Chapter 3

Later

Definition 3.0.1. *$x \in M$ is an interior point of M if there exists $\epsilon > 0$ such that $U_\epsilon(x) \subseteq M$. (We denote $Int(M)$ as the set of all interior points of M.)*

Definition 3.0.2. *M is open if $M = Int(M)$.*

Theorem 3.0.1. *$U \subseteq \mathbb{R}$ is open $\iff$ U is a union of at most countably many disjoint open intervals.*

Example 3.0.1 (discrete metric)**.** *Let (X, d) be a metric space where*

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

for all $x, y \in X$. Then every subset of X is open(e.g. a set of a single point, its neighborhood has radius smaller than 1).

Remark In $\mathbb{R}^n$, any "reasonable" set of strict inequalities defines an open set. For example, $\{x \in \mathbb{R}^n | x_1 > 0\}$ or $\{(x,y) \in \mathbb{R}^2 | 2x > y, y^2 + y > 3x^2 - 2\}$ are open. Another example: $X = \mathbb{R}^{n^2} \backsimeq Mat(n)$. Then the set $\{A \subseteq Mat(n) | det(A) \neq 0\}$ is open where A is non-degenerate. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad det(A) \neq 0 \tag{3.1}$$

Metric on $Mat(n)$ is defined: $||A|| = \max(|a_{ij}|)$. Then we have

$$\begin{aligned} U_\epsilon(A) &= \{B \in Mat(n) | ||A - B|| < \epsilon\} \\ &= \{B \in Mat(n) | \max(|a_{ij} - b_{ij}|) < \epsilon\} \\ &= \{\begin{pmatrix} a_{11}+\epsilon_{11} & a_{12}+\epsilon_{12} & \cdots & a_{1n}+\epsilon_{1n} \\ a_{21}+\epsilon_{21} & a_{22}+\epsilon_{22} & \cdots & a_{2n}+\epsilon_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+\epsilon_{n1} & a_{n2}+\epsilon_{n2} & \cdots & a_{nn}+\epsilon_{nn} \end{pmatrix} | \epsilon_{ij} < \epsilon\} \end{aligned} \tag{3.2}$$

Then we have

$$det(B) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} (a_{i\sigma(i)} + \epsilon_{i\sigma(i)}) \tag{3.3}$$

Let ϵ be small enough such that $\epsilon < ||A||$. Then We can estimate one summand of $det(B)$ as follows:

$$|(a_{11} + \epsilon_{11})(a_{12} + \epsilon_{12}) \cdots (a_{1n} + \epsilon_{1n}) - a_{11}a_{12} \cdots a_{1n}| < ||A||^{n-1} \cdot (2^n - 1)\epsilon \tag{3.4}$$

Sum up all estimates, we have

$$|det(A) - det(B)| < n!||A||^{n-1} \cdot (2^n - 1)\epsilon \tag{3.5}$$

Taking ϵ small enough such that

$$n!||A||^{n-1} \cdot (2^n - 1)\epsilon < |det(A)| \tag{3.6}$$

ensures that $det(B) > 0$. Thus, $U_\epsilon(A) \subseteq \{A \in Mat(n) | det(A) \neq 0\}$. So $\{A \in Mat(n) | det(A) \neq 0\}$ is open.

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Definition 3.0.3. *A point $x \in X$ (not neccessarily in M) is a limit point of M if for every $\epsilon > 0$, the punctured neighborhood $U_\epsilon^*(x)$ contains a point of M ($U_\epsilon^*(x) \cap M \neq \emptyset$) or (the neighborhood $U_\epsilon(x)$ contains infinitely many points of M).*

Definition 3.0.4. *$x \in M$ is isolated if $\exists \epsilon > 0$ such that $U_\epsilon(x) \cap M = \{x\}$.*

Definition 3.0.5. *The closure of M, denoted by $\overline{M}$, is the union of the set of all limit points of M and M itself.*

Definition 3.0.6. *A set M is closed if M contains all its limit points.*

Property 3.0.1. *$M = \overline{M} \iff M$ is closed*

Exercise Show that for any $M \subseteq X$, we have $\overline{M} = \overline{\overline{M}}$.

Property 3.0.2. *If M is open in X, then $CM = X \setminus M$ is closed in X. Conversely, if M is closed in X, then CM is open in X.*

Exercise Prove the above property.

