NATIONAL TECHNICAL UNIVERSITY
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LECTURE NOTES

“MAGNETISM”

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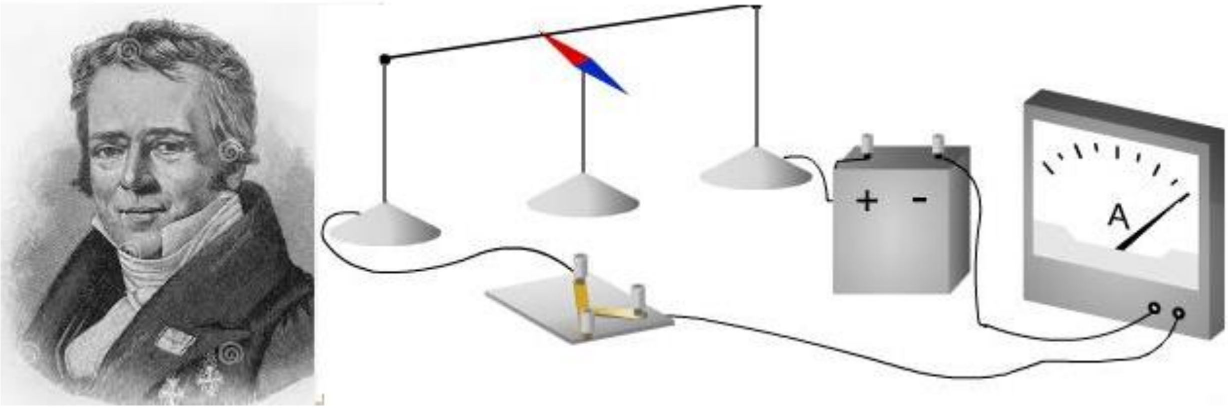
Chapter 5. MAGNETIC FIELD IN VACUUM

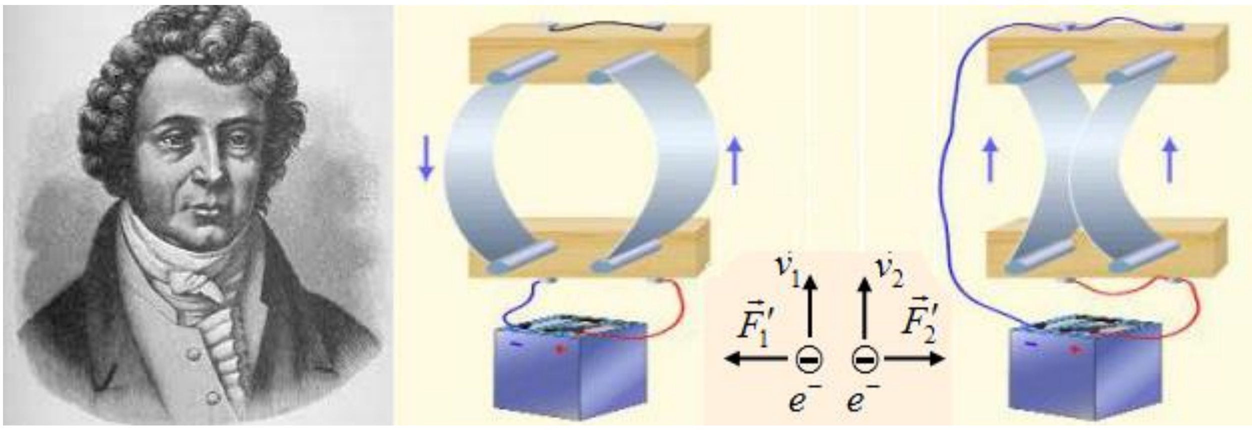
I. MAGNETIC FIELD. BIOT-SAVART LAW

1. Magnetic field

The magnetic and electrical phenomena were considered for a long time to have a different nature and were unrelated to each other. For the first time the link between them was established by Danish scientist Hans Christian Oersted (1977-1851). In 1920, he found out that a pivoted magnetic needle placed parallel to a wire carrying an electric current made a great oscillation. The needle deflected one way for one direction of the current and the opposite way for the other direction.



