ELCE 705
DIGITAL SIGNAL PROCESSING

Frequency Response

5.0 - 5.3

5.4 - 5.7 (Brief introduction)

Contents

- Frequency Response of LTI System
- System functions characterized by LCCD
- Frequency response for Rational system functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear Systems with generalized linear phase

Introduction

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

For LTI systems we can write

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Alternatively, this relationship can be expressed in the z-transform domain as

$$Y(z) = H(z)X(z),$$

where $H(z)$ is the **system function**, or the z-transform of the system impulse response.

Frequency response of LTI systems

The frequency response $H(e^{j\omega})$ of a system is defined as the gain that the system applies to the complex exponential input $e^{j\omega n}$. The Fourier transforms of the system input and output are therefore related by

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}).$$

In terms of magnitude and phase,

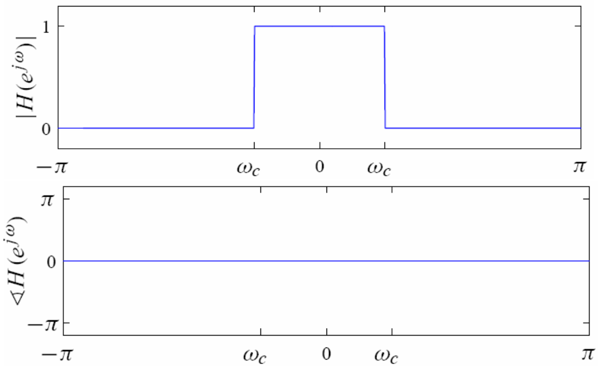
$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$$
$$\sphericalangle Y(e^{j\omega}) = \sphericalangle H(e^{j\omega}) + \sphericalangle X(e^{j\omega}).$$

In this case $|H(e^{j\omega})|$ is referred to as the **magnitude response** or **gain** of the system, and $\sphericalangle H(e^{j\omega})$ is the **phase response** or **phase shift**.

Ideal frequency-selective filters

- Frequency components of the input are suppressed in the output if $|H(e^{j\omega})|$ is small at those frequencies. The **ideal lowpass filter** is defined as the LTI system with frequency response:

$$H_{\mathrm{lp}}(e^{j\omega}) \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

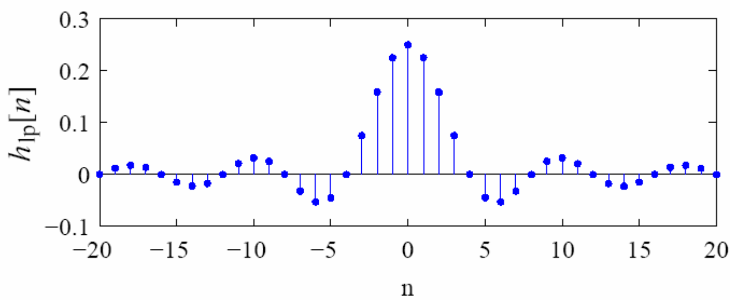


Ideal frequency-selective filters

This response, as for all discrete-time signals, is periodic with period 2π. Its impulse response (for $-\infty < n < \infty$) is

$$h_{\mathrm{lp}}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\omega n}\right]_{-\omega_c}^{\omega_c}$$
$$= \frac{1}{\pi n}\frac{1}{2j}(e^{j\omega_c n} - e^{-j\omega_c n}) = \frac{\sin(\omega_c n)}{\pi n},$$

which for $\omega_c = \pi/4$ is



Ideal frequency-selective filters

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The ideal lowpass filter is noncausal, and its impulse response extends from $-\infty < n < \infty$. The system is therefore not computationally realisable. Also, the phase response of the ideal lowpass filter is specified to be zero — this is a problem in that causal ideal filters have nonzero phase responses.

The **ideal highpass filter** is

$$H_{\text{hp}}(e^{j\omega}) = \begin{cases} 0 & |\omega| \leq \omega_c \\ 1 & \omega_c < |\omega| \leq \pi. \end{cases}$$

Since $H_{\text{hp}}(e^{j\omega}) = 1 - H_{\text{lp}}(e^{j\omega})$, its frequency response is

$$h_{\text{hp}}[n] = \delta[n] - h_{\text{lp}}[n] = \delta[n] - \frac{\sin(\omega_c n)}{\pi n}.$$

Phase distortion and delay

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Consider the ideal delay, with impulse response

$$h_{\text{id}}[n] = \delta[n - n_d]$$

and frequency response

$$H_{\text{id}}(e^{j\omega}) = e^{-j\omega n_d}$$

The magnitude and phase of this response are

$$|H_{\text{id}}(e^{j\omega})| = 1,$$

$$\sphericalangle H_{\text{id}}(e^{j\omega}) = -\omega n_d, \qquad |\omega| < \pi$$

- The phase distortion of the ideal delay is therefore a linear function of ω.
- This is considered to be a rather mild (acceptable) form of phase distortion, since the only effect is to shift the sequence in time.

In designing approximations to ideal filters, we are therefore frequently willing to accept linear phase distortion. The ideal lowpass filter with phase distortion would be defined as

$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d} & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi, \end{cases}$$

with impulse response

$$h_{\mathrm{lp}}[n] = \frac{\sin(\omega_c(n - n_d))}{\pi(n - n_d)}.$$

In other words, a filter with linear phase response can be viewed as a cascade of a zero-phase filter, followed by a time shift or delay.

