Chapter 15

Later

Definition 15.0.1. *Two norms $||\cdot||_a$ and $||\cdot||_b$ are equivalent if there exist positive constants C_1 and C_2 such that for all vectors x,*

$$C_1||x||_a \leq ||x||_b \leq C_2||x||_a.$$

Property 15.0.1. *Equivalence of norms is an equivalence relation.*

Theorem 15.0.1. *All norms on $\mathbb{R}^n$ are equivalent.*

Proof

Given $||\cdot||_a$ and $||\cdot||_b$ are two norms on $\mathbb{R}^n$. We define a function $f : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ by

$$f(x) = \frac{||x||_b}{||x||_a} > 0. \tag{15.1}$$

which is continuous on $\mathbb{R}^n\backslash\{0\}$.

Since $f(x) = f(\lambda x)$ for any $\lambda > 0$, we say that $f(x)$ is conpletely determined by its values on the unit sphere $S = \{x \in \mathbb{R}^n : ||x||_a = 1\}$. Note that S is compact, so $f(x)$ attains its minimum and maximum on S, say m and M. Thus, for any $x \in \mathbb{R}^n\backslash\{0\}$, we have

$$0 < m \leq f(x) \leq M < \infty, \tag{15.2}$$

which implies

$$m||x||_a \leq ||x||_b \leq M||x||_a. \tag{15.3}$$

So the two norms are equivalent. ■

Remark In ∞-dimensional space, the sphere is not compact. However, it is closed and bounded.

Exercise Give an example of norms on l_1 (convergent series) $l_1 = \{(x_1, x_2, \cdots)| \sum |x_i| < \infty\}$.

Now let's come back to uniform continuity. Goal: Prove that any continuous map $f : (X, d_X) \to (Y, d_Y)$ with X compact is uniformly continuous.

Definition 15.0.2. *Let X be a metric space. $A \subseteq X$. Then diam$(A) = \sup\{d(x, y) : x, y \in A\}$ is called the diameter of A.*

Lemma 15.0.1 (the Lebesgue Number Lemma)**.** *Let (X, d) be a compact metric space, and let $\mathcal{A}$ be an open cover of X. Then there exists a positive number $\delta > 0$ (called a Lebesgue number for the cover $\mathcal{A}$) such that for every subset $Y \subseteq X$ with diameter less than δ (i.e., for all $x, y \in Y$, $d(x, y) < \delta$), there exists an open set $A \in \mathcal{A}$ that contains Y.*

Proof of the Lemma

If $X \in \mathcal{A}$, there's nothing to prove.

Otherwise take a finite subcovering $\{A_1, A_2, \cdots, A_n\}$ of $\mathcal{A}$. For each i, let $C_i = X\backslash A_i$ which is closed. We know that a closed subset of a compact set is compact, so each C_i is compact. Define

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, C_i) \tag{15.4}$$

be the average distance from x to the closed sets C_i.

We show that $f(x) > 0$ for all $x \in X$.

For all $x \in X$, we choose A_i such that $x \in A_i$. Choose $\epsilon > 0$ such that $U_\epsilon(x) \subseteq A_i$. Then $d(x, C_i) \geq \epsilon > 0$. So $f(x) \geq \frac{1}{n}d(x, C_i) \geq \frac{\epsilon}{n}$. Let $\delta = \min_{x\in X} f(x) > 0$ (because f is continuous on compact set X). We show that δ is a Lebesgue number.

Take $B \subseteq X$ with $diam(B) < \delta$. Then we can take $x_0 \in B$ with $B \subseteq U_\delta(x_0)$. Then we can take C_m such that

$$\delta \leq f(x_0) \leq d(x_0, C_m). \tag{15.5}$$

for some m. So we have

$$U_\delta(x_0) \subseteq A_m = X \backslash C_m. \tag{15.6}$$

So $B \subseteq A_m$. ■

Example 15.0.1. *Let $X \subseteq \mathbb{R}$ be convered by (a_α, b_α). Take a finite subcovering (a_i, b_i), $i = 1, 2, \cdots, n$. Then $\delta = \min\{b_i - a_i : i = 1, 2, \cdots, n\}$ is a Lebesgue number.*

Definition 15.0.3. *A function between two metric spaces $f : (X, d_X) \to (Y, d_Y)$ is said to be uniformly continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x_1, x_2 \in X$, if $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \epsilon$.*

Theorem 15.0.2. *Let $f : X \to Y$ be a continuous function between two metric spaces. If X is compact, then f is uniformly continuous.*

Proof

Given $\epsilon > 0$, take the open covering of Y by balls $B(y, \frac{\epsilon}{2})$ of radius $\frac{\epsilon}{2}$. Let $\mathcal{A}$ be the open covering of X by the inverse images of these balls under f. Choose δ to be a Lebesgue number for the covering $\mathcal{A}$. Then if x_1 and x_2 are any two points in X with $d_X(x_1, x_2) < \delta$, the set $\{x_1, x_2\}$ has diameter less than δ, so there exists an open set $U = f^{-1}(B(y, \frac{\epsilon}{2}))$ in $\mathcal{A}$ that contains both x_1 and x_2. So $f(x_1)$ and $f(x_2)$ both lie in $B(y, \frac{\epsilon}{2})$, which implies that

$$d_Y(f(x_1), f(x_2)) < \epsilon. \tag{15.7}$$

■

Definition 15.0.4. *If X is a space, a point x of X is said to be an isolated point of X if the one-point set $\{x\}$ is open in X.*

Theorem 15.0.3. *Let X be a non-empty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof

Step 1 We show first that given any non-empty open set U of X and any point x of X, there exists a non-empty open set V contained in U such that $x \notin \bar{V}$.

Choose a point y of U different from x; this is possible if x is in U because x is not an isolated point of X and it is possible if x is not in U simply because U is non-empty. Since X is Hausdorff, there exist disjoint open sets W_x and W_y containing x and y, respectively. Then the open set $V = U \cap W_y$ is non-empty, is contained in U, and its closure $\bar{V}$ is contained in the complement of W_x, so $x \notin \bar{V}$.

Step 2 We show that given $f : \mathbb{Z}_+ \to X$, the function f is not surjective. It follows that X is uncountable.

Let $x_n = f(n)$. Apply Step 1 to the non-empty open set X to choose a non-empty open set $V_1 \subseteq X$ such that $x_1 \notin \bar{V_1}$. In general, given V_{n-1} open and non-empty, choose V_n to be a non-empty open set such that $V_n \subseteq V_{n-1}$ and $x_n \notin \bar{V_n}$. Then we have a sequence of non-empty closed sets $\bar{V_n}$ with

$$\bar{V_1} \supseteq \bar{V_2} \supseteq \bar{V_3} \supseteq \cdots \tag{15.8}$$

Since X is compact, any collection of closed subsets of X with the finite intersection property has non-empty intersection. Thus, there exists a point y in the intersection of all the $\bar{V_n}$. By construction, $y \neq x_n$ for all n, so f is not surjective.

■

Corollary 15.0.1. *$[a, b] \subseteq \mathbb{R}$ is uncountable.*