

Chapter 2

Later

Definition 2.0.1. For $p > 1, q > 1$. p is conjugate to q if $\frac{1}{p} + \frac{1}{q} = 1$.

Remark When $p = 1, q = \infty$; when $p = \infty, q = 1$. And 2 is conjugate to itself.

Lemma 2.0.1 (Jensen's inequality). If $f(t)$ is a strictly concave function on an interval $I \subseteq \mathbb{R}$, then for any $t_1, t_2 \in I$ and any $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$, we have

$$f(\lambda_1 t_1 + \lambda_2 t_2) \geq \lambda_1 f(t_1) + \lambda_2 f(t_2) \quad (2.1)$$

where the equality holds if and only if $t_1 = t_2$.

Proof Obvious.

Theorem 2.0.1 (Young's inequality). For $a, b \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad (2.2)$$

Proof

Let $x = a^p, y = b^q$. Then $a = x^{\frac{1}{p}}, b = y^{\frac{1}{q}}$. Substituting these into the inequality, we need to show that

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}, \forall x, y \geq 0 \quad (2.3)$$

If $x = 0$ or $y = 0$, the inequality holds trivially. Now we assume $x, y > 0$. Let $f(t) = \ln t$, which is strictly concave on $(0, +\infty)$. By Jensen's inequality, we have

$$f\left(\frac{x}{p} + \frac{y}{q}\right) \geq \frac{1}{p}f(x) + \frac{1}{q}f(y) \quad (2.4)$$

Exponentiating both sides, we have

$$\frac{x}{p} + \frac{y}{q} \geq x^{\frac{1}{p}} y^{\frac{1}{q}} \quad (2.5)$$

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Theorem 2.0.2 (Hölder's inequality). For $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, for any $a = (a_i) \in l_p, b = (b_i) \in l_q$ respectively, we have

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \sqrt[p]{\sum_{k=1}^{\infty} a_k^p} \sqrt[q]{\sum_{k=1}^{\infty} b_k^q}$$

Proof

In case when $\|a\|_p = 0$ or $\|b\|_q = 0$, the inequality holds trivially. Now we assume $\|a\|_p > 0$ and $\|b\|_q > 0$. Let

$$x_k = \frac{|a_k|}{\|a\|_p}, \quad y_k = \frac{|b_k|}{\|b\|_q}, \quad \forall k \geq 1 \quad (2.6)$$

We have

$$\sum_{k=1}^{\infty} x_k^p = \frac{|a_k|^p}{\|a\|_p^p} = 1, \quad \sum_{k=1}^{\infty} y_k^q = \frac{|b_k|^q}{\|b\|_q^q} = 1 \quad (2.7)$$

By Young's inequality, we have

$$x_k y_k \leq \frac{x_k^p}{p} + \frac{y_k^q}{q}, \quad \forall k \geq 1 \quad (2.8)$$

Summing over k from 1 to ∞ , we have

$$\sum_{k=1}^{\infty} x_k y_k \leq \frac{1}{p} \sum_{k=1}^{\infty} x_k^p + \frac{1}{q} \sum_{k=1}^{\infty} y_k^q = \frac{1}{p} + \frac{1}{q} = 1 \quad (2.9)$$

Thus, we have

$$\sum_{k=1}^{\infty} |a_k b_k| = \|a\|_p \|b\|_q \sum_{k=1}^{\infty} x_k y_k \leq \|a\|_p \|b\|_q \quad (2.10)$$

For the case when $p = 1$ and $q = \infty$, we have

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \sum_{k=1}^{\infty} |a_k| \|b\|_{\infty} = \|a\|_1 \|b\|_{\infty} \quad (2.11)$$

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Theorem 2.0.3 (Minkowski inequality). *For $p \geq 1$, for any $x, y \in l_p$, we have*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Proof

If $\sum_{k=1}^{\infty} |x_k + y_k|^p = 0$, then $x_k + y_k = 0$ for all k . Thus, $\|x\|_p = \|y\|_p = 0$ and the inequality holds trivially. Now we assume $\sum_{k=1}^{\infty} |x_k + y_k|^p > 0$.

Let us first show that for $x = (x_i), y = (y_i) \in l_p$, we have

$$\sqrt[p]{|x_k + y_k|^p} \leq \sqrt[p]{|x_k|^p} + \sqrt[p]{|y_k|^p}$$

For every summand, we have

$$\begin{aligned} |x_k + y_k|^p &= |x_k + y_k| \cdot |x_k + y_k|^{p-1} \\ &\leq |x_k| \cdot |x_k + y_k|^{p-1} + |y_k| \cdot |x_k + y_k|^{p-1} \end{aligned}$$

So we have

$$\sum_{k=1}^n |x_k + y_k|^p \leq \sum_{k=1}^n |x_k| \cdot |x_k + y_k|^{p-1} + \sum_{k=1}^n |y_k| \cdot |x_k + y_k|^{p-1} \quad (2.12)$$

