Chapter 19

Later

Theorem 19.0.1. *Every second countable regular space is normal.*

Proof

Let X be a second countable regular space with $A, B \subseteq X$ closed and $A \cap B = \emptyset$.

For every $x \in A$, since X is regular, there exist open sets $U \ni x$ such that $U \cap B = \emptyset$.

By the lemma, there exists a open set V such that $x \in V$, $\bar{V} \subseteq U$ and thus $\bar{V} \cap B = \emptyset$.

Let $\mathcal{B}$ be a countable baisis for X. For each $x \in A$, there exists $U_x \in \mathcal{B}$ such that $x \in U_x \subseteq V$, and thus $\bar{U_x} \subseteq \bar{V} \subseteq U$ and $\bar{U_x} \cap B = \emptyset$. Since $\mathcal{B}$ is countable, the collection $\{\bar{U_x} : x \in A\}$ has a countable subcollection $\{\bar{U_n} : n \in \mathbb{N}\}$ covering A such that $\bar{U_n} \cap B = \emptyset$ for each n.

Theorem 19.0.2. *Every metrizable space X is normal.*

Proof Let d be the metric on X. Given two closed disjoint subsets $A, B \subseteq X$.

For each $a \in A$, there exists $r_a > 0$ such that $B(a, r_a) \cap B = \emptyset$.

Similarly, for each $b \in B$, there exists $s_b > 0$ such that $B(b, r_b) \cap A = \emptyset$.

Let $U = \bigcup_{a \in A} B(a, r_{\frac{a}{2}})$ and $V = \bigcup_{b \in B} B(b, r_{\frac{b}{2}})$. Then U and V are open sets containing A and B respectively.

We claim that $U \cap V = \emptyset$. Assume the contrary that there exists $x \in U \cap V$. Then there exist $a \in A$ and $b \in B$ such that $x \in B(a, r_{\frac{a}{2}})$ and $x \in B(b, r_{\frac{b}{2}})$. Without loss of generality, assume that $r_{\frac{a}{2}} \leq r_{\frac{b}{2}}$. Then

$$d(a, b) \leq d(a, x) + d(x, b) < r_{\frac{a}{2}} + r_{\frac{b}{2}} \leq r_a \tag{19.1}$$

This implies that $b \in B(a, r_a)$, which contradicts the choice of r_a. Thus, $U \cap V = \emptyset$. Hence, X is normal.

Theorem 19.0.3. *Every compact Hausdorff space is normal.*

Proof

Let $x \in X$ and $B \subseteq X$ be closed such that $x \notin B$.

For each $y \in B$, since X is Hausdorff, there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Then $\{V_y | y \in B\}$ is an open covering of B. By the compactness of B, there exists a finite subcollection $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$ that covers B. Let $V = \bigcup V_{y_i}$ and $U = \bigcap U_{y_i}$. Then U and V are disjoint open sets containing x and B respectively. So X is regular.

Let $A, B \subseteq X$ be closed and $A \cap B = \emptyset$. For each $a \in A$, since X is regular, there exist disjoint open sets U_a and V_a such that $a \in U_a$ and $B \subseteq V_a$. Then $\{U_a | a \in A\}$ is an open covering of A. By the compactness of A, there exists a finite subcollection $\{U_{a_1}, U_{a_2}, \ldots, U_{a_m}\}$ that covers A. Let $U = \bigcup U_{a_i}$ and $V = \bigcap V_{a_i}$. Then U and V are disjoint open sets containing A and B respectively. So X is normal.

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Lemma 19.0.1 (Urysohn's lemma)**.** *Given a normal space X and two closed disjoint subsets $A, B \subseteq X$, there exists a continuous map $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof

Theorem 19.0.4 (Urysohn's Lemma)**.** *Let X be a normal space. If A and B are disjoint closed subsets of X, then there exists a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

Definition 19.0.1. *Suppose X satisfies (T1). If a point and a closed set can be separated by a continuous function as in Urysohn's lemma, then X is called completely regular(CR). (Sometimes it is called $T_{3\frac{1}{2}}$ space.)*

Remark

Normal $\Rightarrow$ CR $\Rightarrow$ Regular. That's why we call it $T_{3\frac{1}{2}}$ space.

