Chapter 11

Lecture11

Exercise Let $U_\epsilon(x) = \{y \in X | d(x, y) < \epsilon\}$. Prove that the collection of all such $U_\epsilon(x)$ forms a basis for a topology on X.

Definition 11.0.1. *A topological space $(X, \mathcal{T})$ is metrizable if there exists a metric d on X such that the topology induced by d is equal to $\mathcal{T}$.*

Lemma 11.0.1. *Let d and d' be two metrics on X defining the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$ and each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$U_\delta^{d'}(x) \subseteq U_\epsilon^d(x) \tag{11.1}$$

Property 11.0.1. *The metrics d_1, d_2, d_∞ define the same topology on $\mathbb{R}^n$. This is the product topology(or box as n is finite) on $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times).*

Proof Some graphs are shown below to illustrate the idea. See the pictures.

In the following, we will define a metric that is used to bound another metric. Let d be a metric on X. Define

$$\bar{d}(x, y) = \min\{d(x, y), 1\} \tag{11.2}$$

Theorem 11.0.1. *$\bar{d}$ is a metric on X defining the same topology as d.*

First, we need to check that $\bar{d}$ is a metric. The only non-trivial part is the triangle inequality. For any $x, y, z \in X$, we want to show that

$$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z) \tag{11.3}$$

If $\bar{d}(x, y)$ or $\bar{d}(y, z)$ equals 1, then the right hand side is at least 1, and the inequality holds. If both $\bar{d}(x, y)$ and $\bar{d}(y, z)$ are less than 1, then $\bar{d}(x, y) = d(x, y)$ and $\bar{d}(y, z) = d(y, z)$. By the triangle inequality of d, we have

$$\bar{d}(x, z) = \min\{d(x, z), 1\} \leq d(x, z) \leq d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z) \tag{11.4}$$

We want to define a metric on $\mathbb{R}^\omega$.

Definition 11.0.2. *For $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots) \in \mathbb{R}^\omega$, define*

$$\bar{d}(x, y) = \min(\sup(|x_i - y_i|), 1) \tag{11.5}$$

which is called the uniform metric on $\mathbb{R}^\omega$.

Also we can generalize the metric for $\mathbb{R}^J$ for any J. (sup(min) or min(sup) does not matter)

Theorem 11.0.2. *Uniform topology on $\mathbb{R}^J$ is:*

- *(strictly) finer than the direct product topology;*
- *(strictly) coarser than the box topology.*

provided that J is infinite.

Proof

First, we show that the uniform topology is finer than the product topology. Let $x \in \mathbb{R}^J$ and let U be a basic open set in the product topology containing x. Then

$$U = \prod_{\alpha \in J} U_\alpha \tag{11.6}$$

where U_α is an open set in $\mathbb{R}$ and $U_\alpha = \mathbb{R}$ for all but finitely many α. Let K be the finite set of indices α such that $U_\alpha \neq \mathbb{R}$. For each $\alpha \in K$, there exists $1 > \epsilon_\alpha > 0$ such that

$$U_{\epsilon_\alpha}(x_\alpha) \subseteq U_\alpha \tag{11.7}$$

Let $\epsilon = \min_{\alpha \in K} \epsilon_\alpha$. Then the open ball $U_\epsilon^{\bar{d}}(x)$ in the uniform topology is contained in U. Thus the uniform topology is finer than the product topology.

Next, we show that the uniform topology is coarser than the box topology. Let $x \in \mathbb{R}^J$ and let U be a basic open set in the box topology containing x. Then

$$U = \prod_{\alpha \in J} U_\alpha \tag{11.8}$$

where U_α is an open set in $\mathbb{R}$. For each $\alpha \in J$, there exists $1 > \epsilon_\alpha > 0$ such that

$$U_{\epsilon_\alpha}(x_\alpha) \subseteq U_\alpha \tag{11.9}$$

Let $\epsilon = \inf_{\alpha \in J} \epsilon_\alpha$. Then the open ball $U_\epsilon^{\bar{d}}(x)$ in the uniform topology is contained in U. Thus the uniform topology is coarser than the box topology.

For the strictness of the two sides, we can use some examples.

Recall that for two metric spaces (X, d) and (Y, d'), a function $f : X \to Y$ is continuous if and only if for each $x \in X$ and each $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \epsilon \tag{11.10}$$

Theorem 11.0.3 (The sequence lemma)**.** *Let X be a topological space, $A \subseteq X$. If there is a sequence of points $x_n \in A$ with $\lim_{n\to\infty} x_n = x$, then $x \in \bar{A}$. The converse holds if X is metrizable.*

Proof If $x_n \in A$ and $\lim_{n\to\infty} x_n = x$, then for each open set U containing x, we have $U \cap A$ contains all but finitely many x_n, so $U \cap A \neq \emptyset$. Thus $x \in \bar{A}$.

For the converse, suppose X is metrizable with metric d. If $x \in \bar{A}$, then for each $n \in \mathbb{N}$, the open ball $U_{1/n}(x)$ intersects A. Thus we can choose $x_n \in U_{1/n}(x) \cap A$. Then $\lim_{n\to\infty} x_n = x (\forall \epsilon > 0, \exists N > \frac{1}{\epsilon}$, such that $d(x_n, x) < \epsilon$ for all $n > N)$.