Group Delay

- A convenient measure of linearity of the phase is the **group delay**, which relates to the effect of the phase on a narrowband signal.
- Consider the narrowband input $x[n] = s[n]\cos(\omega_0 n)$, where $s[n]$ is the envelope of the signal. Since $X(e^{j\omega})$ is nonzero only around $\omega = \omega_0$, the effect of the phase of the system can be approximated around $\omega = \omega_0$ by

  $$\sphericalangle H(e^{j\omega}) \approx -\phi_0 - \omega n_d$$

- For input $x[n] = s[n]\cos(\omega_0 n)$, output is approximately $y[n] = s[n-n_d]\cos(\omega_0 n - \varphi_0 - \omega_0 n_d)$. The time delay of the envelope $s[n]$ of the narrowband signal $x[n]$ with Fourier transform centered at ω_0 is therefore given by the negative of the slope of the phase at ω_0. The group delay of a system is therefore defined as:

  $$\tau(\omega) = \mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\left\{\arg[H(e^{j\omega})]\right\}$$

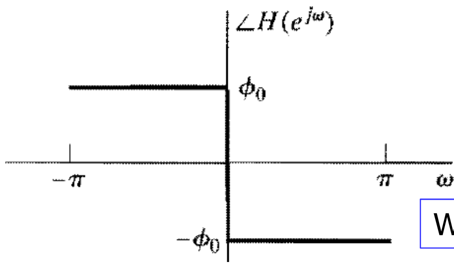
Example

In the system assume that the input can be expressed in the form

$$x[n] = s[n]\cos(\omega_0 n).$$

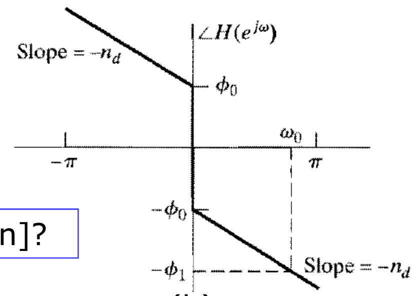
Assume also that $s[n]$ is lowpass and relatively narrowband; i.e., $S(e^{j\omega}) = 0$ for $|\omega| > \Delta$, with Δ very small and $\Delta \ll \omega_0$ so that $X(e^{j\omega})$ is narrowband around $\omega = \pm\omega_0$.

If $|H(e^{j\omega})| = 1$ and $\angle H(e^{j\omega})$ is



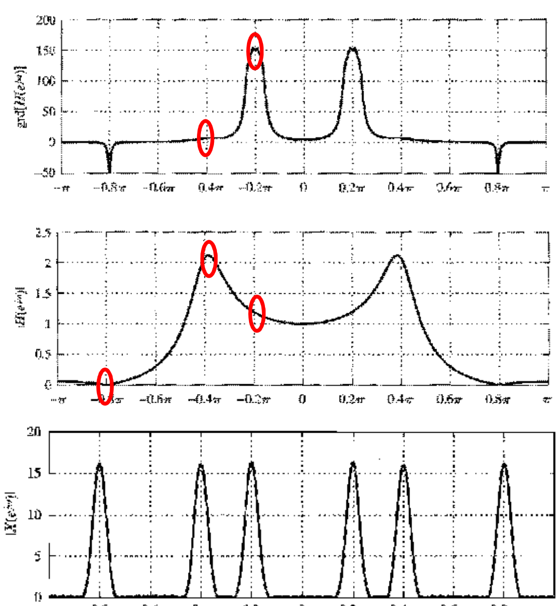
(a)

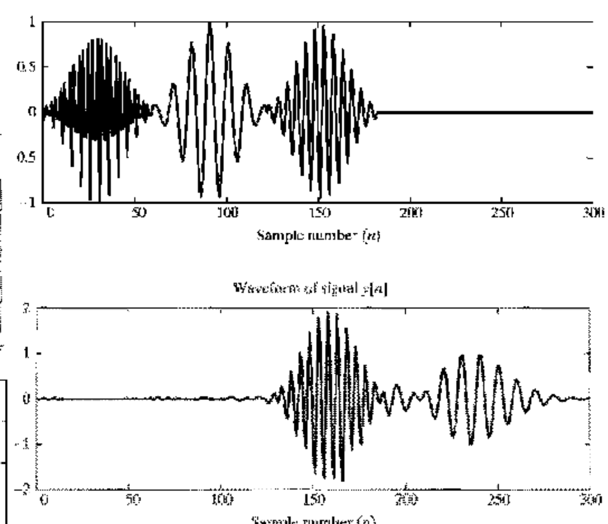
What is y[n]?



(b)

Example





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System response for LCCD systems

Ideal filters cannot be implemented with finite computation. Therefore we need approximations to ideal filters. Systems described by LCCD equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

are useful for providing one class of approximation.

The properties of this class of system are best developed in the z-transform domain. The z-transform of the equation is

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z),$$

or equivalently

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z).$$

The system function for a system that satisfies a difference equation of the required form is therefore

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}.$$

Each factor $(1 - c_k z^{-1})$ in the numerator contributes a zero at $z = c_k$ and a pole at $z = 0$. Each factor $(1 - d_k z^{-1})$ contributes a zero at $z = 0$ and a pole at $z = d_k$.

The difference equation and the algebraic expression for the system function are equivalent, as demonstrated by the next example.

Example: Second order system

Given the system function

$$H(z) = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 + \frac{3}{4}z^{-1})},$$

we can find the corresponding difference equation by noting that

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} = \frac{Y(z)}{X(z)}.$$

Therefore

$$(1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2})Y(z) = (1 + 2z^{-1} + z^{-2})X(z),$$

and the difference equation is

$$y[n] + \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = x[n] + 2x[n-1] + x[n-2]$$

Stability and causality

- A difference equation does not uniquely specify the impulse response of a LTI system.
  - Each possible choice of ROC will lead to a different impulse response.
- If a system is causal
  - Impulse response is a right-sided sequence
  - The ROC of *H(z)* must be outside of the outermost pole.
- If a system is stable
  - ROC include unit circle

the system be stable. $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

For $|z| = 1$ this is identical to the condition $\sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty$

Example: Determine the ROC

The frequency response of the LTI system with difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n]$$

is

$$H(z) = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}.$$

There are three choices for the ROC:

- **Causal:** ROC outside of outermost pole $|z| > 2$ (but then not stable).
- **Stable:** ROC such that $\frac{1}{2} < |z| < 2$ (but then not causal).
- If $|z| < \frac{1}{2}$ then the system is neither causal nor stable.

