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Innovative Applications of O.R.

Measuring competitive balance in sports leagues that award bonus points, with an application to rugby union

Antonio Avila-Cano [a], P. Dorian Owen [b,*], Francisco Triguero-Ruiz [c]

[a] Department of Economics and Economic History, University of Malaga, 29013 Málaga, Spain
[b] Department of Economics, University of Otago, Dunedin 9054, New Zealand
[c] Department of Languages and Computer Science, University of Malaga, 29071 Málaga, Spain

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ABSTRACT

We examine within-season competitive balance measurement in sports leagues where points allocated in each match depend on bonus points awarded as well as the win/draw/loss outcome, as in rugby union. The bilateral nature of matches imposes constraints on the range of competitive balance measures, as teams cannot win matches in which they do not play. Application of normalized competitive balance indices requires knowing the maximum value the index can attain, representing the most unequal distribution of points among teams. However, evaluation of the points distribution consistent with maximum imbalance is non-trivial when the total points allocated are not known in advance, as when bonus points are awarded. We theoretically derive the corresponding perfectly unbalanced distribution of points, characterized as a 'Double Cascade distribution', and provide an Excel tool to obtain it for different parameter values. We apply our results to examine variation in within-season competitive balance in the Six Nations Championship in rugby. The degree of competitive balance, as measured by a normalized concentration index, has improved since the award of bonus points in 2017, primarily because the index's maximum value increases with bonus points. However, with limited available data on the bonus-points regime, it is not possible to establish or reject a causal link, due to induced behavioural changes by teams, between the raw concentration index and bonus points. A relatively high level of imbalance still exists compared to other sports leagues, which will be an important consideration in evaluating future developments of the competition.

1. Introduction

The design of sporting contests, especially professional team sports leagues, has attracted considerable attention in the literature in several disciplines, including economics, operational research and statistics. Szymanski (2003, p.1137) observes that "[d]esigning an optimal contest is both a matter of significant financial concern for the organizers, participating individuals, and teams, and a matter of consuming personal interest for millions of fans". Sports administrators make decisions on multiple dimensions of design and structure, including the rules of the game (Wright, 2014; Kendall & Lenten, 2017), within-match scoring systems (Hogan & Massey, 2017; Scarf et al., 2019), points allocations for different match outcomes (Winchester, 2008; Moschini, 2010; Lenten & Winchester, 2015), league or tournament formats (Scarf et al., 2009; Chater et al., 2021), and scheduling of matches (Rasmussen & Trick, 2008; Ribeiro, 2012; Krumer & Lechner, 2017).[1]

A critical consideration in the design of sports leagues is the nature and extent of competition. This involves a form of cooperative competition (or 'coopetition') to produce a product that is attractive to sports fans. The joint efforts of two teams are required to produce a contest in a single match (Neale, 1964), and multiple teams jointly produce the overall league competition. How evenly these teams are matched in terms of strength or quality defines the degree of competitive balance in individual matches and in the league as a whole. The distribution of strengths and abilities across teams, combined with the inherent random variation involved in sporting competitions, in turn, determines the uncertainty of out-

* Corresponding author.
E-mail addresses: aavila@uma.es (A. Avila-Cano), dorian.owen@otago.ac.nz (P.D. Owen), triguero@uma.es (F. Triguero-Ruiz).

[1] In addition, teams have adopted sports analytics as a means to gain a competitive advantage over rivals, for example, by informing decisions about strategy and measurement of player performance (Oberstone, 2009; Szczepański & McHale, 2016; Kharrat et al., 2020; Craig & Winchester, 2021; McHale & Holmes, 2023).

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come of individual matches and, ultimately, of the end-of-season ranking of teams and the league championship. Matches between teams with very different strengths are more likely to result in wins for the stronger teams, especially in sports with high scoring rates (Scarf et al., 2019, 2022; Baker et al., 2021). Similarly, if there are only a few strong teams in a league, the championship outcome is more predictable than if several teams are of roughly equal strength.

Outcome uncertainty has long been regarded as a central feature of league design due to concerns that if match and championship outcomes are more predictable (less uncertain), this will erode sports fans' interest. Indeed, Rottenberg (1956, p.258) goes so far as to argue that sports leagues are "industries … in which all firms [teams] must be nearly equal if each is to prosper". Concerns about inadequate levels of competitive balance have been widely used to motivate a range of restrictive practices, such as financial 'fair play' rules, player drafts, salary caps, revenue sharing, etc.

Fort and Maxcy (2003) characterize the competitive balance literature as made up of two distinct strands. One strand, 'analysis of competitive balance', is involved with tracking movements in within-season competitive balance over seasons and examining the effects of regulatory changes. This literature primarily uses ex post balance measures based on actual outcomes. The other strand of literature focuses on the implications of varying competitive balance on fan demand for the sporting product, with an emphasis on uncertainty of outcome. Uncertainty of outcome metrics are ideally forward looking although, in practice, backward-looking measures are often used, especially if taking a longer-run perspective (Owen, 2014).

Given its pivotal importance in the analysis of sports leagues, appropriate measurement of competitive balance is, therefore, crucial. However, a common problem with standard measures of dispersion, concentration and inequality applied to teams' wins or points in a season is that they do not fully take into account the constraints imposed on the measures by the design characteristics of the league (Owen, 2014). These include the number of teams and matches, the structure of match schedules (for example, whether the schedules are balanced or unbalanced), and the points allocations for different outcomes; all of these can vary across different sports and leagues, making comparisons difficult. Crucially, such characteristics affect the feasible range of values of standard dispersion measures (Depken, 1999; Utt & Fort, 2002; Lenten, 2008; Owen, 2010; Gayant & Le Pape, 2015).

The effect on different competitive balance measures of varying numbers of teams and matches is now well documented (Owen & King, 2015; Lee et al., 2019; Doria & Nalebuff, 2021; Owen & Owen, 2022) but relatively little attention has been paid to the implications of points allocation systems for competitive balance measurement (Avila-Cano et al., 2021).[2] Most analyses of within-season competitive balance measurement focus on the distribution of win ratios or of points determined by win/draw/loss outcomes. However, in some sports, such as rugby union, points can be awarded based on other match outcomes in addition to the overall win/draw/loss result, for example, bonus points for narrow losses or scoring four or more tries. In general, this implies that the total number of points distributed for the league as a whole is not known in advance, which complicates the measurement of competitive balance.

In this paper, for the first time, we consider appropriate measurement of within-season competitive balance in situations where the number of points allocated in each match can vary depending on whether additional criteria for awarding points, specifically bonus points, are met. Our primary measure of within-season competitive balance is the Herfindahl-Hirschman index (*HHI*), which is based on teams' shares of the total points awarded. This measure has been used in a wide range of different sports, especially those in which draws (ties) are feasible; see Owen and Owen (2022) for an extensive but not exhaustive list of studies. For ease of comparison across leagues and over time, a normalized version of the index is preferred. This takes into consideration changes in the index's maximum and minimum values due to changes in the number of teams or in the points allocation system (Hall & Tideman, 1967; Owen et al., 2007; Gayant & Le Pape, 2015; Triguero Ruiz & Avila-Cano, 2019; Avila-Cano et al., 2021; Owen & Owen, 2022). This requires evaluating *HHI* for the two extreme distributions of points outcomes, corresponding to perfect competitive balance and perfect competitive imbalance (the most unequal distribution of points). The former is easy to implement, i.e., equal shares of points for each team. If the points allocation system generates a fixed number of total points across all teams at the end of the (regular) season, then the perfectly unbalanced distribution of points is also relatively easily specified (Fort & Quirk, 1997; Horowitz, 1997; Utt & Fort, 2002). However, for a given number of teams, if the total number of points allocated is not known at the beginning of the championship, then the final distribution of points becomes 'unstable' (Gayant & Le Pape, 2015). The only unstable distribution so far considered in the literature corresponds to the case where draws (ties) are feasible and points for a draw are worth less than half a win (Avila-Cano et al., 2021). The novelty of our contribution in this paper lies in developing the normalized competitive balance measure for cases in which bonus points are allocated for match outcomes other than for win/draw/loss results. This is particularly relevant for rugby union where bonus points have become the norm in the points allocation systems adopted in most championships.

In Section 2, a basic model is outlined for a league with a scoring system without bonus points. We formalize the concept of a league championship based on: (i) the number of teams in the league, (ii) the points allocation system, and (iii) the number of times that two teams compete against each other. This allows us to characterize the most unequal distributions of points corresponding to perfect competitive imbalance in the simple cases that have so far been considered in the literature. In Section 3, the model is generalized to allow for the award of bonus points. We derive the measurement implications of bonus points for the distribution of points corresponding to perfect imbalance and, hence, for the normalized *HHI* index. We also provide an Excel tool that allows this to be implemented for different bonus points allocations, match outcome points, and league sizes. Section 4 considers the award of bonus points in rugby union and applies the analysis to the Six Nations Championship (SNC). Section 5 summarizes our conclusions.

2. Leagues with a points allocation system without bonuses

Sports leagues typically involve a competition in which each team plays against every other team in the league, usually either once or twice (home and away).[3] Matches between the teams are bilateral and each match generates a result: win, loss and, in some sports, a draw (tie). The two teams are awarded points on the basis of the outcome of the match according to a predetermined points allocation system that is maintained throughout the championship

[2] Some studies (Cain & Haddock, 2006; Fort, 2007; Owen, 2012) examine the implications of different possible points assignments for wins, draws, and losses for the widely used relative standard deviation measure of competitive balance; however, use of this measure is inadvisable because of its excessive sensitivity to season length (Owen & King, 2015; Lee et al., 2019).

[3] Competitions based on an elimination system, such as most association football cup competitions, or a conference format, where schedules are unbalanced and teams do not necessarily play matches against every opposition team, are excluded from our analysis.

competition, although the points system may change between seasons.

A championship competition, C, can be defined by the set of teams that compete in it, T; the points allocation or reward system for teams based on the results they achieve, R; and the scheduling system, L, the number of rounds of matches between the teams. Thus, $C=\{T;\ R;\ L\}$, where $T=\{2,\ \ldots,\ n\}$, $n \in \mathbb{N}$. The number of matches in the championship is $Ln(n-1)/2$; i.e., each of the n teams plays against all the other $(n-1)$ teams a fixed number of times, L, in a pure round-robin schedule of matches (Rasmussen & Trick, 2008). In association football leagues, a double round-robin (home and away) schedule ($L=2$) is common. In rugby union, both double and single round-robin schedules have been adopted (for example, the French Top 14 and (up to 2010) Super Rugby, respectively).