Property 3.0.3. *$M \subseteq X$ is closed $\iff \forall y \in X \setminus M, \exists \epsilon > 0$ such that $U_\epsilon(y) \cap M = \emptyset$. (any point can be seperated from M).*

Proof Suppose $\exists y \in X \setminus M$ that cannot be seperated from M. Then $\forall \epsilon > 0, U_\epsilon(y) \cap M \neq \emptyset$. But $y \notin M$, so $U_\epsilon^*(y) \cap M \neq \emptyset$. So y is a limit point of M which implies that $y \in \overline{M}$. Since M is closed, we have a contradiction.

Suppose every $y \in X \setminus M$ can be seperated from M. Let x be a limit point of M. If $x \notin M$, then by assumption, there exists $\epsilon > 0$ such that $U_\epsilon(x) \cap M = \emptyset$. This contradicts the definition of limit point.

Example 3.0.2. *We will see that $\{det(A) = 0\}$ is closed in $Mat(n)$. So every matrix with $det(A) \neq 0$ can be seperated from the set $\{det(A) = 0\}$.*

Definition 3.0.7. *The function $f : X_1 \to X_2$ between two metric spaces $(X_1, d_1), (X_2, d_2)$ is continuous if for every x, $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x' \in X_1$ with $d_1(x, x') < \delta$, we have $d_2(f(x), f(x')) < \epsilon$.*

Definition 3.0.8 (Alternative definition of continuity)**.** *The function $f : X_1 \to X_2$ between two metric spaces $(X_1, d_1), (X_2, d_2)$ is continuous if for every open set $U \subseteq X_2$, the preimage $f^{-1}(U) = \{x \in X_1 | f(x) \in U\}$ is open in X_1.*

Theorem 3.0.2. *The two definitions of continuity are equivalent.*

Proof Let $f : X_1 \to X_2$ be continuous in the first definition. Let $U \subseteq X_2$ be open. Let $x \in f^{-1}(U)$. Then $f(x) \in U$. Since U is open, there exists $\epsilon > 0$ such that $U_\epsilon(f(x)) \subseteq U$. By continuity of f, there exists $\delta > 0$ such that $\forall x' \in X_1$ with $d_1(x, x') < \delta$, we have $d_2(f(x), f(x')) < \epsilon$, i.e. $f(x') \in U_\epsilon(f(x)) \subseteq U$. Thus, $x' \in f^{-1}(U)$. So we have $U_\delta(x) \subseteq f^{-1}(U)$. Thus, $f^{-1}(U)$ is open in X_1.

Let $f : X_1 \to X_2$ be continuous in the second definition. Let $x \in X_1$. Let $\epsilon > 0$. Consider the open set $U_\epsilon(f(x)) \subseteq X_2$. By continuity of f, the preimage $f^{-1}(U_\epsilon(f(x)))$ is open in X_1. Since $x \in f^{-1}(U_\epsilon(f(x)))$, there exists $\delta > 0$ such that $U_\delta(x) \subseteq f^{-1}(U_\epsilon(f(x)))$. Thus, for every $x' \in X_1$ with $d_1(x, x') < \delta$, we have $f(x') \in U_\epsilon(f(x))$, i.e. $d_2(f(x), f(x')) < \epsilon$. Thus, f is continuous in the first definition.

Question Suppose $f : X \to Y$ is continuous and bijective, is the inverse $f^{-1} : Y \to X$ also continuous?
Answer Not necessarily. For example, let $X = \{0\} \cup (1, 2]$ and $Y = [0, 1]$. Define $f : X \to Y$ by

$$f(x) = \begin{cases} 0, & x = 0 \\ x - 1, & x \in (1, 2] \end{cases}$$

Then f is continuous and bijective. However, $\{0\} \subseteq X$ is open but its preimage $f^{-1}(\{0\}) = \{0\}$ is not open in X.

There's another example: Let $X = [0, 1]$ with discrete metric(i.e., $d(x, y) = 1, \forall x, y \in X$ with $x \neq y$) and $Y = [0, 1]$ with standard metric. Define $f : X \to Y$ by $f(x) = x$. Then f is continuous and bijective. But its inverse is not continuous.