Tensor products and almost split sequences for group rings

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Let G be a finite group and p a prime number dividing the order of G. Let $(K, \mathcal{O}, k)$ be a p-modular system, namely, K is a complete discrete valuation field of characteristic zero with multiplicative valuation ν, $\mathcal{O}$ is a valuation ring of ν with unique maximal ideal $\pi\mathcal{O}$, and $k = \mathcal{O}/\pi\mathcal{O}$ is the residue class field of $\mathcal{O}$ of characteristic p. We assume that k is algebraically closed. We use R to denote $\mathcal{O}$ or k, and we denote by RG the group ring of G over R. RG-lattices means finite generated right RG-modules which are free as R-modules. If V and W are RG-lattices, $V \otimes_R W$ and $\mathrm{Hom}_R(V, W)$ are RG-lattices with the operations of G given by $(x \otimes y)g = xg \otimes yg$ and $[\varphi g](x) = \varphi(xg^{-1})g$ for all $g \in G,\ x \in V,\ y \in W$and $\varphi \in \mathrm{Hom}_R(V, W)$. Also, R_G denotes the trivial RG-lattice and V^* denotes the dual RG-lattice $\mathrm{Hom}_R(V, R_G)$ of V.

For a non-projective indecomposable RG-lattice U, we denote by $\mathscr{A}(U)$ the almost split sequence terminating in U: $0 \longrightarrow \tau U \longrightarrow m(U) \longrightarrow U \longrightarrow 0$. Applying $V \otimes_R -$ to $\mathscr{A}(U)$ for an indecomposable RG-lattice V, we get the exact sequence

$$V \otimes_R \mathscr{A}(U) : 0 \longrightarrow V \otimes_R \tau U \longrightarrow V \otimes_R m(U) \longrightarrow V \otimes_R U \longrightarrow 0.$$

In the case where $U = R_G$, Auslander and Carlson showed that $V \otimes_R \mathscr{A}(R_G)$ is a direct sum of $\mathscr{A}(V)$ and a split sequence if and only if the multiplicity of the trivial RG-lattice R_G in $V \otimes_R V^*$ is one [1].

In this talk, we consider the tensor product sequence $V \otimes_R \mathscr{A}(Sc(Q))$, where Q is a p-subgroup of G and $Sc(Q)$ is the Scott RG-lattice with vertex Q. Then we see that for an indecomposable RG-lattice V with vertex Q, $V \otimes_R \mathscr{A}(Sc(Q))$ is a direct sum of $\mathscr{A}(V)$ and a split sequence if and only if the multiplicity of $Sc(Q)$ in $V \otimes_R V^*$ is one.

In the case $R = \mathcal{O}$, Knorr introduced the notion of virtually irreducible $\mathcal{O}G$-lattices [3]. We also see that for an indecomposable $\mathcal{O}G$-lattice L with vertex Q, the following conditions (i), (ii) and (iii) are equivalent:

(i) L is virtually irreducible and the $\mathcal{O}$-rank of a Q-source of L is not divisible by p;

(ii) $L \otimes_{\mathcal{O}} \mathscr{A}(Sc(Q))$ is a direct sum of $\mathscr{A}(L)$ and a split sequence;

(iii) The multiplicity of $Sc(Q)$ in $L \otimes_{\mathcal{O}} L^*$ is one [2].

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