

# Chapter 16

## Later

**Definition 16.0.1** (def1).  *$X$  is compact if any open covering admits a finite subcovering.*

**Definition 16.0.2** (def2: Frechet compactness).  *$X$  is limit point compact if every infinite subset of  $X$  has a limit point in  $X$ .*

**Definition 16.0.3** (def3: sequential compactness/Bozano-Weierstrass compactness).  *$X$  is sequentially compact if every sequence  $\{x_n\} \subseteq X$  has a convergent subsequence converging to a point in  $X$ .*

**Theorem 16.0.1.** *If  $X$  is a metric space, then  $X$  is compact if and only if it is limit point compact if and only if it is sequentially compact.*

**Proof**

The proof is quite long and technical. We will only some directions in the following.

**Theorem 16.0.2.** *For an arbitrary topological space, compactness implies limit point compactness. The converse is not true in general.*

**Proof**

Let  $X$  be compact, and let  $A \subseteq X$  be infinite.

We assume that  $A$  has no limit point. Then  $A = \bar{A}$ . So  $X \setminus A$  is open. Furthermore, for each  $a \in A$ , we can choose a neighborhood  $U_a$  of  $a$  such that  $U_a \cap (A \setminus \{a\}) = \emptyset$ .

Then  $\{U_a : a \in A\} \cup \{X \setminus A\}$  is an open covering of  $X$ . Since  $X$  is compact, there exists a finite subcovering, say  $\{U_{a_1}, U_{a_2}, \dots, U_{a_n}\} \cup \{X \setminus A\}$ . Thus,

$$X = \bigcup_{i=1}^n U_{a_i} \cup (X \setminus A) = (X \setminus A) \cup \{a_1, a_2, \dots, a_n\},$$

which implies that  $A$  is finite, a contradiction. Hence,  $A$  has a limit point in  $X$ . ■

**Example 16.0.1** (Counterexample for the converse). *Let  $Y = \{p, q\}$  with anti-discrete topology  $J_Y = \{\emptyset, Y\}$ . Let  $X = \mathbb{N} \times Y$  with the product topology where  $\mathbb{N}$  has the discrete topology. Then every non-empty set  $A \subseteq X$  has a limit point. Because if  $(n, p) \in A$ , then any open set containing  $(n, q)$  intersects  $A$  at  $(n, p)$ . So  $X$  is limit point compact. However,  $X$  is not compact. Because  $\{\{n\} \times Y : n \in \mathbb{N}\}$  is an open covering of  $X$  which admits no finite subcovering.*

**Example 16.0.2** (Limit point compact but not sequentially compact). *Let  $X$  be defined as the above example. Consider the sequence  $\{(n, p)\}_{n=1}^{\infty}$ . This sequence has no convergent subsequence. Because for any point  $(m, p)$ , the open set  $\{m\} \times Y$  contains only finitely many terms of the sequence; for any point  $(m, q)$ , the open set  $\{m\} \times Y$  also contains only finitely many terms of the sequence. So  $X$  is not sequentially compact.*

**Theorem 16.0.3.** *For a first-countable topological space  $X$ , limit point compactness implies sequential compactness.*

**Proof**

Take a sequence  $\{x_n\} \subseteq X$ .

If the set of values  $\{x_n : n \in \mathbb{N}\}$  is finite, then there exists a value  $x$  that appears infinitely many times in the sequence. So the subsequence constantly equal to  $x$  converges to  $x$ .

Suppose the set of values  $\{x_n : n \in \mathbb{N}\}$  is infinite. Since  $X$  is first-countable, we can construct a countable basis  $\{U_k\}$  at a limit point  $a$  such that

$$x_{n_1} \in U_1, x_{n_2} \in U_2, \dots, x_{n_k} \in U_k, \dots \quad (16.1)$$

and

$$U_1 \supseteq U_2 \supseteq \dots \supseteq U_k \supseteq \dots \quad (16.2)$$

That is for each open set  $U \ni a$  there exists  $N$  such that  $\forall k \geq N, U_k \subseteq U$ . Then  $x_{n_k} \in U_k \subseteq U$  for all  $k \geq N$ . So  $x_{n_k} \rightarrow a$ . ■

**Remark** For topological spaces, we have the following facts:

1. Compactness  $\Rightarrow$  Limit Point Compactness
2. Sequential Compactness  $\Rightarrow$  Limit Point Compactness
3. But other implications are not true in general.

