Chapter 17

Later

Theorem 17.0.1. *Let X be second countable. Then:*

1. *Every covering of X by open sets has a countable subcovering(X is Lindelöf space).*
2. *There exists a countable dense subset $A \subseteq X$ such that $\bar{A} = X$. (In this case X is called separable.)*

Property 17.0.1. *A second countable space is:*

1. *First countable.*
2. *Separable.*
3. *Lindelöf.*

Example 17.0.1 (non-example). *$\mathbb{R}_l$ with lower limit topology(Sorgengrey line) is first countable, Lindelöf, seperable, but not second countable.*

First-countable: for any $x \in \mathbb{R}_l$, $\{[x, x + \frac{1}{n}) : n \in \mathbb{N}\}$ is a countable local basis at x.

Assume the contrary that $\mathbb{R}_l$ is second countable. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis for $\mathbb{R}_l$. For each $x \in \mathbb{R}$, there exists $B_{n_x} \in \mathcal{B}$ such that $x \in B_{n_x} \subseteq [x, x + 1)$. So $\min B_{n_x} = x$. This implies that the map $f : \mathbb{R} \to \mathcal{B}, f(x) = B_{n_x}$ is injective. However, $\mathbb{R}$ is uncountable while $\mathcal{B}$ is countable, which is a contradiction. So $\mathbb{R}_l$ is not second countable.

Separable: $\mathbb{Q} \subseteq \mathbb{R}_l$ is countalbe and dense in $\mathbb{R}_l$.

Lindelöf: See the below property.

Property 17.0.2 (Lindelöf property of $\mathbb{R}_l$). *Any open cover of $\mathbb{R}_l$ has a countable subcover.*

Proof

Let $\mathcal{A}$ be a covering of $\mathbb{R}_l$ by $[a_\alpha, b_\alpha)$'s. We need to show that there exists a countable subcovering of $\mathcal{A}$ covering $\mathbb{R}_l$.

Let $C = \bigcup (a_\alpha, b_\alpha)$ such that $\mathbb{R} \setminus C$ is not empty. Let $x = a_\beta$ for some β such that $x \notin (a_\alpha, b_\alpha)$. Take $q_x \in (a_\beta, b_\beta) \cap \mathbb{Q}$ thus $x < q_x$.

Take $x, y \in \mathbb{R}$. Then $q_x < q_y$ or we make a contradiction(why?). The map $x \mapsto q_x$ is injective from $\mathbb{R} \setminus C$ to $\mathbb{Q}$. However

Property 17.0.3. *$\mathbb{R}_l \times \mathbb{R}_l$ is not Lindelöf.*

Proof

Example 17.0.2. *Subspace of Lindelöf space is not necessarily Lindelöf.*

17.1 Separation Axioms

Topological space X can satisfy the following separability axioms:

1. $T1$(Frechet): Given two distinct points $x, y \in X$, there exists an open set U such that $x \in U$ and $y \notin U$. (Equivalently, all singleton sets are closed. Easy exercise)
2. $T2$(Hausdorff): Given two distinct points $x, y \in X$, there exist open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.
3. $T3$(Regular): X is $T1$ and given a closed set $F \subseteq X$ and a point $x \notin F$, there exist open sets U, V such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$.

4. $T4$(Normal): X is $T1$ and given two disjoint closed sets $F_1, F_2 \subseteq X$, there exist open sets U, V such that $F_1 \subseteq U, F_2 \subseteq V$ and $U \cap V = \emptyset$.

Lemma 17.1.1. *Let X be a $T1$ topological space. Then*

1. *X is regular if and only if for every point $x \in X$ and every neighborhood U of x, there exists a neighborhood V of x such that $\bar{V} \subseteq U$.*
2. *X is normal if and only if for every closed set $F \subseteq X$ and every neighborhood U of F, there exists a neighborhood V of F such that $\bar{V} \subseteq U$.*

Proof

$\Rightarrow$ Let X be regular.

If $U = X$, there is nothing to prove.

Suppose $U \neq X$. Then $F = X \setminus U$ is closed and $x \notin F$. By regularity of X, there exist open sets V, W such that $x \in V, F \subseteq W$ and $V \cap W = \emptyset$. Now we need to show $\bar{V} \subseteq U$.

If $y \in F$, then $y \in W$. Since $V \cap W = \emptyset$, $y \notin \bar{V}$ because we find a neighborhood W of y such that $W \cap V = \emptyset$. Thus, $\bar{V} \subseteq X \setminus F = U$.

$\Leftarrow$ To be done.

The proof of 2. is similar.

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Theorem 17.1.1.

1. *X is Hausdorff, then $Y \subseteq X$ with subspace topology is Hausdorff.*
2. *X_α is Hausdorff, then $\prod X_\alpha$ with product topology is Hausdorff.*
3. *X is regular, then $Y \subseteq X$ with subspace topology is regular.*
4. *X_α is regular, then $\prod X_\alpha$ with product topology is regular.*

Proof

(a) is obvious.

(b) Let $x = (x_\alpha), y = (y_\alpha) \in \prod X_\alpha$ with $x \neq y$. Then there exists β such that $x_\beta \neq y_\beta$. Since X_β is Hausdorff, there exist open sets $U_\beta, V_\beta \subseteq X_\beta$ such that $x_\beta \in U_\beta, y_\beta \in V_\beta$ and $U_\beta \cap V_\beta = \emptyset$.

(c) is obvious.

(d) To be done.

Remark X is normal does not imply that $Y \subseteq X$ with subspace topology is normal. X_1, X_2 are normal does not imply that $X_1 \times X_2$ with product topology is normal.

Exercise Show that $\mathbb{R}_l$ is normal.

Fact (Without Proof) $\mathbb{R}_l^2$ is not normal. But it is regular(as a product of regular spaces).

Exercise $\mathbb{R}_K$ is a topological space with basis given by all open intervals (a, b) and all sets of the form $(a, b) \setminus K$ where $K = \{1/n | n \in \mathbb{N}\}$. $\mathbb{R}_K$ is Hausdorff. But $\mathbb{R}_K$ is not regular(To be done).