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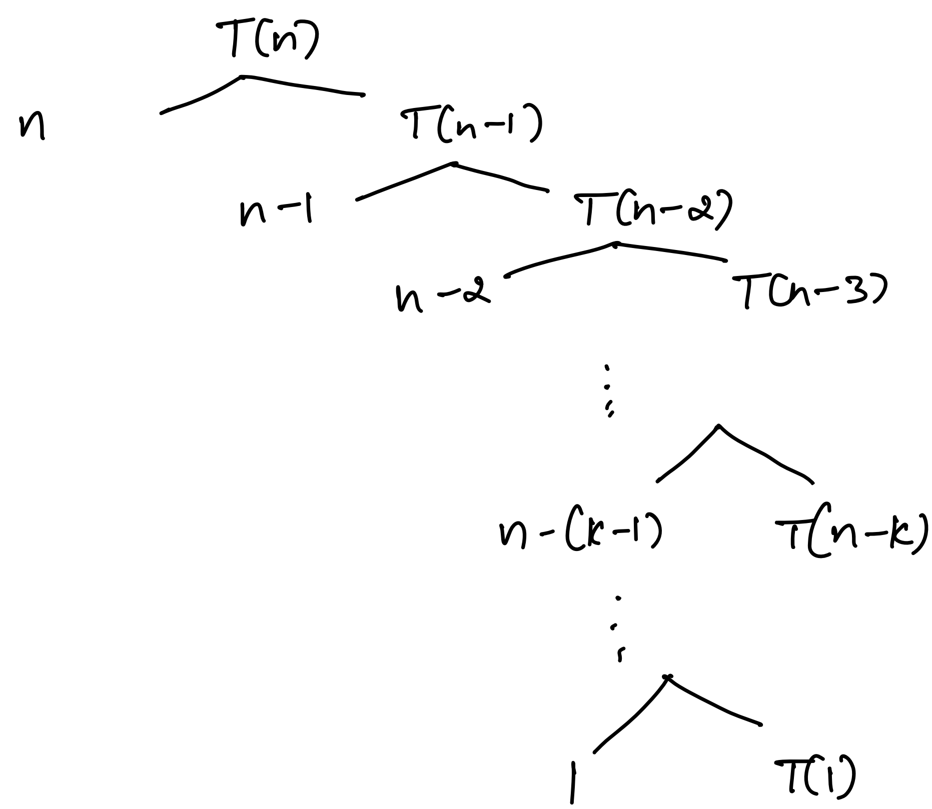
Assignment - 3

4.3-1
Show that the solution of $T(n) = T(n-1) + n$ is $O(n^2)$.

$$T(n) = \begin{cases} 1 & n = 0 \\ T(n-1) + n & n > 0 \end{cases}$$

Recursion tree



Substitution Method

$$T(n) = T(n-1) + n$$
$$= T(n-2) + n-1 + n$$
$$= T(n-3) + n-2 + n-1 + n$$
$$\vdots$$
$$= T(n-k) + n-(k-1) + n-(k-2) + \dots + (n-2) + (n-1) + n$$
$$= T(n-k) + (n-k+1) + (n-k+2) + \dots\dots + (n-2) + (n-1) + n$$

Assume $n-k=0 \Rightarrow n=k$

$$= T(0) + 1 + 2 + 3 + \dots \dots + (n-2) + (n-1) + n$$

$$T(n) = 1 + \frac{n(n+1)}{2} = \frac{n^2+n+2}{2}$$

$\therefore O(n^2)$

4.3-2

Show that the solution of $T(n) = T(\lceil n/2 \rceil) + 1$ is $O(\lg n)$.

