Chapter 8

Lecture 8

Theorem 8.0.1. *Let A be a subset of the topological space X.*

1. $x \in \bar{A} \iff$ every open set $U \subseteq X$ with $x \in U$ satisfies $U \cap A \neq \emptyset$.

2. For a basis $\mathcal{B}$ of $\mathcal{T}$, we have $x \in \bar{A} \iff$ every basis element $B \in \mathcal{B}$ with $x \in B$ satisfies $B \cap A \neq \emptyset$.

Proof to 1

We transform each implication to its contrapositive, thereby obtaining the logically equivalent statement $(\neg P) \iff (\neg Q)$. Written out, it is the follwoing statement. $x \notin \bar{A}$ if and only if there exists an open set U containing x that does not intersect A.

If $x \notin \bar{A}$, then the set $U = X - \bar{A}$ is an open set containing x that does not intersect A.

Conversely, if there exists an open set U containing x which does not intersect A, then $X - U$ is a closed set containing A. Then $\bar{A} \subseteq X - U$. Since $x \notin X - U$, $x \notin \bar{A}$.

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Exercise Finish the proof of 2.

Definition 8.0.1. *For $A \subseteq X$, $x \in X$ is a limit point of A if every open set $U \subseteq X$ containing x contains a point of A different from x itself. We denote A' as the set of all limit points of A.*

Theorem 8.0.2. $\bar{A} = A \cup A'$

Proof

We need to show that $\bar{A} \subseteq A \cup A'$ and $A \cup A' \subseteq \bar{A}$.

($\supseteq$) If $x \in A'$, then every neighborhood U of x intersects A. Then $x \in \bar{A}$ by the previous theorem. If $x \in A$, then obviously $x \in \bar{A}$. Thus $A \cup A' \subseteq \bar{A}$.

($\subseteq$) Let $x \in \bar{A}$. If $x \in A$, then $x \in A \cup A'$. If $x \in \bar{A} \setminus A$, then every neighborhood U of x intersects A at y such that $y \neq x$. Thus $x \in A'$. Therefore, $\bar{A} \subseteq A \cup A'$.

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Corollary 8.0.1. *A set A is closed if and only if $A' \subseteq A$.*

Definition 8.0.2. *For a sequence $\{x_n\}$ in X, we say that a is a limit of the sequence if for every open set U containing a, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$. A sequence is convergent if it has a limit.*

Remark This is equivalent to the following: every open set U containing a contains almost all elements of the sequence except finitely many. And Notice that "all but finitely many" is not the same as "infinitely many".

Example 8.0.1. *Let X be a topological space with the discrete topology. Then a sequence $\{x_n\}$ converges to a if and only if there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n = a$.*

Example 8.0.2. *Let $X = \mathbb{Q}$ with the topology defined by the usual metric. Then the sequence: $3, 3.1, 3.14, 3.141, 3.1415, \dots$ does not converge in X since its limit π is not in $\mathbb{Q}$.*

Example 8.0.3. *Let $X = \mathbb{R}_l$ with the lower limit topology($[a,b)$ are basis open sets). Then we say that a sequence converges in $\mathbb{R}^l$ if and only if it converges to a in $\mathbb{R}$ with an extra condition: $x_n \geq a, \forall n \geq N$ for some $N \in \mathbb{N}$.*

Exercise Show that if x_n converges in $\mathcal{T}$, then x_n converges in any coarser topology $\mathcal{T}' \subseteq \mathcal{T}$.

Example 8.0.4. *$X = \mathbb{N}$, the basis $U_m = \{0, 1, 2 \cdots, m\}$. $\mathcal{T} = \{U_m | m \in \mathbb{N}\} \cup \{\mathbb{N}, \emptyset\}$. Then x_n converges to a if and only if a is essential upper bound of $\{x_n\}$. In this case, the limit is not unique.*

Example 8.0.5. *Finite complement topology on* $\mathbb{R}$.
   For example take $\mathcal{T} = \{\mathbb{R}, \emptyset\} \cup \{\mathbb{R} \setminus \{p_1, p_2, \cdots, p_m\} | p_1, p_2, \cdots, p_m \in \mathbb{R}, m \in \mathbb{N}\}$.

1. *If* x_n *is eventually constant with value* a, *then* x_n *converges to* a. *It can't not converge to* $b \neq a$ *because we can take the open set* $\mathbb{R} \setminus \{a\}$ *which contains* b *but does not contain almost all elements of the sequence.*

2. *If* $x_n = (-1)^n$, *then* x_n *does not converge. If* $x_n = a$ *for infinitely many* n *and* $x_n = b$ *for infinitely many* n $(a \neq b)$, *then* x_n *does not converge.*

3. *If* $x_n = \dfrac{(-1)^n}{n}$, *then* x_n *converges to any real number. Because for any open set* U, U *contains all but finitely many real numbers. So* U *contains almost all elements of the sequence.*

4. *If* x_n *assumes every value finitely many times, then* x_n *converges to any real number.*