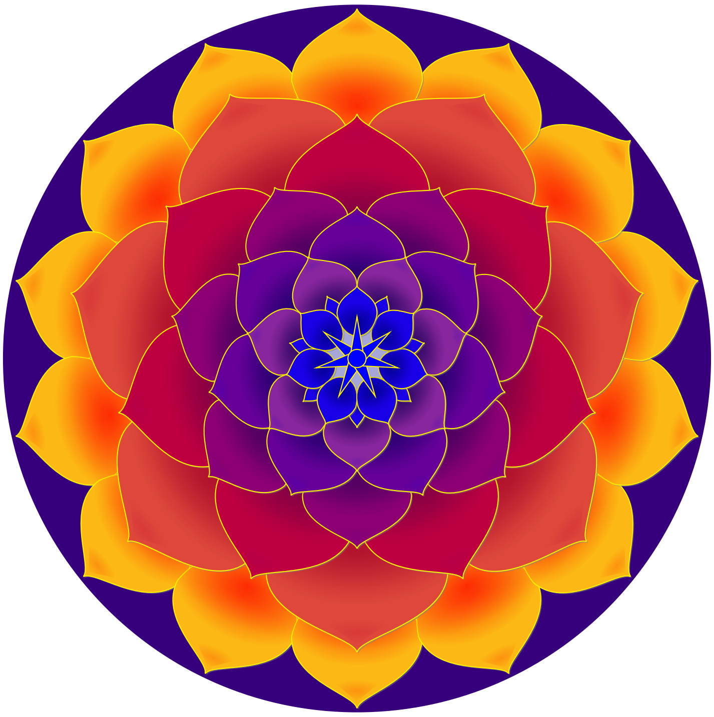

Metric and Topological Space

Mathematics With Computer Science

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Contents

Chapter 1

Metric Spaces

Definition 1.0.1. *A* metric space *is a set X with a function $d : X \times X \to \mathbb{R}_{\geq 0}$(d is the metric, or the distance function) satisfying the following properties for all $x, y, z \in X$:*

- ***Non-negativity:*** *$d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.*
- ***Symmetry:*** *$d(x, y) = d(y, x)$.*
- ***Triangle inequality:*** *$d(x, z) \leq d(x, y) + d(y, z)$.*

Example 1.0.1. *$X = \mathbb{R}$, $d(x, y) = \|x - y\|$*

Example 1.0.2. *X is finite, $d(x, y) = 1$ if $x \neq y$, $d(x, x) = 0$ (discrete metric)*

Example 1.0.3. *X is a finite set of vertices of a connected graph, $d(x, y)$ is the length of the shortest path between x and y (graph metric)*

Example 1.0.4. *$X = \mathbb{R}^n$, $d(x, y) = \|x - y\|$ (Euclidean metric)*

Example 1.0.5. *$X = \mathbb{R}^n$, $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ (manhattan metric)*

Example 1.0.6. *$X = \mathbb{R}^n$, $d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ (sup metric)*

Exercise Prove that the above examples are indeed metric spaces(You may use Cauchy-Schwarz inequality which is stated later).

Example 1.0.7. *$X = \mathbb{R}^2$, $P = (x, y), P' = (x', y')$*

$$d(P, P') = \begin{cases} |x - x'| + |y| + |y'| & \textit{if } x \neq x' \\ |y - y'| & \textit{if } x = x' \end{cases}$$

Example 1.0.8. *$X = C[0, 1]$ (the set of continuous functions on $[0, 1]$), $d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|$ (sup metric)*

Example 1.0.9. *$X = C[0, 1]$, $d(f, g) = \int_0^1 |f(x) - g(x)| dx$ (L^1 metric)*

Remark We cannot replace X by the set of all integrable functions, because the distance between two functions may be zero even if they are not equal (they may differ on a set of measure zero).

Definition 1.0.2 (Normed Space)**.** *Let V be a vector space over $\mathbb{R}$. A* norm *on V is a function $\| \cdot \| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $u, v \in V$ and all $a \in \mathbb{R}$:*

- ***Non-negativity:*** *$\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.*
- ***Absolute homogeneity:*** *$\|av\| = |a|\|v\|$.*
- ***Triangle inequality:*** *$\|u + v\| \leq \|u\| + \|v\|$.*

Remark A norm defines a metric by $d(u, v) = \|u - v\|$ on V.

Example 1.0.10. *$\|v\| = \sqrt{\sum_{i=1}^{n} |v_i|^2}$ (Euclidean norm)*

Example 1.0.11. *$\|v\| = \sum_{i=1}^{n} |v_i|$ (manhattan norm)*

Example 1.0.12. *$\|v\| = \max_{1 \leq i \leq n} |v_i|$ (sup norm)*

Example 1.0.13. *$V = C[0,1]$, $\|f\| = \max_{x\in[0,1]} |f(x)|$ (sup norm)*

Example 1.0.14. *$V = C[0,1]$, $\|f\| = \int_0^1 |f(x)|dx$ (L^1 norm)*

Definition 1.0.3 (l_p-Spaces)**.** *Let $1 \le p < \infty$.*

- *$l_\infty = \{x = (x_1, x_2, \ldots) : \sup_i |x_i| < \infty\}$ with norm $\|x\|_\infty = \sup_i |x_i|$.*
- *$l_1 = \{x = (x_1, x_2, \ldots) : \sum_i |x_i| < \infty\}$ with norm $\|x\|_1 = \sum_i |x_i|$.*
- *$l_2 = \{x = (x_1, x_2, \ldots) : \sum_i |x_i|^2 < \infty\}$ with norm $\|x\|_2 = \sqrt{\sum_i |x_i|^2}$.*
- *$l_p = \{x = (x_1, x_2, \ldots) : \sum_i |x_i|^p < \infty\}$ with norm $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$.*

Property 1.0.1. *$l_p \subseteq l_q$ if $p < q$*

Example 1.0.15. *$x_n = \dfrac{1}{n}$. $\sum_1^\infty \dfrac{1}{n} = \infty$ but $\sum_1^\infty \dfrac{1}{n^2} < \infty$. So $x = (1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots) \in l_2$ but $x \notin l_1$.*

Definition 1.0.4 (Euclidean Space/Vector Space with Inner Product)**.** *Let V be a vector space over $\mathbb{R}$. An* inner product *on V is a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ satisfying the following properties for all $u, v, w \in V$ and all $a \in \mathbb{R}$:*

- ***Positivity:*** *$\langle v, v\rangle \ge 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$.*
- ***Symmetry:*** *$\langle u, v\rangle = \langle v, u\rangle$.*
- ***Bilinearity:*** *$\langle au + w, v\rangle = a\langle u, v\rangle + \langle w, v\rangle$.*

Remark. We always use Euclidean Space as the default example of inner product space.

Theorem 1.0.1 (Cauchy-Schwarz Inequality)**.** *For all $u, v \in V$, $|\langle u, v\rangle| \le \|u\|\|v\|$.*

Proof Let $f(t) = \|u + tv\|^2 = \langle u + tv, u + tv\rangle$. Then using the properties of quadratic polynomials. ■

Property 1.0.2. *$\|v\| = \sqrt{\langle v, v\rangle}$ is a norm on V.*

Proof. Use Cauchy-Schwarz inequality: $|\langle u, v\rangle| \le \|u\|\|v\|$. ■

Corollary 1.0.1. *l_2-norm satisfies the triangle inequality.*

Proof. l_2 is an Euclidean Space(infinite dimenstions) with $\langle x, y\rangle = \sum_i x_i y_i$ and $\|x\|_2 = \sqrt{\sum_i |x_i|^2}$. ■

Remark. $\|x\|_p$ does not come from an inner product if $p \ne 2$.

Theorem 1.0.2. *l_1 and l_∞ are normed spaces.*

Proof
The case for l_1 is left as an exercise. For l_∞, we observe that $\|x\|_\infty = \sup_i |x_i|$.

Theorem 1.0.3. *l_p are normed spaces.*

Proof. This will be proved later in the course.

Chapter 2

Later

Definition 2.0.1. *For $p > 1, q > 1$. p is conjugate to q if $\frac{1}{p} + \frac{1}{q} = 1$.*

Remark When $p = 1$, $q = \infty$; when $p = \infty$, $q = 1$. And 2 is conjugate to itself.

Lemma 2.0.1 (Jensen's inequality)**.** *If $f(t)$ is a strictly concave function on an interval $I \subseteq \mathbb{R}$, then for any $t_1, t_2 \in I$ and any $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$, we have*

$$f(\lambda_1 t_1 + \lambda_2 t_2) \geq \lambda_1 f(t_1) + \lambda_2 f(t_2) \tag{2.1}$$

where the equality holds if and only if $t_1 = t_2$.

Proof Obviuos.

Theorem 2.0.1 (Young's inequality)**.** *For $a, b \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.2}$$

Proof
Let $x = a^p, y = b^q$. Then $a = x^{\frac{1}{p}}, b = y^{\frac{1}{q}}$. Substituting these into the inequality, we need to show that

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}, \forall x, y \geq 0 \tag{2.3}$$

If $x = 0$ or $y = 0$, the inequality holds trivially. Now we assume $x, y > 0$. Let $f(t) = \ln t$, which is strictly concave on $(0, +\infty)$. By Jensen's inequality, we have

$$f\left(\frac{x}{p} + \frac{y}{q}\right) \geq \frac{1}{p} f(x) + \frac{1}{q} f(y) \tag{2.4}$$

Exponentiating both sides, we have

$$\frac{x}{p} + \frac{y}{q} \geq x^{\frac{1}{p}} y^{\frac{1}{q}} \tag{2.5}$$

■

Theorem 2.0.2 (Hölder's inequality)**.** *For $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, for any $a = (a_i) \in l_p, b = (b_i) \in l_q$ respectively, we have*

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \sqrt[\frac{1}{p}]{\sum a_k^p} \sqrt[\frac{1}{q}]{\sum b_k^q}$$

Proof
In case when $\|a\|_p = 0$ or $\|b\|_q = 0$, the inequality holds trivially. Now we assume $\|a\|_p > 0$ and $\|b\|_q > 0$. Let

$$x_k = \frac{|a_k|}{\|a\|_p}, \quad y_k = \frac{|b_k|}{\|b\|_q}, \quad \forall k \geq 1 \tag{2.6}$$

We have

$$\sum_{k=1}^{\infty} x_k^p = \frac{|a_k|^p}{\|a\|_p^p} = 1, \quad \sum_{k=1}^{\infty} y_k^q = \frac{|b_k|^q}{\|b\|_q^q} = 1 \tag{2.7}$$

By Young's inequality, we have

$$x_k y_k \leq \frac{x_k^p}{p} + \frac{y_k^q}{q}, \quad \forall k \geq 1 \tag{2.8}$$

Summing over k from 1 to ∞, we have

$$\sum_{k=1}^{\infty} x_k y_k \leq \frac{1}{p} \sum_{k=1}^{\infty} x_k^p + \frac{1}{q} \sum_{k=1}^{\infty} y_k^q = \frac{1}{p} + \frac{1}{q} = 1 \tag{2.9}$$

Thus, we have

$$\sum_{k=1}^{\infty} |a_k b_k| = \|a\|_p \|b\|_q \sum_{k=1}^{\infty} x_k y_k \leq \|a\|_p \|b\|_q \tag{2.10}$$

For the case when $p = 1$ and $q = \infty$, we have

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \sum_{k=1}^{\infty} |a_k| \|b\|_\infty = \|a\|_1 \|b\|_\infty \tag{2.11}$$

■

Theorem 2.0.3 (Minkowski inequality). *For $p \geq 1$, for any $x, y \in l_p$, we have*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Proof

If $\sum_{k=1}^{\infty} |x_k + y_k|^p = 0$, then $x_k + y_k = 0$ for all k. Thus, $\|x\|_p = \|y\|_p = 0$ and the inequality holds trivially. Now we assume $\sum_{k=1}^{\infty} |x_k + y_k|^p > 0$.

Let us first show that for $x = (x_i), y = (y_i) \in l_p$, we have

$$\sqrt[\frac{1}{k}]{|x_k + y_k|^p} \leq \sqrt[\frac{1}{k}]{|x_k|^p} + \sqrt[\frac{1}{k}]{|y_k|^p}$$

For every summand, we have

$$\begin{aligned} |x_k + y_k|^p &= |x_k + y_k| \cdot |x_k + y_k|^{p-1} \\ &\leq |x_k| \cdot |x_k + y_k|^{p-1} + |y_k| \cdot |x_k + y_k|^{p-1} \end{aligned}$$

So we have

$$\sum_{k=1}^{n} |x_k + y_k|^p \leq \sum_{k=1}^{n} |x_k| \cdot |x_k + y_k|^{p-1} + \sum_{k=1}^{n} |y_k| \cdot |x_k + y_k|^{p-1} \tag{2.12}$$

Let $a_k = |x_k|, b_k = |x_k + y_k|^{p-1}$, then $b_k^q = |x_k + y_k|^p$. By Hölder's inequality, we have

$$\sum_{k=1}^{n} |x_k| \cdot |x_k + y_k|^{p-1} \leq \sqrt[\frac{1}{p}]{\sum_{k=1}^{n} |x_k|^p} \sqrt[\frac{1}{q}]{\sum_{k=1}^{n} |x_k + y_k|^p}$$

On the other hand, let $a_k = |y_k|, b_k = |x_k + y_k|^{p-1}$, we have

$$\sum_{k=1}^{n} |y_k| \cdot |x_k + y_k|^{p-1} \leq \sqrt[\frac{1}{p}]{\sum_{k=1}^{n} |y_k|^p} \sqrt[\frac{1}{q}]{\sum_{k=1}^{n} |x_k + y_k|^p}$$

Combining these two inequalities, we have

$$\sum_{k=1}^{n} |x_k + y_k|^p \leq \left(\sqrt[\frac{1}{p}]{\sum_{k=1}^{n} |x_k|^p} + \sqrt[\frac{1}{p}]{\sum_{k=1}^{n} |y_k|^p} \right) \sqrt[\frac{1}{q}]{\sum_{k=1}^{n} |x_k + y_k|^p} \tag{2.13}$$

Let $S_n = \sum_{k=1}^{n} |x_k + y_k|^p$, then $S_n \leq C S_n^{\frac{1}{q}}$ If $S_n = 0$, then $S_n \leq C^p$. If $S_n > 0$, then $S_n^{1-\frac{1}{q}} = S_n^{\frac{1}{p}} \leq C$ which implies $S_n \leq C^p$. So S_n is increasing and bounded above. Thus, $\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} |x_k + y_k|^p$ exists and is finite.

Dividing both sides by $\sqrt[\frac{1}{p}]{\sum_{k=1}^{n}|x_k+y_k|^p}$, we have

$$\sqrt[\frac{1}{p}]{\sum_{k=1}^{n}|x_k+y_k|^p} \leq \sqrt[\frac{1}{p}]{\sum_{k=1}^{n}|x_k|^p} + \sqrt[\frac{1}{p}]{\sum_{k=1}^{n}|y_k|^p} \tag{2.14}$$

It's easy to show that each term on the right-hand side converges as $n \to \infty$ because $x, y \in l_p$ and we know that a monotonically increasing sequence bounded above converges. So we have that the both sides converge as $n \to \infty$.

Taking limit $n \to \infty$, by continuity of n-th root, we have the desired result.

■

Let X be a metric space. Let $x \in X$. We have the following definitions.

Definition 2.0.2. *We define a neighborhood of x to be a set of the form*

$$U_\epsilon(x) = \{y \in X | d(x,y) < \epsilon\}$$

for some $\epsilon > 0$.

Definition 2.0.3. *We define a punctured neighborhood of x to be a set of the form*

$$U_\epsilon^*(x) = \{y \in X | 0 < d(x,y) < \epsilon\} = U_\epsilon(x) \setminus \{x\}$$

for some $\epsilon > 0$.

Definition 2.0.4. *We say that $M \subseteq X$ is open in X if for every $x \in M$, there exists $\epsilon > 0$ such that $U_\epsilon(x) \subseteq M$.*

Remark $\emptyset, X$ are open in X by definition.

Example 2.0.1. *Is it possible that in a metric space X, a ball is contained properly inside a ball with smaller radius? That is, is there $x \in X$ and $0 < r < s$ such that $U_s(x) \subsetneq U_r(x)$? [Hint: If $Y \subseteq X$ and (X,d) is a metric space, then (Y,d) is also a metric space.]*

Solution Yes. Let $X = (-1,1)$ with the usual metric. Then $U_{\frac{3}{2}}(\dfrac{4}{3}) = (-\frac{4}{3}, 1) \subsetneq U_1(0) = (-1,1)$.

Example 2.0.2. *Draw balls centered at 0 in $\mathbb{R}^2$ with norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$.*

Example 2.0.3. *"Amazon Metric" on $\mathbb{R}^2$ is given by*

$$d((x_1,y_1),(x_2,y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2 \end{cases}$$

We will solve these problems later in the course.

Theorem 2.0.4.

1. *The intersection of finitely many open sets is open, that is, $U_1 \cap U_2 \cap \cdots \cap U_n$ is open where each U_i is open in X.*

2. *The union of any collection of open sets is open, that is, if $\{U_i\}_{i \in I}$ is a collection of open sets in X, then $\bigcup_{i \in I} U_i$ is open.*

Proof to 1.

If $V = U_1 \cap U_2 \cap \cdots \cap U_n = \emptyset$, then V is open by definition.

If $V \neq \emptyset$, let $x \in V$. Since $x \in U_i$ for each $i = 1, 2, \ldots, n$, there exists $\epsilon_i > 0$ such that $U_{\epsilon_i}(x) \subseteq U_i$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. Then $U_\epsilon(x) \subseteq U_i$ for each i, so $U_\epsilon(x) \subseteq V$. Thus, V is open.

Proof to 2.

Let $x \in U = \bigcup_{i \in I} U_i$. Then there exists some $j \in I$ such that $x \in U_j$. Since U_j is open, there exists $\epsilon > 0$ such that $U_\epsilon(x) \subseteq U_j \subseteq U$. Thus, U is open.

Chapter 3

Later

Definition 3.0.1. *$x \in M$ is an interior point of M if there exists $\epsilon > 0$ such that $U_\epsilon(x) \subseteq M$. (We denote $Int(M)$ as the set of all interior points of M.)*

Definition 3.0.2. *M is open if $M = Int(M)$.*

Theorem 3.0.1. *$U \subseteq \mathbb{R}$ is open $\Longleftrightarrow$ U is a union of at most countably many disjoint open intervals.*

Example 3.0.1 (discrete metric)**.** *Let (X, d) be a metric space where*

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

for all $x, y \in X$. Then every subset of X is open(e.g. a set of a single point, its neighborhood has radius smaller than 1).

Remark In $\mathbb{R}^n$, any "reasonable" set of strict inequalities defines an open set. For example, $\{x \in \mathbb{R}^n | x_1 > 0\}$ or $\{(x,y) \in \mathbb{R}^2 | 2x > y, y^2 + y > 3x^2 - 2\}$ are open. Another example: $X = \mathbb{R}^{n^2} \backsimeq Mat(n)$. Then the set $\{A \subseteq Mat(n) | det(A) \neq 0\}$ is open where A is non-degenerate. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad det(A) \neq 0 \tag{3.1}$$

Metric on $Mat(n)$ is defined: $||A|| = \max(|a_{ij}|)$. Then we have

$$\begin{aligned} U_\epsilon(A) &= \{B \in Mat(n) | ||A - B|| < \epsilon\} \\ &= \{B \in Mat(n) | \max(|a_{ij} - b_{ij}|) < \epsilon\} \\ &= \{ \begin{pmatrix} a_{11}+\epsilon_{11} & a_{12}+\epsilon_{12} & \cdots & a_{1n}+\epsilon_{1n} \\ a_{21}+\epsilon_{21} & a_{22}+\epsilon_{22} & \cdots & a_{2n}+\epsilon_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+\epsilon_{n1} & a_{n2}+\epsilon_{n2} & \cdots & a_{nn}+\epsilon_{nn} \end{pmatrix} | \epsilon_{ij} < \epsilon\} \end{aligned} \tag{3.2}$$

Then we have

$$det(B) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} (a_{i\sigma(i)} + \epsilon_{i\sigma(i)}) \tag{3.3}$$

Let ϵ be small enough such that $\epsilon < ||A||$. Then We can estimate one summand of $det(B)$ as follows:

$$|(a_{11} + \epsilon_{11})(a_{12} + \epsilon_{12}) \cdots (a_{1n} + \epsilon_{1n}) - a_{11}a_{12} \cdots a_{1n}| < ||A||^{n-1} \cdot (2^n - 1)\epsilon \tag{3.4}$$

Sum up all estimates, we have

$$|det(A) - det(B)| < n! ||A||^{n-1} \cdot (2^n - 1)\epsilon \tag{3.5}$$

Taking ϵ small enough such that

$$n! ||A||^{n-1} \cdot (2^n - 1)\epsilon < |det(A)| \tag{3.6}$$

ensures that $det(B) > 0$. Thus, $U_\epsilon(A) \subseteq \{A \in Mat(n) | det(A) \neq 0\}$. So $\{A \in Mat(n) | det(A) \neq 0\}$ is open.

■

Definition 3.0.3. *A point $x \in X$ (not neccessarily in M) is a limit point of M if for every $\epsilon > 0$, the punctured neighborhood $U^*_\epsilon(x)$ contains a point of M ($U^*_\epsilon(x) \cap M \neq \emptyset$) or (the neighborhood $U_\epsilon(x)$ contains infinitely many points of M).*

Definition 3.0.4. *$x \in M$ is isolated if $\exists \epsilon > 0$ such that $U_\epsilon(x) \cap M = \{x\}$.*

Definition 3.0.5. *The closure of M, denoted by $\overline{M}$, is the union of the set of all limit points of M and M itself.*

Definition 3.0.6. *A set M is closed if M contains all its limit points.*

Property 3.0.1. *$M = \overline{M} \iff M$ is closed*

Exercise Show that for any $M \subseteq X$, we have $\overline{M} = \overline{\overline{M}}$.

