Eigen Decomposition

Let A be $n \times n$ matrix, then is λ an eigenvalue and $\mathbf{v}$ is an eigenvector of A if $A\mathbf{v} = \lambda\mathbf{v}$

- Where $\mathbf{v}$ is a non-zero vector and λ is a scalar.

$$A\mathbf{v} = \lambda\mathbf{v} \rightarrow A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \rightarrow (A - I\lambda)\mathbf{v} = \mathbf{0}$$

All vectors that satisfy $A\mathbf{v} = \lambda\mathbf{v}$ form an eigenspace of A. λ is also called the characteristic value associated with vector $\mathbf{v}$.

Example: $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

Find the determinant of $(A - I\lambda) = 0$ and solve for λ

$$\begin{vmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 - \lambda - 6 = 0 \rightarrow \lambda_1 = 3, \lambda_2 = -2$$

Now find the eigenvectors corresponding to these eigenvalues

$$\lambda_1 = 3: \begin{vmatrix} 2-3 & -4 \\ -1 & -1-3 \end{vmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \rightarrow \begin{cases} -v_{11} - 4v_{12} = 0 \\ -v_{11} - 4v_{12} = 0 \end{cases}$$

Let $v_{11} = 1$ or $v_{11} = -1$ or $v_{12} = 1$ or $v_{12} = -1$

If $v_{12} = 1 \rightarrow v_{11} = -4 \rightarrow \mathbf{v}_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

Normalize $\mathbf{v}_1 \rightarrow \mathbf{v}_1 = \begin{bmatrix} -4/\sqrt{(-4)^2 + 1^2} \\ 1/\sqrt{(-4)^2 + 1^2} \end{bmatrix} = \begin{bmatrix} -4/\sqrt{17} \\ 1/\sqrt{17} \end{bmatrix} = \begin{bmatrix} -0.9701 \\ 0.2425 \end{bmatrix}$

$$\lambda_2 = -2: \begin{vmatrix} 2-(-2) & -4 \\ -1 & -1-(-2) \end{vmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \rightarrow \begin{cases} 4v_{21} - 4v_{22} = 0 \\ -v_{21} + v_{22} = 0 \end{cases}$$

Let $v_{22} = 1 \rightarrow v_{21} = 1 \rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow$ Normalize $\rightarrow \mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$

Eigen matrix: $\mathbf{V} = \begin{bmatrix} -4/\sqrt{17} & 1/\sqrt{2} \\ 1/\sqrt{17} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -0.9701 & 0.7071 \\ 0.2425 & 0.7071 \end{bmatrix}$

Notice that $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit vectors

Orthogonality of eigenvectors

```
import numpy as np
from numpy import linalg as LA

A = np.array([[2,-4],[-1,-1]])
w,v = LA.eig(A)
print(w)
print(v)
print('The eigenvectors of A are not necessarily orthogonal to each other:',
      np.dot(v[:,0],v[:,1]))
```

```
[ 3. -2.]
[[ 0.9701425   0.70710678]
 [-0.24253563  0.70710678]]
The eigenvectors of A are not necessarily orthogonal to each other: 0.5144957554275265
```

```
C = np.cov(A)
wc,vc = LA.eig(np.cov(A))
print(wc)
print(vc)
print('The eigenvectors of covariance of A are orthogonal:',
      np.dot(vc[:,0],vc[:,1]))
```

```
[18.  0.]
[[1. 0.]
 [0. 1.]]
The eigenvectors of covariance of A are orthogonal: 0.0
```