Chapter 17

***NP*-Completeness**

Composite satellite image of the United States at night, 1996. U.S. government image. NOAA/NGDC DMSP.

Contents

Printed circuit boards connect electronic components, with the motherboard in a personal computer being a well-known example that connects memory chips, input/output ports, and a computer's CPU. The connections in a circuit board are made by an etching process that creates wires on its surface. Then holes are drilled into the board, allowing for the insertion of electronic components, which are subsequently soldered into the holes. Drilling all of the holes in such a board is a relatively slow step in this manufacturing process, and drilling time is wasted during the time the drill must move from one hole to the next. Thus, in order to manufacture a large number of identical printed circuit boards, it is worthwhile to minimize the total amount of time that a drill travels in order to drill all of the holes in a board.

Looking at this process as a computational problem, we can see that this manufacturing problem is actually an instance of a classic algorithmic problem—the ***traveling salesperson problem*** (***TSP***). In the traveling salesperson problem, we are given a set of "cities" that a traveling salesperson needs to visit. In addition, between every pair of cities, v and w, we are given "distance," $d(v, w)$, that is a number representing the cost (in time, miles, money, etc.) for traveling between city v and city w. The TSP objective is to find a tour that visits all of the cities and minimizes the total cost of all the traveling the salesperson needs to do. The problem of drilling all the holes in a printed circuit board is an instance of the traveling salesperson problem. Each of the holes represents a "city" and the distance between two of these cities is the time it would take to move a robotic drill from one hole to another, including the time it would take to change drill bits if the two holes are of different sizes. (See Figure 17.1.)

In addition to the example of optimizing the process of drilling holes in a printed circuit board, there are many other applications of the traveling salesperson problem. For instance, optimizing the delivery of packages in a UPS or FedEx truck, or the route that one should take to see all the points of interest on a vacation, are also traveling salesperson problems. Thus, it would be useful if we had a fast and simple algorithm for computing an optimal solution to any instance of the traveling salesperson problem. Unfortunately, this is a very difficult problem to always solve optimally.

Some computational problems, like the traveling salesperson problem, are hard. Moreover, even after lots of different researchers have worked on designing efficient algorithms for solving them, we may still not have a method that runs in polynomial time. Ideally, in such cases, we would like to prove that it is impossible to find a polynomial-time solution, so that we can clearly establish the difficulty of such a problem. Such a proof would be a great relief when an efficient algorithm evades us, for then we could take comfort from the fact that no efficient algorithm exists for this problem. Unfortunately, such proofs are typically also very difficult to discover.

Still, we can prove that certain problems are computationally as difficult as other problems, which is an indirect way of showing a problem is computationally difficult. In particular, the concept of ***NP-completeness*** allows us to rigorously show that finding an efficient algorithm for a certain problem is at least as hard as finding efficient algorithms for ***all*** the problems in a large class of problems called "***NP***." The formal notion of "efficient" we use here is that a problem has an algorithm running in time proportional to a polynomial function of its input size, n. (Recall that this notion of efficiency was already mentioned in Section 1.1.5.) That is, we consider an algorithm "efficient," for the discussion in this chapter, if it runs in time $O(n^k)$ on any input of size n, for some constant $k > 0$. The class ***NP*** contains some extremely difficult problems, for which polynomial-time solutions have eluded researchers for decades. Therefore, while showing that a problem is ***NP***-complete is admittedly not the same as proving that an efficient algorithm for the problem is impossible, it is nevertheless a powerful statement. Basically, showing that a problem L is ***NP***-complete says that, although we have been unable to find an efficient algorithm for L, neither has any computer scientist who has ever lived! Indeed, most computer scientists strongly believe it is impossible to solve any ***NP***-complete problem in polynomial time.

In this chapter, we formally define the class ***NP*** and its related class ***P***, and we show how to prove that some problems are ***NP***-complete. We also discuss some of the best-known ***NP***-complete problems, showing that each one is at least as hard as every other problem in ***NP***. These problems include satisfiability, vertex cover, knapsack, and traveling salesperson.

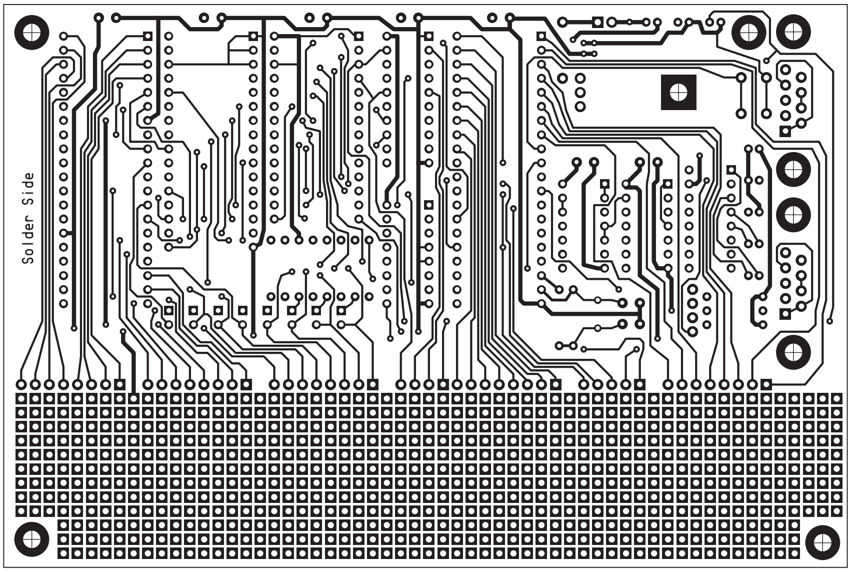


Figure 17.1: Artwork for a printed circuit board. Each circle in this diagram represents a hole that needs to be drilled. Optimizing the route that a drill should take to make these holes is an instance of the traveling salesperson problem (TSP). 8051 Development System Circuit Board, Paul Stoffregen, 2005, public domain image.

17.1 *P* and *NP*

In this section, we define ***P*** and ***NP***, which give us the basic tools for formally saying a problem is computationally difficult. But before we discuss the issue of intractability of computational problems, like the traveling salesperson problem, we need to revisit the definition of the running time of an algorithm.

In order to study ***NP***-completeness, we need to be more precise about running time. Namely, instead of the informal notion of input size as the number of "items" that form the input (see Chapter 1), we define the ***input size***, n, of a problem to be the number of bits used to encode an input instance. We also assume that characters and numbers in the input are encoded using a reasonable ***binary encoding*** scheme, so that each character uses a constant number of bits and each integer $M > 0$ is represented with at most $c \log M$ bits, for some constant $c > 0$. In particular, we disallow inputs that are specified using a ***unary encoding***, where an integer M is represented with M 1's.

Recall that we have, for most of the other chapters of this book, defined the input size, n, to be the number of "items" or "elements" in an input. In this chapter, however, we take n to be the number of bits used to represent an input, as mentioned above. Formally, we define the worst-case ***running time*** of an algorithm A to be the worst-case time taken by A as a function of n, taken over all inputs (with valid encodings) having n bits.

From the standpoint of polynomial-time algorithms, we don't actually lose much by focusing on bit-length as our definition of input size. For instance, if an algorithm has a running time that is polynomial in the number of input bits, n, then it also runs in polynomial time in the number of "items," N, in that same input, for $\sqrt{n} \leq N \leq n$. Likewise, any "reasonable" algorithm that runs in polynomial time in terms of the number of input items, N, will also run in polynomial time in terms of the number of input bits, n, where by "reasonable" we mean that numbers used by the algorithm can be represented using $O(\log N)$ bits or arrays of $O(\log N)$-bit numbers. Thus, under this restriction, n is $O(N \log N)$; hence, a reasonable algorithm with a running time that is polynomial in N will also have a running time that is polynomial in n. (See Figure 17.2 and also Exercise C-17.2.)

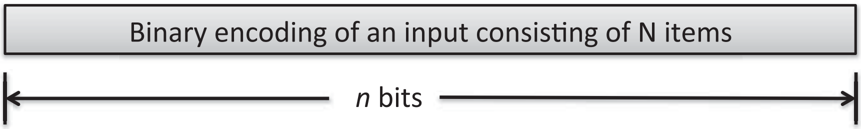


Figure 17.2: Viewing input size in terms of bits.

17.1.1 Defining the Complexity Classes ***P*** and ***NP***

By the notion of a "reasonable" algorithm described above, we can easily show that, for the problems discussed elsewhere in this book, such as graph problems, text processing, or sorting, the polynomial-time algorithms given in those other chapters translate into polynomial-time algorithms using the notion of running time considered in this chapter.

Moreover, the class of polynomials is closed under addition, multiplication, and composition. That is, if $p(n)$ and $q(n)$ are polynomials, then so are $p(n) + q(n)$, $p(n) \cdot q(n)$, and $p(q(n))$. Thus, we can combine or compose polynomial-time algorithms to construct new polynomial-time algorithms. For example, if an algorithm, A, takes an input of size n and produces an output of size $q(n)$ in $O(q(n))$ time, and then an algorithm, B, takes the output of A as its input, where B runs in $O(p(m))$ time on inputs of size m, then this combined algorithm runs in time $O(q(n) + p(q(n)))$, which is polynomial if p and q are polynomials.

Decision Problems and Languages

To simplify our discussion, let us restrict our attention for the time being to ***decision problems***, that is, to computational problems for which the intended output is either "yes" or "no." In other words, a decision problem's output is a single bit, which is either 0 or 1. For example, each of the following are decision problems:

- Given a string T and a string P, does P appear as a substring of T?
- Given two sets S and T, do S and T contain the same set of elements?
- Given a graph G with integer weights on its edges, and an integer k, does G have a minimum spanning tree of weight at most k?

As last problem illustrates, we can often turn an ***optimization problem***, where we are trying to minimize or maximize some value, into a decision problem. We can introduce a parameter k and ask if the optimal value for the optimization problem is at most or at least k. Note that if we can show that a decision problem is hard, then its related optimization version must also be hard.

We say that an algorithm A ***accepts*** an input string x if A outputs "yes" on input x, and A ***rejects*** x if A outputs "no" on input x. Thus, we can view a ***decision problem*** as actually being just a set L of strings—the strings that should be accepted by an algorithm that correctly solves the problem. Indeed, we often use the letter "L" to denote such a decision problem, because a set of strings is often referred to as a ***language***. We can extend this language-based viewpoint further to say that an algorithm A ***decides*** a language L if, for each string x, A outputs "yes" if x is in L and "no" otherwise. Throughout this chapter, we assume that if x is in an improper syntax, then an algorithm given x will output "no." In addition, we restrict our attention to algorithms, that is, computations that terminate on every input after a finite number of steps.

The Million-Dollar $\boldsymbol{P} = \boldsymbol{NP}$ Question

The ***complexity class P*** is the set of all decision problems (or languages) L that can be decided in worst-case polynomial time. That is, there is an algorithm A that, on input x, runs in $p(n)$ time, where n is the size of x and $p(n)$ is a polynomial, and, if $x \in L$, then A outputs "yes," and otherwise A outputs "no." The latter case refers to the ***complement*** of a language L, which consists of all binary strings that are not in L. Given an algorithm A that decides a language L in polynomial time, $p(n)$, we can easily construct a polynomial-time algorithm that decides the complement of L. In particular, given an input x, we can construct a complement algorithm B that simply runs A for $p(n)$ steps, where n is the size of x. If A outputs "yes," then B outputs "no," and if A outputs "no," then B outputs "yes." In either case, the complement algorithm, B, runs in polynomial time. Therefore, if a language L, representing some decision problem, is in ***P***, then the complement of L is also in ***P***.

