

Chapter 21

Later

Definition 21.0.1. A compactification of a Hausdorff space X is a Hausdorff space $Y \supseteq X$ such that $Y = \overline{X}$. We say that compactifications Y_1 and Y_2 are equivalent if there exists a homeomorphism $h : Y_1 \rightarrow Y_2$ such that $h|_X = id_X$.

Example 21.0.1. $X = (0, 1)$, $Y = S^1 \subseteq \mathbb{R}^2$, $X \ni x \mapsto (\cos 2\pi x, \sin 2\pi x) \in Y$ is a compactification of X .

Example 21.0.2. $X = (0, 1)$, $Y = [0, 1] \subseteq \mathbb{R}$.

Property 21.0.1. Let $X \subseteq Y$ and Y be a compact Hausdorff space. Then X is completely regular.

Proof

Since Y is compact Hausdorff, Y is normal. Then Y is completely regular. Thus X is completely regular as well. ■

Claim If X is completely regular, it has a compactification.

Theorem 21.0.1. Let X be T_1 space. Suppose X has an indexed family of continuous functions $\{f_\alpha : X \rightarrow \mathbb{R}\}_{\alpha \in J}$ such that $\forall x_0 \in X$, and $U \ni x_0$ open, there exists a function f_α such that $f_\alpha(x_0) = 1$ and $f_\alpha(X \setminus U) \equiv 0$. Then $F : X \rightarrow \mathbb{R}^J$ defined by $F(x) = (f_\alpha(x))_{\alpha \in J}$ is an embedding $X \hookrightarrow \mathbb{R}^J$. If, additionally, $f_\alpha : X \rightarrow [0, 1]$ for all $\alpha \in J$, then F is an embedding $X \hookrightarrow [0, 1]^J$.

Proof To be done.

Theorem 21.0.2. X is completely regular if and only if it is homeomorphic to a subspace of a cube $[0, 1]^J$ for some index set J .

Proof Not shown in the lecture.

Lemma 21.0.1. Let X be a Hausdorff space. Let $h : X \hookrightarrow Z$ be an embedding with Z compact Hausdorff. Then there exists a compactification Y of X such that there is an embedding $H : Y \hookrightarrow Z$ with $H|_X = h$. Such a compactification is uniquely determined.

Example 21.0.3. Let $X = (0, 1)$. Consider the embedding $h : X \hookrightarrow \mathbb{R}^2$ defined by $h(x) = (x, \sin \frac{1}{x})$. Let $A = \{0\} \times [-1, 1] \cup \{(1, \sin(1))\}$. Then $Y = h(X) \cup A$ is a compactification of $h(X)$.

There's something.

Theorem 21.0.3 (Stone–Čech compactification). Let X be a completely regular space. Then there exists a compactification Y of X such that every bounded continuous function $f : X \rightarrow \mathbb{R}$ can be extended uniquely to a continuous function $\bar{f} : Y \rightarrow \mathbb{R}$. (Y is called the Stone–Čech compactification of X)