Chapter 12

Later

Definition 12.0.1. *A collection $\mathcal{A}$ of subsets of a space X is called "a covev" or "a covering" if the union of the elements of $\mathcal{A}$ is X; it is called an open covering if all these sets are open.*

Definition 12.0.2. *X is called a compact space if every open covering $\mathcal{A}$ contains a finite subcollection which also covers X.*

Remark Being compact depends on the topology: For example, consider X with the antidiscrete topology, i.e., $\mathcal{T} = \{\emptyset, X\}$. Then X is compact.

Example 12.0.1. $\mathbb{R}$ *with the standard topology is not compact. Consider the open covering* $\mathcal{A} = \{(-n, n)|n \in \mathbb{N}\}$.

Example 12.0.2. $(0, 1]$ *with the standard topology is not compact. Consider the open covering* $\mathcal{A} = \{(1/n, 1]|n \in \mathbb{N}\}$.

Example 12.0.3.

- $\{0\} \cup \{\frac{1}{n}|n \in \mathbb{Z}^+\} \subseteq \mathbb{R}$ *is compact because* $\{0\}$ *is the limit point and any open covering must contain an open set containing* 0*, which covers all but finite points.*
- $\{\frac{1}{n}|n \in \mathbb{Z}^+\}$ *is not compact. Consider the open covering* $\mathcal{A} = \{(1/n - \epsilon, 1/n + \epsilon)|n \in \mathbb{N}\}$. *(Similarly, for infinite X with discrete topology, X is not compact.)*

Definition 12.0.3. *$Y \subseteq X$ is compact if every open covering of Y by sets open in X contains a finite subcollection covering Y.*

Lemma 12.0.1. *$Y \subseteq X$ is compact if and only if every open covering of Y by sets open in Y contains a finite subcollection covering Y.*

Proof
Trivial.

Theorem 12.0.1. *Every closed subspace of a compact space is compact.*

Proof Let X be a compact space and let Y be a closed subspace of X. Let $\mathcal{A}$ be an open covering of Y by sets open in X. Since Y is closed, $X \setminus Y$ is open in X. Thus, $\mathcal{A} \cup \{X \setminus Y\}$ is an open covering of X. By the compactness of X, there exists a finite subcollection $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathcal{A}' \cup \{X \setminus Y\}$ covers X. Therefore, $\mathcal{A}'$ covers Y. Hence, Y is compact.

Theorem 12.0.2. *Every compact subspace of a Hausdorff space is closed.*

Proof
It suffices to show that for any $x \in X \setminus Y$, there exists an open set U containing x such that $U \cap Y = \emptyset$. For each $y \in Y$, since X is Hausdorff, there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Then $\{V_y|y \in Y\}$ is an open covering of Y. By the compactness of Y, there exists a finite subcollection $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$ that covers Y. Let

$$U = \bigcap_{i=1}^{n} U_{y_i} \tag{12.1}$$

Then U is an open set containing x. Moreover

$$U \cap Y = (\bigcap_{i=1}^{n} U_{y_i}) \cap Y = \emptyset \tag{12.2}$$

Thus, $U \cap Y = \emptyset$. Hence, Y is closed.

■

Lemma 12.0.2. *Y is a compact subspace of Hausdorff space X. And $x \in X \setminus Y$. Then there exist open sets U and V such that $x \in U$, $Y \subseteq V$ and $U \cap V = \emptyset$.*

Proof Proved inside the proof of the theorem.

Theorem 12.0.3. *The image of a compact set under a continuous map is compact.*

Proof Let $f : X \to Y$ be a continuous map. Consider any open covering $\mathcal{A}$ of $f(X)$ by sets open in Y. Then $\{f^{-1}(U) | U \in \mathcal{A}\}$ is an open covering of X by sets open in X. By the compactness of X, there exists a finite subcollection $\{f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_n)\}$ that covers X. Thus, $\{U_1, U_2, \ldots, U_n\}$ is a finite subcollection of $\mathcal{A}$ that covers $f(X)$. Hence, $f(X)$ is compact.

■

Theorem 12.0.4. *Let $f : X \to Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof

We just combine the above theorems.

It suffices to show that images of closed sets under f are closed. Consider any closed set D in X. By the theorem, D is compact. Thus, by the previous theorem, $f(D)$ is compact. Since Y is Hausdorff, by another theorem, $f(D)$ is closed. Hence, f is a homeomorphism.

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Lemma 12.0.3 (tube lemma). *Let $x_0 \in X$, suppose $\{x_0\} \times Y$ is convered by open sets W_i in $X \times Y$. Then one can choose a finite subcovering $W_1, W_2, \ldots, W_n$ of $\{x_0\} \times Y$ and find an open neighbor U of x_0 such that $U \times Y \subseteq \bigcup_{i=1}^{n} W_i$.*

Proof One may assume that $W_i = U_i \times V_i$ where U_i is open in X and V_i is open in Y. Then $\{V_i\}$ is an open covering of Y. By the compactness of Y, there exists a finite subcollection $\{V_1, V_2, \ldots, V_n\}$ that covers Y.

To be done.

Theorem 12.0.5. *The product of finitely many compact spaces is compact.*

Proof It is enough to prove for two sets. Let X and Y be compact spaces.

We can cover $X \times Y$ by finitly many tubes by the tube lemma. And each tube can be covered by finitly many open sets. Thus $X \times Y$ is compact.

To be done.

Definition 12.0.4. *A collection $\mathcal{C}$ of sets has the finite intersection property if for any finite subcollection $\{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{C}$, we have*

$$C_1 \cap C_2 \cap \ldots \cap C_n \neq \emptyset \tag{12.3}$$

Theorem 12.0.6. *X is a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X with the finite intersection property, we have*

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset \tag{12.4}$$

Proof

Let $\mathcal{A}$ be a collection of open sets in X. Then $\mathcal{C} = \{X \setminus A | A \in \mathcal{A}\}$ is a collection of closed sets in X.

Then $\mathcal{A}$ is an open covering of X if and only if $\bigcap_{C \in \mathcal{C}} C = \emptyset$.

Then a finite subcollection of $\mathcal{A}$ covers X if and only if the corresponding finite subcollection of $\mathcal{C}$ has empty intersection.

Now the theorem follows directly.

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