Chapter 7

Lecture7

7.1 Subspace Topology

Definition 7.1.1. *Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. The* ***subspace topology*** *on Y is defined as*

$$\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}.$$

Property 7.1.1. *$(Y, \mathcal{T}_Y)$ is a topological space.*

Lemma 7.1.1. *If $\mathcal{B}$ is a basis for the topology $\mathcal{T}$ on X, then the collection*

$$\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$$

is a basis for the subspace topology $\mathcal{T}_Y$ on Y.

Proof
Let $U \in \mathcal{T}$ and $y \in U \cap Y$. Then there exists $B \in \mathcal{B}$ such that $y \in B \subseteq U$. Then $y \in B \cap Y \subseteq U \cap Y$.
Remark The set which is open in the subspace topology may not be open in the original topology.

Property 7.1.2. *If Y is open in $(X, \mathcal{T})$ and $U \subseteq Y$ is open in $(Y, \mathcal{T}_Y)$, then U is open in $(X, \mathcal{T})$.*

Proof Since $U \subseteq Y$ is open in $(Y, \mathcal{T}_Y)$, there exists $V \in \mathcal{T}$ such that $U = V \cap Y$. Since Y is open in $(X, \mathcal{T})$, we have $U = V \cap Y$ is open in $(X, \mathcal{T})$.

Theorem 7.1.1. *Let $A \subseteq X$ and $B \subseteq Y$ such that $A \times B \subseteq X \times Y$. Then the product of subspace topologies $\mathcal{T}_A \times \mathcal{T}_B$ is equal to the subspace topology $\mathcal{T}_{A\times B}$ on $A \times B$.*

Proof
Let $U \subseteq X$ and $V \subseteq Y$ be open sets in $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ respectively. Then the products of the form $U \times V$ form a basis for the product topology on $X \times Y$. And we have

$$(U \times V) \cap (A \times B) \tag{7.1}$$

form a basis for the subspace topology on $A \times B$. Note that

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B) \tag{7.2}$$

So the basis for the product of subspace topologies is equal to the basis for the subspace topology on $A \times B$. ■

Let $(X, <)$ be an ordered set. Let $a_0 = min(X)$ and $b_0 = max(X)$ if they exists. Then

$$\mathcal{B} = \{(a, b) : a_0 < a < b < b_0\} \cup \{[a_0, b) : b < b_0\} \cup \{(a, b_0] : a > a_0\} \tag{7.3}$$

is a basis for the order topology on X. Notice that $[a_0, b)$ refers to the set $\{x \in X : x < b\}$ when a_0 does not exist.
Remark Let $Y \subseteq X$ be a subset inheriting the order from X. Then it may happen that the order topology on Y is different from the subspace topology on Y.

Example 7.1.1. *Let $X = \mathbb{R}$ with the usual order and $Y = [0, 1) \cup \{2\}$ be a subset of X. Then $\{2\}$ is open in the subspace topology on Y since $\{2\} = (1.5, 2.5) \cap Y$. However, $\{2\}$ is not open in the order topology on Y.*

Exercise Show that for the order topology $[0, 1) \cup \{2\}$ is connected.

Example 7.1.2. *Let $X = Y = \mathbb{R}$ with the usual order. Then the product of order topologies is the standard topology on $\mathbb{R}^2$. We take the lexicographic order as the order on $\mathbb{R}^2$.*

Remark Let $I \times I = [0,1] \times [0,1] \subseteq X \times Y$. Then the order topology on $[0,1] \times [0,1]$ is different from the subspace topology inherited from the order topology on $\mathbb{R}^2$. The set $\{1/2\} \times (1/2, 1]$ is not open in the order topology on $I \times I$ but it is open in the subspace topology inherited from $\mathbb{R}^2$.

Definition 7.1.2. *Y is convex if for any $a, b \in Y$, the interval $(a, b) \subseteq Y$.*

Theorem 7.1.2. *Let $Y \subseteq X$ be a convex subset of $(X, <)$. Then the the restriction of the order topology on X to Y is equal to the order topology on Y.*

Proof

Take $(a, +\infty), (-\infty, b) \subseteq X$ which form a basis for the order topology on X. Take $Y \subseteq X$ convex.

1. If $a \in Y$, then $(a, +\infty) \cap Y = \{y | y \in Y \text{ and } a < y\}$.
2. If $a \notin Y$, then there are only two cases because Y is convex:
   (a) If a is a lower bound for Y, then $(a, +\infty) \cap Y = Y$
   (b) If a is an upper bound for Y, then $(a, +\infty) \cap Y = \emptyset$

Then we obtain the basis for subspace topology on Y from the basis of order topology on X. And also this is the basis for the order topology on Y.

Definition 7.1.3. *$A \subseteq X$ is closed if and only if $X \setminus A$ is open.*

Property 7.1.3.
1. X and $\emptyset$ are closed.
2. The intersection of any collection of closed sets is closed.
3. The union of finitely many closed sets is closed.

Theorem 7.1.3. *Let X be a topological space. And $Y \subseteq X$ has a subspace topology. Then $A \subseteq Y$ is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

Proof

($\Rightarrow$) Since A is closed in Y, then $Y \setminus A$ is open in Y. So there exists an open set U in X such that $Y \setminus A = U \cap Y$. Let $C = X \setminus U$. Then C is closed in X and

$$A = Y \setminus (Y \setminus A) = Y \setminus (U \cap Y) = Y \cap (X \setminus U) = Y \cap C.$$

($\Leftarrow$) Let $A = C \cap Y$ where C is closed in X. Then $X \setminus C$ is open in X. So we have $(X \setminus C) \cap Y$ is open in Y. But we have

$$(X \setminus C) \cap Y = Y \setminus (C \cap Y) = Y \setminus A. \tag{7.4}$$

which is open in Y. So A is closed in Y.

Definition 7.1.4. *If $U \subseteq X$ is open and $x \in U$ then U is a neighborhood of x.*

Definition 7.1.5. *$Int(A) = \bigcup\{U \subseteq X | U \text{ is open and } U \subseteq A\}$ is the interior of A.*
$\bar{A} = \bigcap\{C \subseteq X | C \text{ is closed and } A \subseteq C\}$ is the closure of A.

Remark $Int(A) \subseteq A \subseteq \bar{A}$.
Remark If $A \subseteq Y \subseteq X$, then the closure of A in Y and the closure of A in X may be different. For example let $X = \mathbb{R}$ and $Y = [0, 1)$. Let $A = (0, 1) \subseteq Y$. Then the closure of A in Y is $[0, 1)$ while the closure of A in X is $[0, 1]$.

Theorem 7.1.4. *Let $Y \subseteq X$ with the subspace topology. Let $\bar{A}$ be the closure of A in X. Then the closure of A in Y is equal to $\bar{A} \cap Y$.*

Proof

Let B be the closure of A in Y. We prove the two inclusions.

Let B be the closure of A in Y. $\bar{A}$ is closed in X. So $\bar{A} \cap Y$ is closed in Y. Then $\bar{A} \cap Y \supseteq A$(Since we are talking about the closure in Y, we must have $A \subseteq Y$). So $B \subseteq \bar{A} \cap Y$($B$ is the smallest closed set containing A).

Since B is closed in Y, there exists a closed set C in X such that $B = C \cap Y$. Hence $C \supseteq A$ which is closed in X. So $C \supseteq \bar{A}$. Then $\bar{A} \cap Y \subseteq C \cap Y = B$.

Hence the two inclusions imply that $B = \bar{A} \cap Y$.

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