29.1-1

> If we express the linear program in (29.24)–(29.28) in the compact notation of (29.19)–(29.21), what are n, m, A, b, and c?

$$
\begin{aligned}
n = m &= 3, \\
A &= \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -2 & 2 \end{pmatrix}, \\
b &= \begin{pmatrix} 7 \\ -7 \\ 4 \end{pmatrix}, \\
c &= \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}.
\end{aligned}
$$

29.1-2

> Give three feasible solutions to the linear program in (29.24)–(29.28). What is the objective value of each one?

1. $(x_1, x_2, x_3) = (6, 1, 0)$ with objective value 9.
2. $(x_1, x_2, x_3) = (5, 2, 0)$ with objective value 4.
3. $(x_1, x_2, x_3) = (4, 3, 0)$ with objective value -1.

> Show that the following linear program is infeasible:
>
> $$\begin{array}{lrcrcr} \text{minimize} & 3x_1 & - & 2x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & \le & 2 \\ & -2x_1 & - & 2x_2 & \le & -10 \\ & & x_1, x_2 & & \ge & 0 \end{array} .$$

By dividing the second constraint by 2 and adding to the first, we have $0 \le -3$, which is impossible. Therefore there linear program is unfeasible.

29.1-7

> Show that the following linear program is unbounded:
>
> $$\begin{array}{lrcrcr} \text{minimize} & x_1 & - & x_2 & & \\ \text{subject to} & & & & & \\ & -2x_1 & + & x_2 & \le & -1 \\ & -x_1 & - & 2x_2 & \le & -2 \\ & & x_1, x_2 & & \ge & 0 \end{array} .$$

For any number $r > 1$, we can set $x_1 = 2r$ and $x_2 = r$. Then, the restaints become

$$\begin{array}{rcrcr} -2(2r) & + & r = -3r & \le & -1 \\ -2r & - & 2r = -4r & \le & -2 \\ 2r, r & & & \ge & 0 \end{array} .$$

All of these inequalities are clearly satisfied because of our initial restriction in selecting r. Now, we look to the objective function, it is $2r - r = r$. So, since we can select r to be arbitrarily large, and still satisfy all of the constraints, we can achieve an arbitrarily large value of the objective function.

29.3-5

> Solve the following linear program using SIMPLEX:
>
> $$\begin{array}{lrcrcr} \text{maximize} & 18x_1 & + & 12.5x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & \le & 20 \\ & x_1 & & & \le & 12 \\ & & & x_2 & \le & 16 \\ & & x_1, x_2 & & \ge & 0 \end{array} .$$

First, we rewrite the linear program into it's slack form

$$\begin{array}{lrl} \text{maximize} & 18x_1 + 12.5x_2 & \\ \text{subject to} & & \\ & x_3 & = 20 - x_1 - x_2 \\ & x_4 & = 12 - x_1 \\ & x_5 & = 16 - x_2 \\ & x_1, x_2, x_3, x_4, x_5 & \ge 0. \end{array}$$

We now stop since no more non-basic variables appear in the objective with a positive coefficient. Our solution is $(12, 8, 0, 0, 8)$ and has a value of 316. Going back to the standard form we started with, we just disregard the values of x_3 through x_5 and have the solution that $x_1 = 12$ and $x_2 = 8$. We can check that this is both feasible and has the objective achieve 316.

29.3-6

> Solve the following linear program using SIMPLEX:
>
> $$\begin{array}{lrcrcl} \text{maximize} & 5x_1 & - & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & - & x_2 & \le & 1 \\ & 2x_1 & + & x_2 & \le & 2 \\ & & x_1, x_2 & & \ge & 0 \ . \end{array}$$

First, we convert the linear program into it's slack form

$$\begin{array}{rcrcrcr} z & = & & & 5x_1 & - & 3x_2 \\ x_3 & = & 1 & - & x_1 & + & x_2 \\ x_4 & = & 2 & - & 2x_1 & - & x_2 \ . \end{array}$$

The nonbasic variables are x_1 and x_2. Of these, only x_1 has a positive coefficient in the objective function, so we must choose $x_e = x_1$. Both equations limit x_1 by 1, so we'll choose the first one to rewrite x_1 with. Using $x_1 = 1 - x_3 + x_2$ we obtain the new system

$$\begin{array}{rcrcrcr} z & = & 5 & - & 5x_3 & + & 2x_2 \\ x_1 & = & 1 & - & x_3 & + & x_2 \\ x_4 & = & & & 2x_3 & - & 3x_2 \ . \end{array}$$

Now x_2 is the only nonbasic variable with positive coefficient in the objective function, so we set $x_e = x_2$. The last equation limits x_2 by 0 which is most restrictive, so we set $x_2 = \frac{2}{3}x_3 - \frac{1}{3}x_4$. Rewriting, our new system becomes

$$\begin{array}{rcrcrcr} z & = & 5 & - & \frac{11}{3}x_3 & - & \frac{2}{3}x_4 \\ x_1 & = & 1 & - & \frac{1}{3}x_3 & - & \frac{1}{3}x_4 \\ x_2 & = & & & \frac{2}{3}x_3 & - & \frac{1}{3}x_4 \ . \end{array}$$

Every nonbasic variable now has negative coefficient in the objective function, so we take the basic solution $(x_1, x_2, x_3, x_4) = (1, 0, 0, 0)$. The objective value this achieves is 5.

29.3-8

> In the proof of Lemma 29.5, we argued that there are at most $\binom{m+n}{n}$ ways to choose a set B of basic variables. Give an example of a linear program in which there are strictly fewer than $\binom{m+n}{n}$ ways to choose the set B.