The points allocation system is $R=\{(P_w,\ P_d,\ P_l),\ \boldsymbol{b}\}$, where P_w, P_d and P_l are the points assigned to teams in each match for a win (w), draw (d) or loss (l), respectively. In turn, $\boldsymbol{b}$ represents a vector of possible additional points assigned for match outcomes other than the overall win/draw/loss result; for example, in several rugby union championships, one bonus point is awarded for scoring four or more tries (the tries bonus), and one bonus point for losing the match by seven or fewer points (the near-loss bonus). We consider that $P_w, P_d, P_l \in \mathbb{N}$,and, as an incentive-compatible requirement, that $P_w > P_d \geq P_l$. Usually, $P_l=0$ and $P_w \geq 2P_d$. In addition, the components of $\boldsymbol{b}$ will be non-negative, so that $\mathbf{b} = \{b_j\}_{j=1}^{m},\ b_j \geq 0$, where m is the number of different types of possible bonus. Note that the points allocation system is discrete and fixed for the duration of each season's competition.

The points total of each team at the end of the championship is the aggregation of the points awarded in each match. Thus, p_i, $i \in \{1,\ 2,\ \ldots,\ n\}$, is the points total accumulated by team i by the end of the championship, and the vector $\boldsymbol{p}=(p_1,\ p_2,\ \ldots,\ p_n)$ represents the final points allocations in the championship, with subscript i indicating the position of each team in the final league table.

Given the vector $\boldsymbol{p}$, we represent the final configuration of points in the championship, without bonuses, in terms of the vector of each participating team's share of the overall points allocated, as $\boldsymbol{s}=(s_1,\ s_2,\ \ldots,\ s_n)$, where $s_i = p_i / \sum_{i=1}^{n} p_i$. This vector belongs to the $(n-1)$-dimensional simplex:

$$S^{n-1} = \left\{ \boldsymbol{s}=(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n,\ \text{ for every } i\ 1 \geq s_i \geq 0,\ \sum_{i=1}^{n} s_i = 1 \right\}$$

Note that the admissible solutions make up a discrete subspace included in the simplex.

If the points allocation system, R, is such that at the end of the championship the number of points, $\sum_{i=1}^{n} p_i$, is independent of the results of the matches, and therefore constant, then the points allocation system is said to satisfy the *stability condition* (Gayant & Le Pape, 2015). This condition is met in a scoring system without bonuses, $R=\{(P_w,\ P_d,\ P_l),\ \mathbf{0}\}$, if:

$$\begin{aligned} &\text{(i)}\ \ \text{there is no possibility of draws}: R = \{(P_w, -, P_l), \mathbf{0}\};\ \text{ or} \\ &\text{(ii)}\ \ P_d = (P_w + P_l)/2 \end{aligned} \tag{1}$$

The measure of competitive balance that we focus on is the Herfindahl-Hirschman concentration index (*HHI*), which, in its simplest form, is defined as the sum of the squares of teams' shares of total points allocated in the league (Depken, 1999), i.e., $HHI = \sum_{i=1}^{n} s_i^2$. We use a normalized version of *HHI*, which allows for changes in both the minimum (HHI_{min}) and maximum (HHI_{max}) values of *HHI* due to changes in the number of teams and/or the points allocation system (Hall & Tideman, 1967; Owen et al., 2007; Gayant & Le Pape, 2015; Triguero Ruiz & Avila-Cano, 2019; Avila-Cano et al., 2021; Owen & Owen, 2022). The normalized index, $HHI_n=(HHI - HHI_{min})/(HHI_{max} - HHI_{min})$, lies in the interval [0, 1], with zero representing perfect balance and one representing maximum imbalance. Other things equal, an increase in the concentration of points among the stronger teams in the competition, i.e., a decrease in competitive balance (increase in competitive imbalance), is reflected in increases in the value of *HHI* and HHI_n. Triguero Ruiz and Avila-Cano's (2019) 'Distance to Competitive Balance' (*DCB*) measure is equivalent to $\sqrt{HHI_n}$ and, being based on a distance function, satisfies the property of cardinality. Cardinality has the practical implication that the *DCB* index is scale-preserving and allows differences between any two values of the index to be interpreted in terms of proportions (or percentages); see Triguero Ruiz and Avila-Cano (2019, section III) for a more detailed discussion.

2.1. The perfect balance distribution and minimum value of HHI

The minimum concentration of points among the teams corresponds to a situation of maximum competitive balance (minimum imbalance). For an ex post outcome index such as *HHI*, this means all the teams have the same points total and, hence, equal shares of total points (Depken, 1999). This maximum competitive balance outcome corresponds to the barycentre of the simplex: $\boldsymbol{s^{max}}=(1/n,\ \ldots,\ 1/n)$. This implies that *HHI* reaches its perfect-balance minimum value at $1/n$, since $HHI_{min}(\boldsymbol{s}) = HHI(\boldsymbol{s^{max}}) = \sum_{i=1}^{n} (s_i^{max})^2 = 1/n$. Depken proposes an *HHI* measure that adjusts for its minimum value: $dHHI=(HHI-1/n)$, i.e. the deviation of *HHI* from its perfect-balance minimum value. This adjustment implies $dHHI=0$ for perfect balance and increased imbalance is represented by increases in the value of *dHHI*.

2.2. The most unequal distribution and maximum value of HHI

A key implication of the bilateral nature of matches in most sports leagues is that (for $n > 2$) no team in the league can accumulate all the points awarded in the championship; teams cannot accrue points from matches in which they do not play. Consequently, there is an upper bound on the degree of imbalance, the minimum of competitive balance, represented by the distribution of points furthest from that corresponding to perfect balance. We denote this most unequal distribution $\boldsymbol{s^{min}}$, and associated with it is a maximum value of *HHI*, i.e., $HHI_{max}(\boldsymbol{s}) = HHI(\boldsymbol{s^{min}}) = \sum_{i=1}^{n} (s_i^{min})^2$. The form of $\boldsymbol{s^{min}}$ and the corresponding $HHI_{max}(\boldsymbol{s})$ depend on the nature of the points allocation system.

2.2.1. The Complete Cascade distribution

The 'most unequal' distribution is usually characterized as a situation in which the strongest team wins all its matches, the second strongest team wins all except its match(es) against the first team, and so on down to the weakest team, which loses all its matches (Fort & Quirk, 1997; Horowitz, 1997). This distribution has been widely used as a benchmark to identify the most unbalanced configuration of outcomes (Utt & Fort, 2002; Larsen et al., 2006; Owen et al., 2007; Gayant & Le Pape, 2012, 2015). Formally, the points awarded to the teams follow the pattern:

$$p_i = LP_w(n-i) \text{ for each } i \tag{2}$$

If the teams are ordered by their end-of-season positions, from strongest to weakest, then, graphically, the distribution of points represents a 'cascade', in which each team has fewer points than the team above them in the ranking. Avila-Cano et al. (2021) label this a '*Complete Cascade distribution*' (CCD). For points allocations such as $R=\{(2,\ 1,\ 0),\ \mathbf{0}\}$ or, if draws are not possible, $R=\{(1,\ –,\ 0),\ \mathbf{0}\}$, the total points allocated are fixed for given n and L. Thus, the denominator of *HHI*, equal to the (squared) total points in the league, is fixed in advance of any matches being played.

The number of matches in which a team plays restricts the maximum of matches it can win and therefore the maximum number of points it can accumulate (the '*Saturation Principle*'). When $\sum_{i=1}^{n} p_i$ is constant, Avila-Cano et al. (2021) show that the marginal effect on *HHI* of an increase in s_i is an increasing function of the points already earned by team *i*. Therefore, to obtain the maximum concentration (minimum competitive balance), each point should be assigned to the team with the highest quota of points, subject to the restrictions imposed by the bilateral competition. Owen et al. (2007) show that, for this most unequal distribution, the maximum value of *HHI*, $HHI_{max}(\boldsymbol{s}) = HHI(\boldsymbol{s}^{min}) = 2(2n-1)/[3n(n-1)]$, assuming a round-robin schedule of matches, regardless of the number of rounds, *L*. *L* is a cancelling multiplicative factor in both the numerator and denominator of HHI_{max}; therefore, without loss of generality, we assume $L = 1$ in the rest of our analysis.

However, although the CCD constitutes the perfectly unbalanced distribution if $P_d = (P_w + P_l)/2$, this is not the case when there is the possibility of draws and the points reward for draws is less than the average of points for wins and losses (Gayant & Le Pape, 2015). This is relevant for most association football leagues, for which $R = \{(3, 1, 0), \boldsymbol{0}\}$. In this case, two points are allocated (one point to each team) in the event of a draw, whereas, for a win, three points are awarded to the winner. The points distribution therefore becomes unstable, with the total points allocated in the league depending on the number of drawn matches. Some authors propose recalculating the points vector using a $\{(2, 1, 0), \boldsymbol{0}\}$ system "to measure competitive imbalance properly" even though this "does not conform to the real point award system and its corresponding incentive properties" (Gayant & Le Pape, 2015, p.110). However, Avila-Cano et al. (2021) show that recalculating the allocated points in this way is not neutral from either a cardinal or even an ordinal perspective.

2.2.2. The Truncated Cascade distribution

Avila-Cano et al. (2021) construct the perfectly unbalanced distribution when the scoring system is such that $P_d < (P_w + P_l)/2$, starting from the consideration that, conceptually, the minimum level of competitive balance coincides with the maximum concentration of points. They define the Euclidean distance from a point configuration, $\boldsymbol{s}$, to the barycentre, $\boldsymbol{s^{max}}$. Finding the configuration furthest away from the barycentre is equivalent to maximizing the distance function, $d(\boldsymbol{s}, \boldsymbol{s^{max}})$, subject to the implicit restrictions in a competition: (i) bilaterality; (ii) a discrete space of admissible solutions; and (iii) a fixed and discrete points allocation system. The maximization problem is:

$$\max_{\{\boldsymbol{s^{min}}\}} d\left(\boldsymbol{s},\ \boldsymbol{s^{max}}\right)$$

where

$$d(\boldsymbol{s},\ \boldsymbol{s^{max}}) = ||\boldsymbol{s} - \boldsymbol{s^{max}}|| = \sqrt{\sum_{i=1}^{n} \left(s_i - \tfrac{1}{n}\right)^2} = \sqrt{\left(\sum_{i=1}^{n} s_i^2\right) - \tfrac{1}{n}}.$$

As $HHI(\boldsymbol{s}) = \sum_{i=1}^{n} s_i^2$, given n, maximizing the distance to the barycentre requires finding the maximum value that the index can reach, $HHI_{max}(\boldsymbol{s})$. Avila-Cano et al. (2021) show that the marginal contribution to the maximum value of *HHI* increases with the number of points accumulated by a team; therefore, the solution requires wins to be accumulated by the top teams in the ranking, with the rest of the matches drawn. This problem is solved by: $\max_{\{q*\}} HHI(n,\ q)$, where q^* is an integer that indicates the position from which the cascade distribution of points is truncated. The teams that occupy the first q^* positions win all their matches except for those played against teams above them in the table (i.e., 'in cascade', as in Section 2.2.1); the teams occupying the remaining $(n - q^*)$ positions draw all their matches other than those they lose against the first q^* teams.

