Chapter 10

Lecture10

10.1 Product topology on $X \times Y$

Theorem 10.1.1. *Let $f : A \to X \times Y$, let $f = (f_1, f_2)$ where $f_1 : A \to X$ and $f_2 : A \to Y$. Then f is continuous if and only if both f_1 and f_2 are continuous. We denote $f(a) = (f_1(a), f_2(a))$.*

Proof
Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projection maps defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Then $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$. Since π_1 and π_2 are continuous, if f is continuous, then both f_1 and f_2 are continuous.

Conversely, suppose both f_1 and f_2 are continuous. Let U, V be open. Then $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$ for any open set U in X and V in Y. Since f_1 and f_2 are continuous, $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in A. Thus, $f^{-1}(U \times V)$ is open in A.

■

There are two ways to introduce product topology.

1. Take $U_1 \subseteq X_1, U_2 \subseteq X_2, \cdots$. Then we can define the topology on $X_1 \times X_2 \times \cdots$ by the basis $\{U_1 \times U_2 \times \cdots\}$. This is called the **box topology**.

2. Take $U_1 \subseteq X_1, U_2 \subseteq X_2, \cdots$ but only finitely many of them are not equal to the whole space. Then we can define the topology on $X_1 \times X_2 \times \cdots$ by the basis $\{U_1 \times U_2 \times \cdots\}$ where only finitely many U_i are not equal to X_i. This is called the **product topology**.

Definition 10.1.1. *Let J be an arbitrary set. A J-tuple of elements from X is a function $x : J \to X$. So $\alpha \in J \mapsto x(\alpha) = x_\alpha \in X$. And sometimes we denote the J-tuple by $(x_\alpha)_{\alpha \in J}$.*

Definition 10.1.2. *Let $(A_\alpha)_{\alpha \in J}$ be an indexed family of sets.*

$$X = \bigcup_{\alpha \in J} A_\alpha \tag{10.1}$$

The Cartesian product of the family $(A_\alpha)_{\alpha \in J}$ is denoted by

$$\prod_{\alpha \in J} A_\alpha \tag{10.2}$$

which is defined as the set of all J-tuples of elements in X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$, that is, the set of all functions

$$x : J \to X \quad \text{such that} \quad x(\alpha) \in A_\alpha \forall \alpha \in J \tag{10.3}$$

When $A_\alpha = X$, we have $\prod_{\alpha \in J} A_\alpha = X^J$, the set of all functions from J to X.

Definition 10.1.3. *Let $(X_\alpha)_{\alpha \in J}$ be an indexed family of topological spaces. The box topology on $\prod_{\alpha \in J} X_\alpha$ is given by the basis*

$$\left\{ \prod_{\alpha \in J} U_\alpha : U_\alpha \text{ is open in } X_\alpha \forall \alpha \in J \right\} \tag{10.4}$$

Then taking two basis elements $\prod_{\alpha \in J} U_\alpha$ and $\prod_{\alpha \in J} V_\alpha$, their intersection is

$$\left(\prod_{\alpha \in J} U_\alpha \right) \cap \left(\prod_{\alpha \in J} V_\alpha \right) = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha) \tag{10.5}$$

which is also a basis element.

Definition 10.1.4. *A collection $\mathcal{S}$ of subsets of topological space X is a **subbasis** if*

$$\bigcup_{S \in \mathcal{S}} S = X \tag{10.6}$$

Property 10.1.1. *Let $\mathcal{S}$ be a subbasis for a space X. Then the collection of all finite intersections of elements of $\mathcal{S}$ forms a basis for the topology on X. Then the topology generated by $\mathcal{S}$ is the collection of all unions of finite intersections of elements of $\mathcal{S}$.*

Exercise Prove the above property.

Definition 10.1.5. *For a given $\beta \in J$, we denote $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ the projection map defined by*

$$x \mapsto x_\beta \tag{10.7}$$

Then let $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ is open in } X_\beta\}$. Then

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta \tag{10.8}$$

*is a subbasis(Proved yourself). The topology defined by this subbasis is called the **product topology**. The basis $\mathcal{B}$ is given by finite intersections of elements of $\mathcal{S}$ and*

$$\mathcal{B} \ni B = \prod_{\alpha \in J} U_\alpha \quad \text{where } U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \tag{10.9}$$

Notice that if $|J| < \infty$, then the box topology and the product topology are the same. And also notice that the product topology is coarser than the box topology.