For a causal and stable system the ROC must be outside the outermost pole and include the unit circle. This is only possible if all the poles are inside the unit circle.

Inverse System

The system $H_i(z)$ is the inverse system to $H(z)$ if

$$G(z) = H(z)H_i(z) = 1,$$

which implies that

$$H(z) = \frac{1}{H_i(z)}.$$

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}$$

The time-domain equivalent is

$$g[n] = h[n] * h_i[n] = \delta[n].$$

The question of which ROC to associate with $H_i(z)$ is answered by the convolution theorem — for the previous equation to hold the regions of convergence of $H(z)$ and $H_i(z)$ must overlap.

- The frequency response of the inverse system, if it exists, is

$$H(e^{j\omega}) = \frac{1}{H_i(e^{j\omega})}$$

- Not all systems have an inverse. For example, there is no way to recover the frequency components above the cutoff frequency that were set to zero by the action of the lowpass filter.

Minimum phase Systems

- A LTI system is stable and causal with a stable and causal inverse if and only if both the poles and zeros of *H(z)* are inside the unit circle — such systems are called **minimum phase** systems.

Impulse response for rational system functions

- If a system has a rational transfer function, with at least one non-zero pole that is not cancelled by a zero, then there will always be a term corresponding to an infinite length sequence in the impulse response. Such systems are called **infinite impulse response (IIR)** systems.
- If a system has no poles except at $z = 0$ (that is, $N = 0$ in the LCCDE expression), then

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

  - The system is determined to within a constant multiplier by its zeros,

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

  - The system is called a **finite impulse response (FIR) system.**

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Frequency Response for rational system functions

If a stable LTI system has a rational system function, then its frequency response has the form

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}.$$

We want to know the magnitude and phase associated with the frequency response. To this end, it is useful to express $H(e^{j\omega})$ in terms of the poles and zeros of $H(z)$:

$$H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - d_k e^{-j\omega})}.$$

It follows that

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} |1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N} |1 - d_k e^{-j\omega}|}.$$

Frequency response

- The **gain** in dB of $|H(e^{j\omega})|$ also called the **log magnitude**, is given by

$$\text{Gain in dB} = 20 \log_{10} |H(e^{j\omega})|$$

$$20 \log_{10} |H(e^{j\omega})| = 20 \log_{10} \left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20 \log_{10} |1 - c_k e^{-j\omega}| - \sum_{k=1}^{N} 20 \log_{10} |1 - d_k e^{-j\omega}|.$$

Also

$$\text{Attenuation in dB} = -\text{Gain in dB}.$$

Thus a 60dB attenuation at frequency ω corresponds to $|H(e^{j\omega})| = 0.001$.

$$20 \log_{10} |Y(e^{j\omega})| = 20 \log_{10} |H(e^{j\omega})| + 20 \log_{10} |X(e^{j\omega})|$$

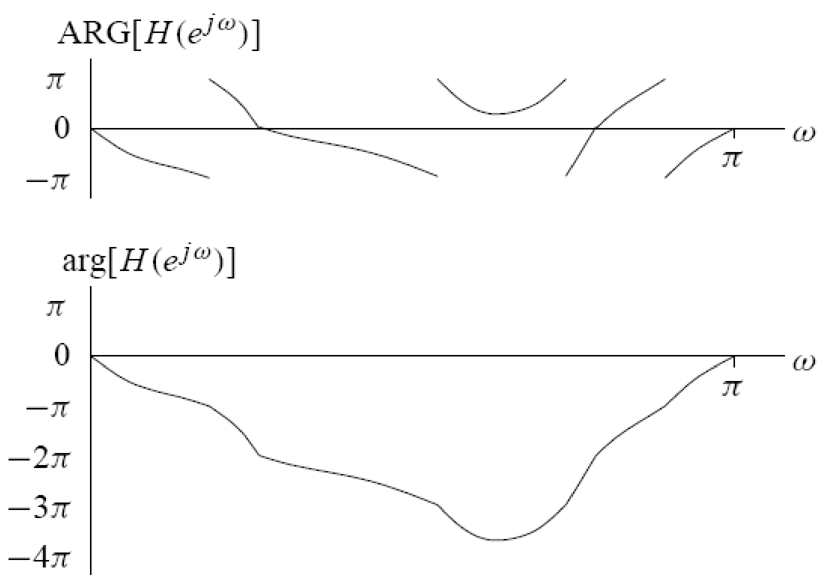
Phase response

- The phase response for a rational system function is

$$\arg\left[H(e^{j\omega})\right] = \arg\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \arg\left[1 - c_k e^{-j\omega}\right] - \sum_{k=1}^{N} \arg\left[1 - d_k e^{-j\omega}\right]$$

  - The phase of each term is ambiguous, since any integer multiple of 2π can be added at each value of ωwithout changing the value of the complex number.
  - **Principal** value ARG[$H(e^{j\omega})$]lies in the range **$-\pi$~+ π**.
    - Appropriate multiples of 2 π can be added or subtracted, if required, to yield the continuous phase function.
- The group delay is:

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{N} \frac{d}{d\omega}(\arg[1 - d_k e^{-j\omega}]) - \sum_{k=1}^{M} \frac{d}{d\omega}(\arg[1 - c_k e^{-j\omega}])$$