**Theorem 16.0.4.** *Sequential compactness implies compactness for metric spaces.*

**Proof** This is harder than other directions.

**Exercise** Prove that  $X$  is sequentially compact, then it is limit point compact.

## 16.1 Locally Compact

**Definition 16.1.1.**  $X$  is said to be locally compact at  $x \in X$  if there exists an neighborhood  $U$  of  $x$  and there exists a compact subspace  $C$  of  $X$  containing  $U$ .  $X$  is locally compact if it's locally compact at each  $x \in X$ .

**Example 16.1.1.**  $\mathbb{R}$  is locally compact. Because for any  $x \in \mathbb{R}$ , take  $U = (x-1, x+1)$  and  $C = [x-1, x+1]$  which is compact.

**Example 16.1.2.**  $\mathbb{R}^n$  is locally compact.

**Example 16.1.3** (non example).  $\mathbb{Q} \subseteq \mathbb{R}$  is not locally compact.

**Exercise** Show the above example.

**Example 16.1.4** (non example).  $\mathbb{R}^\omega$  with product topology is not locally compact. Let  $U = (a_1, b_1) \times (a_2, b_2) \times \dots$ . If  $U \subseteq C$  where  $C$  is compact, then  $\bar{U} = [a_1, b_1] \times [a_2, b_2] \times \dots \subseteq C$ . We know a closed subset of a compact set is compact, so  $\bar{U}$  is compact. But this is not true.

**Exercise** Show the above example.

**Theorem 16.1.1.** Let  $X$  be a topological space. Then  $X$  is Hausdorff locally compact if and only if there exists a space  $Y$  such that

1.  $X$  is a subspace of  $Y$ .
2.  $Y \setminus X$  contains exactly one point  $\{p\}$ .
3.  $Y$  is compact Hausdorff.

This  $Y$  is unique in the following sense: If  $Y, Y'$  are two spaces with the above properties and  $Y = X \cup \{p\}, Y' = X \cup \{q\}$ , then there exists a homeomorphism  $h : Y \rightarrow Y'$  such that  $h(x) = x$  for all  $x \in X$  and  $h(p) = q$ .

**Proof**

**Uniqueness** Let  $Y = X \cup \{p\}, Y' = X \cup \{q\}$  satisfies the above properties. Define  $h : Y \rightarrow Y'$  as follows:  $h(x) = x$  for all  $x \in X$  and  $h(p) = q$ . We show that  $h$  is continuous. But the function is symmetric, so it is enough to show that  $h(U)$  is open in  $Y'$  for all open set  $U$  in  $Y$ .

Take  $U$  be open. There are two cases.

If  $U \subseteq X$ , then we are done.

Suppose  $p \in U$ . Then  $C = Y \setminus U$  is closed in  $Y$ . So  $C$  is compact.

**Construction** We introduce the topology on  $Y = X \cup \{\infty\}$  as follows:

There are two types of open sets in  $Y$ :

1.  $U \subseteq X \subseteq Y$  is open in  $X$ .
2.  $U = Y \setminus C$  where  $C$  is compact.

We need to check that this is a topology.

For the intersection of two open sets of type 1, we have  $U_1 \cap U_2$  is open in  $X$  thus is open in  $Y$ .

For the intersection of type 1 and type 2, we have  $U_1 \cap (Y \setminus C) = U \cap (X \setminus C)$  which is the union of open sets in  $X$  thus is open in  $Y$ .

For the intersection of two open sets of type

It remains to show  $Y$  is compact Hausdorff and if  $X \subseteq Y$  with  $Y$  satisfied the three properties, then  $X$  is locally compact.

We can show that  $X$  is a subspace of  $Y$  because:

1. If  $U \subseteq X$  and  $\infty \notin U$ , then
2.  $(Y \setminus C) \cap X = X \setminus C$  where  $C$  is compact in  $X$ .

We show that  $Y$  is compact.

Let  $\mathcal{A}$  be an open covering of  $Y$ . Then there exists compact  $C$  such that  $Y \setminus C \in \mathcal{A}$ . The rest of the covering  $\mathcal{A}' = \mathcal{A} \setminus \{Y \setminus C\}$  is an open covering of  $C$ . Since  $C$  is compact, there exists a finite subcovering of  $\mathcal{A}'$  covering  $C$ . Thus, adding  $Y \setminus C$  gives a finite subcovering of  $\mathcal{A}$  covering  $Y$ . So  $Y$  is compact.

We show that  $Y$  is Hausdorff.

Take two distinct points  $x, y \in Y$ . There are two cases.

