

Chapter 16

Later

Definition 16.0.1 (def1). *X is compact if any open covering admits a finite subcovering.*

Definition 16.0.2 (def2: Frechet compactness). *X is limit point compact if every infinite subset of X has a limit point in X.*

Definition 16.0.3 (def3: sequential compactness/Bozano-Weierstrass compactness). *X is sequentially compact if every sequence $\{x_n\} \subseteq X$ has a convergent subsequence converging to a point in X.*

Theorem 16.0.1. *If X is a metric space, then X is compact if and only if it is limit point compact if and only if it is sequentially compact.*

Proof

The proof is quite long and technical. We will only some directions in the following.

Theorem 16.0.2. *For an arbitrary topological space, compactness implies limit point compactness. The converse is not true in general.*

Proof

Let X be compact, and let $A \subseteq X$ be infinite.

We assume that A has no limit point. Then $A = \bar{A}$. So $X \setminus A$ is open. Furthermore, for each $a \in A$, we can choose a neighborhood U_a of a such that $U_a \cap (A \setminus \{a\}) = \emptyset$.

Then $\{U_a : a \in A\} \cup \{X \setminus A\}$ is an open covering of X. Since X is compact, there exists a finite subcovering, say $\{U_{a_1}, U_{a_2}, \dots, U_{a_n}\} \cup \{X \setminus A\}$. Thus,

$$X = \bigcup_{i=1}^n U_{a_i} \cup (X \setminus A) = (X \setminus A) \cup \{a_1, a_2, \dots, a_n\},$$

which implies that A is finite, a contradiction. Hence, A has a limit point in X. ■

Example 16.0.1 (Counterexample for the converse). *Let $Y = \{p, q\}$ with anti-discrete topology $J_Y = \{\emptyset, Y\}$. Let $X = \mathbb{N} \times Y$ with the product topology where \mathbb{N} has the discrete topology. Then every non-empty set $A \subseteq X$ has a limit point. Because if $(n, p) \in A$, then any open set containing (n, q) intersects A at (n, p) . So X is limit point compact. However, X is not compact. Because $\{\{n\} \times Y : n \in \mathbb{N}\}$ is an open covering of X which admits no finite subcovering.*

Example 16.0.2 (Limit point compact but not sequentially compact). *Let X be defined as the above example. Consider the sequence $\{(n, p)\}_{n=1}^\infty$. This sequence has no convergent subsequence. Because for any point (m, p) , the open set $\{m\} \times Y$ contains only finitely many terms of the sequence; for any point (m, q) , the open set $\{m\} \times Y$ also contains only finitely many terms of the sequence. So X is not sequentially compact.*

Theorem 16.0.3. *For a first-countable topological space X, limit point compactness implies sequential compactness.*

Proof

Take a sequence $\{x_n\} \subseteq X$.

If the set of values $\{x_n : n \in \mathbb{N}\}$ is finite, then there exists a value x that appears infinitely many times in the sequence. So the subsequence constantly equal to x converges to x .

Suppose the set of values $\{x_n : n \in \mathbb{N}\}$ is infinite. Since X is first-countable, we can construct a countable basis $\{U_k\}$ at a limit point a such that

$$x_{n_1} \in U_1, x_{n_2} \in U_2, \dots, x_{n_k} \in U_k, \dots \quad (16.1)$$

and

$$U_1 \supseteq U_2 \supseteq \dots \supseteq U_k \supseteq \dots \quad (16.2)$$

That is for each open set $U \ni a$ there exists N such that $\forall k \geq N, U_k \subseteq U$. Then $x_{n_k} \in U_k \subseteq U$ for all $k \geq N$. So $x_{n_k} \rightarrow a$. ■

Remark For topological spaces, we have the following facts:

1. Compactness \Rightarrow Limit Point Compactness
2. Sequential Compactness \Rightarrow Limit Point Compactness
3. But other implications are not true in general.

Theorem 16.0.4. *Sequential compactness implies compactness for metric spaces.*

Proof This is harder than other directions.

Exercise Prove that X is sequentially compact, then it is limit point compact.

16.1 Locally Compact

Definition 16.1.1. *X is said to be locally compact at $x \in X$ if there exists an neighborhood U of x and there exists a compact subspace C of X containing U . X is locally compact if it's locally compact at each $x \in X$.*

Example 16.1.1. \mathbb{R} is locally compact. Because for any $x \in \mathbb{R}$, take $U = (x - 1, x + 1)$ and $C = [x - 1, x + 1]$ which is compact.

Example 16.1.2. \mathbb{R}^n is locally compact.

Example 16.1.3 (non example). $\mathbb{Q} \subseteq \mathbb{R}$ is not locally compact.

Exercise Show the above example.

Example 16.1.4 (non example). \mathbb{R}^ω with product topology is not locally compact. Let $U = (a_1, b_1) \times (a_2, b_2) \times \dots$. If $U \subseteq C$ where C is compact, then $\overline{U} = [a_1, b_1] \times [a_2, b_2] \times \dots \subseteq C$. We know a closed subset of a compact set is compact, so \overline{U} is compact. But this is not true.

Exercise Show the above example.

Theorem 16.1.1. *Let X be a topological space. Then X is Hausdorff locally compact if and only if there exists a spcace Y such that*

1. X is a subspace of Y .
2. $Y \setminus X$ contains exactly one point $\{p\}$.
3. Y is compact Hausdorff.

This Y is unique in the following sense: If Y, Y' are two spaces with the above properties and $Y = X \cup \{p\}, Y' = X \cup \{q\}$, then there exists a homeomorphism $h : Y \rightarrow Y'$ such that $h(x) = x$ for all $x \in X$ and $h(p) = q$.

