Polynomial Dedekind Domains and Stacked Pseudo-convergent Sequences

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A Polynomial Dedekind Domain over the ring of integers $\mathbb{Z}$ is a Dedekind domain R contained between the polynomial rings $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. We provide a full characterization of such Dedekind domains, showing that they can be represented by means of rings of integer-valued polynomials. More precisely, for a prime $p \in \mathbb{Z}$, let $\mathbb{C}_p$ be the completion of the algebraic closure $\overline{\mathbb{Q}_p}$ of the field of p-adic numbers and let v_p denote the unique valuation on $\mathbb{C}_p$ extending the classical p-adic valuation. Given a Polynomial Dedekind Domain R over $\mathbb{Z}$, we show that for each prime $p \in \mathbb{Z}$ there exists a finite subset E_p of $\mathbb{C}_p$, whose elements are transcendental over $\mathbb{Q}$, such that the polynomials in R are precisely those which are simultaneously integer-valued over E_p for each prime p, that is, $R = \{f \in \mathbb{Q}[X] \mid v_p(f(E_p)) \geq 0, \forall \text{ prime } p\}$. We show that for each group G which is a direct sum of a countable family of finitely generated abelian groups, there exists a Polynomial Dedekind domain R over $\mathbb{Z}$ with class group G. In particular, we also obtain a characterization of the PIDs between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.

This result is obtained by a characterization of residually algebraic torsion extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$ by means of a suitable kind of pseudo-convergent sequence in $\overline{\mathbb{Q}_p}$ called stacked. In particular, if W is a DVR of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$ such that the residue field extension is algebraic, there exists $\alpha \in \mathbb{C}_p$, transcendental over $\mathbb{Q}$, such that $W = \{\phi \in \mathbb{Q}(X) \mid v_p(\phi(\alpha)) \geq 0\}$. The residue field extension of W over $\mathbb{Z}_{(p)}$ is finite if and only if $\alpha \in \overline{\mathbb{Q}_p}$.

References

[1] G. Peruginelli, Stacked Pseudo-Convergent Sequences and Polynomial Dedekind Domains, preprint, arXiv: https://arxiv.org/abs/2303.11740

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