Formulas, Demostraciones y Derivación Implícita

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Demostraciones

Consideremos un cambio de variable,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Sea $h = x - x_0$, entonces

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Si $x \to x_0$, entonces $h \to 0$.

1) Demostración de $\frac{d}{dx}\sin(x) = \cos(x)$.

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{\sin(x_0 + h) - \sin(x_0)}{h}$$

Recordemos que $\sin(A + B) = \sin(A)\cos(B) + \sin(B)\cos(A)$,

$$= \frac{\sin(x_0)\cos(h) + \sin(h)\cos(x_0) - \sin(x_0)}{h} = \frac{\sin(x_0)(\cos(h) - 1) + \sin(h)\cos(x_0)}{h}$$

$$= \frac{\sin(x_0)(\cos(h) - 1)}{h} \cdot \frac{\cos(h) + 1}{\cos(h) + 1} + \frac{\sin(h)}{h}\cos(x_0)$$

$$\frac{-\sin(x_0)\sin^2(h)}{h(\cos(h) + 1)} + \frac{\sin(h)}{h} \cdot \cos(x_0)$$

$$= \frac{\sin(h)}{h}[\frac{-\sin(x_0)\sin(h)}{\cos(h) + 1}] + \frac{\sin(h)}{h} \cdot \cos(x)$$

Entonces,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{\sin(h)}{h} \cdot \lim_{h \to 0} \frac{-\sin(x_0)\sin(h)}{\cos(h) + 1} + \lim_{h \to 0} \frac{\sin(h)}{h} \cdot \lim_{h \to 0} \cos(x)$$

$$= (1)(0) + (1)(\cos(x_0)) = \cos(x_0)$$

Funciones polinómicas

$$\frac{d}{dx}c = 0$$

$$\frac{d}{dx}x = 1$$

$$\frac{d}{dx}x^n = nx^{n-1},\ n \in \mathbb{R}$$

$$\frac{d}{dx}v^n = nv^{n-1} \cdot \frac{dv}{dx}$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

$$\frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}(uv) = uv' + u'v$$

$$\frac{d}{dx}(\frac{u}{v}) = \frac{vu' - uv'}{v^2}$$

Donde u y v son funciones y c es una constante.

Exampli gratia:

1)

$$y = 7\sqrt{x} - \frac{2}{\sqrt{x}} + \frac{3}{4}\sqrt[3]{x^5} = 7x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + \frac{3}{4}x^{\frac{5}{3}}$$

$$y' = 7 \cdot \frac{1}{2}x^{-\frac{1}{2}} - 2 \cdot (-\frac{1}{2})x^{-\frac{3}{2}} + \frac{3}{4} \cdot \frac{5}{3}x^{\frac{2}{3}} = \frac{7}{2\sqrt{x}} + \frac{1}{\sqrt{x^3}} + \frac{5}{4}\sqrt[3]{x^2}$$

2)

$$y = \sqrt[\pi]{9} - \sqrt[\sqrt{2}]{4x}$$

$$y' = \sqrt[\sqrt{2}]{4} \cdot 1 = \sqrt[\sqrt{2}]{4}$$

3)

$$y = (2x-3)^{2027}$$

$$y' = 2027(2x-3)^{2026} \cdot 2 = 4054(2x-3)^{2026}$$

4)

$$y = \cos(x^2)$$

$$y' = (-\sin(x^2))2x = -2x\sin(x^2)$$

5)

$$y = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$y' = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$= (\frac{1}{\cos^2(x)})^2 = (\sec(x))^2 = \sec^2(x)$$

6)

$$y = \sqrt{4x-3} = (4x-3)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(4x-3)^{-\frac{1}{2}} \cdot 4 = \frac{2}{\sqrt{4x-3}}$$

7)

$$y = (8x-3)^5(3x+7)^8$$

$$u = (8x-3)^5 \Rightarrow u' = 5(8x-3)^4 \cdot 8 = 40(8x-3)^4$$

$$v = (3x+7)^8 \Rightarrow v' = 8(3x+7)^7 \cdot 3 = 24(3x+7)^7$$

$$y' = (8x-3)^5 \cdot 24(3x+7)^7 + 40(8x-3)^4 \cdot (3x+7)^8 = 8(8x-3)^4(3x+7)^7[3(8x-3)+5(3x+7)]$$

$$= 8(8x-3)^4(3x+7)^7(39x+26) = 104(8x-3)^4(3x+7)^7(3x+2)$$

Nota: Grandville y Spivak para estas demostraciones.

