Lagrange Duality

Primal optimization problem in *standard form*:

$$\min_x f(x)$$

$$\text{subject to } g_i(x) \le 0, \quad i = 1, \dots, m$$

$$h_i(x) = 0, \quad i = 1, \dots, p$$

where $g_i(x)$ = inequality constraints, $h_i(x)$ = equality constraints

The generalized Lagrangian function:

$$L(x, \alpha, \beta) = f(x) + \sum_{i=1}^{m} \alpha_i g_i(x) + \sum_{i=1}^{p} \beta_i h_i(x)$$

where α_i, β_i = Lagrange multipliers

Define the primal function:

$$\theta_P(x) = \max_{\alpha,\beta} L(x, \alpha, \beta)$$

That is,

$$\theta_P(x) = \begin{cases} f(x) & \text{if } x \text{ satisfies primal constraints} \\ \infty & \text{otherwise } (g_i(x) > 0 \text{ or } h_i(x) \neq 0) \end{cases}$$

Minimization problem:

$$\min_x \theta_P(x) = \min_x \max_{\alpha,\beta} L(x, \alpha, \beta)$$

Define p^* as the optimal value of the primal problem's objective:

$$p^* = \min_x \theta_P(x)$$

Now define the dual function:

$$\theta_D(\alpha, \beta) = \min_x L(x, \alpha, \beta)$$

Dual optimization problem:

$$\max_{\alpha,\beta} \theta_D(\alpha, \beta) = \max_{\alpha,\beta} \min_x L\,(x, \alpha, \beta)$$

Define d^* to be the optimal value of the objective:

$$d^* = \max_{\alpha,\beta} \theta_D(\alpha, \beta) = \max_{\alpha,\beta} \min_x L\,(x, \alpha, \beta)$$

The primal problem and the dual problem are similar except the *max* and the *min* are exchanged.

Optimization is performed with respect to x and α, β in the primal and dual problem, respectively.

In general, $d^* \leq p^*$

However, if the following conditions are satisfied:

$$f(x), g_i(x) = \text{convex}$$

$$h_i(x) = \text{affine}$$

$$g(x) = \text{feasible, i.e.,} \exists\, x \rightarrow g_i(x) < 0\ \forall\, i$$

then

$$\exists\, x^*, \alpha^*, \beta^* \text{ so that}$$

$$x^* \text{ is the solution to the primal problem}$$

$$\alpha^*, \beta^* \text{ are the solution to the dual problem}$$

$$p^* = d^* = L\,(x^*, \alpha^*, \beta^*)$$

Moreover,

$$x^*, \alpha^*, \beta^* \text{ satisfy the Karush} - \text{Kuhn} - \text{Tucker (KKT) conditions:}$$

$$\nabla_x\, L(x^*, \alpha^*, \beta^*) = 0$$

$$\nabla_{\beta_i} L(x^*, \alpha^*, \beta^*) = 0, \qquad i = 1, \dots, p$$

$$\alpha_i^* g_i(x^*) = 0, \qquad i = 1, \dots, m$$

$$g_i(x^*) \leq 0, \qquad i = 1, \dots, m$$

$$\alpha_i^* \geq 0, \qquad i = 1, \dots, m$$

$\alpha_i^* g_i(x^*) = 0$ is called the KKT dual complementarity condition

Theorem: if x, α, β satisfy the KKT conditions, then they are the solution to the primal and dual problems.

References:

- A. Ng, *CS 229 Lecture Notes*
- S. Boyd, *Convex Optimization*
- T. Rockarfeller, *Convex Analysis*