Chapter 9

Lecture9

Definition 9.0.1. *A topological space X is Hausdorff or T2 if for every pair of distinct points $x, y \in X$, there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.*

Property 9.0.1. *If X is a metric space, then X is Hausdorff.*

Proof Let $d = d(x, y) > 0$. Take $U = B\left(x, \frac{d}{2}\right)$ and $V = B\left(y, \frac{d}{2}\right)$. Then $U \cap V = \emptyset$.

Property 9.0.2. *For a Hausdorff space X, for every $x_0 \in X$, the singleton set $\{x_0\}$ is closed.*

Let $x \in X$ with $x \neq x_0$. Since X is Hausdorff, there exist open sets $U, V \subseteq X$ such that $x_0 \in U$, $x \in V$, and $U \cap V = \emptyset$. Thus $V \subseteq X \setminus \{x_0\}$. Therefore, $X \setminus \{x_0\}$ is open, which means $\{x_0\}$ is closed.

■

Question If $\forall x_0 \in X$, $\{x_0\}$ is closed, is X Hausdorff?
Answer No. Consider the finite complement topology.

Definition 9.0.2. *If every point $x_0 \in X$ is closed, then X is called a Frechet space or T1 space.*

Remark If X is Hausdorff(T2), then X is T1. The converse is not true.

Property 9.0.3. *X is T1 if and only if $\forall x, y \in X, \exists U \ni x$ a open set such that $y \notin U$. That is, for all $x, y \in X$, there exists an open set containing x but not containing y.*

Exercise Prove this property.
Exercise Let X satisfy the following: for $x, y \in X$ with $x \neq y$, either $\exists U \ni x$ open set such that $y \notin U$, or $\exists V \ni y$ open set such that $x \notin V$. Is it true that X is T1?
Solution No. Consider $X = \{a, b\}$ with the topology $\mathcal{T} = \{\emptyset, X, \{a\}\}$.

Theorem 9.0.1. *Let X be T1, and $A \subseteq X$. Then $x \in A'$ if and only if every open set $U \ni x$ contains infinitely many points of A.*

Proof
($\Rightarrow$) Assume the contrary that only finitely many points are in the intersection. Suppose $x \in A'$. Then every open set $U \ni x$ contains a point of A different from x, i.e., $\exists y \in U \setminus \{x\} \cap A$. So we have that $U \cap A = \{y_1, y_2, \cdots, y_m\}$. Since X is T1, then $\{y_i\}$ is closed for each $i = 1, 2, \cdots, m$. Thus $V =: U \setminus \{y_1, y_2, \cdots, y_m\}$ is open and contains x. Then $V \setminus \{x\} \cap A = \emptyset$, which contradicts the assumption that $x \in A'$.
($\Leftarrow$) Obvious.

Theorem 9.0.2. *A sequence (x_n) in a Hausdorff space X converges to at most one point.*

Proof Suppose a_1, a_2 are two distinct limits of the sequence (x_n). Since X is Hausdorff, there exist open sets $U, V \subseteq X$ such that $a_1 \in U$, $a_2 \in V$, and $U \cap V = \emptyset$. Since a_1 is a limit of the sequence, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $x_n \in U$. Similarly, since a_2 is a limit of the sequence, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $x_n \in V$. Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, $x_n \in U$ and $x_n \in V$, which implies $x_n \in U \cap V$. This contradicts the fact that $U \cap V = \emptyset$. Therefore, the sequence (x_n) converges to at most one point.

■

Theorem 9.0.3.
1. *$(X, <)$ with the order topology is Hausdorff.*
2. *If X is Hausdorff, then the set with subspace topology $Y \subseteq X$ is also Hausdorff.*
3. *If X_1, X_2 are Hausdorff, then the product space $X_1 \times X_2$ with the product topology is also Hausdorff.*

Exercise Prove this theorem.

Definition 9.0.3. *Let X, Y be topological spaces. A function $f : X \to Y$ is called continuous if for every open set $V \subseteq Y$, the preimage $f^{-1}(V)$ is an open set in X.*

Remark If the topology on Y is given by the basis $\mathcal{B}$, then f is continuous if and only if for every basis element $B \in \mathcal{B}$, $f^{-1}(B)$ is open in X.

Example 9.0.1. *Consider the identity map for topological spaces $\mathbb{R}$ with the usual topology and $\mathbb{R}_l$ with the lower limit topology. The identity map $id : \mathbb{R} \to \mathbb{R}_l$ is not continuous, but the inverse $id : \mathbb{R}_l \to \mathbb{R}$ is continuous. Note that $(a, b) = \bigcup_{n=1}^{\infty}[a + \frac{1}{n}, b)$.*

Observation If $\mathcal{T}, \mathcal{T}'$ are topologies on X with $\mathcal{T} \subseteq \mathcal{T}'$, then the identity map $id : (X, \mathcal{T}') \to (X, \mathcal{T})$ is continuous and the map $id : (X, \mathcal{T}) \to (X, \mathcal{T}')$ is not continuous.

Theorem 9.0.4. *The following are equivalent:*

1. *$f : X \to Y$ is continuous.*
2. *$\forall A \subseteq X, f(\bar{A}) \subseteq \overline{f(A)}$.*
3. *For every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.*
4. *$\forall x \in X$, for every open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(U) \subseteq V$.*

Proof

(1) $\Rightarrow$ (2) Let $f : X \to Y$ be continuous and $A \subseteq X$. Let $x \in \bar{A}$. We need to show that $f(x) \in \overline{f(A)}$. If $x \in A$, then $f(x) \in f(A) \subseteq \overline{f(A)}$. If $x \in \bar{A} \setminus A$, take $V \ni f(x)$ open in Y. Since f is continuous, $f^{-1}(V)$ is open in X and contains x. Thus $f^{-1}(V) \cap A \neq \emptyset$. Let $y \in f^{-1}(V) \cap A$. Then $f(y) \in V \cap f(A) \neq \emptyset$. Therefore, $f(x) \in \overline{f(A)}$.