Let $a_k = |x_k|, b_k = |x_k + y_k|^{p-1}$, then $b_k^q = |x_k + y_k|^p$. By Hölder's inequality, we have

$$\sum_{k=1}^n |x_k| \cdot |x_k + y_k|^{p-1} \leq \sqrt[p]{\sum_{k=1}^n |x_k|^p} \sqrt[q]{\sum_{k=1}^n |x_k + y_k|^p}$$

On the other hand, let $a_k = |y_k|, b_k = |x_k + y_k|^{p-1}$, we have

$$\sum_{k=1}^n |y_k| \cdot |x_k + y_k|^{p-1} \leq \sqrt[p]{\sum_{k=1}^n |y_k|^p} \sqrt[q]{\sum_{k=1}^n |x_k + y_k|^p}$$

Combining these two inequalities, we have

$$\sum_{k=1}^n |x_k + y_k|^p \leq \left(\sqrt[p]{\sum_{k=1}^n |x_k|^p} + \sqrt[p]{\sum_{k=1}^n |y_k|^p} \right) \sqrt[q]{\sum_{k=1}^n |x_k + y_k|^p} \quad (2.13)$$

Let $S_n = \sum_{k=1}^n |x_k + y_k|^p$, then $S_n \leq C S_n^{\frac{1}{q}}$. If $S_n = 0$, then $S_n \leq C^p$. If $S_n > 0$, then $S_n^{1-\frac{1}{q}} = S_n^{\frac{1}{p}} \leq C$ which implies $S_n \leq C^p$. So S_n is increasing and bounded above. Thus, $\lim_{n \rightarrow \infty} S_n = \sum_{k=1}^{\infty} |x_k + y_k|^p$ exists and is finite.

Dividing both sides by $\sqrt[p]{\sum_{k=1}^n |x_k + y_k|^p}$, we have

$$\sqrt[p]{\sum_{k=1}^n |x_k + y_k|^p} \leq \sqrt[p]{\sum_{k=1}^n |x_k|^p} + \sqrt[p]{\sum_{k=1}^n |y_k|^p} \quad (2.14)$$

It's easy to show that each term on the right-hand side converges as $n \rightarrow \infty$ because $x, y \in l_p$ and we know that a monotonically increasing sequence bounded above converges. So we have that the both sides converge as $n \rightarrow \infty$.

Taking limit $n \rightarrow \infty$, by continuity of n -th root, we have the desired result. ■

Let X be a metric space. Let $x \in X$. We have the following definitions.

Definition 2.0.2. We define a neighborhood of x to be a set of the form

$$U_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$$

for some $\epsilon > 0$.

Definition 2.0.3. We define a punctured neighborhood of x to be a set of the form

$$U_\epsilon^*(x) = \{y \in X \mid 0 < d(x, y) < \epsilon\} = U_\epsilon(x) \setminus \{x\}$$

for some $\epsilon > 0$.

Definition 2.0.4. We say that $M \subseteq X$ is open in X if for every $x \in M$, there exists $\epsilon > 0$ such that $U_\epsilon(x) \subseteq M$.

Remark \emptyset, X are open in X by definition.

Example 2.0.1. Is it possible that in a metric space X , a ball is contained properly inside a ball with smaller radius? That is, is there $x \in X$ and $0 < r < s$ such that $U_s(x) \subsetneq U_r(x)$? [Hint: If $Y \subseteq X$ and (X, d) is a metric space, then (Y, d) is also a metric space.]

Solution Yes. Let $X = (-1, 1)$ with the usual metric. Then $U_{\frac{3}{2}}(\frac{4}{3}) = (-\frac{4}{3}, 1) \subsetneq U_1(0) = (-1, 1)$.

Example 2.0.2. Draw balls centered at 0 in \mathbb{R}^2 with norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$.

Example 2.0.3. "Amazon Metric" on \mathbb{R}^2 is given by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2 \end{cases}$$

We will solve these problems later in the course.

Theorem 2.0.4. 1. The intersection of finitely many open sets is open, that is, $U_1 \cap U_2 \cap \dots \cap U_n$ is open where each U_i is open in X .

2. The union of any collection of open sets is open, that is, if $\{U_i\}_{i \in I}$ is a collection of open sets in X , then $\bigcup_{i \in I} U_i$ is open.

Proof to 1.

If $V = U_1 \cap U_2 \cap \dots \cap U_n = \emptyset$, then V is open by definition.

If $V \neq \emptyset$, let $x \in V$. Since $x \in U_i$ for each $i = 1, 2, \dots, n$, there exists $\epsilon_i > 0$ such that $U_{\epsilon_i}(x) \subseteq U_i$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$. Then $U_\epsilon(x) \subseteq U_i$ for each i , so $U_\epsilon(x) \subseteq V$. Thus, V is open.

Proof to 2.

Let $x \in U = \bigcup_{i \in I} U_i$. Then there exists some $j \in I$ such that $x \in U_j$. Since U_j is open, there exists $\epsilon > 0$ such that $U_\epsilon(x) \subseteq U_j \subseteq U$. Thus, U is open.