

## Chapter 2

# Later

**Definition 2.0.1.** For  $p > 1, q > 1$ .  $p$  is conjugate to  $q$  if  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Remark** When  $p = 1, q = \infty$ ; when  $p = \infty, q = 1$ . And 2 is conjugate to itself.

**Lemma 2.0.1** (Jensen's inequality). If  $f(t)$  is a strictly concave function on an interval  $I \subseteq \mathbb{R}$ , then for any  $t_1, t_2 \in I$  and any  $\lambda_1, \lambda_2 > 0$  with  $\lambda_1 + \lambda_2 = 1$ , we have

$$f(\lambda_1 t_1 + \lambda_2 t_2) \geq \lambda_1 f(t_1) + \lambda_2 f(t_2) \quad (2.1)$$

where the equality holds if and only if  $t_1 = t_2$ .

**Proof** Obvious.

**Theorem 2.0.1** (Young's inequality). For  $a, b \geq 0$  and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad (2.2)$$

**Proof**

Let  $x = a^p, y = b^q$ . Then  $a = x^{\frac{1}{p}}, b = y^{\frac{1}{q}}$ . Substituting these into the inequality, we need to show that

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}, \forall x, y \geq 0 \quad (2.3)$$

If  $x = 0$  or  $y = 0$ , the inequality holds trivially. Now we assume  $x, y > 0$ . Let  $f(t) = \ln t$ , which is strictly concave on  $(0, +\infty)$ . By Jensen's inequality, we have

$$f\left(\frac{x}{p} + \frac{y}{q}\right) \geq \frac{1}{p} f(x) + \frac{1}{q} f(y) \quad (2.4)$$

Exponentiating both sides, we have

$$\frac{x}{p} + \frac{y}{q} \geq x^{\frac{1}{p}} y^{\frac{1}{q}} \quad (2.5)$$

■

**Theorem 2.0.2** (Hölder's inequality). For  $p, q \geq 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , for any  $a = (a_i) \in l_p, b = (b_i) \in l_q$  respectively, we have

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \sqrt[p]{\sum_{k=1}^{\infty} a_k^p} \sqrt[q]{\sum_{k=1}^{\infty} b_k^q}$$

**Proof**

In case when  $\|a\|_p = 0$  or  $\|b\|_q = 0$ , the inequality holds trivially. Now we assume  $\|a\|_p > 0$  and  $\|b\|_q > 0$ . Let

$$x_k = \frac{|a_k|}{\|a\|_p}, \quad y_k = \frac{|b_k|}{\|b\|_q}, \quad \forall k \geq 1 \quad (2.6)$$

We have

$$\sum_{k=1}^{\infty} x_k^p = \frac{|a_k|^p}{\|a\|_p^p} = 1, \quad \sum_{k=1}^{\infty} y_k^q = \frac{|b_k|^q}{\|b\|_q^q} = 1 \quad (2.7)$$

By Young's inequality, we have

$$x_k y_k \leq \frac{x_k^p}{p} + \frac{y_k^q}{q}, \quad \forall k \geq 1 \quad (2.8)$$

Summing over  $k$  from 1 to  $\infty$ , we have

$$\sum_{k=1}^{\infty} x_k y_k \leq \frac{1}{p} \sum_{k=1}^{\infty} x_k^p + \frac{1}{q} \sum_{k=1}^{\infty} y_k^q = \frac{1}{p} + \frac{1}{q} = 1 \quad (2.9)$$

Thus, we have

$$\sum_{k=1}^{\infty} |a_k b_k| = \|a\|_p \|b\|_q \sum_{k=1}^{\infty} x_k y_k \leq \|a\|_p \|b\|_q \quad (2.10)$$

For the case when  $p = 1$  and  $q = \infty$ , we have

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \sum_{k=1}^{\infty} |a_k| \|b\|_{\infty} = \|a\|_1 \|b\|_{\infty} \quad (2.11)$$

■

**Theorem 2.0.3** (Minkowski inequality). *For  $p \geq 1$ , for any  $x, y \in l_p$ , we have*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

**Proof**

If  $\sum_{k=1}^{\infty} |x_k + y_k|^p = 0$ , then  $x_k + y_k = 0$  for all  $k$ . Thus,  $\|x\|_p = \|y\|_p = 0$  and the inequality holds trivially. Now we assume  $\sum_{k=1}^{\infty} |x_k + y_k|^p > 0$ .

Let us first show that for  $x = (x_i), y = (y_i) \in l_p$ , we have

$$\sqrt[p]{|x_k + y_k|^p} \leq \sqrt[p]{|x_k|^p} + \sqrt[p]{|y_k|^p}$$

For every summand, we have

$$\begin{aligned} |x_k + y_k|^p &= |x_k + y_k| \cdot |x_k + y_k|^{p-1} \\ &\leq |x_k| \cdot |x_k + y_k|^{p-1} + |y_k| \cdot |x_k + y_k|^{p-1} \end{aligned}$$

So we have

$$\sum_{k=1}^n |x_k + y_k|^p \leq \sum_{k=1}^n |x_k| \cdot |x_k + y_k|^{p-1} + \sum_{k=1}^n |y_k| \cdot |x_k + y_k|^{p-1} \quad (2.12)$$

