# Chapter 18

# Later

## 18.1 Countability

**Definition 18.1.1.** *$X$ is first countable if for every $x \in X$, there exists a countable basis at $x$. That is, there exists $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ a collection of open sets containing $x$ such that for any open set $U$ containing $x$, there exists $B_n \in \mathcal{B}$ such that $B_n \subseteq U$.*

**Definition 18.1.2.** *$X$ is second countable if there exists a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ for the topology of $X$. That is, for every $x$ and every open set $U$ containing $x$, there exists $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq U$.*

**Exercise** $\mathbb{R}^n$ with standard topology, $B_n = \{U_\epsilon(x) | x \in \mathbb{Q}^n, \epsilon \in \mathbb{Q}_{>0}\}$. $B_n$ is a countable basis. So $\mathbb{R}^n$ is second countable. Show the details.
**Exercise** Show that if $X_n$'s are first(second) countable, then $\prod X_n$ with product topology is first(second) countable.

**Theorem 18.1.1.** *Let $X$ be second countable. Then*

1. *Every open cover of $X$ has a countable subcovering($X$ is Lindelöf space).*
2. *There is a countable subset $A \subseteq X$ such that $\bar{A} = X$($A \subseteq X$ is called dense and $X$ is called separable).*

**Proof to 1**
Let $\mathcal{A}$ be an open cover of $X$. Let $\mathcal{A}$ be an open covering of $X$. For each positive integer $n$ for which it is possible, choose an element $A_n \in \mathcal{A}$ such that $B_n \subseteq A_n$. The collection $\mathcal{A}'$ of all such possible $A_n$ is countable since it is indexed with a subset $J$ of the positive integers.
Furthermore, $\mathcal{A}'$ covers $X$. To see this, let $x \in X$. Since $\mathcal{A}$ is a covering of $X$, there exists $A \in \mathcal{A}$ such that $x \in A$. Since $\mathcal{B}$ is a basis for the topology of $X$, there exists $B_m \in \mathcal{B}$ such that $x \in B_m \subseteq A$. By construction of $\mathcal{A}'$, there exists $A_m \in \mathcal{A}'$ such that $B_m \subseteq A_m$. Thus, $x \in A_m$. So $\mathcal{A}'$ is a countable subcovering of $\mathcal{A}$.
**Proof to 2**
From each non-empty basis element $B_n$, choose a point $x_n \in B_n$. Let $D$ be the set of all such points $x_n$. Then $D$ is countable.
Given any point $x \in X$, every basis element containing $x$ contains a point of $D$, so $x$ is in the closure of $D$. Thus, $\bar{D} = X$.

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**Property 18.1.1.** *A second countable space is:*

1. *First countable.*
2. *Separable.*
3. *Lindelöf.*

**Example 18.1.1** (non-example). *$\mathbb{R}_l$ with lower limit topology(Sorgengrey line) is first countable, Lindelöf, seperable, but not second countable.*
*First-countable: for any $x \in \mathbb{R}_l$, $\{[x, x + \frac{1}{n}) : n \in \mathbb{N}\}$ is a countable local basis at $x$.*
*Assume the contrary that $\mathbb{R}_l$ is second countable. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis for $\mathbb{R}_l$. For each $x \in \mathbb{R}$, there exists $B_{n_x} \in \mathcal{B}$ such that $x \in B_{n_x} \subseteq [x, x+1)$. So $\min B_{n_x} = x$. This implies that the map $f : \mathbb{R} \to \mathcal{B}, f(x) = B_{n_x}$ is injective. However, $\mathbb{R}$ is uncountable while $\mathcal{B}$ is countable, which is a contradiction. So $\mathbb{R}_l$ is not second countable.*
*Separable: $\mathbb{Q} \subseteq \mathbb{R}_l$ is countalbe and dense in $\mathbb{R}_l$.*
*Lindelöf: See the below property.*

**Property 18.1.2** (Lindelöf property of $\mathbb{R}_l$). *Any open cover of $\mathbb{R}_l$ has a countable subcover.*

**Proof**

Let $\mathcal{A}$ be a covering of $\mathbb{R}_l$ by $[a_\alpha, b_\alpha)$'s. We need to show that there exists a countable subcovering of $\mathcal{A}$ covering $\mathbb{R}_l$.