Frequency response of a single pole or zero

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Consider a single zero factor of the form

$$(1 - re^{j\theta}e^{-j\omega})$$

in the frequency response. The magnitude squared of this factor is

$$\begin{aligned}|1 - re^{j\theta}e^{-j\omega}|^2 &= (1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{j\omega}) \\ &= 1 + r^2 - 2r\cos(\omega - \theta),\end{aligned}$$

so the log magnitude in dB is

$$20\log_{10}|1 - re^{j\theta}e^{-j\omega}| = 10\log_{10}[1 + r^2 - 2r\cos(\omega - \theta)].$$

The principle value of the phase for the factor is

$$\text{ARG}[1 - re^{j\theta}e^{-j\omega}] = \arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right].$$

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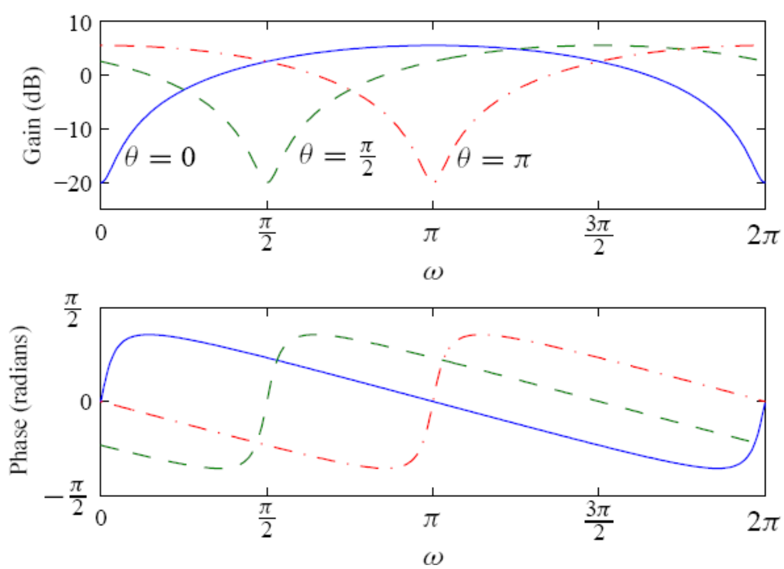
The following plot shows the frequency response for $r = 0.9$ and three different values of θ:



- The gain dips at $\omega = \theta$. As θ changes, the frequency at which the dip occurs changes.
- The gain is maximised for $\omega - \theta = \pi$, and for $r = 0.9$ the magnitude of the resulting gain is

$$10\log_{10}(1 + r^2 + 2r) = 20\log_{10}(1 + r) = 5.57\text{dB}.$$

- The gain is minimised for $\omega = \theta$, and for $r = 0.9$ the resulting gain is

$$10\log_{10}(1 + r^2 - 2r) = 20\log_{10}|1 - r| = -20\text{dB}.$$

- The phase is zero at $\omega = \theta$.

Note that if the factor $(1 - re^{j\theta}e^{j\omega})$ occurs in the denominator, thereby representing a pole factor, then the entire analysis holds with the exception that the sign of the log magnitude and the phase changes.

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Relation between Magnitude and Phase

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- For general LTI system
  - Knowledge about magnitude doesn't provide any information about phase
  - Knowledge about phase doesn't provide any information about magnitude
- For linear constant-coefficient difference equations
  - There is some constraint between magnitude and phase
  - If magnitude and number of pole-zeros are known
    - Only a finite number of choices for phase
  - If phase and number of pole-zeros are known
    - Only a finite number of choices for magnitude (ignoring scale)
- A class of systems called minimum-phase
  - Magnitude specifies phase uniquely
  - Phase specifies magnitude uniquely

Square Magnitude System Function

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- Explore possible choices of system function of the form

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})^* H(e^{j\omega}) = H^*(1/z^*)H(z)\Big|_{z=e^{j\omega}}$$

- Restricting the system to be rational

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \qquad H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k^* z)}$$

- The square system function

$$C(z) = H(z)H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)^2\frac{\prod_{k=1}^{M}(1-c_k z^{-1})(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k z^{-1})(1-d_k^* z)}$$

- Given $\left|H(e^{j\omega})\right|^2$ we can get C(z)
- What information on H(z) can we get from C(z)?

Poles and Zeros of Magnitude Square System Function

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$$C(z) = H(z)H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1-c_k z^{-1})(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k z^{-1})(1-d_k^* z)}$$

- For every pole d_k in H(z) there is pole of C(z) at d_k and $(1/d_k)^*$
- For every zero c_k in H(z) there is zero of C(z) at c_k and $(1/c_k)^*$
- Poles and zeros of C(z) occur in conjugate reciprocal pairs
- If one of the pole/zero is inside the unit circle the reciprocal will be outside
  - Unless they are both on the unit circle
- If H(z) is stable all poles have to be inside the unit circle
  - We can infer which poles of C(z) belong to H(z)
- However, zeros cannot be uniquely determined
  - Example to follow

Example

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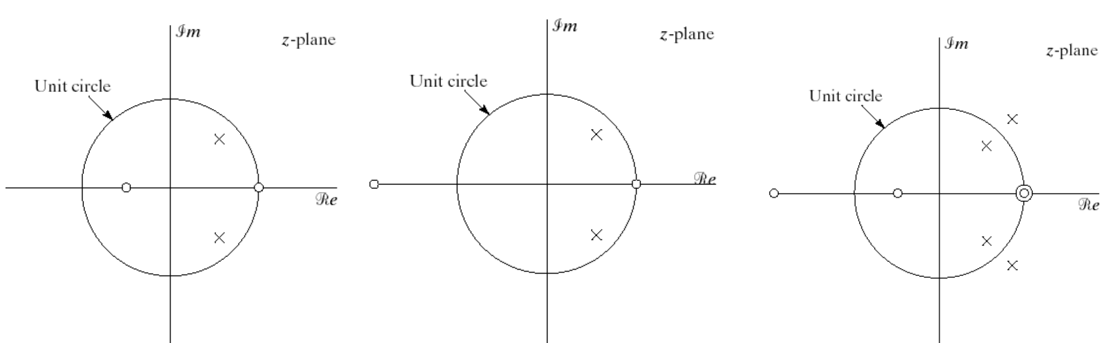
- Two systems with

$$H_1(z) = \frac{2(1-z^{-1})(1+0.5z^{-1})}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})} \qquad H_2(z) = \frac{(1-z^{-1})(1+2z^{-})}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})}$$

