Chapter 4

Fixed point iteration. [2.2]

Definition 4.0.1. *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. A point $p \in D$ is called a **fixed point** of f if*

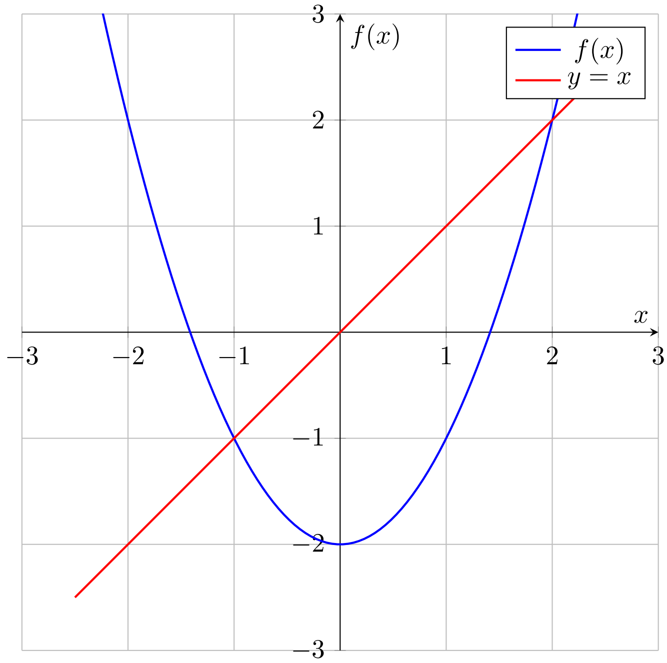
$$f(p) = p \tag{4.1}$$

Example 4.0.1.

1. *Any point $p \in \mathbb{R}$ is a fixed point of $f(x) = x$, the identity function.*
2. *Let $f(x) = x^2 - 2$. Then fixed points of f satisfy*

$$p = f(p) = p^2 - 2 \iff p^2 - p - 2 = (p+1)(p-2) = 0 \iff p = -1 \text{ or } p = 2 \tag{4.2}$$

which can be graphically shown as follows:



Theorem 4.0.1 (Existence and Uniqueness).

1. *If $f : [a,b] \to [a,b]$ is continuous, then f has at least one fixed point in $[a,b]$.*
2. *If $f'(x)$ exists on (a,b) and there exists a constant $0 < k < 1$ such that $|f'(x)| \le k$ for all $x \in (a,b)$, then f has one unique fixed point in $[a,b]$.*

Proof

If $f(a) = a$ or $f(b) = b$, then we are done.

We may assume that $f(a) > a$ and $f(b) < b$. Define $h(x) = f(x) - x$ which is continuous on $[a,b]$, with $h(a) = f(a) - a > 0$ and $h(b) = f(b) - b < 0$. By Bolzano's Intermediate Value Theorem, there exists $c \in (a,b)$ such that $h(c) = 0$, i.e. $f(c) = c$. This proves part 1.

Suppose that $|f'(x)| \le k < 1$, and that p, q are both fixed points of f. We need to show that $p = q$.

We assume the contrary that $p \neq q$. Without loss of generality, we may assume that $p < q$. By the Mean Value Theorem, there exists $\xi \in (p, q)$ such that

$$f'(\xi) = \frac{f(q) - f(p)}{q - p} \tag{4.3}$$

therefore, we have:

$$|p - q| = |f(p) - f(q)| \overset{(*)}{=} |f'(\xi)||p - q| \le k|p - q| < |p - q| \tag{4.4}$$

which makes a contradiction. It follows that $p = q$, i.e., the fixed point is unique.

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Example 4.0.2. *Show that* $f(x) = \dfrac{x^2 - 1}{3}$ *has a unique fixed point on the interval* $[-1, 1]$.

Solution We solve the example by verifying the conditions of the theorem above. Because f is continuous on $[-1, 1]$, it attains its maximum and minimum on $[-1, 1]$. In fact, ther extremal values are attained at $x = \pm 1$ and $x = 0$ because, $x = 0$ is the only zero of $f'(x) = \dfrac{2x}{3}$.

For $x = \pm 1, f(\pm 1) = 0$. For $x = 0, f(0) = -\dfrac{1}{3}$. Thus the max value of f is 0, and the minimum value of f is $-\dfrac{1}{3}$. It follows that $f([-1, 1]) \subseteq [-1, 1]$. So by part 1 of the theorem, f has at least one fixed point on $[-1, 1]$.

Moreover, $|f'(x)| \le |\dfrac{2x}{3}| \le \dfrac{2}{3} < 1$, for each $x \in (-1, 1)$. So by part 2 of the theorem, f has a unique fixed point on $[-1, 1]$.

Remark We may find this fixed point p by solving

$$p = f(p) = \frac{p^2 - 1}{3} \Longleftrightarrow p^2 - 3p - 1 = 0 \Longleftrightarrow p = \frac{3 \pm \sqrt{13}}{2} \tag{4.5}$$

Note that

$$p = \frac{3 - \sqrt{13}}{2} \approx -0.383 \in [-1, 1], \quad p = \frac{3 + \sqrt{13}}{2} \approx 3.31 \notin [-1, 1] \tag{4.6}$$

So the fixed point is indeed unique.