French physicist Andre Marie Ampere (1775-1836) found In the same year the that the two conductors arranged parallel to each other feel a mutual attraction when currents flow in the same direction, and repulsion when the currents flow in the opposite directions. Ampere called this phenomenon of currents interaction the ***electrodynamic effect***.



This strong interaction is carried out at a distance, and therefore must be associated with a ***force field***. But it is not associated with an electric or gravitational interactions. Therefore, it is necessary to introduce other type of field. We call it the ***magnetic field***.

As with any force field, the magnetic field must be characterized by a vector quantity. This characteristic is ***magnetic field*** $\vec{B}$ (***magnetic induction*** or ***magnetic flux density***).

$[B]$ = Tesla = T.

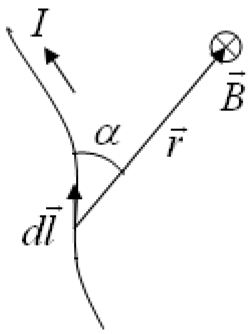
Magnetic fields are produced by electric currents (or moving charges) and affect only the currents (or moving charges).

For magnetic fields the ***principle of superposition*** of fields is valid: the field generated by several sources (charges or currents) is the vector sum of the magnetic fields generated by them separately,

$$\vec{B} = \sum_i \vec{B}_i \ . \tag{5.1}$$

2. Biot-Savart law and its applications

Biot-Savart law relates magnetic fields to the currents which are their sources. In a similar manner, Coulomb's low relates electric fields to the point charges which are their sources. Finding the magnetic field resulting from a current distribution involves the vector product, and is inherently a calculus problem when the distance from the current to the field point is continuously changing.



Each infinitesimal current element makes a contribution to the magnetic field at a chosen point which is perpendicular to the current element, and perpendicular to the vector from the current to this field point. If $d\vec{B}$ is the magnetic field contribution from the current element $Id\vec{l}$, the relationship between the magnetic field contribution and its source current element is called the <u>Biot-Savart law,</u>

$$d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I\left[d\vec{l}, \vec{r}\right]}{r^3} = \frac{\mu_0}{4\pi} \cdot \frac{I\left[d\vec{l}, \vec{e}_r\right]}{r^2} \ , \tag{5.2}$$

where $d\vec{l}$ is an infinitesimal length of the conductor carrying electric current I, $\vec{r}$ is the vector distance from the current to the field point, $\vec{e}_r$ is unit vector to specify the vector $\vec{r}$, $\mu_0 = 4\pi \cdot 10^{-7}$ (H/m) is the ***magnetic permeability*** of free space.

The direction of the magnetic field contribution follows the ***right hand rule***. This direction arises from the vector product nature of the dependence upon electric current.

The magnitude of magnetic field vector is

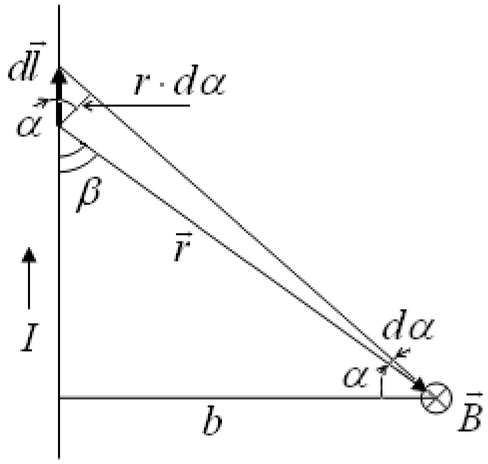
$$dB = \frac{\mu_0}{4\pi} \cdot \frac{I \cdot dl \cdot \sin\alpha}{r^2}. \tag{5.3}$$

According to the principle of superposition the magnetic field at any point is

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\left[d\vec{l}, \vec{r}\right]}{r^3}. \tag{5.4}$$

Applications of Biot-Savart law:

a) ***Magnetic field due to a single, isolated, infinitely long straight wire carrying a current***



Start with the Biot-Savart law

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\left[d\vec{l}, \vec{r}\right]}{\vec{r}^3}.$$

