VECTOR SPACE

REFERENCE: INTRODUCTION TO LINEAR ALGEBRA (GILBERT STRANG)

VECTOR SPACE

Any subset $E \subseteq \mathbb{C}^N$ coupled with addition and multiplication operations that satisfies:

- Commutation: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- Association: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ $\quad (\alpha\beta)\mathbf{x} = \alpha(\beta\boldsymbol{x})$
- Distribution: $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ $\quad (\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
- Additive identity: $\mathbf{x} + \mathbf{0} = \mathbf{x}$
- Additive inverse: $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
- Multiplicative identity: $\mathbf{1}.\,\mathbf{x} = \mathbf{x}$

where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E \; and \; \alpha, \beta \in \mathbb{C}$

SUBSPACE

- Any set $M \subseteq E$ for which the following properties hold:
  - Closure under addition: $\mathbf{x}, \mathbf{y} \in M \to \mathbf{x} + \mathbf{y} \in M$
  - Closure under scalar multiplication: $\mathbf{x} \in M, \alpha \in \mathbb{C} \to \alpha\mathbf{x} \in M$

LINEAR INDEPENDENCE

- Given $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in vector space V
- S is a linearly independent set if $\sum_i \alpha_i \mathbf{v}_i = 0$ is true only when $\alpha_i = 0 \ \forall \ i$

SPAN

- Given $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in vector space V
- S spans V if every vector in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

BASIS

- $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for subspace V iff
    - S spans V
    - S is linearly independent
- Orthogonal basis: $< \mathbf{v}_i, \mathbf{v}_j > = \begin{cases} \neq 0 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$
- Orthonormal basis is an orthogonal basis where $\|\mathbf{v}_i\| = 1 \ \forall \ i$
    - Hence, $< \mathbf{v}_i, \mathbf{v}_j > = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$
    - Good for coordinate systems