The ***complexity class NP*** is defined to include the complexity class ***P*** but allow for the inclusion of languages that may not be in ***P***. Specifically, with ***NP*** problems, we allow algorithms to perform an additional operation:

- choose(b): this operation chooses in a nondeterministic way a bit (that is, a value that is either 0 or 1) and assigns it to b.

When an algorithm A uses the choose primitive operation, then we say A is ***nondeterministic***. We state that an algorithm A ***nondeterministically accepts*** a string x if there exists a set of outcomes to the choose calls that A could make on input x such that A would ultimately output "yes." In other words, it is as if we consider all possible outcomes to choose calls and only select those that lead to acceptance if there is such a set of outcomes.

The complexity class ***NP*** is the set of every decision problem (or language), L, that can be nondeterministically accepted in polynomial time, where we define the running time, $p(n)$, of a nondeterministic algorithm to be the maximum running time for A taken over all possible outcomes to its choose calls on an input of size n. That is, L is in ***NP*** if there is a nondeterministic algorithm A and polynomial $p(n)$ such that, on an input x of size n, if $x \in L$, then there is a set of outcomes to the choose calls in A so that it outputs "yes" and A runs in $p(n)$ time. If x is not in L, then every possible outcome to the choose calls in A results in A outputting "no."

Interestingly, unlike as was the case with ***P***, just because a language L is in ***NP*** does not necessarily imply that the complement of L is also in ***NP***. Indeed, there is a complexity class, called **co**-***NP***, that consists of all languages whose complement is in ***NP***, and many researchers believe **co-*NP*** $\neq$ ***NP***.

The Clay Mathematics Institute has offered \$1 million to the first person who proves whether ***P*** $=$ ***NP***. Although no one has, as of this writing, succeeded in claiming this prize, the majority of computer scientists believe that ***P*** is different than ***NP***. That is, most computer scientists believe that the answer to the "***P*** $=$ ***NP***?" question is "no."

An Alternative Definition of ***NP***

There is actually another way to define the complexity class ***NP***, which might be more intuitive for some readers. This alternative definition of ***NP*** is based on deterministic verification, instead of nondeterministic acceptance. We say that a language L can be ***verified*** by an algorithm A if, given any string x in L as input, there is another string y such that A outputs "yes" on input $z = x + y$, where we use the symbol "+" to denote concatenation. The string y is called a ***certificate*** for membership in L, for it helps us certify that x is indeed in L. Note that we make no claims about verifying when a string is not in L.

This notion of verification allows us to give an alternative definition of the complexity class ***NP***. Namely, we can define ***NP*** to be the set of all languages L, defining decision problems, such that L can be verified in polynomial time. That is, there is a (deterministic) algorithm A that, for any x in L, verifies using some certificate y that x is indeed in L in polynomial time, $p(n)$, including the time A takes to read its input $z = x + y$, where n is the size of x. Note that this definition implies that the size of y is less than $p(n)$. As the following theorem shows, this verification-based definition of ***NP*** is equivalent to the nondeterminism-based definition given above.

Theorem 17.1: *A language L can be (deterministically) verified in polynomial time if and only if L can be nondeterministically accepted in polynomial time.*

Proof: Suppose there is a deterministic algorithm A that can verify in polynomial time, $p(n)$, that a string x is in L when given a polynomial-length certificate y. We can construct a nondeterministic algorithm B that takes the string x as input and calls the choose method to assign the value of each bit in y. After B has constructed a string $z = x + y$, it then calls A to verify that $x \in L$ given the certificate y. If there exists a certificate y such that A accepts z, then there is clearly a set of nondeterministic choices for B that result in B outputting "yes" itself. In addition, B will run in $O(p(n))$ steps.

Next, suppose that there is a nondeterministic algorithm A that, given a string x in L, performs $p(n)$ steps, which may include choose steps, such that, for some sequence of outcomes to these choose steps, A will output "yes." There is a deterministic verification algorithm B that, given x in L, uses as its certificate y the ordered concatenation of all the outcomes to choose calls that A makes on input x in order to ultimately output "yes." Since A runs in $p(n)$ steps, where n is the size of x, the algorithm B will also run in $O(p(n))$ steps given input $z = x + y$. ■

The practical implication of this theorem is that, since both definitions of ***NP*** are equivalent, we can use either one for showing that a problem is in ***NP***. That is, Theorem 17.1 implies that we can structure a nondeterministic algorithm so that all of its choose steps are performed first and the rest of the algorithm is just a verification. We illustrate this approach by showing some interesting decision problems to be in ***NP*** in the next subsection.

17.1.2 Some Interesting Problems in *NP*

Our first example problem in ***NP*** is the traveling salesperson problem, which we discussed above. Recall that in this problem we are given a set of N "cities" together with a distance function, $d(v,w)$, which assigns an integer cost to each pair of cities (so that $d(v,w) = d(w,v)$), and we are asked to find a tour of all the cities that has minimum total cost. Viewing this as a decision problem, or language TSP, we assume we are also given an integer k, and we are asked whether there is a cycle that visits each city exactly once, returning to the starting city, such that the total cost of the tour is at most k. (See Figure 17.3.)

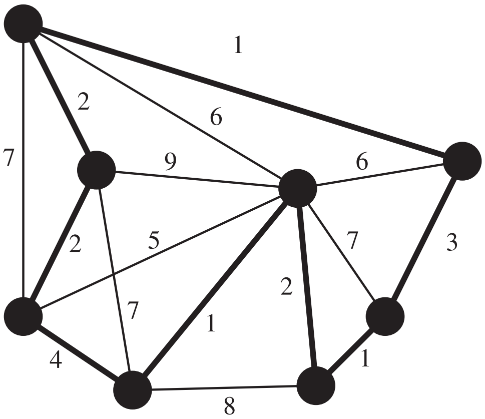


Figure 17.3: An example instance of the TSP decision problem, where $k = 18$ and the answer is "yes." City pairs with a finite cost between them are drawn as an edge along with its integer cost; missing edges are for city pairs with infinite cost between them. A tour with total cost at most k is drawn with bold edges.

Lemma 17.2: TSP *is in* ***NP***.

Proof: Let us define a nondeterministic algorithm that accepts instances of TSP. Assume that the cities are numbered 1 to N. We iteratively call the choose method to determine a sequence S of $N+1$ numbers from 1 to N. Then, we check that each number from 1 to N appears exactly once in S (for example, by sorting S), except for the first and last numbers in S, which should be the same. Then, we verify that the sequence S defines a cycle of cities and that the total cost of the tour defined by S is at most k. This algorithm clearly runs in polynomial time.

Observe that if there is a tour that visits each city exactly once, returning to the starting city, with total cost at most k, then our nondeterministic algorithm will output "yes." Likewise, if our algorithm outputs "yes," then it has found a tour visiting each city exactly once, returning to the starting city, with total cost at most k. Since this algorithm runs in polynomial time, this implies that TSP is in ***NP***. ■

Our next example is a problem related to circuit design testing. A ***Boolean circuit*** is a directed graph where each node, called a ***logic gate***, corresponds to a simple Boolean function, AND, OR, or NOT. The incoming edges for a logic gate correspond to inputs for its Boolean function and the outgoing edges correspond to outputs, which will all be the same value, of course, for that gate. (See Figure 17.4.) Vertices with no incoming edges are ***input*** nodes, and a vertex with no outgoing edges is an ***output*** node.

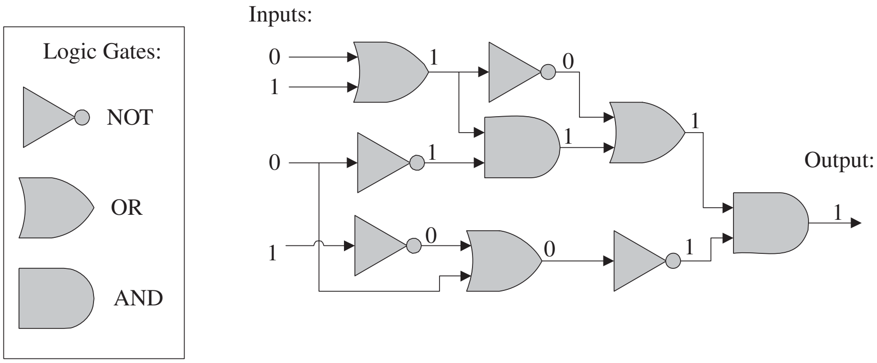


Figure 17.4: An example Boolean circuit.

CIRCUIT-SAT is the problem that takes as input a Boolean circuit with a single output node, and asks whether there is an assignment of values to the circuit's inputs so that its output value is "1." Such an assignment of values is called a ***satisfying assignment***.

Lemma 17.3: CIRCUIT-SAT *is in* ***NP***.

Proof: We construct a nondeterministic algorithm for accepting CIRCUIT-SAT in polynomial time. We first use the choose method to "guess" the values of the input nodes as well as the output value of each logic gate. Then, we simply visit each logic gate g in C, that is, each vertex with at least one incoming edge. We then check that the "guessed" value for the output of g is in fact the correct value for g's Boolean function, be it an AND, OR, or NOT, based on the given values for the inputs for g. This evaluation process can easily be performed in polynomial time. If any check for a gate fails, or if the "guessed" value for the output is 0, then we output "no." If, on the other hand, the check for every gate succeeds and the output is 1, the algorithm outputs "yes." Thus, if there is indeed a satisfying assignment of input values for C, then there is a possible collection of outcomes to the choose statements so that the algorithm will output "yes" in polynomial time. Likewise, if there is a collection of outcomes to the choose statements so that the algorithm outputs "yes" in polynomial-time algorithm, there must be a satisfying assignment of input values for C. Therefore, CIRCUIT-SAT is in ***NP***. ■

The next example problem is for a network monitoring problem. Suppose we are given a computer network, which is modeled using a graph G such that each vertex in G is a computer and each edge in G is a network connection between a pair of computers. We would like to monitor all of these connections by installing special monitoring devices on some of the computers, where a monitoring device placed on a computer can continuously check if all the network connections to that computer are working correctly. But these devices are relatively expensive, so we would like to minimize the number of such devices that we need to deploy. Viewed as a decision problem, which is known as VERTEX-COVER, we are given a graph G and an integer k, and we are asked whether there is a subset C of k vertices such that, for every edge (v, w) of G, $v \in C$ or $w \in C$ (possibly both). Such a subset is known as a ***vertex cover***. In other words, VERTEX-COVER is the decision problem that takes a graph G and an integer k as input, and asks whether there is a vertex cover for G containing at most k vertices. (See Figure 17.5.)

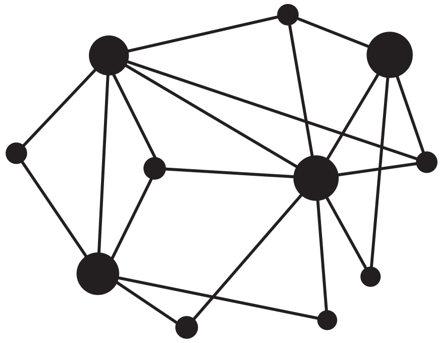


Figure 17.5: An instance of the VERTEX-COVER decision problem, where $k = 4$ and the answer is "yes." The vertices in the vertex cover are drawn as large disks.

Lemma 17.4: VERTEX-COVER *is in* ***NP***.