Let $C = \bigcup(a_\alpha, b_\alpha)$ such that $\mathbb{R} \setminus C$ is not empty. Let $x \in \mathbb{R} \setminus C$. Then $x = a_\beta$ for some $\beta$ such that $x \notin (a_\beta, b_\beta)$. Take $q_x \in (a_\beta, b_\beta) \cap \mathbb{Q}$ thus $x < q_x$.

Take $x, y \in \mathbb{R}$ with $x < y$. Then $q_x < q_y$ or we make a contradiction(why?). The map $x \mapsto q_x$ is injective from $\mathbb{R} \setminus C$ to $\mathbb{Q}$. So $\mathbb{R} \setminus C$ is countable.

Now we show that some countable subcollection of $\mathcal{A}$ covers $\mathbb{R}$. To begin, choose for each element $\mathbb{R} \setminus C$ an interval $[a_\alpha, b_\alpha)$ of $\mathcal{A}$ that contains it. Then we have chosen a countable subcollection $\mathcal{A}'$ of $\mathcal{A}$ covering $\mathbb{R} \setminus C$.

Now take the set $C$ and topologize it as a subspace of $\mathbb{R}$($C$ is open in $\mathbb{R}$, topologizing means taking the intersection of $C$ with open sets in $\mathbb{R}$ as the open sets of $C$). Then $C$ is second countable(since $\mathbb{R}$ is second countable). Now $C$ is covered by the open intervals $(a_\alpha, b_\alpha)$ which are open in $\mathbb{R}$ and hence open in $C$. So there exists a countable subcovering of $C$ by $(a_\alpha, b_\alpha)$'s. Suppose this subcollection is indexed by $\alpha = \alpha_1, \alpha_2, \cdots$. Then the collection

$$\mathcal{A}'' = \{[a_\alpha, b_\alpha) : \alpha = \alpha_1, \alpha_2, \cdots\} \tag{18.1}$$

is a countable subcollection of $\mathcal{A}$ that covers $C$.

Hence the countable collection $\mathcal{A}' \cup \mathcal{A}''$ is a countable subcollection of $\mathcal{A}$ that covers $\mathbb{R}_l$.

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**Property 18.1.3.** *$\mathbb{R}_l \times \mathbb{R}_l$ is not Lindelöf. The space $\mathbb{R}_l \times \mathbb{R}_l$ with the product topology is called the Sorgenfrey plane. (The proof can be shown by a picture not drawn here.)*

**Proof** The space $\mathbb{R}_l^2$ has as basis all sets of the form $[a, b) \times [c, d)$ where $a < b$ and $c < d$. To show that $\mathbb{R}_l^2$ is not Lindelöf, we consider the subspace

$$L = \{(x, -x) | x \in \mathbb{R}_l\} \subseteq \mathbb{R}_l^2. \tag{18.2}$$

It is easy to check that $L$ is closed in $\mathbb{R}_l^2$. Let's cover $\mathbb{R}_l^2$ by the open set $\mathbb{R}_l^2 \setminus L$ and by all basis elements of the form

$$[a, b) \times [-a, d) \tag{18.3}$$

Each of these open sets intersects $L$ in at most one point. Since $L$ is uncountable, no countable subcollection covers $\mathbb{R}_l^2$. Thus, $\mathbb{R}_l^2$ is not Lindelöf.

