Chapter 14

Lecture14

Theorem 14.0.1. $\mathbb{R}^\omega$ *is metrizable in the product topology.*

Proof Define

$$\bar{d}(x_i, y_i) = \min(|x_i - y_i|, 1) \tag{14.1}$$

Let $x, y \in \mathbb{R}^\omega$. Then we define

$$D(x, y) = \sup\{\frac{\bar{d}(x_i, y_i)}{i} | i \in \mathbb{Z}_{>0}\} \tag{14.2}$$

Then we can show that D is a metric on $\mathbb{R}^\omega$ because we have

$$\bar{d}(x_i, z_i) \leq \bar{d}(x_i, y_i) + \bar{d}(y_i, z_i) \Longrightarrow D(x, z) \leq D(x, y) + D(y, z) \tag{14.3}$$

Why does it define the product topology?

($\Rightarrow$) Let U be open in the metric topology. Then there exists $\epsilon > 0$ such that $B_D(x, \epsilon) \subseteq U$. Let N be such that $\frac{1}{N} < \epsilon$. Then let

$$V = \prod_{i=1}^{N}(x_i - \epsilon, x_i + \epsilon) \times \prod_{i=N+1}^{\infty} \mathbb{R} \tag{14.4}$$

Then let $y \in V$. We have

1. If $i \leq N$, then $\bar{d}(x_i, y_i) < \epsilon$.
2. If $i > N$, then $\bar{d}(x_i, y_i) \leq 1$.

Thus, we have $D(x, y) = \sup\{\frac{\bar{d}(x_i, y_i)}{i}\} \leq \max\{\epsilon, \frac{1}{N}\} = \epsilon$. So $y \in B_D(x, \epsilon) \subseteq U$. Hence, $V \subseteq U$. So U is open in the product topology.

($\Leftarrow$) Let U be open in the product topology. Then

$$U = \prod U_i \tag{14.5}$$

where U_i is open if $i \in \{\alpha_1, \cdots, \alpha_n\}$ and $U_i = \mathbb{R}$ otherwise. We want V open in the metric topology such that $V \subseteq U$. Let

$$x \in U \tag{14.6}$$

Then

$$x_i \in U_i \tag{14.7}$$

for $i \in \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Then there exists $\epsilon_i > 0$ such that

$$(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i \tag{14.8}$$

Let

$$\epsilon = \min(\{\frac{\epsilon_i}{i} | i \in \{\alpha_1, \alpha_2, \cdots, \alpha_n\}\} \cup \{1\}) \tag{14.9}$$

Then we claim that

$$U_D(x, \epsilon) \subseteq U \tag{14.10}$$

Let $y \in U_D(x, \epsilon)$. Then we have

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \epsilon \tag{14.11}$$

If $i \in \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, then

$$\frac{\bar{d}(x_i, y_i)}{i} < \epsilon \leq \frac{\epsilon_i}{i} \tag{14.12}$$

which implies that

$$y_i \in (x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i \tag{14.13}$$

So $y \in U$. Hence, $U_D(x, \epsilon) \subseteq U$. So U is open in the metric topology.

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Definition 14.0.1. $(X, \leq)$ *has a least upper bound property or supremum property if every non-empty subset of X that is bounded above has a least upper bound in X. (For example, $\mathbb{R}$ has the least upper bound property)*

Let X be a set with order topology and supremum property.

Theorem 14.0.2. $\forall a \leq b \in X$*, we have* $[a, b]$ *is compact.*

Proof Let $\mathcal{A}$ be an open covering of $[a, b]$. We need that there exists a finite subcovering of $\mathcal{A}$ covering $[a, b]$.
Step 1 Let $x \in [a, b]$ with $x \neq b$, there exists $y > x$ such that at most two elements of $\mathcal{A}$ can cover $[x, y]$.

