Chapter 24

Later

Lemma 24.0.1. *Let $A \subseteq X$. Let $f : A \to Z$ be continuous and Z be Hausdorff. Then there exists at most one extension $g : \bar{A} \to Z, g|_A = f$.*

Theorem 24.0.1. *Let X be completely regular and Y be a compactification of X such that Y satisfies the Stone–Čech property. Then any continuous map $f : X \to C$ where C is a compact Hausdorff space can be extended to a continuous map $g : Y \to C$.*

Theorem 24.0.2. *Let Y_1, Y_2 be compactifications of a completely regular space X. Suppose both Y_1 and Y_2 satisfy the Stone–Čech property. Then there exist homeomorphisms $h : Y_1 \to Y_2$.*

24.1 Complete Metric Spaces

Definition 24.1.1. *Let (X, d) be a metric space. A sequence $\{x_n\} \subseteq X$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(x_n, x_m) < \varepsilon$.*

Property 24.1.1. *Every convergent sequence is a Cauchy sequence.*

Proof To be done.

Definition 24.1.2. *A metric space (X, d) is called complete if every Cauchy sequence in X converges to a limit in X.*

Example 24.1.1 (non-example)**.** *$(0, 1)$ with the usual metric. Then the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence that does not converge in $(0, 1)$.*

Example 24.1.2 (non-example)**.** *$\mathbb{Q}$ with the usual metric is not complete. For example, the sequence defined by the decimal approximations of $\sqrt{2}$ is a Cauchy sequence in $\mathbb{Q}$ that does not converge to a rational number.*

Property 24.1.2. *If $A \subseteq X$ is closed and (X, d) is complete, then A is complete.*

Property 24.1.3. *Let (X, d) be a metric space. Take $\bar{d} = \min\{d, 1\}$. Then (X, d) is complete implies $(X, \bar{d})$ is complete.*

Lemma 24.1.1. *X is complete if and only if every Cauchy sequence has a convergent subsequence.*

Proof

"$\Rightarrow$": Trivial.

"$\Leftarrow$": To be done.

Theorem 24.1.1. *$\mathbb{R}^k$ is complete in d_1, d_2, d_∞.*

Proof

We show this for d_∞.

Let $\{x_n\} \subseteq \mathbb{R}^k$ be a Cauchy sequence in d_∞. Then $\{x_n\}$ is bounded, i.e., there exists $M > 0$ such that for all n, $d_\infty(x_n, 0) < M$. So such sequence lies in a compact set $[-M, M]^k$. We know in metric spaces, compactness is equivalent to sequential compactness. Thus there exists a convergent subsequence $\{x_{n_k}\}$ that converges to some $x \in [-M, M]^k$. By the lemma, $\{x_n\}$ converges to x.

All these metrics are equivalent, so $\mathbb{R}^k$ is complete in d_1, d_2 as well.

■

Lemma 24.1.2. *Let $X = \prod_{\alpha \in J} X_\alpha$. Let (x_n) be a sequence in X. Then (x_n) converges to $x \in X$ if and only if for all $\alpha \in J$, the sequence of α-th coordinates $(\pi_\alpha(x_n))$ converges to $\pi_\alpha(x)$ in X_α.*

Proof

"$\Rightarrow$": π_α is continuous for all $\alpha \in J$.

"$\Leftarrow$": Suppose $(\pi_\alpha(x_n))$ converges to $\pi_\alpha(x)$ for all $\alpha \in J$. Let U be an open neighborhood of x. Then there exists a finite set $K \subseteq J$ and open sets $U_\alpha \subseteq X_\alpha$ for all $\alpha \in K$ such that $x \in \prod_{\alpha \in K} U_\alpha \times \prod_{\alpha \in J \setminus K} X_\alpha \subseteq U$. Since $(\pi_\alpha(x_n))$ converges to $\pi_\alpha(x)$, for each $\alpha \in K$, there exists $N_\alpha \in \mathbb{N}$ such that for all $n \geq N_\alpha$, $\pi_\alpha(x_n) \in U_\alpha$. Let $N = \max_{\alpha \in K} N_\alpha$. Then for all $n \geq N$, $x_n \in U$. Thus (x_n) converges to x.

Theorem 24.1.2. *There is a metric on $\mathbb{R}^\omega$ that makes it complete.*

Proof

Define $\bar{d}(a,b) = \min\{1, |a-b|\}$ for all $a, b \in \mathbb{R}$. Define $D(x,y) = \sup_{n \in \mathbb{N}}\{\frac{\bar{d}(x_n,y_n)}{n}\}$ for all $x = (x_n), y = (y_n) \in \mathbb{R}^\omega$. Then D is a metric on $\mathbb{R}^\omega$.

Let $\{x^m\} \subseteq \mathbb{R}^\omega$ be a Cauchy sequence in D. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l \geq N$, $D(x^m, x^l) < \varepsilon$. In particular, for all $n \in \mathbb{N}$, $\bar{d}(x_n^m, x_n^l) \leq nD(x^m, x^l) < n\varepsilon$. Thus for each fixed n, the sequence $\{x_n^m\}_{m=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $x_n \in \mathbb{R}$ such that x_n^m converges to x_n as $m \to \infty$. (might not be true, you need to check this)

24.2 Uniform Metric

Let (Y, d) be a metric space. Let $\bar{d} = \min\{d, 1\}$. We take the set of functions Y^J. There is a metric $\bar{\rho}$ on Y^J defined by $\bar{\rho}(x,y) = \sup_{\alpha \in J}\{\bar{d}(x_\alpha, y_\alpha)\}$. This is a metric(exercise). It is called the uniform metric.

Example 24.2.1. *Let $J = [0,1]$ and $Y = \mathbb{R}$. Define $\bar{d}(f,g) = \sup_{x \in [0,1]}\{\min(1, |f(x) - g(x)|)\}$.*

Theorem 24.2.1. *If Y is complete with respect to d, then Y^J is complete with respect to the uniform metric $\bar{\rho}$.*