For each $0 \leq q \leq n-1$, with $n \geq 2$ and $q \in \mathbb{N}$, Avila-Cano et al. (2021) obtain a generalized distribution they call the '*Truncated Cascade distribution*' (TCD). The distribution obtained, $\boldsymbol{s^{min}}$, is defined formally in terms of the points accumulated by the teams as:

$$p_i = \begin{cases} P_w(n-i) \text{ for all } i = 1, \ldots, q^* \\ P_d(n - q^* - 1) \text{ for all } i = q^* + 1, \ldots, n \end{cases} \quad (3)$$

The truncation, q^*, arises from the relatively higher returns to winning matches. Since a draw produces fewer total points for the two teams than a win/loss, this distribution minimizes the number of points for teams at the bottom end of the league and maximizes the distance to the perfectly balanced distribution. If $q^* = n - 1$, we obtain the CCD in Section 2.2.1 as a special case, with $p_i = P_w(n - i)$ for all *i*, confirming the CCD as the appropriate perfectly unbalanced distribution for a $\{(2, 1, 0; \boldsymbol{0}\}$ points allocation system. However, the CCD is not the correct reference distribution if $P_d < (P_w + P_l)/2$.

3. Leagues with a points allocation system with bonuses

The TCD derived by Avila-Cano et al. (2021) is limited to leagues with a points allocation system that rewards each team with a fixed number of points determined by the overall result (win/draw/loss) achieved by the team in each match. It does not necessarily provide the perfectly unbalanced distribution in competitions in which the points awarded for the same overall result (win/draw/loss) are variable. This can occur if, for example, there are additional bonus points for try scoring, narrow losses, or, at the end of the championship, for a team that wins all its matches, as in the case of rugby union.

In this section, we therefore generalize the derivation of the most unbalanced distribution of points to allow for a broader set of points allocation systems, including bonus points. Thus, we extend the analysis to a wider range of sports that involve a series of bilateral matches. We assume there is a consistent implicit restriction that the reward for draws is unique. We distinguish between systems in which $P_d < (P_w + P_l)/2$ (e.g., $R = \{(3, 1, 0), \boldsymbol{b}\}$, and systems in which $P_d \geq (P_w + P_l)/2$ (e.g., $R = \{(3, 2, 0), \boldsymbol{b}\}$ or $\{(2, 1, 0), \boldsymbol{b}\}$.

Let $B_i \geq 0$ represent the bonus points allocated to team *i*. The maximum number of bonus points is determined by the specific rules of the sport. We assume (in section 3.2) when deriving the most concentrated points configuration that, for each *i*, $p_{i+1} + B_{i+1} < p_i + B_i$. This condition, which is imposed only on the configuration of points that generates the maximum value of *HHI*, prevents the award of bonus points for HHI_{max} from yielding total points that equal or exceed the total points earned by a team that wins a greater number of matches.

In a competition $C = \{T;\ R;\ L\}$ with the possibility of draws and bonuses, *HHI* will depend on the number of teams that win matches and the extent to which each team accrues points for bonuses, $B_i \geq 0$. Let $p_i^T = p_i + B_i$ be the total points, including bonus points, accumulated by team *i* by the end of the season. $TP_k^{(q)} = \sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i$ is the total of points earned by all teams at the end of the season, where *q* is the truncation value (the number of teams that win following a cascade pattern) and *k* the number of teams that obtain bonus points. $U_k^{(q)} = \sum_{i=1}^{k} \left(p_i^T\right)^2 + \sum_{i=k+1}^{n} p_i^2$ is the sum of the squares of the points earned by each team at the end of the season, and is the numerator of *HHI*. $D_k^{(q)} = \left(TP_k^{(q)}\right)^2$ is the square of the total points obtained by all teams at the end of the competition and is the denominator of *HHI*.

The points earned by team *i*, are:

$$p_i^T = p_i + B_i \text{ for all } i = 1, \ldots, n$$

$$p_i = \begin{cases} p_i^{(w)} = P_w(n-i) + P_l(i-1) \text{ for all } i = 1, \ldots, q \\ p_i^{(d)} = P_d(n-q-1) + P_l q \text{ for all } i = q+1, \ldots, n \end{cases} \tag{4}$$

Note that Eq. (4) generalizes Eqs. (2) and (3) to allow for cases where the payoff for losses is not zero. Substituting from the expressions in Eq. (4) into $HHI^{(q)} = \sum_{i=1}^{n} s_i^2 = U^{(q)}/D^{(q)}$ demonstrates that $HHI^{(q)}$ is a function of q and n (see Appendix A in the Supplementary Materials).

Before examining the effects of adding bonus points, we first consider Proposition 3.1, which characterizes the growth of total points in a championship without bonus points.

Proposition 3.1. Given $C=\{T; R; L\}$, let $TP(n, q; P_w, P_d, P_l) = \sum_{i=1}^{q} p_i^{(w)} + \sum_{i=q+1}^{n} p_i^{(d)}$, then $TP(n, q; P_w, P_d, P_l)$ is $\begin{cases} \text{decreasing in } q \text{ iff } P_w + P_l < 2P_d \\ \text{constant for any } q \text{ iff } P_w + P_l = 2P_d \\ \text{increasing in } q \text{ iff } P_w + P_l > 2P_d \end{cases}$

Proof. See Supplementary Materials, Appendix B.

The sequence of lemmas, propositions and theorems in the following sections allows us to characterize and calculate the maximum values of *HHI* and derive the corresponding points distributions that generate them. Here we provide a brief overview of this sequence; in addition, we present a summary in the Supplementary Materials (Table A1).

In a scoring system with bonus points, Lemma 3.1 provides a necessary and sufficient condition for the maximum value of *HHI* to increase as bonuses are added. Lemma 3.2 is a generalization of this result. Both are required to prove Theorem 3.1, a version of what we call the 'Saturation Principle'. This tells us that the maximum of *HHI* grows up to the availability of bonuses. Theorem 3.2 assures us that *HHI* has a unique minimum and maximum.

If the scoring system is such that $P_d < (P_w + P_l)/2$, we show, in Theorem 3.3, that the maximum value of *HHI* is obtained by adding bonuses to teams that win matches. In addition, we find that the level of truncation with bonuses does not exceed truncation without bonuses. For its proof we need Lemma 3.3, which defines a necessary and sufficient condition to identify when the introduction of bonuses increases the maximum of *HHI*.

If the scoring system is such that $P_d \geq (Pw + P_l)/2$, we show, in Proposition 3.2, that the truncation where the maximum *HHI* is reached with bonuses is the same as without bonuses, i.e., $q^* = n - 1$. Furthermore, we show that bonuses must be added to a number of teams smaller than q^*. We label the corresponding most-unbalanced distribution a Truncated Double Cascade distribution (TDCD).

In Theorem 3.4. we identify that, in such a bonus scoring system, the maximum *HHI* is reached when the marginal proportional increase in the numerator is less than the marginal proportional increase in the denominator of *HHI*. For its proof we need Lemma 3.4, which defines a necessary and sufficient condition.

3.1. Existence and uniqueness of the minimum and maximum values of HHI

For any truncation value, q, let $HHI_k^{(q)}$ be the maximum value of *HHI* with added bonuses up to team k; i.e., $B_i = 0$, for $i > k$. Thus:

$$HHI_{k+1}^{(q)} = \frac{\sum_{i=1}^{k+1} \left(p_i^T\right)^2 + \sum_{i=k+2}^{n} p_i^2}{\left(\sum_{i=1}^{k+1} p_i^T + \sum_{i=k+2}^{n} p_i\right)^2}$$
$$= \left(HHI_k^{(q)} + \frac{2p_{k+1}B_{k+1} + B_{k+1}^2}{\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right)^2} \right) \cdot \frac{\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right)^2}{\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i + B_{k+1}\right)^2}. \tag{5}$$

Remark 3.1. Note that, for $k=0$, the value $HHI_0^{(q)} = HHI^{(q)}$ corresponds, for each q, to the maximum value of *HHI* obtained without any bonus points.

Lemma 3.1 establishes a necessary and sufficient condition for the maximum value of *HHI* to increase as a result of awarding bonus points to the $(k+1)$th team.

Lemma 3.1. For $B_{k+1} > 0$, $HHI_{k+1}^{(q)} > HHI_k^{(q)}$ if and only if

$$HHI_k^{(q)} < \frac{2p_{k+1} + B_{k+1}}{2\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right) + B_{k+1}} \tag{6}$$

Proof. If the award of bonus points to team $k+1$, i.e., $B_{k+1} > 0$, generates a greater maximum *HHI* value, then $HHI_{k+1}^{(q)} > HHI_k^{(q)}$. Thus, from Eq. (5):

$$HHI_k^{(q)} < HHI_{k+1}^{(q)} = \left(HHI_k^{(q)} + \frac{2p_{k+1}B_{k+1} + B_{k+1}^2}{\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right)^2} \right) \cdot \frac{\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right)^2}{\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i + B_{k+1}\right)^2}$$

Rearranging, we obtain:

$$2p_{k+1}B_{k+1} + B_{k+1}^2 > HHI_k^{(q)} \left(\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i + B_{k+1} \right)^2 - \left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i \right)^2 \right)$$
$$= HHI_k^{(q)} \left(2B_{k+1} \left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i \right) + B_{k+1}^2 \right)$$

Thus, the inequality in (6) is a necessary condition for the awarding of bonus points to the $(k+1)$th team to lead to a higher maximum value of *HHI*.

In reverse, if the inequality in (6) is fulfilled, from the property of fractions

$$\frac{a+b}{c+d} > \frac{a}{c} \Leftrightarrow \frac{b}{d} > \frac{a}{c} \tag{7}$$

then:

$$HHI_{k+1}^{(q)} = \frac{\left[\sum_{i=1}^{k} \left(p_i^T\right)^2 + \sum_{i=k+1}^{n} p_i^2\right] + 2p_{k+1}B_{k+1} + B_{k+1}^2}{\left[\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right)^2\right] + 2B_{k+1}\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right) + B_{k+1}^2}$$
$$> \frac{\sum_{i=1}^{k} \left(p_i^T\right)^2 + \sum_{i=k+1}^{n} p_i^2}{\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right)^2} = HHI_k^{(q)} \tag{8}$$

Therefore, the inequality in (6) is also a sufficient condition. □

Remark 3.2. Note that if $B_1 > 0$, by the inequality in (8), for $k=0$, $HHI_1^{(q)} > HHI_0^{(q)}$. Consequently, by introducing bonus points, the degree of concentration that can potentially be achieved is greater than for the otherwise equivalent competition without bonus points.