(2) $\Rightarrow$ (3) Let $C \subseteq Y$ be closed. We need to show that $f^{-1}(C) := A$ is closed in X, i.e., $\bar{A} = A$. We have $f(A) = f(f^{-1}(C)) = C$. So if $x \in \bar{A}, f(x) \in f(\bar{A}) \subseteq \overline{f(A)} = \overline{C} = C$. So $x \in f^{-1}(C) = A$. Thus $\bar{A} \subseteq A$. The other direction is obvious.

(3) $\Rightarrow$ (1) Obvious(take the complement).

(1) $\Rightarrow$ (4) Let $f(x) \in V$. Take $x \in f^{-1}(V) =: U$ which is open in X. Then $f(U) \subseteq V$.

(4) $\Rightarrow$ (1) Let $V \subseteq Y$ be open. We need to show that $f^{-1}(V)$ is open in X. Let $x \in f^{-1}(V)$. Then $\exists x \in U_x \subseteq X$ such that $f(U_x) \subseteq V$. Take $U = \bigcup_{f(x)\in V} U_x$ which is open in X. So $f^{-1}(V) \subseteq U$. But $U \subseteq f^{-1}(V)$.

9.1 Constructing continuous functions

Theorem 9.1.1. *Let X, Y, Z be topological spaces.*

1. *$f : X \to Y$ is a constant function, i.e., $f(X) = y$. Then f is continuous.*
2. *Let $A \subseteq X$ be a set with subspace topology. Then the embedding map $j : A \hookrightarrow X$ defined by $j(x) = x$ is continuous.*
3. *Let $f : X \to Y$, $g : Y \to Z$ be continuous, then $g \circ f : X \to Z$ is continous.*
4. *Restriction of domain: Let $f : X \to Y$ be continuous and $A \subseteq X$, then $f|_A : A \to Y$ is continuous.*
5. *$f : X \to Y$ and $Z \subseteq Y$ with $f(X) \subseteq Z$. Then $f : X \to Z$ is continuous.*
6. *If $f : X \to Y, Y \subseteq Z$, then $f : X \to Z$ is continuous.*
7. *$f : X \to Y$ is continuous if $X = \bigcup U_\alpha$ (U_α is required to be open in X) and $f|_{U_\alpha} : U_\alpha \to Y$ is continuous.*

Poof

1. Let $V \subseteq Y$ be open. If $y \in V$, then $f^{-1}(V) = X$ which is open. If $y \notin V$, then $f^{-1}(V) = \emptyset$ which is open.
2. Let $U \subseteq X$ be open. Then $j^{-1}(U) = U \cap A$ which is open in A.
3. $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ which is open in X.
4. $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$ which is open in A(Subspace topology).

5. Let W be open in Z. By subspace topology, $\exists V$ open in Y such that $W = V \cap Z$. Then $f^{-1}(W) = f^{-1}(V \cap Z) = f^{-1}(V)$(because $f(X) \subseteq Z$) which is open in X.

6. Let $f : X \to Y$ be continuous and $Y \subseteq Z$(Z has an topology $\mathcal{T}_Z$ and Y has the subspace topology $\mathcal{T}_Y = \{U \cap Y | U \in \mathcal{T}_Z\}$). Then we have for every open set $W \subseteq Z$, $f^{-1}(W) = f^{-1}(W \cap Y)$ which is open in X.

7. Let $X = \bigcup U_\alpha$ and U_α is open in X. And for each α, $f|_{U_\alpha} : U_\alpha \to Y$ is continuous.

   Take $V \subseteq Y$ open. Then

   $$f^{-1}(V) = f^{-1}(V) \cap X = f^{-1}(V) \cap \left(\bigcup U_\alpha\right) = \bigcup \left(f^{-1}(V) \cap U_\alpha\right) = \bigcup (f|_{U_\alpha})^{-1}(V) \tag{9.1}$$

   Since $(f|_{U_\alpha})^{-1}$ is continuous, then $(f|_{U_\alpha})^{-1}(V)$ is open in U_α. And since U_α has the subspace topology and U_α is open in X, then $(f|_{U_\alpha})^{-1}(V)$ is open in X. Thus $f^{-1}(V)$ is open in X.

Theorem 9.1.2. *Let $X = A \cup B$ where A and B are both open in X or both closed in X. And $f : A \to Y$, $g : B \to Y$ be continuous and $f(x) = g(x)$ for $x \in A \cap B$. Then define $h : X \to Y$ as*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

Then h is continuous.

Proof Let's assume that A, B are both open. Let $V \subseteq Y$ be open. Then

$$h^{-1}(V) = \{x \in X | h(x) \in V\} = \{x \in A | f(x) \in V\} \cup \{x \in B | g(x) \in V\} = f^{-1}(V) \cup g^{-1}(V) \tag{9.2}$$

Since f, g are continuous, by the subspace topology, $f^{-1}(V) = A \cap U_1$ where U_1 is open in X, and $g^{-1}(V) = B \cap U_2$ where U_2 is open in X. Thus

$$h^{-1}(V) = (A \cap U_1) \cup (B \cap U_2) \tag{9.3}$$

is open in X since A, B, U_1, U_2 are all open in X.

For the case where A, B are both closed, we have a similar argument.

■

Exercise Prove the above theorem.