Property 19.0.1. *If X is completely regular, then it is regular.*

Proof

Take $x_0 \in X$ and a closed set A such that $x_0 \notin A$. Since X is CR, there exists a continuous function $f : X \to [0,1]$ such that $f(x_0) = 0$ and $f(a) = 1$ for all $a \in A$. Let $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$. Then U and V are open sets because f is continuous and $[0, \frac{1}{2})$, $(\frac{1}{2}, 1]$ are open in $[0,1]$. So $x_0 \in U$, $A \subseteq V$ and $U \cap V = \varnothing$. Thus X is regular.

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Theorem 19.0.5. *1. Subspace of CR space is CR.*

2. If X_α is CR, then $\prod X_\alpha$ with product topology is CR.

Proof

Let X be CR, and $Y \subseteq X$ with subspace topology. Take $y_0 \in Y$ and a closed set $A \subseteq Y$ such that $y_0 \notin A$. Let $\bar{A}$ be the closure of A in X. Then $\bar{A}$ is closed in X and $y_0 \notin \bar{A}$. Since X is CR, there exists a continuous function $f : X \to [0,1]$ such that $f(y_0) = 0$ and $f(a) = 1$ for all $a \in \bar{A}$. Restrict f to Y, we get a continuous function $f|_Y : Y \to [0,1]$ such that $f|_Y(y_0) = 0$ and $f|_Y(a) = 1$ for all $a \in A$. Thus Y is CR.

Take $A \subseteq \prod X_\alpha$ closed and $x = (x_\alpha) \notin A$. Then there exists a basic open set $U = \prod U_\alpha$ such that $x \in U$ and $U \cap A = \emptyset$, where U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many α. Let those finitely many α be $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since X_{α_i} is CR, there exists a continuous function $f_{\alpha_i} : X_{\alpha_i} \to [0,1]$ such that $f_{\alpha_i}(x_{\alpha_i}) = 0$ and $f_{\alpha_i}(a) = 1$ for all $a \in X_{\alpha_i} \setminus U_{\alpha_i}$. So we can define a continuous function $f : \prod X_\alpha \to [0,1]$ by

$$f(x) = \prod_{i=1}^{n} f_{\alpha_i}(x_{\alpha_i}).$$

Then $f(x) = 0$ and for all $a \in A$, $f(a) = 1$. Thus $\prod X_\alpha$ is CR. (To be checked)

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Example 19.0.1. *$\mathbb{R}_l$ is normal then it is CR. So $\mathbb{R}_l^2$ is CR. But it is not normal as we have shown.*

Fact(no proof) There exist regular spaces that are not CR.
Remark Proof of Urysohn's lemma for metric spaces(exercise):

Let X be metric space with metric $d : X \times X \to \mathbb{R}$. Given two disjoint closed sets $A, B \subseteq X$, define $d_B(x) = d(x, B)$ which vanishes exactly on B. Then define $d_A(x) = d(x, A)$ which vanishes exactly on A. Now define

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}.$$

Then $f(A) = \{0\}$ and $f(B) = \{1\}$ and f is continuous because we've proved that distance from a point to a closed set is continuous.

Theorem 19.0.6 (Urysohn Metrization Theorem)**.** *Every regular space with contable basis is metrizable.*

Idea of Proof Embede X into $[0,1]^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$. This $\mathbb{R}^{\mathbb{N}}$ can be with product topology or uniform metric topology.
Step 1 Construct a family $f_n : X \to [0,1]$ of functions such that $\forall x_0 \in X$ and $\forall$ neighborhood U of x_0, there exists n such that $f_n(x_0) > 1$ and $f_n(x) = 0$ for all $x \notin U$.

For each x_0 and neighborhood U of x_0, since X is regular, by Urysohn's lemma, there exists such functions. But we want to get countably many of them.

Given $x_0 \in U$ there exists B_m such that $x_0 \in B_m \subseteq U$. Since X is regular, there exists B_n such that $x_0 \in B_n$ and $\bar{B_n} \subseteq B_m$. By Urysohn's lemma, there exist