- Both share the same magnitude square system function



$H_1(z)$ $\qquad$ $H_2(z)$

$$C(z) = H_1(z)H_1^*(1/z^*) = H_2(z)H_2^*(1/z^*)$$

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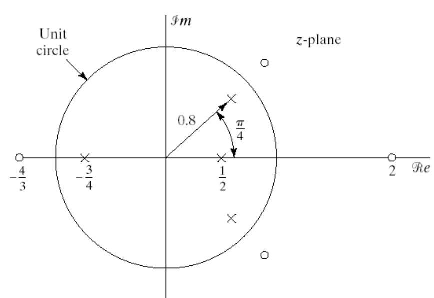
All-Pass System

- A system with frequency response magnitude constant
- Important uses such as compensating for phase distortion
- Simple all-pass system

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$



- Magnitude response constant

$$H_{ap}\left(e^{j\omega}\right) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

- Most general form with real impulse response

$$H_{ap}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1} - d_k}{1 - d_k z^{-1}}\prod_{k=1}^{M_c}\frac{\left(z^{-1} - c_k^*\right)\left(z^{-1} - c_k\right)}{\left(1 - c_k z^{-1}\right)\left(1 - c_k^* z^{-1}\right)}$$

- A: positive constant, d_k: real poles, c_k: complex poles

Phase of All-Pass Systems

$$H_{ap}\left(e^{j\omega}\right) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

- Let's write the phase with a represented in polar form

$$\angle\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = -\omega - 2\arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]$$

- The group delay of this system can be written as

$$\mathrm{grd}\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = \frac{1 - r^2}{1 - 2r\cos(\omega - \theta) + r^2} = \frac{1 - r^2}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

- For stable and causal system $|r|<1$
  - Group delay of all-pass systems is always positive
  - Phase between 0 and π is always negative

$$\arg\left[H_{ap}\left(e^{j\omega}\right)\right] \le 0 \quad \text{for} \quad 0 \le \omega < \pi$$

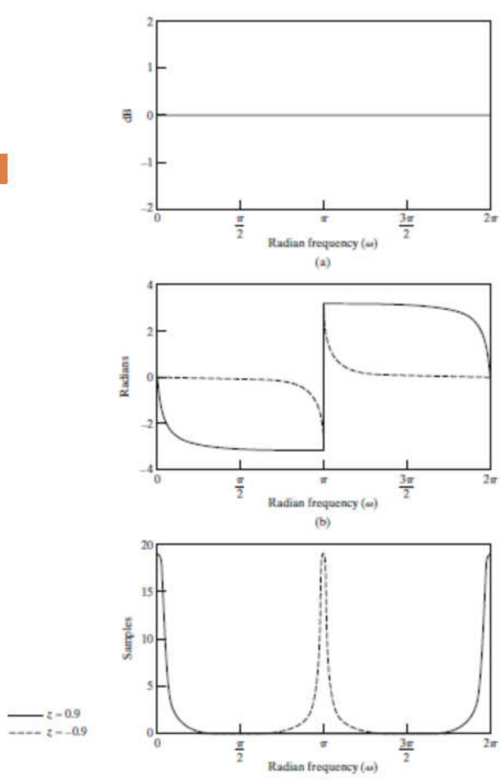


Figure 5.22 Frequency response for all-pass filters with real poles at $z = 0.9$ (solid line) and $z = -0.9$ (dashed line). (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

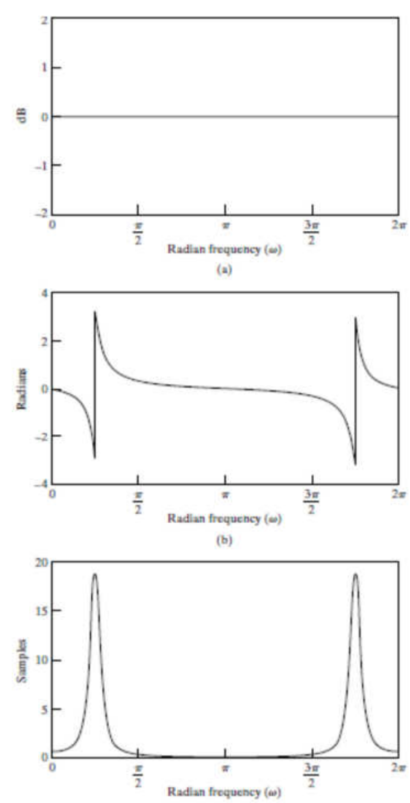


Figure 5.23 Frequency response of second-order all-pass system with poles at $z = 0.9e^{\pm j\pi/4}$. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

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Minimum-Phase System

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- A system with all poles and zeros inside the unit circle
- Both the system function and the inverse is causal and stable
- "minimum-phase" comes from the property of the phase
  - Not obvious to see with the given definition
  - Will look into it
- Given a magnitude square system function that is minimum phase
  - The original system is uniquely determined
- Minimum-phase and All-pass decomposition
  - Any rational system function can be decomposed as

$$H(z) = H_{min}(z) H_{ap}(z)$$

Example 1: Minimum-Phase System

- Consider the following system $H_1(z) = \dfrac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}}$
- One pole inside the unit circle:
  - Make part of minimum-phase system
- One zero outside the unit circle:
  - Add an all-pass system to reflect this zero inside the unit circle

$$H_1(z) = \frac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}} = 3\frac{1}{1+\frac{1}{2}z^{-1}}\left(z^{-1}+\frac{1}{3}\right) = 3\frac{1}{1+\frac{1}{2}z^{-1}}\left(z^{-1}+\frac{1}{3}\right)\frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}$$

$$H_1(z) = \left(3\frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{z^{-1}+\frac{1}{3}}{1+\frac{1}{3}z^{-1}}\right) = H_{\min}(z)H_{ap}(z)$$