Let the angle between vectors $d\vec{l}$ and $\vec{r}$ be β (instead of α). Then the magnitude of the field is

$$dB = \frac{\mu_0}{4\pi} \cdot \frac{I \cdot dl \cdot \sin\beta}{r^2}.$$

$$\left.\begin{array}{l} \sin\beta = \cos\alpha \\ dl \cdot \sin\beta = dl \cdot \cos\alpha = r \cdot d\alpha \\ r = b/\cos\alpha \end{array}\right\} \Rightarrow \; dl \cdot \cos\alpha = \frac{b}{\cos\alpha} d\alpha.$$

Consequently, $dB = \frac{\mu_0}{4\pi} \cdot \frac{I \cdot dl \cdot \cos\alpha}{r^2} = \frac{\mu_0}{4\pi} \cdot \frac{I \cdot \cos\alpha \cdot d\alpha}{b}$.

All vectors $d\vec{B}$ are of the same direction so we may summarize their magnitudes. Integrating dB in the limits $\left(-\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}\right)$ we obtain

$$B = \int_{-\pi/2}^{\pi/2} dB = \frac{\mu_0}{4\pi} \cdot \frac{I}{b} \int_{-\pi/2}^{\pi/2} \cos\alpha \cdot d\alpha = \frac{\mu_0}{4\pi} \cdot \frac{2I}{b}.$$

So, the magnetic field due to a long straight wire is

$$B = \frac{\mu_0}{4\pi} \cdot \frac{2I}{b} = \frac{\mu_0}{2\pi} \cdot \frac{I}{b}. \qquad (5.5)$$

The direction is given by a ***right-hand rule***: point the thumb of your right hand in the direction of the current, and your fingers indicate the direction of the circular magnetic field lines around the wire.

b) ***Magnetic field on the axis of a circular loop of radius R***

The total vector of magnetic field $\vec{B}$ directs upwards and is equal to the sum of vertical projections $d\vec{B}_Z$ (the sum of horizontal projections $d\vec{B}_X$ is equal to zero). The angle between vectors $d\vec{l}$ and $\vec{r}$ is a right angle, therefore, $dB = \frac{\mu_0}{4\pi} \cdot \frac{I \cdot dl}{r^2}$.

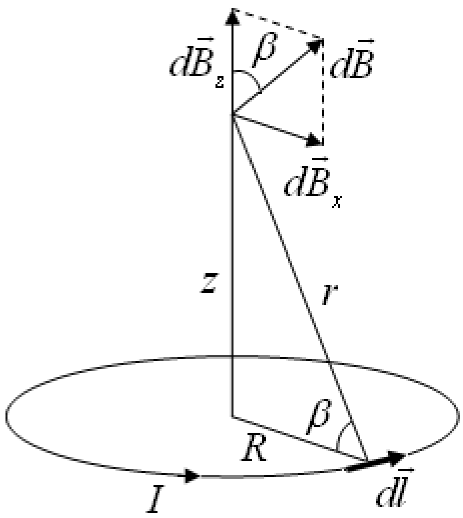
$$dB_Z = dB \cdot \cos\beta = \frac{\mu_0}{4\pi} \cdot \frac{I \cdot dl \cdot \cos\beta}{r^2}.$$

Since $\cos\beta = \frac{R}{r} = \frac{R}{\left(z^2 + R^2\right)^{\frac{1}{2}}}$, then

$$dB_Z = \frac{\mu_0}{4\pi} \cdot \frac{I \cdot dl \cdot R}{r^3} = \frac{\mu_0}{4\pi} \cdot \frac{I \cdot dl \cdot R}{\left(z^2 + R^2\right)^{\frac{3}{2}}}.$$

$$B = \oint dB_Z = \frac{\mu_0}{4\pi} \cdot \frac{I \cdot R}{\left(z^2 + R^2\right)^{\frac{3}{2}}} \oint dl =$$

$$= \frac{\mu_0}{4\pi} \cdot \frac{I \cdot R}{\left(z^2 + R^2\right)^{\frac{3}{2}}} \cdot 2\pi R = \frac{\mu_0}{4\pi} \cdot \frac{2\pi R^2 I}{\left(z^2 + R^2\right)^{\frac{3}{2}}}.$$