Idea To approximate the fixed point α of a function f, start from an initial guess p_0 and define a sequence $\{p_n\}$ by

$$p_n = f(p_{n-1}), \quad n = 1, 2, \ldots \tag{4.7}$$

Then we have

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} f(p_{n-1}) = f\left(\lim_{n\to\infty} p_{n-1}\right) = f(p) \tag{4.8}$$

so the solution of $x = f(x)$ is obtained by the process under suitable assumptions. Note that

$$\lim_{n\to\infty} f(p_{n-1}) = f\left(\lim_{n\to\infty} p_{n-1}\right) \tag{4.9}$$

requires f to be continuous.

Theorem 4.0.2 (The fixed point theorem)**.** *Let f be continuously differentiable on $[a, b]$ such that $f([a, b]) \subseteq [a, b]$ and $|f'(x)| \le k < 1$ for all $x \in [a, b]$. Then for any initial guess $p_0 \in [a, b]$, the sequence defined by*

$$p_n = f(p_{n-1}), \quad n = 1, 2, \ldots \tag{4.10}$$

converges to the unique fixed point p of f in $[a, b]$.

Proof By **??**, there exists a unique fixed point p of f in $[a, b]$. Because $|f'(x)| \le k$, it follows from the Mean Value Theorem that

$$\forall n \ge 1, \quad |p_n - p| = |f(p_{n-1}) - f(p)| = |f'(\xi)||p_{n-1} - p| \le k|p_{n-1} - p| \tag{4.11}$$

where ξ is between p_{n-1} and p ($\xi_{n-1} \in (a, b)$).

Applying this estimate to all $n = 1, 2, \ldots$, yields

$$|p_n - p| \le k^n |p_0 - p| \tag{4.12}$$

It follows that

$$\lim_{n\to\infty} |p_n - p| \le \lim_{n\to\infty} k^n |p_0 - p| = 0 \tag{4.13}$$

because $0 < k < 1$. Thus $\lim_{n\to\infty} p_n = p$.

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Corollary 4.0.1. *Under hypothesis of* **??**, *the bounds for error in fixed point iteration are given by*

$$|p_n - p| \le k^n \cdot \max(\{p_0 - a, b - p_0\}) \tag{4.14}$$

and

$$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0| \tag{4.15}$$

for all $n \ge 1$.

Proof The estimate **??** gives

$$|p_n - p| \le k^n |p_0 - p| \tag{4.16}$$

Combining this with the fact that $|p_0 - p| \le \max(\{p_0 - a, b - p_0\})$, we obtain the first bound.

For $n \ge 1$, it follows as before that

$$|p_{n+1} - p_n| = |f(p_n) - f(p_{n-1})| \le k|p_n - p_{n-1}| \le \cdots \le k^n |p_1 - p_0| \tag{4.17}$$

Therefore, for $m > n \ge 1$, we have

$$\begin{aligned} |p_m - p_n| &= |p_m - p_{m-1} + p_{m-1} - \cdots - p_{n+1} + p_{n+1} - p_n| \\ &\le \sum_{j=n}^{m-1} |p_{j+1} - p_j| \\ &\le \sum_{j=n}^{m-1} k^j |p_1 - p_0| \\ &= k^n |p_1 - p_0| \sum_{j=0}^{m-n-1} k^j \end{aligned} \tag{4.18}$$

By **??**, $\lim_{n\to\infty} p_n = p$, so taking $m \to \infty$ gives

$$|p - p_n| = \lim_{m\to\infty} |p_m - p_n| \le \lim_{m\to\infty} k^n |p_1 - p_0| \sum_{j=0}^{m-n-1} k^j \le k^n |p_1 - p_0| \sum_{j=0}^{\infty} k^j = \frac{k^n}{1-k}|p_1 - p_0| \tag{4.19}$$

because the geometric series $\sum_{j=0}^{\infty} k^j$ converges to $\dfrac{1}{1-k}$ for $0 < k < 1$.

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Example 4.0.3. *Approximate solution of* $\cos(x) = 3x - 1$ *by using fixed point iteration.*

Solution Let $f(x) = \cos(x) - 3x + 1$. We need to find p_0 that can be used as an initial guess. One can use the bisection method, i.e., finding an interval $[a, b]$ such that $f(a)f(b) < 0$. Note that $f(0) = 2 > 0$ and $f(\frac{\pi}{2}) = -\frac{3\pi}{2} - 1 < 0$, so roots are on the interval(including endpoints are reasonable).

Then function to be iterated is

$$g(x) := \frac{\cos(x) + 1}{3} \tag{4.20}$$

because $g(x) = x \iff \cos(x) = 3x - 1$. Note that for this function

$$g'(x) = -\frac{\sin(x)}{3} \tag{4.21}$$

and

$$|g'(x)| < 1 \tag{4.22}$$

at $x = 0$. We use $p_0 = 0$ as the initial guess and iterates to get

$$p_n \to p \approx 0.6071 \tag{4.23}$$

(8 iterations are required for 4 decimal places accuracy).