Spectra of triangulated categories and their applications

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For a tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$, Balmer ([1]) introduced the ringed space $\mathsf{Spec}_{\otimes}(\mathcal{T})$, which is called the tensor triangular spectrum of $\mathcal{T}$. As tensor triangulated categories are ubiquitous in mathematics, this gives a unified approach to studying commutative algebra, algebraic geometry, representation theory, and so on via algebro-geometric methods. This theory is called tensor triangular geometry. For example, Balmer [(1)] proved the following reconstruction theorems in algebraic geometry and representation theory:

• Let X be a noetherian scheme. Then the tensor triangular spectrum of the perfect derived category $\mathsf{D}^{\mathsf{perf}}(X)$ of X is isomorphic to X: $\mathsf{Spec}_{\otimes}(\mathsf{D}^{\mathsf{perf}}(X)) \cong X$.

• Let G be a finite group and k a field. Then the tensor triangular spectrum of the stable category $\mathsf{stmod}(kG)$ of kG is isomorphic to $\mathsf{Proj}\,\mathsf{H}^{\bullet}(G;k)$: $\mathsf{Spec}_{\otimes}(\mathsf{stmod}(kG)) \cong \mathsf{Proj}\,\mathsf{H}^{\bullet}(G;k)$.

Tensor triangular geometry has been successful and well-studied so far in many areas, whereas it cannot be directly applied to triangulated categories without tensor structures. Therefore it is a natural and important problem to develop an analogous theory of tensor triangular geometry for triangulated categories without using tensor structures. In one of these attempts, I have recently introduced the ringed space $\mathsf{Spec}_{\triangle}(\mathcal{T})$, which is called the triangular spectrum of $\mathcal{T}$ for a triangulated category $\mathcal{T}$ ([2,3]).

In this talk, we develop the theory of triangular spectra and apply it to representation theory, algebraic geometry, and commutative algebra. The results include several known or unknown reconstruction theorems.

References

[1] P. Balmer, The spectrum of prime ideals in tensor triangulated categories, J. Reine Angew. Math. 588 (2005), 149-168.

[2] H. Matsui, Prime thick subcategories and spectra of derived and singularity categories of noetherian schemes, Pacific J. Math. 313 (2021), no. 2, 433-457.

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