Lemma 3.2 serves to verify that the expression used in Lemma 3.1 (the right-hand side of the inequality in (6)) is an increasing function of the points obtained by bonuses; this helps demonstrate that an increase in the maximum value of HHI is guaranteed with the introduction of bonus points.

Lemma 3.2. Let f_j be a function, defined for each $j \leq k$ as $f_j(m) = \frac{2p_j + m}{2(\sum_{i=1}^{j-1} p_i^T + \sum_{i=j}^{n} p_i) + m}$, $m \in \mathbb{Z}_+$. Then $f_j(m)$ is increasing in m.

Proof. From the construction of f_j, $f_j(m) < 1$, for any value of $z \in \mathbb{Z}_+$. From (7):

$$f_j(m+z) = \frac{2p_j + m + z}{2\left(\sum_{i=1}^{j-1} p_i^T + \sum_{i=j}^{n} p_i\right) + m + z} > \frac{2p_j + m}{2\left(\sum_{i=1}^{j-1} p_i^T + \sum_{i=j}^{n} p_i\right) + m} = f_j(m)$$

Hence, function f_j is increasing in m for each j. □

Theorem 3.1 (Saturation Principle)**.** If a team j satisfies the condition of Lemma 3.1, then the associated maximum value of HHI increases as a result of awarding bonus points until the availability of bonuses for that team is saturated.

Proof. By Lemma 3.1, for $B_j = 1$, the condition for an increase in the maximum of HHI is that $HHI_{j-1}^{(q)} < f_j(1)$. Once the condition for the first bonus has been satisfied, it will continue to satisfy the condition for Lemma 3.2, with the addition of further bonus points for team j limited only by the maximum number of possible bonus points for that team. □

In Theorem 3.2, we show that there are unique values for the maximum and minimum of HHI in each competition.

Theorem 3.2. HHI has a unique minimum and maximum for each competition, $C = \{T;\ R;\ L\}$.

Proof. HHI is defined in a discrete subspace of S^{n-1}, so it has at least a maximum and a minimum. Due to its construction, the numerator and the denominator of HHI increase or decrease together. Therefore, except for different points allocation systems that are proportional, there cannot be more than one maximum or minimum configuration; hence, the maximum and minimum values are unique. □

3.2. The truncation value associated with HHI_{max}

In the following Lemma 3.3, we establish a new, more easily calculable, necessary and sufficient condition to identify when the introduction of bonuses decreases competitive balance. This condition is based on the 'pseudo-share' value of each team.

Lemma 3.3. Given $C = \{T;\ R;\ L\}$, with $P_d < (P_w + P_l)/2$, and for every $B_{k+1} > 0$, such that $p_k + B_k > p_{k+1} + B_{k+1}$, then $HHI_{k+1}^{(q^*)} > HHI_k^{(q^*)}$ if and only if

$$HHI_k^{(q^*)} < \frac{p_{k+1}^{(w)}}{\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i}$$

where $p_{k+1}^{(w)}$ represents the points if team $k+1$ wins its matches against teams $k+2$ up to n.

Proof. If $HHI_{k+1}^{(q^*)} > HHI_k^{(q^*)}$ by Lemma 3.1, for every $B_k > 0$

$$HHI_k^{(q^*)} < \frac{2p_{k+1}^{(w)} + B_{k+1}}{2\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right) + B_{k+1}}$$

$$\Rightarrow HHI_k^{(q^*)} \leq \operatorname*{Infimum}_{B_k \to 0}\left(\frac{2p_{k+1}^{(w)} + B_{k+1}}{2\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right) + B_{k+1}}\right) = \frac{p_{k+1}^{(w)}}{\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i}$$

In reverse, if $HHI_k^{(q^*)}$ fulfills

$$HHI_k^{(q^*)} < \frac{p_{k+1}^{(w)}}{\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i}$$

then, by (7),

$$HHI_k^{(q^*)} < \frac{p_{k+1}^{(w)}}{\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i} < \frac{2p_{k+1}^{(w)} + B_{k+1}}{2\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right) + B_{k+1}}$$

and therefore, by Lemma 3.1, $HHI_{k+1}^{(q^*)} > HHI_k^{(q^*)}$. □

Remark 3.3. Note that, for a given q, the introduction of bonuses increases the maximum value of HHI only if awarded to winning teams, i.e., for $0 < k \leq q$.

Remark 3.4. Note that to ensure that the introduction of bonuses does not generate any undesirable ordinal effects for the points configuration for HHI_{max}, we introduce a bound on bonuses such that: $p_k + B_k > p_{k+1} + B_{k+1}$. So, for each k: $B_k < p_1^T - p_k = B_1 + p_1 - p_k$. This does not preclude this condition being violated for observed points distributions for values of $HHI < HHI_{max}$ (as indeed has occurred in some competitions, e.g., Super Rugby in each season between 2005 and 2009).

Remark 3.5. Note that for $k \leq q$, $HHI_k^{(q)} > \frac{p_{q+1}^{(d)}}{\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{q} p_i + \sum_{i=q+1}^{n} p_i}$ because

$$HHI_k^{(q)} - \frac{p_{q+1}^{(d)}}{\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i} = \frac{\left(\sum_{i=1}^{k}\left(\left(p_i^T\right)^2 - p_i^T p_{q+1}^{(d)}\right) + \sum_{i=k+1}^{q}\left(\left(p_i^{(w)}\right)^2 - p_i^{(w)} p_{q+1}^{(d)}\right) + \sum_{i=q+1}^{n}\left(\left(p_{q+1}^{(d)}\right)^2 - p_{q+1}^{(d)} p_{q+1}^{(d)}\right)\right)}{\left(\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i\right)^2}$$

As $\sum_{i=q+1}^{n}((p_{q+1}^{(d)})^2 - p_{q+1}^{(d)} p_{q+1}^{(d)}) = 0$, $p_{q+1}^{(d)} < p_i^{(w)}$ for i = k+1 to q, and $p_{q+1}^{(d)} < p_i^T$ for i = 1 to k, then $HHI_k^{(q)} - \frac{p_{q+1}^{(d)}}{\sum_{i=1}^{k} p_i^T + \sum_{i=k+1}^{n} p_i} > 0$.

To obtain $\boldsymbol{s^{min}}$, it is necessary to identify the number of teams that obtain bonus points. The conditions enabling a team to win bonus points are determined by the competition regulations.

Theorem 3.3. Given $C = \{T;\ R;\ L\}$, with $P_d < (P_w + P_l)/2$, and $p_k + B_k > p_{k+1} + B_{k+1}$ then:

(i) If q^* is the truncation of winning teams, then the maximum value of HHI is reached at $HHI_{q^*}^{(q^*)}$.
(ii) The truncation with bonuses will be less than or equal to the truncation without bonuses.

Proof. i.a) From Remark 3.3, the maximum value of HHI increases only when adding bonuses to the winning teams. If $HHI_{q^*}^{(q^*)} > HHI_{q^*+1}^{(q^*)}$, by Theorem 3.2, when adding bonuses to teams q^*+1 up to n, i.e., for $k > q^*$, $HHI_k^{(q^*)} < HHI_{q^*}^{(q^*)}$.

i.b) On the other hand, by Lemma 3.3, if $k < q^*$, then the maximum value of HHI increases by adding bonuses up to q^*.

Therefore, by i.a) and i.b), $HHI_k^{(q^*)} < HHI_{q^*}^{(q^*)}$, for $k < q^*$ and $k > q^*$. That is, the truncation value of the distribution that generates the minimum competitive balance coincides with the number of teams to which bonuses are added, i.e., $q^* = k^*$.

(ii) See Supplementary Materials, Appendix C. □

Theorem 3.3 defines the corresponding $\boldsymbol{s^{min}}$, which we call the '*Truncated Cascade* distribution' (TCD). Again, if the teams are ordered by their end-of-season positions, from strongest to weakest, then, graphically, the distribution of points represents a 'cascade', which is 'truncated' in k^*. It is a variant of a TCD in which k^* is smaller, i.e., fewer teams accumulate more points than in the non-bonus system.

Given the instability generated by the introduction of bonus points, we need to find the truncation value that generates the highest value of *HHI*. If $P_d < (P_w + P_l)/2$, the initial value for q is 0. Adding the possible bonuses for each team, we find the truncation for the value from which *HHI* decreases. In Proposition 3.2, we show that, if $P_d \geq (P_w + P_l)/2$, the truncation value is $n-1$, obtaining a maximum with the introduction of bonuses for a value of k less than n.

Proposition 3.2. Given $C = \{T;\ R;\ L\}$, where $R = \{(P_w,\ P_d,\ P_l),\ \boldsymbol{b}\}$ and $P_d \geq (P_w + P_l)/2$, then $q^* = n - 1$.

Proof. See Supplementary Materials, Appendix D. □

Remark 3.6. Note that, for $\boldsymbol{b} = \boldsymbol{0}$, Proposition 3.2 is also applicable to a competition without bonuses.

Proposition 3.2 establishes the corresponding $\boldsymbol{s^{min}}$, which we call the '*Truncated Double Cascade* distribution' (TDCD). If the teams are ordered by their end-of-season positions, from strongest to weakest, then, graphically, the distribution of points represents two 'cascades', and they are 'truncated' in k^*.

With the results of this section, we can specify a simple procedure to obtain the exact truncation that generates HHI_{max} with bonuses for any scoring pattern:

```
Given n and scoring pattern {Pw, Pd, Pl}, with Pw > Pd > Pl
Input(n; Pw, Pd, Pl)
Declare q; k; HHInew; HHIback; qmax; kmax
If Pd < (Pw + Pl)/2,
  q = 0
  HHInew = 1/n
  Repeat
    q = q + 1
    HHIback = HHInew
    Calculate HHInew with bonuses up to team q
  until HHInew < HHIback
  qmax = q − 1
  kmax = qmax
else
  qmax = n − 1
  k = 0
  HHInew = n*(n − 1)/2
  Repeat
    k = k + 1
    HHIback = HHInew
    Calculate HHInew with bonuses up to team k
  until HHInew < HHIback
  kmax = k − 1
end if
Output(Truncation #qmax; bonuses #kmax)
```

Additionally, a Microsoft Excel tool has been developed to obtain the configuration of points that yields the maximum *HHI* value, either with or without the awarding of bonuses. This tool, which integrates the results obtained in Avila-Cano et al. (2021) and the results in this paper, is available as Supplementary Material.

3.3. Obtaining $\boldsymbol{s^{min}}$ and HHI_{max}

Based on the results in Section 3.2, we are able to calculate $\boldsymbol{s^{min}}$.