Let  $a_k = |x_k|, b_k = |x_k + y_k|^{p-1}$ , then  $b_k^q = |x_k + y_k|^p$ . By Hölder's inequality, we have

$$\sum_{k=1}^n |x_k| \cdot |x_k + y_k|^{p-1} \leq \sqrt[p]{\sum_{k=1}^n |x_k|^p} \sqrt[q]{\sum_{k=1}^n |x_k + y_k|^p}$$

On the other hand, let  $a_k = |y_k|, b_k = |x_k + y_k|^{p-1}$ , we have

$$\sum_{k=1}^n |y_k| \cdot |x_k + y_k|^{p-1} \leq \sqrt[p]{\sum_{k=1}^n |y_k|^p} \sqrt[q]{\sum_{k=1}^n |x_k + y_k|^p}$$

Combining these two inequalities, we have

$$\sum_{k=1}^n |x_k + y_k|^p \leq \left( \sqrt[p]{\sum_{k=1}^n |x_k|^p} + \sqrt[p]{\sum_{k=1}^n |y_k|^p} \right) \sqrt[q]{\sum_{k=1}^n |x_k + y_k|^p} \quad (2.13)$$

Let  $S_n = \sum_{k=1}^n |x_k + y_k|^p$ , then  $S_n \leq C S_n^{\frac{1}{q}}$ . If  $S_n = 0$ , then  $S_n \leq C^p$ . If  $S_n > 0$ , then  $S_n^{1-\frac{1}{q}} = S_n^{\frac{1}{p}} \leq C$  which implies  $S_n \leq C^p$ . So  $S_n$  is increasing and bounded above. Thus,  $\lim_{n \rightarrow \infty} S_n = \sum_{k=1}^{\infty} |x_k + y_k|^p$  exists and is finite.

Dividing both sides by  $\sqrt[n]{\sum_{k=1}^n |x_k + y_k|^p}$ , we have

$$\sqrt[n]{\sum_{k=1}^n |x_k + y_k|^p} \leq \sqrt[n]{\sum_{k=1}^n |x_k|^p} + \sqrt[n]{\sum_{k=1}^n |y_k|^p} \quad (2.14)$$

It's easy to show that each term on the right-hand side converges as  $n \rightarrow \infty$  because  $x, y \in l_p$  and we know that a monotonically increasing sequence bounded above converges. So we have that the both sides converge as  $n \rightarrow \infty$ .

Taking limit  $n \rightarrow \infty$ , by continuity of  $n$ -th root, we have the desired result. ■

Let  $X$  be a metric space. Let  $x \in X$ . We have the following definitions.

**Definition 2.0.2.** We define a neighborhood of  $x$  to be a set of the form

$$U_\epsilon(x) = \{y \in X | d(x, y) < \epsilon\}$$

for some  $\epsilon > 0$ .

**Definition 2.0.3.** We define a punctured neighborhood of  $x$  to be a set of the form

$$U_\epsilon^*(x) = \{y \in X | 0 < d(x, y) < \epsilon\} = U_\epsilon(x) \setminus \{x\}$$

for some  $\epsilon > 0$ .

**Definition 2.0.4.** We say that  $M \subseteq X$  is open in  $X$  if for every  $x \in M$ , there exists  $\epsilon > 0$  such that  $U_\epsilon(x) \subseteq M$ .

**Remark**  $\emptyset, X$  are open in  $X$  by definition.

**Example 2.0.1.** Is it possible that in a metric space  $X$ , a ball is contained properly inside a ball with smaller radius? That is, is there  $x \in X$  and  $0 < r < s$  such that  $U_s(x) \subsetneq U_r(x)$ ? [Hint: If  $Y \subseteq X$  and  $(X, d)$  is a metric space, then  $(Y, d)$  is also a metric space.]

**Solution** Yes. Let  $X = (-1, 1)$  with the usual metric. Then  $U_{\frac{2}{3}}(\frac{4}{3}) = (-\frac{4}{3}, 1) \subsetneq U_1(0) = (-1, 1)$ .

**Example 2.0.2.** Draw balls centered at 0 in  $\mathbb{R}^2$  with norms  $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ .

**Example 2.0.3.** "Amazon Metric" on  $\mathbb{R}^2$  is given by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2 \end{cases}$$

We will solve these problems later in the course.

**Theorem 2.0.4.** 1. The intersection of finitely many open sets is open, that is,  $U_1 \cap U_2 \cap \cdots \cap U_n$  is open where each  $U_i$  is open in  $X$ .

2. The union of any collection of open sets is open, that is, if  $\{U_i\}_{i \in I}$  is a collection of open sets in  $X$ , then  $\bigcup_{i \in I} U_i$  is open.

**Proof to 1.**

If  $V = U_1 \cap U_2 \cap \cdots \cap U_n = \emptyset$ , then  $V$  is open by definition.

If  $V \neq \emptyset$ , let  $x \in V$ . Since  $x \in U_i$  for each  $i = 1, 2, \dots, n$ , there exists  $\epsilon_i > 0$  such that  $U_{\epsilon_i}(x) \subseteq U_i$ . Let  $\epsilon = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ . Then  $U_\epsilon(x) \subseteq U_i$  for each  $i$ , so  $U_\epsilon(x) \subseteq V$ . Thus,  $V$  is open.

**Proof to 2.**

Let  $x \in U = \bigcup_{i \in I} U_i$ . Then there exists some  $j \in I$  such that  $x \in U_j$ . Since  $U_j$  is open, there exists  $\epsilon > 0$  such that  $U_\epsilon(x) \subseteq U_j \subseteq U$ . Thus,  $U$  is open.