



Metric and Topological Space

Mathematics With Computer Science

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Chapter 1

Metric Spaces

Definition 1.0.1. A metric space is a set X with a function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ (d is the metric, or the distance function) satisfying the following properties for all $x, y, z \in X$:

- **Non-negativity:** $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
- **Symmetry:** $d(x, y) = d(y, x)$.
- **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$.

Example 1.0.1. $X = \mathbb{R}$, $d(x, y) = \|x - y\|$

Example 1.0.2. X is finite, $d(x, y) = 1$ if $x \neq y$, $d(x, x) = 0$ (discrete metric)

Example 1.0.3. X is a finite set of vertices of a connected graph, $d(x, y)$ is the length of the shortest path between x and y (graph metric)

Example 1.0.4. $X = \mathbb{R}^n$, $d(x, y) = \|x - y\|$ (Euclidean metric)

Example 1.0.5. $X = \mathbb{R}^n$, $d(x, y) = \sum_{i=1}^n |x_i - y_i|$ (manhattan metric)

Example 1.0.6. $X = \mathbb{R}^n$, $d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ (sup metric)

Exercise Prove that the above examples are indeed metric spaces (You may use Cauchy-Schwarz inequality which is stated later).

Example 1.0.7. $X = \mathbb{R}^2$, $P = (x, y)$, $P' = (x', y')$

$$d(P, P') = \begin{cases} |x - x'| + |y| + |y'| & \text{if } x \neq x' \\ |y - y'| & \text{if } x = x' \end{cases}$$

Example 1.0.8. $X = C[0, 1]$ (the set of continuous functions on $[0, 1]$), $d(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|$ (sup metric)

Example 1.0.9. $X = C[0, 1]$, $d(f, g) = \int_0^1 |f(x) - g(x)| dx$ (L^1 metric)

Remark We cannot replace X by the set of all integrable functions, because the distance between two functions may be zero even if they are not equal (they may differ on a set of measure zero).

Definition 1.0.2 (Normed Space). Let V be a vector space over \mathbb{R} . A norm on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties for all $u, v \in V$ and all $a \in \mathbb{R}$:

- **Non-negativity:** $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.
- **Absolute homogeneity:** $\|av\| = |a|\|v\|$.
- **Triangle inequality:** $\|u + v\| \leq \|u\| + \|v\|$.

Remark A norm defines a metric by $d(u, v) = \|u - v\|$ on V .

Example 1.0.10. $\|v\| = \sqrt{\sum_{i=1}^n |v_i|^2}$ (Euclidean norm)

Example 1.0.11. $\|v\| = \sum_{i=1}^n |v_i|$ (manhattan norm)

Example 1.0.12. $\|v\| = \max_{1 \leq i \leq n} |v_i|$ (sup norm)

Example 1.0.13. $V = C[0, 1]$, $\|f\| = \max_{x \in [0, 1]} |f(x)|$ (*sup norm*)

Example 1.0.14. $V = C[0, 1]$, $\|f\| = \int_0^1 |f(x)| dx$ (L^1 norm)

Definition 1.0.3 (l_p -Spaces). Let $1 \leq p < \infty$.

- $l_\infty = \{x = (x_1, x_2, \dots) : \sup_i |x_i| < \infty\}$ with norm $\|x\|_\infty = \sup_i |x_i|$.
- $l_1 = \{x = (x_1, x_2, \dots) : \sum_i |x_i| < \infty\}$ with norm $\|x\|_1 = \sum_i |x_i|$.
- $l_2 = \{x = (x_1, x_2, \dots) : \sum_i |x_i|^2 < \infty\}$ with norm $\|x\|_2 = \sqrt{\sum_i |x_i|^2}$.
- $l_p = \{x = (x_1, x_2, \dots) : \sum_i |x_i|^p < \infty\}$ with norm $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$.

Property 1.0.1. $l_p \subseteq l_q$ if $p < q$

Example 1.0.15. $x_n = \frac{1}{n}$. $\sum_1^\infty \frac{1}{n} = \infty$ but $\sum_1^\infty \frac{1}{n^2} < \infty$. So $x = (1, \frac{1}{2}, \frac{1}{3}, \dots) \in l_2$ but $x \notin l_1$.

Definition 1.0.4 (Euclidean Space/Vector Space with Inner Product). Let V be a vector space over \mathbb{R} . An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ satisfying the following properties for all $u, v, w \in V$ and all $a \in \mathbb{R}$:

- **Positivity:** $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$.
- **Symmetry:** $\langle u, v \rangle = \langle v, u \rangle$.
- **Bilinearity:** $\langle au + w, v \rangle = a\langle u, v \rangle + \langle w, v \rangle$.

Remark. We always use Euclidean Space as the default example of inner product space.

Theorem 1.0.1 (Cauchy-Schwarz Inequality). For all $u, v \in V$, $|\langle u, v \rangle| \leq \|u\| \|v\|$.

Proof Let $f(t) = \|u + tv\|^2 = \langle u + tv, u + tv \rangle$. Then using the properties of quadratic polynomials. ■

Property 1.0.2. $\|v\| = \sqrt{\langle v, v \rangle}$ is a norm on V .

Proof. Use Cauchy-Schwarz inequality: $|\langle u, v \rangle| \leq \|u\| \|v\|$. ■

Corollary 1.0.1. l_2 -norm satisfies the triangle inequality.

Proof. l_2 is an Euclidean Space(infinite dimensions) with $\langle x, y \rangle = \sum_i x_i y_i$ and $\|x\|_2 = \sqrt{\sum_i |x_i|^2}$. ■

Remark. $\|x\|_p$ does not come from an inner product if $p \neq 2$.

Theorem 1.0.2. l_p are normed spaces.

Proof. This will be proved later in the course.