Lemma 3.4. Given $C = \{T;\ R;\ L\}$, $HHI_k^{(q^*)}$ is the maximum *HHI* if and only if: $\frac{HHI_{k^*+1}^{(q^*)}}{HHI_{k^*}^{(q^*)}} < 1$.

Proof. This follows directly from the existence and uniqueness of the maximum of *HHI* (Theorem 3.2) and the Saturation Principle (Theorem 3.1). □

In Theorem 3.4, we establish the condition to obtain the perfectly unbalanced distribution of points.

Theorem 3.4. The maximum value of *HHI* is reached for the value $k^* = q^*$ that satisfies:

$$\frac{U_{k^*+1}^{(k^*)} - U_{k^*}^{(k^*)}}{U_{k^*}^{(k^*)}} < \frac{D_{k^*+1}^{(k^*)} - D_{k^*}^{(k^*)}}{D_{k^*}^{(k^*)}} \tag{9}$$

Proof.

$$\frac{HHI_{k^*+1}^{(q^*)}}{HHI_{k^*}^{(q^*)}} = \frac{\frac{U_{k^*}^{(k^*)} + \left(U_{k^*+1}^{(k^*)} - U_{k^*}^{(k^*)}\right)}{D_{k^*}^{(k^*)} + \left(D_{k^*+1}^{(k^*)} - D_{k^*}^{(k^*)}\right)}}{\frac{U_{k^*}^{(k^*)}}{D_{k^*}^{(k^*)}}} = \frac{1 + \frac{U_{k^*+1}^{(k^*)} - U_{k^*}^{(k^*)}}{U_{k^*}^{(k^*)}}}{1 + \frac{D_{k^*+1}^{(k^*)} - D_{k^*}^{(k^*)}}{D_{k^*}^{(k^*)}}}$$

From Lemma 3.4:

$$\frac{HHI_{k^*+1}^{(q^*)}}{HHI_{k^*}^{(q^*)}} < 1$$

Thus, condition (9) is satisfied. □

In summary, for any number of teams in a bilateral competition, we have characterized HHI_{max} in a points allocation system with bonuses and established an exact solution procedure for obtaining the maximum value of *HHI*. To execute this recurring procedure, we add available bonuses for each team until the condition to increase the maximum value of *HHI* does not hold.

This allows us to categorize the perfectly unbalanced distributions corresponding to different points allocation systems (see Table 1). If there are no bonus points and draws are either not feasible or rewarded with the average of the points attributed to a win and a loss, then the most unequal points distribution is a CCD, as summarized in Section 2.2; this type of distribution is relevant for sports such as association football under the previous (2,1,0) points system. With no bonus points, but an unstable points system in which draws are rewarded with less than the average of the points attributed to a win and a loss, the maximum value of *HHI* is generated by a TCD, as summarized in Section 2.2.2. This is the relevant most unequal distribution for most association football leagues, which implement a {3, 1, 0), 0} points system. With possible bonus points and draws worth the average of a win and a loss, the relevant distribution of points is a TDCD. This applies to most current rugby union competitions. For the final combination, with bonus points and draws worth less than the average of a win and a loss, the maximum value of *HHI* corresponds to a TCD; we are unaware of any sport that has applied this scheme to date. The different types of distributions are illustrated graphically (for $n = 20$) in Figs. 1 to 4. Note that the bonus points allocated include a 3-points bonus for the first team in the ranking (Figs. 3 and 4) to correspond to the 'Grand Slam' bonus, which is available in the SNC, as discussed in Section 4.

The CCD (Fig. 1) is the most unbalanced distribution when the scoring system, without bonus points, satisfies the stability condition in Eq. (1). The TCD (Fig. 2) corresponds to the case in which

Table 1
Classification of perfectly unbalanced distributions depending on the points allocation system.

	No draws or $P_d = (P_w + P_l)/2$	Draws and $P_d < (P_w + P_l)/2$
Without bonus points ($\boldsymbol{b} = 0$)	(i) Complete Cascade (CCD) – Fig. 1	(ii) Truncated Cascade (TCD) – Fig. 2
With bonus points ($\boldsymbol{b} > 0$)	(iii) Truncated Double Cascade (TDCD) – Fig. 3	(iv) Truncated Cascade (TCD) – Fig. 4

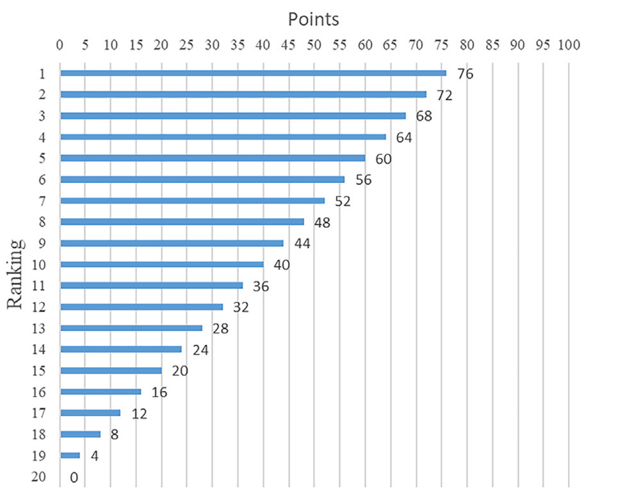

Fig. 1. Complete Cascade distribution with $n = 20$, $R = \{(4, 2, 0), (0, 0, 0)\}$, $L = 1$ ($q^* = 19$, $k^* = 0$, and $HHI_{max} = 0.068421$).

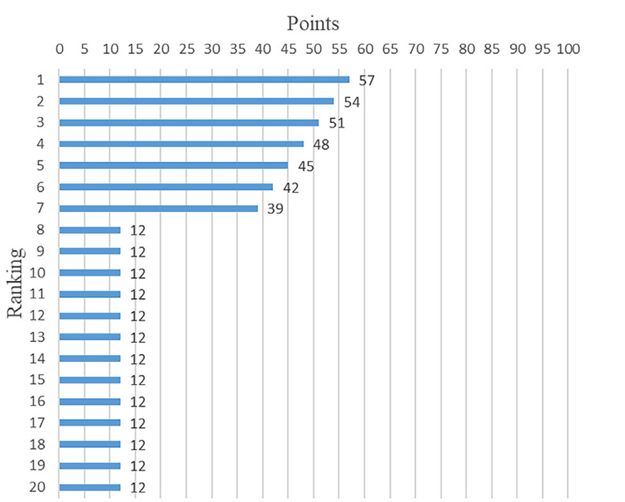

Fig. 2. Truncated Cascade distribution with $n = 20$, $R = \{(3, 1, 0), (0, 0, 0)\}$, $L = 1$ ($q^* = 7$, $k^* = 0$, and $HHI_{max} = 0.075402$).

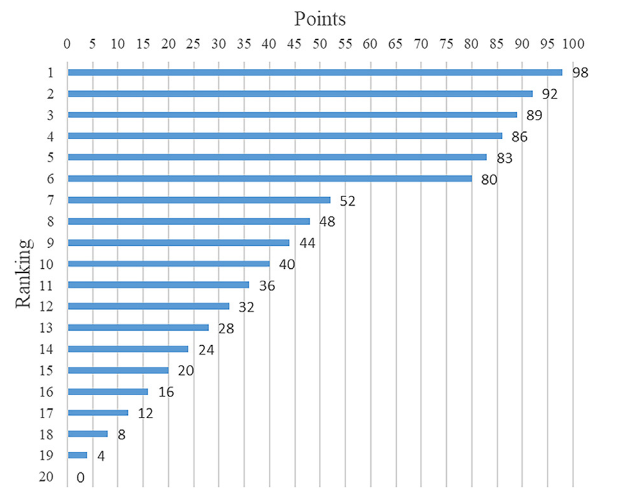

Fig. 3. Truncated Double Cascade distribution with $n = 20$, $R = \{(4, 2, 0), (3, 1, 1)\}$, $L = 1$ ($q^* = 19$, $k^* = 6$, and $HHI_{max} = 0.075130$).

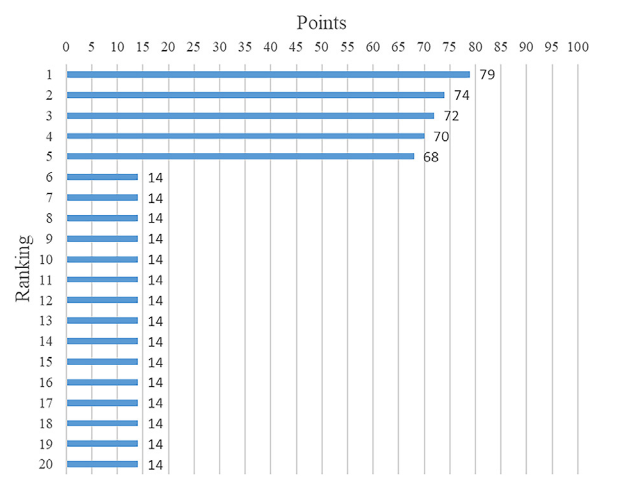

Fig. 4. Truncated Cascade distribution with $n = 20$, $R = \{(3, 1, 0), (3, 1, 1)\}$, $L = 1$. ($q^* = 5$, $k^* = 5$, and $HHI_{max} = 0.089438$).

Eq. (1) is not satisfied. The TDCD (Fig. 3) generalizes the CCD in response to instability due to the introduction of bonuses. In the case of the TCD (Fig. 4), both sources of instability are combined, hence generalizing the TCD.

In all cases, instability arises when the points allocation rules are altered, either by rewarding draws with fewer points than the average of a win and a loss, or by introducing bonuses that additionally reward more attractive play by teams. The introduction of any of these modifications in the incentive mechanism generates a greater level of maximum concentration. In particular, the maximum values of *HHI* that can be achieved in a league with 20 teams, are similar when either of the two modifications that introduce instability is introduced. By introducing both modifications together, the maximum value of *HHI* increases considerably.

Maximum concentration of allocated points is represented by “the control of a few” (Hirschman, 1945, p.158). How many teams control the competition in the perfectly unbalanced distribution? In the case of a CCD, there is no dominant block of teams, whereas (for $n = 20$) the number of teams that constitute a dominant block decreases successively for the TCD without bonus points (7 teams), TDCD (6 teams) and TCD with bonus points (5 teams).

4. The case of rugby union

As we have seen, deriving the perfectly unbalanced distribution, $\boldsymbol{s^{min}}$, is complicated by the instability generated in competitions where the sum of the points allocated by the end of the championship is not constant. In this section, we focus on rugby union, in which the source of instability is the awarding of bonus points. First, we consider the nature of bonus points systems in rugby union and discuss existing literature on the effects of their introduction. Next, we consider the characterization of HHI_k and $\boldsymbol{s^{min}}$ for generic rugby competitions with bonus points. We then apply these results to measure HHI_n and DCB for the SNC (which is contested by the men's national teams from England, France, Ireland, Italy, Scotland and Wales). In line with the 'analysis of competitive balance' strand of the competitive balance literature, we track changes in these normalized measures before and after the introduction of bonus points in this competition.