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**Example 18.1.2.** *Subspace of Lindelöf space is not necessarily Lindelöf. The ordered square $I_0^2$ is compact thus is Lindelöf. But the subspace $A = I_0 \times (0, 1)$ is not Lindelöf. For $A$ is the union of disjoint sets $U_x = \{x\} \times (0, 1)$, each of which is open in $A$. This collection of sets $\{U_x | x \in I_0\}$ is uncountable, and no proper subcollection covers $A$.*

## 18.2 Separation Axioms

Topological space $X$ can satisfy the following separability axioms:

1. $T1$(Frechet): Given two distinct points $x, y \in X$, there exists an open set $U$ such that $x \in U$ and $y \notin U$. (Equivalently, all singleton sets are closed. Easy exercise)

2. $T2$(Hausdorff): Given two distinct points $x, y \in X$, there exist open sets $U, V$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

3. $T3$(Regular): $X$ is $T1$ and given a closed set $F \subseteq X$ and a point $x \notin F$, there exist open sets $U, V$ such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$.

4. $T4$(Normal): $X$ is $T1$ and given two disjoint closed sets $F_1, F_2 \subseteq X$, there exist open sets $U, V$ such that $F_1 \subseteq U, F_2 \subseteq V$ and $U \cap V = \emptyset$.

**Lemma 18.2.1.** *Let $X$ be a $T1$ topological space. Then*

1. *$X$ is regular if and only if for every point $x \in X$ and every neighborhood $U$ of $x$, there exists a neighborhood $V$ of $x$ such that $\bar{V} \subseteq U$.*

2. *$X$ is normal if and only if for every closed set $F \subseteq X$ and every neighborhood $U$ of $F$, there exists a neighborhood $V$ of $F$ such that $\bar{V} \subseteq U$.*

**Proof**

$\Rightarrow$ Let $X$ be regular.

If $U = X$, there is nothing to prove.

Suppose $U \neq X$. Then $F = X \setminus U$ is closed and $x \notin F$. By regularity of $X$, there exist open sets $V, W$ such that $x \in V, F \subseteq W$ and $V \cap W = \emptyset$. Now we need to show $\bar{V} \subseteq U$.

If $y \in F$, then $y \in W$. Since $V \cap W = \emptyset$, $y \notin \bar{V}$ because we find a neighborhood $W$ of $y$ such that $W \cap V = \emptyset$. Thus, $\bar{V} \subseteq X \setminus F = U$.

$\Leftarrow$ Suppose that the point $x$ and the closed set $B$ not containing $X$ are given. Let $U = X \setminus B$. By hypothesis, there is a neighborhood $V$ of $x$ such that $\bar{V} \subseteq U$. The open sets $V$ and $X \setminus \bar{V}$ then separate $x$ and $B$.

The proof of 2. is similar.

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**Theorem 18.2.1.** *1. $X$ is Hausdorff, then $Y \subseteq X$ with subspace topology is Hausdorff.*

*2. $X_\alpha$ is Hausdorff, then $\prod X_\alpha$ with product topology is Hausdorff.*

*3. $X$ is regular, then $Y \subseteq X$ with subspace topology is regular.*

*4. $X_\alpha$ is regular, then $\prod X_\alpha$ with product topology is regular.*

**Proof**

(a) is obvious.

(b) Let $x = (x_\alpha), y = (y_\alpha) \in \prod X_\alpha$ with $x \neq y$. Then there exists $\beta$ such that $x_\beta \neq y_\beta$. Since $X_\beta$ is Hausdorff, there exist open sets $U_\beta, V_\beta \subseteq X_\beta$ such that $x_\beta \in U_\beta, y_\beta \in V_\beta$ and $U_\beta \cap V_\beta = \emptyset$. Then $\pi_\beta^{-1}(U_\beta), \pi_\beta^{-1}(V_\beta)$ are open in $\prod X_\alpha$, $x \in \pi_\beta^{-1}(U_\beta), y \in \pi_\beta^{-1}(V_\beta)$ and $\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \emptyset$. So we are done.

