Chapter 20

Later

Theorem 20.0.1 (Urysohn's Lemma(Recalling))**.** *Let X be a normal space. If A and B are disjoint closed subsets of X, then there exists a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

Definition 20.0.1. *Suppose X satisfies (T1). If a point and a closed set can be separated by a continuous function as in Urysohn's lemma, then X is called completely regular(CR). (Sometimes it is called $T_{3\frac{1}{2}}$ space.)*

Property 20.0.1. *Every normal space is completely regular.*

Proof By Urysohn's lemma.

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Remark

Normal ⇒ CR ⇒ Regular. That's why we call it $T_{3\frac{1}{2}}$ space.

Property 20.0.2. *If X is completely regular, then it is regular.*

Proof

Take $x_0 \in X$ and a closed set A such that $x_0 \notin A$. Since X is CR, there exists a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 0$ and $f(a) = 1$ for all $a \in A$. Let $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$. Then U and V are open sets because f is continuous and $[0, \frac{1}{2})$, $(\frac{1}{2}, 1]$ are open in $[0, 1]$. So $x_0 \in U$, $A \subseteq V$ and $U \cap V = \varnothing$. Thus X is regular.

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Theorem 20.0.2. *1. Subspace of CR space is CR.*

2. If X_α is CR, then $\prod X_\alpha$ with product topology is CR.

Proof

Let X be CR, and $Y \subseteq X$ with subspace topology. Take $y_0 \in Y$ and a closed set $A \subseteq Y$ such that $y_0 \notin A$. Let $\bar{A}$ be the closure of A in X so $A = \bar{A} \cap Y$. Then $\bar{A}$ is closed in X and $y_0 \notin \bar{A}$. Since X is CR, there exists a continuous function $f : X \to [0, 1]$ such that $f(y_0) = 0$ and $f(a) = 1$ for all $a \in \bar{A}$. Restrict f to Y, we get a continuous function $f|_Y : Y \to [0, 1]$ such that $f|_Y(y_0) = 0$ and $f|_Y(a) = 1$ for all $a \in A$. Thus Y is CR.

Take $A \subseteq \prod X_\alpha$ closed and $x = (x_\alpha) \notin A$. Then there exists a basic open set $U = \prod U_\alpha$ such that $x \in U$ and $U \cap A = \emptyset$, where U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many α. Let those finitely many α be $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since X_{α_i} is CR, there exists a continuous function $f_{\alpha_i} : X_{\alpha_i} \to [0, 1]$ such that $f_{\alpha_i}(x_{\alpha_i}) = 1$ and $f_{\alpha_i}(a) = 0$ for all $a \in X_{\alpha_i} \setminus U_{\alpha_i}$. So we can define a continuous function $f : \prod X_\alpha \to [0, 1]$ by

$$f(x) = \prod_{i=1}^{n} f_{\alpha_i}(\pi_{\alpha_i}(x)).$$

Then $f(x) = 1$ and for all $a \in A$, $f(a) = 0$. Thus $\prod X_\alpha$ is CR.

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Example 20.0.1. *$\mathbb{R}_l$ is normal then it is CR. So $\mathbb{R}_l^2$ is CR. But it is not normal as we have shown.*

Fact(no proof) There exist regular spaces that are not CR.
Remark Proof of Urysohn's lemma for metric spaces(exercise):

Let X be metric space with metric $d : X \times X \to \mathbb{R}$. Given two disjoint closed sets $A, B \subseteq X$, define $d_B(x) = d(x, B)$ which vanishes exactly on B. Then define $d_A(x) = d(x, A)$ which vanishes exactly on A. Now define

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}.$$

Then $f(A) = \{0\}$ and $f(B) = \{1\}$ and f is continuous because we've proved that distance from a point to a closed set is continuous.

Theorem 20.0.3 (Urysohn Metrization Theorem). *Every regular space with contable basis is metrizable.*

Idea of Proof Embede X into $[0,1]^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$. This $\mathbb{R}^{\mathbb{N}}$ can be with product topology or uniform metric topology.
Step 1 Construct a family $f_n : X \to [0,1]$ of functions such that $\forall x_0 \in X$ and $\forall$ neighborhood U of x_0, there exists n such that $f_n(x_0) > 1$ and $f_n(x) = 0$ for all $x \notin U$.

For each x_0 and neighborhood U of x_0, since X is regular, by Urysohn's lemma, there exists such functions. But we want to get countably many of them.

Given $x_0 \in U$ there exists B_m such that $x_0 \in B_m \subseteq U$. Since X is regular, there exists B_n such that $x_0 \in B_n$ and $\bar{B_n} \subseteq B_m$. By Urysohn's lemma, there exist