Stat 102

Introduction to Business Statistics
Class 9

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Today's module

Topics to be covered in this module:

- Last time
- Defining the log transform
- Prospecting for a log relationship
- Fitting log models
- Optimizing with log models
- Four model possibilities and the interpretation of the slope in each one
- Summary
- Next time

Last time

- Prospecting for curvature
- Data transformations
- The power function
- Comparing regressions on the original and transformed scale

Log transformations

The most frequently used of all transformations:

1. Empirical: Lots of data is severely right skewed. The log transform draws in the tail.
2. Theoretical: many relationships show *diminishing returns to scale* and the log is a good function to capture this structure.
3. Interpretative: logs give rise to simple interpretations in terms of percent change.

Defining the log transform

- Think of logarithms as exponents going backward (in math language we call that the *inverse* of a function).
- Ask yourself "to what power do I have to raise the number b to get to y?"
- In other words you need to solve the equation:

$$b^{?} = y.$$

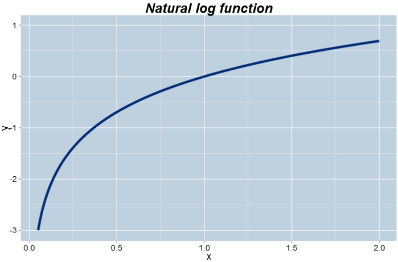
- The question mark is defined as the logarithm of y base b.

A typical log relationship

Here's what the log (base e) curve looks like. It is useful for capturing relationships that display *diminishing returns to scale*. Describe the idea of diminishing returns in English:

> *It's an increasing relationship, but it grows at an ever-decreasing rate.*

Figure 1: A graph of the natural log function



Logs base 10

- We can get a good understanding of what the log function is doing by considering the special case when the base is 10. Consider the equation:

$$10^{?} = 1000.$$

  In English the question is: to what power do I have to raise 10 to get 1000?
  The logarithm of 1000 base 10 is 3 because $10^3 = 1000$.

- Here is another example where we are tying to find the log base ten of 1,000,000:

$$10^{?} = 1{,}000{,}000.$$

  The logarithm of 1,000,000 base 10 is 6 because $10^6 = 1{,}000{,}000$.

- Finally, working with a billion:

$$10^{?} = 1{,}000{,}000{,}000.$$

  The logarithm of 1,000,000,000 base 10 is 9.

Logs base 2

- Logs base 2 make an appearance in calculations to do with computers and binary representations of numbers. Consider the equation:

$$2^? = 256.$$

- Because $2^8 = 256$ the logarithm of 256 base 2 is 8. This means that with eight *bits* (zeros or ones) you can represent 256 different numbers.
- Solving for the question mark in the equation:

$$2^? = 65{,}536,$$

  implies that the logarithm of 65,536 base 2 is 16 because $2^{16} = 65{,}536$.
- Solving for the question mark in the equation

$$2^? = 4{,}294{,}967{,}296,$$

  implies that the logarithm of 4,294,967,296 base 2 is 32.

A special base for logs is the log base e.

- This log function has several different notations, the most common being:

$$\ln(x) = \log_e(x).$$

- From the definition of the logarithm function, if we can solve for the question mark in:

$$e^{?} = x$$

  then we have found the natural logarithm of x.

- So by definition:

$$e^{\ln(x)} = e^{\log_e(x)} = x.$$

  It is also the case that:

$$\ln(e^x) = \log_e(e^x) = x.$$

  That is, taking natural logs undoes "e" to the power.

In STAT 102 we will only use the natural log function.

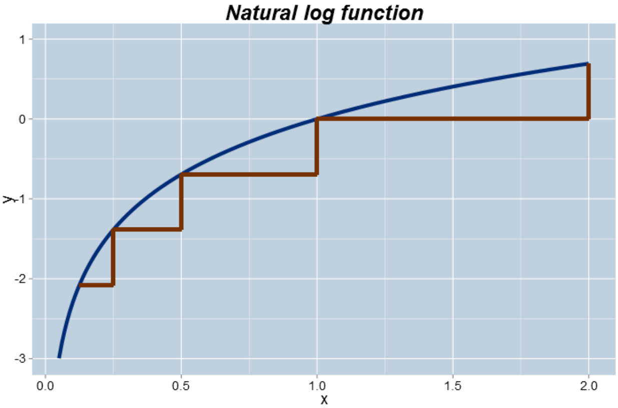
Log facts (true for any base b)