Example 2: Minimum-Phase System

- Consider the following system

$$H_2(z) = \frac{\left(1+\frac{3}{2}e^{j\pi/4}z^{-1}\right)\left(1+\frac{3}{2}e^{-j\pi/4}z^{-1}\right)}{1-\frac{1}{3}z^{-1}}$$

- One pole inside the unit circle:
- Complex conjugate zero pair outside the unit circle

$$H_2(z) = \frac{\left(1+\frac{3}{2}e^{j\pi/4}z^{-1}\right)\left(1+\frac{3}{2}e^{-j\pi/4}z^{-1}\right)}{1-\frac{1}{3}z^{-1}}$$

$$= \frac{\frac{3}{2}e^{j\pi/4}\frac{3}{2}e^{-j\pi/4}\left(\frac{2}{3}e^{-j\pi/4}+z^{-1}\right)\left(\frac{2}{3}e^{j\pi/4}+z^{-1}\right)}{1-\frac{1}{3}z^{-1}}$$

Example 2 Cont'd

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$$H_2(z) = \frac{\frac{9}{4}\left(\frac{2}{3}e^{-j\pi/4} + z^{-1}\right)\left(\frac{2}{3}e^{j\pi/4} + z^{-1}\right)}{1 - \frac{1}{3}z^{-1}} \cdot \frac{\left(1 + \frac{2}{3}e^{-j\pi/4}z^{-1}\right)\left(1 + \frac{2}{3}e^{j\pi/4}z^{-1}\right)}{\left(1 + \frac{2}{3}e^{-j\pi/4}z^{-1}\right)\left(1 + \frac{2}{3}e^{j\pi/4}z^{-1}\right)}$$

$$H_2(z) = \frac{\frac{9}{4}\left(1 + \frac{2}{3}e^{-j\pi/4}z^{-1}\right)\left(1 + \frac{2}{3}e^{j\pi/4}z^{-1}\right)}{1 - \frac{1}{3}z^{-1}} \cdot \frac{\left(\frac{2}{3}e^{-j\pi/4} + z^{-1}\right)\left(\frac{2}{3}e^{j\pi/4} + z^{-1}\right)}{\left(1 + \frac{2}{3}e^{-j\pi/4}z^{-1}\right)\left(1 + \frac{2}{3}e^{j\pi/4}z^{-1}\right)}$$

$$H_2(z) = H_{min}(z)H_{ap}(z)$$

Frequency-Response Compensation

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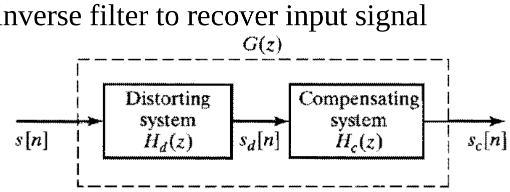
- In many applications a signal is distorted by an LTI system
- Could filter with inverse filter to recover input signal



- Make use of minimum-phase all-pass decomposition
  - Invert minimum phase part
- Assume a distorting system $H_d(z)$
- Decompose it into $H_d(z) = H_{d,min}(z)H_{d,ap}(z)$
- Define compensating system as $$H_c(z) = \frac{1}{H_{d,min}(z)}$$

Cascade of the distorting system and compensating system

$$G(z) = H_c(z)H_d(z) = H_{d,min}(z)H_{d,ap}(z)\frac{1}{H_{d,min}(z)} = H_{d,ap}(z)$$

Properties of Minimum-Phase Systems

- Minimum Phase-Lag Property
  - Continuous phase of a non-minimum-phase system

$$\arg\left[H_d\left(e^{j\omega}\right)\right] = \arg\left[H_{min}\left(e^{j\omega}\right)\right] + \arg\left[H_{ap}\left(e^{j\omega}\right)\right]$$

  - All-pass systems have negative phase between 0 and π
  - So any non-minimum phase system will have a more negative phase compared to the minimum-phase system
  - The negative of the phase is called the phase-lag function
  - The name minimum-phase comes from minimum phase-lag
- Minimum Group-Delay Property

$$\mathrm{grd}\left[H_d\left(e^{j\omega}\right)\right] = \mathrm{grd}\left[H_{min}\left(e^{j\omega}\right)\right] + \mathrm{grd}\left[H_{ap}\left(e^{j\omega}\right)\right]$$

  - Group-delay of all-pass systems is positive
  - Any non-minimum-phase system will always have greater group delay

Contents

Linear Phase System

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- Ideal Delay System $$H_{id}\left(e^{j\omega}\right) = e^{-j\omega\alpha} \qquad |\omega| < \pi$$
- Magnitude, phase, and group delay
$$\left|H_{id}\left(e^{j\omega}\right)\right| = 1$$
$$\angle H_{id}\left(e^{j\omega}\right) = -\omega\alpha$$
$$\mathrm{grd}\left[H_{id}\left(e^{j\omega}\right)\right] = \alpha$$
- Impulse response $$h_{id}[n] = \frac{\sin(\pi(n-\alpha))}{\pi(n-\alpha)}$$
- If $\alpha = n_d$ is integer $$h_{id}[n] = \delta[n - n_d]$$
- For integer α linear phase system delays the input
$$y[n] = x[n] * h_{id}[n] = x[n] * \delta[n - n_d] = x[n - n_d]$$