4.1. Bonus points in rugby union competitions

Different bonus points systems have been adopted in the different major rugby competitions. The most common system awards four points for a win, two points for a draw, one point for scoring four or more tries, and one point for losing by seven or fewer points (seven points being the value of a converted try). This system was adopted in the inaugural Super Rugby competition, a Southern Hemisphere competition involving regional teams from Australia, New Zealand and South Africa, in 1996 (until 2015), followed by the Rugby World Cup in 2003. It was not until 2017 that bonus points were included in the SNC; they follow the standard system but with the addition of three bonus competition points for a 'Grand Slam' (one team beating all the other teams). This is to ensure that a team achieving a Grand Slam cannot be overtaken in the final rankings by a team with fewer wins accruing bonuses from try scoring or narrow losses.[4]

The introduction of bonus points in rugby and the details of the points system implemented have multiple effects that have been analysed in the literature in economics, operational research and statistics. Lenten and Winchester (2015) analyse the effects of the four-tries bonus in Super Rugby (2002 to 2012) in encouraging attacking play and more tries; they find evidence that significantly more tries are scored in the final stages of matches by teams that needed an additional try to gain the four-tries bonus point, but only when the overall result was in little doubt. Butler et al. (2020), based on data for group-stage matches in the European Rugby Cup from 1996/97 to 2013/14, also find moderate support for a positive incentive effect of a four-tries bonus on try-scoring outcomes, especially for home teams.

Winchester (2008, 2014), using a prediction model of score margins in rugby matches, examines the extent to which awarding bonus points helps more accurately rank teams according to their relative strengths, compared to awarding points purely for wins (and draws, which are relatively rare in rugby). He finds strong evidence that awarding a narrow-loss bonus point significantly improves predictive accuracy of score margins based on team strengths; the four-tries bonus is not significantly correlated with team strengths, but there is evidence of a significant correlation between team strengths and a net-tries bonus. These results were influential in the subsequent change to the bonus points system in Super Rugby and The Rugby Championship (Winchester, 2016). Fioravanti et al. (2021), in a game-theoretical model of effort expended by rugby teams, show results consistent with Winchester's findings: a three-net-tries bonus induces teams to exert more effort, compared to the four-tries bonus, which in turn dominates the no-bonus scenario.

From the perspective of ranking teams from strongest to weakest, bonus points that reward stronger teams, therefore, appear to be desirable. However, bonus points may also have implications for the outcome uncertainty of the competition. Scarf et al. (2019) simulate outcomes for the Rugby World Cup 2015 with scores for matches between teams generated by independent Poisson random variables. They show that increasing scoring rates, motivated by the desire to incentivize entertaining attacking play, may reduce the uncertainty of outcome of individual matches and hence of competition outcomes.[5] With high scoring rates, strong teams are more likely to beat weaker teams, and by larger points margins. To the extent that bonus points encourage higher scoring rates and accentuate the superiority of stronger teams, there may be a concern that inequality in playing strengths will further reduce uncertainty of outcome and increase measures of the concentration of points. We discuss this further in section 4.4, where we consider the different effects on HHI_n and DCB of (a) increasing HHI_{max} due to the introduction of bonus points and (b) variations in the raw HHI measure induced by changes in teams' behaviour in response to the bonus points.

4.2. Bonus points and characterization of HHI_k

In terms of the notation established earlier, a generic rugby union points allocation system can be represented as $R = \{(P_w, P_d, P_l), \mathbf{b}\}$, with $\boldsymbol{b} = \{GS, b_t, b_{nl}\}$ where GS denotes a Grand Slam bonus, b_t the number of points assigned for either a four-tries or net-tries bonus, and b_{nl} the number of points assigned to a narrow-loss bonus. Currently, most rugby championships set $R = \{(4, 2, 0), \mathbf{b}\}$, with $\boldsymbol{b} = \{0, 1, 1\}$, except for the SNC in which $GS = 3$. However, the results in this section apply for any points allocation value for each type of bonus (including 0 if no points are allocated to that type of bonus). The Excel tool provided allows anyone to obtain the results for HHI_{max}, q^* and k^* for different values of the elements of R.

Remark 4.1. Given that $P_w = 2P_d$, in the absence of bonus points a CCD is obtained for $\boldsymbol{s^{min}}$ $(q^* = n - 1)$.

Due to the Saturation Principle, $\boldsymbol{s^{min}}$ is obtained by assigning the maximum bonus to the first k teams in the ranking. So, for team i, the total points obtained would be:

$$p_i^T = P_w(n-i) + B_i \text{ for } i = 1, \dots, n \tag{10}$$

and the points that could be obtained for bonuses would be:

$$B_i = \begin{cases} GS + (n-1)b_t + (i-1)b_{nl} \text{ for } i = 1 \\ (n-1)b_t + (i-1)b_{nl} \text{ for } i > 1 \end{cases} \tag{11}$$

Note that Eqs. (10) and (11) correspond to the TDCD represented in Fig. 3.

From Proposition 3.1, $q^* = n - 1$. Therefore, hereafter, we will denote $HHI_k = HHI_{k^*}^{(n-1)}$. Analogously, $U_k = U_{k^*}^{(n-1)}$; $D_k = D_{k^*}^{(n-1)}$ and $TP_k = TP_{k^*}^{(n-1)}$. Focusing on the expression for the maximum value of HHI:

$$HHI_k = \frac{\sum_{i=1}^{k} [P_w(n-i) + B_i]^2 + \sum_{i=k+1}^{n} [P_w(n-i)]^2}{\left[\sum_{i=1}^{k} [P_w(n-i) + B_i] + \sum_{i=k+1}^{n} (P_w(n-i))\right]^2}$$

[4] Other variations to the bonus points allocations in rugby union include replacing the four-tries bonus with a net-tries bonus for scoring at last three more tries than the opposing team. This was introduced in 2007/08 by the French Professional League (Ligue Nationale de Rugby) and replaced the four-tries bonus in Super Rugby and the Rugby Championship (an annual competition contested by the men's national teams from Argentina, Australia, New Zealand and South Africa) in 2016.

[5] Currently, in rugby union matches, each team's within-match score is based on accumulated points for the following scoring actions: five points for a try, two for a conversion, three for a penalty goal and three for a drop goal. Points scoring rates in individual rugby matches have increased over time, partly due to increases in points assignments for the different scoring actions, but also due to the number of scoring actions per match (Hogan & Massey, 2017; Scarf et al., 2019).

Remark 4.2. Note that for $k=0$ (no bonuses awarded to any team), the value HHI_0 corresponds to the HHI associated with $\boldsymbol{s^{min}}$ without introducing bonuses.

4.3. Obtaining $\boldsymbol{s^{min}}$

The $\boldsymbol{s^{min}}$ distribution is obtained from a TDC configuration. For the case in which the first k teams obtain bonus points, the corresponding values for the components of HHI_{max} are:

$$TP_k = \sqrt{D_k} = [P_w(n-1) + GS + (n-1)b_t] + \sum_{i=2}^{k}[P_w(n-i) + (n-1)b_t + (i-1)b_{nl}] + \sum_{i=k+1}^{n}[P_w(n-i)]$$

and

$$U_k = [P_w(n-1) + GS + (n-1)b_t]^2 + \sum_{i=2}^{k}[P_w(n-i) + (n-1)b_t + (i-1)b_{nl}]^2 + \sum_{i=k+1}^{n}[P_w(n-i)]^2$$

Remark 4.3. For $k=0$, $TP_0 = \sum_{i=1}^{n}[P_w(n-i)] = P_w n(n-1)/2$ and $U_0 = \sum_{i=1}^{n}[P_w(n-i)]^2$. Therefore, if no bonus points are introduced for any teams: $HHI_0 = 2(2n-1)/[3n(n-1)]$, which is the result obtained by Owen et al. (2007) referred to in Section 2.2.1.

For $k > 1$, $HHI_k = U_k/D_k$. Given the current regulations of the SNC, R={(4, 2, 0), $\boldsymbol{b}$}, with $\boldsymbol{b}$ = {3, 1, 1}, we obtain:

$$TP_k = \sqrt{D_k} = \sum_{i=1}^{n}[4(n-i)] + 3 + \sum_{i=1}^{k}[(n-1) + (i-1)] = 2n(n-1) + 3 + (n-1)k + \frac{(k-1)k}{2} = {}^{k^2}\!/_2 + \left[n - {}^3\!/_2\right]k + 2n^2 - 2n + 3$$

$$U_k = (4(n-1)+(n-1)+3)^2 + \sum_{i=2}^{k}[4(n-i)+(n-1)+(i-1)]^2 + \sum_{i=k+1}^{n}[4(n-i)]^2 = (5n-2)^2 + \sum_{i=2}^{k}[5n-2-3i]^2 + 16\sum_{i=k+1}^{n}[n-i]^2 = -{}^7\!/_3k^3 + \left(n + {}^5\!/_2\right)k^2 + \left[9n^2 - 19n + {}^{53}\!/_6\right]k + \frac{8n(n-1)(2n-1)}{3} + 3[10n-7]$$

$$D_k = \left({}^{k^2}\!/_2 + \left[n - {}^3\!/_2\right]k + 2n^2 - 2n + 3\right)^2 = {}^{k^4}\!/_4 + \left(n - {}^3\!/_2\right)k^3 + \left[3n^2 - 5n + {}^{21}\!/_4\right]k^2 + \left[2\left(n - {}^3\!/_2\right)\left(2n^2 - 2n + 3\right)\right]k + \left(2n^2 - 2n + 3\right)^2$$

The increases in the components of HHI that occur when introducing bonuses for the marginal (k+1)th team (compared to the values with bonuses up to the kth team) are:

$$\Delta_k TP = TP_{k+1} - TP_k = n + k - 1$$

$$\Delta_k U = U_{k+1} - U_k = -7k^2 + 2(n-1)k + 9(n-1)^2$$

$$\Delta_k D = D_{k+1} - D_k = 2TP_k B_{k+1} + B_{k+1}^2 = 2\left[{}^{k^2}\!/_2 + \left(n - {}^3\!/_2\right)k + 2n^2 - 2n + 3\right][(n-1) + (k+1-1)] + [(n-1) + (k+1-1)]^2 = k^3 + 3(n-1)k^2 + \left(6n^2 - 7n + 7\right)k + 4n^3 - 7n^2 + 8n - 5$$

For the case of the SNC, for $i=1$, we must add the Grand Slam (GS) 3 points bonus. So we obtain:

$$TP_0 = \sqrt{D_0} = P_w\frac{n(n-1)}{2} = 2n(n-1) \text{ and}$$
$$U_0 = \frac{16n(n-1)(2n-1)}{6}$$

$$\Delta_0 TP = n - 1 + GS = n + 2$$

$$\Delta_0 U = 9(n-1)^2 + 2GS[4(n-1) + (n-1)] = 9(n-1)^2 + 9 + 6[(n-1) + 4(n-1)] = 9(n-1)^2 + 30(n-1) + 9$$

$$\Delta_0 D = 4n^3 - 7n^2 + 8n - 5 + 2P_1 GS + GS^2 = 4n^3 - 7n^2 + 8n - 5 + 12n(n-1) + 9$$

The condition that the values associated with $\boldsymbol{s^{min}}$ must satisfy, according to Theorem 3.4, is:

$$\frac{\Delta_k U}{U_k} < \frac{\Delta_k D}{D_k}$$

where $\Delta_k U = U_{k*+1}^{(n-1)} - U_{k*}^{(n-1)}$ and $\Delta_k D = D_{k*+1}^{(n-1)} - D_{k*}^{(n-1)}$.

