Chapter 16

Later

Definition 16.0.1 (def1)**.** *X is compact if any open covering admits a finite subcovering.*

Definition 16.0.2 (def2: Frechet compactness)**.** *X is limit point compact if every infinite set $A \subseteq X$ has a limit point in X.*

Definition 16.0.3 (def3: sequential compactness/Bozano-Weierstrass compactness)**.** *X is sequentially compact if every sequence $\{x_n\} \subseteq X$ has a convergent subsequence converging to a point in X.*

Theorem 16.0.1. *If X is a metric space, then X is compact if and only if it is limit point compact if and only if it is sequentially compact.*

Theorem 16.0.2. *For an arbitrary topological space, compactness implies limit point compactness. The converse is not true in general.*

Proof

Let X be compact, and let $A \subseteq X$ be infinite.

We assume that A has no limit point. Then $A = \bar{A}$. So $X \setminus A$ is open.

Let $a \in A$ such that $U_a = X \setminus (A \setminus \{a\})$ is open and contains a. Then $\{U_a : a \in A\} \cup \{X \setminus A\}$ is an open covering of X. Since X is compact, there exists a finite subcovering, say $\{U_{a_1}, U_{a_2}, \cdots, U_{a_n}\} \cup \{X \setminus A\}$. Thus,

$$X = \bigcup_{i=1}^{n} U_{a_i} \cup (X \setminus A) = X \setminus (A \setminus \{a_1, a_2, \cdots, a_n\}),$$

which implies that A is finite, a contradiction. Hence, A has a limit point in X. ■

Example 16.0.1 (Counterexample for the converse)**.** *Let $Y = \{p, q\}$ with anti-dicrete topology $J_Y = \{\emptyset, Y\}$. Let $X = \mathbb{N} \times Y$ with the product topology where $\mathbb{N}$ has the discrete topology. Then every non-empty set $A \subseteq X$ has a limit point. Because if $(n, p) \in A$, then any open set containing (n, q) intersects A at (n, p). So X is limit point compact. However, X is not compact. Because $\{\{n\} \times Y : n \in \mathbb{N}\}$ is an open covering of X which admits no finite subcovering.*

Example 16.0.2 (Limit point compact but not sequentially compact)**.** *Let $X = \mathbb{N} \times \{p\}$ has no convergent subsequence. So X is not sequentially compact.*

Theorem 16.0.3. *For a first-countable topological space X, limit point compactness implies sequential compactness.*

Proof

Take a sequence $\{x_n\} \subseteq X$.

If the set of values $\{x_n : n \in \mathbb{N}\}$ is finite, then there exists a value x that appears infinitely many times in the sequence. So the subsequence constantly equal to x converges to x.

Suppose the set of values $\{x_n : n \in \mathbb{N}\}$ is infinite. Since X is first-countable, we can construct a countable basis $\{U_k\}$ at a limit point a such that

$$x_{n_1} \in U_1, x_{n_2} \in U_2, \cdots, x_{n_k} \in U_k, \cdots \tag{16.1}$$

and

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k \supseteq \cdots \tag{16.2}$$

That is for each open set $U \ni a$ there exists N such that $\forall k \geq N$, $U_k \subseteq U$. Then $x_{n_k} \in U_k \subseteq U$ for all $k \geq N$. So $x_{n_k} \to a$. ■

Remark For topological spaces, we have the following facts:

1. Compactness $\Rightarrow$ Limit Point Compactness
2. Sequential Compactness $\Rightarrow$ Limit Point Compactness
3. But other implications are not true in general.

Theorem 16.0.4. *Sequential compactness implies compactness for metric spaces.*

Proof This is harder than other directions.
Exercise Prove that X is sequentially compact, then it is limit point compact.

16.1 Locally Compact

Definition 16.1.1. *X is said to be locally compact at $x \in X$ if there exists an neighborhood U of x and there exists a compact subspace C of X containing U. X is locally compact if it's locally compact at each $x \in X$.*

Example 16.1.1. *$\mathbb{R}$ is locally compact. Because for any $x \in \mathbb{R}$, take $U = (x-1, x+1)$ and $C = [x-1, x+1]$ which is compact.*

Example 16.1.2. *$\mathbb{R}^n$ is locally compact.*

Example 16.1.3 (non example)**.** *$\mathbb{Q} \subseteq \mathbb{R}$ is not locally compact.*

Exercise Show the above example.

Example 16.1.4 (non example)**.** *$\mathbb{R}^\omega$ with product topology is not locally compact. Let $U = (a_1, b_1) \times (a_2, b_2) \times \cdots$. If $U \subseteq C$ where C is compact, then $\overline{U} = [a_1, b_1] \times [a_2, b_2] \times \cdots \subseteq C$. We know a closed subset of a compact set is compact, so $\overline{U}$ is compact. But this is not true.*

Exercise Show the above example.

Theorem 16.1.1. *Let X be a topological space. Then X is Hausdorff locally compact if and only if there exists a spcace Y such that*

1. *X is a subspace of Y.*
2. *$Y \setminus X$ contains exactly one point $\{p\}$.*
3. *Y is compact Hausdorff.*

This Y is unique in the following sense: If Y, Y' are two spaces with the above properties and $Y = X \cup \{p\}, Y' = X \cup \{q\}$, then there exists a homeomorphism $h : Y \to Y'$ such that $h(x) = x$ for all $x \in X$ and $h(p) = q$.