Proof

Uniqueness Let $Y = X \cup \{p\}, Y' = X \cup \{q\}$ satisfies the above properties. Define $h : Y \rightarrow Y'$ as follows: $h(x) = x$ for all $x \in X$ and $h(p) = q$. We show that h is continuous. But the function is symmetric, so it is enough to show that $h(U)$ is open in Y' for all open set U in Y .

Take U be open. There are two cases.

If $U \subseteq X$, then we are done.

Suppose $p \in U$. Then $C = Y \setminus U$ is closed in Y . So C is compact.

Construction We introduce the topology on $Y = X \cup \{\infty\}$ as follows:

There are two types of open sets in Y :

1. $U \subseteq X \subseteq Y$ is open in X .
2. $U = Y \setminus C$ where C is compact.

We need to check that this is a topology.

For the intersection of two open sets of type 1, we have $U_1 \cap U_2$ is open in X thus is open in Y .

For the intersection of type 1 and type 2, we have $U_1 \cap (Y \setminus C) = U \cap (X \setminus C)$ which is the union of open sets in X thus is open in Y .

For the intersection of two open sets of type

It remains to show Y is compact Hausdorff and if $X \subseteq Y$ with Y satisfied the three properties, then X is locally compact.

We can show that X is a subspace of Y because:

1. If $U \subseteq X$ and $\infty \notin U$, then
2. $(Y \setminus C) \cap X = X \setminus C$ where C is compact in X .

We show that Y is compact.

Let \mathcal{A} be an open covering of Y . Then there exists compact C such that $Y \setminus C \in \mathcal{A}$. The rest of the covering $\mathcal{A}' = \mathcal{A} \setminus \{Y \setminus C\}$ is an open covering of C . Since C is compact, there exists a finite subcovering of \mathcal{A}' covering C . Thus, adding $Y \setminus C$ gives a finite subcovering of \mathcal{A} covering Y . So Y is compact.

We show that Y is Hausdorff.

Take two distinct points $x, y \in Y$. There are two cases.

1. If $x, y \in X$, since X is Hausdorff, there exist disjoint open sets U, V in X such that $x \in U, y \in V$. Then U, V are open in Y and disjoint.
2. If one of them is ∞ , say $y = \infty$. Since X is locally compact at x , there exists an open neighborhood U of x and a compact set C such that $U \subseteq C$. Then $V = Y \setminus C$ is an open neighborhood of ∞ . Clearly, $U \cap V = \emptyset$.

We prove the other direction.

To be done.

■

Theorem 16.1.2. Let X be a Hausdorff space. Then X is locally compact if and only if for any $x \in X$ and any neighborhood U of x , there exists a neighborhood V of x such that \bar{V} is compact and $\bar{V} \subseteq U$.

Proof

\Leftarrow $x \in V \subseteq \bar{V} \subseteq U$ and \bar{V} is compact. So X is locally compact at x .

\Rightarrow Let Y be the one-point compactification(compact and Hausdorff) of X . Let U be a neighborhood of x in Y . Then $Y \setminus U := C$ is a closed subset of Y . So C is compact. By lemma 12.0.2, there exists two open sets V, W in Y such that $x \in V, C \subseteq W$ and $V \cap W = \emptyset$. Then \bar{V} is compact and $\bar{V} \cap C = \emptyset$. So $\bar{V} \subseteq U$. ■

Corollary 16.1.1. Let X be locally compact Hausdorff space. $A \subseteq X$ is open or closed. Then A is locally compact.

Proof

Let $A \subseteq X$ be closed. Given $x \in A$, since X is locally compact, there exists an open neighborhood U of x in X and a compact set C such that $x \in U \subseteq C$. Then $U \cap A$ is an open neighborhood of x in A and $C \cap A$ is compact in A . So A is locally compact at x .

Let $A \subseteq X$ be open. Given $x \in A$, since X is locally compact and X is Hausdorff, by the previous theorem, there exists a neighborhood V of x such that \bar{V} is compact and $\bar{V} \subseteq A$. So A is locally compact at x . ■

Corollary 16.1.2. X is locally compact Hausdorff if and only if X is homeomorphic to an open subspace of a compact Hausdorff space.

Exercise Show the above corollary.

16.2 Urysohn's Metrization Theorem

Theorem 16.2.1. Every X that is regular(T_3) and second-countable is metrizable.

16.3 Countability

Definition 16.3.1. X is first countable if for every $x \in X$, there exists a countable basis at x . That is, there exists $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ a collection of open sets containing x such that for any open set U containing x , there exists $B_n \in \mathcal{B}$ such that $B_n \subseteq U$.

Definition 16.3.2. X is second countable if there exists a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ for the topology of X . That is, for every x and every open set U containing x , there exists $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq U$.

Exercise \mathbb{R}^n with standard topology, $B_n = \{U_\epsilon(x) | x \in \mathbb{Q}^n, \epsilon \in \mathbb{Q}_{>0}\}$. B_n is a countable basis. So \mathbb{R}^n is second countable. Show the details.

Exercise Show that if X_n 's are first(second) countable, then $\prod X_n$ with product topology is first(second) countable.

Theorem 16.3.1. Let X be second countable. Then

1. Every open cover of X has a countable subcovering (X is Lindelöf space).
2. There is a countable subset $A \subseteq X$ such that $\bar{A} = X$ ($A \subseteq X$ is dense).

Proof