4.4. Application to the Six Nations Championship

Table 2 presents the implications of the SNC points system for the truncation values that parameterize the without-bonuses CCD (q^*) and the number of teams with bonuses (k^*) for hypothetical leagues with different numbers of teams. We consider different hypothetical values of n to illustrate that a given points allocation system can have varying effects on the components of HHI_n that depend on the size of the league, n.[6] For each n, we report the minimum and maximum values of HHI and its implied theoretical range, (HHI_{max} – HHI_{min}), for points allocation systems with and without bonus points. The ratio between the ranges with and without bonuses indicates the potential extent of overestimation of the concentration if the without-bonuses HHI_{max} value, $HHI_{max}{}^0$, is used for leagues with bonus points. For $n > 2$, the overestimation of maximum concentration when mistakenly using the no-bonuses HHI_{max} exceeds 35%. The relevant results for the SNC, with $n=6$, are presented in bold. Figs. 5 and 6 show the distributions generating the minimum level of competitive balance for the SNC without and with bonuses, respectively.

Table 3 contains the points totals obtained by each team in the 2015 to 2022 seasons of the SNC, according to the points allocation system current in each season. In addition, in the last six columns, the results for seasons using the current {(4, 2, 0); $\boldsymbol{b}$} points system are reconstructed using the previous {(2, 1, 0); $\boldsymbol{0}$} system.

The change of scoring system highlights the need to normalize the HHI index, since the maximum value that can be attained varies with each scoring system. In effect, the change in the points allocation system means that a CCD is not the most unbalanced distribution of results; higher values of the corresponding HHI can be found for a different distribution of results. The use of

[6] We consider values of n up to 14, historically a commonly observed competition size in several (domestic) rugby union competitions (e.g., the French Top 14, the English Gallagher Premiership, and Pro 14). This can be extended to larger n using the Excel tool provided as Supplementary Material.

Table 2
Implications of the Six Nations Championship bonus points allocation system for components of HHI_n as n varies.

n	HHI_{min}	q^*	HHI_{max} without bonuses ($HHI_{max}{}^0$)	k^*	HHI_{max} with bonuses ($HHI_{max}{}^w$)	Range without bonuses (R^0)	Range with bonuses (R^w)	R^w/R^0
2	0.50000	1	1.00000	1	1.00000	0.50000	0.50000	1.00000
3	0.33333	2	0.55556	1	0.64014	0.22222	0.30681	1.38062
4	0.25000	3	0.38889	1	0.44889	0.13889	0.19889	1.43200
5	0.20000	4	0.30000	1	0.34088	0.10000	0.14088	1.40878
6	**0.16667**	**5**	**0.24444**	**1**	**0.27336**	**0.07778**	**0.10669**	**1.37173**
7	0.14286	6	0.20635	2	0.22980	0.06349	0.08694	1.36935
8	0.12500	7	0.17857	2	0.19811	0.05357	0.07311	1.36465
9	0.11111	8	0.15741	2	0.17378	0.04630	0.06267	1.35366
10	0.10000	9	0.14074	3	0.15528	0.04074	0.05528	1.35693
11	0.09091	10	0.12727	3	0.14029	0.03636	0.04938	1.35796
12	0.08333	11	0.11616	3	0.12780	0.03283	0.04446	1.35446
13	0.07692	12	0.10684	4	0.11750	0.02991	0.04058	1.35645
14	0.07143	13	0.09890	4	0.10876	0.02747	0.03733	1.35891

Notes: $R = HHI_{max} - HHI_{min}$; superscripts ^{0} and ^{w} denote without and with bonuses, respectively; n is the number of teams, q^* the truncation value without bonuses, and k^* the number of teams with bonuses. The bolded entries for $n = 6$ apply to the SNC.

Table 3
The Six Nations Championship points totals, 2015-2022.

	Original points								Reconstructed points {(2, 1, 0); 0}					
	{(2, 1, 0), 0}		{(4, 2, 0), (3, 1, 1)}											
	2015	2016	2017	2018	2019	2020	2021	2022	2017	2018	2019	2020	2021	2022
England	8	10	19	10	18	18	10	10	8	4	7	8	4	4
France	4	4	14	11	10	18	16	25	6	4	4	8	6	10
Ireland	8	5	14	26	14	14	15	21	6	10	6	6	6	8
Italy	2	0	0	1	0	0	0	4	0	0	0	0	0	2
Scotland	0	4	14	13	9	14	15	10	6	6	3	6	6	4
Wales	8	7	10	15	23	8	20	7	4	6	10	2	8	2
Total	30	30	71	76	74	72	76	77	30	30	30	30	30	30

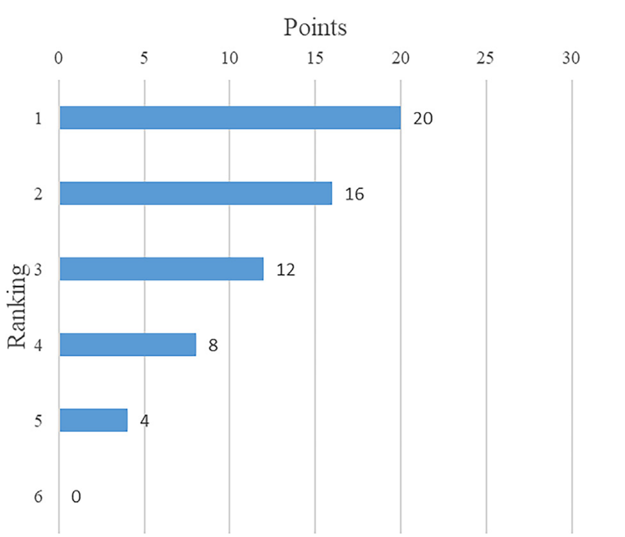

Fig. 5. Complete Cascade distribution with $n = 6$, $R = \{(4, 2, 0),(0, 0, 0)\}$, and L=1 ($q^* = 5$, $k^* = 0$, and $HHI_{max} = 0.244444$).

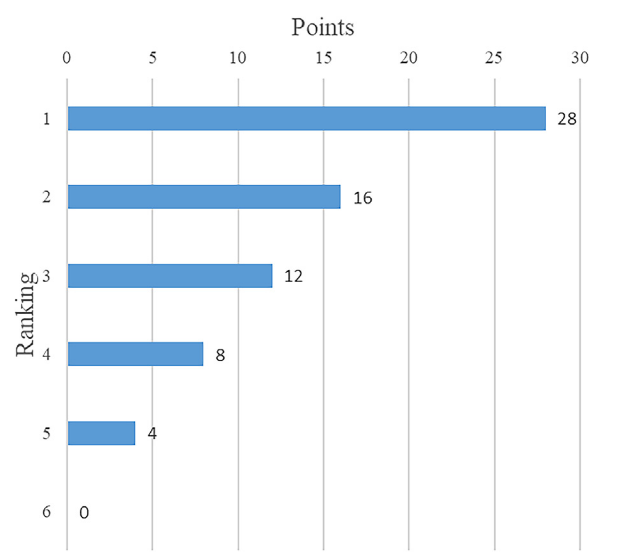

Fig. 6. Truncated Double Cascade distribution with $n = 6$, $R = \{(4, 2, 0),(3, 1, 1)\}$, and $L = 1$ ($q^* = 5$, $k^* = 1$, and HHI_{max} =0.273356).

HHI_n takes this into account. To compare results from the different points allocation systems, Gayant and Le Pape's (2015) suggestion is to apply the CCD to the reconstructed points totals. However, such a recalculation can change the rankings of teams, and alters the values of the competitive balance index (Avila-Cano et al., 2021).[7]

Table 4 presents the HHI values calculated according to the original and reconstructed points totals. The minimum value that the index can reach remains constant (at $1/n$), given that the number of teams does not vary. The maximum possible values vary, reflecting the change in the points allocation system for the seasons from 2017 onwards.

The impact of introducing bonus points on the value of the normalized HHI index, reported in Table 4, can be conveniently decomposed into two effects: (a) an increase in HHI_{max} (TDCD), in line with the theoretical results in section 3, which, for an unchanged value of the raw concentration index, HHI and unchanged HHI_{min}, will, by definition, increase the denominator of HHI_n and so decrease HHI_n, implying an improvement in competitive bal-

[7] For example, in 2018, with bonus points, Wales ranks second and Scotland third, but they are ranked equally on reconstructed points.

Table 4
The Six Nations Championship: Competitive balance indices.

	2015	2016	2017	2018	2019	2020	2021	2022
HHI (O)	0.2356	0.2289	0.2081	0.2237	0.2246	0.2130	0.2088	0.2245
HHI (R)	-	-	0.2089	0.2267	0.2333	0.2267	0.2089	0.2267
HHI_{min}	0.1667	0.1667	0.1667	0.1667	0.1667	0.1667	0.1667	0.1667
HHI_{max} (CCD)	0.2444	0.2444	0.2444	0.2444	0.2444	0.2444	0.2444	0.2444
HHI_{max} (TDCD)	0.2444	0.2444	0.2734	0.2734	0.2734	0.2734	0.2734	0.2734
HHI_n (CCD)	0.8857	0.8000	0.5326	0.7331	0.7451	0.5953	0.5417	0.7434
HHI_n (R)	0.8857	0.8000	0.5429	0.7714	0.8572	0.7715	0.5429	0.7714
HHI_n (TDCD)	0.8857	0.8000	0.3883	0.5344	0.5431	0.4339	0.3949	0.5420
DCB (CCD)	0.9411	0.8944	0.7298	0.8562	0.8632	0.7715	0.7360	0.8622
DCB (R)	0.9411	0.8944	0.7368	0.8783	0.9258	0.8783	0.7368	0.8783
DCB (TDCD)	0.9411	0.8944	0.6231	0.7310	0.7370	0.6587	0.6284	0.7362

Notes: O: Original points; R: Reconstructed points; *DCB* = Distance to Competitive Balance

ance; and (b) changes in the raw *HHI* measure, compared to a without-bonuses counterfactual, in response to the existence of bonus points. Unlike (a), which is completely predictable given the measurement system and the theoretical analysis, (b) is dependent on induced changes in teams' behaviour, cannot be signed unambiguously, and is ultimately an empirical matter. As noted in the earlier discussion of the existing literature on bonus points in rugby (Winchester, 2008, 2014), if bonus points help strong teams accumulate more points and rise to the top of the standings while weak teams do the opposite and fall to the bottom, then we might expect this to be reflected in increased concentration of overall points, i.e., an increase in the raw *HHI*; in that case, the overall result for HHI_n depends on the balance of these influences.

