Chapter 13

Later

13.1 Metrizability

Recall the following theorem.

Theorem 13.1.1 (Sequence Lemma)**.** *Let X be a topological space. Let $A \subseteq X$. If $\exists\{x_n\} \subseteq A$ such that $x_n \to x$ in X, then $x \in \overline{A}$. The converse is true if X is metrizable.*

Theorem 13.1.2 (Heine's definition of limit)**.** *Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y. The converse is true if X is metrizable.*

Poof

($\Rightarrow$) Let f be continuous. Let $V \ni f(x)$ be open in Y. Then $f^{-1}(V)$ is open in X and contains x. Since $x_n \to x$, $\exists N$ such that $\forall n \geq N$, $x_n \in f^{-1}(V)$. Thus, $\forall n \geq N$, $f(x_n) \in V$. Hence, $f(x_n) \to f(x)$.

($\Leftarrow$) Recall that f is continuous if and only if for all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

Let $A \subseteq X$. Let $x \in \bar{A}$.

Definition 13.1.1. *X is first-countable if it has a countable basis at each $x \in X$. Given $x \in X$, there exists a countable collection of open sets $\{U_n\}$ such that for any open set U containing x, $\exists n$ such that $U_n \subseteq U$. (From this we can construct $\tilde{U_n} = \bigcap_{i=1}^{n} U_i$ such that $\{\tilde{U_n}\}$ is also a countable basis at x with $\tilde{U}_{n+1} \subseteq \tilde{U_n}$.)*

Definition 13.1.2. *X is second-countable if it has a countable basis for the topology. There exists countable a basis $\mathcal{B}$ for X such that $\forall x \in X, \forall U$ open in X containing x, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Property 13.1.1. *If X is second-countable, then X is first-countable.*

Example 13.1.1. *$\mathbb{R}^n$ has a countable basis. For instance, the set of all open balls with rational radii and centers at points with rational coordinates forms a countable basis for the standard topology on $\mathbb{R}^n$.*

Example 13.1.2. *$\mathbb{R}$ with finite complement topology is not first-countable. Let $U_1, U_2, \ldots$ be a countable open sets containing x. Then $\bigcap_{n=1}^{\infty} U_n \setminus \{x\}$ is not empty, so there exists $y \neq x$ such that $y \in U_n$ for all n. Take $U = \mathbb{R} \setminus \{y\}$, which is open and contains x. However, there is no U_n such that $U_n \subseteq U$. Hence, $\mathbb{R}$ with finite complement topology is not first-countable.*

Example 13.1.3. *X is uncountable with discrete topology. Then $\forall x \in X$, the set $\{\{x\}\}$ is open. So any basis of X must contain $\{x\}$ for all $x \in X$. So X is not second-countable. But X is metrizable thus first-countable.*

Example 13.1.4. *$\mathbb{R}^2$ with "Amazon River metric". Define*

$$d((x,y),(x',y')) = \begin{cases} |y-y'|, & x = x' \\ |y| + |y'| + |x-x'|, & x \neq x' \end{cases} \tag{13.1}$$

Then $\{(x,y) | x = x_0, y \in (y_0 - \epsilon, y_0 + \epsilon)\}$ with $\epsilon < |y_0|$ is an open ball centered at (x_0, y_0). There are uncountable many such disjoint open sets. So $\mathbb{R}^2$ with Amazon River metric is not second-countable. But it is metrizable thus first-countable.

Example 13.1.5. *Let $\mathbb{R}_l$ be the set of real numbers with the lower limit topology. Then $\mathbb{R}_l$ is not second-countable. Suppose $\mathcal{B}$ is a countable basis for $\mathbb{R}_l$. For each $x \in \mathbb{R}$, there exists a open set $[x, +\infty)$ containing x. Thus, there exists a basis element $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, +\infty)$ which means $minB_x = x$. Since $\mathcal{B}$ is countable, the set of minimums $\{minB | B \in \mathcal{B}\}$ is also countable. This contradicts the uncountability of $\mathbb{R}$. Hence, $\mathbb{R}_l$ is not second-countable. However, $\mathbb{R}_l$ is first-countable since for each $x \in \mathbb{R}$, the collection of basis elements $\{[x, x+1/n) | n \in \mathbb{N}\}$ forms a countable basis at x.*

Property 13.1.2. $\mathbb{R}^\omega$ *with box topology is not metrizable. (Hence* $\mathbb{R}^J$ *with box topology is not metrizable for infinite* J*.)*

Proof Let's prove that there doesn't exist a first-countable basis at 0.

We assume the contrary that there exists a countable basis $\{V_n\}$ at 0. Since each V_m is open in box topology, $\exists$ open intervals $U_{m,i}$ in $\mathbb{R}$ such that

$$\prod_{i=1}^{\infty} U_{m,i} \subseteq V_m \tag{13.2}$$

We can take $a_{m,i} > 0$ such that $(-a_{m,i}, a_{m,i}) \subseteq U_{m,i}$.

But we can construct an open set $U = \prod_{i=1}^{\infty}(-b_i, b_i)$ containing 0 such that

$$b_i = \frac{a_{i,i}}{2} \tag{13.3}$$

Then

$$\forall m, \prod_{i=1}^{\infty}(-a_{m,i}, a_{m,i}) \nsubseteq \prod_{i=1}^{\infty}(-b_i, b_i) \Longrightarrow V_m \nsubseteq U \tag{13.4}$$

This contradicts the assumption that $\{V_n\}$ is a basis at 0. Hence, $\mathbb{R}^\omega$ with box topology is not first-countable, thus not metrizable.

Property 13.1.3. *Let* J *be uncountable. Then* $\mathbb{R}^J$ *with product topology is not metrizable.*

Proof Let's prove that there doesn't exist a countable basis at 0.

We assume the contrary that there exists a countable basis $\{V_n\}$ at 0.

To be completed.