Chapter 16

Later

Definition 16.0.1 (def1). *X is compact if any open covering admits a finite subcovering.*

Definition 16.0.2 (def2: Frechet compactness). *X is limit point compact if every infinite subset of X has a limit point in X.*

Definition 16.0.3 (def3: sequential compactness/Bozano-Weierstrass compactness). *X is sequentially compact if every sequence $\{x_n\} \subseteq X$ has a convergent subsequence converging to a point in X.*

Theorem 16.0.1. *If X is a metric space, then X is compact if and only if it is limit point compact if and only if it is sequentially compact.*

Proof

The proof is quite long and technical. We will only some directions in the following.

Theorem 16.0.2. *For an arbitrary topological space, compactness implies limit point compactness. The converse is not true in general.*

Proof

Let X be compact, and let $A \subseteq X$ be infinite.

We assume that A has no limit point. Then $A = \bar{A}$. So $X \setminus A$ is open. Furthermore, for each $a \in A$, we can choose a neighborhood U_a of a such that $U_a \cap (A \setminus \{a\}) = \emptyset$.

Then $\{U_a : a \in A\} \cup \{X \setminus A\}$ is an open covering of X. Since X is compact, there exists a finite subcovering, say $\{U_{a_1}, U_{a_2}, \cdots, U_{a_n}\} \cup \{X \setminus A\}$. Thus,

$$X = \bigcup_{i=1}^{n} U_{a_i} \cup (X \setminus A) = (X \setminus A) \cup \{a_1, a_2, \cdots, a_n\},$$

which implies that A is finite, a contradiction. Hence, A has a limit point in X.

∎

Example 16.0.1 (Counterexample for the converse). *Let $Y = \{p, q\}$ with anti-dicrete topology $J_Y = \{\emptyset, Y\}$. Let $X = \mathbb{N} \times Y$ with the product topology where $\mathbb{N}$ has the discrete topology. Then every non-empty set $A \subseteq X$ has a limit point. Because if $(n, p) \in A$, then any open set containing (n, q) intersects A at (n, p). So X is limit point compact. However, X is not compact. Because $\{\{n\} \times Y : n \in \mathbb{N}\}$ is an open covering of X which admits no finite subcovering.*

Example 16.0.2 (Limit point compact but not sequentially compact). *Let X be defined as the above example. Consider the sequence $\{(n, p)\}_{n=1}^{\infty}$. This sequence has no convergent subsequence. Because for any point (m, p), the open set $\{m\} \times Y$ contains only finitely many terms of the sequence; for any point (m, q), the open set $\{m\} \times Y$ also contains only finitely many terms of the sequence. So X is not sequentially compact.*

Theorem 16.0.3. *For a first-countable topological space X, limit point compactness implies sequential compactness.*

Proof

Take a sequence $\{x_n\} \subseteq X$.

If the set of values $\{x_n : n \in \mathbb{N}\}$ is finite, then there exists a value x that appears infinitely many times in the sequence. So the subsequence constantly equal to x converges to x.

Suppose the set of values $\{x_n : n \in \mathbb{N}\}$ is infinite. Since X is first-countable, we can construct a countable basis $\{U_k\}$ at a limit point a such that

$$x_{n_1} \in U_1, x_{n_2} \in U_2, \cdots, x_{n_k} \in U_k, \cdots \tag{16.1}$$

and

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k \supseteq \cdots \tag{16.2}$$

That is for each open set $U \ni a$ there exists N such that $\forall k \geq N$, $U_k \subseteq U$. Then $x_{n_k} \in U_k \subseteq U$ for all $k \geq N$. So $x_{n_k} \to a$. ∎

Remark For topological spaces, we have the following facts:

1. Compactness $\Rightarrow$ Limit Point Compactness

2. Sequential Compactness $\Rightarrow$ Limit Point Compactness

3. But other implications are not true in general.

Theorem 16.0.4. *Sequential compactness implies compactness for metric spaces.*

Proof This is harder than other directions.

Exercise Prove that X is sequentially compact, then it is limit point compact.

16.1 Locally Compact

Definition 16.1.1. *A space X is said to be locally compact at x if there is some compact subspace C of X containing a neighborhood U of x. If X is locally compact at each of its points, then X is said to be locally compact.*

Example 16.1.1. $\mathbb{R}$ *is locally compact. Because for any $x \in \mathbb{R}$, take $U = (x-1, x+1)$ and $C = [x-1, x+1]$ which is compact.*

Example 16.1.2. $\mathbb{R}^n$ *is locally compact.*

Example 16.1.3 (non example). $\mathbb{Q} \subseteq \mathbb{R}$ *is not locally compact.*

Exercise Show the above example.

