

# Chapter 14

## Lecture 14

**Theorem 14.0.1.**  $\mathbb{R}^\omega$  is metrizable in the product topology.

**Proof** Define

$$\bar{d}(x_i, y_i) = \min(|x_i - y_i|, 1) \quad (14.1)$$

Let  $x, y \in \mathbb{R}^\omega$ . Then we define

$$D(x, y) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i} \mid i \in \mathbb{Z}_{>0}\right\} \quad (14.2)$$

Then we can show that  $D$  is a metric on  $\mathbb{R}^\omega$  because we have

$$\bar{d}(x_i, z_i) \leq \bar{d}(x_i, y_i) + \bar{d}(y_i, z_i) \implies D(x, z) \leq D(x, y) + D(y, z) \quad (14.3)$$

Why does it define the product topology?

( $\Rightarrow$ ) Let  $U$  be open in the metric topology. Then there exists  $\epsilon > 0$  such that  $B_D(x, \epsilon) \subseteq U$ . Let  $N$  be such that  $\frac{1}{N} < \epsilon$ . Then let

$$V = \prod_{i=1}^N (x_i - \epsilon, x_i + \epsilon) \times \prod_{i=N+1}^{\infty} \mathbb{R} \quad (14.4)$$

Then let  $y \in V$ . We have

1. If  $i \leq N$ , then  $\bar{d}(x_i, y_i) < \epsilon$ .
2. If  $i > N$ , then  $\bar{d}(x_i, y_i) \leq 1$ .

Thus, we have  $D(x, y) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} \leq \max\{\epsilon, \frac{1}{N}\} = \epsilon$ . So  $y \in B_D(x, \epsilon) \subseteq U$ . Hence,  $V \subseteq U$ . So  $U$  is open in the product topology.

( $\Leftarrow$ ) Let  $U$  be open in the product topology. Then

$$U = \prod U_i \quad (14.5)$$

where  $U_i$  is open if  $i \in \{\alpha_1, \dots, \alpha_n\}$  and  $U_i = \mathbb{R}$  otherwise. We want  $V$  open in the metric topology such that  $V \subseteq U$ . Let

$$x \in U \quad (14.6)$$

Then

$$x_i \in U_i \quad (14.7)$$

for  $i \in \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ . Then there exists  $\epsilon_i > 0$  such that

$$(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i \quad (14.8)$$

Let

$$\epsilon = \min\left(\left\{\frac{\epsilon_i}{i} \mid i \in \{\alpha_1, \alpha_2, \dots, \alpha_n\}\right\} \cup \{1\}\right) \quad (14.9)$$

Then we claim that

$$U_D(x, \epsilon) \subseteq U \quad (14.10)$$

Let  $y \in U_D(x, \epsilon)$ . Then we have

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \epsilon \quad (14.11)$$

If  $i \in \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ , then

$$\frac{\bar{d}(x_i, y_i)}{i} < \epsilon \leq \frac{\epsilon_i}{i} \quad (14.12)$$

which implies that

$$y_i \in (x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i \quad (14.13)$$

So  $y \in U$ . Hence,  $U_D(x, \epsilon) \subseteq U$ . So  $U$  is open in the metric topology.

**Definition 14.0.1.**  $(X, \leq)$  has a least upper bound property or supremum property if every non-empty subset of  $X$  that is bounded above has a least upper bound in  $X$ . (For example,  $\mathbb{R}$  has the least upper bound property)

Let  $X$  be a set with order topology and supremum property.

**Theorem 14.0.2.**  $\forall a \leq b \in X$ , we have  $[a, b]$  is compact.

**Proof** Let  $\mathcal{A}$  be an open covering of  $[a, b]$ . We need that there exists a finite subcovering of  $\mathcal{A}$  covering  $[a, b]$ .

**Step 1** Let  $x \in [a, b]$  with  $x \neq b$ , there exists  $y > x$  such that at most two elements of  $\mathcal{A}$  can cover  $[x, y]$ .