Special case is when $z = 0$ (***magnetic field in the center of a circular loop***):

$$B_0 = \frac{\mu_0}{4\pi} \cdot \frac{2\pi I}{R}. \qquad (5.6)$$

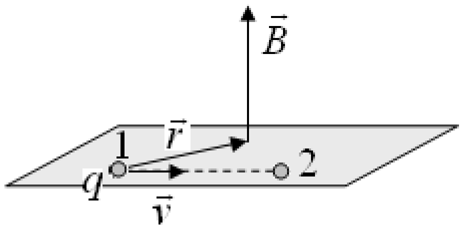
A ***right-hand rule for circular loop***: curl the fingers of your right hand in the direction of the current flow, and your thumb points in the direction of the magnetic field inside the loop.

3. The magnetic field of a moving charge. Relationship between $\vec{B}$ and $\vec{E}$

The current is the charge flow. In conductors the current is the directed flow of electrons moving with a velocity $\vec{v}$ (drift velocity). Then

$I = j \cdot S = n \cdot e \cdot v \cdot S$,

where S is a cross-sectional area of a conductor, n is a number of carriers in a unit volume.



So, taking into account that vectors $\vec{v}$ and $d\vec{l}$ are of the same direction, $S \cdot dl$ is the infinitesimal volume of an element of a conductor, and n is the total number of carriers in this volume, we have:

$$d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I\left[d\vec{l}, \vec{r}\right]}{r^3} = \frac{\mu_0}{4\pi} \cdot \frac{n \cdot e \cdot v \cdot S\left[d\vec{l}, \vec{r}\right]}{r^3} = \frac{\mu_0}{4\pi} \cdot \frac{n \cdot e \cdot S \cdot dl\left[\vec{v}, \vec{r}\right]}{r^3} = \frac{\mu_0}{4\pi} \cdot \frac{dq\left[\vec{v}, \vec{r}\right]}{r^3}.$$

The ***magnetic field of a moving charge*** is

$$\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{q\left[\vec{v}, \vec{r}\right]}{r^3}. \qquad (5.7)$$

Any moving electrical charge generates both electric and magnetic fields. Let's find the relationship between the electric and magnetic fields.

Electric field of a point charge is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} \cdot \vec{e}_r,$$

hence, $q = 4\pi\varepsilon_0 r^2 \left(\vec{E}, \vec{e}_r\right)$. Then,

$$\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{q\left[\vec{v}, \vec{r}\right]}{r^3} = \frac{\mu_0}{4\pi} \cdot \frac{4\pi\varepsilon_0 r^2 \left(\vec{E}, \vec{e}_r\right)\left[\vec{v}, \vec{r}\right]}{r^3} = \frac{\mu_0\varepsilon_0 \left(\vec{E}, \vec{e}_r\right)\left[\vec{v}, \vec{r}\right]}{r} = \qquad (5.8)$$

$$= \frac{\mu_0\varepsilon_0 \left(\vec{r}, \vec{e}_r\right)\left[\vec{v}, \vec{E}\right]}{r} = \frac{\mu_0\varepsilon_0 r\left[\vec{v}, \vec{E}\right]}{r} = \mu_0\varepsilon_0\left[\vec{v}, \vec{E}\right] = \frac{\left[\vec{v}, \vec{E}\right]}{c^2},$$

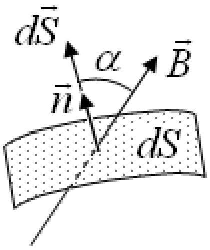
$c = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}} = 3 \cdot 10^8$ m/s is an ***electrodynamics constant*** (***speed of light in free space***), ε_0 is electric permittivity of free space, μ_0 is magnetic permeability of free space.

II. FUNDAMENTAL LAWS OF A MAGNETIC FIELD

Magnetic field, as well as electrical field, has two important properties. They are related to the flux and circulation of the vector $\vec{B}$.

Magnetic field can be mapped out by the lines of the vector $\vec{B}$ which may be defined as the lines whose tangents coincide with the vectors of magnetic field. The number of these lines per unit area is proportional to the magnitude of the vector $\vec{B}$.

