Chapter 21

Later

21.1 The tychonoff theorem

Theorem 21.1.1. *If X, Y are compact sets, then $X \times Y$ is also compact.*

Theorem 21.1.2. *If X_α's with $\alpha \in J$ are compact sets, then $\prod_{\alpha \in J} X_\alpha$ is also compact.*

21.2 Closed set formulation of compactness

Let X be a topological space. Let $\mathcal{C}$ be a family of closed sets in X.

Definition 21.2.1. *We say $\mathcal{C}$ has the finite intersection property if for any finite closed sets $C_1, C_2, \ldots, C_n \in \mathcal{C}$, we have*

$$C_1 \cap C_2 \cap \ldots \cap C_n \neq \varnothing.$$

Theorem 21.2.1. *A topological space X is compact if and only if for any family $\mathcal{C}$ of closed sets in X with the finite intersection property, we have*

$$\bigcap_{C \in \mathcal{C}} C \neq \varnothing.$$

Exercise Prove the above theorem(It has been proved long time ago).

Example 21.2.1. *Let $X = (0,1), C_n = (0, \frac{1}{n}]$ for $n \in \mathbb{N}$. Then $\{C_n\}_{n=1}^{\infty}$ has the finite intersection property, but*

$$\bigcap_{n=1}^{\infty} C_n = \varnothing.$$

Recall Zorn's lemma

Theorem 21.2.2 (Zorn's lemma)**.** *Let $(A, \leq)$ be a partially ordered set. If every chain in A has an upper bound in A, then A contains at least one maximal element.*

Lemma 21.2.1. *Let X be a set, $\mathcal{A}$ be a family of subsets of X with the finite intersection property. Then there exists a family $\mathcal{D}$ of subsets of X such that*

1. *$\mathcal{A} \subseteq \mathcal{D}$;*
2. *$\mathcal{D}$ has the finite intersection property;*
3. *$\mathcal{D}$ is maximal with respect to (ii), i.e., if $\mathcal{E}$ is a family of subsets of X such that $\mathcal{D} \subsetneq \mathcal{E}$, then $\mathcal{E}$ does not have the finite intersection property.*

Proof

Let $\mathcal{A}$ be a family of subsets of X with the finite intersection property. Let

$$\mathbb{A} = \{\mathcal{B} \supseteq \mathcal{A} | \mathcal{B} \text{ has finite intersection property } \} \tag{21.1}$$

Let $\subsetneq$ be the partial order on $\mathbb{A}$. Let $\mathbb{B} \subseteq \mathbb{A}$ be a chain. We need to show that $\mathbb{B}$ has an upper bound in $\mathbb{A}$. Let

$$\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}. \tag{21.2}$$

To be done.

Lemma 21.2.2. *Let X be a set, $\mathcal{D}$ be a maximal family of subsets of X with the finite intersection property(obtain by the previous lemma). Then we have*

1. *$\mathcal{D}$ is closed under finite intersections, i.e., for any finite subsets $D_1, D_2, \ldots, D_n \in \mathcal{D}$, we have*

$$D_1 \cap D_2 \cap \ldots \cap D_n \in \mathcal{D};$$

2. *If $A \subseteq X$ satisfies $A \cap D \neq \varnothing$ for all $D \in \mathcal{D}$, then $A \in \mathcal{D}$.*

Proof

Take $C_1, C_2, \ldots, C_n \in \mathcal{D}$. Since $\mathcal{D}$ has the finite intersection property, we have

$$C_1 \cap C_2 \cap \ldots \cap C_n \neq \varnothing.$$

Suppose $C_1 \cap C_2 \cap \ldots \cap C_n \notin \mathcal{D}$. Let $\mathcal{E} = \mathcal{D} \cup \{C_1 \cap C_2 \cap \ldots \cap C_n\}$.

Take $D_1', D_2', \ldots, D_m' \in \mathcal{D}$. Then

$$(C_1 \cap C_2 \cap \ldots \cap C_n) \cap D_1' \cap D_2' \cap \ldots \cap D_m' \neq \varnothing.$$

Thus $\mathcal{E}$ has the finite intersection property, contradicting the maximality of $\mathcal{D}$. Thus $C_1 \cap C_2 \cap \ldots \cap C_n \in \mathcal{D}$.