

Chapter 19

Later

Theorem 19.0.1 (Urysohn's Lemma). *Let X be a normal space. If A and B are disjoint closed subsets of X , then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

Definition 19.0.1. *Suppose X satisfies (T1). If a point and a closed set can be separated by a continuous function as in Urysohn's lemma, then X is called completely regular (CR). (Sometimes it is called $T_{3\frac{1}{2}}$ space.)*

Remark

Normal \Rightarrow CR \Rightarrow Regular. That's why we call it $T_{3\frac{1}{2}}$ space.

Property 19.0.1. *If X is completely regular, then it is regular.*

Proof

Take $x_0 \in X$ and a closed set A such that $x_0 \notin A$. Since X is CR, there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(x_0) = 0$ and $f(a) = 1$ for all $a \in A$. Let $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$. Then U and V are open sets because f is continuous and $[0, \frac{1}{2})$, $(\frac{1}{2}, 1]$ are open in $[0, 1]$. So $x_0 \in U$, $A \subseteq V$ and $U \cap V = \emptyset$. Thus X is regular. ■

Theorem 19.0.2. 1. *Subspace of CR space is CR.*

2. *If X_α is CR, then $\prod X_\alpha$ with product topology is CR.*

Proof

Let X be CR, and $Y \subseteq X$ with subspace topology. Take $y_0 \in Y$ and a closed set $A \subseteq Y$ such that $y_0 \notin A$. Let \bar{A} be the closure of A in X . Then \bar{A} is closed in X and $y_0 \notin \bar{A}$. Since X is CR, there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(y_0) = 0$ and $f(a) = 1$ for all $a \in \bar{A}$. Restrict f to Y , we get a continuous function $f|_Y : Y \rightarrow [0, 1]$ such that $f|_Y(y_0) = 0$ and $f|_Y(a) = 1$ for all $a \in A$. Thus Y is CR.

Take $A \subseteq \prod X_\alpha$ closed and $x = (x_\alpha) \notin A$. Then there exists a basic open set $U = \prod U_\alpha$ such that $x \in U$ and $U \cap A = \emptyset$, where U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many α . Let those finitely many α be $\alpha_1, \alpha_2, \dots, \alpha_n$. Since X_{α_i} is CR, there exists a continuous function $f_{\alpha_i} : X_{\alpha_i} \rightarrow [0, 1]$ such that $f_{\alpha_i}(x_{\alpha_i}) = 0$ and $f_{\alpha_i}(a) = 1$ for all $a \in X_{\alpha_i} \setminus U_{\alpha_i}$. So we can define a continuous function $f : \prod X_\alpha \rightarrow [0, 1]$ by

$$f(x) = \prod_{i=1}^n f_{\alpha_i}(x_{\alpha_i}).$$

Then $f(x) = 0$ and for all $a \in A$, $f(a) = 1$. Thus $\prod X_\alpha$ is CR. (To be checked) ■

Example 19.0.1. \mathbb{R}_l is normal then it is CR. So \mathbb{R}_l^2 is CR. But it is not normal as we have shown.

Fact(no proof) There exist regular spaces that are not CR.

Remark Proof of Urysohn's lemma for metric spaces(exercise):

Let X be metric space with metric $d : X \times X \rightarrow \mathbb{R}$. Given two disjoint closed sets $A, B \subseteq X$, define $d_B(x) = d(x, B)$ which vanishes exactly on B . Then define $d_A(x) = d(x, A)$ which vanishes exactly on A . Now define

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}.$$

Then $f(A) = \{0\}$ and $f(B) = \{1\}$ and f is continuous because we've proved that distance from a point to a closed set is continuous.

Theorem 19.0.3 (Urysohn Metrization Theorem). *Every regular space with countable basis is metrizable.*

Idea of Proof Embed X into $[0, 1]^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$. This $\mathbb{R}^{\mathbb{N}}$ can be with product topology or uniform metric topology.

Step 1 Construct a family $f_n : X \rightarrow [0, 1]$ of functions such that $\forall x_0 \in X$ and \forall neighborhood U of x_0 , there exists n such that $f_n(x_0) > 1$ and $f_n(x) = 0$ for all $x \notin U$.