

Chapter 13

Later

13.1 Metrizability

Recall the following theorem.

Theorem 13.1.1 (Sequence Lemma). *Let X be a topological space. Let $A \subseteq X$. If $\exists \{x_n\} \subseteq A$ such that $x_n \rightarrow x$ in X , then $x \in \bar{A}$. The converse is true if X is metrizable.*

Theorem 13.1.2 (Heine's definition of limit). *Let $f : X \rightarrow Y$. If f is continuous, then for every convergent sequence $x_n \rightarrow x$ in X , we have $f(x_n) \rightarrow f(x)$ in Y . The converse is true if X is metrizable.*

Poof

(\Rightarrow) Let f be continuous. Let $V \ni f(x)$ be open in Y . Then $f^{-1}(V)$ is open in X and contains x . Since $x_n \rightarrow x$, $\exists N$ such that $\forall n \geq N$, $x_n \in f^{-1}(V)$. Thus, $\forall n \geq N$, $f(x_n) \in V$. Hence, $f(x_n) \rightarrow f(x)$.

(\Leftarrow) Recall that f is continuous if and only if for all $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$.

Let $A \subseteq X$. Let $x \in \bar{A}$. Since X is metrizable, by the sequence lemma, there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \rightarrow x$. By the assumption, $f(x_n) \rightarrow f(x)$. Thus, by the sequence lemma again, $f(x) \in \overline{f(A)}$. So $f(\bar{A}) \subseteq \overline{f(A)}$. Hence, f is continuous. ■

Definition 13.1.1. X is first-countable if it has a countable basis at each $x \in X$. Given $x \in X$, there exists a countable collection of open sets $\{U_n\}$ such that for any open set U containing x , $\exists n$ such that $U_n \subseteq U$. (From this we can construct $\tilde{U}_n = \bigcap_{i=1}^n U_i$ such that $\{\tilde{U}_n\}$ is also a countable basis at x with $\tilde{U}_{n+1} \subseteq \tilde{U}_n$.)

Definition 13.1.2. X is second-countable if it has a countable basis for the topology. There exists countable a basis \mathcal{B} for X such that $\forall x \in X, \forall U$ open in X containing x , $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Property 13.1.1. If X is second-countable, then X is first-countable.

Example 13.1.1. \mathbb{R}^n has a countable basis. For instance, the set of all open balls with rational radii and centers at points with rational coordinates forms a countable basis for the standard topology on \mathbb{R}^n .

Example 13.1.2. \mathbb{R} with finite complement topology is not first-countable. Let U_1, U_2, \dots be a countable open sets containing x . Then $\bigcap_{n=1}^{\infty} U_n \setminus \{x\}$ is not empty, so there exists $y \neq x$ such that $y \in U_n$ for all n . Take $U = \mathbb{R} \setminus \{y\}$, which is open and contains x . However, there is no U_n such that $U_n \subseteq U$. Hence, \mathbb{R} with finite complement topology is not first-countable.

Example 13.1.3. X is uncountable with discrete topology. Then $\forall x \in X$, the set $\{x\}$ is open. So any basis of X must contain $\{x\}$ for all $x \in X$. So X is not second-countable. But X is metrizable thus first-countable.

Example 13.1.4. \mathbb{R}^2 with "Amazon River metric". Define

$$d((x, y), (x', y')) = \begin{cases} |y - y'|, & x = x' \\ |y| + |y'| + |x - x'|, & x \neq x' \end{cases} \quad (13.1)$$

Then $\{(x, y) | x = x_0, y \in (y_0 - \epsilon, y_0 + \epsilon)\}$ with $\epsilon < |y_0|$ is an open ball centered at (x_0, y_0) . There are uncountable many such disjoint open sets. So \mathbb{R}^2 with Amazon River metric is not second-countable. But it is metrizable thus first-countable.

Example 13.1.5. Let \mathbb{R}_l be the set of real numbers with the lower limit topology. Then \mathbb{R}_l is not second-countable. Suppose \mathcal{B} is a countable basis for \mathbb{R}_l . For each $x \in \mathbb{R}$, there exists a open set $[x, +\infty)$ containing x . Thus, there exists a basis element $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, +\infty)$ which means $\min B_x = x$. Since \mathcal{B} is countable, the set of minimums $\{\min B | B \in \mathcal{B}\}$ is also countable. This contradicts the uncountability of \mathbb{R} . Hence, \mathbb{R}_l is not second-countable. However, \mathbb{R}_l is first-countable since for each $x \in \mathbb{R}$, the collection of basis elements $\{[x, x + 1/n) | n \in \mathbb{N}\}$ forms a countable basis at x .

Property 13.1.2. \mathbb{R}^ω with box topology is not metrizable. (Hence \mathbb{R}^J with box topology is not metrizable for infinite J .)

Proof

We let

$$A = \{(x_1, x_2, \dots) | x_i > 0, \forall i\} \quad (13.2)$$

Then $0 \in \bar{A}$ in the box topology.

We let

$$B = (a_1, b_1) \times (a_2, b_2) \times \dots \quad (13.3)$$

such that $a_i < 0 < b_i$ for all i . Then B is a basis element in the box topology containing 0. And the intersection of B and A is not empty.

We assert that there is no sequence in A converging to 0. Let $\{a_n\}_{n=1}^\infty$ be a sequence in A , where

$$a_n = (a_n^{(1)}, a_n^{(2)}, \dots) \quad (13.4)$$

with $a_n^{(i)} > 0$ for all i, n . We can construct a basis element B' in the box topology containing 0:

$$B' = (-a_1^{(1)}, a_1^{(1)}) \times (-a_2^{(2)}, a_2^{(2)}) \times \dots \quad (13.5)$$

Then B' does not contain any a_n . Thus $\{a_n\}$ does not converge to 0.

So the sequence lemma tells us that \mathbb{R}^ω with box topology is not metrizable. ■

Property 13.1.3. Let J be uncountable. Then \mathbb{R}^J in product topology is not metrizable.

Proof

Take

$$A = \{(x_1, x_2, \dots) | x_\alpha = 1 \text{ for all but finitely many coordinates}\} \quad (13.6)$$

Then $0 \in \bar{A}$ in the product topology (Easy to verify).

Given n , let J_n denotes the subset of J consisting of those indices α such that the α -th coordinate of a_n is different from 1. Since J_n is finite for each n , then the union $\bigcup_{n=1}^\infty J_n$ is countable. However, J is uncountable, so there exists $\beta \in J$ such that $\beta \notin \bigcup_{n=1}^\infty J_n$. That is, for all n , the β -th coordinate of a_n is equal to 1.

Let $\{a_n\}_{n=1}^\infty$ be a sequence in A , where there exists $\beta \in J$ such that $a_\beta^{(n)} = 1$ for all n . Then we can take

$$U = \pi_\beta^{-1}((-1, 1)) \quad (13.7)$$

which is a basis element in the product topology containing 0. However, U does not contain any a_n . Thus $\{a_n\}$ does not converge to 0.

By the sequence lemma, \mathbb{R}^J in product topology is not metrizable. ■