Proof: Suppose we are given an integer k and a graph G, with its vertices numbered 1 to N. We use repeated calls to the choose method to form a collection C of k numbers that range from 1 to N. As a verification, we insert all the numbers of C into a dictionary, and then we examine each of the edges in G to make sure that, for each edge (v, w) in G, v is in C or w is in C. If we ever find an edge with neither of its end-vertices in G, then we output "no." If we run through all the edges of G so that each has an end-vertex in C, then we output "yes." Such a computation clearly runs in polynomial time. Note that if G has a vertex cover of size at most k, then there is an assignment of numbers to define the collection C so that each edge of G passes our test and our algorithm outputs "yes." Likewise, if our algorithm outputs "yes," then there must be a subset C of the vertices of size at most k, such that C is a vertex cover. Thus, VERTEX-COVER is in ***NP***. ■

17.2 *NP*-Completeness

The notion of nondeterministic acceptance of a decision problem (or language) is admittedly strange. There is, after all, no conventional computer that can efficiently perform a nondeterministic algorithm with many calls to the choose method. Indeed, to date no one has shown how even an unconventional computer, such as a quantum computer or DNA computer, can efficiently simulate any nondeterministic polynomial-time algorithm using a polynomial amount of resources. Certainly, we can deterministically simulate a nondeterministic algorithm by trying out, one by one, all possible outcomes to the choose statements that the algorithm makes. But this simulation becomes an exponential-time computation for any nondeterministic algorithm that makes at least n^ϵ calls to the choose method, for any fixed constant $\epsilon > 0$. Indeed, there are hundreds of problems in the complexity class ***NP*** for which most computer scientists strongly believe there is no conventional deterministic method for solving them in polynomial time.

The usefulness of the complexity class ***NP***, therefore, is that it formally captures a host of problems that many believe to be computationally difficult. In fact, there are some problems that are provably at least as hard as every other problem in ***NP***, as far as polynomial-time solutions are concerned. This notion of hardness is based on the concept of polynomial-time reducibility, which we now discuss.

17.2.1 Polynomial-Time Reducibility and *NP*-Hardness

We say that a language L, defining some decision problem, is ***polynomial-time reducible*** to a language M, if there is a function f computable in polynomial time, that takes an input x to L, and transforms it to an input $f(x)$ of M, such that $x \in L$ if and only if $f(x) \in M$. In addition, we use a shorthand notation, saying $L \xrightarrow{\text{poly}} M$ to signify that language L is polynomial-time reducible to language M.

We say that a language M, defining some decision problem, is ***NP-hard*** if every other language L in ***NP*** is polynomial-time reducible to M. In more mathematical notation, M is ***NP***-hard, if, for every $L \in$ ***NP***, $L \xrightarrow{\text{poly}} M$. If a language M is ***NP***-hard and it is also in the class ***NP*** itself, then M is ***NP-complete***. Thus, an ***NP***-complete problem is, in a very formal sense, one of the hardest problems in ***NP***, as far as polynomial-time computability is concerned. For, if anyone ever shows that an ***NP***-complete problem L is solvable in polynomial time, then that immediately implies that every other problem in the entire class ***NP*** is solvable in polynomial time. For, in this case, we could accept any other ***NP*** language M by reducing it to L and then running the algorithm for L. In other words, if anyone finds a deterministic polynomial-time algorithm for even one ***NP***-complete problem, then ***P*** = ***NP***.

17.2.2 The Cook-Levin Theorem

At first, it might appear that the definition of ***NP***-completeness is too strong. Still, as the following theorem shows, there is at least one ***NP***-complete problem.

Theorem 17.5 (The Cook-Levin Theorem): CIRCUIT-SAT *is **NP**-complete.*

Rather than give a formal proof of this theorem, which is somewhat cumbersome, let us instead provide a sketch of this proof, which highlights the main ideas. To begin, note that Lemma 17.3 shows that CIRCUIT-SAT is in ***NP***. Thus, we have yet to show this problem is ***NP***-hard. That is, we need to show that every problem in ***NP*** is polynomial-time reducible to CIRCUIT-SAT.

So, consider a language L, representing some decision problem that is in ***NP***. Since L is in ***NP***, there is a deterministic algorithm D that accepts any x in L in polynomial-time $p(n)$, given a polynomial-sized certificate y, where n is the size of x. The main idea of the proof is to build a large, but polynomial-sized, circuit C that simulates the algorithm D on an input x in such a way that C is satisfiable if and only if there is a certificate y such that D outputs "yes" on input $z = x + y$, where "$+$" denotes concatenation.

Configurations of a Computation

Recall (from Section 1.1.2) that any deterministic algorithm, such as D, can be implemented on a simple computational model (called the Random Access Machine, or RAM) that consists of a CPU and a bank M of addressable memory cells. In our case, the memory M contains the input, x, the certificate, y, the working storage, W, that D needs to perform its computations, and the code for the algorithm D itself. The working storage W for D includes all the registers used for temporary calculations and the stack frames for the procedures that D calls during its execution. The topmost such stack frame in W contains the program counter (PC) that identifies where D currently is in its program execution. Thus, there are no memory cells in the CPU itself. In performing each step of D, the CPU reads the next instruction i, which is pointed to by the PC, and performs the calculation indicated by i, be it a comparison, arithmetic operation, a conditional jump, a step in procedure call, etc., and then updates the PC to point to the next instruction to be performed. Thus, the current state of D is completely characterized by the contents of its memory cells. Moreover, since D accepts an x in L in a polynomial $p(n)$ number of steps, where n is the size of x, then the entire effective collection of its memory cells can be assumed to consist of just $p(n)$ bits. For in $p(n)$ steps, D can access at most $p(n)$ memory cells. Note also that the size of D's code is constant with respect to the sizes of x, y, and even W. We refer to the $p(n)$-sized collection M of memory cells for an execution of D as the ***configuration*** of the algorithm D.

Boolean Circuits Can Perform Computations

The heart of the reduction of L to CIRCUIT-SAT depends on our constructing a Boolean circuit that simulates the workings of the CPU in our computational model. We omit the details of such a construction in this proof sketch, but it is well known that a CPU can be designed as a Boolean circuit consisting of AND, OR, and NOT gates. For example, such constructions are studied in depth in courses on computer architecture. Moreover, let us further take for granted that this circuit, including its address unit for connecting to a memory of $p(n)$ bits, can be designed to take a configuration of D as input and provide as output the configuration resulting from processing the next computational step. In addition, let us assume that this simulation circuit, which we will call S, can be constructed to consist of at most $cp(n)^2$ AND, OR, and NOT gates, for some constant $c > 0$. We are admittedly making an important assumption here, which would be established formally in an actual proof of the Cook-Levin Theorem, but this assumption should at least be intuitive, for if it were not the case, then the CPUs that come inside modern computers and smartphones would not be as small as they are.

The Simulation

To then simulate the entire $p(n)$ steps of D, we make $p(n)$ copies of S, with the output from one copy serving as the input for the next. (See Figure 17.6.) Part of the input for the first copy of S consists of "hard-wired" values for the program for D, the value of x, the initial stack frame (complete with PC pointing to the first instruction of D), and the remaining working storage (initialized to all 0's). The only unspecified true inputs to the first copy of S are the cells of D's configuration for the certificate y. These are the true inputs to our circuit. Likewise, we ignore all the outputs from the final copy of S, except the single output that indicates the answer from D, with "1" for "yes" and "0" for "no." The total size of the circuit C is $O(p(n)^3)$, which of course is still polynomial in the size of x.

Completing the Proof Sketch

Consider an input x that D accepts for some certificate y after $p(n)$ steps. Then there is an assignment of values to the input to C corresponding to y, such that, by having C simulate D on this input and the hard-wired values for x, we will ultimately have C output a 1. Thus, C is satisfiable in this case. Conversely, consider a case when C is satisfiable. Then there is a set of inputs, which correspond to the certificate y, such that C outputs a 1. But, since C exactly simulates the algorithm D, this implies that there is an assignment of values to the certificate y, such that D outputs "yes." Thus, D will verify x in this case. Therefore, D accepts x with certificate y if and only if C is satisfiable.

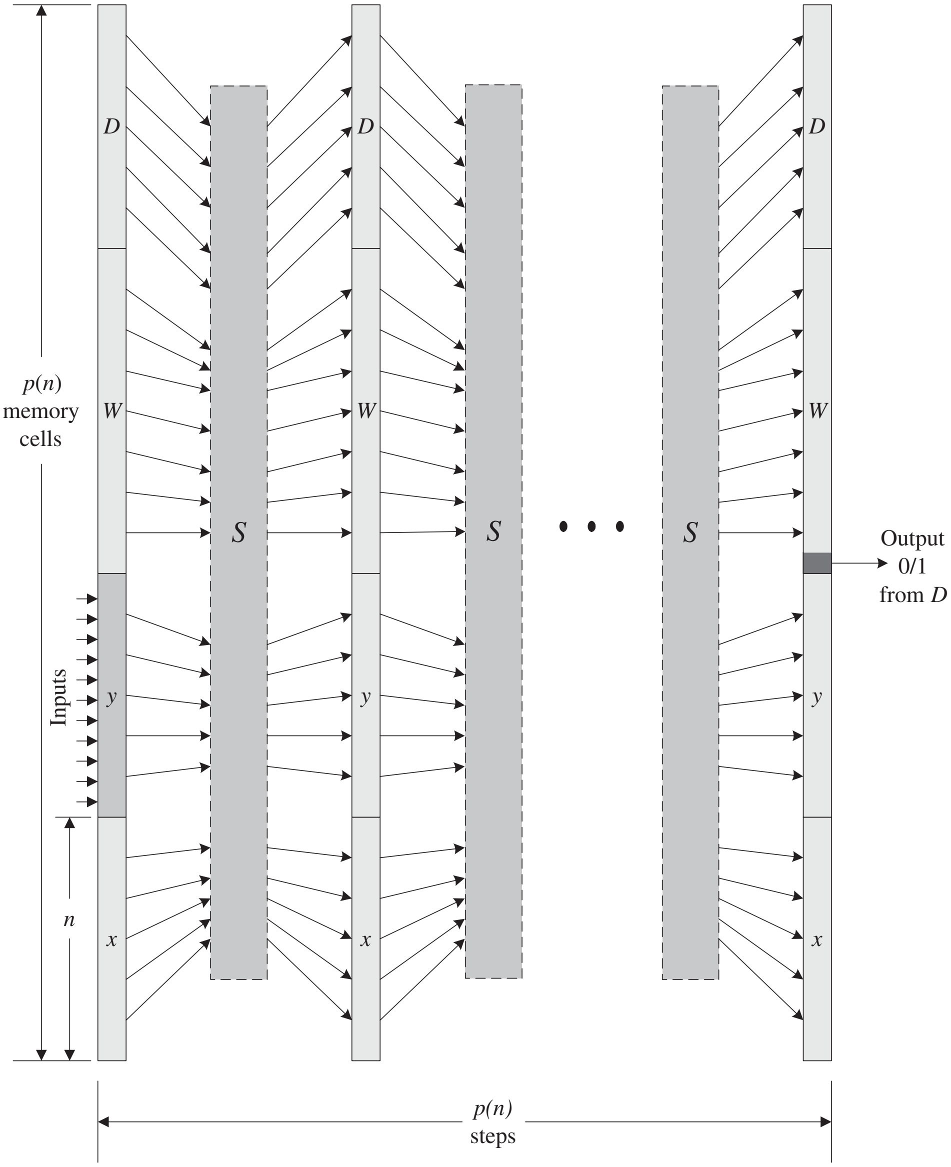


Figure 17.6: An illustration of the circuit used to prove that CIRCUIT-SAT is ***NP***-hard. The only true inputs correspond to the certificate, y. The problem instance, x, the working storage, W, and the program code, D, are initially "hard-wired" values. The only output is the bit that determines whether the algorithm accepts x.