$$
\begin{aligned}
\log_b(x \times y) &= \log_b(x) + \log_b(y). \\
\log_b\left(\frac{x}{y}\right) &= \log_b(x) - \log_b(y). \\
\log_b(x^m) &= m \log_b(x). \\
\log_b(0) &= -\infty. \\
\log_b(1) &= 0. \\
\log_b(b) &= 1. \\
\log_b(b^x) &= x \log_b(b) = x \times 1 = x.
\end{aligned}
$$

The last statement shows that the log base b undoes "*b to the power x*".

The key feature of the log function

Figure 2: Stepping up the log function

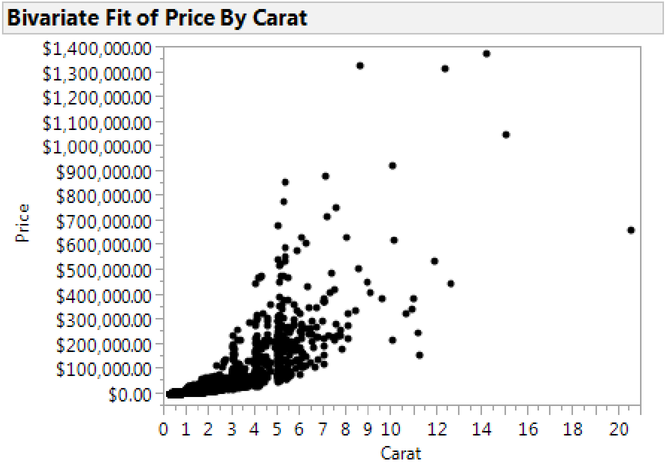


For each step you have to go twice as far forward, just to rise by the exact **same** amount.

Prospecting for logs

The big diamonds data on the original scale:

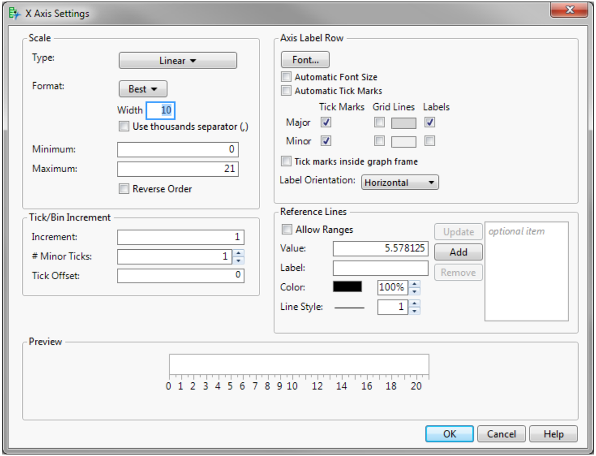
Figure 3: Diamonds relationship with curvature



Changing the scale of an axis

Double click on an axis to reveal:

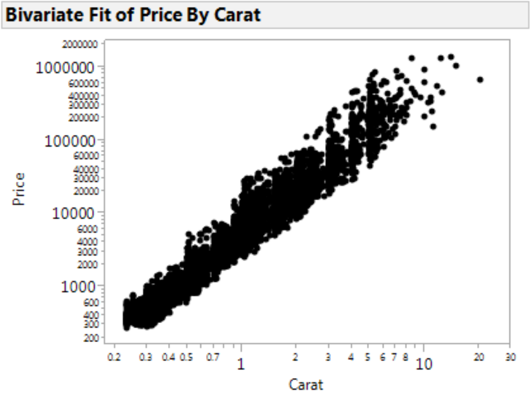
Figure 4: Axis dialog



Plotting on the log-log scale

Rescaling to the log on both axes shows a clear log-log relationship.