Solution Let C be a compact subspace of $\mathbb{Q}$ containing a neighborhood U of $x \in \mathbb{Q}$. Since U is open in $\mathbb{Q}$, there exists an interval $(a, b) \subseteq \mathbb{R}$ such that $U = (a, b) \cap \mathbb{Q}$. We know Q is dense in $\mathbb{R}$, so there exists an irrational number $y \in (a, b)$. Let $\{q_n\} \subseteq (a, b) \cap \mathbb{Q}$ be a sequence converging to y. Since C is compact, so $\{q_n\}$ has a subsequence converging to a point in C. But a convergent sequence in $\mathbb{R}$ has a unique limit, so the subsequence converges to y, which is not in $\mathbb{Q}$ thus not in C. This is a contradiction. Hence, $\mathbb{Q}$ is not locally compact. ∎

Example 16.1.4 (non example). $\mathbb{R}^\omega$ *with product topology is not locally compact. If U is open, then U contains a basis element $(a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots$. Then U is not contained in any compact set C (C contains one factor of $\mathbb{R}$, then the projection to some factor space is $\mathbb{R}$, but a projection is continuous, which means $\mathbb{R}$ is compact. This is definitely not true).*

Theorem 16.1.1. *Let X be a topological space. Then X is Hausdorff locally compact if and only if there exists a space Y such that*

1. *X is a subspace of Y.*

2. *$Y \setminus X$ contains exactly one point p.*

3. *Y is compact Hausdorff.*

This Y is unique in the following sense: If Y, Y' are two spaces with the above properties and $Y = X \cup \{p\}, Y' = X \cup \{q\}$, then there exists a homeomorphism $h : Y \to Y'$ such that $h(x) = x$ for all $x \in X$ and $h(p) = q$.

Proof

Uniqueness Let $Y = X \cup \{p\}$, $Y' = X \cup \{q\}$ satisfies the above properties. Define $h : Y \to Y'$ as follows: $h(x) = x$ for all $x \in X$ and $h(p) = q$. We show that h is continuous. But the function is symmetric, so it is enough to show that $h(U)$ is open in Y' for all open set U in Y.

Take U be open. There are two cases.

If $U \subseteq X$, then we are done.

Suppose $p \in U$. Then $C = Y \setminus U$ is closed in Y. So C is compact.

Construction We introduce the topology on $Y = X \cup \{\infty\}$ as follows:

There are two types of open sets in Y:

1. $U \subseteq X \subseteq Y$ is open in X.

2. $U = Y \setminus C$ where C is compact subspace of X.

We need to check that this is a topology.

For the intersection of two open sets of type 1, we have $U_1 \cap U_2$ is open in X thus is open in Y.

For the intersection of type 1 and type 2, we have $U_1 \cap (Y \setminus C) = U_1 \cap (X \setminus C)$ which is the union of open sets in X thus is open in Y because C is closed in X(as a compact subspace of a Hausdorff space).

For the intersection of two open sets of type 2, we have $(Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2)$ which is open in Y because $C_1 \cup C_2$ is compact.

Similarly, one checks that the union of any collection of open sets is open. We have

1. $\bigcup U_\alpha = U$ is of type 1.

2. $\bigcup (Y \setminus C_\alpha) = Y \setminus \bigcap C_\alpha = Y \setminus C$ where $C = \bigcap C_\alpha$ is compact(Easy to check). This is of type 2.

3. $\bigcup U_\alpha \cup \bigcup (Y \setminus C_\beta) = U \cup (Y \setminus C) = Y \setminus (C \setminus U)$. This is of type 2 because $C \setminus U = C \cap (X \setminus U)$ is closed in C thus is compact because a closed subset of a compact set is compact.

Now we show that X is a subspace of Y. Given any open set V of Y, we show its intersection with X is open in X. If V is of type 1, then $V \cap X = V$ which is open in X. If V is of type 2, then $V \cap X = (Y \setminus C) \cap X = X \setminus C$ which is open in X because C is closed in X as a compact subspace of a Hausdorff space. Conversely, given any open set U of X, U is an open set of Y of type 1. So X is a subspace of Y.

We show that Y is compact.