17.2.3 How to Prove Problems to be ***NP***-Complete

Now that we are armed with one ***NP***-complete problem, we can prove other problems are ***NP***-complete using simple polynomial-time reductions. We explore a number of such reductions in the remainder of this chapter.

Given just a single ***NP***-complete problem, we can now use polynomial-time reducibility to show other problems to be ***NP***-complete. We will make repeated use of the following important lemma about polynomial-time reducibility.

Lemma 17.6: *If* $L_1 \xrightarrow{\textit{poly}} L_2$ *and* $L_2 \xrightarrow{\textit{poly}} L_3$*, then* $L_1 \xrightarrow{\textit{poly}} L_3$.

Proof: Since $L_1 \xrightarrow{\text{poly}} L_2$, any instance, x, of size n, for L_1, can be converted in polynomial-time, $p(n)$, into an instance $f(x)$ for L_2, such that $x \in L_1$ if and only if $f(x) \in L_2$. Likewise, since $L_2 \xrightarrow{\text{poly}} L_3$, any instance, y, of size m, for L_2, can be converted in polynomial-time, $q(m)$, into an instance $g(y)$ for L_3, such that $y \in L_2$ if and only if $g(y) \in L_3$. Combining these two constructions, any instance, x, of size n, for L_1 can be converted in time, $p(n) + q(k)$, into an instance, $g(f(x))$, for L_3, such that $x \in L_1$ if and only if $g(f(x)) \in L_3$, where k is the size of $f(x)$. But, $k \le p(n)$, since $f(x)$ is constructed in $p(n)$ steps. Thus, $q(k) \le q(p(n))$. Since the composition of two polynomials always results in another polynomial, this inequality implies that $L_1 \xrightarrow{\text{poly}} L_3$. ■

In this section we establish several important problems to be ***NP***-complete, using this lemma. All of the proofs have the same general structure. Given a new problem L, we first prove that L is in ***NP***. Then, we reduce a known ***NP***-complete problem to L in polynomial time, showing L to be ***NP***-hard. Thus, we show L to be in ***NP*** and also ***NP***-hard; hence, L has been shown to be ***NP***-complete. (Why not do the reduction in the other direction?) These reductions generally take one of three forms:

- ***Restriction***: This form shows a problem L is ***NP***-hard by noting that a known ***NP***-complete problem M is actually just a special case of L.
- ***Local replacement***: This forms reduces a known ***NP***-complete problem M to L by dividing instances of M and L into "basic units," and then showing how each basic unit of M can be locally converted into a basic unit of L.
- ***Component design***: This form reduces a known ***NP***-complete problem M to L by building components for an instance of L that will enforce important structural functions for instances of M. For example, some components might enforce a "choice" while others enforce an "evaluation" function.

The last of the three above forms tends to be the most difficult to construct; it is the form used, for example, by the proof of the Cook-Levin Theorem (17.5).

In Figure 17.7, we illustrate the problems we prove are ***NP***-complete, together with the problems they are reduced from and the technique used in each polynomial-time reduction.

In the remainder of this chapter we study some important ***NP***-complete problems. We treat most of them in pairs, with each pair addressing an important class of problems, including problems involving Boolean formulas, graphs, sets, and numbers. We begin with two problems involving Boolean formulas.

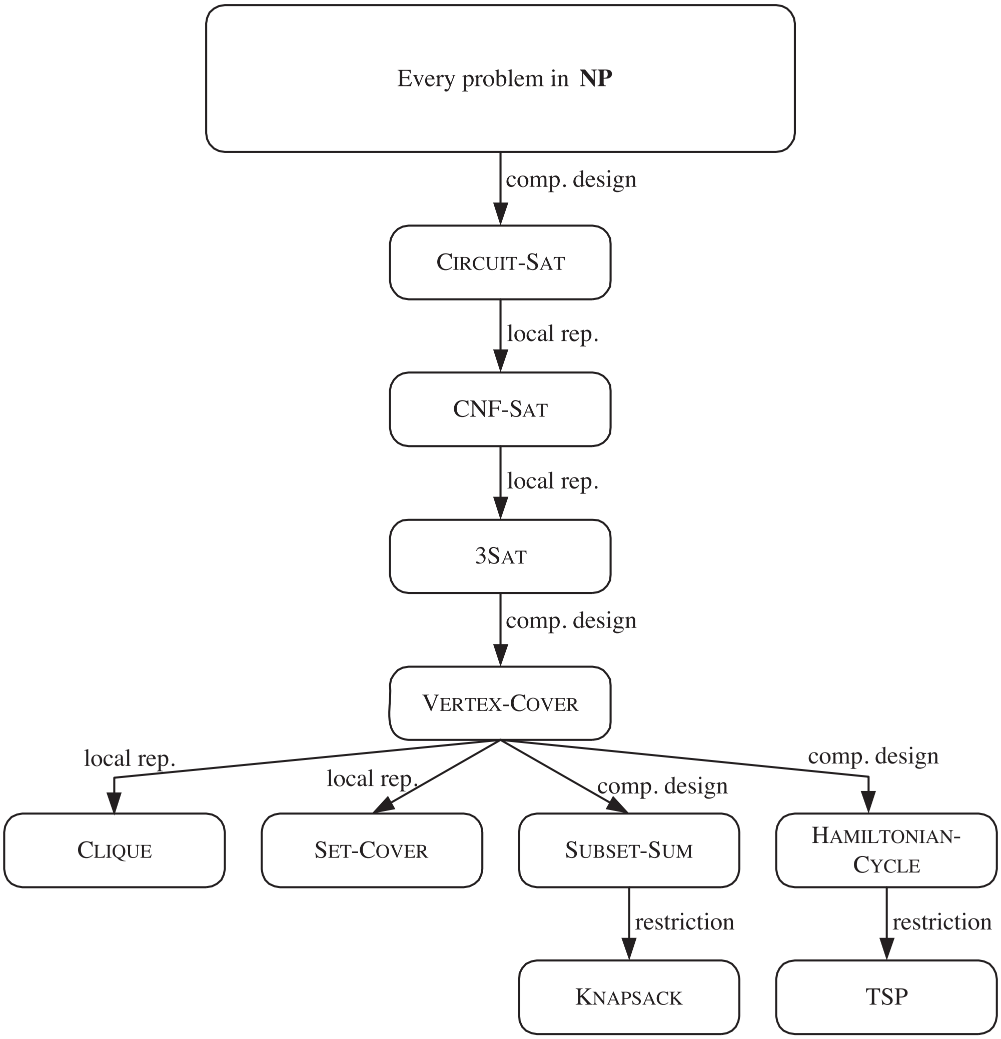


Figure 17.7: Illustration of the reductions used in some fundamental ***NP***-completeness proofs. Each directed edge denotes a polynomial-time reduction, with the label on the edge indicating the primary form of that reduction. The top-most reduction is the Cook-Levin Theorem.

17.3 CNF-SAT and 3SAT

The first reductions we present are for problems involving Boolean formulas. A Boolean formula is a parenthesized expression that is formed from Boolean variables using Boolean operations, such as OR ($+$), AND ($\cdot$), NOT (drawn as a bar over the negated subexpression), IMPLIES ($\rightarrow$), and IF-AND-ONLY-IF ($\leftrightarrow$). A Boolean formula is in ***conjunctive normal form*** (CNF) if it is formed as a collection of subexpressions, called ***clauses***, that are combined using AND, with each clause formed as the OR of Boolean variables or their negation, called ***literals***. For example, the following Boolean formula is in CNF:

$$(\overline{x_1} + x_2 + x_4 + \overline{x_7})(x_3 + \overline{x_5})(\overline{x_2} + x_4 + \overline{x_6} + x_8)(x_1 + x_3 + x_5 + \overline{x_8}).$$

This formula evaluates to 1 if x_2, x_3, and x_4 are 1, where we use 0 for **false** and 1 for **true**. CNF is called a "normal" form, because any Boolean formula can be converted into this form.

CNF-SAT

Problem CNF-SAT takes a Boolean formula in CNF form as input and asks whether there is an assignment of Boolean values to its variables so that the formula evaluates to 1. It is easy to show that CNF-SAT is in ***NP***, for, given a Boolean formula S, we can construct a simple nondeterministic algorithm that first "guesses" an assignment of Boolean values for the variables in S and then evaluates each clause of S in turn. If all the clauses of S evaluate to 1, then S is satisfied; otherwise, it is not.

To show that CNF-SAT is ***NP***-hard, we will reduce the Circuit-SAT problem to it in polynomial time. So, suppose we are given a Boolean circuit, C. Without loss of generality, we assume that each AND and OR gate has two inputs and each NOT gate has one input. To begin the construction of a formula S equivalent to C, we create a variable x_i for each input for the entire circuit C. One might be tempted to limit the set of variables to just these x_i's and immediately start constructing a formula for C by combining subexpressions for inputs, but it is not clear that this approach would take polynomial time. (See Exercise C-17.5.) Instead, we create a variable y_i for each output of a gate in C. Then we create a short formula B_g that corresponds to each gate g in C as follows:

- If g is an AND gate with inputs a and b (which could be either x_i's or y_i's) and output c, then $B_g = (c \leftrightarrow (a \cdot b))$.
- If g is an OR gate with inputs a and b and output c, then $B_g = (c \leftrightarrow (a + b))$.
- If g is a NOT gate with input a and output b, then $B_g = (b \leftrightarrow \overline{a})$.

We wish to create our formula S by taking the AND of all of these B_g's, but such a formula would not be in CNF. So our method is to first convert each B_g to be in

a	b	c	$B = (c \leftrightarrow (a \cdot b))$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	1	1
0	0	1	0
0	0	0	1

DNF formula for $\overline{B} = a \cdot b \cdot \overline{c} + a \cdot \overline{b} \cdot c + \overline{a} \cdot b \cdot c + \overline{a} \cdot \overline{b} \cdot c$

CNF formula for $B = (\overline{a} + \overline{b} + c) \cdot (\overline{a} + b + \overline{c}) \cdot (a + \overline{b} + \overline{c}) \cdot (a + b + \overline{c})$.

Figure 17.8: A truth table for a Boolean formula B over variables a, b, and c. The equivalent formula for $\overline{B}$ in DNF, and equivalent formula for B in CNF.

CNF, and then combine all of these transformed B_g's by AND operations to define the CNF formula S.

