

Proof:

$$\begin{aligned}
\mathcal{F}\{x(-n)\} &= \sum_{n=-\infty}^{\infty} x(\underbrace{-n}_{m})e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m)e^{j\omega m} = X(-\omega) \\
\mathcal{F}^{-1}\{X(-\omega)\} &= \mathcal{F}^{-1}\{\mathcal{F}\{\, x(-n)\}\} = x(-n)
\end{aligned}$$

Properties of DTFT (c)

Convolution theorem:

$$\text{If} \quad X(\omega) = \mathcal{F}\{x(n)\} \quad , \quad H(\omega) = \mathcal{F}\{h(n)\} \quad ,$$

$$\text{and} \quad y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \{x(n)\} * \{h(n)\}$$

$$\text{then} \quad Y(\omega) = \mathcal{F}\{y(n)\} = X(\omega)H(\omega)$$

Convolution of sequences in time-domain is equivalent to multiplication of the corresponding Fourier transforms in frequency domain.

Proof of convolution theorem:

$$\begin{aligned}
Y(\omega) &= \mathcal{F}\{y(n)\} = \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x(k)h(\underbrace{n-k}_{m}) \right\} e^{-j\omega n} \\
&= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)h(m)e^{-j\omega(m+k)} \\
&= \left\{ \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} \right\} \left\{ \sum_{m=-\infty}^{\infty} h(m)e^{-j\omega m} \right\} \\
&= X(\omega)H(\omega)
\end{aligned}$$

Properties of DTFT (c)

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Windowing theorem:

$$\text{If} \quad X(\omega) = \mathcal{F}\{x(n)\} \quad , \quad W(\omega) = \mathcal{F}\{w(n)\} \quad ,$$

$$\text{and} \quad y(n) = x(n)w(n)$$

$$\text{then} \quad Y(\omega) = \mathcal{F}\{y(n)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\nu)W(\omega-\nu)\,d\nu$$

Multiplication of sequences in time-domain is equivalent to periodic convolution of the corresponding Fourier transforms in frequency domain.

Proof: by means of direct substitution, similarly to the proof of convolution theorem.

Summary of main properties of DTFT

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Sequence $x(n)$	Fourier Transform $X(\omega)$
$a\,x(n)+b\,y(n)$	$a\,X(\omega)+b\,Y(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x(n-m)$	$e^{-j\omega m}X(\omega)$
$e^{j\nu n}x(n)$	$X(\omega-\nu)$
$x(-n)$	$X(-\omega)$
$n\,x(n)$	$j\,\frac{dX(\omega)}{d\omega}$
$\{x(n)\}*\{h(n)\}$	$X(\omega)H(\omega)$
$x(n)w(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\nu)W(\omega-\nu)\,d\nu$
$\sum_{n=-\infty}^{\infty}\|x(n)\|^2$	$\frac{1}{2\pi}\int_{-\pi}^{\pi}\|X(\omega)\|^2\,d\omega$
$\sum_{n=-\infty}^{\infty}x(n)y^*(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\omega)Y^*(\omega)\,d\omega$

The End