Chapter 1

Introduction, Horner's Method

Motivation:

- Sometimes problems require too much time, effort, etc. to be praticallly solved wtih a computer.
- Some problems cannot be solved analytically.
- For applications, numerical solutions are often sufficient.

Example 1.0.1. *Can you solve :*

1. $\sin(0.67)$
2. $\int_0^1 e^{-x^2} \mathrm{d}x$
3. $x^2 - \sin(x) - 1 = 0, x = ?$

Numerical problems lead to new mathematical questions.

Example 1.0.2. *Suppose $Ax = b, A \in \mathbb{R}^{n\times n}, det(A) \neq 0$, and A is symmetric. Find $det(A)$.*

Approach I: Use Sarrus' rule $det(A) = \sum_{\sigma\in S_n} sgn(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$. For each permutation σ, we have $n!$ permutations. So we have $n \cdot n!$ product operations. Furthermore, we have $n! - 1$ additions, meaning that the method requires $n(n!) + n! - 1$ operations. This approach works fine for small n, but not for large n.

Approach II: Use the diagonalization method. Since A is symmetric, we can find an orthogonal matrix Q such that $Q^T AQ = D$, where D is a diagonal matrix. Then, we have $det(A) = det(Q)det(D) = det(D)$. We can compute $det(A)$ in $cn^3 + n - 1$ operations where c is a constant.

Definition 1.0.1. *Consider sequences (x_n) and (α_n), where $n = 0, 1, 2, \ldots$. We say that (x_n) is in $O(\alpha_n)$ if there exist constants C, N such that*

$$|x_n| \leq C|\alpha_n|, \quad \forall n \geq N$$

Example 1.0.3. $\dfrac{n+1}{n^2}$ *is in* $O(\frac{1}{n})$. $Cn^3 + n - 1$ *is in* $O(n^3)$. $n(n!)$ *is in* $O(n(n!))$.

Remark It is also true that $Cn^3 + n - 1$ is in $O(n(n!))$.

Definition 1.0.2. *Let $(x_n), (\alpha_n)$ be sequences, where $n = 0, 1, 2, \ldots$. We say that (x_n) is in $\Theta(\alpha_n)$ if there exist constants C_1, C_2, N such that*

$$C_1|\alpha_n| \leq |x_n| \leq C_2|\alpha_n|, \quad \forall n \geq N$$

Example 1.0.4. $n(n!) \leq n(n!) + n! - 1 \leq 2n(n!)$ *for* $n \geq 1$. *So* $n(n!) + n! - 1$ *is in* $\Theta(n(n!))$.

Example 1.0.5. $Cn^3 \leq Cn^3 + n - 1 \leq (C + 1)n^3$ *for* $n \geq 1$. *So* $Cn^3 + n - 1$ *is in* $\Theta(n^3)$.

Question How many operations are required to evaluate a polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ at a point z_0 where $a_n, a_{n-1}, \ldots, a_1, a_0, z_0 \in \mathbb{R}$?

The simplest approach computes $a_k z^k$ by using k multiplications, and then sums them up. This requires $n + (n - 1) + \cdots + 1 + n = \dfrac{n(n+1)}{2} + n$ which is in $\Theta(n^2)$.

The Horner's method is based on the remaider theorem, which reduce complexity of evaluation of a polynomial. It only requires $\Theta(n)$ operations.

Theorem 1.0.1 (Horner's Method). *Let $p(z)$ be a polynomial of degree n, with real coefficients, and $z_0 \in \mathbb{R}$. Then there exists $r \in \mathbb{R}$ and a polynomial $q(z)$ of degree $n-1$ such that*

$$p(z) = r + (z - z_0)q(z)$$

Proof. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Our goal is to find coefficients $b_{n-1}, b_{n-2}, \ldots, b_1, b_0$ and a constant r such that

$$p(z) = r + (z - z_0)(b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_1 z + b_0)$$

Equating coefficients of like powers of z on both sides, we get a system of equations:

$$\begin{aligned} a_n &= b_{n-1} \\ a_{n-1} &= b_{n-2} - z_0 b_{n-1} \\ a_{n-2} &= b_{n-3} - z_0 b_{n-2} \\ &\vdots \\ a_1 &= b_0 - z_0 b_1 \\ a_0 &= r - z_0 b_0 \end{aligned}$$

Solving the equations from the top the the bottom recursively leads to a unique solution:

$$\begin{aligned} b_{n-1} &= a_n \\ b_{n-2} &= a_{n-1} + z_0 b_{n-1} \\ b_{n-3} &= a_{n-2} + z_0 b_{n-2} \\ &\vdots \\ b_0 &= a_1 + z_0 b_1 \\ r &= a_0 + z_0 b_0 \end{aligned}$$

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Remark. It is convenient to write $b_{-1} = r$ which leads to equations

$$b_{n-1} = a_n$$

$$a_k = b_{k-1} - z_0 b_k, \quad k = n-1, \ldots, 0$$

These equations can be graphically represented as:

$$\begin{array}{c|cccccc} & a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ z_0 & & -z_0 b_{n-1} & -z_0 b_{n-2} & \cdots & -z_0 b_1 & -z_0 b_0 \\ \hline & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_0 & b_{-1} \end{array}$$

Remark We have n equations, each of which requires one addition and one multiplication. Overall complexity of computing $p(z_0)$ is in $\Theta(n)$.

Remark The algorithm can be used for finding all roots of $p(z)$ by starting with a root z_0, then writing $p(z) = (z - z_0)q(z)$, and finding z_1 and etc. We will later learn how to find these roots and how to apply the algorithm.

Remark Note that $p'(z) = q(z) + (z - z_0)q'(z)$, so $p'(z_0) = q(z_0)$ which can be evaluated by Horner again.

Example 1.0.6. *Let $p(z) = z^4 - 4z^3 + 7z^2 - 5z - 2$. We compute $p(3)$ and $p'(3)$.*

We set up the following table:

$$\begin{array}{c|ccccc} & 1 & -4 & 7 & -5 & -2 \\ 3 & & 3 & -3 & 12 & 21 \\ \hline & 1 & -1 & 4 & 7 & 19 \end{array}$$

So we have $p(3) = r = 19$. Then we set up the following table to compute $p'(3) = q(3)$:

$$\begin{array}{c|cccc} & 1 & -1 & 4 & 7 \\ & & & & \\ 3 & & 3 & 6 & 30 \\ & & & & \\ \hline & 1 & 2 & 10 & 37 \end{array}$$

So we have $p'(3) = q(3) = 37$.