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lceil n/2 \rceil) + 1 & n > 1 \end{cases}$$

$$T(n) = T(\lceil n/2 \rceil) + 1 \qquad T(\lceil n/2 \rceil) = T(n/2 + 1)$$

$$= T(n/2 + 1) + 1$$

$$= T(n/2^2 + 1) + 1 + 1$$

$$= T(n/2^3 + 1) + 1 + 1 + 1$$

$$\vdots$$

$$T(n) = T(n/2^k + 1) + 1 + 1 + \cdots\cdots + 1 = T(n/2^k + 1) + k$$

Assume $\frac{n}{2^k} = 1 \rightarrow n = 2^k$

$$k = \log_2 n$$

$$= T(1) + k(1)$$

$$T(n) = 1 + \log_2 n$$

$$\therefore \quad O(\log_2 n)$$

4.3-3

We saw that the solution of $T(n) = 2T(\lfloor n/2 \rfloor) + n$ is $O(n \lg n)$. Show that the solution of this recurrence is also $\Omega(n \lg n)$. Conclude that the solution is $\Theta(n \lg n)$.

lets assume $T(n) \leq cn \log_2 n$

$$T(n) \leq 2c \lfloor n/2 \rfloor \log_2 \lfloor n/2 \rfloor + n$$

$$\leq cn \log_2 (n/2) + n$$

$$\leq cn \log n - cn \log 2 + n$$

$$\le cn\log n + (1-c)n$$
$$\le cn\log_2 n$$
for $c \ge 0$

Next, we guess $T(n) \ge c(n+2)\log_2(n+2)$

$$
\begin{aligned}
T(n) &\ge 2c(\lfloor n/2\rfloor+2)\log_2(\lfloor n/2\rfloor+2)+n \\
&\ge 2c(n/2-1+2)\log_2(n/2-1+2)+n \\
&\ge 2c(n/2+1)\log_2(n/2+1)+n \\
&\ge \cancel{2}c\left(\frac{n+2}{\cancel{2}}\right)\log_2\left(\frac{n+2}{2}\right)+n \\
&\ge c(n+2)\log_2(n+2) - c(n+2)\log_2 2 + n \\
&\ge c(n+2)\log_2(n+2) + (1-c)n - 2c \\
&\ge c(n+2)\log_2(n+2) \quad \text{for } n \ge \frac{2c}{1-c},
\end{aligned}
$$

$0<c<1$

4.4-6

Argue that the solution to the recurrence $T(n) = T(n/3) + T(2n/3) + cn$, where c is a constant, is $\Omega(n \lg n)$ by appealing to a recursion tree.

The above recurrence has 2 branches:

1) Growing at a rate of $n/3$
2) Growing at a rate of $2n/3$

Lower bound is decided by one that terminates faster. So, $T(n/3)$ will terminate faster than $2T(n/3)$

At depth $i = 0, 1, 2, \ldots \log_3 n$ of the tree, the the leaves cost cn

$\therefore$ Total cost of tree $= T(n) = \sum_{i=0}^{\log_3 n} cn$

$$T(n) = (\log_3 n + 1)\, cn$$
$$= cn \log_3 n + cn$$
$$= \frac{cn \log n}{\log 3} + cn$$
$$\geq cn \log n \quad \text{where } cn \geq \log 3$$
$$= \Omega(n \log n)$$

4.4-9

Use a recursion tree to give an asymptotically tight solution to the recurrence $T(n) = T(\alpha n) + T((1-\alpha)n) + cn$, where α is a constant in the range $0 < \alpha < 1$ and $c > 0$ is also a constant.

Assume $0 < \alpha < 1/2$ and $1 - \alpha = \beta$

So, depth of tree is $\log_{1/\alpha} n$ and each level costs cn

Lets guess that leaves are $\Theta(n)$

$$T(n) = \sum_{i=0}^{\log_{1/\alpha} n} cn + \Theta(n)$$

$$= cn \log_{1/\alpha} n + \Theta(n)$$

$$= \Theta(n \log n)$$

(or) <u>using Substitution method</u>

Let $T(n) \leq dn\log_2 n$ for $d > 0$ to prove upper bound

$$T(n) = T(\alpha n) + T((1-\alpha)n) + cn$$

$$\leq d\alpha n \log_2(\alpha n) + d(1-\alpha)\log_2 (1-\alpha)n + cn$$

$$= d\alpha n \log_2 n + d\alpha n \log_2 \alpha + d(1-\alpha) n \log_2 (1-\alpha) + cn$$