Property 3.0.2. *If M is open in X, then $CM = X \setminus M$ is closed in X. Conversely, if M is closed in X, then CM is open in X.*

Exercise Prove the above property.

Property 3.0.3. *$M \subseteq X$ is closed $\iff \forall y \in X \setminus M, \exists \epsilon > 0$ such that $U_\epsilon(y) \cap M = \emptyset$. (any point can be seperated from M).*

Proof Suppose $\exists y \in X \setminus M$ that cannot be seperated from M. Then $\forall \epsilon > 0, U_\epsilon(y) \cap M \neq \emptyset$. But $y \notin M$, so $U^*_\epsilon(y) \cap M \neq \emptyset$. So y is a limit point of M which implies that $y \in \overline{M}$. Since M is closed, we have a contradiction.

Suppose every $y \in X \setminus M$ can be seperated from M. Let x be a limit point of M. If $x \notin M$, then by assumption, there exists $\epsilon > 0$ such that $U_\epsilon(x) \cap M = \emptyset$. This contradicts the definition of limit point.

Example 3.0.2. *We will see that $\{det(A) = 0\}$ is closed in $Mat(n)$. So every matrix with $det(A) \neq 0$ can be seperated from the set $\{det(A) = 0\}$.*

Definition 3.0.7. *The function $f : X_1 \to X_2$ between two metric spaces $(X_1, d_1), (X_2, d_2)$ is continuous if for every x, $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x' \in X_1$ with $d_1(x, x') < \delta$, we have $d_2(f(x), f(x')) < \epsilon$.*

Definition 3.0.8 (Alternative definition of continuity)**.** *The function $f : X_1 \to X_2$ between two metric spaces $(X_1, d_1), (X_2, d_2)$ is continuous if for every open set $U \subseteq X_2$, the preimage $f^{-1}(U) = \{x \in X_1 | f(x) \in U\}$ is open in X_1.*

Theorem 3.0.2. *The two definitions of continuity are equivalent.*

Proof Let $f : X_1 \to X_2$ be continuous in the first definition. Let $U \subseteq X_2$ be open. Let $x \in f^{-1}(U)$. Then $f(x) \in U$. Since U is open, there exists $\epsilon > 0$ such that $U_\epsilon(f(x)) \subseteq U$. By continuity of f, there exists $\delta > 0$ such that $\forall x' \in X_1$ with $d_1(x, x') < \delta$, we have $d_2(f(x), f(x')) < \epsilon$, i.e. $f(x') \in U_\epsilon(f(x)) \subseteq U$. Thus, $x' \in f^{-1}(U)$. So we have $U_\delta(x) \subseteq f^{-1}(U)$. Thus, $f^{-1}(U)$ is open in X_1.

Let $f : X_1 \to X_2$ be continuous in the second definition. Let $x \in X_1$. Let $\epsilon > 0$. Consider the open set $U_\epsilon(f(x)) \subseteq X_2$. By continuity of f, the preimage $f^{-1}(U_\epsilon(f(x)))$ is open in X_1. Since $x \in f^{-1}(U_\epsilon(f(x)))$, there exists $\delta > 0$ such that $U_\delta(x) \subseteq f^{-1}(U_\epsilon(f(x)))$. Thus, for every $x' \in X_1$ with $d_1(x, x') < \delta$, we have $f(x') \in U_\epsilon(f(x))$, i.e. $d_2(f(x), f(x')) < \epsilon$. Thus, f is continuous in the first definition.

Question Suppose $f : X \to Y$ is continuous and bijective, is the inverse $f^{-1} : Y \to X$ also continuous?

Answer Not necessarily. For example, let $X = \{0\} \cup (1, 2]$ and $Y = [0, 1]$. Define $f : X \to Y$ by

$$f(x) = \begin{cases} 0, & x = 0 \\ x - 1, & x \in (1, 2] \end{cases}$$

Then f is continuous and bijective. However, $\{0\} \subseteq X$ is open but its preimage $f^{-1}(\{0\}) = \{0\}$ is not open in X.

There's another example: Let $X = [0, 1]$ with discrete metric(i.e., $d(x, y) = 1, \forall x, y \in X$ with $x \neq y$) and $Y = [0, 1]$ with standard metric. Define $f : X \to Y$ by $f(x) = x$. Then f is continuous and bijective. But its inverse is not continuous.

Chapter 4

Later

Definition 4.0.1. *A continuous bijective map $f : X \to Y$ is called a homeomorphism if f^{-1} is also continuous.*

Remark "homeomorphism" is not the same as "homomorphism".

Definition 4.0.2. *$X \simeq Y$ are homeomorphic or topological equivalent if there exists a homeomorphism $f : X \to Y$*

Definition 4.0.3. *Two metric spaces (X, d_X) and (Y, d_Y) are called isometric if there exists a bijective map $f : X \to Y$ such that for all $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$. ($f$ is called an isometry)*

Exercise Prove that every isometry is a homeomorphism, but some homeomorphisms are not isometries.

Example 4.0.1. *Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ be the unit disk in $\mathbb{R}^2$ with the Euclidean metric. Let $S^1 = \{(x, y) \in \mathbb{R}^2 | \max |x|, |y| = 1\}$ be the unit square in $\mathbb{R}^2$ with the Euclidean metric. Then D and S^1 are homeomorphic but not isometric.*

Proof

The idea of prove homeomorphism is to map every radius of the disk to the corresponding line segment of the square.

Assume there exists an isometry $f : D \to S^1$. Note that the diameter of D is 2, while the diameter of S^1 is $2\sqrt{2}$. This contradicts the definition of isometry. Thus, no such isometry exists.

Example 4.0.2. *Another example of homeomorphism is the linear map from the interval $(0, 1)$ to (a, b) (by stretching and compressing).*

Example 4.0.3. $(0, 1) \simeq (-\frac{\pi}{2}, \frac{\pi}{2}) \simeq \mathbb{R}$ *by the tangent function.*

The above example can be described graphically by a bowl(a circle centered at $(0, \frac{1}{2})$ with radius $\frac{1}{2}$ without the upper half) with radius $\frac{1}{2}$ on the real line.

Property 4.0.1. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Corollary 4.0.1. *If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then $g \circ f : X \to Z$ is a homeomorphism.*

Property 4.0.2. *Homeomorphism $\simeq$ is an equivalence relation.*

Definition 4.0.4. *X is said to be path-connected if any two points $x, y \in X$ can be joined by a path: there exists a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.*

Definition 4.0.5. *Let X be a metric space. X is said to be connected if one of the following equivalent conditions holds:*

1. *X cannot be represetned as $X = U_1 \sqcup U_2$ where U_1, U_2 are non-empty open subsets of X.*
2. *X cannot be represented as $X = V_1 \sqcup V_2$ where V_1, V_2 are non-empty closed subsets of X.*
3. *There is no proper non-empty subset $U \subseteq X$ which is both open and closed in X.*

Property 4.0.3. *If X is path-connected and $f : X \to Y$ is a homeomorphism, then Y is also path-connected.*

Proof Let $y_1, y_2 \in Y$. Since f is bijective, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is path-connected, there exists a continuous map $g : [0,1] \to X$ such that $g(0) = x_1$ and $g(1) = x_2$. Consider the map $h = f \circ g : [0,1] \to Y$. Since both f and g are continuous, h is continuous. Moreover, $h(0) = f(g(0)) = f(x_1) = y_1$ and $h(1) = f(g(1)) = f(x_2) = y_2$. Thus, there exists a continuous path in Y connecting y_1 and y_2, proving that Y is path-connected.

■

Theorem 4.0.1 (Intermediate Value Theorem). *Let $f : [a,b] \to \mathbb{R}$ be continuous. For any c between $f(a)$ and $f(b)$, there exists some $x \in [a,b]$ such that $f(x) = c$.*

Example 4.0.4. $\mathbb{R} \setminus \{0\}$ *is not path-connected.*

Proof By IVT.

Theorem 4.0.2 (Intermediate Value Theorem for Path-Connectedness). *Let X be a path-connected space and $f : X \to \mathbb{R}$ be continuous. Suppose there exist $x_1, x_2 \in X$ such that $f(x_1) = c_1$ and $f(x_2) = c_2$. Then for any c between c_1 and c_2, there exists some $x \in X$ such that $f(x) = c$.*

Proof Since X is path-connected, there exists a continuous map $g : [0,1] \to X$ such that $g(0) = x_1$ and $g(1) = x_2$. Consider the map $h = f \circ g : [0,1] \to \mathbb{R}$. Since both f and g are continuous, h is continuous. Moreover, $h(0) = f(g(0)) = f(x_1) = c_1$ and $h(1) = f(g(1)) = f(x_2) = c_2$. By the Intermediate Value Theorem, for any c between c_1 and c_2, there exists some $t \in [0,1]$ such that $h(t) = c$. Let $x = g(t)$. Then $f(x) = f(g(t)) = h(t) = c$. Thus, there exists some $x \in X$ such that $f(x) = c$.

■

Property 4.0.4. *Let X be a metric space. Then IVT for X holds if and only if there is no continuous and surjective map $f : X \to \{0,1\}$.*

Proof

We assume that X contains more than one point.

One direction is trivial.

Let's assume that the IVT doesn't hold for X.

Since the IVT does not hold for X, there exists $y_1, y_2 \in \mathbb{R}$ such that there exists y_3 between y_1 and y_2 which is not in the range of f. Then $f : X \to \mathbb{R} \setminus \{y_3\}$. Now we can easily define a continuous and surjective map $\tilde{f} : X \to \{0,1\}$.

■

Property 4.0.5. *Let X be a metric space. X is connected if and only if IVT for X holds.*

Proof

The statement is equivalent to: X is connected if and only if there is no continuous and surjective map $f : X \to \{0,1\}$.

Suppose there is a continuous and surjective map $f : X \to \{0,1\}$. Then $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are non-empty, disjoint, open subsets of X whose union is X. Thus, X is not connected.

Suppose that there is no continuous and surjective map $f : X \to \{0,1\}$. If X is not connected, then there exist non-empty, disjoint, open subsets U_1, U_2 of X such that $X = U_1 \sqcup U_2$. We can define a map $f : X \to \{0,1\}$ by setting $f(x) = 0$ if $x \in U_1$ and $f(x) = 1$ if $x \in U_2$. This map is continuous and surjective, contradicting our assumption. Therefore, X must be connected.

■

Corollary 4.0.2. *Let X be a metric space. If X is path-connected, then X is connected.*

Proof Since X is path-connected, IVT for X holds. Thus, X is connected.

■

Remark IVT holds for X doesn't implies X is path-connected.

Example 4.0.5. *This is an example of a connected space but not path-connected:*

$$X = \{(x, \sin\frac{1}{x})|x > 0\} \cup (\{0\} \times [-1,1]) \tag{4.1}$$

Chapter 5

Later

Theorem 5.0.1. *The following are equivalent:*

1. *The Intermediate Value Theorem holds for X: for any continuous map $f : X \to \mathbb{R}$, and any c between $f(x_1)$ and $f(x_2)$ for some $x_1, x_2 \in X$, there exists some $x \in X$ such that $f(x) = c$.*
2. *There is no continuous surjection from X onto a discrete two-point space $\{0, 1\}$.*
3. *X cannot be partitioned into two nonempty disjoint open sets.*
4. *X cannot be partitioned into two nonempty disjoint closed sets.*
5. *Every proper subset of X that is both open and closed is either $\emptyset$ or X itself.*

Remark If X is path-connected, then it is connected. The converse is not necessarily true.

Lemma 5.0.1. *The closure of a connected set is connected.*

Proof Let A be a connected set and $\bar{A}$ be its closure. Suppose $\bar{A} = U_1 \cup U_2$ where U_1 and U_2 are nonempty disjoint open sets in $\bar{A}$. Then $A = (A \cap U_1) \cup (A \cap U_2)$ is non-trivial: let $U_1 \cap A = \emptyset$, then $U_1 \subseteq \bar{A} \setminus A$ which is impossible.

Example 5.0.1. *Let $X = \{(x, y)|y = \sin(\frac{1}{x}), x > 0\} \cup \{(0, y)|y \in [-1, 1]\} \subset \mathbb{R}^2$ with the Euclidean metric. Then X is connected but not path-connected.*

Proof

First we prove that X is not path-connected. Let $p_1, p_2 \in X$ such that $P_1 = (0, 0)$ and $P_2 = (1, \sin(1))$. Let $\gamma : [0, 1] \to X$ with $\gamma(t) = (x(t), y(t))$ such that $\gamma(0) = P_1$ and $\gamma(1) = P_2$. Since γ is continuous, both $x(t)$ and $y(t)$ are continuous.

Let $U = \{\tau | x(t) = 0\} \subseteq [0, 1]$. Since U is a preimage of a closed set under a continuous map, U is closed. Thus, $t_0 = \sup U$ exists. Note that $t_0 < 1$ since $x(1) = 1$.

Let $\Phi(t) = \dfrac{1}{x(t)}$ which is well-defined on for $t > t_0$. Notice that $\Phi(t) \to +\infty$ as $t \to t_0^+$. Then, for $t > t_0$, $y(t) = \sin(\Phi(t))$. Take the interval $(t_0, t_0 + \epsilon)$, then $y(t)$ oscillates between -1 and 1 infinitely many times. So $y(t)$ is not continuous at t_0. This is a contradiction. Thus, no such path γ exists and X is not path-connected.

Second, we prove that X is connected. Let $X = Y \cup \{(0, y)|y \in [-1, 1]\}$. We know that $\bar{Y} = X$ and Y is path-connected (thus connected). Then using the property that the closure of a connected set is connected, we have X is connected.

Remark Also, we can prove it using 2 in the 5.0.1.

5.0.1 Connected Components

For any $x, y \in X$, we say that $x \sim y$ if $\exists \gamma : [0, 1] \to X$ continuous such that $\gamma(0) = x$ and $\gamma(1) = y$. It is easy to check that $\sim$ is an equivalence relation. We call the equivalence classes the connected components of X.

Exercise Define connected components for the notion of connected (not just path-connected) spaces.

Observation The number of connected components is preserved under homeomorphisms.

Example 5.0.2. $(0, 1) \cup (2, 3)$ *is not homeomorphic to* $(0, 1)$ *since the former has 2 connected components while the latter has 1 connected component.*

Example 5.0.3. *The interval* $[0, 1]$ *is not homeomorphic to* $[0, 1] \times [0, 1]$. *Removing a point(not in the boundary) from* $[0, 1]$ *results in a disconneted space, while removing a point from* $[0, 1] \times [0, 1]$ *still results in a connected space. Thus, they are not homeomorphic.*

Example 5.0.4. $(0,1) \not\approx [0,1] \not\approx [0,1)$. *Removing a boundary point from* $[0,1]$ *still results in a connected space, while removing a boundary point from* $(0,1)$ *results in a disconnected space. Thus, they are not homeomorphic.*

Example 5.0.5. *We can split the alphabet(capital letter) into nine homeomorphism classes:*

- *A*
- *B*
- *C, I, J, L, M, N, S, U, V, W, Z*
- *D, O*
- *E, F, G, T, Y*
- *H*
- *K, X, Z*
- *P*
- *Q, R*

Example 5.0.6 (First homology group)**.**

$$2026 \rightsquigarrow 2$$
$$2025 \rightsquigarrow 1$$
$$1949 \rightsquigarrow 2$$
$$1982 \rightsquigarrow 3$$
$$1988 \rightsquigarrow 5$$

Property 5.0.1. $U \subseteq \mathbb{R}^2$ *open in one of the metrics defined by the norms listed below iff it is open in the others:*

- *Euclidean norm:* $||(x,y)||_2 = \sqrt{x^2+y^2}$
- *Supremum norm:* $||(x,y)||_\infty = \max |x|, |y|$
- *Diamond norm:* $||(x,y)||_{1/2} = \sqrt{|x|} + \sqrt{|y|}$

In other words, these metrics define the same topology. So they have the same open sets.

Definition 5.0.1. *Two norms on* V $||\cdot||_a$ *and* $||\cdot||_b$ *are called equivalent if* $\exists c_1, c_2$ *such that:*

$$c_1||v||_a \leq ||v||_b \leq c_2||v||_a \tag{5.1}$$

Exercise

1. This is an equivalent relation.
2. Equivalent norms define the same topology.
3. $||\cdot||_1$, $||\cdot||_2$, and $||\cdot||_\infty$ on $\mathbb{R}^n$ are equivalent.

* All norms on $\mathbb{R}^n$ are equivalent.

The notes below is not for this lecture.

Theorem 5.0.2. *All norms on* $\mathbb{R}^n$ *are equivalent.*

Proof

Given $\|\cdot\|_a, \|\cdot\|_b$ two norms on $\mathbb{R}^n$. We define a function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by $f(x) = \dfrac{||x||_a}{||x||_b}$ which is a continuous function. And $f(x) = f(\lambda x)$ for any $\lambda > 0$. So f is completely determined by its values on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n | ||x||_a = 1\}$. Since S^{n-1} is compact, f attains its maximum and minimum on S^{n-1}, denoted as m and M. Thus, for any $x \neq 0$, we have:

$$0 < m \leq \frac{||x||_a}{||x||_b} \leq M < +\infty \tag{5.2}$$

Therefore, $\|x\|_b$ and $\|x\|_a$ are equivalent.

Remark In infinite-dimensional vector spaces, S^{n-1} is not compact(However, it is closed and bounded).
Exercise Give an example of norms on l_1(convergent series, i.e., $l_1 = \{(a_n) | \sum_{n=1}^{\infty} |a_n| < +\infty\}$).
Solution The trivial one is $||x|| = \sum_{n=1}^{\infty} |a_n|$. It is easy to verify as it is convergent.

Chapter 6

Lecture5

6.1 Topological Spaces in general

Definition 6.1.1. *A **topological space** is a pair $(X, \mathcal{U})$ where X is a set and $\mathcal{U} \subseteq 2^X$ such that:*

1. $\emptyset, X \in \mathcal{U}$.
2. *If $U_\alpha \in \mathcal{U}$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{U}$.*
3. *If $U_1, U_2, \ldots, U_n \in \mathcal{U}$, then $\bigcap_{i=1}^n U_i \in \mathcal{U}$.*

*$\mathcal{U}$ is called a **topology** on X. The elements of $\mathcal{U}$ are called **open sets**.*

Example 6.1.1. *Any metric space (X, d) is a topological space, with $\mathcal{U}$ defined as:*

$$U \in \mathcal{U} \iff \forall x \in U, \exists r > 0 \text{ such that } B(x, r) \subseteq U. \tag{6.1}$$

Open sets defined in this topology are the same as open sets defined by the metric.

Example 6.1.2. *Let X be any non-empty set. The **discrete topology** on X is defined as $\mathcal{U} = 2^X$. Every subset of X is open.*

Example 6.1.3. *Let X be any non-empty set. The **anti-discrete topology** on X is defined as $\mathcal{U} = \{\emptyset, X\}$. Only $\emptyset$ and X are open. **This example is not defined by a metric.***

Example 6.1.4 (Finite complement topology)**.** *Let X be an infinite set, say, $\mathbb{R}$. Define*

$$U \in \mathcal{U} \iff X \setminus U \text{ is finite or } U = \emptyset. \tag{6.2}$$

Property 6.1.1. *Let (X, d) be a metric space. Let $x, y \in X$, then there exists open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Definition 6.1.2. *Let $(X, \mathcal{U})$ is a Hausdorff space(separable space) if for any $x, y \in X$, there exists open sets $U, V \in \mathcal{U}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Remark Finite complement topology in 6.1.4 is not Hausdorff. If we take any two non-empty open sets U, V, then both $X \setminus U$ and $X \setminus V$ are finite. Thus, $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$ is also finite, which means $U \cap V \neq \emptyset$. So it is not Hausdorff.
Remark Also we see that 6.1.3 is not Hausdorff.

Example 6.1.5. *Let $X = \mathbb{N}$. Let $U_n = \{i \in X | i \leq n\}$ for some n. Let $\mathcal{U} = \{\emptyset, X\} \cup \{U_n | n \in \mathbb{N}\}$. Then $(X, \mathcal{U})$ is a topological space.*

For finite Union, let $U_{n_1}, U_{n_2}, \ldots, U_{n_k} \in \mathcal{U}$, then

$$\bigcup_{i=1}^k U_{n_i} = U_{\max\{n_1, n_2, \ldots, n_k\}} \in \mathcal{U}. \tag{6.3}$$

For infinite Union, let $\{U_{n_\alpha}\}_{\alpha \in A} \subseteq \mathcal{U}$, then

$$\bigcup_{\alpha \in A} U_{n_\alpha} = \mathbb{N} \tag{6.4}$$

For finite Intersection, let $U_{n_1}, U_{n_2}, \dots, U_{n_k} \in \mathcal{U}$, then

$$\bigcap_{i=1}^{k} U_{n_i} = U_{\min\{n_1,n_2,\dots,n_k\}} \in \mathcal{U}. \tag{6.5}$$

This set is not Hausdorff.

Alternatively, topology can be defined by closed sets: $Y \subseteq X$ is closed if $X \setminus Y$ is open. So we can rewrite the definition of Topology space:

1. $\emptyset, X$ are closed.
2. If F_α is closed for all $\alpha \in A$, then $\bigcap_{\alpha\in A} F_\alpha$ is closed.
3. If $F_1, F_2, \dots, F_n$ are closed, then $\bigcup_{i=1}^{n} F_i$ is closed.

Question Is there an infinite collection of open sets such that their intersection is not open?

