

# Chapter 21

## Later

**Idea** For the idea of this lecture, refer to Munkres' Topology, P230 or pdf P247.

**Definition 21.0.1.** Let  $(A, \leq)$  be a partially ordered set. A subset  $B \subseteq A$  is called a chain if for any  $a, b \in B$ , we have either  $a \leq b$  or  $b \leq a$ .

**Theorem 21.0.1** (Zorn's lemma). Let  $(A, \leq)$  be a partially ordered set. If every chain in  $A$  has an upper bound in  $A$ , then  $A$  contains at least one maximal element.

**Lemma 21.0.1.** Let  $X$  be a set; let  $\mathcal{A}$  be a collection of subsets of  $X$  having the finite intersection property. Then there is a collection  $\mathcal{D}$  of subsets of  $X$  such that  $\mathcal{D}$  contains  $\mathcal{A}$ , and  $\mathcal{D}$  has the finite intersection property, and no collections of subsets of  $X$  properly containing  $\mathcal{D}$  has the finite intersection property.

### Proof

For purposes of this proof, we shall call a set whose elements are collections of subsets of  $X$  a "superset" and shall denote it by an outline letter. To summarize the notation:

- $c$  is an element of  $X$ ;
- $C$  is a subset of  $X$ ;
- $\mathcal{C}$  is a collection of subsets of  $X$ ;
- $\mathbb{C}$  is a superset whose elements are collections of subsets of  $X$ .

Now by hypothesis,  $\mathcal{A}$  is a collection of subsets of  $X$  having the finite intersection property. Let  $\mathbb{A}$  be the superset consisting of all collections of subsets of  $X$  such that  $\forall \mathcal{B} \in \mathbb{A}$ , we have  $\mathcal{B} \supseteq \mathcal{A}$  and  $\mathcal{B}$  has the finite intersection property. To prove the lemma, we need to show that  $\mathbb{A}$  contains a maximal element with respect to the partial order  $\subseteq$ .

In order to apply Zorn's lemma, we must show that if  $\mathbb{B}$  is a "subsuperset" of  $\mathbb{A}$  that is a chain, then  $\mathbb{B}$  has an upper bound in  $\mathbb{A}$ . Let

$$\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}. \quad (21.1)$$

Certainly  $\mathcal{C}$  contains  $\mathcal{A}$ , since each  $\mathcal{B} \in \mathbb{B}$  contains  $\mathcal{A}$ . Let  $C_1, C_2, \dots, C_n$  be finitely many elements of  $\mathcal{C}$ . Then for each  $i = 1, 2, \dots, n$ , there exists  $\mathcal{B}_i \in \mathbb{B}$  such that  $C_i \in \mathcal{B}_i$ . Since  $\mathbb{B}$  is a chain, there exists  $\mathcal{B}_j \in \mathbb{B}$  such that  $\mathcal{B}_i \subseteq \mathcal{B}_j$  for all  $i = 1, 2, \dots, n$ . Thus  $C_i \in \mathcal{B}_j$  for all  $i = 1, 2, \dots, n$ . Since  $\mathcal{B}_j$  has the finite intersection property, we have

$$C_1 \cap C_2 \cap \dots \cap C_n \neq \emptyset. \quad (21.2)$$

Thus  $\mathcal{C}$  has the finite intersection property. And so  $\mathcal{C} \in \mathbb{A}$ . Moreover, for each  $\mathcal{B} \in \mathbb{B}$ , we have  $\mathcal{B} \subseteq \mathcal{C}$ . Thus  $\mathcal{C}$  is an upper bound of  $\mathbb{B}$  in  $\mathbb{A}$ . By Zorn's lemma,  $\mathbb{A}$  contains a maximal element, which we denote by  $\mathcal{D}$ . This completes the proof of the lemma. ■

**Lemma 21.0.2.** Let  $X$  be a set; let  $\mathcal{D}$  be a collection of subsets of  $X$  that is maximal with respect to the finite intersection property. Then

1. Any finite intersection of elements of  $\mathcal{D}$  is an element of  $\mathcal{D}$ ;
2. If  $A$  is a subset of  $X$  that intersects every element of  $\mathcal{D}$ , then  $A$  is an element of  $\mathcal{D}$ .