The results for 2015 and 2016, before bonus points were introduced, show the relatively high level of competitive imbalance in the SNC, with HHI_n and *DCB* values at over 80% and 90%, respectively, of their maximum possible value. After the introduction of bonus points, the raw *HHI* values exhibit a slight decrease (regardless of whether original, *HHI* (O), or reconstructed points (*HHI* (R)) are considered). Because the reduction in HHI_n due to a larger HHI_{max} is not offset by a behaviour-induced increase in *HHI*, the normalized HHI_n and *DCB* measures indicate improvements in competitive balance for bonus-point seasons. However, the use of reconstructed points or a maximum value of *HHI* based on the inappropriate CCD implies a higher level of concentration (more imbalance) compared to the relevant TDCD-based maximum value. Application of the true maximum value of *HHI*, corresponding to a TDCD, implies a quantitatively significant decrease in the relative concentration of points.

To consider the magnitude of the differences, we focus on Triguero Ruiz and Avila-Cano's (2019) *DCB* measure, which is the square root of HHI_n. As *DCB* complies with the cardinality property, the differences between two values can be interpreted in terms of percentage points (Avila-Cano & Triguero Ruiz, 2021). *DCB* indicates that in the 2018 season, for example: (i) 73.1% of the maximum concentration of points was reached, compared to 85.6% if the invalid CCD maximum value is considered, or 87.8% if the points are reconstructed using the previous points allocation system; (ii) concentration increased 10.8 percentage points from the previous season, compared to 12.6 for *DCB* (CCD) or 14.2 for *DCB* (R); (iii) failure to consider the relevant maximum concentration achievable implies an overvaluation of concentration in 2018 of 12.5 percentage points (using DCB (CCD) or 14.7 percentage points (using DCB (R)).

Sports administrators are generally concerned about inadequate levels of competitive balance reducing the attractiveness of their leagues. The policy reactions by sport administrators to perceptions of inadequate balance (in comparison with other leagues in the same sport or different sports) can be potentially far-reaching. In general, these depend on the sporting and legal context but can include anti-competitive practices in the labour and product markets (such as salary caps, revenue sharing, financial 'fair play' rules, player drafts, etc.), changes in the teams included in the competition (through restructuring or promotion/relegation), structural changes to league formats (e.g., introducing playoffs), or making adjustments to the rules of the game, within-match scoring systems, and/or points allocation systems for match outcomes.

In Fig. 7, we extend the results for *DCB* back to 2000, when Italy joined the Championship. The plot illustrates dramatically that this competition has historically had high levels of within-season imbalance. Indeed, the maximum possible level of imbalance, an extremely rare occurrence in most high-profile sports leagues, was observed for four successive seasons between 2002 and 2005. Consideration of the maximum concentration relevant for the points allocation system with bonus points yields a somewhat more favourable picture of the level of competitive balance under the new points system, compared to the measures using the inappropriate maximum values of *HHI*. Based on the appropriate normalized balance measure, the introduction of bonus points has moved the competition further from its perfectly unbalanced ceiling, with, as far as it is possible to tell after only eight seasons, no obvious offsetting effect due to induced behavioural changes by teams. However, the competition still exhibits relatively high levels of within-season imbalance compared to many other sporting competitions.[8] This is partly related to the high scoring rates in rugby union making it more likely that strong teams will beat weaker teams, and partly due to Italy's high loss ratio (with only 11 wins and one draw in 115 matches over 23 seasons).

This provides an important dimension of the context for policy makers' decisions on future developments of the SNC. For example, there has been much speculation about whether South Africa will join from 2025, either replacing Italy or as part of an expanded competition with seven teams.[9] Replacing Italy with South Africa would enhance the competitiveness of the competition by removing the weakest team. With $n = 6$ and a strong team joining the strongest five teams from most previous seasons, k^* remains at 1, HHI_{max} is unchanged, and raw concentration is likely to decrease, increasing competitive balance. If South Africa is added, $n = 7$, k^* increases to 2, HHI_{max} decreases and there is more chance that

[8] For example, Triguero-Ruiz and Avila-Cano (2019, Table 2) report the results of calculating *DCB* for the five major association football leagues between 1997/98 and 2016/17. Over this span of seasons, these leagues have a mean value of *DCB* of approximately 40% (0.4042), with the largest value (0.5485) in the Spanish Primera División's 2014/15 season. Such values of *DCB* are significantly lower than what is observed in the SNC.

[9] https://www.dailymail.co.uk/sport/rugbyunion/article-10520605/Rugby-South-Africa-set-join-Six-Nations-2025-Italy-facing-boot.html and https://www.theguardian.com/sport/2022/feb/18/six-nations-deny-south-africa-to-join-rugby-union

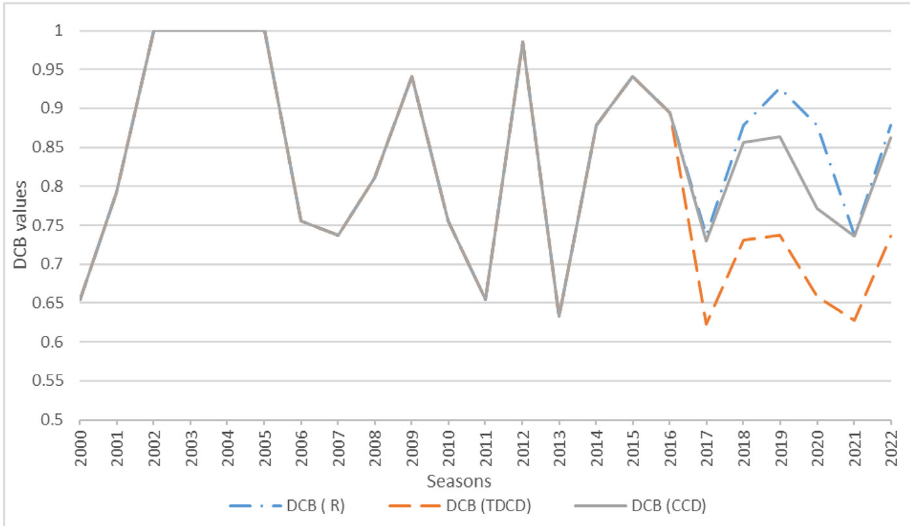

Fig. 7. Distance to Competitive Balance indices for the Six Nations Championship.

competitive balance will decrease.[10] While beyond the scope of this paper, our generalization of the normalized balance measure makes it feasible to compare competitive balance in the existing SNC with that in other rugby competitions. Sports administrators could then use this information, in conjunction with other relevant considerations, such as expected future changes to competitive balance, expectations of additional broadcast revenue, and global scheduling constraints to assess the pros and cons of changes to the structure of the competition.

5. Conclusions

Measurement of competitive balance of a championship using normalized indices, such as HHI_n or DCB, requires knowing the maximum value that the index can reach, corresponding to the distribution of results of the bilateral confrontations that generates the maximum concentration, i.e., the minimum level of competitive balance (maximum level of imbalance).

For bilateral competitions in which total points allocated across all the teams in the championship are constant and known in advance of the matches, this 'most unequal' distribution of points is well known. The first team in the ranking of points wins all its matches, the second, all its matches except those against the first team, and so on until the last, which loses all its games (the Complete Cascade distribution). If draws are feasible and rewarded with less points than the average score of a win and a loss, then the stability condition of a fixed number of points to allocate is not met, as in most association football leagues. In this case, the most unequal distribution involves a set of q teams winning in cascade, while the remaining $(n-q)$ teams draw all the games they have not lost against the first q teams (a Truncated Cascade distribution) (Avila-Cano et al., 2021).

In this paper, we consider an additional source of instability: a points allocation system that would otherwise keep total points allocated constant is complemented with the addition of bonus points, as in most rugby union competitions. Given that the number of bonus points awarded in each game is unforeseeable in advance of playing the matches, the total number of points to be allocated in the league is not constant. The distribution that generates the maximum concentration of points involves a Truncated Double Cascade distribution (TDCD); compared to a standard Cascade distribution, there is a set of k teams that accumulate bonuses, also in cascade, while the $(n - k)$ remaining teams do not.

If additional bonuses are introduced in a context where the total points allocated in the championship are already not constant because of the relative rewards of wins and draws, then this generates two sources of instability. In this case, the most unequal distribution involves bonus points accumulating to a group of k teams that also accumulate the wins, while the rest of the teams draw the games they do not lose and do not accumulate any bonus points. This generates a more extreme form of the Truncated Cascade distribution.

Based on the derivations of the most unequal distributions of points, and the corresponding maximum values of HHI, we apply the TDCD and calculation of a normalized HHI index to the Six Nations Championship, an annual international rugby union competition. The introduction of bonuses increases the feasible maximum degree of concentration of points and, therefore, generates a larger theoretical range of the concentration index. This implies that, for equal values of the raw concentration index, lower values of the normalized index of competitive imbalance are achieved.

Historically, the SNC exhibits a high level of within-season competitive imbalance, with several seasons producing the maximum feasible degree of within-season competitive imbalance. This is consistent with the view that in sports with high scoring rates, such as rugby, differences in team strengths are more likely to be reflected in match outcomes than is the case for sports with low scoring rates, such as association football. Despite concerns that bonus points could exacerbate competitive imbalance, the degree of within-season balance has improved since their introduction in 2017. However, with only eight observations currently available on the bonus-points regime, it is not possible to establish or reject a causal link, due to induced behavioural changes by teams, between the raw concentration index and bonus points.

Declaration of Competing Interest

None.

[10] Some commentators have even argued that including South Africa, one of world rugby's strongest international teams, could mean that the probability of Ireland, Scotland or Wales winning the competition would be reduced to near zero (https://punditarena.com/rugby/eoin-harte/matt-williams-celtic-nations-never-win-six-nations-south-africa/).

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejor.2023.01.064.

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