6.2 Zariski Topology

Let $X = \mathbb{C}^n$(or $\mathbb{K}^n$). $Y \subseteq \mathbb{C}^n$ is closed if Y is a solution of

$$\begin{cases} f_1(x_1,x_2,\dots,x_n) = 0 \\ f_2(x_1,x_2,\dots,x_n) = 0 \\ \vdots \\ f_m(x_1,x_2,\dots,x_n) = 0 \\ \vdots \end{cases} \tag{6.6}$$

X is defined by $0 = 0$. $\emptyset$ is defined by $1 = 0$. Any points in $\mathbb{C}^n$ is defined by

$$\begin{cases} x_1 - a_1 = 0 \\ x_2 - a_2 = 0 \\ \vdots \\ x_n - a_n = 0 \end{cases} \tag{6.7}$$

A circle in $\mathbb{C}^2$ is defined by $x^2 + y^2 - 1 = 0$.

Any intersection of closed sets is the set of solutions of all the polynomials that define each closed set, so it is still closed.

Any union of finite closed sets is also closed since we can use the product of all polynomials that define each closed set to define the union.

For $n = 1$, since any polynomial with degree at least 1 has finitely many roots, recalling the finite complement topology in 6.1.4, we see that Zariski topology on $\mathbb{C}$ is the finite complement topology. But this is not Hausdorff.

6.3 Basis of a topology

Definition 6.3.1. *Let X be a set. A **basis** $\mathcal{B}$ is a is a set in 2^X such that*

1. *For any $x \in X$, $\exists B \in \mathcal{B}$ such that $x \in B$.*
2. *For any $x \in X$, if $x \in B_1$ and $x \in B_2$ where $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Property 6.3.1. *The collection of all unions of elements in $\mathcal{B}$ forms a topology on X, called the topology **generated** by $\mathcal{B}$.*

Proof

1. $\emptyset, X$ are in the topology generated by $\mathcal{B}$.
2. Let $\{U_\alpha\}_{\alpha\in A}$ be a collection of open sets in the topology generated by $\mathcal{B}$. Then for each α, we have $U_\alpha = \bigcup_{i\in I_\alpha} B_i$ where $B_i \in \mathcal{B}$. Thus,

$$\bigcup_{\alpha\in A} U_\alpha = \bigcup_{\alpha\in A} \bigcup_{i\in I_\alpha} B_i, \tag{6.8}$$

which is still a union of elements in $\mathcal{B}$. So $\bigcup_{\alpha\in A} U_\alpha$ is in the topology generated by $\mathcal{B}$.

3. Let $U_1, U_2 \in \mathcal{U}$ be two open sets in the topology generated by $\mathcal{B}$. Then let $x \in U_1 \cap U_2$, we have $\exists B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. By the second property of basis, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Similarly, for any $x \in U_1 \cap U_2$, we can find such B_3. Thus, $U_1 \cap U_2$ is also in the topology generated by $\mathcal{B}$.

Example 6.3.1. *For a metric space (X, d), the collection of all open balls $\{B(x, r)|x \in X, r > 0\}$ is a basis of the topology induced by d.*

Example 6.3.2. *All the rectangles in $\mathbb{R}^2$ with sides parallel to the axes form a basis of the standard topology on $\mathbb{R}^2$.*

Example 6.3.3. *Let X be any set. Let $\mathcal{B} = \{\{x\}|x \in X\}$. Then $\mathcal{B}$ is a basis of the discrete topology on X.*

Theorem 6.3.1. *If $\mathcal{B}$ is a basis of a topology $\mathcal{U}$ on X, then for any open set $U \in \mathcal{U}$, we have $U = \bigcup_{B \in \mathcal{B}, B \subseteq U} B$. (The Proof is based on the wrong definition of basis in the previous version, but the right version is not hard.)*

Proof

"$\subseteq$": For any $x \in U$, since $\mathcal{B}$ is a basis, there exists $B \in \mathcal{B}$ such that $x \in B$. There exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B \cap U$. Thus, $B' \subseteq U$ and $x \in \bigcup_{B \in \mathcal{B}, B \subseteq U} B$.

"$\supseteq$": For any $x \in \bigcup_{B \in \mathcal{B}, B \subseteq U} B$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Thus, $x \in U$.

Example 6.3.4.

1. *Balls in d_1, d_2, d_∞ metrics in $\mathbb{R}^n$ form bases of the standard topology on $\mathbb{R}^n$.*
2. *$\{U_\epsilon(x)|\epsilon > 0, x \in \mathbb{Q}^n\}$is a countable basis of the standard topology on $\mathbb{R}^n$.*

Exercise Find a metric space not admitting countable basis of topology.

Lemma 6.3.1. *Let X be a topological space. Let $\mathcal{C}$ be a collection of open sets of X such that: let x be a point in X, then for any open set U containing x, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis of the topology on X.*

Proof

1. For any $x \in X$, since X is open and contains x, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq X$.
2. Let $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$. Since $C_1 \cap C_2$ is open and contains x, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$.
3. Let $\mathcal{T}$ be the topology generated by $\mathcal{C}$. Let $\mathcal{U}$ be the initial topology on X. For any $U \in \mathcal{U}$, for any $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. So,
$$U = \bigcup_{C \in \mathcal{C}, C \subseteq U} C, \tag{6.9}$$
which means $U \in \mathcal{T}$. Thus, $\mathcal{U} \subseteq \mathcal{T}$.
4. And we also have $\mathcal{T} \subseteq \mathcal{U}$ because: since $\mathcal{C} \subseteq \mathcal{U}$, any union of elements in $\mathcal{C}$ is also in $\mathcal{U}$. Thus, $\mathcal{T} \subseteq \mathcal{U}$.
5. Combining the above two results, we have $\mathcal{T} = \mathcal{U}$.

Definition 6.3.2. *If $\mathcal{T}$,$\mathcal{U}$ are two topologies on X, we say that $\mathcal{T}$ is* ***finer*** *than $\mathcal{U}$ (or $\mathcal{U}$ is* ***coarser*** *than $\mathcal{T}$) if $\mathcal{U} \subseteq \mathcal{T}$.*

Example 6.3.5. *Let X be any set. The discrete topology on X is the finest topology on X. The anti-discrete topology on X is the coarsest topology on X.*

Lemma 6.3.2. *Let $\mathcal{B}$,$\mathcal{C}$ be two bases of topologies $\mathcal{T}$,$\mathcal{U}$ on X respectively. Then the following are equivalent:*

1. *$\mathcal{U}$ is finer than $\mathcal{T}$.*
2. *For any $x \in X$, for any $B \in \mathcal{B}$ containing x, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq B$.*

Proof

1. "(1) $\Rightarrow$ (2)": Let $x \in X$. Let $B \in \mathcal{B}$ such that $x \in B$. Since $\mathcal{U}$ is finer than $\mathcal{T}$, we have $B \in \mathcal{U}$. Since $\mathcal{C}$ is a basis of $\mathcal{U}$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq B$.
2. "(2) $\Rightarrow$ (1)": Let $U \in \mathcal{T}$. For any $x \in U$, since $\mathcal{B}$ is a basis of $\mathcal{T}$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. By (2), there exists $C \in \mathcal{C}$ such that $x \in C \subseteq B \subseteq U$. Thus, for any $x \in U$, we can find such C. By the previous lemma($\mathcal{C}$ is a basis of $\mathcal{U}$), we have $U \in \mathcal{U}$. Therefore, $\mathcal{U}$ is finer than $\mathcal{T}$.

Example 6.3.6. *By the above lemma, we see that the topology generated by the basis in 6.3.1 is finer than the topology generated by the basis in 6.3.2. And similarly, the topology generated by the basis in 6.3.2 is finer than the topology generated by the basis in 6.3.1. Thus, they are the same topology.*

Example 6.3.7 (Lower limit topology on $\mathbb{R}$)**.** *Let $\mathcal{B} = \{[a, b)|a < b, a, b \in \mathbb{R}\}$. Then $\mathcal{B}$ is a basis of a topology on $\mathbb{R}$, called the **lower limit topology** called $\mathbb{R}_l$.*

One can show that the lower limit topology is finer than the standard topology on $\mathbb{R}$ since for $a < x < b$, we have $x \in [x, b) \subseteq (a, b)$. But the lower limit topology is not equal to the standard topology since take $0 \in [0, 1)$, there is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.

We say that $\mathbb{R}_l$ is strictly finer than the standard topology on $\mathbb{R}$.

Exercise Prove that $\mathbb{R}_l$ does not have a countable basis.

Example 6.3.8 (Order Topology)**.** *Let X be a simply ordered set.*

$$\mathcal{B} = \{(a, b)|a < b, a, b \in X\} \cup \{[a_0, b)|a_0 < b\} \cup \{(a, b_0]|a < b_0\} \tag{6.10}$$

*where a_0 is the smallest element of X if it exists, and b_0 is the largest element of X if it exists. Then $\mathcal{B}$ is a basis of a topology on X, called the **order topology**.*

Remark Let $(X,\mathcal{U})$, $(Y, \mathcal{Y})$ be two topological spaces. Then $\mathcal{U} \times \mathcal{V}$ may not be a topology on $X \times Y$. For example, the union of two rectangles may not be a rectangle. The correct idea is to define the basis of the product topology as:

$$\mathcal{B} = \{U \times V|U \in \mathcal{U}, V \in \mathcal{V}\}. \tag{6.11}$$

Furthermore, we see that for $U_1, U_2 \in \mathcal{U}$ and $V_1, V_2 \in \mathcal{V}$, we have

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2), \tag{6.12}$$

Definition 6.3.3. *The topology on $X \times Y$ defined by basis $\mathcal{U} \times \mathcal{V}$ is called the product topology on $X \times Y$.*

Theorem 6.3.2. *Let $\mathcal{B} \subseteq \mathcal{U}$ be a basis of $\mathcal{U}$ and let $\mathcal{C} \subseteq \mathcal{V}$ be a basis of $\mathcal{V}$. Then $\mathcal{D} = \{B \times C|B \in \mathcal{B}, C \in \mathcal{C}\}$ is a basis of the product topology on $X \times Y$.*

Proof

We need to show: for every open set $W \subseteq X \times Y$ and for every $(x, y) \in W$, there exists $B \times C$ with $B \in \mathcal{B}$, $C \in \mathcal{C}$ such that $(x, y) \in B \times C \subseteq W$.

Since W is open in the product topology, there exists $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $(x, y) \in U \times V \subseteq W$. Since $\mathcal{B}$ is a basis of $\mathcal{U}$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Since $\mathcal{C}$ is a basis of $\mathcal{V}$, there exists $C \in \mathcal{C}$ such that $y \in C \subseteq V$. Thus, we have $(x, y) \in B \times C \subseteq U \times V \subseteq W$.

Example 6.3.9. *Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projection maps defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Then we have*

$$\pi_1^{-1}(U) = U \times Y \text{ for any } U \in \mathcal{U}, \tag{6.13}$$

$$\pi_2^{-1}(V) = X \times V \text{ for any } V \in \mathcal{V}. \tag{6.14}$$

which means that both π_1 and π_2 are continuous. And we have

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V). \tag{6.15}$$

is open in the product topology. And we have

$$S = \{\pi_1^{-1}(U)|U \in \mathcal{U}\} \cup \{\pi_2^{-1}(V)|V \in \mathcal{V}\} \tag{6.16}$$

is a subbasis of the product topology on $X \times Y$ (Note: We have not defined subbasis).

Chapter 7

Lecture7

7.1 Subspace Topology

Definition 7.1.1. *Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. The **subspace topology** on Y is defined as*

$$\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}.$$

Property 7.1.1. *$(Y, \mathcal{T}_Y)$ is a topological space.*

Lemma 7.1.1. *If $\mathcal{B}$ is a basis for the topology $\mathcal{T}$ on X, then the collection*

$$\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$$

is a basis for the subspace topology $\mathcal{T}_Y$ on Y.

Proof
Let $U \in \mathcal{T}$ and $y \in U \cap Y$. Then there exists $B \in \mathcal{B}$ such that $y \in B \subseteq U$. Then $y \in B \cap Y \subseteq U \cap Y$.
Remark The set which is open in the subspace topology may not be open in the original topology.

Property 7.1.2. *If Y is open in $(X, \mathcal{T})$ and $U \subseteq Y$ is open in $(Y, \mathcal{T}_Y)$, then U is open in $(X, \mathcal{T})$.*

Proof Since $U \subseteq Y$ is open in $(Y, \mathcal{T}_Y)$, there exists $V \in \mathcal{T}$ such that $U = V \cap Y$. Since Y is open in $(X, \mathcal{T})$, we have $U = V \cap Y$ is open in $(X, \mathcal{T})$.

Theorem 7.1.1. *Let $A \subseteq X$ and $B \subseteq Y$ such that $A \times B \subseteq X \times Y$. Then the product of subspace topologies $\mathcal{T}_A \times \mathcal{T}_B$ is equal to the subspace topology $\mathcal{T}_{A \times B}$ on $A \times B$.*

Proof
Let $U \subseteq X$ and $V \subseteq Y$ be open sets in $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ respectively. Then the products of the form $U \times V$ form a basis for the product topology on $X \times Y$. And we have

$$(U \times V) \cap (A \times B) \tag{7.1}$$

form a basis for the subspace topology on $A \times B$. Note that

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B) \tag{7.2}$$

So the basis for the product of subspace topologies is equal to the basis for the subspace topology on $A \times B$. ■

Let $(X, <)$ be an ordered set. Let $a_0 = min(X)$ and $b_0 = max(X)$ if they exists. Then

$$\mathcal{B} = \{(a, b) : a_0 < a < b < b_0\} \cup \{[a_0, b) : b < b_0\} \cup \{(a, b_0] : a > a_0\} \tag{7.3}$$

is a basis for the order topology on X. Notice that $[a_0, b)$ refers to the set $\{x \in X : x < b\}$ when a_0 does not exist.
Remark Let $Y \subseteq X$ be a subset inheriting the order from X. Then it may happen that the order topology on Y is different from the subspace topology on Y.

Example 7.1.1. *Let $X = \mathbb{R}$ with the usual order and $Y = [0, 1) \cup \{2\}$ be a subset of X. Then $\{2\}$ is open in the subspace topology on Y since $\{2\} = (1.5, 2.5) \cap Y$. However, $\{2\}$ is not open in the order topology on Y.*

Exercise Show that for the order topology $[0, 1) \cup \{2\}$ is connected.

Example 7.1.2. *Let $X = Y = \mathbb{R}$ with the usual order. Then the product of order topologies is the standard topology on $\mathbb{R}^2$. We take the lexicographic order as the order on $\mathbb{R}^2$.*

Remark Let $I \times I = [0, 1] \times [0, 1] \subseteq X \times Y$. Then the order topology on $[0, 1] \times [0, 1]$ is different from the subspace topology inherited from the order topology on $\mathbb{R}^2$. The set $\{1/2\} \times (1/2, 1]$ is not open in the order topology on $I \times I$ but it is open in the subspace topology inherited from $\mathbb{R}^2$.

Definition 7.1.2. *Y is convex if for any $a, b \in Y$, the interval $(a, b) \subseteq Y$.*

Theorem 7.1.2. *Let $Y \subseteq X$ be a convex subset of $(X, <)$. Then the the restriction of the order topology on X to Y is equal to the order topology on Y.*

Proof

Take $(a, +\infty), (-\infty, b) \subseteq X$ which form a basis for the order topology on X. Take $Y \subseteq X$ convex.

1. If $a \in Y$, then $(a, +\infty) \cap Y = \{y | y \in Y \text{ and } a < y\}$.
2. If $a \notin Y$, then there are only two cases because Y is convex:
   (a) If a is a lower bound for Y, then $(a, +\infty) \cap Y = Y$
   (b) If a is an upper bound for Y, then $(a, +\infty) \cap Y = \emptyset$

Then we obtain the basis for subspace topology on Y from the basis of order topology on X. And also this is the basis for the order topology on Y.

Definition 7.1.3. *$A \subseteq X$ is closed if and only if $X \setminus A$ is open.*

Property 7.1.3. *1. X and $\emptyset$ are closed.*

2. The intersection of any collection of closed sets is closed.

3. The union of finitely many closed sets is closed.

Theorem 7.1.3. *Let X be a topological space. And $Y \subseteq X$ has a subspace topology. Then $A \subseteq Y$ is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

Proof

($\Rightarrow$) Since A is closed in Y, then $Y \setminus A$ is open in Y. So there exists an open set U in X such that $Y \setminus A = U \cap Y$. Let $C = X \setminus U$. Then C is closed in X and

$$A = Y \setminus (Y \setminus A) = Y \setminus (U \cap Y) = Y \cap (X \setminus U) = Y \cap C.$$

($\Leftarrow$) Let $A = C \cap Y$ where C is closed in X. Then $X \setminus C$ is open in X. So we have $(X \setminus C) \cap Y$ is open in Y. But we have

$$(X \setminus C) \cap Y = Y \setminus (C \cap Y) = Y \setminus A. \tag{7.4}$$

which is open in Y. So A is closed in Y.

Definition 7.1.4. *If $U \subseteq X$ is open and $x \in U$ then U is a neighborhood of x.*

Definition 7.1.5. *$Int(A) = \bigcup\{U \subseteq X | U$ is open and $U \subseteq A\}$ is the interior of A.*
$\bar{A} = \bigcap\{C \subseteq X | C$ is closed and $A \subseteq C\}$ is the closure of A.

Remark $Int(A) \subseteq A \subseteq \bar{A}$.
Remark If $A \subseteq Y \subseteq X$, then the closure of A in Y and the closure of A in X may be different. For example let $X = \mathbb{R}$ and $Y = [0, 1)$. Let $A = (0, 1) \subseteq Y$. Then the closure of A in Y is $[0, 1)$ while the closure of A in X is $[0, 1]$.

Theorem 7.1.4. *Let $Y \subseteq X$ with the subspace topology. Let $\bar{A}$ be the closure of A in X. Then the closure of A in Y is equal to $\bar{A} \cap Y$.*

Proof

Let B be the closure of A in Y. We prove the two inclusions.

Let B be the closure of A in Y. $\bar{A}$ is closed in X. So $\bar{A} \cap Y$ is closed in Y. Then $\bar{A} \cap Y \supseteq A$(Since we are talking about the closure in Y, we must have $A \subseteq Y$). So $B \subseteq \bar{A} \cap Y$($B$ is the smallest closed set containing A).

Since B is closed in Y, there exists a closed set C in X such that $B = C \cap Y$. Hence $C \supseteq A$ which is closed in X. So $C \supseteq \bar{A}$. Then $\bar{A} \cap Y \subseteq C \cap Y = B$.

Hence the two inclusions imply that $B = \bar{A} \cap Y$.

■

Chapter 8

Lecture 8

Theorem 8.0.1. *Let A be a subset of the topological space X.*

1. $x \in \bar{A} \iff$ *every open set $U \subseteq X$ with $x \in U$ satisfies $U \cap A \neq \emptyset$.*
2. *For a basis $\mathcal{B}$ of $\mathcal{T}$, we have $x \in \bar{A} \iff$ every basis element $B \in \mathcal{B}$ with $x \in B$ satisfies $B \cap A \neq \emptyset$.*

Proof to 1

We transform each implication to its contrapositive, thereby obtaining the logically equivalent statement $(\neg P) \iff (\neg Q)$. Written out, it is the follwoing statement. $x \notin \bar{A}$ if and only if there exists an open set U containing x that does not intersect A.

If $x \notin \bar{A}$, then the set $U = X - \bar{A}$ is an open set containing x that does not intersect A.

Conversely, if there exists an open set U containing x which does not intersect A, then $X - U$ is a closed set containing A. Then $\bar{A} \subseteq X - U$. Since $x \notin X - U$, $x \notin \bar{A}$.

■

Exercise Finish the proof of 2.

Definition 8.0.1. *For $A \subseteq X$, $x \in X$ is a limit point of A if every open set $U \subseteq X$ containing x contains a point of A different from x itself. We denote A' as the set of all limit points of A.*

Theorem 8.0.2. $\bar{A} = A \cup A'$

Proof

We need to show that $\bar{A} \subseteq A \cup A'$ and $A \cup A' \subseteq \bar{A}$.

($\supseteq$) If $x \in A'$, then every neighborhood U of x intersects A. Then $x \in \bar{A}$ by the previous theorem. If $x \in A$, then obviously $x \in \bar{A}$. Thus $A \cup A' \subseteq \bar{A}$.

($\subseteq$) Let $x \in \bar{A}$. If $x \in A$, then $x \in A \cup A'$. If $x \in \bar{A} \setminus A$, then every neighborhood U of x intersects A at y such that $y \neq x$. Thus $x \in A'$. Therefore, $\bar{A} \subseteq A \cup A'$.

■

Corollary 8.0.1. *A set A is closed if and only if $A' \subseteq A$.*

Definition 8.0.2. *For a sequence $\{x_n\}$ in X, we say that a is a limit of the sequence if for every open set U containing a, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$. A sequence is convergent if it has a limit.*

Remark This is equivalent to the following: every open set U containing a contains almost all elements of the sequence except finitely many. And Notice that "all but finitely many" is not the same as "infinitely many".

Example 8.0.1. *Let X be a topological space with the discrete topology. Then a sequence $\{x_n\}$ converges to a if and only if there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n = a$.*

Example 8.0.2. *Let $X = \mathbb{Q}$ with the topology defined by the usual metric. Then the sequence:* $3, 3.1, 3.14, 3.141, 3.1415, \ldots$ *does not converge in X since its limit π is not in $\mathbb{Q}$.*

Example 8.0.3. *Let $X = \mathbb{R}_l$ with the lower limit topology($[a, b)$ are basis open sets). Then we say that a sequence converges in $\mathbb{R}^l$ if and only if it converges to a in $\mathbb{R}$ with an extra condition: $x_n \geq a, \forall n \geq N$ for some $N \in \mathbb{N}$.*

Exercise Show that if x_n converges in $\mathcal{T}$, then x_n converges in any coarser topology $\mathcal{T}' \subseteq \mathcal{T}$.

