Chapter 22

Lecture 22

Definition 22.0.1. *A compactification of a Hausdorff space X is a compact Hausdorff space $Y \supseteq X$ such that $Y = \overline{X}$. We say that compactifications Y_1 and Y_2 are equivalent if there exists a homeomorphism $h : Y_1 \to Y_2$ such that $h|_X = id_X$.*

Example 22.0.1. $X = (0, 1)$, $Y = [0, 1] \subseteq \mathbb{R}$.

Theorem 22.0.1 (Imbedding theorem)**.** *Let X be a T_1 space. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous functions $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each point x_0 of X and each open set U containing x_0, there exists an index α such that $f_\alpha(x_0)$ is positive and f_α vanishes on $X \setminus U$. Then the function $F : X \to \mathbb{R}^J$ defined by*

$$F(x) = (f_\alpha(x))_{\alpha \in J} \tag{22.1}$$

is an imbedding of X in $\mathbb{R}^J$. If $f_\alpha : X \to [0, 1]$ for each $\alpha \in J$, then F is an imbedding of X in the cube $[0, 1]^J$.

Proof The proof is similar to the proof of Urysohn metrization theorem.

Here T_1 is required to guarantee that singletons are closed, so that if $x \neq y$, there exists f_α such that $f_\alpha(x) \neq f_\alpha(y)$.

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Theorem 22.0.2. *A space is completely regular if and only if it is homeomorphic to a subspace of a cube $[0, 1]^J$ for some index set J.*

Proof

($\Rightarrow$) Let X be completely regular. For each $x_0 \in X$ and each open set U containing x_0, there exists a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and f vanishes on $X \setminus U$. Let $\{f_\alpha\}_{\alpha \in J}$ be the collection of all such continuous functions. Then by the previous theorem, the function $F : X \to [0, 1]^J$ defined by $F(x) = (f_\alpha(x))_{\alpha \in J}$ is an embedding. Thus X is homeomorphic to the subspace $F(X)$ of the cube $[0, 1]^J$.

($\Leftarrow$) Recall that every metrizable space is normal thus is completely regular. And recall that product of completely regular spaces is completely regular. Since $[0, 1]$ is metrizable, it is completely regular. Thus the cube $[0, 1]^J$ is completely regular.

Since X is homeomorphic to a subspace of $[0, 1]^J$, X is completely regular as well because subspace of completely regular space is completely regular.

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Lemma 22.0.1. *Let X be a Hausdorff space. Let $h : X \to Z$ be an imbedding of X in the compact Hausdorff space Z. Then there exists a corresponding compactification Y of X such that there is an imbedding $H : Y \to Z$ with $H|_X = h$. Such a compactification Y is uniquely determined up to equivalence.*

Proof To be done(P255 of the pdf).

Example 22.0.2. $X = (0, 1)$, $Y = S^1 \subseteq \mathbb{R}^2$, $X \ni x \mapsto (\cos 2\pi x, \sin 2\pi x) \in Y$ *is a compactification of X. (Not the Usual sense of Compactification, see the above lemma.)*

Property 22.0.1. *Let $X \subseteq Y$ and Y be a compact Hausdorff space. Then X is completely regular.*

Proof

Since Y is compact Hausdorff, Y is normal. Then Y is completely regular. Thus X is completely regular as well.

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Claim If X is completely regular, it has a compactification.

Proof Since X is completely regular, by Theorem **??**, there exists an embedding $h : X \to [0, 1]^J$ for some index set J. Since $[0, 1]^J$ is compact Hausdorff, by Lemma **??**, there exists a compactification Y of X.

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Example 22.0.3. *Let $X = (0,1)$. Consider the embedding $h : X \hookrightarrow \mathbb{R}^2$ defined by $h(x) = (x, \sin\frac{1}{x})$. Let $A = \{0\} \times [-1,1] \cup \{(1, \sin(1))\}$. Then $Y = h(X) \cup A$ is a compactification of $h(X)$.*

Let Y be a compactification of X in $\mathbb{R}^2$ as above. Then the continuous function $f : X \to \mathbb{R}$ defined by $f(x) = \sin\frac{1}{x}$ can be extended to a continuous function $\overline{f} : Y \to \mathbb{R}$ by defining $\overline{f} = \pi_2 \circ H$ where $H : Y \to \mathbb{R}^2$ is the imbedding and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ is the projection to the second coordinate.

Theorem 22.0.3 (Stone–Čech compactification)**.** *Let X be a completely regular space. Then there exists a compactification Y of X such that every bounded continuous function $f : X \to \mathbb{R}$ extends uniquely to a continuous function $\overline{f} : Y \to \mathbb{R}$. ($Y$ is called the Stone–Čech compactification of X)*

Proof

We prove the existence first and the uniqueness will be discussed in the next lecture.

To be done(Hint: The imbedding Theorem).