1. Gauss' law for magnetic field $\vec{B}$



Magnetic flux is the dot product of the average magnetic flux density $(\vec{B})$ times the area $(d\vec{S})$ that it penetrates, or the product of the magnetic field (B) times the perpendicular area $(dS_{\perp})$ that it penetrates. The contribution to magnetic flux $d\Phi$ for a given area is equal to the area times the component of magnetic field perpendicular to the area.

$$d\Phi = (\vec{B}, d\vec{S}) = \vec{B} \cdot d\vec{S} = B \cdot dS \cdot \cos\alpha = B \cdot dS_{\perp} = B_{\perp} \cdot dS . \qquad (5.9)$$

$[\Phi] = \text{Weber} = \text{Wb}$.

Gauss' law: For closed surface, the sum of magnetic fluxes is always equal to zero

$$\oint_S \vec{B} \cdot d\vec{S} = 0 . \qquad (5.10)$$

Thinking of the force lines as representing a kind of fluid flow, the so-called "magnetic flux", we see that for a closed surface, as much magnetic flux flows into the surface as flows out. Gauss' law is the mathematical expression of the fact that no magnetic monopoles have ever been discovered (the sources of magnetic fields are currents).

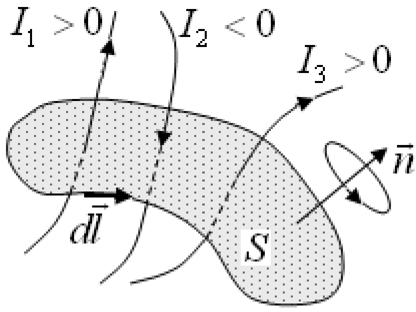
2. Circulation theorem for the vector $\vec{B}$ (for the magnetic field of direct currents in free space)

Circulation theorem*. Circulation of a vector $\vec{B}$ along any path is equal to product μ_0 and an algebraic sum of the currents enclosed by this path. Mathematically,

$$\oint \vec{B} d\vec{l} = \mu_0 I , \qquad (5.11)$$

where $I = \sum_i I_i$ ($I > 0$, if its direction connected with the direction of going around the contour by a right screw rule).

In some textbooks this theorem is named “Ampère's Law” or “***Ampère’s circuital Law***”: sum of the magnetic fields along any closed path is proportional to the current that passes through. We use the name “Ampère's Law” for the law describing the forces between current-carrying wires in magnetic field, which is named “Ampère's force Law”.



If circulation of a vector $\vec{B}$ is not equal to zero, the magnetic field is not potential (like electrostatic). Such fields are called ***rotational*** or ***vorticity*** field.

Like Gauss's Law, the circulation theorem can be used to find the magnetic field of a sufficiently symmetric current.

a) ***Magnetic field due to a long (infinite) straight wire of radius*** a**.**

Outside of a conductor (contour Γ_1) ($\cos\alpha = 1$, as $\vec{B} \parallel d\vec{l}$)

$\oint \vec{B} d\vec{l} = B\oint dl \cdot \cos\alpha = B\oint dl = B \cdot 2\pi r = \mu_0 I$.