Example 8.0.4. *$X = \mathbb{N}$, the basis $U_m = \{0, 1, 2 \cdots, m\}$. $\mathcal{T} = \{U_m | m \in \mathbb{N}\} \cup \{\mathbb{N}, \emptyset\}$. Then x_n converges to a if and only if a is essential upper bound of $\{x_n\}$. In this case, the limit is not unique.*

Example 8.0.5. *Finite complement topology on* $\mathbb{R}$.
For example take $\mathcal{T} = \{\mathbb{R}, \emptyset\} \cup \{\mathbb{R} \setminus \{p_1, p_2, \cdots, p_m\} | p_1, p_2, \cdots, p_m \in \mathbb{R}, m \in \mathbb{N}\}$.

1. *If* x_n *is eventually constant with value* a*, then* x_n *converges to* a*. It can't not converge to* $b \neq a$ *because we can take the open set* $\mathbb{R} \setminus \{a\}$ *which contains* b *but does not contain almost all elements of the sequence.*

2. *If* $x_n = (-1)^n$*, then* x_n *does not converge. If* $x_n = a$ *for infinitely many* n *and* $x_n = b$ *for infinitely many* n $(a \neq b)$*, then* x_n *does not converge.*

3. *If* $x_n = \dfrac{(-1)^n}{n}$*, then* x_n *converges to any real number. Because for any open set* U*,* U *contains all but finitely many real numbers. So* U *contains almost all elements of the sequence.*

4. *If* x_n *assumes every value finitely many times, then* x_n *converges to any real number.*

Chapter 9

Lecture9

Definition 9.0.1. *A topological space X is Hausdorff or T2 if for every pair of distinct points $x, y \in X$, there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.*

Property 9.0.1. *If X is a metric space, then X is Hausdorff.*

Proof Let $d = d(x, y) > 0$. Take $U = B\left(x, \frac{d}{2}\right)$ and $V = B\left(y, \frac{d}{2}\right)$. Then $U \cap V = \emptyset$.

Property 9.0.2. *For a Hausdorff space X, for every $x_0 \in X$, the singleton set $\{x_0\}$ is closed.*

Let $x \in X$ with $x \neq x_0$. Since X is Hausdorff, there exist open sets $U, V \subseteq X$ such that $x_0 \in U$, $x \in V$, and $U \cap V = \emptyset$. Thus $V \subseteq X \setminus \{x_0\}$. Therefore, $X \setminus \{x_0\}$ is open, which means $\{x_0\}$ is closed.

■

Question If $\forall x_0 \in X$, $\{x_0\}$ is closed, is X Hausdorff?
Answer No. Consider the finite complement topology.

Definition 9.0.2. *If every point $x_0 \in X$ is closed, then X is called a Frechet space or T1 space.*

Remark If X is Hausdorff(T2), then X is T1. The converse is not true.

Property 9.0.3. *X is T1 if and only if $\forall x, y \in X, \exists U \ni x$ a open set such that $y \notin U$. That is, for all $x, y \in X$, there exists an open set containing x but not containing y.*

Exercise Prove this property.
Exercise Let X satisfy the following: for $x, y \in X$ with $x \neq y$, either $\exists U \ni x$ open set such that $y \notin U$, or $\exists V \ni y$ open set such that $x \notin V$. Is it true that X is T1?
Solution No. Consider $X = \{a, b\}$ with the topology $\mathcal{T} = \{\emptyset, X, \{a\}\}$.

Theorem 9.0.1. *Let X be T1, and $A \subseteq X$. Then $x \in A'$ if and only if every open set $U \ni x$ contains infinitely many points of A.*

Proof
($\Rightarrow$) Assume the contrary that only finitely many points are in the intersection. Suppose $x \in A'$. Then every open set $U \ni x$ contains a point of A different from x, i.e., $\exists y \in U \setminus \{x\} \cap A$. So we have that $U \cap A = \{y_1, y_2, \cdots, y_m\}$. Since X is T1, then $\{y_i\}$ is closed for each $i = 1, 2, \cdots, m$. Thus $V =: U \setminus \{y_1, y_2, \cdots, y_m\}$ is open and contains x. Then $V \setminus \{x\} \cap A = \emptyset$, which contradicts the assumption that $x \in A'$.
($\Leftarrow$) Obvious.

Theorem 9.0.2. *A sequence (x_n) in a Hausdorff space X converges to at most one point.*

Proof Suppose a_1, a_2 are two distinct limits of the sequence (x_n). Since X is Hausdorff, there exist open sets $U, V \subseteq X$ such that $a_1 \in U$, $a_2 \in V$, and $U \cap V = \emptyset$. Since a_1 is a limit of the sequence, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $x_n \in U$. Similarly, since a_2 is a limit of the sequence, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $x_n \in V$. Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, $x_n \in U$ and $x_n \in V$, which implies $x_n \in U \cap V$. This contradicts the fact that $U \cap V = \emptyset$. Therefore, the sequence (x_n) converges to at most one point.

■

Theorem 9.0.3.

1. *$(X, <)$ with the order topology is Hausdorff.*
2. *If X is Hausdorff, then the set with subspace topology $Y \subseteq X$ is also Hausdorff.*
3. *If X_1, X_2 are Hausdorff, then the product space $X_1 \times X_2$ with the product topology is also Hausdorff.*

Exercise Prove this theorem.

Definition 9.0.3. *Let X, Y be topological spaces. A function $f : X \to Y$ is called continuous if for every open set $V \subseteq Y$, the preimage $f^{-1}(V)$ is an open set in X.*

Remark If the topology on Y is given by the basis $\mathcal{B}$, then f is continuous if and only if for every basis element $B \in \mathcal{B}$, $f^{-1}(B)$ is open in X.

Example 9.0.1. *Consider the identity map for topological spaces $\mathbb{R}$ with the usual topology and $\mathbb{R}_l$ with the lower limit topology. The identity map $id : \mathbb{R} \to \mathbb{R}_l$ is not continuous, but the inverse $id : \mathbb{R}_l \to \mathbb{R}$ is continuous. Note that $(a, b) = \bigcup_{n=1}^{\infty}[a + \frac{1}{n}, b)$.*

Observation If $\mathcal{T}, \mathcal{T}'$ are topologies on X with $\mathcal{T} \subseteq \mathcal{T}'$, then the identity map $id : (X, \mathcal{T}') \to (X, \mathcal{T})$ is continuous and the map $id : (X, \mathcal{T}) \to (X, \mathcal{T}')$ is not continuous.

Theorem 9.0.4. *The following are equivalent:*

1. *$f : X \to Y$ is continuous.*
2. *$\forall A \subseteq X, f(\bar{A}) \subseteq \overline{f(A)}$.*
3. *For every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.*
4. *$\forall x \in X$, for every open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(U) \subseteq V$.*

Proof

(1) $\Rightarrow$ (2) Let $f : X \to Y$ be continuous and $A \subseteq X$. Let $x \in \bar{A}$. We need to show that $f(x) \in \overline{f(A)}$. If $x \in A$, then $f(x) \in f(A) \subseteq \overline{f(A)}$. If $x \in \bar{A} \setminus A$, take $V \ni f(x)$ open in Y. Since f is continuous, $f^{-1}(V)$ is open in X and contains x. Thus $f^{-1}(V) \cap A \neq \emptyset$. Let $y \in f^{-1}(V) \cap A$. Then $f(y) \in V \cap f(A) \neq \emptyset$. Therefore, $f(x) \in \overline{f(A)}$.

(2) $\Rightarrow$ (3) Let $C \subseteq Y$ be closed. We need to show that $f^{-1}(C) := A$ is closed in X, i.e., $\bar{A} = A$. We have $f(A) = f(f^{-1}(C)) = C$. So if $x \in \bar{A}, f(x) \in f(\bar{A}) \subseteq \overline{f(A)} = \overline{C} = C$. So $x \in f^{-1}(C) = A$. Thus $\bar{A} \subseteq A$. The other direction is obvious.

(3) $\Rightarrow$ (1) Obvious(take the complement).

(1) $\Rightarrow$ (4) Let $f(x) \in V$. Take $x \in f^{-1}(V) =: U$ which is open in X. Then $f(U) \subseteq V$.

(4) $\Rightarrow$ (1) Let $V \subseteq Y$ be open. We need to show that $f^{-1}(V)$ is open in X. Let $x \in f^{-1}(V)$. Then $\exists x \in U_x \subseteq X$ such that $f(U_x) \subseteq V$. Take $U = \bigcup_{f(x)\in V} U_x$ which is open in X. So $f^{-1}(V) \subseteq U$. But $U \subseteq f^{-1}(V)$.

9.1 Constructing continuous functions

Theorem 9.1.1. *Let X, Y, Z be topological spaces.*

1. *$f : X \to Y$ is a constant function, i.e., $f(X) = y$. Then f is continuous.*
2. *Let $A \subseteq X$ be a set with subspace topology. Then the embedding map $j : A \hookrightarrow X$ defined by $j(x) = x$ is continuous.*
3. *Let $f : X \to Y$, $g : Y \to Z$ be continuous, then $g \circ f : X \to Z$ is continous.*
4. *Restriction of domain: Let $f : X \to Y$ be continuous and $A \subseteq X$, then $f|_A : A \to Y$ is continuous.*
5. *$f : X \to Y$ and $Z \subseteq Y$ with $f(X) \subseteq Z$. Then $f : X \to Z$ is continuous.*
6. *If $f : X \to Y, Y \subseteq Z$, then $f : X \to Z$ is continuous.*
7. *$f : X \to Y$ is continuous if $X = \bigcup U_\alpha$ (U_α is required to be open in X) and $f|_{U_\alpha} : U_\alpha \to Y$ is continuous.*

Poof

1. Let $V \subseteq Y$ be open. If $y \in V$, then $f^{-1}(V) = X$ which is open. If $y \notin V$, then $f^{-1}(V) = \emptyset$ which is open.
2. Let $U \subseteq X$ be open. Then $j^{-1}(U) = U \cap A$ which is open in A.
3. $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ which is open in X.
4. $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$ which is open in A(Subspace topology).

5. Let W be open in Z. By subspace topology, $\exists V$ open in Y such that $W = V \cap Z$. Then $f^{-1}(W) = f^{-1}(V \cap Z) = f^{-1}(V)$(because $f(X) \subseteq Z$) which is open in X.

6. Let $f : X \to Y$ be continuous and $Y \subseteq Z$(Z has an topology $\mathcal{T}_Z$ and Y has the subspace topology $\mathcal{T}_Y = \{U \cap Y | U \in \mathcal{T}_Z\}$). Then we have for every open set $W \subseteq Z$, $f^{-1}(W) = f^{-1}(W \cap Y)$ which is open in X.

7. Let $X = \bigcup U_\alpha$ and U_α is open in X. And for each α, $f|_{U_\alpha} : U_\alpha \to Y$ is continuous.

   Take $V \subseteq Y$ open. Then

$$f^{-1}(V) = f^{-1}(V) \cap X = f^{-1}(V) \cap \left(\bigcup U_\alpha\right) = \bigcup \left(f^{-1}(V) \cap U_\alpha\right) = \bigcup (f|_{U_\alpha})^{-1}(V) \tag{9.1}$$

   Since $(f|_{U_\alpha})^{-1}$ is continuous, then $(f|_{U_\alpha})^{-1}(V)$ is open in U_α. And since U_α has the subspace topology and U_α is open in X, then $(f|_{U_\alpha})^{-1}(V)$ is open in X. Thus $f^{-1}(V)$ is open in X.

Theorem 9.1.2. *Let $X = A \cup B$ where A and B are both open in X or both closed in X. And $f : A \to Y$, $g : B \to Y$ be continuous and $f(x) = g(x)$ for $x \in A \cap B$. Then define $h : X \to Y$ as*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

Then h is continuous.

Proof Let's assume that A, B are both open. Let $V \subseteq Y$ be open. Then

$$h^{-1}(V) = \{x \in X | h(x) \in V\} = \{x \in A | f(x) \in V\} \cup \{x \in B | g(x) \in V\} = f^{-1}(V) \cup g^{-1}(V) \tag{9.2}$$

Since f, g are continuous, by the subspace topology, $f^{-1}(V) = A \cap U_1$ where U_1 is open in X, and $g^{-1}(V) = B \cap U_2$ where U_2 is open in X. Thus

$$h^{-1}(V) = (A \cap U_1) \cup (B \cap U_2) \tag{9.3}$$

is open in X since A, B, U_1, U_2 are all open in X.

For the case where A, B are both closed, we have a similar argument.

■

Exercise Prove the above theorem.

Chapter 10

Lecture10

10.1 Product topology on $X \times Y$

Theorem 10.1.1. *Let $f : A \to X \times Y$, let $f = (f_1, f_2)$ where $f_1 : A \to X$ and $f_2 : A \to Y$. Then f is continuous if and only if both f_1 and f_2 are continuous. We denote $f(a) = (f_1(a), f_2(a))$.*

Proof

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projection maps defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Then $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$. Since π_1 and π_2 are continuous, if f is continuous, then both f_1 and f_2 are continuous.

Conversely, suppose both f_1 and f_2 are continuous. Let U, V be open. Then $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$ for any open set U in X and V in Y. Since f_1 and f_2 are continuous, $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in A. Thus, $f^{-1}(U \times V)$ is open in A.

■

There are two ways to introduce product topology.

1. Take $U_1 \subseteq X_1, U_2 \subseteq X_2, \cdots$. Then we can define the topology on $X_1 \times X_2 \times \cdots$ by the basis $\{U_1 \times U_2 \times \cdots\}$. This is called the **box topology**.

2. Take $U_1 \subseteq X_1, U_2 \subseteq X_2, \cdots$ but only finitely many of them are not equal to the whole space. Then we can define the topology on $X_1 \times X_2 \times \cdots$ by the basis $\{U_1 \times U_2 \times \cdots\}$ where only finitely many U_i are not equal to X_i. This is called the **product topology**.

Definition 10.1.1. *Let J be an arbitrary set. A J-tuple of elements from X is a function $x : J \to X$. So $\alpha \in J \mapsto x(\alpha) = x_\alpha \in X$. And sometimes we denote the J-tuple by $(x_\alpha)_{\alpha \in J}$.*

Definition 10.1.2. *Let $(A_\alpha)_{\alpha \in J}$ be an indexed family of sets.*

$$X = \bigcup_{\alpha \in J} A_\alpha \tag{10.1}$$

The Cartesian product of the family $(A_\alpha)_{\alpha \in J}$ is denoted by

$$\prod_{\alpha \in J} A_\alpha \tag{10.2}$$

which is defined as the set of all J-tuples of elements in X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$, that is, the set of all functions

$$x : J \to X \quad \text{such that} \quad x(\alpha) \in A_\alpha \forall \alpha \in J \tag{10.3}$$

When $A_\alpha = X$, we have $\prod_{\alpha \in J} A_\alpha = X^J$, the set of all functions from J to X.

Definition 10.1.3. *Let $(X_\alpha)_{\alpha \in J}$ be an indexed family of topological spaces. The box topology on $\prod_{\alpha \in J} X_\alpha$ is given by the basis*

$$\left\{ \prod_{\alpha \in J} U_\alpha : U_\alpha \text{ is open in } X_\alpha \forall \alpha \in J \right\} \tag{10.4}$$

Then taking two basis elements $\prod_{\alpha \in J} U_\alpha$ and $\prod_{\alpha \in J} V_\alpha$, their intersection is

$$\left(\prod_{\alpha \in J} U_\alpha \right) \cap \left(\prod_{\alpha \in J} V_\alpha \right) = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha) \tag{10.5}$$

which is also a basis element.

Definition 10.1.4. *A collection $\mathcal{S}$ of subsets of topological space X is a **subbasis** if*

$$\bigcup_{S\in\mathcal{S}} S = X \tag{10.6}$$

Property 10.1.1. *Let $\mathcal{S}$ be a subbasis for a space X. Then the collection of all finite intersections of elements of $\mathcal{S}$ forms a basis for the topology on X. Then the topology generated by $\mathcal{S}$ is the collection of all unions of finite intersections of elements of $\mathcal{S}$.*

Exercise Prove the above property.

Definition 10.1.5. *For a given $\beta \in J$, we denote $\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta$ the projection map defined by*

$$x \mapsto x_\beta \tag{10.7}$$

Then let $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta$ is open in $X_\beta\}$. Then

$$\mathcal{S} = \bigcup_{\beta\in J} \mathcal{S}_\beta \tag{10.8}$$

*is a subbasis(Proved yourself). The topology defined by this subbasis is called the **product topology**. The basis $\mathcal{B}$ is given by finite intersections of elements of $\mathcal{S}$ and*

$$\mathcal{B} \ni B = \prod_{\alpha\in J} U_\alpha \quad \text{where } U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \tag{10.9}$$

Notice that if $|J| < \infty$, then the box topology and the product topology are the same. And also notice that the product topology is coarser than the box topology.

Theorem 10.1.2. *Let $\mathcal{J}$ be a set of indices. Let X_α be a topological space for each $\alpha \in \mathcal{J}$. Let $A_\alpha \subseteq X_\alpha$ with subspace topology for each $\alpha \in \mathcal{J}$. Then the product/box topology on $\prod_{\alpha\in\mathcal{J}} A_\alpha$ is the subspace topology inherited from the product/ box topology on $\prod_{\alpha\in\mathcal{J}} X_\alpha$.*

In other words, let A_α be a subspace of X_α for each $\alpha \in \mathcal{J}$. Then $\prod_{\alpha\in\mathcal{J}} A_\alpha$ is a subspace of $\prod_{\alpha\in\mathcal{J}} X_\alpha$ with the product/box topology.

Theorem 10.1.3. *Let $\mathcal{J}$ be a set of indices. Let X_α be a Hausdorff topological space for each $\alpha \in \mathcal{J}$. Then the product/ box topology on $\prod_{\alpha\in\mathcal{J}} X_\alpha$ is Hausdorff.*

Theorem 10.1.4. *Let $\mathcal{J}$ be a set of indices. Let X_α be a topological space for each $\alpha \in \mathcal{J}$. Let $A_\alpha \subseteq X_\alpha$ with subspace topology for each $\alpha \in \mathcal{J}$. Then $\prod_{\alpha\in\mathcal{J}} \bar{A}_\alpha = \overline{\prod_{\alpha\in\mathcal{J}} A_\alpha}$ with the product/ box topology.*

Proof

We give a proof for the box topology case. The product topology case is similar.

Take $(x_\alpha)_{\alpha\in\mathcal{J}} = x \in \prod_{\alpha\in\mathcal{J}} \bar{A}_\alpha$. Let $\prod_{\alpha\in\mathcal{J}} U_\alpha$ be a basis open set containing x where U_α is open in X_α. Since $x_\alpha \in \bar{A}_\alpha$, there exists $y_\alpha \in U_\alpha \cap A_\alpha$ for each $\alpha \in \mathcal{J}$. Thus, $(y_\alpha)_{\alpha\in\mathcal{J}} \in \prod_{\alpha\in\mathcal{J}} U_\alpha \cap \prod_{\alpha\in\mathcal{J}} A_\alpha$. Therefore, $x \in \overline{\prod_{\alpha\in\mathcal{J}} A_\alpha}$. So we have $\prod_{\alpha\in\mathcal{J}} \bar{A}_\alpha \subseteq \overline{\prod_{\alpha\in\mathcal{J}} A_\alpha}$.

Conversely, take $x = (x_\alpha)_{\alpha\in\mathcal{J}} \in \overline{\prod_{\alpha\in\mathcal{J}} A_\alpha}$. Let β be an arbitrary index in $\mathcal{J}$. Take V_β be an open set containing x_β. Then its preimage $\pi_\beta^{-1}(V_\beta)$ is open in $\prod_{\alpha\in\mathcal{J}} X_\alpha$. Since $x \in \overline{\prod_{\alpha\in\mathcal{J}} A_\alpha}$, there exists $y = (y_\alpha)_{\alpha\in\mathcal{J}} \in \pi_\beta^{-1}(V_\beta) \cap \prod_{\alpha\in\mathcal{J}} A_\alpha$. So $y_\beta \in V_\beta \cap A_\beta$. Thus, $x_\beta \in \bar{A}_\beta$. Since β is arbitrary, we have $x \in \prod_{\alpha\in\mathcal{J}} \bar{A}_\alpha$. Therefore, $\overline{\prod_{\alpha\in\mathcal{J}} A_\alpha} \subseteq \prod_{\alpha\in\mathcal{J}} \bar{A}_\alpha$.

Two inclusions together give the desired equality.

■

Theorem 10.1.5. *Let $f : A \to \prod_{\alpha\in\mathcal{J}} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha\in\mathcal{J}}$ where $f_\alpha : A \to X_\alpha$ for each $\alpha \in \mathcal{J}$. Then f is continuous if and only if each f_α is continuous.*

Proof

Let π_β be the projecction of the product onto its β-th factor. Then function π_β is continuous because for any open set U_β in X_β, we have $\pi_\beta^{-1}(U_\beta)$ is a subbaisis element for the product topology. Since $f_\beta = \pi_\beta \circ f$, if f is continuous, then each f_β is continuous.

Conversely, To prove f is continuous, it suffices to show that the preimage of each subbasis element is open in A. Take any subbasis element $\pi_\beta^{-1}(U_\beta)$ where U_β is open in X_β. Then

$$f^{-1}(\pi_\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta) \tag{10.10}$$

Since f_β is continuous, $f_\beta^{-1}(U_\beta)$ is open in A. Thus, f is continuous.