1. If  $x$  has an immediate successor  $s(x)$ , then  $[x, s(x)] = \{x, s(x)\}$  is covered by two elements from  $\mathcal{A}$ .
2. If not. Take  $x \in A$  with  $A \in \mathcal{A}$  such that  $\exists c \in X, x < c \leq b, [x, c] \subseteq A$ . Take any  $y \in (x, c)$ . Then  $[x, y] \subseteq A$  is covered by one element from  $\mathcal{A}$ .

**Step 2** Let  $C$  be the set of points  $y$  with  $a \leq y \leq b$  such that  $[a, y]$  has a finite subcovering from  $\mathcal{A}$ . Since  $a \in C$ ,  $C$  is non-empty. Let  $c = \sup C$ .

**Step 3** Show that  $c \in C$ . Let  $c \in A, A \in \mathcal{A}, \exists d \in A$  such that  $d < c, [d, c] \subseteq A$ . Since  $[a, d]$  can be covered by finite elements(say  $N$ ) from  $\mathcal{A}$ ,  $[a, c]$  can be covered by  $N + 1$  elements from  $\mathcal{A}$ . So  $c \in C$ .

**Step 4** Show that  $c = b$ . If not, by Step 1,  $\exists e > c$  such that  $[c, e]$  can be covered by two elements from  $\mathcal{A}$ . Since  $[a, c]$  can be covered by finite elements(say  $M$ ) from  $\mathcal{A}$ ,  $[a, e]$  can be covered by at most  $M + 2$  elements from  $\mathcal{A}$ . So  $e \in C$  which contradicts the assumption that  $c = \sup C$ . Hence,  $c = b$ .

**Corollary 14.0.1.**  $[a, b] \subseteq \mathbb{R}$  is compact.

**Corollary 14.0.2.**  $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$  is compact.

**Theorem 14.0.3.**  $A \subseteq \mathbb{R}^n$  is compact if and only if  $A$  is closed and bounded.

**Proof**

( $\Rightarrow$ ) Take a covering by

$$\bigcup U_N(0) \quad (14.14)$$

which is the union of open balls centered at 0 with radius  $N$  for  $N \in \mathbb{Z}_{>0}$ . Since  $A$  is compact, there exists a finite subcovering. Thus,  $A \subseteq U_{N_0}(0)$  for  $N_0 = \max\{N_1, N_2, \dots, N_k\}$ . So  $A$  is bounded.

( $\Leftarrow$ ) Since  $A$  is bounded, there exists  $n$  such that

$$A \subseteq [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] \quad (14.15)$$

We know that the closed subset of a compact set is compact. So  $A$  is compact. ■

**Theorem 14.0.4** (Extreme value theorem). Let  $f : X \rightarrow Y$  be continuous where  $X$  is compact and  $Y$  is an ordered set with the order topology. Then  $\exists a, b \in X$  such that  $\forall x \in X, f(a) \leq f(x) \leq f(b)$ .

**Proof**

We know that  $A = f(X)$  is compact. We want to show that  $A$  has a largest element.

We assume the contrary that  $A$  has no largest element. Then  $\forall a \in A, \exists b \in A$  such that  $b > a$ . So

$$A = \bigcup_{a \in A} (a, +\infty) \quad (14.16)$$

which is an open covering of  $A$ . Since  $A$  is compact, there exists a finite subcovering. Let the largest element among the finite elements be  $a_0$ . Then  $a_0$  is not covered. This is a contradiction. So  $A$  has a largest element. Similarly,  $A$  has a smallest element. ■

**Definition 14.0.2.** Let  $(X, d)$  be a metric space,  $A \subseteq X, x \in X$ . Let  $d(x, A) = \inf\{d(x, a) | a \in A\}$ .

**Property 14.0.1.**  $d(x, A)$  is a continuous function for  $A$  fixed.

**Proof**

We know

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a) \quad (14.17)$$

Then

$$d(x, A) - d(x, y) \leq \inf\{d(y, a) | a \in A\} = d(y, A) \quad (14.18)$$

So

$$d(x, A) - d(y, A) \leq d(x, y) \quad (14.19)$$

■