Consider the simple program

$$\begin{array}{rcrcr} z & = & & - & x_1 \\ x_2 & = & 1 & - & x_1 \ . \end{array}$$

In this case, we have $m = n = 1$, so $\binom{m+n}{n} = \binom{2}{1} = 2$, however, since the only coefficients of the objective function are negative, we can't make any other choices for basic variable. We must immediately terminate with the basic solution $(x_1, x_2) = (0, 1)$, which is optimal.

> Solve the following linear program using SIMPLEX:
>
> $$\begin{array}{lrcrcrcr} \text{minimize} & x_1 & + & x_2 & + & x_3 & & \\ \text{subject to} & & & & & & & \\ & 2x_1 & + & 7.5x_2 & + & 3x_3 & \geq & 10000 \\ & 20x_1 & & 5x_2 & + & 10x_3 & \geq & 30000 \\ & & x_1, x_2, x_3 & & & & \geq & 0 \;. \end{array}$$

First, we convert this equation to the slack form. Doing so doesn't change the objective, but the constraints become

$$\begin{array}{rcrcrcrcr} z & = & & - & x_1 & - & x_2 & - & x_3 \\ x_4 & = & -10000 & + & 2x_1 & + & 7.5x_2 & + & 3x_3 \\ x_5 & = & -30000 & + & 20x_1 & + & 5x_2 & + & 10x_3 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 & . & & & & & \end{array}$$

Also, since the objective is to minimize a given function, we'll change it over to maximizing the negative of that function. In particular maximize $-x_1 - x_2 - x_3$. Now, we note that the initial basic solution is not feasible, because it would leave x_4 and x_5 being negative. This means that finding an initial solution requires using the method of section 29.5. The auxiliary linear program in slack form is

$$\begin{array}{rcrcrcrcrcr} z & = & & - & x_0 & & & & & & \\ x_4 & = & -10000 & + & x_0 & + & 2x_1 & + & 7.5x_2 & + & 3x_3 \\ x_5 & = & -30000 & + & x_0 & + & 20x_1 & + & 5x_2 & + & 10x_3 \\ x_0, x_1, x_2, x_3, x_4, x_5 & \geq & 0 & . & & & & & & & \end{array}$$

We choose x_0 as the entering variable and x_5 as the leaving variable, since it is the basic variable whose value in the basic solution is most negative. After pivoting, we have the slack form

$$\begin{array}{rcrcrcrcrcr} z & = & -30000 & + & 20x_1 & + & 5x_2 & + & 10x_3 & - & x_5 \\ x_0 & = & 30000 & - & 20x_1 & - & 5x_2 & - & 10x_3 & + & x_5 \\ x_4 & = & 20000 & - & 18x_1 & + & 2.5x_2 & - & 7x_3 & + & x_5 \\ x_0, x_1, x_2, x_3, x_4, x_5 & \geq & 0 & . & & & & & & & \end{array}$$

$$
\begin{array}{rcrcrcrcrcr}
x_4 & = & 20000 & - & 18x_1 & + & 2.5x_2 & - & 7x_3 & + & x_5 \\
x_0, x_1, x_2, x_3, x_4, x_5 & \geq & 0 & . & & & & & & &
\end{array}
$$

The associated basic solution is feasible, so now we just need to repeatedly call PIVOT until we obtain an optimal solution to L_{aux}. We'll choose x_2 as our entering variable. This gives

$$
\begin{array}{rcrcrcrcrcr}
z & = & & - & x_0 & & & & & & \\
x_2 & = & 6000 & - & 0.2x_0 & - & 4x_1 & - & 2x_3 & + & 0.2x_5 \\
x_4 & = & 35000 & - & 0.5x_0 & - & 28x_1 & - & 12x_3 & + & 1.5x_5 \\
x_0, x_1, x_2, x_3, x_4, x_5 & \geq & 0 & . & & & & & & &
\end{array}
$$

This slack form is the final solution to the auxiliary problem. Since this solution has $x_0 = 0$, we know that our initial problem was feasible. Furthermore, since $x_0 = 0$, we can just remove it from the set of constraints. We then restore the original objective function, with appropriate substitutions made to include only the nonbasic variables. This yields

$$
\begin{array}{rcrcrcrcr}
z & = & -6000 & + & 3x_1 & + & x_3 & - & 0.2x_5 \\
x_2 & = & 6000 & - & 4x_1 & - & 2x_3 & + & 0.2x_5 \\
x_4 & = & 35000 & - & 28x_1 & - & 12x_3 & + & 1.5x_5 \\
x_1, x_2, x_3, x_4, x_5 & \geq & 0 & . & & & & &
\end{array}
$$

This slack form has a feasible basic solution, and we can return it to SIMPLEX. We choose x_1 as our entering variable. This gives

$$
\begin{array}{rcrcrcrcr}
z & = & -2250 & - & \frac{2}{7}x_3 & - & \frac{3}{28}x_4 & - & \frac{11}{280}x_5 \\
x_1 & = & 1250 & - & \frac{3}{7}x_3 & - & \frac{1}{28}x_4 & + & \frac{15}{280}x_5 \\
x_2 & = & 1000 & - & \frac{2}{7}x_3 & + & \frac{4}{28}x_4 & - & \frac{4}{280}x_5 \\
x_1, x_2, x_3, x_4, x_5 & \geq & 0 & . & & & & &
\end{array}
$$

At this point, all coefficients in the objective function are negative, so the basic solution is an optimal solution. This solution is $(x_1, x_2, x_3) = (1250, 1000, 0)$.