Chapter 2

Later

Definition 2.0.1. *For $p > 1, q > 1$. p is conjugate to q if $\frac{1}{p} + \frac{1}{q} = 1$.*

Remark When $p = 1$, $q = \infty$; when $p = \infty$, $q = 1$. And 2 is conjugate to itself.

Theorem 2.0.1 (Young's inequality)**.**

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1 \tag{2.1}$$

Proof $\ln(x)$ is concave, just use Jensen's inequality.

Theorem 2.0.2 (Hölder's inequality)**.** *For $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, for any $a = (a_i) \in l_p, b = (b_i) \in l_q$ respectively, we have*

$$\sum_{k=1}^{\infty} a_k b_k \leq \sqrt[\frac{1}{p}]{\sum a_k^p} \sqrt[\frac{1}{q}]{\sum b_k^q}$$

Proof

Theorem 2.0.3 (Minkowski inequality)**.** *For $p \geq 1$, for any $x, y \in l_p$, we have*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Proof

Let us first show that for $x = (x_i), y = (y_i) \in l_p$, we have

$$\sqrt[\frac{1}{p}]{|x_k + y_k|^p} \leq \sqrt[\frac{1}{p}]{|x_k|^p} + \sqrt[\frac{1}{p}]{|y_k|^p}$$

For every summand, we have

$$\begin{aligned} |x_k + y_k|^p &= |x_k + y_k| \cdot |x_k + y_k|^{p-1} \\ &\leq |x_k| \cdot |x_k + y_k|^{p-1} + |y_k| \cdot |x_k + y_k|^{p-1} \end{aligned}$$

So we have

$$\sum_{k=1}^{n} |x_k + y_k|^p \leq \sum_{k=1}^{n} |x_k| \cdot |x_k + y_k|^{p-1} + \sum_{k=1}^{n} |y_k| \cdot |x_k + y_k|^{p-1} \tag{2.2}$$

Let $a_k = |x_k|, b_k = |x_k + y_k|^{p-1}$, then $b_k^q = |x_k + y_k|^p$. By Hölder's inequality, we have

$$\sum_{k=1}^{n} |x_k| \cdot |x_k + y_k|^{p-1} \leq \sqrt[\frac{1}{p}]{\sum_{k=1}^{n} |x_k|^p} \sqrt[\frac{1}{q}]{\sum_{k=1}^{n} |x_k + y_k|^p}$$

On the other hand, let $a_k = |y_k|, b_k = |x_k + y_k|^{p-1}$, we have

$$\sum_{k=1}^{n} |y_k| \cdot |x_k + y_k|^{p-1} \leq \sqrt[\frac{1}{p}]{\sum_{k=1}^{n} |y_k|^p} \sqrt[\frac{1}{q}]{\sum_{k=1}^{n} |x_k + y_k|^p}$$

Combining these two inequalities, we have

$$\sum_{k=1}^{n}|x_k+y_k|^p \leq \left(\sqrt[\frac{1}{p}]{\sum_{k=1}^{n}|x_k|^p}+\sqrt[\frac{1}{p}]{\sum_{k=1}^{n}|y_k|^p}\right)\sqrt[\frac{1}{q}]{\sum_{k=1}^{n}|x_k+y_k|^p} \tag{2.3}$$

Dividing both sides by $\sqrt[\frac{1}{q}]{\sum_{k=1}^{n}|x_k+y_k|^p}$, we have

$$\sqrt[\frac{1}{p}]{\sum_{k=1}^{n}|x_k+y_k|^p} \leq \sqrt[\frac{1}{p}]{\sum_{k=1}^{n}|x_k|^p}+\sqrt[\frac{1}{p}]{\sum_{k=1}^{n}|y_k|^p} \tag{2.4}$$

Taking limit $n \to \infty$, we have the desired result.

■

Let X be a metric space. Let $x \in X$. We have the following definitions.

Definition 2.0.2. *We define a neighborhood of x to be a set of the form*

$$U_\epsilon(x) = \{y \in X | d(x,y) < \epsilon\}$$

for some $\epsilon > 0$.

Definition 2.0.3. *We define a punctured neighborhood of x to be a set of the form*

$$U_\epsilon^*(x) = \{y \in X | 0 < d(x,y) < \epsilon\} = U_\epsilon(x) \setminus \{x\}$$

for some $\epsilon > 0$.

Definition 2.0.4. *We say that $M \subseteq X$ is open in X if for every $x \in M$, there exists $\epsilon > 0$ such that $U_\epsilon(x) \subseteq M$.*

Remark $\emptyset, X$ are open in X by definition.

Example 2.0.1. *Is it possible that in a metric space X, a ball is contained properly inside a ball with smaller radius? That is, is there $x \in X$ and $0 < r < s$ such that $U_s(x) \subsetneq U_r(x)$? [Hint: If $Y \subseteq X$ and (X, d) is a metric space, then (Y, d) is also a metric space.]*

Solution Yes. Let $X = (-1, 1)$ with the usual metric. Then $U_{\frac{3}{2}}(\dfrac{4}{3}) = (-\frac{4}{3}, 1) \subsetneq U_1(0) = (-1, 1)$.

Example 2.0.2. *Draw balls centered at 0 in $\mathbb{R}^2$ with norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$.*

Example 2.0.3. *"Amazon Metric" on $\mathbb{R}^2$ is given by*

$$d((x_1,y_1),(x_2,y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2 \end{cases}$$

We will solve these problems later in the course.

Theorem 2.0.4. *1. The intersection of finitely many open sets is open, that is, $U_1 \cap U_2 \cap \cdots \cap U_n$ is open where each U_i is open in X.*

2. The union of any collection of open sets is open, that is, if $\{U_i\}_{i \in I}$ is a collection of open sets in X, then $\bigcup_{i\in I} U_i$ is open.

Proof to 1.
If $V = U_1 \cap U_2 \cap \cdots \cap U_n = \emptyset$, then V is open by definition.
If $V \neq \emptyset$, let $x \in V$. Since $x \in U_i$ for each $i = 1, 2, \ldots, n$, there exists $\epsilon_i > 0$ such that $U_{\epsilon_i}(x) \subseteq U_i$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. Then $U_\epsilon(x) \subseteq U_i$ for each i, so $U_\epsilon(x) \subseteq V$. Thus, V is open.
Proof to 2.
Let $x \in U = \bigcup_{i\in I} U_i$. Then there exists some $j \in I$ such that $x \in U_j$. Since U_j is open, there exists $\epsilon > 0$ such that $U_\epsilon(x) \subseteq U_j \subseteq U$. Thus, U is open.