Figure 5: Rescaled axes



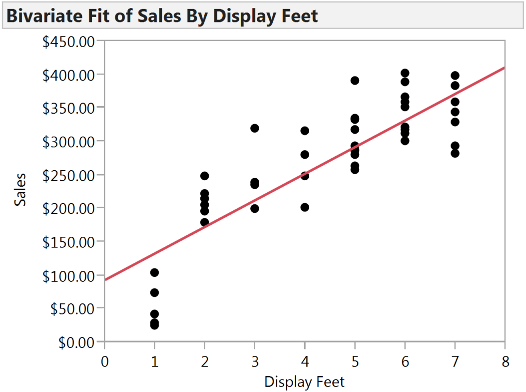
This suggests a log-log regression model will be a very good starting point.

Diminishing returns to scale

- A liquor store company wants to maximize its sales. It has a new wine to sell. It has an alternative product available that will guarantee \$50 per linear shelf foot.
- How many feet of the new wine should it put out for sale?

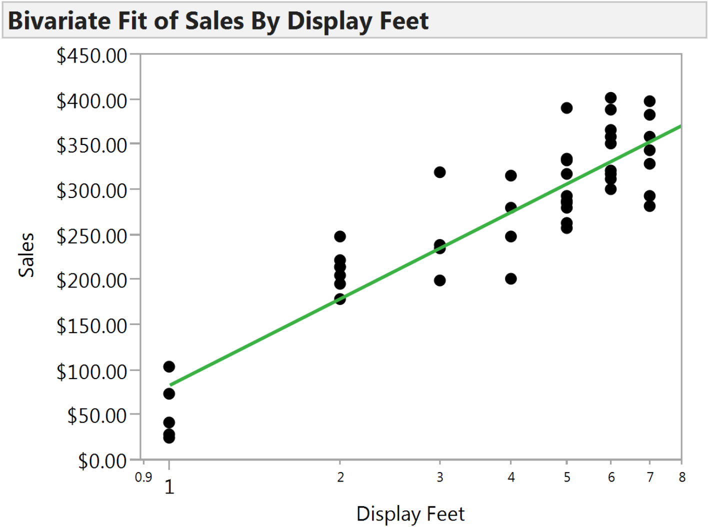
Diminishing returns to scale

The linear fit doesn't do a very good job.



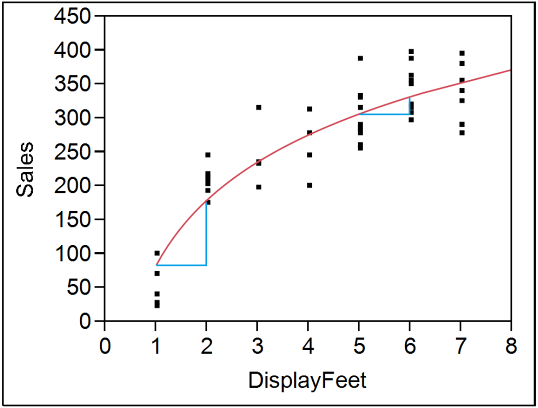
Diminishing returns to scale

Plotted on a log(x) against y scale we see the straight line that was fit by the method of least squares.



Diminishing returns to scale

A log curve is a better fit. It captures the [1]curvature.



[1]Use the Fit special dialog from the Fit Y by X titlebar to try various transformations.

Interpreting the coefficients

- The equation of the curve is $\text{Sales} = 83.56 + 138.62 \log(\text{Display Feet})$
- Interpretation:
  - The intercept: when Display Feet equals 1, the sales are expected to be \$83.56.
  - The slope: for every one percent increase in display feet we expect a \$1.39 (138.56/100) absolute increase in sales.

Solving the problem

- The equation of the log curve is:

$$y = 83.56 + 138.62 \log(x).$$

- It makes sense to increase the display of the new wine until its incremental sales for each extra linear shelf foot first falls beneath $50.
- This is because the alternative product should be substituted in at this point.
- Mathematically, the incremental change in sales is the slope of the curve which we can find explicitly by taking the derivative.
- The solution to the optimality problem is to find the value of x, (display feet) at which the slope of the curve equals 50.

