

# Chapter 15

## Later

**Definition 15.0.1.** Two norms  $\|\cdot\|_a$  and  $\|\cdot\|_b$  are equivalent if there exist positive constants  $C_1$  and  $C_2$  such that for all vectors  $x$ ,

$$C_1\|x\|_a \leq \|x\|_b \leq C_2\|x\|_a.$$

**Property 15.0.1.** Equivalence of norms is an equivalence relation.

**Theorem 15.0.1.** All norms on  $\mathbb{R}^n$  are equivalent.

### Proof

Given  $\|\cdot\|_a$  and  $\|\cdot\|_b$  are two norms on  $\mathbb{R}^n$ . We define a function  $f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$  by

$$f(x) = \frac{\|x\|_b}{\|x\|_a} > 0. \quad (15.1)$$

which is continuous on  $\mathbb{R}^n \setminus \{0\}$ .

Since  $f(x) = f(\lambda x)$  for any  $\lambda > 0$ , we say that  $f(x)$  is completely determined by its values on the unit sphere  $S = \{x \in \mathbb{R}^n : \|x\|_a = 1\}$ . Note that  $S$  is compact, so  $f(x)$  attains its minimum and maximum on  $S$ , say  $m$  and  $M$ . Thus, for any  $x \in \mathbb{R}^n \setminus \{0\}$ , we have

$$0 < m \leq f(x) \leq M < \infty, \quad (15.2)$$

which implies

$$m\|x\|_a \leq \|x\|_b \leq M\|x\|_a. \quad (15.3)$$

So the two norms are equivalent. ■

**Remark** In  $\infty$ -dimensional space, the sphere is not compact. However, it is closed and bounded.

**Exercise** Give an example of norms on  $l_1$  (convergent series)  $l_1 = \{(x_1, x_2, \dots) | \sum |x_i| < \infty\}$ .

Back to uniform Continuity.

Goal: Prove that any continuous map  $f : (X, d_X) \rightarrow (Y, d_Y)$  with  $X$  compact is uniformly continuous.

**Definition 15.0.2.** Let  $X$  be a metric space.  $A \subseteq X$ . Then  $\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}$  is called the diameter of  $A$ .

**Lemma 15.0.1** (the Lebesgue Number Lemma). Let  $(X, d)$  be a compact metric space, and let  $\mathcal{A}$  be an open cover of  $X$ . Then there exists a positive number  $\delta > 0$  (called a Lebesgue number for the cover  $\mathcal{A}$ ) such that for every subset  $Y \subseteq X$  with diameter less than  $\delta$  (i.e., for all  $x, y \in Y$ ,  $d(x, y) < \delta$ ), there exists an open set  $A \in \mathcal{A}$  that contains  $Y$ .

**Example 15.0.1.** Let  $X \subseteq \mathbb{R}$  be covered by  $(a_\alpha, b_\alpha)$ . Take a finite subcovering  $(a_i, b_i)$ ,  $i = 1, 2, \dots, n$ . Then  $\delta = \min\{b_i - a_i : i = 1, 2, \dots, n\}$  is a Lebesgue number.

### Proof of the Lemma

If  $X \in \mathcal{A}$ , there's nothing to prove.

Otherwise take a finite subcovering  $\{A_1, A_2, \dots, A_n\}$  of  $\mathcal{A}$ . For each  $i$ , let  $C_i = X \setminus A_i$  which is closed. We know that a closed subset of a compact set is compact, so each  $C_i$  is compact. So  $\forall x \in X$ , we have  $d(x, C_i) \geq 0$ . Define

$$f(x) = \frac{1}{n} \sum_{i=1}^n d(x, C_i) \quad (15.4)$$

be the average distance from  $x$  to the closed sets  $C_i$ .

We show that  $f(x) > 0$  for all  $x \in X$ .

For all  $x \in X$ , we choose  $A_i$  such that  $x \in A_i$ . Choose  $\epsilon > 0$  such that  $U_\epsilon(x) \subseteq A_i$  and  $d(x, C_i) \geq \epsilon > 0$ . So  $f(x) \geq \frac{1}{n}d(x, C_i) \geq \frac{\epsilon}{n}$ . Let  $\delta = \min_{x \in X} f(x) > 0$  (because  $f$  is continuous on compact set  $X$ ). Then  $\delta$  is a Lebesgue number.

Take  $B \subseteq X$  with  $\text{diam}(B) < \delta$ . Then we can take  $x_0 \in B$  with  $B \subseteq U_\delta(x_0)$ . Then we can take  $C_m$  such that

$$\delta \leq f(x_0) \leq d(x_0, C_m). \quad (15.5)$$

for some  $m$ . So we have

$$U_\delta(x_0) \subseteq A_m = X \setminus C_m. \quad (15.6)$$

So  $B \subseteq A_m$ . ■

**Definition 15.0.3.**  $f : (X, d_X) \rightarrow (Y, d_Y)$ . A function between two metric spaces is said to be uniformly continuous if for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x_1, x_2 \in X$ , if  $d_X(x_1, x_2) < \delta$ , then  $d_Y(f(x_1), f(x_2)) < \epsilon$ .

**Theorem 15.0.2.** Let  $f : X \rightarrow Y$  be a continuous function between two metric spaces. If  $X$  is compact, then  $f$  is uniformly continuous.

**Proof** To be done.

**Theorem 15.0.3.** Let  $X$  be a non-empty compact Hausdorff space without isolated points. Then  $X$  is uncountable.

**Proof** To be done.

**Corollary 15.0.1.**  $[a, b] \subseteq \mathbb{R}$  is uncountable.