Chapter 1

Metric Spaces

Definition 1.0.1. *A* metric space *is a set X with a function $d : X \times X \to \mathbb{R}_{\geq 0}$(d is the metric, or the distance function) satisfying the following properties for all $x, y, z \in X$:*

- ***Non-negativity:*** *$d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.*
- ***Symmetry:*** *$d(x, y) = d(y, x)$.*
- ***Triangle inequality:*** *$d(x, z) \leq d(x, y) + d(y, z)$.*

Example 1.0.1. *$X = \mathbb{R}$, $d(x, y) = \|x - y\|$*

Example 1.0.2. *X is finite, $d(x, y) = 1$ if $x \neq y$, $d(x, x) = 0$ (discrete metric)*

Example 1.0.3. *X is a finite set of vertices of a connected graph, $d(x, y)$ is the length of the shortest path between x and y (graph metric)*

Example 1.0.4. *$X = \mathbb{R}^n$, $d(x, y) = \|x - y\|$ (Euclidean metric)*

Example 1.0.5. *$X = \mathbb{R}^n$, $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ (manhattan metric)*

Example 1.0.6. *$X = \mathbb{R}^n$, $d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ (sup metric)*

Exercise Prove that the above examples are indeed metric spaces(You may use Cauchy-Schwarz inequality which is stated later).

Example 1.0.7. *$X = \mathbb{R}^2$, $P = (x, y), P' = (x', y')$*

$$d(P, P') = \begin{cases} |x - x'| + |y| + |y'| & \textit{if } x \neq x' \\ |y - y'| & \textit{if } x = x' \end{cases}$$

Example 1.0.8. *$X = C[0, 1]$ (the set of continuous functions on $[0, 1]$), $d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|$ (sup metric)*

Example 1.0.9. *$X = C[0, 1]$, $d(f, g) = \int_0^1 |f(x) - g(x)| dx$ (L^1 metric)*

Remark We cannot replace X by the set of all integrable functions, because the distance between two functions may be zero even if they are not equal (they may differ on a set of measure zero).

Definition 1.0.2 (Normed Space)**.** *Let V be a vector space over $\mathbb{R}$. A* norm *on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $u, v \in V$ and all $a \in \mathbb{R}$:*

- ***Non-negativity:*** *$\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.*
- ***Absolute homogeneity:*** *$\|av\| = |a|\|v\|$.*
- ***Triangle inequality:*** *$\|u + v\| \leq \|u\| + \|v\|$.*

Remark A norm defines a metric by $d(u, v) = \|u - v\|$ on V.

Example 1.0.10. *$\|v\| = \sqrt{\sum_{i=1}^{n} |v_i|^2}$ (Euclidean norm)*

Example 1.0.11. *$\|v\| = \sum_{i=1}^{n} |v_i|$ (manhattan norm)*

Example 1.0.12. *$\|v\| = \max_{1 \leq i \leq n} |v_i|$ (sup norm)*

Example 1.0.13. $V = C[0,1]$, $\|f\| = \max_{x\in[0,1]} |f(x)|$ *(sup norm)*

Example 1.0.14. $V = C[0,1]$, $\|f\| = \int_0^1 |f(x)|dx$ *(L^1 norm)*

Definition 1.0.3 (l_p-Spaces)**.** *Let $1 \le p < \infty$.*

- $l_\infty = \{x = (x_1, x_2, \ldots) : \sup_i |x_i| < \infty\}$ *with norm* $\|x\|_\infty = \sup_i |x_i|$.
- $l_1 = \{x = (x_1, x_2, \ldots) : \sum_i |x_i| < \infty\}$ *with norm* $\|x\|_1 = \sum_i |x_i|$.
- $l_2 = \{x = (x_1, x_2, \ldots) : \sum_i |x_i|^2 < \infty\}$ *with norm* $\|x\|_2 = \sqrt{\sum_i |x_i|^2}$.
- $l_p = \{x = (x_1, x_2, \ldots) : \sum_i |x_i|^p < \infty\}$ *with norm* $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$.

Property 1.0.1. $l_p \subseteq l_q$ *if* $p < q$

Example 1.0.15. $x_n = \dfrac{1}{n}$. $\sum_1^\infty \dfrac{1}{n} = \infty$ *but* $\sum_1^\infty \dfrac{1}{n^2} < \infty$. *So* $x = (1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots) \in l_2$ *but* $x \notin l_1$.

Definition 1.0.4 (Euclidean Space/Vector Space with Inner Product)**.** *Let V be a vector space over $\mathbb{R}$. An* inner product *on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfying the following properties for all $u, v, w \in V$ and all $a \in \mathbb{R}$:*

- ***Positivity:*** $\langle v, v\rangle \ge 0$ *and* $\langle v, v\rangle = 0$ *if and only if* $v = 0$.
- ***Symmetry:*** $\langle u, v\rangle = \langle v, u\rangle$.
- ***Bilinearity:*** $\langle au + w, v\rangle = a\langle u, v\rangle + \langle w, v\rangle$.

Remark. We always use Euclidean Space as the default example of inner product space.

Theorem 1.0.1 (Cauchy-Schwarz Inequality)**.** *For all $u, v \in V$, $|\langle u, v\rangle| \le \|u\|\|v\|$.*

Proof Let $f(t) = \|u + tv\|^2 = \langle u + tv, u + tv\rangle$. Then using the properties of quadratic polynomials. ∎

Property 1.0.2. $\|v\| = \sqrt{\langle v, v\rangle}$ *is a norm on V.*

Proof. Use Cauchy-Schwarz inequality: $|\langle u, v\rangle| \le \|u\|\|v\|$. ∎

Corollary 1.0.1. *l_2-norm satisfies the triangle inequality.*

Proof. l_2 is an Euclidean Space(infinite dimenstions) with $\langle x, y\rangle = \sum_i x_i y_i$ and $\|x\|_2 = \sqrt{\sum_i |x_i|^2}$. ∎

Remark. $\|x\|_p$ does not come from an inner product if $p \ne 2$.

Theorem 1.0.2. *l_1 and l_∞ are normed spaces.*

Proof
The case for l_1 is left as an exercise. For l_∞, we observe that $\|x\|_\infty = \sup_i |x_i|$.

Theorem 1.0.3. *l_p are normed spaces.*

Proof. This will be proved later in the course.