1. If x has an immediate successor $s(x)$, then $[x, s(x)] = \{x, s(x)\}$ is covered by two elements from $\mathcal{A}$.
2. If not. Take $x \in A$ with $A \in \mathcal{A}$ such that $\exists c \in X, x < c \leq b, [x, c) \subseteq A$. Take any $y \in (x, c)$. Then $[x, y] \subseteq A$ is covered by one element from $\mathcal{A}$.

Step 2 Let C be the set of points y with $a \leq y \leq b$ such that $[a, y]$ has a finite subcovering from $\mathcal{A}$. Since $a \in C$, C is non-empty. Let $c = \sup C$.
Step 3 Show that $c \in C$.

We know from the first step that $c \neq a$.

Choose $A \in \mathcal{A}$ such that $c \in A$. Then $\exists d \in [a, b]$ such that $(d, c] \subseteq A$. If $c \notin C$, then there must be some point $z \in (d, c)$ with $z \in C$. Then $[a, z]$ can be covered by finite elements(say M) from $\mathcal{A}$. Thus, $[a, c] \subseteq [a, z] \cup (d, c]$ can be covered by at most $M + 1$ elements from $\mathcal{A}$. So $c \in C$ which contradicts the assumption that $c \notin C$. Hence, $c \in C$.
Step 4 Show that $c = b$. If not, by Step 1, $\exists e > c$ such that $[c, e]$ can be covered by two elements from $\mathcal{A}$. Since $[a, c]$ can be covered by finite elements(say M) from $\mathcal{A}$, $[a, e]$ can be covered by at most $M + 2$ elements from $\mathcal{A}$. So $e \in C$ which contradicts the assumption that $c = \sup C$. Hence, $c = b$.

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Corollary 14.0.1. $[a, b] \subseteq \mathbb{R}$ *is compact.*

Corollary 14.0.2. $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ *is compact.*

Theorem 14.0.3. $A \subseteq \mathbb{R}^n$ *is compact if and only if A is closed and bounded.*

Proof

($\Rightarrow$) Take a covering by

$$\bigcup U_N(0) \tag{14.14}$$

which is the union of open balls centered at 0 with radius N for $N \in \mathbb{Z}_{>0}$. Since A is compact, there exists a finite subcovering. Thus, $A \subseteq U_{N_0}(0)$ for $N_0 = \max\{N_1, N_2, \cdots, N_k\}$. So A is bounded.

We know that $\mathbb{R}^n$ is a Hausdorff space. So A is closed as a compact subset of a Hausdorff space.

($\Leftarrow$) Since A is bounded, there exists n such that

$$A \subseteq [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \tag{14.15}$$

We know that the closed subset of a compact set is compact. So A is compact.

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Theorem 14.0.4 (Extreme value theorem)**.** *Let $f : X \to Y$ be continuous where X is compact and Y is an ordered set with the order topology. Then $\exists a, b \in X$ such that $\forall x \in X$, $f(a) \leq f(x) \leq f(b)$.*

Proof

We know that $A = f(X)$ is compact. We want to show that A has a largest element.

We assume the contrary that A has no largest element. Then $\forall a \in A$, $\exists b \in A$ such that $b > a$. So

$$A = \bigcup_{a \in A} (-\infty, a) \tag{14.16}$$

which is an open covering of A. Since A is compact, there exists a finite subcovering. Let the largest element among the finite elements be a_0. Then a_0 is not covered. This is a contradiction. So A has a largest element. Similarly, A has a smallest element.

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Definition 14.0.2. *Let (X, d) be a metric space, $A \subseteq X$, $x \in X$. Let $d(x, A) = \inf\{d(x, a) | a \in A\}$.*

Property 14.0.1. *$d(x, A)$ is a continuous function for A fixed.*

Proof

We know
$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a) \tag{14.17}$$
for any $a \in A$. Then
$$d(x, A) - d(x, y) \leq \inf\{d(y, a) | a \in A\} = d(y, A) \tag{14.18}$$
So
$$d(x, A) - d(y, A) \leq d(x, y) \tag{14.19}$$

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