Chapter 18

Later

Theorem 18.0.1. *Every second countable regular space is normal.*

Proof

Let X be a second countable regular space with $A, B \subseteq X$ closed and $A \cap B = \emptyset$.

For every $x \in A$, since X is regular, there exist open sets $U \ni x$ such that $U \cap B = \emptyset$.

By the lemma, there exists a open set V such that $x \in V$, $\bar{V} \subseteq U$ and thus $\bar{V} \cap B = \emptyset$.

Let $\mathcal{B}$ be a countable baisis for X. For each $x \in A$, there exists $U_x \in \mathcal{B}$ such that $x \in U_x \subseteq V$, and thus $\bar{U_x} \subseteq \bar{V} \subseteq U$ and $\bar{U_x} \cap B = \emptyset$. Since $\mathcal{B}$ is countable, the collection $\{\bar{U_x} : x \in A\}$ has a countable subcollection $\{\bar{U_n} : n \in \mathbb{N}\}$ covering A such that $\bar{U_n} \cap B = \emptyset$ for each n.

Theorem 18.0.2. *Every metrizable space X is normal.*

Proof Let d be the metric on X. Given two closed disjoint subsets $A, B \subseteq X$.

For each $a \in A$, there exists $r_a > 0$ such that $B(a, r_a) \cap B = \emptyset$.

Similarly, for each $b \in B$, there exists $s_b > 0$ such that $B(b, r_b) \cap A = \emptyset$.

Let $U = \bigcup_{a \in A} B(a, r_{\frac{a}{2}})$ and $V = \bigcup_{b \in B} B(b, r_{\frac{b}{2}})$. Then U and V are open sets containing A and B respectively.

We claim that $U \cap V = \emptyset$. Assume the contrary that there exists $x \in U \cap V$. Then there exist $a \in A$ and $b \in B$ such that $x \in B(a, r_{\frac{a}{2}})$ and $x \in B(b, r_{\frac{b}{2}})$. Without loss of generality, assume that $r_{\frac{a}{2}} \leq r_{\frac{b}{2}}$. Then

$$d(a, b) \leq d(a, x) + d(x, b) < r_{\frac{a}{2}} + r_{\frac{b}{2}} \leq r_a \tag{18.1}$$

This implies that $b \in B(a, r_a)$, which contradicts the choice of r_a. Thus, $U \cap V = \emptyset$. Hence, X is normal.

Theorem 18.0.3. *Every compact Hausdorff space is normal.*

Proof

Let $x \in X$ and $B \subseteq X$ be closed such that $x \notin B$.

For each $y \in B$, since X is Hausdorff, there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Then $\{V_y | y \in B\}$ is an open covering of B. By the compactness of B, there exists a finite subcollection $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$ that covers B. Let $V = \bigcup V_{y_i}$ and $U = \bigcap U_{y_i}$. Then U and V are disjoint open sets containing x and B respectively. So X is regular.

Let $A, B \subseteq X$ be closed and $A \cap B = \emptyset$. For each $a \in A$, since X is regular, there exist disjoint open sets U_a and V_a such that $a \in U_a$ and $B \subseteq V_a$. Then $\{U_a | a \in A\}$ is an open covering of A. By the compactness of A, there exists a finite subcollection $\{U_{a_1}, U_{a_2}, \ldots, U_{a_m}\}$ that covers A. Let $U = \bigcup U_{a_i}$ and $V = \bigcap V_{a_i}$. Then U and V are disjoint open sets containing A and B respectively. So X is normal.

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