**Proof to 1**

Let  $B$  be any finite intersection of elements of  $\mathcal{D}$ . Let  $\mathcal{E} = \mathcal{D} \cup \{B\}$ . We can show that  $\mathcal{E}$  has the finite intersection property (Check it). By maximality of  $\mathcal{D}$ , we have  $\mathcal{E} = \mathcal{D}$ , so  $B \in \mathcal{D}$ .

**Proof to 2**

Let  $A$  be a subset of  $X$  that intersects every element of  $\mathcal{D}$ . Let  $\mathcal{E} = \mathcal{D} \cup \{A\}$ . Take finitely many elements from  $\mathcal{E}$ . If none of them is  $A$ , then their intersection is not empty because  $\mathcal{D}$  has the finite intersection property, otherwise, it is of the form

$$A \cap D_1 \cap D_2 \cap \dots \cap D_n, \quad (21.3)$$

Now,  $D_1 \cap D_2 \cap \dots \cap D_n \in \mathcal{D}$  by part 1, so  $A$  intersects it by hypothesis. Thus the intersection is not empty. So  $\mathcal{E}$  has the finite intersection property. By maximality of  $\mathcal{D}$ , we have  $\mathcal{E} = \mathcal{D}$ , so  $A \in \mathcal{D}$ . ■

**Theorem 21.0.2** (Tychonoff theorem). *Any arbitrary product of compact spaces is compact in the product topology.*

**Proof**

Let

$$X = \prod_{\alpha \in J} X_{\alpha}, \quad (21.4)$$

where each  $X_{\alpha}$  is compact. Let  $\mathcal{A}$  be a collection of subsets of  $X$  having the finite intersection property. We prove that the intersection  $\bigcap_{A \in \mathcal{A}} A$  is non-empty. Compactness of  $X$  follows.

By 21.0.1, we choose  $\mathcal{D}$  containing  $\mathcal{A}$  that is maximal with respect to the finite intersection property. It suffices to show that the intersection

$$\bigcap_{D \in \mathcal{D}} \bar{D} \quad (21.5)$$

is non-empty.

Given any  $\alpha \in J$ , we consider the collection

$$\{\pi_{\alpha}(D) \mid D \in \mathcal{D}\} \quad (21.6)$$

of subsets of  $X_{\alpha}$ . This collection has the finite intersection property because  $\mathcal{D}$  does. By compactness of  $X_{\alpha}$ , we can choose a point  $x_{\alpha} \in X_{\alpha}$  such that

$$x_{\alpha} \in \bigcap_{D \in \mathcal{D}} \overline{\pi_{\alpha}(D)}. \quad (21.7)$$

Now the point  $x = (x_{\alpha})$  is defined in  $X$ .

Let  $U_{\beta}$  be a neighborhood of  $x_{\beta}$  in  $X_{\beta}$ . Since  $x_{\beta} \in \overline{\pi_{\beta}(D)}$ , we have that  $\exists y \in D, \pi_{\beta}(y) \in U_{\beta} \cap \pi_{\beta}(D)$ . Thus  $y \in \pi_{\beta}^{-1}(U_{\beta}) \cap D$ . Since this is true for any  $D \in \mathcal{D}$ , by the previous lemma, we have  $\pi_{\beta}^{-1}(U_{\beta}) \in \mathcal{D}$ . So every subbasis element containing  $x$  is in  $\mathcal{D}$ . And then it follows from the same lemma that every basis (finite intersection of subbasis) element containing  $x$  is in  $\mathcal{D}$ . Thus any open neighborhood of  $x$  intersects every  $D \in \mathcal{D}$ . Therefore,  $x \in \bar{D}$  for all  $D \in \mathcal{D}$ . ■