To convert a Boolean formula B into CNF, we construct a truth table for B, as shown in Figure 17.8. We then construct a short formula D_i for each table row that evaluates to 0. Each D_i consists of the AND of the variables for the table, with the variable negated if and only if its value in that row is 0. We create a formula D by taking the OR of all the D_i's. Such a formula, which is the OR of formulas that are the AND of variables or their negation, is said to be in ***disjunctive normal form***, or ***DNF***. In this case, we have a DNF formula D that is equivalent to $\overline{B}$, since it evaluates to 1 if and only if B evaluates to 0. To convert D into a CNF formula for B, we apply, to each D_i, De Morgan's laws, which establish that

$$\overline{(a+b)} = \overline{a} \cdot \overline{b} \quad \text{and} \quad \overline{(a \cdot b)} = \overline{a} + \overline{b}.$$

From Figure 17.8, we can replace each B_g that is of the form $(c \leftrightarrow (a \cdot b))$, by

$$(\overline{a} + \overline{b} + c)(\overline{a} + b + \overline{c})(a + \overline{b} + \overline{c})(a + b + \overline{c}),$$

which is in CNF. Likewise, for each B_g that is of the form $(b \leftrightarrow \overline{a})$, we can replace B_g by the equivalent CNF formula

$$(\overline{a} + \overline{b})(a + b).$$

We leave the CNF substitution for a B_g of the form $(c \leftrightarrow (a + b))$ as an exercise (R-17.2). Substituting each B_g in this way results in a CNF formula S' that corresponds exactly to each input and logic gate of the circuit, C. To construct the final Boolean formula S, then, we define $S = S' \cdot y$, where y is the variable that is associated with the output of the gate that defines the value of C itself. Thus, C is satisfiable if and only if S is satisfiable. Moreover, the construction from C to S builds a constant-sized subexpression for each input and gate of C; hence, this construction runs in polynomial time. Therefore, this local-replacement reduction gives us the following.

Theorem 17.7: CNF-SAT *is* ***NP****-complete.*

3SAT

Consider the 3SAT problem, which takes a Boolean formula S that is in conjunctive normal form (CNF) with each clause in S having exactly three literals, and asks whether S is satisfiable. Recall that a Boolean formula is in CNF if it is formed by the AND of a collection of clauses, each of which is the OR of a set of literals. For example, the following formula could be an instance of 3SAT:

$$(\overline{x_1} + x_2 + \overline{x_7})(x_3 + \overline{x_5} + x_6)(\overline{x_2} + x_4 + \overline{x_6})(x_1 + x_5 + \overline{x_8}).$$

Thus, the 3SAT problem is a restricted version of the CNF-SAT problem. (Note that we cannot use the restriction form of ***NP***-hardness proof, however, for this proof form only works for reducing a restricted version to its more general form.) In this subsection, we show that 3SAT is ***NP***-complete, using the local-replacement form of proof. Interestingly, the 2SAT problem, in which every clause has exactly two literals, can be solved in polynomial time. (See Exercises C-17.6 and C-17.7.)

Note that 3SAT is in ***NP***, for we can construct a nondeterministic polynomial-time algorithm that takes a CNF formula S with 3-literals per clause, guesses an assignment of Boolean values for S, and then evaluates S to see if it is equal to 1.

To prove that 3SAT is ***NP***-hard, we reduce the CNF-SAT problem to it in polynomial time. Let C be a given Boolean formula in CNF. We perform the following local replacement for each clause C_i in C:

- If $C_i = (a)$, that is, it has one term, which may be a negated variable, then we replace C_i with $S_i = (a + b + c) \cdot (a + \overline{b} + c) \cdot (a + b + \overline{c}) \cdot (a + \overline{b} + \overline{c})$, where b and c are new variables not used anywhere else.
- If $C_i = (a + b)$, that is, it has two terms, then we replace C_i with the subformula $S_i = (a + b + c) \cdot (a + b + \overline{c})$, where c is a new variable not used anywhere else.
- If $C_i = (a + b + c)$, that is, it has three terms, then we set $S_i = C_i$.
- If $C_i = (a_1 + a_2 + a_3 + \cdots + a_k)$, that is, it has $k > 3$ terms, then we replace C_i with $S_i = (a_1 + a_2 + b_1) \cdot (\overline{b_1} + a_3 + b_2) \cdot (\overline{b_2} + a_4 + b_3) \cdots (\overline{b_{k-3}} + a_{k-1} + a_k)$, where $b_1, b_2, \ldots, b_{k-1}$ are new variables not used anywhere else.

Notice that the value assigned to the newly introduced variables is completely irrelevant. No matter what we assign them, the clause C_i is 1 if and only if the small formula S_i is also 1. Thus, the original clause C is 1 if and only if S is 1. Moreover, note that each clause increases in size by at most a constant factor and that the computations involved are simple substitutions. Therefore, we have shown how to reduce an instance of the CNF-SAT problem to an equivalent instance of the 3SAT problem in polynomial time. This, together with the earlier observation about 3SAT belonging to ***NP***, gives us the following theorem.

Theorem 17.8: 3SAT *is* ***NP****-complete.*

17.4 VERTEX-COVER, CLIQUE, and SET-COVER

In the VERTEX-COVER problem, we are given a graph G and an integer k and asked whether there is a vertex cover for G containing at most k vertices. That is, VERTEX-COVER asks whether there is a subset C of vertices of size at most k, such that for each edge (v, w), we have $v \in C$ or $w \in C$. We showed, in Lemma 17.4, that VERTEX-COVER is in ***NP***.

VERTEX-COVER is *NP*-Complete

Given that VERTEX-COVER is in ***NP***, to show that VERTEX-COVER is ***NP***-complete, we will show that VERTEX-COVER is ***NP***-hard, by reducing the 3SAT problem to it in polynomial time. This reduction is interesting in two respects. First, it shows an example of reducing a logic problem to a graph problem. Second, it illustrates an application of the component-design proof technique.

Let S be a given instance of the 3SAT problem, that is, a CNF formula such that each clause has exactly three literals. We construct a graph G and an integer k such that G has a vertex cover of size at most k if and only if S is satisfiable. We begin our construction by adding the following:

- For each variable x_i used in the formula S, we add two vertices in G, one that we label with x_i and the other we label with $\overline{x_i}$. We also add the edge $(x_i, \overline{x_i})$ to G. (Note: These labels are for our own benefit; after we construct the graph G, we can always relabel vertices with integers if that is what an instance of the VERTEX-COVER problem should look like.)

Each edge $(x_i, \overline{x_i})$ is a "truth-setting" component, for, with this edge in G, a vertex cover must include at least one of x_i or $\overline{x_i}$. In addition, we add the following:

- For each clause $C_i = (a + b + c)$ in S, we form a triangle consisting of three vertices, $i1$, $i2$, and $i3$, and three edges, $(i1, i2)$, $(i2, i3)$, and $(i3, i1)$.

Note that any vertex cover will have to include at least two of the vertices in $\{i1, i2, i3\}$ for each such triangle. Each such triangle is a "satisfaction-enforcing" component. We then connect these two types of components, by adding, for each clause $C_i = (a + b + c)$, the edges $(i1, a)$, $(i2, b)$, and $(i3, c)$. (See Figure 17.9.) Finally, we set the integer parameter $k = n + 2m$, where n is the number of variables in S and m is the number of clauses. Thus, if there is a vertex cover of size at most k, it must have size exactly k. This completes the construction of an instance of the VERTEX-COVER problem. This construction clearly runs in polynomial time, so let us consider its correctness.

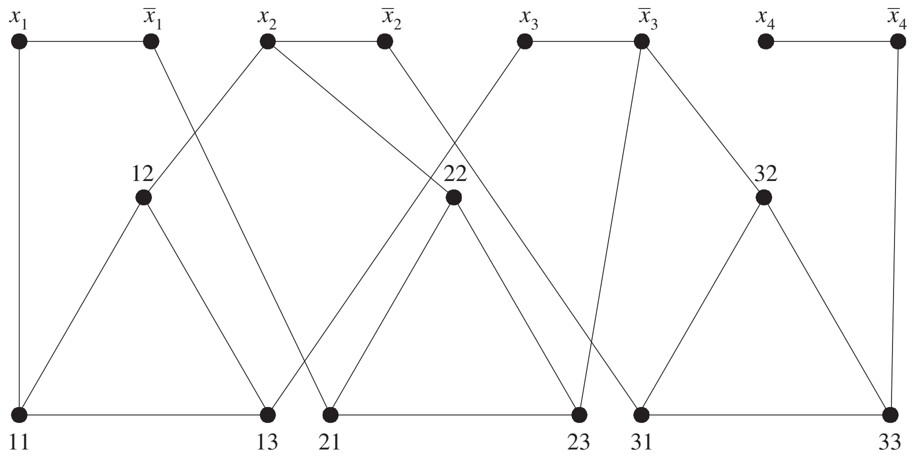


Figure 17.9: Example graph G as an instance of the VERTEX-COVER problem constructed from the formula $S = (x_1 + x_2 + x_3) \cdot (\overline{x_1} + x_2 + \overline{x_3}) \cdot (\overline{x_2} + \overline{x_3} + \overline{x_4})$.

Suppose there is an assignment of Boolean values to variables in S so that S is satisfied. From the graph G constructed from S, we can build a subset of vertices C that contains each literal a (in a truth-setting component) that is assigned 1 by the satisfying assignment. Likewise, for each clause $C_i = (a + b + c)$, the satisfying assignment sets at least one of a, b, or c to 1. Whichever one of a, b, or c is 1 (picking arbitrarily if there are ties), we include the other two in our subset C. This C is of size $n + 2m$. Moreover, notice that each edge in a truth-setting component and clause-satisfying component is covered, and two of every three edges incident on a clause-satisfying component are also covered. In addition, notice that an edge incident to a component associated clause C_i that is not covered by a vertex in the component must be covered by the node in C labeled with a literal, for the corresponding literal in C_i is 1.

Suppose then the converse, namely, that there is a vertex cover C of size at most $n+2m$. By construction, this set must have size exactly $n+2m$, for it must contain one vertex from each truth-setting component and two vertices from each clause-satisfying component. This leaves one edge incident to a clause-satisfying component that is not covered by a vertex in the clause-satisfying component; hence, this edge must be covered by the other endpoint, which is labeled with a literal. Thus, we can assign the literal in S associated with this node 1 and each clause in S is satisfied; hence, all of S is satisfied. Therefore, S is satisfiable if and only if G has a vertex cover of size at most k. This gives us the following.

Theorem 17.9: VERTEX-COVER *is* ***NP****-complete.*

As mentioned before, the above reduction illustrates the component-design technique. We constructed truth-setting and clause-satisfying components in our graph G to enforce important properties in the clause S.

CLIQUE

As with the VERTEX-COVER problem, there are several other problems that involve selecting a subset of objects from a larger set so as to optimize the size the subset can have while still satisfying an important property. The next such problem we consider is the CLIQUE problem.

A ***clique*** in a graph G is a subset C of vertices such that, for each v and w in C, with $v \neq w$, (v, w) is an edge. That is, there is an edge between every pair of distinct vertices in C. Problem CLIQUE takes a graph G and an integer k as input and asks whether there is a clique in G of size at least k.

We leave as a simple exercise (R-17.7) to show that CLIQUE is in ***NP***. To show CLIQUE is ***NP***-hard, we reduce the VERTEX-COVER problem to it. Therefore, let (G, k) be an instance of the VERTEX-COVER problem. For the CLIQUE problem, we construct the complement graph G^c, which has the same vertex set as G, but has the edge (v, w), with $v \neq w$, if and only if (v, w) is not in G. We define the integer parameter for CLIQUE as $n-k$, where k is the integer parameter for VERTEX-COVER. This construction runs in polynomial time and serves as a reduction, for G^c has a clique of size at least $n - k$ if and only if G has a vertex cover of size at most k. (See Figure 17.10.)