Solving the problem

- The equation of the log curve is:

$$y = 83.56 + 138.62 \log(x).$$

- The [2]derivative is

$$\frac{dy}{dx} = \frac{138.62}{x}.$$

- Setting the derivative equal to 50 gives:

$$50 = \frac{138.62}{x},$$

  and so

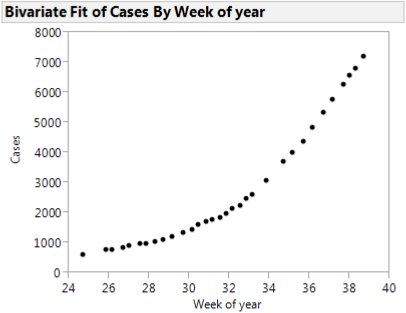
$$x = \frac{138.62}{50} = 2.77.$$

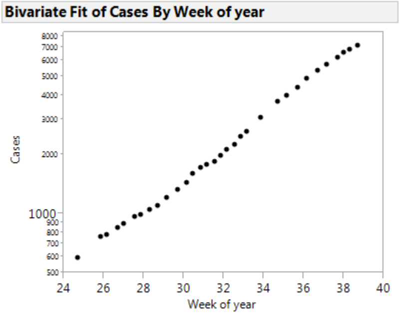
- You can verify that it is a maximum by establishing that the second derivative at this point is negative.

[2]The derivative of the natural log is $1/x$.

Exponential growth

Using the initial outbreak Ebola data we have



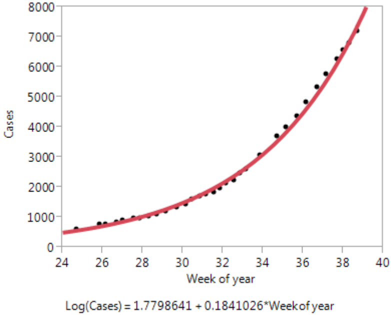


It looks much more linear on the $\log(y)$ scale.

Formulating the exponential model

- Based on the rescaling of the y axis consider the model $\log(y) = b_0 + b_1 x$, where x is the week of the year.
- Exponentiate each side of the equation to get: $y = e^{b_0} e^{b_1 x}$.
- Note that y is related to x through the exponential function.
- The growth rate is given by b_1:
  - If b_1 is positive it is termed *exponential growth*
  - If b_1 is negative it is termed *exponential decay*
- The multiplicative constant e^{b_0} is the value of y at time $x = 0$.

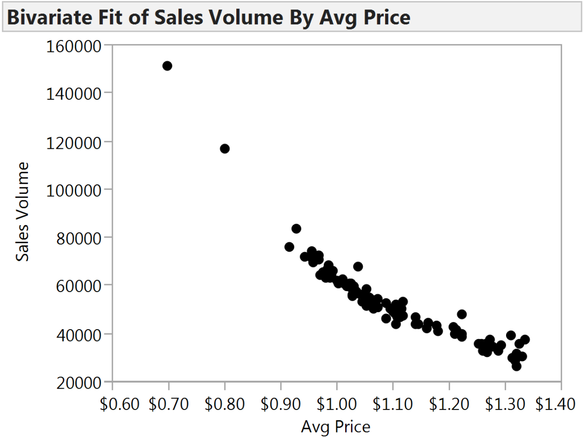
Fitting the Ebola curve



Interpretation: the slope in this regression is 0.184. This means that on a weekly basis there is an approximate 18% growth rate in the number of cases.

A demand function

Consider the relationship between the price of a can of cat food and the quantity sold.



A demand function

- A mathematically convenient way to model this relationship is with the multiplicative function:

$$q = \alpha p^{\beta_1},$$

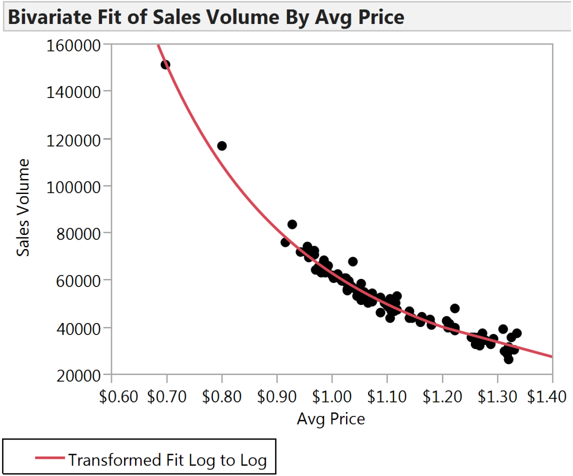
where q is the quantity demanded and p is the price.