Funciones trigonométricas

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\sin(v) = v'\cos(v)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\cos(v) = -v'\sin(v)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\tan(v) = v'\sec^2(v)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\frac{d}{dx}\cot(v) = -v'\csc^2(v)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\sec(v) = v'\sec(v)\tan(v)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\csc(v) = -v'\csc(v)\cot(v)$$

Donde v es el argumento.

Funciones logaritmicas

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^v = v'e^v$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\ln(v) = \frac{v'}{v}$$

8)

$$y = \sin(\sqrt{x})$$

$$y' = \frac{1}{2\sqrt{x}} \cdot \cos\sqrt{x} = \frac{\cos(\sqrt{x})}{2\sqrt{x}}$$

9)

$$y = e^{-\frac{1}{3}x}$$

$$y' = -\frac{1}{3}e^{-\frac{1}{3}x}$$

10)

$$y = \ln(\cos(x))$$

$$y' = \frac{1}{\cos(x)} \cdot (-\sin(x)) = -\frac{\sin(x)}{\cos(x)} = -\tan(x)$$

11)

$$y = \ln(\ln(\ln(\ln(x))))$$

$$y' = \frac{[\ln(\ln(\ln(x)))]'}{\ln(\ln(\ln(x)))} = \frac{1}{\ln(\ln(\ln(x)))} \cdot \frac{[\ln(\ln(x))]'}{\ln(\ln(x))} = \frac{1}{\ln(\ln(\ln(x)))} \cdot \frac{1}{\ln(\ln(x))} \cdot \frac{(\ln(x))'}{\ln(x)}$$

$$= \frac{1}{\ln(\ln(\ln(x)))} \cdot \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

Recordemos que $a\ln(x) = \ln(x^a)$,

$$y' = \frac{1}{x\ln(x)\ln(\ln(x))\ln(\ln(\ln(x)))} = \frac{1}{\ln(\ln(\ln(x)))^{\ln(\ln(x))^{\ln(x^x)}}}$$

Derivación implícita

$$x^2 + y^2 = 1 \to y = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}} \Rightarrow y' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

La derivación implícita requiere de derivar con la expresión *en crudo*.

Exampli gratia: Derivemos ambos miembros de la ecuación $x^2 + y^2 = 1$,

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

$$2x + 2y \cdot y' = 0$$

$$2yy' = -2x$$

$$y' = \frac{-2x}{2y} = -\frac{x}{y}$$

Pero $y = \sqrt{1-x^2}$,

$$y' = \frac{-x}{\sqrt{1-x^2}}$$

Demostraciones de derivación implícita

Exampli gratia:

1) Demostremos que $\frac{d}{dx}e^x = e^x$,

Sea $y = e^x$,

$$\ln(y) = \ln(e^x)$$

$$\ln(y) = x\ln(e)$$

$$\ln(y) = x \cdot 1$$

$$\ln(y) = x$$

Derivamos implícitamente:

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}x$$

$$\frac{y'}{y} = 1$$

$$y' = y$$

$$y' = e^x$$

2) $\frac{d}{dx}a^v = a^v \ln(a) \cdot v'$.

Demostración:

Sea $y = a^v$,

$$\ln(y) = \ln(a^v)$$

$$\ln(y) = v\ln(a)$$

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}(\ln(a) \cdot v)$$

$$\frac{y'}{y} = \ln(a) \cdot v'$$

$$y' = \ln(a)v'y$$

$$y' = \ln(a)v'a^v$$