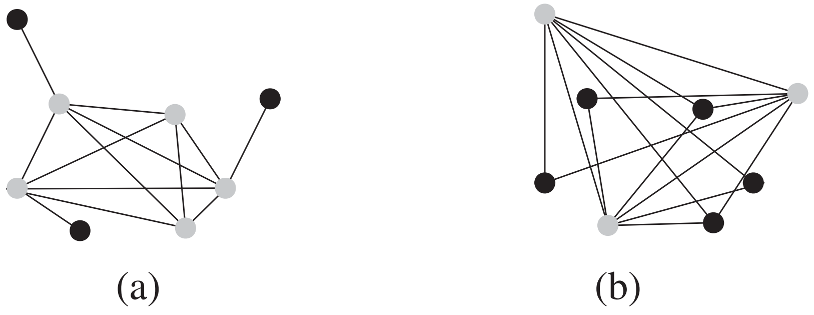


Figure 17.10: A graph G illustrating the proof that CLIQUE is ***NP***-hard. (a) Shows the graph G with the nodes of a clique of size 5 shaded in gray. (b) Shows the graph G^c with the nodes of a vertex cover of size 3 shaded in gray.

Therefore, we have the following.

Theorem 17.10: CLIQUE *is* ***NP***-*complete.*

Note how simple the above proof by local replacement is. Interestingly, the next reduction, which is also based on the local-replacement technique, is even simpler.

SET-COVER

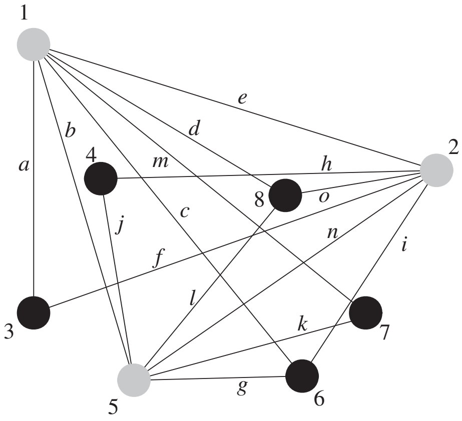
Problem SET-COVER takes a collection of m sets S_1, S_2, …, S_m and an integer parameter k as input and asks whether there is a subcollection of k sets S_{i_1}, S_{i_2},

. . ., S_{i_k}, such that

$$\bigcup_{i=1}^{m} S_i = \bigcup_{j=1}^{k} S_{i_j}.$$

That is, the union of the subcollection of k sets includes every element in the union of the original m sets.

We leave it to an exercise (R-17.14) to show Set-Cover is in ***NP***. As to the reduction, we note that we can define an instance of Set-Cover from an instance G and k of Vertex-Cover. Namely, for each vertex v of G, there is set S_v, which contains the edges of G incident on v. Clearly, there is a set cover among these sets S_v's of size k if and only if there is a vertex cover of size k in G. (See Figure 17.11.)



$$\begin{aligned}
S_1 &= \{a, b, c, m, d, e\}\\
S_2 &= \{e, h, o, n, i\}\\
S_3 &= \{a, f\}\\
S_4 &= \{j, m\}\\
S_5 &= \{b, j, l, n, k, g\}\\
S_6 &= \{g, c, i\}\\
S_7 &= \{k, m\}\\
S_8 &= \{d, o, l\}
\end{aligned}$$

(a) (b)

Figure 17.11: A graph G illustrating the proof that Set-Cover is ***NP***-hard. The vertices are numbered 1 through 8 and the edges are given letter labels a through o. (a) Shows the graph G with the nodes of a vertex cover of size 3 shaded in gray. (b) Shows the sets associated with each vertex in G, with the subscript of each set identifying the associated vertex. Note that $S_1 \cup S_2 \cup S_5$ contains all the edges of G.

Thus we have the following.

Theorem 17.11: Set-Cover *is **NP**-complete.*

This reduction illustrates how easily we can covert a graph problem into a set problem. In the next subsection, we show how we can actually reduce graph problems to number problems.

17.5 SUBSET-SUM and KNAPSACK

Some hard problems involve only numbers. In such cases, we must take extra care to use the size of the input in bits, for some numbers can be very large. To clarify the role that the size of numbers can make, researchers say that a problem L is ***strongly NP-hard*** if L remains ***NP***-hard even when we restrict the value of each number in the input to be bounded by a polynomial in the size (in bits) of the input. An input x of size n would satisfy this condition, for example, if each number i in x was represented using $O(\log n)$ bits. Interestingly, the number problems we study in this section are not strongly ***NP***-hard. (See Exercises C-17.14 and C-17.15.)

In the SUBSET-SUM problem, we are given a set S of n integers and an integer k, and we are asked whether there is a subset of integers in S that sum to k. This problem could arise, for example, as in the following.

Example 17.12: *Suppose we have an Internet web server, and we are presented with a collection of download requests. For each download request we can easily determine the size of the requested file. Thus, we can abstract each web request simply as an integer—the size of the requested file. Given this set of integers, we might be interested in determining a subset of them that exactly sums to the bandwidth our server can accommodate in one minute. Unfortunately, this problem is an instance of SUBSET-SUM. Moreover, because it is **NP**-complete, this problem will actually become harder to solve as our web server's bandwidth and request-handling ability improves.*

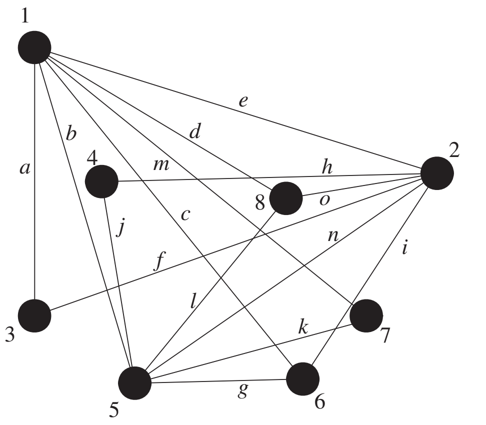
SUBSET-SUM might at first seem easy, and indeed showing that it belongs to ***NP*** is straightforward. (See Exercise R-17.15.) Unfortunately, it is ***NP***-complete, as we now show. Let G and k be given as an instance of the VERTEX-COVER problem. Number the vertices of G from 1 to n and the edges G from 1 to m, and construct the ***incidence matrix*** H for G, defined so that $H[i, j] = 1$ if and only if the edge numbered j is incident on the vertex numbered i; otherwise, $H[i, j] = 0$. (See Figure 17.12.)

We use H to define some admittedly large (but still polynomial-sized) numbers to use as inputs to the SUBSET-SUM problem. Namely, for each row i of H, which encodes all the edges incident on vertex i, we construct the number

$$a_i = 4^{m+1} + \sum_{j=1}^{m} H[i, j]4^j.$$

Note that this number adds in a different power of 4 for each 1-entry in the ith row of $H[i, j]$, plus a larger power of 4 for good measure. The collection of a_i's defines an "incidence component" to our reduction, for each power of 4 in an a_i, except for the largest, corresponds to a possible incidence between vertex i and some edge.

In addition to the above incidence component, we also define an "edge-covering



(a)

H	1	2	3	4	5	6	7	8
a	1	0	1	0	0	0	0	0
b	1	0	0	0	1	0	0	0
c	1	0	0	0	0	1	0	0
d	1	0	0	0	0	0	0	1
e	1	1	0	0	0	0	0	0
f	0	1	1	0	0	0	0	0
g	0	0	0	0	1	1	0	0
h	0	1	0	1	0	0	0	0
i	0	1	0	0	0	1	0	0
j	0	0	0	1	1	0	0	0
k	0	0	0	0	1	0	1	0
l	0	0	0	0	1	0	0	1
m	0	1	0	1	0	0	0	0
n	0	1	0	0	1	0	0	0
o	0	1	0	0	0	0	0	1

(b)

Figure 17.12: A graph G illustrating the proof that SUBSET-SUM is ***NP***-hard. The vertices are numbered 1 through 8 and the edges are given letter labels a through o. (a) Shows the graph G; (b) shows the incidence matrix H for G. Note that there is a 1 for each edge in one or more of the columns for vertices 1, 2, and 5.

component," where, for each edge j, we define a number

$$b_j = 4^j.$$

We then set the sum we wish to attain with a subset of these numbers as

$$k' = k4^{m+1} + \sum_{j=1}^{m} 2 \cdot 4^j,$$

where k is the integer parameter for the VERTEX-COVER instance.

Let us consider how this reduction, which clearly runs in polynomial time, actually works. Suppose graph G has a vertex cover $C = \{i_1, i_2, \ldots, i_k\}$, of size k. Then we can construct a set of values adding to k' by taking every a_i with an index in C, that is, each a_{i_r} for $r = 1, 2, \ldots, k$. In addition, for each edge numbered j in G, if only one of j's endpoints is included in C, then we also include b_j in our subset. This set of numbers sums to k', for it includes k values of 4^{m+1} plus 2 values of each 4^j (either from two a_{i_r}'s such that this edge has both endpoints in C or from one a_{i_r} and one b_j if C contains just one endpoint of edge j).

Suppose there is a subset of numbers suming to k'. Since k' contains k values of 4^{m+1}, it must include exactly k a_i's. Let us include vertex i in our cover for each such a_i. Such a set is a cover, for each edge j, which corresponds to a power 4^j, must contribute two values to this sum. Since only one value can come from a b_j, one must have come from at least one of the chosen a_i's. Thus we have the following:

Theorem 17.13: SUBSET-SUM *is **NP**-complete.*

Knapsack

In the Knapsack problem, illustrated in Figure 17.13, we are given a set S of items, numbered 1 to n. Each item i has an integer size, s_i, and worth, w_i. We are also given two integer parameters, s, and w, and are asked whether there is a subset, T, of S such that

$$\sum_{i \in T} s_i \leq s, \quad \text{and} \quad \sum_{i \in T} w_i \geq w.$$

Problem Knapsack defined above is the decision version of the optimization problem "0-1 knapsack" discussed in Section 12.6.

We can motivate the Knapsack problem with the following Internet application.

Example 17.14: *Suppose we have s widgets that we are interested in selling at an Internet auction website. A prospective buyer i can bid on multiple lots by saying that he or she is interested in buying s_i widgets at a total price of w_i dollars. If multiple-lot requests, such as this, cannot be broken up (that is, buyer i wants exactly s_i widgets), then determining if we can earn w dollars from this auction gives rise to the Knapsack problem. (If lots can be broken up, then our auction optimization problem gives rise to the fractional knapsack problem, which can be solved efficiently using the greedy method of Section 10.1.)*

The Knapsack problem is in ***NP***, for we can construct a nondeterministic polynomial-time algorithm that guesses the items to place in our subset T and then verifies that they do not violate the s and w constraints, respectively.

Knapsack is also ***NP***-hard, as it actually contains the Subset-Sum problem as a special case. In particular, any instance of numbers given for the Subset-Sum problem can correspond to the items for an instance of Knapsack with each $w_i = s_i$ set to a value in the Subset-Sum instance and the targets for the size s and worth w both equal to k, where k is the integer we wish to sum to for the Subset-Sum problem. Thus, by the restriction proof technique, we have the following.

Theorem 17.15: Knapsack *is* ***NP***-*complete.*

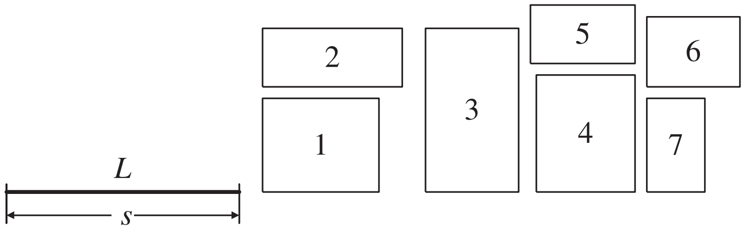


Figure 17.13: A geometric view of the Knapsack problem. Given a line L of length s, and a collection of n rectangles, can we translate a subset of the rectangles to have their bottom edge on L so that the total area of the rectangles touching L is at least w? Here, the width of rectangle i is s_i and its area is w_i.