■

Example 10.1.1. *For box topology, the above theorem may fail.*

Consider $\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself. Let us define $f : \mathbb{R} \to \mathbb{R}^\omega$ by

$$f(t) = (t, t, t, \ldots) \tag{10.11}$$

Then each coordinate function $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(t) = t$ is continuous; therefore the function f is continuous given the product topology on $\mathbb{R}^\omega$. However, f is not continuous when $\mathbb{R}^\omega$ is given the box topology. Consider, the basis element

$$B = \prod_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right) \tag{10.12}$$

for the box topology on $\mathbb{R}^\omega$.

We assert that $f^{-1}(B)$ is not open in $\mathbb{R}$. If it were open, then there would exist $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq f^{-1}(B)$. This would mean $f((-\epsilon, \epsilon)) \subseteq B$, so that

$$f_n((-\epsilon, \epsilon)) = (-\epsilon, \epsilon) \subseteq \left(-\frac{1}{n}, \frac{1}{n} \right) \tag{10.13}$$

for each n. But this is impossible for n sufficiently large that $\frac{1}{n} < \epsilon$. Thus, $f^{-1}(B)$ is not open, and f is not continuous.

Chapter 11

Lecture11

Exercise Let $U_\epsilon(x) = \{y \in X | d(x, y) < \epsilon\}$. Prove that the collection of all such $U_\epsilon(x)$ forms a basis for a topology on X.

Definition 11.0.1. *A topological space $(X, \mathcal{T})$ is metrizable if there exists a metric d on X such that the topology induced by d is equal to $\mathcal{T}$.*

Lemma 11.0.1. *Let d and d' be two metrics on X defining the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$ and each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$U_\delta^{d'}(x) \subseteq U_\epsilon^d(x) \tag{11.1}$$

Property 11.0.1. *The metrics d_1, d_2, d_∞ define the same topology on $\mathbb{R}^n$. This is the product topology(or box as n is finite) on $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times).*

Proof This can be shown using graphs.

In the following, we will define a metric that is used to bound another metric. Let d be a metric on X. Define

$$\bar{d}(x, y) = \min\{d(x, y), 1\} \tag{11.2}$$

Theorem 11.0.1. *$\bar{d}$ is a metric on X defining the same topology as d.*

First, we need to check that $\bar{d}$ is a metric. The only non-trivial part is the triangle inequality. For any $x, y, z \in X$, we want to show that

$$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z) \tag{11.3}$$

If $\bar{d}(x, y)$ or $\bar{d}(y, z)$ equals 1, then the right hand side is at least 1, and the inequality holds. If both $\bar{d}(x, y)$ and $\bar{d}(y, z)$ are less than 1, then $\bar{d}(x, y) = d(x, y)$ and $\bar{d}(y, z) = d(y, z)$. By the triangle inequality of d, we have

$$\bar{d}(x, z) = \min\{d(x, z), 1\} \leq d(x, z) \leq d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z) \tag{11.4}$$

Now we note that in any metric space, the collection of ϵ-balls with $\epsilon < 1$ forms a basis for the topology induced by the metric. It follows that $\bar{d}$ and d induce the same topology on X, because the collections of ϵ-balls with $\epsilon < 1$ are the same for both metrics.

■

We want to define a metric on $\mathbb{R}^\omega$.

Definition 11.0.2. *For $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots) \in \mathbb{R}^\omega$, define*

$$\bar{d}(x, y) = \min(\sup(|x_i - y_i|), 1) \tag{11.5}$$

which is called the uniform metric on $\mathbb{R}^\omega$. And the topology it induces is called the uniform topology.

Also we can generalize the metric for $\mathbb{R}^J$ for any J. (sup(min) or min(sup) does not matter)

Theorem 11.0.2. *Uniform topology on $\mathbb{R}^J$ is:*

- *(strictly) finer than the direct product topology;*
- *(strictly) coarser than the box topology.*

provided that J is infinite.

Proof

First, we show that the uniform topology is finer than the product topology. Let $x \in \mathbb{R}^J$ and let U be a basic open set in the product topology containing x. Then

$$U = \prod_{\alpha \in J} U_\alpha \tag{11.6}$$

where U_α is an open set in $\mathbb{R}$ and $U_\alpha = \mathbb{R}$ for all but finitely many α. Let K be the finite set of indices α such that $U_\alpha \neq \mathbb{R}$. For each $\alpha \in K$, there exists $1 > \epsilon_\alpha > 0$ such that

$$U_{\epsilon_\alpha}(x_\alpha) \subseteq U_\alpha \tag{11.7}$$

Let $\epsilon = \min_{\alpha \in K} \epsilon_\alpha$. Then the open ball $U_\epsilon^{\bar{d}}(x)$ in the uniform topology is contained in U. Thus the uniform topology is finer than the product topology.

On the other hand, let

$$U = \prod_{\alpha \in J} (x_\alpha - \epsilon, x_\alpha + \epsilon) \tag{11.8}$$

with $\epsilon < 1$ which is a basis element in the uniform topology. And U is also a basis element in the box topology. Thus the uniform topology is coarser than the box topology.

For the strictness of the two sides, we can use some examples.

Since the uniform topology is finer than the product topology and the the box topology is finer than the uniform topology, we have that for a sequence (x_n) in $\mathbb{R}^J$, if $x_n \to x$ in the box topology, then $x_n \to x$ in the uniform topology, and if $x_n \to x$ in the uniform topology, then $x_n \to x$ in the product topology.

So for the strictness, it suffices to find a sequence that converges in one topology but not in the other.

Let $\{w_i\}_{i=1}^\infty$ be a sequence in $\mathbb{R}^\omega$ defined as

$$w_1 = (1,1,1,\ldots), w_2 = (0,2,2,\ldots), w_3 = (0,0,3,3,\ldots), \ldots \tag{11.9}$$

Then $w_n \to 0$ in the product topology. But it diverges in the uniform topology and the box topology. Thus the uniform topology is strictly finer than the product topology.

Let $\{x_i\}_{i=1}^\infty$ be a sequence in $\mathbb{R}^\omega$ defined as

$$x_1 = (1,1,1,\ldots), x_2 = (0, \frac{1}{2}, \frac{1}{2}, \ldots), x_3 = (0,0,\frac{1}{3},\frac{1}{3},\ldots), \ldots \tag{11.10}$$

Then w.l.o.g. if $\epsilon < 1$, the ball in the uniform topology

$$U_\epsilon^{\bar{d}}(0) = \{y \in \mathbb{R}^\omega | \sup(|y_i - 0|) < \epsilon\} = \{y \in \mathbb{R}^\omega | |y_i| < \epsilon, \forall i\} \tag{11.11}$$

contains all but finitely many x_n. Thus $x_n \to 0$ in the uniform topology. However, for the box topology, consider

$$(-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots \tag{11.12}$$

which is a basis element in the box topology containing 0. This set does not contain any x_n for all n. Thus x_n does not converge to 0 in the box topology. Therefore, the uniform topology is strictly coarser than the box topology.

Consider the sequence $\{y_n\}_{n=1}^\infty$ in $\mathbb{R}^\omega$ defined as

$$y_1 = (1,0,0,0,\ldots), y_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \ldots), y_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \ldots), \ldots \tag{11.13}$$

Then $y_n \to 0$ in all three topologies.

■

Recall that for two metric spaces (X,d) and (Y,d'), a function $f : X \to Y$ is continuous if and only if for each $x \in X$ and each $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x,y) < \delta \Rightarrow d'(f(x), f(y)) < \epsilon \tag{11.14}$$

Problem If $x_n \to 0$ in the box topology on $\mathbb{R}^\omega$, what can be said about x_n? Is the guess: $\exists M, \forall m \geq M, x_n^{(m)} = 0$ correct?

Answer Not sure.

Theorem 11.0.3 (The sequence lemma)**.** *Let X be a topological space, $A \subseteq X$. If there is a sequence of points $x_n \in A$ with $\lim_{n\to\infty} x_n = x$, then $x \in \bar{A}$. The converse holds if X is metrizable.*

Proof If $x_n \in A$ and $\lim_{n\to\infty} x_n = x$, then for each open set U containing x, we have $U \cap A$ contains all but finitely many x_n, so $U \cap A \neq \emptyset$. Thus $x \in \bar{A}$.

For the converse, suppose X is metrizable with metric d. If $x \in \bar{A}$, then for each $n \in \mathbb{N}$, the open ball $U_{1/n}(x)$ intersects A. Thus we can choose $x_n \in U_{1/n}(x) \cap A$. Then $\lim_{n\to\infty} x_n = x(\forall \epsilon > 0, \exists N > \frac{1}{\epsilon}$, such that $d(x_n, x) < \epsilon$ for all $n > N)$.

■

Chapter 12

Later

Definition 12.0.1. *A collection $\mathcal{A}$ of subsets of a space X is called "a covev" or "a covering" if the union of the elements of $\mathcal{A}$ is X; it is called an open covering if all these sets are open.*

Definition 12.0.2. *X is called a compact space if every open covering $\mathcal{A}$ contains a finite subcollection which also covers X.*

Remark Being compact depends on the topology: For example, consider X with the antidiscrete topology, i.e., $\mathcal{T} = \{\emptyset, X\}$. Then X is compact.

Example 12.0.1. $\mathbb{R}$ *with the standard topology is not compact. Consider the open covering* $\mathcal{A} = \{(-n, n)|n \in \mathbb{N}\}$.

Example 12.0.2. $(0, 1]$ *with the standard topology is not compact. Consider the open covering* $\mathcal{A} = \{(1/n, 1]|n \in \mathbb{N}\}$.

Example 12.0.3.

- $\{0\} \cup \{\frac{1}{n}|n \in \mathbb{Z}^+\} \subseteq \mathbb{R}$ *is compact because* $\{0\}$ *is the limit point and any open covering must contain an open set containing* 0*, which covers all but finite points.*
- $\{\frac{1}{n}|n \in \mathbb{Z}^+\}$ *is not compact. Consider the open covering* $\mathcal{A} = \{(1/n - \epsilon, 1/n + \epsilon)|n \in \mathbb{N}\}$. *(Similarly, for infinite X with discrete topology, X is not compact.)*

Definition 12.0.3. *$Y \subseteq X$ is compact if every open covering of Y by sets open in X contains a finite subcollection covering Y.*

Lemma 12.0.1. *$Y \subseteq X$ is compact if and only if every open covering of Y by sets open in Y contains a finite subcollection covering Y.*

Proof
Trivial.

Theorem 12.0.1. *Every closed subspace of a compact space is compact.*

Proof Let X be a compact space and let Y be a closed subspace of X. Let $\mathcal{A}$ be an open covering of Y by sets open in X. Since Y is closed, $X \setminus Y$ is open in X. Thus, $\mathcal{A} \cup \{X \setminus Y\}$ is an open covering of X. By the compactness of X, there exists a finite subcollection $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathcal{A}' \cup \{X \setminus Y\}$ covers X. Therefore, $\mathcal{A}'$ covers Y. Hence, Y is compact.

Theorem 12.0.2. *Every compact subspace of a Hausdorff space is closed.*

Proof
It suffices to show that for any $x \in X \setminus Y$, there exists an open set U containing x such that $U \cap Y = \emptyset$. For each $y \in Y$, since X is Hausdorff, there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Then $\{V_y|y \in Y\}$ is an open covering of Y. By the compactness of Y, there exists a finite subcollection $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$ that covers Y. Let

$$U = \bigcap_{i=1}^{n} U_{y_i} \tag{12.1}$$

Then U is an open set containing x. Moreover

$$U \cap Y = (\bigcap_{i=1}^{n} U_{y_i}) \cap Y = \emptyset \tag{12.2}$$

Thus, $U \cap Y = \emptyset$. Hence, Y is closed.

■

Lemma 12.0.2. *Y is a compact subspace of Hausdorff space X. And $x \in X \setminus Y$. Then there exist open sets U and V such that $x \in U$, $Y \subseteq V$ and $U \cap V = \emptyset$.*

Proof Proved inside the proof of the theorem.

Theorem 12.0.3. *The image of a compact set under a continuous map is compact.*

Proof Let $f : X \to Y$ be a continuous map. Consider any open covering $\mathcal{A}$ of $f(X)$ by sets open in Y. Then $\{f^{-1}(U) | U \in \mathcal{A}\}$ is an open covering of X by sets open in X. By the compactness of X, there exists a finite subcollection $\{f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_n)\}$ that covers X. Thus, $\{U_1, U_2, \ldots, U_n\}$ is a finite subcollection of $\mathcal{A}$ that covers $f(X)$. Hence, $f(X)$ is compact.

■

Theorem 12.0.4. *Let $f : X \to Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof

We just combine the above theorems.

It suffices to show that images of closed sets under f are closed. Consider any closed set D in X. By the theorem, D is compact. Thus, by the previous theorem, $f(D)$ is compact. Since Y is Hausdorff, by another theorem, $f(D)$ is closed. Hence, f is a homeomorphism.

■

Lemma 12.0.3 (tube lemma)**.** *Let $x_0 \in X$, suppose $\{x_0\} \times Y$ is convered by open sets W_i in $X \times Y$. Then one can choose a finite subcovering $W_1, W_2, \ldots, W_n$ of $\{x_0\} \times Y$ and find an open neighbor U of x_0 such that $U \times Y \subseteq \bigcup_{i=1}^n W_i$.*

Proof One may assume that $W_i = U_i \times V_i$ where U_i is open in X and V_i is open in Y. Then $\{V_i\}$ is an open covering of Y. By the compactness of Y, there exists a finite subcollection $\{V_1, V_2, \ldots, V_n\}$ that covers Y.

To be done.

Theorem 12.0.5. *The product of finitely many compact spaces is compact.*

Proof It is enough to prove for two sets. Let X and Y be compact spaces.

We can cover $X \times Y$ by finitly many tubes by the tube lemma. And each tube can be covered by finitly many open sets. Thus $X \times Y$ is compact.

To be done.

Definition 12.0.4. *A collection $\mathcal{C}$ of sets has the finite intersection property if for any finite subcollection $\{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{C}$, we have*

$$C_1 \cap C_2 \cap \ldots \cap C_n \neq \emptyset \tag{12.3}$$

Theorem 12.0.6. *X is a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X with the finite intersection property, we have*

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset \tag{12.4}$$

Proof

Let $\mathcal{A}$ be a collection of open sets in X. Then $\mathcal{C} = \{X \setminus A | A \in \mathcal{A}\}$ is a collection of closed sets in X.

Then $\mathcal{A}$ is an open covering of X if and only if $\bigcap_{C \in \mathcal{C}} C = \emptyset$.

Then a finite subcollection of $\mathcal{A}$ covers X if and only if the corresponding finite subcollection of $\mathcal{C}$ has empty intersection.

Now the theorem follows directly.

■

Chapter 13

Later

13.1 Metrizability

Recall the following theorem.

Theorem 13.1.1 (Sequence Lemma)**.** *Let X be a topological space. Let $A \subseteq X$. If $\exists \{x_n\} \subseteq A$ such that $x_n \to x$ in X, then $x \in \overline{A}$. The converse is true if X is metrizable.*

Theorem 13.1.2 (Heine's definition of limit)**.** *Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y. The converse is true if X is metrizable.*

Poof

($\Rightarrow$) Let f be continuous. Let $V \ni f(x)$ be open in Y. Then $f^{-1}(V)$ is open in X and contains x. Since $x_n \to x$, $\exists N$ such that $\forall n \geq N$, $x_n \in f^{-1}(V)$. Thus, $\forall n \geq N$, $f(x_n) \in V$. Hence, $f(x_n) \to f(x)$.

($\Leftarrow$) Recall that f is continuous if and only if for all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

Let $A \subseteq X$. Let $x \in \bar{A}$. Since X is metrizable, by the sequence lemma, there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to x$. By the assumption, $f(x_n) \to f(x)$. Thus, by the sequence lemma again, $f(x) \in \overline{f(A)}$. So $f(\overline{A}) \subseteq \overline{f(A)}$. Hence, f is continuous.

■

Definition 13.1.1. *X is first-countable if it has a countable basis at each $x \in X$. Given $x \in X$, there exists a countable collection of open sets $\{U_n\}$ such that for any open set U containing x, $\exists n$ such that $U_n \subseteq U$. (From this we can construct $\tilde{U}_n = \bigcap_{i=1}^{n} U_i$ such that $\{\tilde{U}_n\}$ is also a countable basis at x with $\tilde{U}_{n+1} \subseteq \tilde{U}_n$.)*

Definition 13.1.2. *X is second-countable if it has a countable basis for the topology. There exists countable a basis $\mathcal{B}$ for X such that $\forall x \in X, \forall U$ open in X containing x, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Property 13.1.1. *If X is second-countable, then X is first-countable.*

Example 13.1.1. *$\mathbb{R}^n$ has a countable basis. For instance, the set of all open balls with rational radii and centers at points with rational coordinates forms a countable basis for the standard topology on $\mathbb{R}^n$.*

Example 13.1.2. *$\mathbb{R}$ with finite complement topology is not first-countable. Let $U_1, U_2, \ldots$ be a countable open sets containing x. Then $\bigcap_{n=1}^{\infty} U_n \setminus \{x\}$ is not empty, so there exists $y \neq x$ such that $y \in U_n$ for all n. Take $U = \mathbb{R} \setminus \{y\}$, which is open and contains x. However, there is no U_n such that $U_n \subseteq U$. Hence, $\mathbb{R}$ with finite complement topology is not first-countable.*

Example 13.1.3. *X is uncountable with discrete topology. Then $\forall x \in X$, the set $\{x\}$ is open. So any basis of X must contain $\{x\}$ for all $x \in X$. So X is not second-countable. But X is metrizable thus first-countable.*

Example 13.1.4. *$\mathbb{R}^2$ with "Amazon River metric". Define*

$$d((x,y),(x',y')) = \begin{cases} |y-y'|, & x = x' \\ |y|+|y'|+|x-x'|, & x \neq x' \end{cases} \tag{13.1}$$

Then $\{(x,y) | x = x_0, y \in (y_0-\epsilon, y_0+\epsilon)\}$ with $\epsilon < |y_0|$ is an open ball centered at (x_0,y_0). There are uncountable many such disjoint open sets. So $\mathbb{R}^2$ with Amazon River metric is not second-countable. But it is metrizable thus first-countable.

Example 13.1.5. *Let $\mathbb{R}_l$ be the set of real numbers with the lower limit topology. Then $\mathbb{R}_l$ is not second-countable. Suppose $\mathcal{B}$ is a countable basis for $\mathbb{R}_l$. For each $x \in \mathbb{R}$, there exists a open set $[x, +\infty)$ containing x. Thus, there exists a basis element $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, +\infty)$ which means $minB_x = x$. Since $\mathcal{B}$ is countable, the set of minimums $\{minB|B \in \mathcal{B}\}$ is also countable. This contradicts the uncountability of $\mathbb{R}$. Hence, $\mathbb{R}_l$ is not second-countable. However, $\mathbb{R}_l$ is first-countable since for each $x \in \mathbb{R}$, the collection of basis elements $\{[x, x + 1/n)|n \in \mathbb{N}\}$ forms a countable basis at x.*

Property 13.1.2. $\mathbb{R}^\omega$ *with box topology is not metrizable. (Hence $\mathbb{R}^J$ with box topology is not metrizable for infinite J.)*

Proof

We let
$$A = \{(x_1, x_2, \ldots)|x_i > 0, \forall i\} \tag{13.2}$$
Then $0 \in \bar{A}$ in the box topology.

We let
$$B = (a_1, b_1) \times (a_2, b_2) \times \cdots \tag{13.3}$$
such that $a_i < 0 < b_i$ for all i. Then B is a basis element in the box topology containing 0. And the intersection of B and A is not empty.

We assert that there is no sequence in A converging to 0. Let $\{a_n\}_{n=1}^\infty$ be a sequence in A, where
$$a_n = (a_n^{(1)}, a_n^{(2)}, \ldots) \tag{13.4}$$
with $a_n^{(i)} > 0$ for all i, n. We can construct a basis element B' in the box topology containing 0:
$$B' = (-a_1^{(1)}, a_1^{(1)}) \times (-a_2^{(2)}, a_2^{(2)}) \times \cdots \tag{13.5}$$
Then B' does not contain any a_n. Thus $\{a_n\}$ does not converge to 0.

So the sequence lemma tells us that $\mathbb{R}^\omega$ with box topology is not metrizable.

■

Property 13.1.3. *Let J be uncountable. Then $\mathbb{R}^J$ in product topology is not metrizable.*

Proof

Take
$$A = \{(x_1, x_2, \cdots)|x_\alpha = 1 \text{ for all but finitely many coordinates}\} \tag{13.6}$$
Then $0 \in \bar{A}$ in the product topology(Easy to verify).

Given n, let J_n denotes the subset of J consisting of those indices α such that the α-th coordinate of a_n is different from 1. Since J_n is finite for each n, then the union $\bigcup_{n=1}^\infty J_n$ is countable. However, J is uncountable, so there exists $\beta \in J$ such that $\beta \notin \bigcup_{n=1}^\infty J_n$. That is, for all n, the β-th coordinate of a_n is equal to 1.

Let $\{a_n\}_{n=1}^\infty$ be a sequence in A, where there exists $\beta \in J$ such that $a_\beta^{(n)} = 1$ for all n. Then we can take
$$U = \pi_\beta^{-1}((-1, 1)) \tag{13.7}$$
which is a basis element in the product topology containing 0. However, U does not contain any a_n. Thus $\{a_n\}$ does not converge to 0.

By the sequence lemma, $\mathbb{R}^J$ in product topology is not metrizable.

■

Chapter 14

Lecture14

Theorem 14.0.1. $\mathbb{R}^\omega$ *is metrizable in the product topology.*

Proof Define

$$\bar{d}(x_i, y_i) = \min(|x_i - y_i|, 1) \tag{14.1}$$

Let $x, y \in \mathbb{R}^\omega$. Then we define

$$D(x, y) = \sup\{\frac{\bar{d}(x_i, y_i)}{i} | i \in \mathbb{Z}_{>0}\} \tag{14.2}$$

Then we can show that D is a metric on $\mathbb{R}^\omega$ because we have

$$\bar{d}(x_i, z_i) \leq \bar{d}(x_i, y_i) + \bar{d}(y_i, z_i) \Longrightarrow D(x, z) \leq D(x, y) + D(y, z) \tag{14.3}$$

Why does it define the product topology?

