Chapter 19

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Definition 19.0.1. *Matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$ are called similar if there exists an invertible matrix $P \in \mathbb{C}^{n\times m}$ such that*

$$A = PBP^{-1} \tag{19.1}$$

We write $A \sim B$.

Property 19.0.1. *Similarity is an equivalence relation on the set of all square matrices.*

Theorem 19.0.1. *If $A \sim B$, then A and B have the same characteristic polynomial, and thus the same eigenvalues with the same algebraic multiplicities.*

Proof
We have

$$P(\lambda) = \det(\lambda I - A) = \det(\lambda I - PBP^{-1}) = \det(P(\lambda I - B)P^{-1}) = \det(\lambda I - B) \tag{19.2}$$

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19.1 Schur's decomposition

Idea The idea of diagonalization may be problematic as:

- Diagonalization is not always possible.
- It is neccessary to compute the inverse of P.
- Computing the decompostition is much easier if P is unitary(called orthogonal if real).

Theorem 19.1.1 (Shur's decomposition)**.** *For all $A \in \mathbb{C}^{n\times n}$, there exists an upper triangular $T \in \mathbb{C}^{n\times n}$ and a unitary $U \in \mathbb{C}^{n\times n}$ such that*

$$A = UTU^* \tag{19.3}$$

Proof
Let λ_1 be an eigenvalue of A. We choose orthogonal bases $\{q_1, q_2, \ldots, q_k\} = E_{\lambda_1}$ of the eigenspace of λ_1 and $E_{\lambda_1}^{\perp}$ respectively. Let $Q_1 = [q_1, q_2, \ldots, q_n]$ and thus Q_1 is unitary. We obtain

$$A = Q_1 \begin{bmatrix} \lambda_1 I_k & B_1 \\ 0 & A_1 \end{bmatrix} Q_1^* \tag{19.4}$$

Repeating the process on A_1, we obtain a unitary Q_2 such that

$$A_1 = Q_2 \begin{bmatrix} \lambda_2 I_{k_2} & B_2 \\ 0 & A_2 \end{bmatrix} Q_2^* \tag{19.5}$$

Continuing this process, we eventually obtain

$$A = UTU^* \tag{19.6}$$

where $U = Q_1 Q_2 \cdots Q_m$ is unitary and T is upper triangular.

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Definition 19.1.1. *A matrix $A \in \mathbb{C}^{n\times n}$ is called normal if $AA^* = A^*A$.*

Remark A Hermitian matrix is normal since $A^*A = A^2 = AA^*$. A unitary matrix is normal since $AA^* = I = A^*A$.

Theorem 19.1.2. *A matrix $A \in \mathbb{C}^{n\times n}$ is normal if and only if it is unitarity diagonalizable, i.e., there exists a unitary matrix U and a diagonal matrix Λ such that*

$$A = U\Lambda U^* \tag{19.7}$$

Proof

Suppose $A = UTU^*$ is the Schur decomposition of A. If A is normal, then

$$(UTU^*)(UTU^*)^* = (UTU^*)^*(UTU^*) \tag{19.8}$$

This implies that $TT^* = T^*T$. Since T is upper triangular, T must be diagonal. Thus, A is unitarily diagonalizable.

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Remark A matrix that is diagonalizable but not unitarily diagonalizable is

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \tag{19.9}$$

The eigenvalues of A are 1 and 2, but $AA^* \neq A^*A$. (Exercise)

Corollary 19.1.1. *For a normal matrix, eigenvectors corresponding to distinct eigenvalues are orthogonal, or can be normalized to be orthonormal. The eigenvectors related to the same eigenvalue can be orthogonalized by using Gram-Schmidt process.*

19.2 Singular Value Decomposition

Theorem 19.2.1 (Singular Value Decomposition)**.** *For any matrix $A \in \mathbb{C}^{m\times n}$, it can be represented in the form*

$$A = U\Sigma V^* \tag{19.10}$$

where $U \in \mathbb{C}^{m\times p}$ and $V \in \mathbb{C}^{n\times p}$ have orthonormal columns(so U is unitary if $m = p$ and V is unitary if $n = p$), $\Sigma \in \mathbb{C}^{p\times p}$ satisfies

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix} \tag{19.11}$$

where $p = \min\{m, n\}$, such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$ are the singular values of A .