Chapter 5

Later

Theorem 5.0.1. *The following are equivalent:*

1. *The Intermediate Value Theorem holds for X: for any continuous map $f : X \to \mathbb{R}$, and any c between $f(x_1)$ and $f(x_2)$ for some $x_1, x_2 \in X$, there exists some $x \in X$ such that $f(x) = c$.*
2. *There is no continuous surjection from X onto a discrete two-point space $\{0, 1\}$.*
3. *X cannot be partitioned into two nonempty disjoint open sets.*
4. *X cannot be partitioned into two nonempty disjoint closed sets.*
5. *Every proper subset of X that is both open and closed is either $\emptyset$ or X itself.*

Remark If X is path-connected, then it is connected. The converse is not necessarily true.

Lemma 5.0.1. *The closure of a connected set is connected.*

Proof Let A be a connected set and $\bar{A}$ be its closure. Suppose $\bar{A} = U_1 \cup U_2$ where U_1 and U_2 are nonempty disjoint open sets in $\bar{A}$. Then $A = (A \cap U_1) \cup (A \cap U_2)$ is non-trivial: let $U_1 \cap A = \emptyset$, then $U_1 \subseteq \bar{A} \setminus A$ which is impossible.

Example 5.0.1. *Let $X = \{(x, y) | y = \sin(\frac{1}{x}), x > 0\} \cup \{(0, y) | y \in [-1, 1]\} \subset \mathbb{R}^2$ with the Euclidean metric. Then X is connected but not path-connected.*

Proof

First we prove that X is not path-connected. Let $p_1, p_2 \in X$ such that $P_1 = (0, 0)$ and $P_2 = (1, \sin(1))$. Let $\gamma : [0, 1] \to X$ with $\gamma(t) = (x(t), y(t))$ such that $\gamma(0) = P_1$ and $\gamma(1) = P_2$. Since γ is continuous, both $x(t)$ and $y(t)$ are continuous.

Let $U = \{\tau | x(t) = 0\} \subseteq [0, 1]$. Since U is a preimage of a closed set under a continuous map, U is closed. Thus, $t_0 = \sup U$ exists. Note that $t_0 < 1$ since $x(1) = 1$.

Let $\Phi(t) = \dfrac{1}{x(t)}$ which is well-defined on for $t > t_0$. Notice that $\Phi(t) \to +\infty$ as $t \to t_0^+$. Then, for $t > t_0$, $y(t) = \sin(\Phi(t))$. Take the interval $(t_0, t_0 + \epsilon)$, then $y(t)$ oscillates between -1 and 1 infinitely many times. So $y(t)$ is not continuous at t_0. This is a contradiction. Thus, no such path γ exists and X is not path-connected.

Second, we prove that X is connected. Let $X = Y \cup \{(0, y) | y \in [-1, 1]\}$. We know that $\bar{Y} = X$ and Y is path-connected (thus connected). Then using the property that the closure of a connected set is connected, we have X is connected.

Remark Also, we can prove it using 2 in the 5.0.1.

5.0.1 Connected Components

For any $x, y \in X$, we say that $x \sim y$ if $\exists \gamma : [0, 1] \to X$ continuous such that $\gamma(0) = x$ and $\gamma(1) = y$. It is easy to check that $\sim$ is an equivalence relation. We call the equivalence classes the connected components of X.

Exercise Define connected components for the notion of connected (not just path-connected) spaces.

Observation The number of connected components is preserved under homeomorphisms.

Example 5.0.2. $(0, 1) \cup (2, 3)$ *is not homeomorphic to* $(0, 1)$ *since the former has 2 connected components while the latter has 1 connected component.*

Example 5.0.3. *The interval* $[0, 1]$ *is not homeomorphic to* $[0, 1] \times [0, 1]$. *Removing a point(not in the boundary) from* $[0, 1]$ *results in a disconneted space, while removing a point from* $[0, 1] \times [0, 1]$ *still results in a connected space. Thus, they are not homeomorphic.*

Example 5.0.4. $(0,1) \not\approx [0,1] \not\approx [0,1)$. *Removing a boundary point from* $[0,1]$ *still results in a connected space, while removing a boundary point from* $(0,1)$ *results in a disconnected space. Thus, they are not homeomorphic.*

Example 5.0.5. *We can split the alphabet(capital letter) into nine homeomorphism classes:*

- *A*
- *B*
- *C, I, J, L, M, N, S, U, V, W, Z*
- *D, O*
- *E, F, G, T, Y*
- *H*
- *K, X, Z*
- *P*
- *Q, R*

Example 5.0.6 (First homology group)**.**

$$2026 \rightsquigarrow 2$$
$$2025 \rightsquigarrow 1$$
$$1949 \rightsquigarrow 2$$
$$1982 \rightsquigarrow 3$$
$$1988 \rightsquigarrow 5$$

Property 5.0.1. $U \subseteq \mathbb{R}^2$ *open in one of the metrics defined by the norms listed below iff it is open in the others:*

- *Euclidean norm:* $||(x,y)||_2 = \sqrt{x^2+y^2}$
- *Supremum norm:* $||(x,y)||_\infty = \max |x|, |y|$
- *Diamond norm:* $||(x,y)||_{1/2} = \sqrt{|x|} + \sqrt{|y|}$

In other words, these metrics define the same topology. So they have the same open sets.

Definition 5.0.1. *Two norms on* V $||\cdot||_a$ *and* $||\cdot||_b$ *are called equivalent if* $\exists c_1, c_2$ *such that:*

$$c_1||v||_a \leq ||v||_b \leq c_2||v||_a \tag{5.1}$$

Exercise

1. This is an equivalent relation.
2. Equivalent norms define the same topology.
3. $||\cdot||_1$, $||\cdot||_2$, and $||\cdot||_\infty$ on $\mathbb{R}^n$ are equivalent.

* All norms on $\mathbb{R}^n$ are equivalent.

The notes below is not for this lecture.

Theorem 5.0.2. *All norms on* $\mathbb{R}^n$ *are equivalent.*

Proof

Given $\|\cdot\|_a, \|\cdot\|_b$ two norms on $\mathbb{R}^n$. We define a function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by $f(x) = \dfrac{||x||_a}{||x||_b}$ which is a continuous function. And $f(x) = f(\lambda x)$ for any $\lambda > 0$. So f is completely determined by its values on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n | ||x||_a = 1\}$. Since S^{n-1} is compact, f attains its maximum and minimum on S^{n-1}, denoted as m and M. Thus, for any $x \neq 0$, we have:

$$0 < m \leq \frac{||x||_a}{||x||_b} \leq M < +\infty \tag{5.2}$$

Therefore, $\|x\|_b$ and $\|x\|_a$ are equivalent.

Remark In infinite-dimensional vector spaces, S^{n-1} is not compact(However, it is closed and bounded).

Exercise Give an example of norms on l_1(convergent series, i.e., $l_1 = \{(a_n) | \sum_{n=1}^\infty |a_n| < +\infty\}$).

Solution The trivial one is $||x|| = \sum_{n=1}^\infty |a_n|$. It is easy to verify as it is convergent.