Chapter 8

Lecture 8

Theorem 8.0.1. *1. $x \in \bar{A} \iff$ every open set $U \subseteq X$ with $x \in U$ satisfies $U \cap A \neq \emptyset$.*

2. For a basis $\mathcal{B}$ of $\mathcal{T}$, we have $x \in \bar{A} \iff$ every basis element $B \in \mathcal{B}$ with $x \in B$ satisfies $B \cap A \neq \emptyset$.

Proof

1. See P113.
2. $\cdots$

Exercise Finish the proof of (2).

Definition 8.0.1. *For $A \subseteq X$, $x \in X$ is a limit point of A if every open set $U \subseteq X$ containing x contains a point of A different from x itself. We denote A' as the set of all limit points of A.*

Theorem 8.0.2. $\bar{A} = A \cup A'$

Proof
We need to show that $\bar{A} \subseteq A \cup A'$ and $A \cup A' \subseteq \bar{A}$.
($\supseteq$) If $x \in A'$, then every neighborhood U of x intersects A. Then $x \in \bar{A}$ by the previous theorem. If $x \in A$, then obviously $x \in \bar{A}$. Thus $A \cup A' \subseteq \bar{A}$.
($\subseteq$) Let $x \in \bar{A}$. If $x \in A$, then $x \in A \cup A'$. If $x \in \bar{A} \setminus A$, then every neighborhood U of x intersects A at y such that $y \neq x$. Thus $x \in A'$. Therefore, $\bar{A} \subseteq A \cup A'$.

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Corollary 8.0.1. *A set A is closed if and only if $A' \subseteq A$.*

Definition 8.0.2. *For a sequence $\{x_n\}$ in X, we say that a is a limit of the sequence if for every open set U containing a, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$. A sequence is convergent if it has a limit.*

Remark This is equivalent to the following: every open set U containing a contains almost all elements of the sequence except finitely many. And Notice that "all but finitely many" is not the same as "infinitely many".

Example 8.0.1. *Let X be a topological space with the discrete topology. Then a sequence $\{x_n\}$ converges to a if and only if there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n = a$.*

Example 8.0.2. *Let $X = \mathbb{Q}$ with the topology defined by the usual metric. Then the sequence:* $3, 3.1, 3.14, 3.141, 3.1415, \ldots$ *does not converge in X since its limit π is not in $\mathbb{Q}$.*

Example 8.0.3. *Let $X = \mathbb{R}_l$ with the lower limit topology($[a, b)$ are basis open sets). Then we say that a sequence converges in $\mathbb{R}^l$ if and only if it converges to a in $\mathbb{R}$ with an extra condition: $x_n \geq a, \forall n \geq N$ for some $N \in \mathbb{N}$.*

Exercise Show that if x_n converges in $\mathcal{T}$, then x_n converges in any coarser topology $\mathcal{T}' \subseteq \mathcal{T}$.

Example 8.0.4. *$X = \mathbb{N}$, the basis $U_m = \{0, 1, 2 \cdots, m\}$. $\mathcal{T} = \{U_m | m \in \mathbb{N}, \mathbb{N}, \emptyset\}$. Then x_n converges to a if and only if a is essential upper bound of $\{x_n\}$. In this case, the limit is not unique.*

Example 8.0.5. *Finite complement topology on $\mathbb{R}$. For example take $\mathcal{T} = \{\mathbb{R}, \emptyset, \mathbb{R} \setminus \{p_1, p_2, \cdots, p_m\}\}$.*

1. *If x_n is eventually constant with value a, then x_n converges to a. It can't not converge to $b \neq a$ because we can take the open set $\mathbb{R} \setminus \{a\}$ which contains b but does not contain almost all elements of the sequence.*

2. *If* $x_n = (-1)^n$*, then* x_n *does not converge. If* $x_n = a$ *for infinitely many* n *and* $x_n = b$ *for infinitely many* n $(a \neq b)$*, then* x_n *does not converge.*

3. *If* $x_n = \dfrac{(-1)^n}{n}$*, then* x_n *converges to any real number. Because for any open set* U*,* U *contains all but finitely many real numbers. So* U *contains almost all elements of the sequence.*

4. *If* x_n *assumes every value finitely many times, then* x_n *converges to any real number.*