17.6 HAMILTONIAN-CYCLE and TSP

The last two ***NP***-complete problems we consider ask about the existence of certain kinds of cycles in a graph. Such problems are useful for optimizing the travel of robots and circuit-board drills, as discussed at the start of this chapter.

HAMILTONIAN-CYCLE

HAMILTONIAN-CYCLE is the problem that takes a graph G and asks whether there is a cycle in G that visits each vertex in G exactly once, returning to its starting vertex. (See Figure 17.14a.) It is relatively easy to show that HAMILTONIAN-CYCLE is in ***NP***—guess a sequence of vertices and verify that each consecutive pair of vertices in this sequence is connected by an edge and that every vertex (other than the starting and ending vertex) is visited exactly once. To show that this problem is ***NP***-complete, we will reduce VERTEX-COVER to it, using a component-design type of reduction.

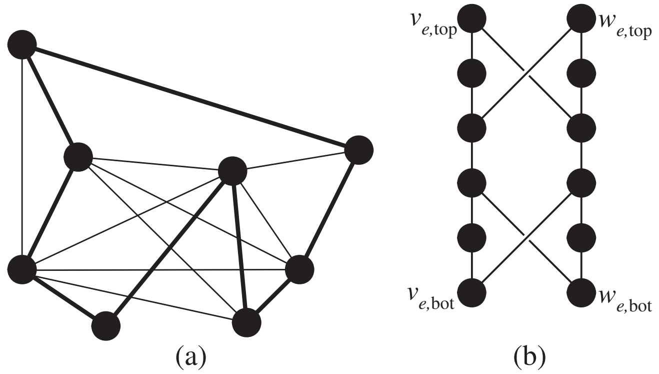


Figure 17.14: Illustrating the HAMILTONIAN-CYCLE problem and its ***NP***-completeness proof. (a) Shows an example graph with a Hamiltonian cycle shown in bold. (b) Illustrates a cover-enforcer subgraph H_e used to show that HAMILTONIAN-CYCLE is ***NP***-hard.

Let G and k be a given instance of the VERTEX-COVER problem. We will construct a graph H that has a Hamiltonian cycle if and only if G has a vertex cover of size k. We begin by including a set of k initially disconnected vertices $X = \{x_1, x_2, \ldots, x_k\}$ to H. This set of vertices will serve as a "cover-choosing" component, for they will serve to identify which nodes of G should be included in a vertex cover. In addition, for each edge $e = (v, w)$ in G we create a "cover-enforcer" subgraph H_e in H. This subgraph H_e has 12 vertices and 14 edges as shown in Figure 17.14b.

Six of the vertices in the cover-enforcer H_e for $e = (v, w)$ correspond to v and the other six correspond to w. Moreover, we label two vertices in cover-enforcer

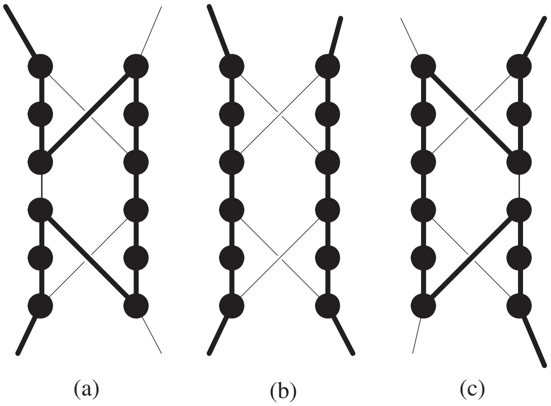


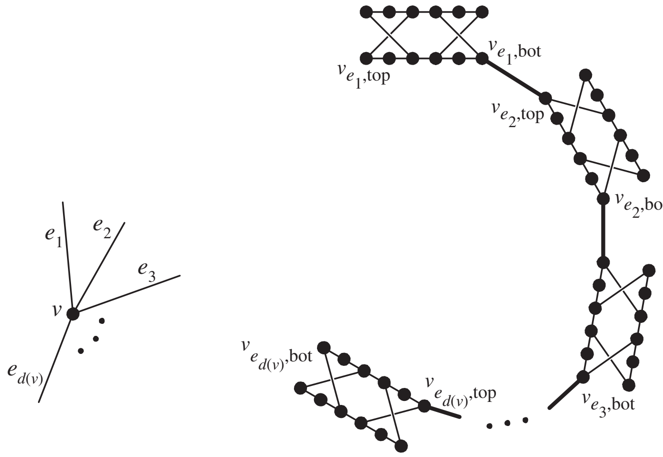
Figure 17.15: The three possible ways that a Hamiltonian cycle can visit the edges in a cover-enforcer H_e.

H_e that correspond to v as $v_{e,\text{top}}$ and $v_{e,\text{bot}}$, and we label two vertices in H_e that correspond to w as $w_{e,\text{top}}$ and $w_{e,\text{bot}}$. These are the only vertices in H_e that will be connected to any other vertices in H outside of H_e. Thus, a Hamiltonian cycle can visit the nodes of H_e in only one of three possible ways, as shown in Figure 17.15.

We join the important vertices in each cover-enforcer H_e to other vertices in H in two ways, one that corresponds to the cover-choosing component and one that corresponds to the cover-enforcing component. For the cover-choosing component, we add an edge from each vertex in X to every vertex $v_{e,\text{top}}$ and every vertex $v_{e,\text{bot}}$. That is, we add $2kn$ edges to H, where n is the number of vertices in G.

For the cover-enforcing component, we consider each vertex v in G in turn. For each such v, let $\{e_1, e_2, \ldots, e_{d(v)}\}$ be a listing of the edges of G that are incident upon v. We use this listing to create edges in H by joining $v_{e_i,\text{bot}}$ in H_{e_i} to $v_{e_{i+1},\text{top}}$ in $H_{e_{i+1}}$, for $i = 1, 2, \ldots, d-1$. (See Figure 17.16.) We refer to the H_{e_i} components joined in this way as belonging to the ***covering thread*** for v. This completes the construction of the graph H. Note that this computation runs in polynomial time in the size of G.

We claim that G has a vertex cover of size k if and only if H has a Hamiltonian cycle. Suppose, first, that G has a vertex cover of size k. Let $C = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ be such a cover. We construct a Hamiltonian cycle in H, by connecting a series of paths P_j, where each P_j starts at x_j and ends at x_{j+1}, for $j = 1, 2, \ldots, k-1$, except for the last path P_k, which starts at x_k and ends at x_1. We form such a path P_j as follows. Start with x_j, and then visit the entire covering thread for v_{i_j} in H, returning to x_{j+1} (or x_1 if $j = k$). For each cover-enforcer subgraph H_e in the covering thread for v_{i_j}, which is visited in this P_j, we write, without loss of generality, e as (v_{i_j}, w). If w is not also in C, then we visit this H_e as in Figure 17.15a or Figure 17.15c (with respect to v_{i_j}). Instead, if w is also in C, then we visit this H_e as in Figure 17.15b. In this way we will visit each vertex in H exactly once, since C is a vertex cover for G. Thus, this cycle is in fact a Hamiltonian cycle.



(a) (b)

Figure 17.16: Connecting the cover-enforcers. (a) A vertex v in G and its set of incident edges $\{e_1, e_2, \ldots, e_{d(v)}\}$. (b) The connections made between the H_{e_i}'s in H for the edges incident upon v.

Suppose, conversely, that H has a Hamiltonian cycle. Since this cycle must visit all the vertices in X, we break this cycle up into k paths, P_1, P_2, . . ., P_k, each of which starts and ends at a vertex in X. Moreover, by the structure of the cover-enforcer subgraphs H_e and the way that we connected them, each P_j must traverse a portion (possibly all) of a covering thread for a vertex v in G. Let C be the set of all such vertices in G. Since the Hamiltonian cycle must include the vertices from every cover-enforcer H_e and every such subgraph must be traversed in a way that corresponds to one (or both) of e's endpoints, C must be a vertex cover in G.

Therefore, G has a vertex cover of size k if and only if H has a Hamiltonian cycle. This gives us the following.

Theorem 17.16: *H* AMILTONIAN-CYCLE *is **NP**-complete.*

TSP

In the ***traveling salesperson problem***, or TSP, we are given an integer parameter k and a graph G, such that each edge e in G is assigned an integer cost $c(e)$, and we are asked whether there is a cycle in G that visits all the vertices in G (possibly more than once) and has total cost at most k. We have already established that TSP is in ***NP***, in Lemma 17.2. Given this fact, showing that TSP is ***NP***-complete is easy, as the TSP problem contains the HAMILTONIAN-CYCLE problem as a special case. Namely, given an instance G of the HAMILTONIAN-CYCLE problem, we can create an instance of TSP by assigning each edge in G the cost $c(e) = 1$ and setting the integer parameter $k = n$, where n is the number of vertices in G. Therefore, using the restriction form of reduction, we get the following.

Theorem 17.17: TSP *is **NP**-complete.*

17.7 Exercises

Reinforcement

R-17.1 Professor Amongus has shown that a decision problem L is polynomial-time reducible to an ***NP***-complete problem M. Moreover, after 80 pages of dense mathematics, he has also just proven that L can be solved in polynomial time. Has he just proven that ***P*** = ***NP*** ? Why, or why not?

R-17.2 Use a truth table to convert the Boolean formula $B = (a \leftrightarrow (b + c))$ into an equivalent formula in CNF. Show the truth table and the intermediate DNF formula for $\overline{B}$.

R-17.3 Show that the problem SAT, which takes an arbitrary Boolean formula S as input and asks whether S is satisfiable, is ***NP***-complete.

R-17.4 Consider the problem DNF-SAT, which takes a Boolean formula S in disjunctive normal form (DNF) as input and asks whether S is satisfiable. Describe a deterministic polynomial-time algorithm for DNF-SAT.

R-17.5 Consider the problem DNF-DISSAT, which takes a Boolean formula S in disjunctive normal form (DNF) as input and asks whether S is dissatisfiable, that is, there is an assignment of Boolean values to the variables of S so that it evaluates to 0. Show that DNF-DISSAT is ***NP***-complete.

R-17.6 Convert the Boolean formula $B = (x_1 \leftrightarrow x_2) \cdot (\overline{x_3} + x_4 x_5) \cdot (\overline{x_1 x_2} + x_3 \overline{x_4})$ into CNF.

R-17.7 Show that the CLIQUE problem is in ***NP***.

R-17.8 Given the CNF formula $B = (x_1) \cdot (\overline{x_2} + x_3 + x_5 + \overline{x_6}) \cdot (x_1 + x_4) \cdot (x_3 + \overline{x_5})$, show the reduction of B into an equivalent input for the 3SAT problem.

R-17.9 Given $B = (x_1 + \overline{x_2} + x_3) \cdot (x_4 + x_5 + \overline{x_6}) \cdot (x_1 + \overline{x_4} + \overline{x_5}) \cdot (x_3 + x_4 + x_6)$, draw the instance of VERTEX-COVER that is constructed by the reduction from 3SAT of the Boolean formula B.

R-17.10 Draw an example of a graph with 10 vertices and 15 edges that has a vertex cover of size 2.

