Chapter 10

Lecture10

10.1 Product topology on $X \times Y$

Theorem 10.1.1. *Let $f : A \to X \times Y$, let $f = (f_1, f_2)$ where $f_1 : A \to X$ and $f_2 : A \to Y$. Then f is continuous if and only if both f_1 and f_2 are continuous. We denote $f(a) = (f_1(a), f_2(a))$.*

Proof

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projection maps defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Then $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$. Since π_1 and π_2 are continuous, if f is continuous, then both f_1 and f_2 are continuous.

Conversely, suppose both f_1 and f_2 are continuous. Let U, V be open. Then $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$ for any open set U in X and V in Y. Since f_1 and f_2 are continuous, $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in A. Thus, $f^{-1}(U \times V)$ is open in A.

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There are two ways to introduce product topology.

1. Take $U_1 \subseteq X_1, U_2 \subseteq X_2, \cdots$. Then we can define the topology on $X_1 \times X_2 \times \cdots$ by the basis $\{U_1 \times U_2 \times \cdots\}$. This is called the **box topology**.

2. Take $U_1 \subseteq X_1, U_2 \subseteq X_2, \cdots$ but only finitely many of them are not equal to the whole space. Then we can define the topology on $X_1 \times X_2 \times \cdots$ by the basis $\{U_1 \times U_2 \times \cdots\}$ where only finitely many U_i are not equal to X_i. This is called the **product topology**.

Definition 10.1.1. *Let J be an arbitrary set. A J-tuple of elements from X is a function $x : J \to X$. So $\alpha \in J \mapsto x(\alpha) = x_\alpha \in X$. And sometimes we denote the J-tuple by $(x_\alpha)_{\alpha \in J}$.*

Definition 10.1.2. *Let $(A_\alpha)_{\alpha \in J}$ be an indexed family of sets.*

$$X = \bigcup_{\alpha \in J} A_\alpha \tag{10.1}$$

The Cartesian product of the family $(A_\alpha)_{\alpha \in J}$ is denoted by

$$\prod_{\alpha \in J} A_\alpha \tag{10.2}$$

which is defined as the set of all J-tuples of elements in X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$, that is, the set of all functions

$$x : J \to X \quad \text{such that} \quad x(\alpha) \in A_\alpha \forall \alpha \in J \tag{10.3}$$

When $A_\alpha = X$, we have $\prod_{\alpha \in J} A_\alpha = X^J$, the set of all functions from J to X.

Definition 10.1.3. *Let $(X_\alpha)_{\alpha \in J}$ be an indexed family of topological spaces. The box topology on $\prod_{\alpha \in J} X_\alpha$ is given by the basis*

$$\left\{ \prod_{\alpha \in J} U_\alpha : U_\alpha \text{ is open in } X_\alpha \forall \alpha \in J \right\} \tag{10.4}$$

Then taking two basis elements $\prod_{\alpha \in J} U_\alpha$ and $\prod_{\alpha \in J} V_\alpha$, their intersection is

$$\left(\prod_{\alpha \in J} U_\alpha \right) \cap \left(\prod_{\alpha \in J} V_\alpha \right) = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha) \tag{10.5}$$

which is also a basis element.

Definition 10.1.4. *A collection $\mathcal{S}$ of subsets of topological space X is a* ***subbasis*** *if*

$$\bigcup_{S\in\mathcal{S}} S = X \tag{10.6}$$

Property 10.1.1. *Let $\mathcal{S}$ be a subbasis for a space X. Then the collection of all finite intersections of elements of $\mathcal{S}$ forms a basis for the topology on X. Then the topology generated by $\mathcal{S}$ is the collection of all unions of finite intersections of elements of $\mathcal{S}$.*

Exercise Prove the above property.

Definition 10.1.5. *For a given $\beta \in J$, we denote $\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta$ the projection map defined by*

$$x \mapsto x_\beta \tag{10.7}$$

Then let $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta$ is open in $X_\beta\}$. Then

$$\mathcal{S} = \bigcup_{\beta\in J} \mathcal{S}_\beta \tag{10.8}$$

is a subbasis(Proved yourself). The topology defined by this subbasis is called the ***product topology****. The basis $\mathcal{B}$ is given by finite intersections of elements of $\mathcal{S}$ and*

$$\mathcal{B} \ni B = \prod_{\alpha\in J} U_\alpha \quad \text{where } U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \tag{10.9}$$

Notice that if $|J| < \infty$, then the box topology and the product topology are the same. And also notice that the product topology is coarser than the box topology.

Theorem 10.1.2. *Let $\mathcal{J}$ be a set of indices. Let X_α be a topological space for each $\alpha \in \mathcal{J}$. Let $A_\alpha \subseteq X_\alpha$ with subspace topology for each $\alpha \in \mathcal{J}$. Then the product/box topology on $\prod_{\alpha\in\mathcal{J}} A_\alpha$ is the subspace topology inherited from the product/ box topology on $\prod_{\alpha\in\mathcal{J}} X_\alpha$.*

In other words, let A_α be a subspace of X_α for each $\alpha \in \mathcal{J}$. Then $\prod_{\alpha\in\mathcal{J}} A_\alpha$ is a subspace of $\prod_{\alpha\in\mathcal{J}} X_\alpha$ with the product/box topology.