- It is convenient because on taking logs of both sides we have:

$$\log(q) = \log(\alpha) + \beta_1 \log(p).$$

- In this log-log relationship, we can estimate β_1 by running a regression of $\log(q)$ against $\log(p)$ and using the least squares estimate of the slope b_1.

Fitting the log-log model



Transformed Fit Log to Log

Log(Sales Volume) = 11.050556 - 2.4420491*Log(Avg Price)

Fitting the log-log model

- Note that the slope is -2.44.
- This is called as the *price elasticity of demand*.
- The interpretation is that:

  For every 1% increase in price there is an expected 2.44% decrease in sales.

- Predict the quantity sold when the price is \$2.00.

$$\log(Sales) = 11.0506 - 2.442 \log(2).$$

$$\log(Sales) = 9.3579.$$

$$Sales = e^{9.3579} = 11,590.$$

Optimal pricing

- An important application of price elasticity is in optimal pricing.
- One can show that if the cost of production is a constant c and the price elasticity is written as γ, then given this model, the price which maximizes profit is:

$$p = \frac{c\gamma}{1+\gamma}.$$

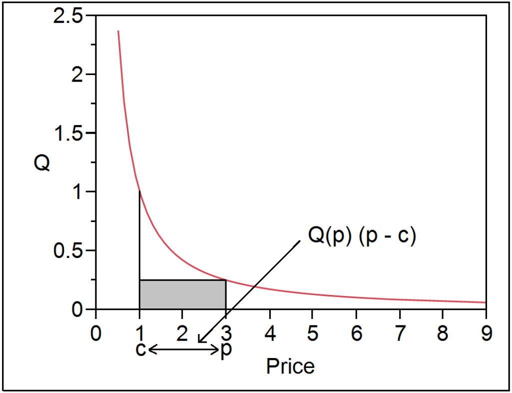
- Assume that costs are \$0.60, then the optimal price is

$$p = \frac{0.6 \times -2.44}{1 + -2.44} = 1.017.$$

Visualizing the optimization

Here's what the demand curve looks like. The profit for a given cost (c) and price (p) corresponds to the shaded rectangle. The goal is to find the price that makes this rectangle have the largest area.

Figure 6: The demand curve with the profit for price, p, shaded in gray.



A note on elasticity

- The definition: the elasticity of y with respect to x is $\frac{dy}{dx}\frac{x}{y}$.
- For the power function: $y = \alpha x^{\beta}$ we have $\frac{dy}{dx} = \alpha\,\beta x^{\beta-1}$ so that

$$\frac{dy}{dx}\frac{x}{y} = \alpha\,\beta x^{\beta-1}\frac{x}{y} = \alpha\,\beta x^{\beta-1}\frac{x}{\alpha x^{\beta}} = \beta$$

- Power functions are unique in having *constant elasticity* (the elasticity does not depend on the value x where it is being calculated).
- If you want constant elasticity then this means using a power function.
- If you want to use a power function then this implies linearity on the log-log scale, so have a look at the data on this scale and see if it is a reasonable assumption.

Log transforms and their interpretations

Note: all percent change interpretations for log transforms are valid only if the percent change considered is small. The smaller it is the better the approximation.

Four cases:

1. $E(Y|X) = \beta_0 + \beta_1 X$.
2. $E(Y|X) = \beta_0 + \beta_1 log(X)$.
3. $E\{log(Y)|X\} = \beta_0 + \beta_1 X$.
4. $E\{log(Y)|X\} = \beta_0 + \beta_1 log(X)$.

Interpretations

Four respective interpretations for β_1:

1. For a 1 unit change in X, the average of Y changes by β_1.
2. For a 1 percent change in X, the average of Y changes by $\beta_1/100$.
3. For a 1 unit change in X, the average of Y changes by 100 β_1 percent.
4. For a 1 percent change in X, the average of Y changes by β_1 percent – the economist's elasticity definition.

See Stines' book, "Behind the Math", Chapter 20 for the derivation.

Module summary

1. Defining the log transform
2. Prospecting for a log relationship
3. Fitting log models
4. Optimizing with log models
5. Four model possibilities and the interpretation of the slope in each one

Next time

- The Simple Regression Model (SRM) defined