1. If  $x, y \in X$ , since  $X$  is Hausdorff, there exist disjoint open sets  $U, V$  in  $X$  such that  $x \in U, y \in V$ . Then  $U, V$  are open in  $Y$  and disjoint.
2. If one of them is  $\infty$ , say  $y = \infty$ . Since  $X$  is locally compact at  $x$ , there exists an open neighborhood  $U$  of  $x$  and a compact set  $C$  such that  $U \subseteq C$ . Then  $V = Y \setminus C$  is an open neighborhood of  $\infty$ . Clearly,  $U \cap V = \emptyset$ .

We prove the other direction.

To be done. ■

**Theorem 16.1.2.** *Let  $X$  be a Hausdorff space. Then  $X$  is locally compact if and only if for any  $x \in X$  and any neighborhood  $U$  of  $x$ , there exists a neighborhood  $V$  of  $x$  such that  $\bar{V}$  is compact and  $\bar{V} \subseteq U$ .*

**Proof**

$\Leftarrow$   $x \in V \subseteq \bar{V} \subseteq U$  and  $\bar{V}$  is compact. So  $X$  is locally compact at  $x$ .

$\Rightarrow$  Let  $Y$  be the one-point compactification(compact and Hausdorff) of  $X$ . Let  $U$  be a neighborhood of  $x$  in  $Y$ . Then  $Y \setminus U := C$  is a closed subset of  $Y$ . So  $C$  is compact. By lemma 12.0.2, there exists two open sets  $V, W$  in  $Y$  such that  $x \in V, C \subseteq W$  and  $V \cap W = \emptyset$ . Then  $\bar{V}$  is compact and  $\bar{V} \cap C = \emptyset$ . So  $\bar{V} \subseteq U$ . ■

**Corollary 16.1.1.** *Let  $X$  be locally compact Hausdorff space.  $A \subseteq X$  is open or closed. Then  $A$  is locally compact.*

**Proof**

Let  $A \subseteq X$  be closed. Given  $x \in A$ , since  $X$  is locally compact, there exists an open neighborhood  $U$  of  $x$  in  $X$  and a compact set  $C$  such that  $x \in U \subseteq C$ . Then  $U \cap A$  is an open neighborhood of  $x$  in  $A$  and  $C \cap A$  is compact in  $A$ . So  $A$  is locally compact at  $x$ .

Let  $A \subseteq X$  be open. Given  $x \in A$ , since  $X$  is locally compact and  $X$  is Hausdorff, by the previous theorem, there exists a neighborhood  $V$  of  $x$  such that  $\bar{V}$  is compact and  $\bar{V} \subseteq A$ . So  $A$  is locally compact at  $x$ . ■

**Corollary 16.1.2.**  *$X$  is locally compact Hausdorff if and only if  $X$  is homeomorphic to an open subspace of a compact Hausdorff space.*

**Exercise** Show the above corollary.

## 16.2 Urysohn's Metrization Theorem

**Theorem 16.2.1.** *Every  $X$  that is regular( $T_3$ ) and second-countable is metrizable.*

### 16.3 Countability

**Definition 16.3.1.**  $X$  is first countable if for every  $x \in X$ , there exists a countable basis at  $x$ . That is, there exists  $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$  a collection of open sets containing  $x$  such that for any open set  $U$  containing  $x$ , there exists  $B_n \in \mathcal{B}$  such that  $B_n \subseteq U$ .

**Definition 16.3.2.**  $X$  is second countable if there exists a countable basis  $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$  for the topology of  $X$ . That is, for every  $x$  and every open set  $U$  containing  $x$ , there exists  $B_n \in \mathcal{B}$  such that  $x \in B_n \subseteq U$ .

**Exercise**  $\mathbb{R}^n$  with standard topology,  $B_n = \{U_\epsilon(x) | x \in \mathbb{Q}^n, \epsilon \in \mathbb{Q}_{>0}\}$ .  $B_n$  is a countable basis. So  $\mathbb{R}^n$  is second countable. Show the details.

**Exercise** Show that if  $X_n$ 's are first(second) countable, then  $\prod X_n$  with product topology is first(second) countable.

**Theorem 16.3.1.** Let  $X$  be second countable. Then

1. Every open cover of  $X$  has a countable subcovering ( $X$  is Lindelöf space).
2. There is a countable subset  $A \subseteq X$  such that  $\bar{A} = X$  ( $A \subseteq X$  is dense).

**Proof**