Symmetry of Linear Phase Impulse Responses

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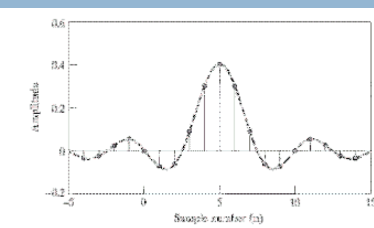
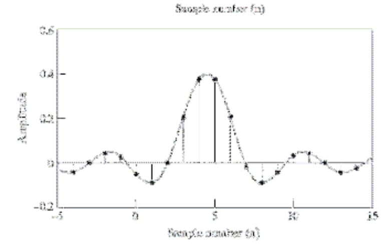
- Linear-phase systems
$$H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{-j\omega\alpha}$$

A zero-phase system output is delayed by α

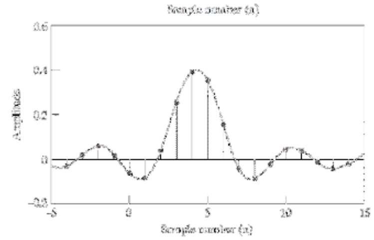
- If 2α is integer
  - Impulse response symmetric
$$h[2\alpha - n] = h[n]$$



$\alpha=5$



$\alpha=4.5$



$\alpha=4.3$

Generalized Linear Phase System

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- Generalized Linear Phase

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega\alpha + j\beta}$$

$A(e^{j\omega})$: Real function of ω
α and β constants

- Has constant group delay

$$\tau(\omega) = \mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}(\arg[H(e^{j\omega})]) = \alpha$$

- And linear phase of general form

$$\arg[H(e^{j\omega})] = \beta - \omega\alpha \qquad 0 \le \omega < \pi$$

Condition for Generalized Linear Phase

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- We can write a generalized linear phase system response as

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h[n]\cos(\omega n) - j\sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)$$

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega\alpha + j\beta} = A(e^{j\omega})\cos(\beta - \omega\alpha) + jA(e^{j\omega})\sin(\beta - \omega\alpha)$$

- The phase angle of this system is

$$\frac{\sin(\beta - \omega\alpha)}{\cos(\beta - \omega\alpha)} = \frac{-\sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)}{\sum_{n=-\infty}^{\infty} h[n]\cos(\omega n)}$$

- Cross multiply to get necessary condition for generalized linear phase

$$\sum_{n=-\infty}^{\infty} h[n]\cos(\omega n)\sin(\beta - \omega\alpha) - \sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)\cos(\beta - \omega\alpha) = 0$$

$$\sum_{n=-\infty}^{\infty} h[n][\cos(\omega n)\sin(\beta - \omega\alpha) - \sin(\omega n)\cos(\beta - \omega\alpha)] = 0$$

$$\sum_{n=-\infty}^{\infty} h[n]\sin(\beta - \omega\alpha + \omega n) = \sum_{n=-\infty}^{\infty} h[n]\sin[\beta + \omega(n - \alpha)] = 0$$

Symmetry of Generalized Linear Phase

- Necessary condition for generalized linear phase

$$\sum_{n=-\infty}^{\infty} h[n]\sin[\beta + \omega(n-\alpha)] = 0$$

- For β=0 or π

$$\sum_{n=-\infty}^{\infty} h[n]\sin[\omega(n-\alpha)] = 0 \longrightarrow h[2\alpha - n] = h[n]$$

- For β= π/2 or 3π/2

$$\sum_{n=-\infty}^{\infty} h[n]\cos[\omega(n-\alpha)] = 0 \longrightarrow h[2\alpha - n] = -h[n]$$

Causal Generalized Linear-Phase System

- If the system is causal and generalized linear-phase

$$h[M-n] = \mp h[n]$$

- Since h[n]=0 for n<0 we get

$$h[n] = 0 \qquad n < 0 \quad \text{and} \quad n > M$$

- An FIR impulse response of length M+1 is generalized linear phase if they are symmetric
- Here M is an even integer

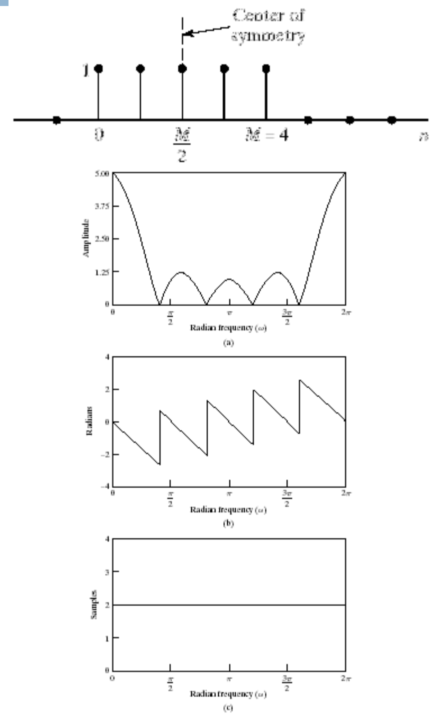
Type I FIR Linear-Phase System

- Type I system is defined with symmetric impulse response
  $$h[n] = h[M-n] \qquad \text{for } 0 \le n \le M$$
  - M is an even integer
- The frequency response can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$
$$= e^{-j\omega M/2}\left[\sum_{n=0}^{M/2} a[n]\cos(\omega n)\right]$$