Theorem 10.1.3. *Let $\mathcal{J}$ be a set of indices. Let X_α be a Hausdorff topological space for each $\alpha \in \mathcal{J}$. Then the product/ box topology on $\prod_{\alpha\in\mathcal{J}} X_\alpha$ is Hausdorff.*

Theorem 10.1.4. *Let $\mathcal{J}$ be a set of indices. Let X_α be a topological space for each $\alpha \in \mathcal{J}$. Let $A_\alpha \subseteq X_\alpha$ with subspace topology for each $\alpha \in \mathcal{J}$. Then $\prod_{\alpha\in\mathcal{J}} \bar{A}_\alpha = \overline{\prod_{\alpha\in\mathcal{J}} A_\alpha}$ with the product/ box topology.*

Proof

We give a proof for the box topology case. The product topology case is similar.

Take $(x_\alpha)_{\alpha\in\mathcal{J}} = x \in \prod_{\alpha\in\mathcal{J}} \bar{A}_\alpha$. Let $\prod_{\alpha\in\mathcal{J}} U_\alpha$ be a basis open set containing x where U_α is open in X_α. Since $x_\alpha \in \bar{A}_\alpha$, there exists $y_\alpha \in U_\alpha \cap A_\alpha$ for each $\alpha \in \mathcal{J}$. Thus, $(y_\alpha)_{\alpha\in\mathcal{J}} \in \prod_{\alpha\in\mathcal{J}} U_\alpha \cap \prod_{\alpha\in\mathcal{J}} A_\alpha$. Therefore, $x \in \overline{\prod_{\alpha\in\mathcal{J}} A_\alpha}$. So we have $\prod_{\alpha\in\mathcal{J}} \bar{A}_\alpha \subseteq \overline{\prod_{\alpha\in\mathcal{J}} A_\alpha}$.

Conversely, take $x = (x_\alpha)_{\alpha\in\mathcal{J}} \in \overline{\prod_{\alpha\in\mathcal{J}} A_\alpha}$. Let β be an arbitrary index in $\mathcal{J}$. Take V_β be an open set containing x_β. Then its preimage $\pi_\beta^{-1}(V_\beta)$ is open in $\prod_{\alpha\in\mathcal{J}} X_\alpha$. Since $x \in \overline{\prod_{\alpha\in\mathcal{J}} A_\alpha}$, there exists $y = (y_\alpha)_{\alpha\in\mathcal{J}} \in \pi_\beta^{-1}(V_\beta) \cap \prod_{\alpha\in\mathcal{J}} A_\alpha$. So $y_\beta \in V_\beta \cap A_\beta$. Thus, $x_\beta \in \bar{A}_\beta$. Since β is arbitrary, we have $x \in \prod_{\alpha\in\mathcal{J}} \bar{A}_\alpha$. Therefore, $\overline{\prod_{\alpha\in\mathcal{J}} A_\alpha} \subseteq \prod_{\alpha\in\mathcal{J}} \bar{A}_\alpha$.

Two inclusions together give the desired equality.

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Theorem 10.1.5. *Let $f : A \to \prod_{\alpha\in\mathcal{J}} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha\in\mathcal{J}}$ where $f_\alpha : A \to X_\alpha$ for each $\alpha \in \mathcal{J}$. Then f is continuous if and only if each f_α is continuous.*

Proof

Let π_β be the projecction of the product onto its β-th factor. Then function π_β is continuous because for any open set U_β in X_β, we have $\pi_\beta^{-1}(U_\beta)$ is a subbaisis element for the product topology. Since $f_\beta = \pi_\beta \circ f$, if f is continuous, then each f_β is continuous.

Conversely, To prove f is continuous, it suffices to show that the preimage of each subbasis element is open in A. Take any subbasis element $\pi_\beta^{-1}(U_\beta)$ where U_β is open in X_β. Then

$$f^{-1}(\pi_\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta) \tag{10.10}$$

Since f_β is continuous, $f_\beta^{-1}(U_\beta)$ is open in A. Thus, f is continuous.

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Example 10.1.1. *For box topology, the above theorem may fail.*

Consider $\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself. Let us define $f : \mathbb{R} \to \mathbb{R}^\omega$ by

$$f(t) = (t, t, t, \ldots) \tag{10.11}$$

Then each coordinate function $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(t) = t$ is continuous; therefore the function f is continuous given the product topology on $\mathbb{R}^\omega$. However, f is not continuous when $\mathbb{R}^\omega$ is given the box topology. Consider, the basis element

$$B = \prod_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) \tag{10.12}$$

for the box topology on $\mathbb{R}^\omega$.

We assert that $f^{-1}(B)$ is not open in $\mathbb{R}$. If it were open, then there would exist $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq f^{-1}(B)$. This would mean $f((-\epsilon, \epsilon)) \subseteq B$, so that

$$f_n((-\epsilon, \epsilon)) = (-\epsilon, \epsilon) \subseteq \left(-\frac{1}{n}, \frac{1}{n}\right) \tag{10.13}$$

for each n. But this is impossible for n sufficiently large that $\frac{1}{n} < \epsilon$. Thus, $f^{-1}(B)$ is not open, and f is not continuous.