(c) (It is easy to verify the subspace of $T_1$ space is $T_1$. )Let $Y \subseteq X$ with subspace topology. Let $y \in Y$ and $B$ be a closed set in $Y$ such that $y \notin B$. Then there exists a closed set $F$ in $X$ such that $B = F \cap Y$. Since $y \notin B$, $y \notin F$. Since $X$ is regular, there exist open sets $U, V \subseteq X$ such that $y \in U, F \subseteq V$ and $U \cap V = \emptyset$. Then $U \cap Y, V \cap Y$ are open in $Y$, $y \in U \cap Y, B \subseteq V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = \emptyset$. So we are done.

(d) Let $\{X_\alpha\}$ be a family of regular spaces. Let $X = \prod X_\alpha$. $X_\alpha$ is regular implies that $X_\alpha$ is Hausdorff. By (b), $X$ is Hausdorff thus is $T_1$.

Let $x = (x_\alpha) \in X$ and let $U$ be a neighborhood of $x$ in $X$. Choose a basis element $\prod U_\alpha \subseteq U$ containing $x$. By the previous lemma, we can choose, for each $\alpha$, a neighborhood $V_\alpha$ of $x_\alpha$ in $X_\alpha$ such that $\bar{V_\alpha} \subseteq U_\alpha$. If it happens that $U_\alpha = X_\alpha$ for some $\alpha$, we choose $V_\alpha = X_\alpha$. Then $V = \prod V_\alpha$ is a neighborhood of $x$ in $X$. Recall from lecture 11, that $\bar{V} = \prod \bar{V_\alpha}$. It follows at once that $\bar{V} \subseteq \prod U_\alpha \subseteq U$. By the previous lemma, $X$ is regular.

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**Remark** $X$ is normal does not imply that $Y \subseteq X$ with subspace topology is normal. $X_1, X_2$ are normal does not imply that $X_1 \times X_2$ with product topology is normal.

**Exercise** Show that $\mathbb{R}_l$ is normal.

**Solution** It is immediate that one-point sets are closed in $\mathbb{R}_l$, since the topology of $\mathbb{R}_l$ is finer than the standard topology on $\mathbb{R}$. Let $A$ and $B$ are disjoint closed sets in $\mathbb{R}_l$. For each $a \in A$, since $B$ is closed and $a \notin B$, there exists a basis element $[a, x_a)$ such that $[a, x_a) \cap B = \emptyset$. For each $b \in B$, since $A$ is closed and $b \notin A$, there exists a basis element $[b, y_b)$ such that $[b, y_b) \cap A = \emptyset$. Then the open sets

$$U = \bigcup_{a \in A} [a, x_a), \quad V = \bigcup_{b \in B} [b, y_b) \tag{18.4}$$

are disjoint open sets containing $A$ and $B$ respectively. So we are done.

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**Fact** (Without Proof, Efforts are required to show this) $\mathbb{R}_l^2$ is not normal. But it is regular( as a product of regular spaces).

**Exercise** $\mathbb{R}_K$ is a topological space with basis given by all open intervals $(a, b)$ and all sets of the form $(a, b) \setminus K$ where $K = \{1/n | n \in \mathbb{N}\}$. $\mathbb{R}_K$ is Hausdorff. But $\mathbb{R}_K$ is not regular.

The set $K$ is closed in $\mathbb{R}_K$ and it does not contain the point 0. Suppose that there exist disjoint open sets $U$ and $V$ such that $0 \in U$ and $K \subseteq V$. Choose a basis element containing 0 and lying in $U$. It must be a basis element of the form $(a, b) \setminus K$, since each basis element of the form $(a, b)$ contains intersects $K$. Choose $1/n \in K$ such that $1/n < b$. Then there exists a basis element of the form $(c, d)$ such that $1/n \in (c, d) \subseteq V$. Finally, the basis elements $(a, b) \setminus K$ and $(c, d)$ intersect, since $c < 1/n < b$. So $U$ and $V$ are not disjoint.

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