

Chapter 21

Later

Idea For the idea of this lecture, refer to Munkres' Topology, P230 or pdf P247.

Definition 21.0.1. Let (A, \leq) be a partially ordered set. A subset $B \subseteq A$ is called a chain if for any $a, b \in B$, we have either $a \leq b$ or $b \leq a$.

Theorem 21.0.1 (Zorn's lemma). Let (A, \leq) be a partially ordered set. If every chain in A has an upper bound in A , then A contains at least one maximal element.

Lemma 21.0.1. Let X be a set; let \mathcal{A} be a collection of subsets of X having the finite intersection property. Then there is a collection \mathcal{D} of subsets of X such that \mathcal{D} contains \mathcal{A} , and \mathcal{D} has the finite intersection property, and no collections of subsets of X properly containing \mathcal{D} has the finite intersection property.

Proof

For purposes of this proof, we shall call a set whose elements are collections of subsets of X a "superset" and shall denote it by an outline letter. To summarize the notation:

- c is an element of X ;
- C is a subset of X ;
- \mathcal{C} is a collection of subsets of X ;
- \mathbb{C} is a superset whose elements are collections of subsets of X .

Now by hypothesis, \mathcal{A} is a collection of subsets of X having the finite intersection property. Let \mathbb{A} be the superset consisting of all collections of subsets of X such that $\forall \mathcal{B} \in \mathbb{A}$, we have $\mathcal{B} \supseteq \mathcal{A}$ and \mathcal{B} has the finite intersection property. To prove the lemma, we need to show that \mathbb{A} contains a maximal element with respect to the partial order \subseteq .

In order to apply Zorn's lemma, we must show that if \mathbb{B} is a "subsupserset" of \mathbb{A} that is a chain, then \mathbb{B} has an upper bound in \mathbb{A} . Let

$$\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}. \quad (21.1)$$

Certainly \mathcal{C} contains \mathcal{A} , since each $\mathcal{B} \in \mathbb{B}$ contains \mathcal{A} . Let C_1, C_2, \dots, C_n be finitely many elements of \mathcal{C} . Then for each $i = 1, 2, \dots, n$, there exists $\mathcal{B}_i \in \mathbb{B}$ such that $C_i \in \mathcal{B}_i$. Since \mathbb{B} is a chain, there exists $\mathcal{B}_j \in \mathbb{B}$ such that $\mathcal{B}_i \subseteq \mathcal{B}_j$ for all $i = 1, 2, \dots, n$. Thus $C_i \in \mathcal{B}_j$ for all $i = 1, 2, \dots, n$. Since \mathcal{B}_j has the finite intersection property, we have

$$C_1 \cap C_2 \cap \dots \cap C_n \neq \emptyset. \quad (21.2)$$

Thus \mathcal{C} has the finite intersection property. And so $\mathcal{C} \in \mathbb{A}$. Moreover, for each $\mathcal{B} \in \mathbb{B}$, we have $\mathcal{B} \subseteq \mathcal{C}$. Thus \mathcal{C} is an upper bound of \mathbb{B} in \mathbb{A} . By Zorn's lemma, \mathbb{A} contains a maximal element, which we denote by \mathcal{D} . This completes the proof of the lemma. ■

Lemma 21.0.2. Let X be a set; let \mathcal{D} be a collection of subsets of X that is maximal with respect to the finite intersection property. Then

1. Any finite intersection of elements of \mathcal{D} is an element of \mathcal{D} ;
2. If A is a subset of X that intersects every element of \mathcal{D} , then A is an element of \mathcal{D} .

Proof to 1

Let B be any finite intersection of elements of \mathcal{D} . Let $\mathcal{E} = \mathcal{D} \cup \{B\}$. We can show that \mathcal{E} has the finite intersection property (Check it). By maximality of \mathcal{D} , we have $\mathcal{E} = \mathcal{D}$, so $B \in \mathcal{D}$.

Proof to 2

Let A be a subset of X that intersects every element of \mathcal{D} . Let $\mathcal{E} = \mathcal{D} \cup \{A\}$. Take finitely many elements from \mathcal{E} . If none of them is A , then their intersection is not empty because \mathcal{D} has the finite intersection property, otherwise, it is of the form

$$A \cap D_1 \cap D_2 \cap \dots \cap D_n, \quad (21.3)$$

Now, $D_1 \cap D_2 \cap \dots \cap D_n \in \mathcal{D}$ by part 1, so A intersects it by hypothesis. Thus the intersection is not empty. So \mathcal{E} has the finite intersection property. By maximality of \mathcal{D} , we have $\mathcal{E} = \mathcal{D}$, so $A \in \mathcal{D}$. ■

Theorem 21.0.2 (Tychonoff theorem). *Any arbitrary product of compact spaces is compact in the product topology.*

Proof

Let

$$X = \prod_{\alpha \in J} X_\alpha, \quad (21.4)$$

where each X_α is compact. Let \mathcal{A} be a collection of subsets of X having the finite intersection property. We prove that the intersection $\bigcap_{A \in \mathcal{A}} \bar{A}$ is non-empty. Compactness of X follows.

By 21.0.1, we choose \mathcal{D} containing \mathcal{A} that is maximal with respect to the finite intersection property. It suffices to show that the intersection

$$\bigcap_{D \in \mathcal{D}} \bar{D} \quad (21.5)$$

is non-empty.

Given any $\alpha \in J$, we consider the collection

$$\{\pi_\alpha(D) | D \in \mathcal{D}\} \quad (21.6)$$

of subsets of X_α . This collection has the finite intersection property because \mathcal{D} does. By compactness of X_α , we can choose a point $x_\alpha \in X_\alpha$ such that

$$x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}. \quad (21.7)$$

Now the point $x = (x_\alpha)$ is defined in X .

Let U_β be a neighborhood of x_β in X_β . Since $x_\beta \in \overline{\pi_\beta(D)}$, we have that $\exists y \in D, \pi_\beta(y) \in U_\beta \cap \pi_\beta(D)$. Thus $y \in \pi_\beta^{-1}(U_\beta) \cap D$. Since this is true for any $D \in \mathcal{D}$, by the previous lemma, we have $\pi_\beta^{-1}(U_\beta) \in \mathcal{D}$. So every subbasis element containing x is in \mathcal{D} . And then it follows from the same lemma that every basis (finite intersection of subbasis) element containing x is in \mathcal{D} . Thus any open neighborhood of x intersects every $D \in \mathcal{D}$. Therefore, $x \in \bar{D}$ for all $D \in \mathcal{D}$. ■