Chapter 19

Later

Theorem 19.0.1. *Every second countable regular space is normal.*

Proof

Let X be a second countable regular space with $A, B \subseteq X$ closed and $A \cap B = \emptyset$.

For every $x \in A$, since X is regular, there exist open sets $U \ni x$ such that $U \cap B = \emptyset$.

By the lemma, there exists a open set V such that $x \in V$, $\bar{V} \subseteq U$ and thus $\bar{V} \cap B = \emptyset$.

Let $\mathcal{B}$ be a countable baisis for X. For each $x \in A$, there exists $U_x \in \mathcal{B}$ such that $x \in U_x \subseteq V$, and thus $\bar{U_x} \subseteq \bar{V} \subseteq U$ and $\bar{U_x} \cap B = \emptyset$. Since $\mathcal{B}$ is countable, the collection $\{U_x : x \in A\}$ has a countable subcollection $\{U_n : n \in \mathbb{N}\}$ covering A such that $\bar{U_n} \cap B = \emptyset$ for each n.

Similarly, we can choose a countable collection $\{V_m : m \in \mathbb{N}\}$ of open sets covering B, such that $\bar{V_m} \cap A = \emptyset$ for each m. Then the sets $U = \bigcup_n U_n$ and $V = \bigcup_m V_m$ are open sets containing A and B respectively. But they need not be disjoint. We perform the following simple trick to construct two open sets that are disjoint. Given n, define

$$U_n' = U_n \setminus \bigcup_{m=1}^{n} \bar{V_m}, \quad V_n' = V_n \setminus \bigcup_{k=1}^{n} \bar{U_k} \tag{19.1}$$

Note that U_n' and V_n' are open. The collection $\{U_n' : n \in \mathbb{N}\}$ covers A because each x in A belongs to some U_n and x belongs to none of the $\bar{V_m}$'s. Similarly, $\{V_n' : n \in \mathbb{N}\}$ covers B.

Finally, the open sets $U' = \bigcup_n U_n'$ and $V' = \bigcup_m V_m'$ are disjoint. For if $x \in U' \cap V'$, then there exist n, m such that $x \in U_n'$ and $x \in V_m'$. Without loss of generality, assume that $n \leq m$. Then

$$x \in U_n' \subseteq U_n \setminus \bigcup_{k=1}^{n} \bar{V_k} \subseteq U_n \setminus \bar{V_m} \tag{19.2}$$

This contradicts the fact that $x \in V_m' \subseteq V_m \subseteq \bar{V_m}$. Thus, $U' \cap V' = \emptyset$. Hence, X is normal.

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Theorem 19.0.2. *Every metrizable space X is normal.*

Proof Let d be the metric on X. Given two closed disjoint subsets $A, B \subseteq X$.

For each $a \in A$, there exists $r_a > 0$ such that $B(a, r_a) \cap B = \emptyset$.

Similarly, for each $b \in B$, there exists $r_b > 0$ such that $B(b, r_b) \cap A = \emptyset$.

Let $U = \bigcup_{a \in A} B(a, \frac{r_a}{2})$ and $V = \bigcup_{b \in B} B(b, \frac{r_b}{2})$. Then U and V are open sets containing A and B respectively.

We claim that $U \cap V = \emptyset$. Assume the contrary that there exists $x \in U \cap V$. Then there exist $a \in A$ and $b \in B$ such that $x \in B(a, \frac{r_a}{2})$ and $x \in B(b, \frac{r_b}{2})$. Without loss of generality, assume that $\frac{r_a}{2} \leq \frac{r_b}{2}$. Then

$$d(a, b) \leq d(a, x) + d(x, b) < \frac{r_a}{2} + \frac{r_b}{2} \leq r_a \tag{19.3}$$

This implies that $b \in B(a, r_a)$, which contradicts the choice of r_a. Thus, $U \cap V = \emptyset$. Hence, X is normal.

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Theorem 19.0.3. *Every compact Hausdorff space is normal.*

Proof

Let $x \in X$ and $B \subseteq X$ be closed such that $x \notin B$.

For each $y \in B$, since X is Hausdorff, there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Then $\{V_y | y \in B\}$ is an open covering of B. Since X is compact and Hausdorff, B is compact as a closed subset of X. By the compactness of B, there exists a finite subcollection $\{V_{y_1}, V_{y_2}, \dots, V_{y_n}\}$ that covers B. Let $V = \bigcup V_{y_i}$ and $U = \bigcap U_{y_i}$. Then U and V are disjoint open sets containing x and B respectively. So X is regular.

Let $A, B \subseteq X$ be closed and $A \cap B = \emptyset$. For each $a \in A$, since X is regular, there exist disjoint open sets U_a and V_a such that $a \in U_a$ and $B \subseteq V_a$. Then $\{U_a | a \in A\}$ is an open covering of A. By the compactness of A, there exists a finite subcollection $\{U_{a_1}, U_{a_2}, \dots, U_{a_m}\}$ that covers A. Let $U = \bigcup U_{a_i}$ and $V = \bigcap V_{a_i}$. Then U and V are disjoint open sets containing A and B respectively. So X is normal.

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Lemma 19.0.1 (Urysohn's lemma)**.** *Given a normal space X and two closed disjoint subsets $A, B \subseteq X$, there exists a continuous map $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof The proof is too long to be included here. Please refer to Munkres' Topology, PDF P 224. It is not required to be reproduced.

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