($\Rightarrow$) Let U be open in the metric topology. Then there exists $\epsilon > 0$ such that $B_D(x, \epsilon) \subseteq U$. Let N be such that $\frac{1}{N} < \epsilon$. Then let

$$V = \prod_{i=1}^{N}(x_i - \epsilon, x_i + \epsilon) \times \prod_{i=N+1}^{\infty} \mathbb{R} \tag{14.4}$$

Then let $y \in V$. We have

1. If $i \leq N$, then $\bar{d}(x_i, y_i) < \epsilon$.
2. If $i > N$, then $\bar{d}(x_i, y_i) \leq 1$.

Thus, we have $D(x, y) = \sup\{\frac{\bar{d}(x_i, y_i)}{i}\} \leq \max\{\epsilon, \frac{1}{N}\} = \epsilon$. So $y \in B_D(x, \epsilon) \subseteq U$. Hence, $V \subseteq U$. So U is open in the product topology.

($\Leftarrow$) Let U be open in the product topology. Then

$$U = \prod U_i \tag{14.5}$$

where U_i is open if $i \in \{\alpha_1, \cdots, \alpha_n\}$ and $U_i = \mathbb{R}$ otherwise. We want V open in the metric topology such that $V \subseteq U$. Let

$$x \in U \tag{14.6}$$

Then

$$x_i \in U_i \tag{14.7}$$

for $i \in \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Then there exists $\epsilon_i > 0$ such that

$$(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i \tag{14.8}$$

Let

$$\epsilon = \min(\{\frac{\epsilon_i}{i} | i \in \{\alpha_1, \alpha_2, \cdots, \alpha_n\}\} \cup \{1\}) \tag{14.9}$$

Then we claim that

$$U_D(x, \epsilon) \subseteq U \tag{14.10}$$

Let $y \in U_D(x, \epsilon)$. Then we have

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \epsilon \tag{14.11}$$

If $i \in \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, then

$$\frac{\bar{d}(x_i, y_i)}{i} < \epsilon \le \frac{\epsilon_i}{i} \tag{14.12}$$

which implies that

$$y_i \in (x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i \tag{14.13}$$

So $y \in U$. Hence, $U_D(x, \epsilon) \subseteq U$. So U is open in the metric topology.

■

Definition 14.0.1. $(X, \le)$ *has a least upper bound property or supremum property if every non-empty subset of* X *that is bounded above has a least upper bound in* X. *(For example,* $\mathbb{R}$ *has the least upper bound property)*

Let X be a set with order topology and supremum property.

Theorem 14.0.2. $\forall a \le b \in X$, *we have* $[a, b]$ *is compact.*

Proof Let $\mathcal{A}$ be an open covering of $[a, b]$. We need that there exists a finite subcovering of $\mathcal{A}$ covering $[a, b]$.
Step 1 Let $x \in [a, b]$ with $x \neq b$, there exists $y > x$ such that at most two elements of $\mathcal{A}$ can cover $[x, y]$.

1. If x has an immediate successor $s(x)$, then $[x, s(x)] = \{x, s(x)\}$ is covered by two elements from $\mathcal{A}$.
2. If not. Take $x \in A$ with $A \in \mathcal{A}$ such that $\exists c \in X, x < c \le b, [x, c) \subseteq A$. Take any $y \in (x, c)$. Then $[x, y] \subseteq A$ is covered by one element from $\mathcal{A}$.

Step 2 Let C be the set of points y with $a \le y \le b$ such that $[a, y]$ has a finite subcovering from $\mathcal{A}$. Since $a \in C$, C is non-empty. Let $c = \sup C$.
Step 3 Show that $c \in C$.
We know from the first step that $c \neq a$.
Choose $A \in \mathcal{A}$ such that $c \in A$. Then $\exists d \in [a, b]$ such that $(d, c] \subseteq A$. If $c \notin C$, then there must be some point $z \in (d, c)$ with $z \in C$. Then $[a, z]$ can be covered by finite elements(say M) from $\mathcal{A}$. Thus, $[a, c] \subseteq [a, z] \cup (d, c]$ can be covered by at most $M + 1$ elements from $\mathcal{A}$. So $c \in C$ which contradicts the assumption that $c \notin C$. Hence, $c \in C$.
Step 4 Show that $c = b$. If not, by Step 1, $\exists e > c$ such that $[c, e]$ can be covered by two elements from $\mathcal{A}$. Since $[a, c]$ can be covered by finite elements(say M) from $\mathcal{A}$, $[a, e]$ can be covered by at most $M + 2$ elements from $\mathcal{A}$. So $e \in C$ which contradicts the assumption that $c = \sup C$. Hence, $c = b$.

■

Corollary 14.0.1. $[a, b] \subseteq \mathbb{R}$ *is compact.*

Corollary 14.0.2. $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ *is compact.*

Theorem 14.0.3. $A \subseteq \mathbb{R}^n$ *is compact if and only if* A *is closed and bounded.*

Proof
($\Rightarrow$) Take a covering by

$$\bigcup U_N(0) \tag{14.14}$$

which is the union of open balls centered at 0 with radius N for $N \in \mathbb{Z}_{>0}$. Since A is compact, there exists a finite subcovering. Thus, $A \subseteq U_{N_0}(0)$ for $N_0 = \max\{N_1, N_2, \cdots, N_k\}$. So A is bounded.
We know that $\mathbb{R}^n$ is a Hausdorff space. So A is closed as a compact subset of a Hausdorff space.
($\Leftarrow$) Since A is bounded, there exists n such that

$$A \subseteq [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \tag{14.15}$$

We know that the closed subset of a compact set is compact. So A is compact.

■

Theorem 14.0.4 (Extreme value theorem)**.** *Let* $f : X \to Y$ *be continuous where* X *is compact and* Y *is an ordered set with the order topology. Then* $\exists a, b \in X$ *such that* $\forall x \in X$, $f(a) \le f(x) \le f(b)$.

Proof
We know that $A = f(X)$ is compact. We want to show that A has a largest element.
We assume the contrary that A has no largest element. Then $\forall a \in A$, $\exists b \in A$ such that $b > a$. So

$$A = \bigcup_{a \in A} (-\infty, a) \tag{14.16}$$

which is an open covering of A. Since A is compact, there exists a finite subcovering. Let the largest element among the finite elements be a_0. Then a_0 is not covered. This is a contradiction. So A has a largest element. Similarly, A has a smallest element.

■

Definition 14.0.2. *Let (X, d) be a metric space, $A \subseteq X$, $x \in X$. Let $d(x, A) = \inf\{d(x, a) | a \in A\}$.*

Property 14.0.1. *$d(x, A)$ is a continuous function for A fixed.*

Proof

We know

$$d(x, A) \le d(x, a) \le d(x, y) + d(y, a) \tag{14.17}$$

for any $a \in A$. Then

$$d(x, A) - d(x, y) \le \inf\{d(y, a) | a \in A\} = d(y, A) \tag{14.18}$$

So

$$d(x, A) - d(y, A) \le d(x, y) \tag{14.19}$$

■

Chapter 15

Later

Definition 15.0.1. *Two norms $||\cdot||_a$ and $||\cdot||_b$ are equivalent if there exist positive constants C_1 and C_2 such that for all vectors x,*

$$C_1||x||_a \leq ||x||_b \leq C_2||x||_a.$$

Property 15.0.1. *Equivalence of norms is an equivalence relation.*

Theorem 15.0.1. *All norms on $\mathbb{R}^n$ are equivalent.*

Proof

Given $||\cdot||_a$ and $||\cdot||_b$ are two norms on $\mathbb{R}^n$. We define a function $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ by

$$f(x) = \frac{||x||_b}{||x||_a} > 0. \tag{15.1}$$

which is continuous on $\mathbb{R}^n \backslash \{0\}$.

Since $f(x) = f(\lambda x)$ for any $\lambda > 0$, we say that $f(x)$ is conpletely determined by its values on the unit sphere $S = \{x \in \mathbb{R}^n : ||x||_a = 1\}$. Note that S is compact, so $f(x)$ attains its minimum and maximum on S, say m and M. Thus, for any $x \in \mathbb{R}^n \backslash \{0\}$, we have

$$0 < m \leq f(x) \leq M < \infty, \tag{15.2}$$

which implies

$$m||x||_a \leq ||x||_b \leq M||x||_a. \tag{15.3}$$

So the two norms are equivalent. ∎

Remark In ∞-dimensional space, the sphere is not compact. However, it is closed and bounded.

Exercise Give an example of norms on l_1 (convergent series) $l_1 = \{(x_1, x_2, \cdots) | \sum |x_i| < \infty\}$.

Now let's come back to uniform continuity. Goal: Prove that any continuous map $f : (X, d_X) \to (Y, d_Y)$ with X compact is uniformly continuous.

Definition 15.0.2. *Let X be a metric space. $A \subseteq X$. Then diam$(A) = \sup\{d(x,y) : x, y \in A\}$ is called the diameter of A.*

Lemma 15.0.1 (the Lebesgue Number Lemma)**.** *Let (X, d) be a compact metric space, and let $\mathcal{A}$ be an open cover of X. Then there exists a positive number $\delta > 0$ (called a Lebesgue number for the cover $\mathcal{A}$) such that for every subset $Y \subseteq X$ with diameter less than δ (i.e., for all $x, y \in Y$, $d(x,y) < \delta$), there exists an open set $A \in \mathcal{A}$ that contains Y.*

Proof of the Lemma

If $X \in \mathcal{A}$, there's nothing to prove.

Otherwise take a finite subcovering $\{A_1, A_2, \cdots, A_n\}$ of $\mathcal{A}$. For each i, let $C_i = X \backslash A_i$ which is closed. We know that a closed subset of a compact set is compact, so each C_i is compact. Define

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, C_i) \tag{15.4}$$

be the average distance from x to the closed sets C_i.

We show that $f(x) > 0$ for all $x \in X$.

For all $x \in X$, we choose A_i such that $x \in A_i$. Choose $\epsilon > 0$ such that $U_\epsilon(x) \subseteq A_i$. Then $d(x, C_i) \geq \epsilon > 0$. So $f(x) \geq \frac{1}{n}d(x, C_i) \geq \frac{\epsilon}{n}$. Let $\delta = \min_{x \in X} f(x) > 0$ (because f is continuous on compact set X). We show that δ is a Lebesgue number.

Take $B \subseteq X$ with $diam(B) < \delta$. Then we can take $x_0 \in B$ with $B \subseteq U_\delta(x_0)$. Then we can take C_m such that

$$\delta \leq f(x_0) \leq d(x_0, C_m). \tag{15.5}$$

for some m. So we have

$$U_\delta(x_0) \subseteq A_m = X \backslash C_m. \tag{15.6}$$

So $B \subseteq A_m$. ■

Example 15.0.1. *Let $X \subseteq \mathbb{R}$ be convered by (a_α, b_α). Take a finite subcovering (a_i, b_i), $i = 1, 2, \cdots, n$. Then $\delta = \min\{b_i - a_i : i = 1, 2, \cdots, n\}$ is a Lebesgue number.*

Definition 15.0.3. *A function between two metric spaces $f : (X, d_X) \to (Y, d_Y)$ is said to be uniformly continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x_1, x_2 \in X$, if $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \epsilon$.*

Theorem 15.0.2. *Let $f : X \to Y$ be a continuous function between two metric spaces. If X is compact, then f is uniformly continuous.*

Proof

Given $\epsilon > 0$, take the open covering of Y by balls $B(y, \frac{\epsilon}{2})$ of radius $\frac{\epsilon}{2}$. Let $\mathcal{A}$ be the open covering of X by the inverse images of these balls under f. Choose δ to be a Lebesgue number for the covering $\mathcal{A}$. Then if x_1 and x_2 are any two points in X with $d_X(x_1, x_2) < \delta$, the set $\{x_1, x_2\}$ has diameter less than δ, so there exists an open set $U = f^{-1}(B(y, \frac{\epsilon}{2}))$ in $\mathcal{A}$ that contains both x_1 and x_2. So $f(x_1)$ and $f(x_2)$ both lie in $B(y, \frac{\epsilon}{2})$, which implies that

$$d_Y(f(x_1), f(x_2)) < \epsilon. \tag{15.7}$$

■

Definition 15.0.4. *If X is a space, a point x of X is said to be an isolated point of X if the one-point set $\{x\}$ is open in X.*

Theorem 15.0.3. *Let X be a non-empty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof

Step 1 We show first that given any non-empty open set U of X and any point x of X, there exists a non-empty open set V contained in U such that $x \notin \bar{V}$.

Choose a point y of U different from x; this is possible if x is in U because x is not an isolated point of X and it is possible if x is not in U simply because U is non-empty. Since X is Hausdorff, there exist disjoint open sets W_x and W_y containing x and y, respectively. Then the open set $V = U \cap W_y$ is non-empty, is contained in U, and its closure $\bar{V}$ is contained in the complement of W_x, so $x \notin \bar{V}$.

Step 2 We show that given $f : \mathbb{Z}_+ \to X$, the function f is not surjective. It follows that X is uncountable.

Let $x_n = f(n)$. Apply Step 1 to the non-empty open set X to choose a non-empty open set $V_1 \subseteq X$ such that $x_1 \notin \bar{V_1}$. In general, given V_{n-1} open and non-empty, choose V_n to be a non-empty open set such that $V_n \subseteq V_{n-1}$ and $x_n \notin \bar{V_n}$. Then we have a sequence of non-empty closed sets $\bar{V_n}$ with

$$\bar{V_1} \supseteq \bar{V_2} \supseteq \bar{V_3} \supseteq \cdots \tag{15.8}$$

Since X is compact, any collection of closed subsets of X with the finite intersection property has non-empty intersection. Thus, there exists a point y in the intersection of all the $\bar{V_n}$. By construction, $y \neq x_n$ for all n, so f is not surjective.

■

Corollary 15.0.1. $[a, b] \subseteq \mathbb{R}$ *is uncountable.*

Chapter 16

Later

Definition 16.0.1 (def1)**.** *X is compact if any open covering admits a finite subcovering.*

Definition 16.0.2 (def2: Frechet compactness)**.** *X is limit point compact if every infinite subset of X has a limit point in X.*

Definition 16.0.3 (def3: sequential compactness/Bozano-Weierstrass compactness)**.** *X is sequentially compact if every sequence $\{x_n\} \subseteq X$ has a convergent subsequence converging to a point in X.*

Theorem 16.0.1. *If X is a metric space, then X is compact if and only if it is limit point compact if and only if it is sequentially compact.*

Proof

The proof is quite long and technical. We will only some directions in the following.

Theorem 16.0.2. *For an arbitrary topological space, compactness implies limit point compactness. The converse is not true in general.*

Proof

Let X be compact, and let $A \subseteq X$ be infinite.

We assume that A has no limit point. Then $A = \bar{A}$. So $X \setminus A$ is open. Furthermore, for each $a \in A$, we can choose a neighborhood U_a of a such that $U_a \cap (A \setminus \{a\}) = \emptyset$.

Then $\{U_a : a \in A\} \cup \{X \setminus A\}$ is an open covering of X. Since X is compact, there exists a finite subcovering, say $\{U_{a_1}, U_{a_2}, \cdots, U_{a_n}\} \cup \{X \setminus A\}$. Thus,

$$X = \bigcup_{i=1}^{n} U_{a_i} \cup (X \setminus A) = (X \setminus A) \cup \{a_1, a_2, \cdots, a_n\},$$

which implies that A is finite, a contradiction. Hence, A has a limit point in X.

■

Example 16.0.1 (Counterexample for the converse)**.** *Let $Y = \{p, q\}$ with anti-dicrete topology $J_Y = \{\emptyset, Y\}$. Let $X = \mathbb{N} \times Y$ with the product topology where $\mathbb{N}$ has the discrete topology. Then every non-empty set $A \subseteq X$ has a limit point. Because if $(n, p) \in A$, then any open set containing (n, q) intersects A at (n, p). So X is limit point compact. However, X is not compact. Because $\{\{n\} \times Y : n \in \mathbb{N}\}$ is an open covering of X which admits no finite subcovering.*

Example 16.0.2 (Limit point compact but not sequentially compact)**.** *Let X be defined as the above example. Consider the sequence $\{(n, p)\}_{n=1}^{\infty}$. This sequence has no convergent subsequence. Because for any point (m, p), the open set $\{m\} \times Y$ contains only finitely many terms of the sequence; for any point (m, q), the open set $\{m\} \times Y$ also contains only finitely many terms of the sequence. So X is not sequentially compact.*

Theorem 16.0.3. *For a first-countable topological space X, limit point compactness implies sequential compactness.*

Proof

Take a sequence $\{x_n\} \subseteq X$.

If the set of values $\{x_n : n \in \mathbb{N}\}$ is finite, then there exists a value x that appears infinitely many times in the sequence. So the subsequence constantly equal to x converges to x.

Suppose the set of values $\{x_n : n \in \mathbb{N}\}$ is infinite. Since X is first-countable, we can construct a countable basis $\{U_k\}$ at a limit point a such that

$$x_{n_1} \in U_1, x_{n_2} \in U_2, \cdots, x_{n_k} \in U_k, \cdots \tag{16.1}$$

and

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k \supseteq \cdots \tag{16.2}$$

That is for each open set $U \ni a$ there exists N such that $\forall k \geq N$, $U_k \subseteq U$. Then $x_{n_k} \in U_k \subseteq U$ for all $k \geq N$. So $x_{n_k} \to a$. ■

Remark For topological spaces, we have the following facts:

1. Compactness ⇒ Limit Point Compactness
2. Sequential Compactness ⇒ Limit Point Compactness
3. But other implications are not true in general.

Theorem 16.0.4. *Sequential compactness implies compactness for metric spaces.*

Proof This is harder than other directions.
Exercise Prove that X is sequentially compact, then it is limit point compact.

16.1 Locally Compact

Definition 16.1.1. *A space X is said to be locally compact at x if there is some compact subspace C of X containing a neighborhood U of x. If X is locally compact at each of its points, then X is said to be locally compact.*

Example 16.1.1. $\mathbb{R}$ *is locally compact. Because for any $x \in \mathbb{R}$, take $U = (x-1, x+1)$ and $C = [x-1, x+1]$ which is compact.*

Example 16.1.2. $\mathbb{R}^n$ *is locally compact.*

Example 16.1.3 (non example)**.** $\mathbb{Q} \subseteq \mathbb{R}$ *is not locally compact.*

Exercise Show the above example.
Solution Let C be a compact subspace of $\mathbb{Q}$ containing a neighborhood U of $x \in \mathbb{Q}$. Since U is open in $\mathbb{Q}$, there exists an interval $(a, b) \subseteq \mathbb{R}$ such that $U = (a, b) \cap \mathbb{Q}$. We know Q is dense in $\mathbb{R}$, so there exists an irrational number $y \in (a, b)$. Let $\{q_n\} \subseteq (a, b) \cap \mathbb{Q}$ be a sequence converging to y. Since C is compact, so $\{q_n\}$ has a subsequence converging to a point in C. But a convergent sequence in $\mathbb{R}$ has a unique limit, so the subsequence converges to y, which is not in $\mathbb{Q}$ thus not in C. This is a contradiction. Hence, $\mathbb{Q}$ is not locally compact. ■

Example 16.1.4 (non example)**.** $\mathbb{R}^\omega$ *with product topology is not locally compact. If U is open, then U contains a basis element $(a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots$. Then U is not contained in any compact set C(C contains one factor of $\mathbb{R}$, then the projection to some factor space is $\mathbb{R}$, but a projection is continuous, which means $\mathbb{R}$ is compact. This is definitely not true).*

Chapter 17

Later

Theorem 17.0.1. *Let X be a topological space. Then X is Hausdorff locally compact if and only if there exists a space Y such that*

1. *X is a subspace of Y.*
2. *$Y \setminus X$ contains exactly one point p.*
3. *Y is compact Hausdorff.*

This Y is unique in the following sense: If Y, Y' are two spaces with the above properties and $Y = X \cup \{p\}, Y' = X \cup \{q\}$, then there exists a homeomorphism $h : Y \to Y'$ such that $h(x) = x$ for all $x \in X$ and $h(p) = q$.

Proof
Uniqueness Let $Y = X \cup \{p\}$, $Y' = X \cup \{q\}$ satisfies the above properties. Define $h : Y \to Y'$ as follows: $h(x) = x$ for all $x \in X$ and $h(p) = q$. We show that h is continuous. But the function is symmetric, so it is enough to show that $h(U)$ is open in Y' for all open set U in Y.

Take U be open. There are two cases.

If $U \subseteq X$, then we are done.

Suppose $p \in U$. Then $C = Y \setminus U$ is closed in Y. So C is compact.
Construction We introduce the topology on $Y = X \cup \{\infty\}$ as follows:

There are two types of open sets in Y:

1. $U \subseteq X \subseteq Y$ is open in X.
2. $U = Y \setminus C$ where C is compact subspace of X.

We need to check that this is a topology.

For the intersection of two open sets of type 1, we have $U_1 \cap U_2$ is open in X thus is open in Y.

For the intersection of type 1 and type 2, we have $U_1 \cap (Y \setminus C) = U_1 \cap (X \setminus C)$ which is the union of open sets in X thus is open in Y because C is closed in X(as a compact subspace of a Hausdorff space).

For the intersection of two open sets of type 2, we have $(Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2)$ which is open in Y because $C_1 \cup C_2$ is compact.