$$= dn\log_2 n + dn\left(\alpha \log_2 \alpha + (1-\alpha)\log_2 (1-\alpha)\right) + cn$$

$$\leq dn \log_2 n$$

where, $d \geq \dfrac{-c}{\alpha \log_2 \alpha + (1-\alpha)\log_2(1-\alpha)}$

Let $T(n) \geq dn\log_2 n$ for $d>0$ to prove lower bound

$$T(n) = T(\alpha n) + T(1-\alpha)n + cn$$

$$\geq d\alpha n \log_2(\alpha n) + d(1-\alpha)\log_2 (1-\alpha)n + cn$$

$$= d\alpha n \log_2 n + d\alpha n \log_2 \alpha + d(1-\alpha) n \log_2 (1-\alpha) + cn$$

$$= dn \log_2 n + dn\left(\alpha \log_2 \alpha + (1-\alpha)\log_2 (1-\alpha)\right) + cn$$

$$\geq dn \log_2 n$$

where, $0 < d \leq \dfrac{-c}{\alpha \log_2 \alpha + (1-\alpha)\log_2(1-\alpha)}$

$$\therefore \quad T(n) = \Theta(n \log_2 n)$$

4.5-1

Use the master method to give tight asymptotic bounds for the following recurrences.

a. $T(n) = 2T(n/4) + 1$.

b. $T(n) = 2T(n/4) + \sqrt{n}$.

c. $T(n) = 2T(n/4) + n$.

d. $T(n) = 2T(n/4) + n^2$.

Master's Theorem for dividing functions

$$T(n) = aT(n/b) + f(n)$$

$a \geq 1 \qquad b > 1 \qquad f(n) = \Theta(n^k \log^p n)$

1) $\log_b a$ or a

2) k or b^k

Case 1 : if $\log_b a > k$ then $\Theta(n^{\log_b a})$

Case 2: if $\log_b a = k$

if $p > -1$, $\theta(n^k \log^{p+1} n)$
if $p = -1$, $\theta(n^k \log \log n)$
if $p < -1$, $\theta(n^k)$

Case 3: if $\log_b a < k$

if $p \geq 0$ $\theta(n^k \log^p n)$
if $p < 0$ $O(n^k)$

a) $T(n) = 2T(n/4) + 1$

$a = 2$ $k = 0$

$b = 4$ $p = 0$

$\log_b a = \log_4 2 = 1/2 > k$

$\therefore$ It is case 1 of Master's Theorem

$$\theta(n^{\log_b a}) = \theta(n^{1/2}) = \theta(\sqrt{n})$$

b) $T(n) = 2T(n/4) + \sqrt{n}$

$a = 2$ $k = 1/2$

$b = 4$ $p = 0$

$\log_b a = \log_4 2 = 1/2 = k$

$\therefore$ It is case 2 of Master's Theorem

$\theta(n^k \log^{p+1} n)$ when $p > -1$ i.e, $\theta(\sqrt{n} \log_2 n)$

c) $T(n) = 2T(n/4) + n$

$a = 2$ $\quad k = 1$

$b = 4$ $\quad p = 0$

$\log_b a = \log_4 2 = 1/2 < k$

$\therefore$ It is case 3 of Master's Theorem

$\theta(n^k \log^p n)$ where $p \geq 0$ i.e, $\theta(n)$

d) $T(n) = 2T(n/4) + n^2$

$a = 2$ $\quad k = 2$

$b = 4$ $\quad p = 0$

$\log_b a = \log_4 2 = 1/2 = k$

$\therefore$ It is case 3 of Master's Theorem

$\theta(n^k \log^p n)$ where $p \geq 0$ i.e, $\theta(n^2)$