Let $\mathcal{A}$ be an open covering of Y. The collection $\mathcal{A}$ must contain an open set containing ∞. So there exists a compact set C such that $U = Y \setminus C$ with $U \in \mathcal{A}$. Take all the members of $\mathcal{A}$ different from U and intersect them with X. This gives a collection of open sets in X which covers C. Since C is compact, there exists a finite subcovering of C, say $\{V_1, V_2, \cdots, V_n\}$. Then $\{U, V_1, V_2, \cdots, V_n\}$ is a finite subcovering of Y. So Y is compact.

We show that Y is Hausdorff.

Take two distinct points $x, y \in Y$. There are two cases.

1. If $x, y \in X$, since X is Hausdorff, there exist disjoint open sets U, V in X such that $x \in U, y \in V$. Then U, V are open in Y and disjoint.

2. If one of them is ∞, say $y = \infty$. Since X is locally compact at x, there exists an open neighborhood U of x and a compact set C such that $U \subseteq C$. So C is closed in X thus in Y. Then $V = Y \setminus C$ is an open neighborhood of ∞. Clearly, $U \cap V = \emptyset$.

Finally we prove the other direction. Suppose a space Y satisfying conditions (1)-(3) exists. Then X is Hausdorff as a subspace of a Hausdorff space. Given any $x \in X$, we show X is locally compact at x. Choose disjoint open sets U, V in Y such that $x \in U$ and $p \in V$. Then $C = Y \setminus V$ is compact as a closed subset of a compact space. Also, $x \in U \subseteq C$. So X is locally compact at x.

■

Theorem 16.1.2. *Let X be a Hausdorff space. Then X is locally compact if and only if for any $x \in X$ and any neighborhood U of x, there exists a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subseteq U$.*

Proof

$\Leftarrow$ $x \in V \subseteq \bar{V} \subseteq U$ and $\bar{V}$ is compact. So X is locally compact at x.

$\Rightarrow$ Let Y be the one-point compactification(compact and Hausdorff) of X. Let U be a neighborhood of x in Y. Then $Y \setminus U := C$ is a closed subset of Y. So C is compact. By lemma 12.0.2, there exists two open sets V, W in Y such that $x \in V$, $C \subseteq W$ and $V \cap W = \emptyset$. Then $\bar{V}$ is compact and $\bar{V} \cap C = \emptyset$. So $\bar{V} \subseteq U$. ■

Corollary 16.1.1. *Let X be locally compact Hausdorff space. $A \subseteq X$ is open or closed. Then A is locally compact.*

Proof

Let $A \subseteq X$ be closed. Given $x \in A$, since X is locally compact, there exists an open neighborhood U of x in X and a compact set C such that $x \in U \subseteq C$. Then $U \cap A$ is an open neighborhood of x in A and $C \cap A$ is compact in A. So A is locally compact at x.

Let $A \subseteq X$ be open. Given $x \in A$, since X is locally compact and X is Hausdorff, by the previous theorem, there exists a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subseteq A$. So A is locally compact at x. ■

Corollary 16.1.2. *X is locally compact Hausdorff if and only if X is homeomorphic to an open subspace of a compact Hausdorff space.*

Exercise Show the above corollary.

16.2   Urysohn's Metrization Theorem

Theorem 16.2.1. *Every X that is regular(T3) and second-countable is metrizable.*

16.3   Countability

Definition 16.3.1. *X is first countable if for every $x \in X$, there exists a countable basis at x. That is, there exists $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ a collection of open sets containing x such that for any open set U containing x, there exists $B_n \in \mathcal{B}$ such that $B_n \subseteq U$.*

Definition 16.3.2. *X is second countable if there exists a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ for the topology of X. That is, for every x and every open set U containing x, there exists $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq U$.*

Exercise $\mathbb{R}^n$ with standard topology, $B_n = \{U_\epsilon(x) | x \in \mathbb{Q}^n, \epsilon \in \mathbb{Q}_{>0}\}$. B_n is a countable basis. So $\mathbb{R}^n$ is second countable. Show the details.

Exercise Show that if X_n's are first(second) countable, then $\prod X_n$ with product topology is first(second) countable.

Theorem 16.3.1. *Let X be second countable. Then*

1. *Every open cover of X has a countable subcovering(X is Lindelöf space).*

2. *There is a countable subset $A \subseteq X$ such that $\bar{A} = X$($A \subseteq X$ is dense).*

Proof