Chapter 17

Later

Theorem 17.0.1. *Let X be a topological space. Then X is Hausdorff locally compact if and only if there exists a space Y such that*

1. *X is a subspace of Y.*
2. *$Y \setminus X$ contains exactly one point p.*
3. *Y is compact Hausdorff.*

This Y is unique in the following sense: If Y, Y' are two spaces with the above properties and $Y = X \cup \{p\}, Y' = X \cup \{q\}$, then there exists a homeomorphism $h : Y \to Y'$ such that $h(x) = x$ for all $x \in X$ and $h(p) = q$.

Proof

Uniqueness Let $Y = X \cup \{p\}$, $Y' = X \cup \{q\}$ satisfies the above properties. Define $h : Y \to Y'$ as follows: $h(x) = x$ for all $x \in X$ and $h(p) = q$. We show that h is continuous. But the function is symmetric, so it is enough to show that $h(U)$ is open in Y' for all open set U in Y.

Take U be open. There are two cases.

If $U \subseteq X$, then we are done.

Suppose $p \in U$. Then $C = Y \setminus U$ is closed in Y. So C is compact.

Construction We introduce the topology on $Y = X \cup \{\infty\}$ as follows:

There are two types of open sets in Y:

1. $U \subseteq X \subseteq Y$ is open in X.
2. $U = Y \setminus C$ where C is compact subspace of X.

We need to check that this is a topology.

For the intersection of two open sets of type 1, we have $U_1 \cap U_2$ is open in X thus is open in Y.

For the intersection of type 1 and type 2, we have $U_1 \cap (Y \setminus C) = U_1 \cap (X \setminus C)$ which is the union of open sets in X thus is open in Y because C is closed in X(as a compact subspace of a Hausdorff space).

For the intersection of two open sets of type 2, we have $(Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2)$ which is open in Y because $C_1 \cup C_2$ is compact.

Similarly, one checks that the union of any collection of open sets is open. We have

1. $\bigcup U_\alpha = U$ is of type 1.
2. $\bigcup (Y \setminus C_\alpha) = Y \setminus \bigcap C_\alpha = Y \setminus C$ where $C = \bigcap C_\alpha$ is compact(Easy to check). This is of type 2.
3. $\bigcup U_\alpha \cup \bigcup (Y \setminus C_\beta) = U \cup (Y \setminus C) = Y \setminus (C \setminus U)$. This is of type 2 because $C \setminus U = C \cap (X \setminus U)$ is closed in C thus is compact because a closed subset of a compact set is compact.

Now we show that X is a subspace of Y. Given any open set V of Y, we show its intersection with X is open in X. If V is of type 1, then $V \cap X = V$ which is open in X. If V is of type 2, then $V \cap X = (Y \setminus C) \cap X = X \setminus C$ which is open in X because C is closed in X as a compact subspace of a Hausdorff space. Conversely, given any open set U of X, U is an open set of Y of type 1. So X is a subspace of Y.

We show that Y is compact.

Let $\mathcal{A}$ be an open covering of Y. The collection $\mathcal{A}$ must contain an open set containing ∞. So there exists a compact set C such that $U = Y \setminus C$ with $U \in \mathcal{A}$. Take all the members of $\mathcal{A}$ different from U and intersect them with X. This gives a collection of open sets in X which covers C. Since C is compact, there exists a finite subcovering of C, say $\{V_1, V_2, \cdots, V_n\}$. Then $\{U, V_1, V_2, \cdots, V_n\}$ is a finite subcovering of Y. So Y is compact.

We show that Y is Hausdorff.

Take two distinct points $x, y \in Y$. There are two cases.

1. If $x, y \in X$, since X is Hausdorff, there exist disjoint open sets U, V in X such that $x \in U, y \in V$. Then U, V are open in Y and disjoint.
2. If one of them is ∞, say $y = \infty$. Since X is locally compact at x, there exists an open neighborhood U of x and a compact set C such that $U \subseteq C$. So C is closed in X thus in Y. Then $V = Y \setminus C$ is an open neighborhood of ∞. Clearly, $U \cap V = \emptyset$.

Finally we prove the other direction. Suppose a space Y satisfying conditions (1)-(3) exists. Then X is Hausdorff as a subspace of a Hausdorff space. Given any $x \in X$, we show X is locally compact at x. Choose disjoint open sets U, V in Y such that $x \in U$ and $p \in V$. Then $C = Y \setminus V$ is compact as a closed subset of a compact space. Also, $x \in U \subseteq C$. So X is locally compact at x.

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Definition 17.0.1. *If Y and X are as in the above theorem, then Y is called the one-point compactification of X.*

Example 17.0.1. *1. The one point compactification of $\mathbb{R}$ is homeomorphic to S^1. We denote $\overline{\mathbb{R}} \cong S^1$.*

2. The one point compactification of $\mathbb{R}^n$ is homeomorphic to S^n. We denote $\overline{\mathbb{R}^n} \cong S^n$.

3. The one point compactification of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (Riemann sphere) is homeomorphic to S^2.

Theorem 17.0.2. *Let X be a Hausdorff space. Then X is locally compact if and only if for any $x \in X$ and any neighborhood U of x, there exists a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subseteq U$.*

Proof

$\Leftarrow$ $x \in V \subseteq \bar{V} \subseteq U$ and $\bar{V}$ is compact. So X is locally compact at x.

$\Rightarrow$ Let Y be the one-point compactification(compact and Hausdorff) of X. Let U be a neighborhood of x in Y. Then $Y \setminus U := C$ is a closed subset of Y. So C is compact subspace of Y(Hausdorff). By lemma 12.0.2, there exists two open sets V, W in Y such that $x \in V$, $C \subseteq W$ and $V \cap W = \emptyset$. Then $\bar{V}$ is compact and $\bar{V} \cap C = \emptyset$. So $\bar{V} \subseteq U$. ■

Corollary 17.0.1. *Let X be locally compact Hausdorff space. $A \subseteq X$ is open or closed. Then A is locally compact.*

Proof

Let $A \subseteq X$ be closed. Given $x \in A$, since X is locally compact, there exists an open neighborhood U of x in X and a compact set C such that $x \in U \subseteq C$. Then $C \cap A$ is closed in C thus is compact, and it contains the neighborhood $U \cap A$ of x in A. So A is locally compact at x.

Let $A \subseteq X$ be open. Given $x \in A$, since X is locally compact and X is Hausdorff, by the previous theorem, there exists a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subseteq A$. Then $C = \bar{V}$ is a compact set containing the neighborhood V of x in A. So A is locally compact at x.

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Corollary 17.0.2. *X is locally compact Hausdorff if and only if X is homeomorphic to an open subspace of a compact Hausdorff space.*

Exercise Show the above corollary by **??** and **??**.

Solution

$\Rightarrow$ Let Y be the one-point compactification of X. Recall that Hausdorff is T_1. So $X = Y \setminus \{p\}$ is open in Y. The identity map $id : X \to X$ is a homeomorphism from X to the open subspace X of Y.

$\Leftarrow$ Let X be homeomorphic to an open subspace U of a compact Hausdorff space Z. Since Z is compact Hausdorff, Z is locally compact Hausdorff. By **??**, U is locally compact Hausdorff. Since X is homeomorphic to U, X is locally compact Hausdorff.

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17.1 Urysohn's Metrization Theorem

Theorem 17.1.1. *Every X that is regular(T3) and second-countable is metrizable.*