Similarly, one checks that the union of any collection of open sets is open. We have

1. $\bigcup U_\alpha = U$ is of type 1.
2. $\bigcup (Y \setminus C_\alpha) = Y \setminus \bigcap C_\alpha = Y \setminus C$ where $C = \bigcap C_\alpha$ is compact(Easy to check). This is of type 2.
3. $\bigcup U_\alpha \cup \bigcup (Y \setminus C_\beta) = U \cup (Y \setminus C) = Y \setminus (C \setminus U)$. This is of type 2 because $C \setminus U = C \cap (X \setminus U)$ is closed in C thus is compact because a closed subset of a compact set is compact.

Now we show that X is a subspace of Y. Given any open set V of Y, we show its intersection with X is open in X. If V is of type 1, then $V \cap X = V$ which is open in X. If V is of type 2, then $V \cap X = (Y \setminus C) \cap X = X \setminus C$ which is open in X because C is closed in X as a compact subspace of a Hausdorff space. Conversely, given any open set U of X, U is an open set of Y of type 1. So X is a subspace of Y.

We show that Y is compact.

Let $\mathcal{A}$ be an open covering of Y. The collection $\mathcal{A}$ must contain an open set containing ∞. So there exists a compact set C such that $U = Y \setminus C$ with $U \in \mathcal{A}$. Take all the members of $\mathcal{A}$ different from U and intersect them with X. This gives a collection of open sets in X which covers C. Since C is compact, there exists a finite subcovering of C, say $\{V_1, V_2, \cdots, V_n\}$. Then $\{U, V_1, V_2, \cdots, V_n\}$ is a finite subcovering of Y. So Y is compact.

We show that Y is Hausdorff.

Take two distinct points $x, y \in Y$. There are two cases.

1. If $x, y \in X$, since X is Hausdorff, there exist disjoint open sets U, V in X such that $x \in U, y \in V$. Then U, V are open in Y and disjoint.

2. If one of them is ∞, say $y = \infty$. Since X is locally compact at x, there exists an open neighborhood U of x and a compact set C such that $U \subseteq C$. So C is closed in X thus in Y. Then $V = Y \setminus C$ is an open neighborhood of ∞. Clearly, $U \cap V = \emptyset$.

Finally we prove the other direction. Suppose a space Y satisfying conditions (1)-(3) exists. Then X is Hausdorff as a subspace of a Hausdorff space. Given any $x \in X$, we show X is locally compact at x. Choose disjoint open sets U, V in Y such that $x \in U$ and $p \in V$. Then $C = Y \setminus V$ is compact as a closed subset of a compact space. Also, $x \in U \subseteq C$. So X is locally compact at x.

■

Definition 17.0.1. *If Y and X are as in the above theorem, then Y is called the one-point compactification of X.*

Example 17.0.1.

1. *The one point compactification of $\mathbb{R}$ is homeomorphic to S^1. We denote $\overline{\mathbb{R}} \cong S^1$.*
2. *The one point compactification of $\mathbb{R}^n$ is homeomorphic to S^n. We denote $\overline{\mathbb{R}^n} \cong S^n$.*
3. *The one point compactification of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$(Riemann sphere) is homeomorphic to S^2.*

Theorem 17.0.2. *Let X be a Hausdorff space. Then X is locally compact if and only if for any $x \in X$ and any neighborhood U of x, there exists a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subseteq U$.*

Proof

$\Leftarrow$ $x \in V \subseteq \bar{V} \subseteq U$ and $\bar{V}$ is compact. So X is locally compact at x.

$\Rightarrow$ Let Y be the one-point compactification(compact and Hausdorff) of X. Let U be a neighborhood of x in Y. Then $Y \setminus U := C$ is a closed subset of Y. So C is compact subspace of Y(Hausdorff). By lemma 12.0.2, there exists two open sets V, W in Y such that $x \in V$, $C \subseteq W$ and $V \cap W = \emptyset$. Then $\bar{V}$ is compact and $\bar{V} \cap C = \emptyset$. So $\bar{V} \subseteq U$. ■

Corollary 17.0.1. *Let X be locally compact Hausdorff space. $A \subseteq X$ is open or closed. Then A is locally compact.*

Proof

Let $A \subseteq X$ be closed. Given $x \in A$, since X is locally compact, there exists an open neighborhood U of x in X and a compact set C such that $x \in U \subseteq C$. Then $C \cap A$ is closed in C thus is compact, and it contains the neighborhood $U \cap A$ of x in A. So A is locally compact at x.

Let $A \subseteq X$ be open. Given $x \in A$, since X is locally compact and X is Hausdorff, by the previous theorem, there exists a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subseteq A$. Then $C = \bar{V}$ is a compact set containing the neighborhood V of x in A. So A is locally compact at x.

■

Corollary 17.0.2. *X is locally compact Hausdorff if and only if X is homeomorphic to an open subspace of a compact Hausdorff space.*

Exercise Show the above corollary by 17.0.1 and 17.0.1.

Solution

$\Rightarrow$ Let Y be the one-point compactification of X. Recall that Hausdorff is T_1. So $X = Y \setminus \{p\}$ is open in Y. The identity map $id : X \to X$ is a homeomorphism from X to the open subspace X of Y.

$\Leftarrow$ Let X be homeomorphic to an open subspace U of a compact Hausdorff space Z. Since Z is compact Hausdorff, Z is locally compact Hausdorff. By 17.0.1, U is locally compact Hausdorff. Since X is homeomorphic to U, X is locally compact Hausdorff.

■

17.1 Urysohn's Metrization Theorem

Theorem 17.1.1. *Every X that is regular(T3) and second-countable is metrizable.*

Chapter 18

Later

18.1 Countability

Definition 18.1.1. *X is first countable if for every $x \in X$, there exists a countable basis at x. That is, there exists $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ a collection of open sets containing x such that for any open set U containing x, there exists $B_n \in \mathcal{B}$ such that $B_n \subseteq U$.*

Definition 18.1.2. *X is second countable if there exists a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ for the topology of X. That is, for every x and every open set U containing x, there exists $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq U$.*

Exercise $\mathbb{R}^n$ with standard topology, $B_n = \{U_\epsilon(x) | x \in \mathbb{Q}^n, \epsilon \in \mathbb{Q}_{>0}\}$. B_n is a countable basis. So $\mathbb{R}^n$ is second countable. Show the details.
Exercise Show that if X_n's are first(second) countable, then $\prod X_n$ with product topology is first(second) countable.

Theorem 18.1.1. *Let X be second countable. Then*

1. *Every open cover of X has a countable subcovering(X is Lindelöf space).*
2. *There is a countable subset $A \subseteq X$ such that $\bar{A} = X$($A \subseteq X$ is called dense and X is called separable).*

Proof to 1
Let $\mathcal{A}$ be an open cover of X. Let $\mathcal{A}$ be an open covering of X. For each positive integer n for which it is possible, choose an element $A_n \in \mathcal{A}$ such that $B_n \subseteq A_n$. The collection $\mathcal{A}'$ of all such possible A_n is countable since it is indexed with a subset J of the positive integers.
Furthermore, $\mathcal{A}'$ covers X. To see this, let $x \in X$. Since $\mathcal{A}$ is a covering of X, there exists $A \in \mathcal{A}$ such that $x \in A$. Since $\mathcal{B}$ is a basis for the topology of X, there exists $B_m \in \mathcal{B}$ such that $x \in B_m \subseteq A$. By construction of $\mathcal{A}'$, there exists $A_m \in \mathcal{A}'$ such that $B_m \subseteq A_m$. Thus, $x \in A_m$. So $\mathcal{A}'$ is a countable subcovering of $\mathcal{A}$.
Proof to 2
From each non-empty basis element B_n, choose a point $x_n \in B_n$. Let D be the set of all such points x_n. Then D is countable.
Given any point $x \in X$, every basis element containing x contains a point of D, so x is in the closure of D. Thus, $\bar{D} = X$.

■

Property 18.1.1. *A second countable space is:*

1. *First countable.*
2. *Separable.*
3. *Lindelöf.*

Example 18.1.1 (non-example)**.** *$\mathbb{R}_l$ with lower limit topology(Sorgengrey line) is first countable, Lindelöf, seperable, but not second countable.*
First-countable: for any $x \in \mathbb{R}_l$, $\{[x, x + \frac{1}{n}) : n \in \mathbb{N}\}$ is a countable local basis at x.
Assume the contrary that $\mathbb{R}_l$ is second countable. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis for $\mathbb{R}_l$. For each $x \in \mathbb{R}$, there exists $B_{n_x} \in \mathcal{B}$ such that $x \in B_{n_x} \subseteq [x, x + 1)$. So $\min B_{n_x} = x$. This implies that the map $f : \mathbb{R} \to \mathcal{B}, f(x) = B_{n_x}$ is injective. However, $\mathbb{R}$ is uncountable while $\mathcal{B}$ is countable, which is a contradiction. So $\mathbb{R}_l$ is not second countable.
Separable: $\mathbb{Q} \subseteq \mathbb{R}_l$ is countalbe and dense in $\mathbb{R}_l$.
Lindelöf: See the below property.

Property 18.1.2 (Lindelöf property of $\mathbb{R}_l$). *Any open cover of $\mathbb{R}_l$ has a countable subcover.*

Proof

Let $\mathcal{A}$ be a covering of $\mathbb{R}_l$ by $[a_\alpha, b_\alpha)$'s. We need to show that there exists a countable subcovering of $\mathcal{A}$ covering $\mathbb{R}_l$.

Let $C = \bigcup(a_\alpha, b_\alpha)$ such that $\mathbb{R} \setminus C$ is not empty. Let $x \in \mathbb{R} \setminus C$. Then $x = a_\beta$ for some β such that $x \notin (a_\beta, b_\beta)$. Take $q_x \in (a_\beta, b_\beta) \cap \mathbb{Q}$ thus $x < q_x$.

Take $x, y \in \mathbb{R}$ with $x < y$. Then $q_x < q_y$ or we make a contradiction(why?). The map $x \mapsto q_x$ is injective from $\mathbb{R} \setminus C$ to $\mathbb{Q}$. So $\mathbb{R} \setminus C$ is countable.

Now we show that some countable subcollection of $\mathcal{A}$ covers $\mathbb{R}$. To begin, choose for each element $\mathbb{R} \setminus C$ an interval $[a_\alpha, b_\alpha)$ of $\mathcal{A}$ that contains it. Then we have chosen a countable subcollection $\mathcal{A}'$ of $\mathcal{A}$ covering $\mathbb{R} \setminus C$.

Now take the set C and topologize it as a subspace of $\mathbb{R}$(C is open in $\mathbb{R}$, topologizing means taking the intersection of C with open sets in $\mathbb{R}$ as the open sets of C). Then C is second countable(since $\mathbb{R}$ is second countable). Now C is covered by the open intervals (a_α, b_α) which are open in $\mathbb{R}$ and hence open in C. So there exists a countable subcovering of C by (a_α, b_α)'s. Suppose this subcollection is indexed by $\alpha = \alpha_1, \alpha_2, \cdots$. Then the collection

$$\mathcal{A}'' = \{[a_\alpha, b_\alpha) : \alpha = \alpha_1, \alpha_2, \cdots\} \tag{18.1}$$

is a countable subcollection of $\mathcal{A}$ that covers C.

Hence the countable collection $\mathcal{A}' \cup \mathcal{A}''$ is a countable subcollection of $\mathcal{A}$ that covers $\mathbb{R}_l$.

■

Property 18.1.3. *$\mathbb{R}_l \times \mathbb{R}_l$ is not Lindelöf. The space $\mathbb{R}_l \times \mathbb{R}_l$ with the product topology is called the Sorgenfrey plane. (The proof can be shown by a picture not drawn here.)*

Proof The space $\mathbb{R}_l^2$ has as basis all sets of the form $[a, b) \times [c, d)$ where $a < b$ and $c < d$. To show that $\mathbb{R}_l^2$ is not Lindelöf, we consider the subspace

$$L = \{(x, -x) | x \in \mathbb{R}_l\} \subseteq \mathbb{R}_l^2. \tag{18.2}$$

It is easy to check that L is closed in $\mathbb{R}_l^2$. Let's cover $\mathbb{R}_l^2$ by the open set $\mathbb{R}_l^2 \setminus L$ and by all basis elements of the form

$$[a, b) \times [-a, d) \tag{18.3}$$

Each of these open sets intersects L in at most one point. Since L is uncountable, no countable subcollection covers $\mathbb{R}_l^2$. Thus, $\mathbb{R}_l^2$ is not Lindelöf.

■

Example 18.1.2. *Subspace of Lindelöf space is not necessarily Lindelöf. The ordered square I_0^2 is compact thus is Lindelöf. But the subspace $A = I_0 \times (0, 1)$ is not Lindelöf. For A is the union of disjoint sets $U_x = \{x\} \times (0, 1)$, each of which is open in A. This collection of sets $\{U_x | x \in I_0\}$ is uncountable, and no proper subcollection covers A.*

18.2 Separation Axioms

Topological space X can satisfy the following separability axioms:

1. $T1$(Frechet): Given two distinct points $x, y \in X$, there exists an open set U such that $x \in U$ and $y \notin U$. (Equivalently, all singleton sets are closed. Easy exercise)

2. $T2$(Hausdorff): Given two distinct points $x, y \in X$, there exist open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

3. $T3$(Regular): X is $T1$ and given a closed set $F \subseteq X$ and a point $x \notin F$, there exist open sets U, V such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$.

4. $T4$(Normal): X is $T1$ and given two disjoint closed sets $F_1, F_2 \subseteq X$, there exist open sets U, V such that $F_1 \subseteq U, F_2 \subseteq V$ and $U \cap V = \emptyset$.

Lemma 18.2.1. *Let X be a $T1$ topological space. Then*

1. *X is regular if and only if for every point $x \in X$ and every neighborhood U of x, there exists a neighborhood V of x such that $\bar{V} \subseteq U$.*

2. *X is normal if and only if for every closed set $F \subseteq X$ and every neighborhood U of F, there exists a neighborhood V of F such that $\bar{V} \subseteq U$.*

Proof

$\Rightarrow$ Let X be regular.

If $U = X$, there is nothing to prove.

Suppose $U \neq X$. Then $F = X \setminus U$ is closed and $x \notin F$. By regularity of X, there exist open sets V, W such that $x \in V, F \subseteq W$ and $V \cap W = \emptyset$. Now we need to show $\bar{V} \subseteq U$.

If $y \in F$, then $y \in W$. Since $V \cap W = \emptyset$, $y \notin \bar{V}$ because we find a neighborhood W of y such that $W \cap V = \emptyset$. Thus, $\bar{V} \subseteq X \setminus F = U$.

$\Leftarrow$ Suppose that the point x and the closed set B not containing X are given. Let $U = X \setminus B$. By hypothesis, there is a neighborhood V of x such that $\bar{V} \subseteq U$. The open sets V and $X \setminus \bar{V}$ then separate x and B.

The proof of 2. is similar.

■

Theorem 18.2.1. *1. X is Hausdorff, then $Y \subseteq X$ with subspace topology is Hausdorff.*

2. X_α is Hausdorff, then $\prod X_\alpha$ with product topology is Hausdorff.

3. X is regular, then $Y \subseteq X$ with subspace topology is regular.

4. X_α is regular, then $\prod X_\alpha$ with product topology is regular.

Proof

(a) is obvious.

(b) Let $x = (x_\alpha), y = (y_\alpha) \in \prod X_\alpha$ with $x \neq y$. Then there exists β such that $x_\beta \neq y_\beta$. Since X_β is Hausdorff, there exist open sets $U_\beta, V_\beta \subseteq X_\beta$ such that $x_\beta \in U_\beta, y_\beta \in V_\beta$ and $U_\beta \cap V_\beta = \emptyset$. Then $\pi_\beta^{-1}(U_\beta), \pi_\beta^{-1}(V_\beta)$ are open in $\prod X_\alpha$, $x \in \pi_\beta^{-1}(U_\beta), y \in \pi_\beta^{-1}(V_\beta)$ and $\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \emptyset$. So we are done.

(c) (It is easy to verify the subspace of T_1 space is T_1.)Let $Y \subseteq X$ with subspace topology. Let $y \in Y$ and B be a closed set in Y such that $y \notin B$. Then there exists a closed set F in X such that $B = F \cap Y$. Since $y \notin B$, $y \notin F$. Since X is regular, there exist open sets $U, V \subseteq X$ such that $y \in U, F \subseteq V$ and $U \cap V = \emptyset$. Then $U \cap Y, V \cap Y$ are open in Y, $y \in U \cap Y, B \subseteq V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = \emptyset$. So we are done.

(d) Let $\{X_\alpha\}$ be a family of regular spaces. Let $X = \prod X_\alpha$. X_α is regular implies that X_α is Hausdorff. By (b), X is Hausdorff thus is T_1.

Let $x = (x_\alpha) \in X$ and let U be a neighborhood of x in X. Choose a basis element $\prod U_\alpha \subseteq U$ containing x. By the previous lemma, we can choose, for each α, a neighborhood V_α of x_α in X_α such that $\bar{V_\alpha} \subseteq U_\alpha$. If it happens that $U_\alpha = X_\alpha$ for some α, we choose $V_\alpha = X_\alpha$. Then $V = \prod V_\alpha$ is a neighborhood of x in X. Recall from lecture 11, that $\bar{V} = \prod \bar{V_\alpha}$. It follows at once that $\bar{V} \subseteq \prod U_\alpha \subseteq U$. By the previous lemma, X is regular.

■

Remark X is normal does not imply that $Y \subseteq X$ with subspace topology is normal. X_1, X_2 are normal does not imply that $X_1 \times X_2$ with product topology is normal.

Exercise Show that $\mathbb{R}_l$ is normal.

Solution It is immediate that one-point sets are closed in $\mathbb{R}_l$, since the topology of $\mathbb{R}_l$ is finer than the standard topology on $\mathbb{R}$. Let A and B are disjoint closed sets in $\mathbb{R}_l$. For each $a \in A$, since B is closed and $a \notin B$, there exists a basis element $[a, x_a)$ such that $[a, x_a) \cap B = \emptyset$. For each $b \in B$, since A is closed and $b \notin A$, there exists a basis element $[b, y_b)$ such that $[b, y_b) \cap A = \emptyset$. Then the open sets

$$U = \bigcup_{a \in A} [a, x_a), \quad V = \bigcup_{b \in B} [b, y_b) \tag{18.4}$$

are disjoint open sets containing A and B respectively. So we are done.

■

Fact (Without Proof, Efforts are required to show this) $\mathbb{R}_l^2$ is not normal. But it is regular(as a product of regular spaces).

Exercise $\mathbb{R}_K$ is a topological space with basis given by all open intervals (a, b) and all sets of the form $(a, b) \setminus K$ where $K = \{1/n | n \in \mathbb{N}\}$. $\mathbb{R}_K$ is Hausdorff. But $\mathbb{R}_K$ is not regular.

The set K is closed in $\mathbb{R}_K$ and it does not contain the point 0. Suppose that there exist disjoint open sets U and V such that $0 \in U$ and $K \subseteq V$. Choose a basis element containing 0 and lying in U. It must be a basis element of the form $(a, b) \setminus K$, since each basis element of the form (a, b) contains intersects K. Choose $1/n \in K$ such that $1/n < b$. Then there exists a basis element of the form (c, d) such that $1/n \in (c, d) \subseteq V$. Finally, the basis elements $(a, b) \setminus K$ and (c, d) intersect, since $c < 1/n < b$. So U and V are not disjoint.

■

Chapter 19

Later

Theorem 19.0.1. *Every second countable regular space is normal.*

Proof

Let X be a second countable regular space with $A, B \subseteq X$ closed and $A \cap B = \emptyset$.

For every $x \in A$, since X is regular, there exist open sets $U \ni x$ such that $U \cap B = \emptyset$.

By the lemma, there exists a open set V such that $x \in V$, $\bar{V} \subseteq U$ and thus $\bar{V} \cap B = \emptyset$.

Let $\mathcal{B}$ be a countable baisis for X. For each $x \in A$, there exists $U_x \in \mathcal{B}$ such that $x \in U_x \subseteq V$, and thus $\bar{U_x} \subseteq \bar{V} \subseteq U$ and $\bar{U_x} \cap B = \emptyset$. Since $\mathcal{B}$ is countable, the collection $\{U_x : x \in A\}$ has a countable subcollection $\{U_n : n \in \mathbb{N}\}$ covering A such that $\bar{U_n} \cap B = \emptyset$ for each n.

Similarly, we can choose a countable collection $\{V_m : m \in \mathbb{N}\}$ of open sets covering B, such that $\bar{V_m} \cap A = \emptyset$ for each m. Then the sets $U = \bigcup_n U_n$ and $V = \bigcup_m V_m$ are open sets containing A and B respectively. But they need not be disjoint. We perform the following simple trick to construct two open sets that are disjoint. Given n, define

$$U'_n = U_n \setminus \bigcup_{m=1}^{n} \bar{V_m}, \quad V'_n = V_n \setminus \bigcup_{k=1}^{n} \bar{U_k} \tag{19.1}$$

Note that U'_n and V'_n are open. The collection $\{U'_n : n \in \mathbb{N}\}$ covers A because each x in A belongs to some U_n and x belongs to none of the $\bar{V_m}$'s. Similarly, $\{V'_n : n \in \mathbb{N}\}$ covers B.

Finally, the open sets $U' = \bigcup_n U'_n$ and $V' = \bigcup_m V'_m$ are disjoint. For if $x \in U' \cap V'$, then there exist n, m such that $x \in U'_n$ and $x \in V'_m$. Without loss of generality, assume that $n \leq m$. Then

$$x \in U'_n \subseteq U_n \setminus \bigcup_{k=1}^{n} \bar{V_k} \subseteq U_n \setminus \bar{V_m} \tag{19.2}$$

This contradicts the fact that $x \in V'_m \subseteq V_m \subseteq \bar{V_m}$. Thus, $U' \cap V' = \emptyset$. Hence, X is normal.