- Where

$$a[0] = h[M/2]$$
$$a[k] = 2h[M/2-k] \qquad \text{for } k = 1,2,\ldots,M/2$$



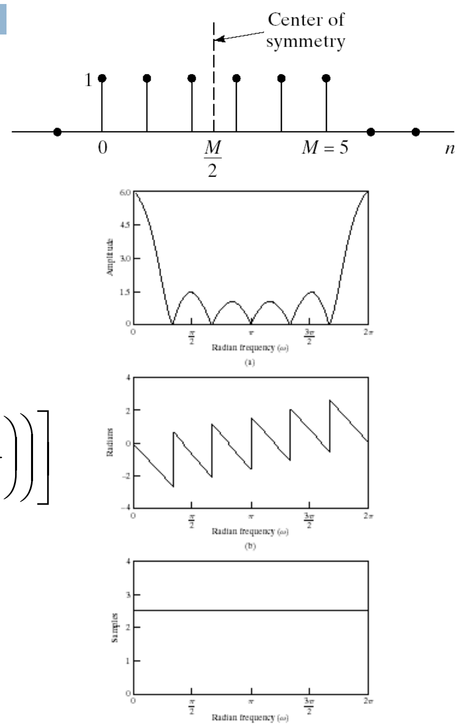
Type II FIR Linear-Phase System

- Type II system is defined with symmetric impulse response
  $$h[n] = h[M-n] \qquad \text{for } 0 \le n \le M$$
  - M is an odd integer
- The frequency response can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$
$$= e^{-j\omega M/2}\left[\sum_{n=1}^{(M+1)/2} b[n]\cos\left(\omega\left(n-\frac{1}{2}\right)\right)\right]$$

- Where

$$b[k] = 2h[(M+1)/2-k]$$
$$\text{for } k = 1,2,\ldots,(M+1)/2$$



Type III FIR Linear-Phase System

- Type III system is defined with symmetric impulse response

$$h[n] = -h[M-n] \qquad \text{for } 0 \le n \le M$$

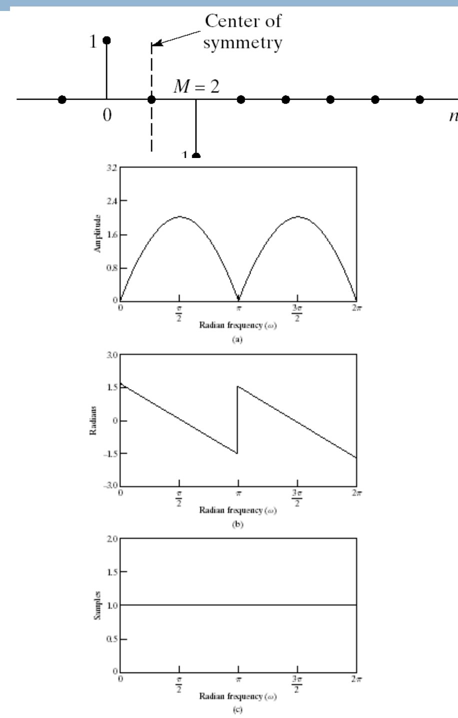
  - M is an even integer
- The frequency response can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$

$$= je^{-j\omega M/2}\left[\sum_{n=1}^{M/2} c[n]\sin(\omega n)\right]$$

- Where

$$c[k] = 2h[M/2-k]$$
$$\text{for } k = 1,2,\ldots,M/2$$



Type IV FIR Linear-Phase System

- Type IV system is defined with symmetric impulse response

$$h[n] = -h[M-n] \qquad \text{for } 0 \le n \le M$$

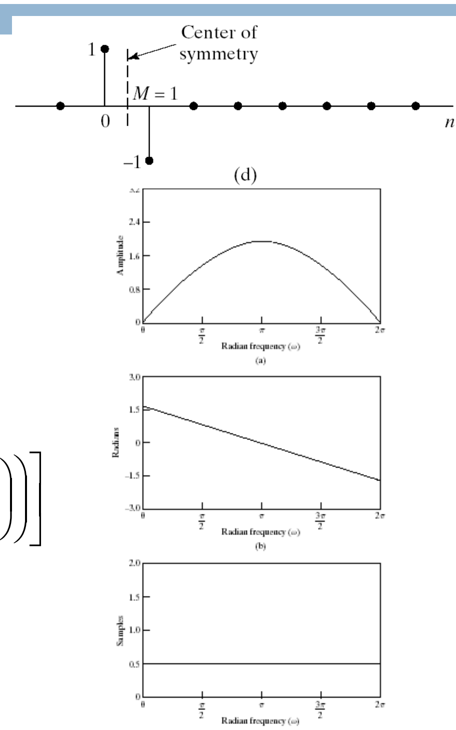
  - M is an odd integer
- The frequency response can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$

$$= je^{-j\omega M/2}\left[\sum_{n=1}^{(M+1)/2} d[n]\sin\left(\omega\left(n-\frac{1}{2}\right)\right)\right]$$

- Where

$$d[k] = 2h[(M+1)/2-k]$$
$$\text{for } k = 1,2,\ldots,(M+1)/2$$



Location of Zeros for Symmetric Cases

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- For type I and II we have

$$h[n] = h[M-n] \xrightarrow{z} H(z) = z^{-M}H(z^{-1})$$

- So if z_0 is a zero $1/z_0$ is also a zero of the system
- If h[n] is real and z_0 is a zero z_0^* is also a zero
- So for real and symmetric h[n] zeros come in sets of four
- Particular importance of z=-1
  - If M is odd implies that $H(-1) = (-1)^M H(-1)$

$$H(-1) = 0$$

  - Cannot design high-pass filter with symmetric FIR filter and M odd

Location of Zeros for Antisymmetric Cases

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- For type III and IV we have

$$h[n] = -h[M-n] \xrightarrow{z} H(z) = -z^{-M}H(z^{-1})$$

- All properties of symmetric systems holds
- Particular importance of both z=+1 and z=-1
  - If z=1

$$H(1) = -H(1) \Rightarrow H(1) = 0$$

    - Independent from M: odd or even
  - If z=-1

$$H(-1) = (-1)^{M+1} H(-1)$$

    - If M+1 is odd implies that

$$H(-1) = 0$$

Thank you !

THE END