Chapter 4

Later

Definition 4.0.1. *A continuous bijective map $f : X \to Y$ is called a homeomorphism if f^{-1} is also continuous.*

Remark ”homeomorphism” is not the same as ”homomorphism”.

Definition 4.0.2. *$X \simeq Y$ are homeomorphic or topological equivalent if there exists a homeomorphism $f : X \to Y$*

Definition 4.0.3. *Two metric spaces (X, d_X) and (Y, d_Y) are called isometric if there exists a bijective map $f : X \to Y$ such that for all $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$. ($f$ is called an isometry)*

Exercise Prove that every isometry is a homeomorphism, but some homeomorphisms are not isometries.

Example 4.0.1. *Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ be the unit disk in $\mathbb{R}^2$ with the Euclidean metric. Let $S^1 = \{(x, y) \in \mathbb{R}^2 | \max |x|, |y| = 1\}$ be the unit square in $\mathbb{R}^2$ with the Euclidean metric. Then D and S^1 are homeomorphic but not isometric.*

Proof

The idea of prove homeomorphism is to map every radius of the disk to the corresponding line segment of the square.

Assume there exists an isometry $f : D \to S^1$. Note that the diameter of D is 2, while the diameter of S^1 is $2\sqrt{2}$. This contradicts the definition of isometry. Thus, no such isometry exists.

Example 4.0.2. *Another example of homeomorphism is the linear map from the interval $(0, 1)$ to (a, b) (by stretching and compressing).*

Example 4.0.3. $(0, 1) \simeq (-\frac{\pi}{2}, \frac{\pi}{2}) \simeq \mathbb{R}$ *by the tangent function.*

The above example can be described graphically by a bowl(a circle centered at $(0, \frac{1}{2})$ with radius $\frac{1}{2}$ without the upper half) with radius $\frac{1}{2}$ on the real line.

Property 4.0.1. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Corollary 4.0.1. *If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then $g \circ f : X \to Z$ is a homeomorphism.*

Property 4.0.2. *Homeomorphism $\simeq$ is an equivalence relation.*

Definition 4.0.4. *X is said to be path-connected if any two points $x, y \in X$ can be joined by a path: there exists a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.*

Definition 4.0.5. *Let X be a metric space. X is said to be connected if one of the following equivalent conditions holds:*

1. *X cannot be represetned as $X = U_1 \sqcup U_2$ where U_1, U_2 are non-empty open subsets of X.*
2. *X cannot be represented as $X = V_1 \sqcup V_2$ where V_1, V_2 are non-empty closed subsets of X.*
3. *There is no proper non-empty subset $U \subseteq X$ which is both open and closed in X.*

Property 4.0.3. *If X is path-connected and $f : X \to Y$ is a homeomorphism, then Y is also path-connected.*

Proof Let $y_1, y_2 \in Y$. Since f is bijective, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is path-connected, there exists a continuous map $g : [0,1] \to X$ such that $g(0) = x_1$ and $g(1) = x_2$. Consider the map $h = f \circ g : [0,1] \to Y$. Since both f and g are continuous, h is continuous. Moreover, $h(0) = f(g(0)) = f(x_1) = y_1$ and $h(1) = f(g(1)) = f(x_2) = y_2$. Thus, there exists a continuous path in Y connecting y_1 and y_2, proving that Y is path-connected.

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Theorem 4.0.1 (Intermediate Value Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous. For any c between $f(a)$ and $f(b)$, there exists some $x \in [a,b]$ such that $f(x) = c$.*

Example 4.0.4. $\mathbb{R} \setminus \{0\}$ *is not path-connected.*

Proof By IVT.

Theorem 4.0.2 (Intermediate Value Theorem for Path-Connectedness)**.** *Let X be a path-connected space and $f : X \to \mathbb{R}$ be continuous. Suppose there exist $x_1, x_2 \in X$ such that $f(x_1) = c_1$ and $f(x_2) = c_2$. Then for any c between c_1 and c_2, there exists some $x \in X$ such that $f(x) = c$.*

Proof Since X is path-connected, there exists a continuous map $g : [0,1] \to X$ such that $g(0) = x_1$ and $g(1) = x_2$. Consider the map $h = f \circ g : [0,1] \to \mathbb{R}$. Since both f and g are continuous, h is continuous. Moreover, $h(0) = f(g(0)) = f(x_1) = c_1$ and $h(1) = f(g(1)) = f(x_2) = c_2$. By the Intermediate Value Theorem, for any c between c_1 and c_2, there exists some $t \in [0,1]$ such that $h(t) = c$. Let $x = g(t)$. Then $f(x) = f(g(t)) = h(t) = c$. Thus, there exists some $x \in X$ such that $f(x) = c$.

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Property 4.0.4. *Let X be a metric space. Then IVT for X holds if and only if there is no continuous and surjective map $f : X \to \{0,1\}$.*

Proof

We assume that X contains more than one point.

One direction is trivial.

Let's assume that the IVT doesn't hold for X.

Since the IVT does not hold for X, there exists $y_1, y_2 \in \mathbb{R}$ such that there exists y_3 between y_1 and y_2 which is not in the range of f. Then $f : X \to \mathbb{R} \setminus \{y_3\}$. Now we can easily define a continuous and surjective map $\tilde{f} : X \to \{0,1\}$.

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Property 4.0.5. *Let X be a metric space. X is connected if and only if IVT for X holds.*

Proof

The statement is equivalent to: X is connected if and only if there is no continuous and surjective map $f : X \to \{0,1\}$.

Suppose there is a continuous and surjective map $f : X \to \{0,1\}$. Then $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are non-empty, disjoint, open subsets of X whose union is X. Thus, X is not connected.

Suppose that there is no continuous and surjective map $f : X \to \{0,1\}$. If X is not connected, then there exist non-empty, disjoint, open subsets U_1, U_2 of X such that $X = U_1 \sqcup U_2$. We can define a map $f : X \to \{0,1\}$ by setting $f(x) = 0$ if $x \in U_1$ and $f(x) = 1$ if $x \in U_2$. This map is continuous and surjective, contradicting our assumption. Therefore, X must be connected.

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Corollary 4.0.2. *Let X be a metric space. If X is path-connected, then X is connected.*

Proof Since X is path-connected, IVT for X holds. Thus, X is connected.

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Remark IVT holds for X doesn't implies X is path-connected.

Example 4.0.5. *This is an example of a connected space but not path-connected:*

$$X = \{(x, \sin\frac{1}{x})|x > 0\} \cup (\{0\} \times [-1,1]) \tag{4.1}$$