■

Theorem 19.0.2. *Every metrizable space X is normal.*

Proof Let d be the metric on X. Given two closed disjoint subsets $A, B \subseteq X$.

For each $a \in A$, there exists $r_a > 0$ such that $B(a, r_a) \cap B = \emptyset$.

Similarly, for each $b \in B$, there exists $r_b > 0$ such that $B(b, r_b) \cap A = \emptyset$.

Let $U = \bigcup_{a \in A} B(a, \frac{r_a}{2})$ and $V = \bigcup_{b \in B} B(b, \frac{r_b}{2})$. Then U and V are open sets containing A and B respectively.

We claim that $U \cap V = \emptyset$. Assume the contrary that there exists $x \in U \cap V$. Then there exist $a \in A$ and $b \in B$ such that $x \in B(a, \frac{r_a}{2})$ and $x \in B(b, \frac{r_b}{2})$. Without loss of generality, assume that $\frac{r_a}{2} \leq \frac{r_b}{2}$. Then

$$d(a, b) \leq d(a, x) + d(x, b) < \frac{r_a}{2} + \frac{r_b}{2} \leq r_a \tag{19.3}$$

This implies that $b \in B(a, r_a)$, which contradicts the choice of r_a. Thus, $U \cap V = \emptyset$. Hence, X is normal.

■

Theorem 19.0.3. *Every compact Hausdorff space is normal.*

Proof

Let $x \in X$ and $B \subseteq X$ be closed such that $x \notin B$.

For each $y \in B$, since X is Hausdorff, there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Then $\{V_y | y \in B\}$ is an open covering of B. Since X is compact and Hausdorff, B is compact as a closed subset of X. By the compactness of B, there exists a finite subcollection $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$ that covers B. Let $V = \bigcup V_{y_i}$ and $U = \bigcap U_{y_i}$. Then U and V are disjoint open sets containing x and B respectively. So X is regular.

Let $A, B \subseteq X$ be closed and $A \cap B = \emptyset$. For each $a \in A$, since X is regular, there exist disjoint open sets U_a and V_a such that $a \in U_a$ and $B \subseteq V_a$. Then $\{U_a | a \in A\}$ is an open covering of A. By the compactness of A, there exists a finite subcollection $\{U_{a_1}, U_{a_2}, \ldots, U_{a_m}\}$ that covers A. Let $U = \bigcup U_{a_i}$ and $V = \bigcap V_{a_i}$. Then U and V are disjoint open sets containing A and B respectively. So X is normal.

■

Lemma 19.0.1 (Urysohn's lemma). *Given a normal space X and two closed disjoint subsets $A, B \subseteq X$, there exists a continuous map $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof The proof is too long to be included here. Please refer to Munkres' Topology, PDF P 224. It is not required to be reproduced.

■

Chapter 20

Later

Theorem 20.0.1 (Urysohn's Lemma(Recalling)). *Let X be a normal space. If A and B are disjoint closed subsets of X, then there exists a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

Definition 20.0.1. *Suppose X satisfies (T1). If a point and a closed set can be separated by a continuous function as in Urysohn's lemma, then X is called completely regular(CR). (Sometimes it is called $T_{3\frac{1}{2}}$ space.)*

Property 20.0.1. *Every normal space is completely regular.*

Proof By Urysohn's lemma.

■

Remark

Normal ⇒ CR ⇒ Regular. That's why we call it $T_{3\frac{1}{2}}$ space.

Property 20.0.2. *If X is completely regular, then it is regular.*

Proof

Take $x_0 \in X$ and a closed set A such that $x_0 \notin A$. Since X is CR, there exists a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 0$ and $f(a) = 1$ for all $a \in A$. Let $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$. Then U and V are open sets because f is continuous and $[0, \frac{1}{2})$, $(\frac{1}{2}, 1]$ are open in $[0, 1]$. So $x_0 \in U$, $A \subseteq V$ and $U \cap V = \varnothing$. Thus X is regular.

■

Theorem 20.0.2.

1. *Subspace of CR space is CR.*
2. *If X_α is CR, then $\prod X_\alpha$ with product topology is CR.*

Proof

Let X be CR, and $Y \subseteq X$ with subspace topology. Take $y_0 \in Y$ and a closed set $A \subseteq Y$ such that $y_0 \notin A$. Let $\bar{A}$ be the closure of A in X so $A = \bar{A} \cap Y$. Then $\bar{A}$ is closed in X and $y_0 \notin \bar{A}$. Since X is CR, there exists a continuous function $f : X \to [0, 1]$ such that $f(y_0) = 0$ and $f(a) = 1$ for all $a \in \bar{A}$. Restrict f to Y, we get a continuous function $f|_Y : Y \to [0, 1]$ such that $f|_Y(y_0) = 0$ and $f|_Y(a) = 1$ for all $a \in A$. Thus Y is CR.

Take $A \subseteq \prod X_\alpha$ closed and $x = (x_\alpha) \notin A$. Then there exists a basic open set $U = \prod U_\alpha$ such that $x \in U$ and $U \cap A = \emptyset$, where U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many α. Let those finitely many α be $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since X_{α_i} is CR, there exists a continuous function $f_{\alpha_i} : X_{\alpha_i} \to [0, 1]$ such that $f_{\alpha_i}(x_{\alpha_i}) = 1$ and $f_{\alpha_i}(a) = 0$ for all $a \in X_{\alpha_i} \setminus U_{\alpha_i}$. So we can define a continuous function $f : \prod X_\alpha \to [0, 1]$ by

$$f(x) = \prod_{i=1}^{n} f_{\alpha_i}(\pi_{\alpha_i}(x)).$$

Then $f(x) = 1$ and for all $a \in A$, $f(a) = 0$. Thus $\prod X_\alpha$ is CR.

■

Example 20.0.1. *$\mathbb{R}_l$ is normal then it is CR. So $\mathbb{R}_l^2$ is CR. But it is not normal as we have shown.*

Fact(no proof) There exist regular spaces that are not CR.
Remark Proof of Urysohn's lemma for metric spaces(exercise):

Let X be metric space with metric $d : X \times X \to \mathbb{R}$. Given two disjoint closed sets $A, B \subseteq X$, define $d_B(x) = d(x, B)$ which vanishes exactly on B. Then define $d_A(x) = d(x, A)$ which vanishes exactly on A. Now define

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}.$$

Then $f(A) = \{0\}$ and $f(B) = \{1\}$ and f is continuous because we've proved that distance from a point to a closed set is continuous.

Theorem 20.0.3 (Urysohn Metrization Theorem)**.** *Every regular space with contable basis is metrizable.*

Proof

You can see the proof in P215 of Munkres' book, P232 of the pdf.

■

Chapter 21

Later

Idea For the idea of this lecture, refer to Munkres' Topology, P230 or pdf P247.

Definition 21.0.1. *Let $(A, \leq)$ be a partially ordered set. A subset $B \subseteq A$ is called a chain if for any $a, b \in B$, we have either $a \leq b$ or $b \leq a$.*

Theorem 21.0.1 (Zorn's lemma)**.** *Let $(A, \leq)$ be a partially ordered set. If every chain in A has an upper bound in A, then A contains at least one maximal element.*

Lemma 21.0.1. *Let X be a set; let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. Then there is a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D}$ contains $\mathcal{A}$, and $\mathcal{D}$ has the finite intersection property, and no collections of subsets of X properly containing $\mathcal{D}$ has the finite intersection property.*

Proof
For purposes of this proof, we shall call a set whose elements are collections of subsets of X a "superset" and shall denote it by an outline letter. To summarize the notation:

- c is an element of X;
- C is a subset of X;
- $\mathcal{C}$ is a collection of subsets of X;
- $\mathbb{C}$ is a superset whose elements are collections of subsets of X.

Now by hypothesis, $\mathcal{A}$ is a collection of subsets of X having the finite intersection property. Let $\mathbb{A}$ be the superset consisting of all collections of subsets of X such that $\forall \mathcal{B} \in \mathbb{A}$, we have $\mathcal{B} \supseteq \mathcal{A}$ and $\mathcal{B}$ has the finite intersection property. To prove the lemma, we need to show that $\mathbb{A}$ contains a maximal element with respect to the partial order $\subseteq$.

In order to apply Zorn's lemma, we must show that if $\mathbb{B}$ is a "subsuperset" of $\mathbb{A}$ that is a chain, then $\mathbb{B}$ has an upper bound in $\mathbb{A}$. Let

$$\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}. \tag{21.1}$$

Certainly $\mathcal{C}$ contains $\mathcal{A}$, since each $\mathcal{B} \in \mathbb{B}$ contains $\mathcal{A}$. Let $C_1, C_2, \ldots, C_n$ be finitely many elements of $\mathcal{C}$. Then for each $i = 1, 2, \ldots, n$, there exists $\mathcal{B}_i \in \mathbb{B}$ such that $C_i \in \mathcal{B}_i$. Since $\mathbb{B}$ is a chain, there exists $\mathcal{B}_j \in \mathbb{B}$ such that $\mathcal{B}_i \subseteq \mathcal{B}_j$ for all $i = 1, 2, \ldots, n$. Thus $C_i \in \mathcal{B}_j$ for all $i = 1, 2, \ldots, n$. Since $\mathcal{B}_j$ has the finite intersection property, we have

$$C_1 \cap C_2 \cap \ldots \cap C_n \neq \varnothing. \tag{21.2}$$

Thus $\mathcal{C}$ has the finite intersection property. And so $\mathcal{C} \in \mathbb{A}$. Moreover, for each $\mathcal{B} \in \mathbb{B}$, we have $\mathcal{B} \subseteq \mathcal{C}$. Thus $\mathcal{C}$ is an upper bound of $\mathbb{B}$ in $\mathbb{A}$. By Zorn's lemma, $\mathbb{A}$ contains a maximal element, which we denote by $\mathcal{D}$. This completes the proof of the lemma.

■

Lemma 21.0.2. *Let X be a set; let $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then*

1. *Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$;*
2. *If A is a subset of X that intersects every element of $\mathcal{D}$, then A is an element of $\mathcal{D}$.*

Proof to 1

Let B be any finite intersection of elements of $\mathcal{D}$. Let $\mathcal{E} = \mathcal{D} \cup \{B\}$. We can show that $\mathcal{E}$ has the finite intersection property(Check it). By maximality of $\mathcal{D}$, we have $\mathcal{E} = \mathcal{D}$, so $B \in \mathcal{D}$.

Proof to 2

Let A be a subset of X that intersects every element of $\mathcal{D}$. Let $\mathcal{E} = \mathcal{D} \cup \{A\}$. Take finitely many elements from $\mathcal{E}$. If none of them is A, then their intersection is not empty because $\mathcal{D}$ has the finite intersection property, otherwise, it is of the form

$$A \cap D_1 \cap D_2 \cap \ldots \cap D_n, \tag{21.3}$$

Now, $D_1 \cap D_2 \cap \ldots \cap D_n \in \mathcal{D}$ by part 1, so A intersects it by hypothesis. Thus the intersection is not empty. So $\mathcal{E}$ has the finite intersection property. By maximality of $\mathcal{D}$, we have $\mathcal{E} = \mathcal{D}$, so $A \in \mathcal{D}$.

■

Theorem 21.0.2 (Tychonoff theorem)**.** *Any arbitrary product of compact spaces is compact in the product topology.*

Proof

Let

$$X = \prod_{\alpha \in J} X_\alpha, \tag{21.4}$$

where each X_α is compact. Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. We prove that the intersection $\bigcap_{A \in \mathcal{A}} \bar{A}$ is non-empty. Compactness of X follows.

By 21.0.1, we choose $\mathcal{D}$ containing $\mathcal{A}$ that is maximal with respect to the finite intersection property. It suffices to show that the intersection

$$\bigcap_{D \in \mathcal{D}} \bar{D} \tag{21.5}$$

is non-empty.

Given any $\alpha \in J$, we consider the collection

$$\{\pi_\alpha(D) | D \in \mathcal{D}\} \tag{21.6}$$

of subsets of X_α. This collection has the finite intersection property because $\mathcal{D}$ does. By compactness of X_α, we can choose a point $x_\alpha \in X_\alpha$ such that

$$x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}. \tag{21.7}$$

Now the point $x = (x_\alpha)$ is defined in X.

Let U_β be a neighborhood of x_β in X_β. Since $x_\beta \in \overline{\pi_\beta(D)}$, we have that $\exists y \in D, \pi_\beta(y) \in U_\beta \cap \pi_\beta(D)$. Thus $y \in \pi_\beta^{-1}(U_\beta) \cap D$. Since this is true for any $D \in \mathcal{D}$, by the previous lemma, we have $\pi_\beta^{-1}(U_\beta) \in \mathcal{D}$. So every subbasis element containing x is in $\mathcal{D}$. And then it follows from the same lemma that every basis(finite intersection of subbasis) element containing x is in $\mathcal{D}$. Thus any open neighborhood of x intersects every $D \in \mathcal{D}$. Therefore, $x \in \bar{D}$ for all $D \in \mathcal{D}$.

■

Chapter 21

Later

Chapter 22

Lecture 22

Definition 22.0.1. *A compactification of a Hausdorff space X is a compact Hausdorff space $Y \supseteq X$ such that $Y = \overline{X}$. We say that compactifications Y_1 and Y_2 are equivalent if there exists a homeomorphism $h : Y_1 \to Y_2$ such that $h|_X = id_X$.*

Example 22.0.1. $X = (0,1)$, $Y = S^1 \subseteq \mathbb{R}^2$, $X \ni x \mapsto (\cos 2\pi x, \sin 2\pi x) \in Y$ *is a compactification of X. (This is not the usual sense of the above definition but there exists a injective homeomorphism from X to Y and the closure of the image is Y.)*

Example 22.0.2. $X = (0,1)$, $Y = [0,1] \subseteq \mathbb{R}$.

Property 22.0.1. *Let $X \subseteq Y$ and Y be a compact Hausdorff space. Then X is completely regular.*

Proof
Since Y is compact Hausdorff, Y is normal. Then Y is completely regular. Thus X is completely regular as well.

■

Claim If X is completely regular, it has a compactification.

Theorem 22.0.1. *Let X be $T1$ space. Suppose X has an indexed family of continuous functions $\{f_\alpha : X \to \mathbb{R}\}_{\alpha\in J}$ such that $\forall x_0 \in X$, and $U \ni x_0$ open, there exists a function f_α such that $f_\alpha(x_0) = 1$ and $f_\alpha(X \setminus U) \equiv 0$. Then $F : X \to \mathbb{R}^J$ defined by $F(x) = (f_\alpha(x))_{\alpha\in J}$ is an embedding $X \hookrightarrow \mathbb{R}^J$. If, additionally, $f_\alpha : X \to [0,1]$ for all $\alpha \in J$, then F is an embedding $X \hookrightarrow [0,1]^J$.*

Proof To be done.

Theorem 22.0.2. *X is completely regular if and only if it is homeomorphic to a subspace of a cube $[0,1]^J$ for some index set J.*

Proof Not shown in the lecture.

Lemma 22.0.1. *Let X be a Hausdorff space. Let $h : X \hookrightarrow Z$ be an embedding with Z compact Hausdorff. Then there exists a compacdification Y of X such that there is an embedding $H : Y \hookrightarrow Z$ with $H|_X = h$. Such a compactification is uniquely determined.*

Example 22.0.3. *Let $X = (0,1)$. Consider the embedding $h : X \hookrightarrow \mathbb{R}^2$ defined by $h(x) = (x, \sin\frac{1}{x})$. Let $A = \{0\} \times [-1,1] \cup \{(1, \sin(1))\}$. Then $Y = h(X) \cup A$ is a compactification of $h(X)$.*

There's something.

Theorem 22.0.3 (Stone–Čech compactification)**.** *Let X be a completely regular space. Then there exists a compactification Y of X such that every bounded continuous function $f : X \to \mathbb{R}$ can be extended uniquely to a continuous function $\overline{f} : Y \to \mathbb{R}$. ($Y$ is called the Stone–Čech compactification of X)*

Proof To be done.

Chapter 24

Later

Lemma 24.0.1. *Let $A \subseteq X$. Let $f : A \to Z$ be continuous and Z be Hausdorff. Then there exists at most one extension $g : \bar{A} \to Z, g|_A = f$.*

Theorem 24.0.1. *Let X be completely regular and Y be a compactification of X such that Y satisfies the Stone–Čech property. Then any continuous map $f : X \to C$ where C is a compact Hausdorff space can be extended to a continuous map $g : Y \to C$.*

Theorem 24.0.2. *Let Y_1, Y_2 be compactifications of a completely regular space X. Suppose both Y_1 and Y_2 satisfy the Stone–Čech property. Then there exist homeomorphisms $h : Y_1 \to Y_2$.*

24.1 Complete Metric Spaces

Definition 24.1.1. *Let (X, d) be a metric space. A sequence $\{x_n\} \subseteq X$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(x_n, x_m) < \varepsilon$.*

Property 24.1.1. *Every convergent sequence is a Cauchy sequence.*

Proof To be done.

Definition 24.1.2. *A metric space (X, d) is called complete if every Cauchy sequence in X converges to a limit in X.*

Example 24.1.1 (non-example)**.** $(0, 1)$ *with the usual metric. Then the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence that does not converge in* $(0, 1)$.

Example 24.1.2 (non-example)**.** $\mathbb{Q}$ *with the usual metric is not complete. For example, the sequence defined by the decimal approximations of $\sqrt{2}$ is a Cauchy sequence in $\mathbb{Q}$ that does not converge to a rational number.*

Property 24.1.2. *If $A \subseteq X$ is closed and (X, d) is complete, then A is complete.*

Property 24.1.3. *Let (X, d) be a metric space. Take $\bar{d} = \min\{d, 1\}$. Then (X, d) is complete implies $(X, \bar{d})$ is complete.*

Lemma 24.1.1. *X is complete if and only if every Cauchy sequence has a convergent subsequence.*

Proof
"$\Rightarrow$": Trivial.
"$\Leftarrow$": To be done.

Theorem 24.1.1. $\mathbb{R}^k$ *is complete in d_1, d_2, d_∞.*

Proof
We show this for d_∞.
Let $\{x_n\} \subseteq \mathbb{R}^k$ be a Cauchy sequence in d_∞. Then $\{x_n\}$ is bounded, i.e., there exists $M > 0$ such that for all n, $d_\infty(x_n, 0) < M$. So such sequence lies in a compact set $[-M, M]^k$. We know in metric spaces, compactness is equivalent to sequential compactness. Thus there exists a convergent subsequence $\{x_{n_k}\}$ that converges to some $x \in [-M, M]^k$. By the lemma, $\{x_n\}$ converges to x.
All these metrics are equivalent, so $\mathbb{R}^k$ is complete in d_1, d_2 as well.

■

Lemma 24.1.2. *Let $X = \prod_{\alpha\in J} X_\alpha$. Let (x_n) be a sequence in X. Then (x_n) converges to $x \in X$ if and only if for all $\alpha \in J$, the sequence of α-th coordinates $(\pi_\alpha(x_n))$ converges to $\pi_\alpha(x)$ in X_α.*

Proof

"$\Rightarrow$": π_α is continuous for all $\alpha \in J$.

"$\Leftarrow$": Suppose $(\pi_\alpha(x_n))$ converges to $\pi_\alpha(x)$ for all $\alpha \in J$. Let U be an open neighborhood of x. Then there exists a finite set $K \subseteq J$ and open sets $U_\alpha \subseteq X_\alpha$ for all $\alpha \in K$ such that $x \in \prod_{\alpha\in K} U_\alpha \times \prod_{\alpha\in J\setminus K} X_\alpha \subseteq U$. Since $(\pi_\alpha(x_n))$ converges to $\pi_\alpha(x)$, for each $\alpha \in K$, there exists $N_\alpha \in \mathbb{N}$ such that for all $n \geq N_\alpha$, $\pi_\alpha(x_n) \in U_\alpha$. Let $N = \max_{\alpha\in K} N_\alpha$. Then for all $n \geq N$, $x_n \in U$. Thus (x_n) converges to x.

Theorem 24.1.2. *There is a metric on $\mathbb{R}^\omega$ that makes it complete.*

Proof

Define $\bar{d}(a, b) = \min\{1, |a - b|\}$ for all $a, b \in \mathbb{R}$. Define $D(x, y) = \sup_{n\in\mathbb{N}}\{\frac{\bar{d}(x_n,y_n)}{n}\}$ for all $x = (x_n), y = (y_n) \in \mathbb{R}^\omega$. Then D is a metric on $\mathbb{R}^\omega$.

Let $\{x^m\} \subseteq \mathbb{R}^\omega$ be a Cauchy sequence in D. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l \geq N$, $D(x^m, x^l) < \varepsilon$. In particular, for all $n \in \mathbb{N}$, $\bar{d}(x^m_n, x^l_n) \leq nD(x^m, x^l) < n\varepsilon$. Thus for each fixed n, the sequence $\{x^m_n\}_{m=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $x_n \in \mathbb{R}$ such that x^m_n converges to x_n as $m \to \infty$. (might not be true, you need to check this)

24.2 Uniform Metric

Let (Y, d) be a metric space. Let $\bar{d} = \min\{d, 1\}$. We take the set of functions Y^J. There is a metric $\bar{\rho}$ on Y^J defined by $\bar{\rho}(x, y) = \sup_{\alpha\in J}\{\bar{d}(x_\alpha, y_\alpha)\}$. This is a metric(exercise). It is called the uniform metric.

Example 24.2.1. *Let $J = [0, 1]$ and $Y = \mathbb{R}$. Define $\bar{d}(f, g) = \sup_{x\in[0,1]}\{\min(1, |f(x) - g(x)|)\}$.*

Theorem 24.2.1. *If Y is complete with respect to d, then Y^J is complete with respect to the uniform metric $\bar{\rho}$.*