R-17.11 Draw an example of a graph with 10 vertices and 15 edges that has a clique of size 6.

R-17.12 Professor Amongus has just designed an algorithm that can take any graph G with n vertices and determine in $O(n^k)$ time whether G contains a clique of size k. Does Professor Amongus deserve the Turing Award for having just shown that ***P*** = ***NP***? Why or why not?

R-17.13 Is there a subset of the numbers in $\{23, 59, 17, 47, 14, 40, 22, 8\}$ that sums to 100? What about 130? Show your work.

R-17.14 Show that the SET-COVER problem is in ***NP***.

R-17.15 Show that the SUBSET-SUM problem is in ***NP***.

R-17.16 Draw an example of a graph with 10 vertices and 20 edges that has a Hamiltonian cycle. Also, draw an example of a graph with 10 vertices and 20 edges that does not have a Hamiltonian cycle.

R-17.17 The ***Manhattan distance*** between two points (a, b) and (c, d) in the plane is $|a - c| + |b - d|$. Using Manhattan distance to define the cost between every pair of points, find an optimal traveling salesperson tour of the following set of points: $\{(1, 1), (2, 8), (1, 5), (3, -4), (5, 6), (-2, -6)\}$.

Creativity

C-17.1 Let n denote the size of an input in bits and N denote the size in a number of items. Define an algorithm to be ***c-incremental*** if any primitive operation involving one or two objects represented with b bits results in an object represented with at most $b + c$ bits, for $c \geq 0$. Show that an algorithm using multiplication as a primitive operation may not be c-incremental for any constant c.

C-17.2 Using the definition of a c-incremental algorithm from the previous exercise, show that, if a c-incremental algorithm A has a worst-case running time $t(N)$ in the RAM model, as a function of the number of input items, N, for some constant $c > 0$, then A has running time $O(n^2 t(n))$, in terms of the number, n, of bits in a standard binary encoding of the input.

C-17.3 Show that we can deterministically simulate in polynomial time any nondeterministic algorithm A that runs in polynomial time and makes at most $O(\log n)$ calls to the choose method, where n is the size of the input to A.

C-17.4 Show that every language L in ***P*** is polynomial-time reducible to the language $M = \{5\}$, that is, the language that simply asks whether the binary encoding of the input is equal to 5.

C-17.5 Show how to construct a Boolean circuit C such that, if we create variables only for the inputs of C and then try to build a Boolean formula that is equivalent to C, then we will create a formula exponentially larger than an encoding of C.

Hint: Use recursion to repeat subexpressions in a way that doubles their size each time they are used.

C-17.6 Show that the backtracking algorithm given in Section 18.4.1 for the CNF-SAT problem runs in polynomial time if every clause in the given Boolean formula has at most two literals. That is, it solves 2SAT in polynomial time.

C-17.7 Consider the 2SAT version of the CNF-SAT problem, in which every clause in the given formula S has exactly two literals. Note that any clause of the form $(a + b)$ can be thought of as two implications, $(\overline{a} \rightarrow b)$ and $(\overline{b} \rightarrow a)$. Consider a graph G from S, such that each vertex in G is associated with a variable, x, in S, or its negation, $\overline{x}$. Let there be a directed edge in G from $\overline{a}$ to b for each clause equivalent to $(\overline{a} \rightarrow b)$. Show that S is not satisfiable if and only if there is a variable x such that there is a path in G from x to $\overline{x}$ and a path from $\overline{x}$ to x. Derive from this rule a polynomial-time algorithm for solving this special case of the CNF-SAT problem. What is the running time of your algorithm?

C-17.8 Suppose an oracle has given you a magic computer, C, that when given any Boolean formula B in CNF will tell you in one step whether B is satisfiable. Show how to use C to construct an actual assignment of satisfying Boolean values to the variables in any satisfiable formula B. How many calls do you need to make to C in the worst case in order to do this?

C-17.9 Define SUBGRAPH-ISOMORPHISM as the problem that takes a graph, G, and another graph, H, and determines if H is isomorphic to a subgraph of G. That is, the problem is to determine whether there is a one-to-one mapping, f, of the vertices in H to a subset of the vertices in G such that, if (v, w) is an edge in H, then $(f(v), f(w))$ is an edge in G. Show that SUBGRAPH-ISOMORPHISM is ***NP***-complete.

C-17.10 Define INDEPENDENT-SET as the problem that takes a graph G and an integer k and asks whether G contains an independent set of vertices of size k. That is, G contains a set I of vertices of size k such that, for any v and w in I, there is no edge (v, w) in G. Show that INDEPENDENT-SET is ***NP***-complete.

C-17.11 Define HYPER-COMMUNITY to be the problem that takes a collection of n web pages and an integer k, and determines if there are k web pages that all contain hyperlinks to each other. Show that HYPER-COMMUNITY is ***NP***-complete.

C-17.12 Define PARTITION as the problem that takes a set $S = \{s_1, s_2, \ldots, s_n\}$ of numbers and asks whether there is a subset T of S such that

$$\sum_{s_i \in T} s_i = \sum_{s_i \in S-T} s_i.$$

That is, it asks whether there is a partition of the numbers into two groups that sum to the same value. Show that PARTITION is ***NP***-complete.

C-17.13 Show that the HAMILTONIAN-CYCLE problem on directed graphs is ***NP***-complete.

C-17.14 Show that the SUBSET-SUM problem is solvable in polynomial time if the input is given in a unary encoding. That is, show that SUBSET-SUM is not strongly ***NP***-hard. What is the running time of your algorithm?

C-17.15 Show that the KNAPSACK problem is solvable in polynomial time if the input is given in a unary encoding. That is, show that KNAPSACK is not strongly ***NP***-hard. What is the running time of your algorithm?

C-17.16 Consider the special case of TSP where the vertices correspond to points in the plane, with the cost defined on an edge for every pair (p, q) being the usual Euclidean distance between p and q. Show that an optimal tour will not have any pair of crossing edges.

C-17.17 Given a graph G and two distinct vertices, v and w in G, define HAMILTONIAN-PATH to be the problem of determining whether there is a path that starts at v and ends at w and visits all the vertices of G exactly once. Show that the HAMILTONIAN-PATH problem is ***NP***-complete.

Applications

A-17.1 Imagine that the annual university job fair is scheduled for next month and it is your job to book companies to host booths in the large Truman Auditorium during the fair. Unfortunately, at last year's job fair, a fight broke out between some people from competing companies, so the university president, Dr. Noah Drama, has issued a rule that prohibits any pair of competing companies from both being invited to this year's event. In addition, he has shown you a website that lists the competitors for every company that might be invited to this year's job fair and he has asked you to invite the maximum number of noncompeting companies as possible. Show that the decision version of the problem Dr. Drama has asked you to solve is ***NP***-complete.

A-17.2 Suppose the football coach for the Anteaters has heard about your abilities to solve challenging problems and has hired you to write a computer program that can decide which of their many trophies to feature on their prized trophy shelf. He is asking that you do this as a computer program, rather than just coming up with a single decision, because the Anteaters are getting new trophies every year. The trophies come in all different shapes and sizes, and the ones on the prized trophy shelf have to be lined up next to one another. So the dimension that matters most is a trophy's width in centimeters, which is given as an integer. In addition, the coach has assigned an integer score to each trophy, so that a very prestigious trophy, like the one for winning the championship, would have a high score, whereas a less prestigious trophy, like the one for having the funniest uniforms, would have a low score. Moreover, given his eccentric nature, these scores can be arbitrarily large. He has asked that, given a listing of all the team's trophies along with their widths and prestige scores, your program should choose the set that maximizes the total prestige score and fits on the team's trophy shelf. Show that the decision version of the problem the coach has given you is ***NP***-complete.

A-17.3 Consider the trophy-choosing problem from the previous exercise, but now suppose that each of the prestige scores is an integer in the range from 1 to 10. Describe how you can solve this version of the problem in polynomial time.

A-17.4 Suppose a friend of yours is rushing for one of the university fraternities, Tau Nu Tau (TNT). His job for this week is to arrange all the bottles in the TNT beer-bottle collection in a circle, subject to the constraint that each pair of consecutive bottles must be for beers that were both drunk in some TNT party. He has been given a listing of the beers in the TNT beer-bottle collection, and, for each beer on the list, he is told which other beers were drunk along with this one at some TNT party. Politely show that your friend has been asked to solve an ***NP***-complete problem.

A-17.5 Suppose you are computer security expert working for a major company, CableClock, any you have just discovered that many of the computers at CableClock are infected with malware that must have come from users visiting unsafe websites. For each infected computer, you are given a log file that lists all websites it has visited since the last time it was scanned for malware. Unfortunately, as you look over these log files, you notice that there isn't a single website that they all visited. You conclude, therefore, that there must be a number of websites that are able to inject this malware, and the most likely candidates would be in a smallest collection that is visited by all the infected computers. Show that the decision version of the problem of determining such a collection is ***NP***-complete.

A-17.6 Imagine that you are a Hollywood movie producer who is trying to decide how your new movie should end. To help you make this decision, you would like to assemble a group of movie-goers together to do a focus group. To avoid biases, you have asked that the group be selected so that no two people in the group has previously seen the same movie. So, among the set of possible focus-group members, you have asked that each one fill out a list of all the movies they have seen, and you will be using these lists to make your decision about who to invite to the focus group. Show that the decision version of the problem of finding the largest set of movie-goers for this focus group such that no two people in the group has previously seen the same movie is ***NP***-complete.

A-17.7 Suppose that you and a friend are both taking a Russian literature course and have agreed to buy all of your books together "fifty-fifty," so that for each book purchased, you paid half and your friend paid half. Suppose now that the course has ended and it is time to sell these books to the used-book buyer, who has posted the used-book values of all of your books on her website. Unfortunately, with your differing social calendars, there is no good time for you and your friend to go to the bookstore together to return your shared books. So you need to divide up the books between the two of you so that the total used-book value of the two sets is the same. Show that determining whether such a division of the books is possible, where there is an arbitrary number of books having arbitrary values, is ***NP***-complete.

Chapter Notes

Computing models are discussed in the textbooks by Lewis and Papadimitriou [143], Savage [184] and Sipser [195].

The proof sketch of the Cook-Levin Theorem (17.5) given in this chapter is an adaptation of a proof sketch of Cormen, Leiserson, and Rivest [50]. Cook's original theorem [48] showed that CNF-SAT was ***NP***-complete, and Levin's original theorem [141] was for a tiling problem. We refer to Theorem 17.5 as the "Cook-Levin" Theorem in honor of these two seminal papers, for their proofs were along the same lines as the proof sketch given for Theorem 17.5. Karp [122] demonstrated several more problems to be ***NP***-complete, and subsequently hundreds of other problems have been shown to be ***NP***-complete. Garey and Johnson [80] give a very nice discussion of ***NP***-completeness as well as a catalog of many important ***NP***-complete and ***NP***-hard problems.

The reductions given in this chapter that use local replacement and restriction are well known in the computer science literature; for example, see Garey and Johnson [80] or Aho, Hopcroft, and Ullman [8]. The component-design proof that VERTEX-COVER is ***NP***-complete is an adaptation of a proof of Garey and Johnson [80], as is the component-design proof that HAMILTONIAN-CYCLE is ***NP***-complete, which itself is a combination of two reductions by Karp [122]. The component-design proof that SUBSET-SUM is ***NP***-complete is an adaptation of a proof of Cormen, Leiserson, and Rivest [50].