Proof
Uniqueness Let $Y = X \cup \{p\}$, $Y' = X \cup \{q\}$ satisfies the above properties. Define $h : Y \to Y'$ as follows: $h(x) = x$ for all $x \in X$ and $h(p) = q$. We show that h is continuous. But the function is symmetric, so it is enough to show that $h(U)$ is open in Y' for all open set U in Y.
Take U be open. There are two cases.
If $U \subseteq X$, then we are done.
Suppose $p \in U$. Then $C = Y \setminus U$ is closed in Y. So C is compact.
Construction We introduce the topology on $Y = X \cup \{\infty\}$ as follows:
There are two types of open sets in Y:

1. $U \subseteq X \subseteq Y$ is open in X.
2. $U = Y \setminus C$ where C is compact.

We need to check that this is a topology.
For the intersection of two open sets of type 1, we have $U_1 \cap U_2$ is open in X thus is open in Y.
For the intersection of type 1 and type 2, we have $U_1 \cap (Y \setminus C) = U \cap (X \setminus C)$ which is the union of open sets in X thus is open in Y.
For the intersection of two open sets of type
It remains to showw Y is compact Hausdorff and if $X \subseteq Y$ with Y satisfied the three properties, then X is locally compact.
We can show that X is a subspace of Y because:

1. If $U \subseteq X$ and $\infty \notin U$, then

2. $(Y \setminus C) \cap X = X \setminus C$ where C is compact in X.

We show that Y is compact.

Let $\mathcal{A}$ be an open covering of Y. Then there exists compact C such that $Y \setminus C \in \mathcal{A}$. The rest of the covering $\mathcal{A}' = \mathcal{A} \setminus \{Y \setminus C\}$ is an open covering of C. Since C is compact, there exists a finite subcovering of $\mathcal{A}'$ covering C. Thus, adding $Y \setminus C$ gives a finite subcovering of $\mathcal{A}$ covering Y. So Y is compact.

We show that Y is Hausdorff.

Take two distinct points $x, y \in Y$. There are two cases.

1. If $x, y \in X$, since X is Hausdorff, there exist disjoint open sets U, V in X such that $x \in U, y \in V$. Then U, V are open in Y and disjoint.

2. If one of them is ∞, say $y = \infty$. Since X is locally compact at x, there exists an open neighborhood U of x and a compact set C such that $U \subseteq C$. Then $V = Y \setminus C$ is an open neighborhood of ∞. Clearly, $U \cap V = \emptyset$.

We prove the other direction.

To be done.

■

Theorem 16.1.2. *Let X be a Hausdorff space. Then X is locally compact if and only if for any $x \in X$ and any neighborhood U of x, there exists a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subseteq U$.*

Proof

$\Leftarrow$ $x \in V \subseteq \bar{V} \subseteq U$ and $\bar{V}$ is compact. So X is locally compact at x.

$\Rightarrow$ Let Y be the one-point compactification(compact and Hausdorff) of X. Let U be a neighborhood of x in Y. Then $Y \setminus U := C$ is a closed subset of Y. So C is compact. By lemma 12.0.2, there exists two open sets V, W in Y such that $x \in V$, $C \subseteq W$ and $V \cap W = \emptyset$. Then $\bar{V}$ is compact and $\bar{V} \cap C = \emptyset$. So $\bar{V} \subseteq U$. ■

Corollary 16.1.1. *Let X be locally compact Hausdorff space. $A \subseteq X$ is open or closed. Then A is locally compact.*

Proof

Let $A \subseteq X$ be closed. Given $x \in A$, since X is locally compact, there exists an open neighborhood U of x in X and a compact set C such that $x \in U \subseteq C$. Then $U \cap A$ is an open neighborhood of x in A and $C \cap A$ is compact in A. So A is locally compact at x.

Let $A \subseteq X$ be open. Given $x \in A$, since X is locally compact and X is Hausdorff, by the previous theorem, there exists a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subseteq A$. So A is locally compact at x. ■

Corollary 16.1.2. *X is locally compact Hausdorff if and only if X is homeomorphic to an open subspace of a compact Hausdorff space.*

Exercise Show the above corollary.

16.2 Urysohn's Metrization Theorem

Theorem 16.2.1. *Every X that is regular(T3) and second-countable is metrizable.*

16.3 Countability

Definition 16.3.1. *X is first countable if for every $x \in X$, there exists a countable basis at x. That is, there exists $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ a collection of open sets containing x such that for any open set U containing x, there exists $B_n \in \mathcal{B}$ such that $B_n \subseteq U$.*

Definition 16.3.2. *X is second countable if there exists a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ for the topology of X. That is, for every x and every open set U containing x, there exists $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq U$.*

Exercise $\mathbb{R}^n$ with standard topology, $B_n = \{U_\epsilon(x) | x \in \mathbb{Q}^n, \epsilon \in \mathbb{Q}_{>0}\}$. B_n is a countable basis. So $\mathbb{R}^n$ is second countable. Show the details.

Exercise Show that if X_n's are first(second) countable, then $\prod X_n$ with product topology is first(second) countable.

Theorem 16.3.1. *Let X be second countable. Then*

1. *Every open cover of X has a countable subcovering(X is Lindelöf space).*

2. *There is a countable subset $A \subseteq X$ such that $\bar{A} = X$($A \subseteq X$ is dense).*

Proof