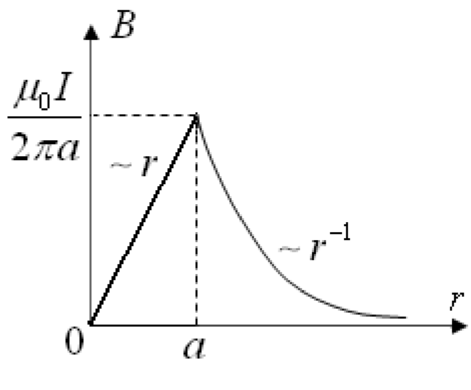
Consequently, for $r > a$,

$$B = \frac{\mu_0 I}{2\pi r}. \qquad (5.12)$$

Outside of a conductor (contour Γ_2), taking into account the fact that

$\frac{I_r}{I} = \frac{\pi r^2}{\pi a^2} = \left(\frac{r}{a}\right)^2$,

$I_r = I\left(\frac{r}{a}\right)^2$,



we obtain

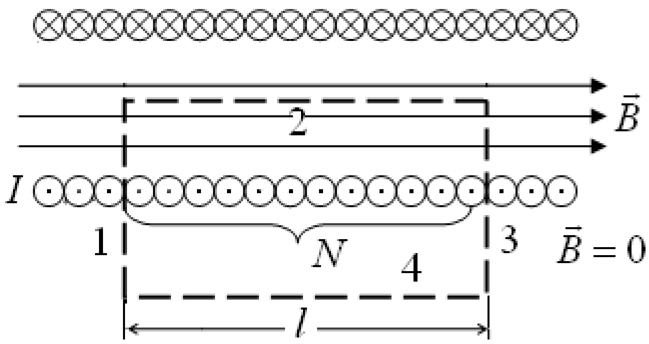
$B \cdot 2\pi r = \mu_0 I_r = \mu_0 I\left(\frac{r}{a}\right)^2$.

Consequently, for $r < a$,

$$B = \frac{\mu_0 I \cdot r}{2\pi a^2}. \qquad (5.13)$$

b) ***Magnetic field inside a straight, infinite, air core solenoid***

Solenoid (a long straight coil of wire) can be used to generate a nearly uniform magnetic field similar to that of a bar magnet. Such solenoids have an enormous number of practical applications. The field can be greatly strengthened by adding an iron core. The magnetic field is concentrated into a nearly uniform field in the center of a long solenoid. The field outside is weak and divergent.



Apply the circulation theorem for the path indicated. The current enclosed is $I_{enclosed} = NI$, where N is the number of turns inside the path and I is the current in the wire. The path integral of the field can be broken into four parts, one for each side of the path

$$\oint \vec{B}d\vec{l} = \int_1 \vec{B}d\vec{l} + \int_2 \vec{B}d\vec{l} + \int_3 \vec{B}d\vec{l} + \int_4 \vec{B}d\vec{l} .$$

Along side 4 the field is close to zero. Along sides 1 and 3 the field is perpendicular to the path so, $\oint \vec{B}d\vec{l} = 0 + \int_2 \vec{B}d\vec{l} + 0 + 0 = Bl$. Assume that the solenoid is so long that symmetry requires the magnetic field to be constant along the path. Using the circulation theorem, $\oint \vec{B}d\vec{l} = \mu_0 I_{enclosed}$. $\Rightarrow$ $Bl = \mu_0 NI$.

Consequently,

$$B = \mu_0 \cdot \frac{N}{l} \cdot I = \mu_0 \cdot n \cdot I , \tag{5.14}$$

where n is an amount of turns per unit length of the solenoid.

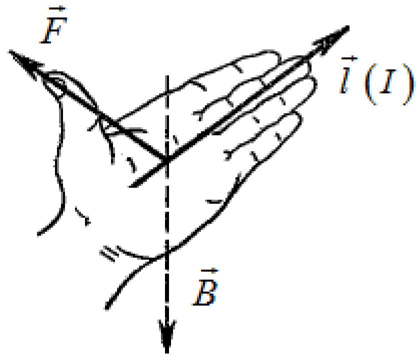
III. AMPÈRE'S LAW

1. Ampère's force

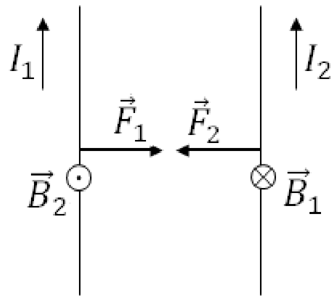
When a conductor carrying a current I is placed into magnetic field $\vec{B}$ the force on an infinitesimal segment of it $\left(d\vec{l}\right)$ is given by

$$d\vec{F} = I\left[d\vec{l},\vec{B}\right]. \tag{5.15}$$

This is the ***Ampère's force***. This force acting on a conductor is always at the right angle to the plane which contains both the conductor and the direction of the field in which it is placed. The direction of the force $\vec{F}$ may be visualized by the ***left hand rule***: you orient your left hand so that the outstretched fingers point in the direction of the current and the magnetic field lines enter into your palm. If your hand is oriented in this way, then the extended thumb points in the direction of the magnetic force on the conductor.



Let us consider the magnetic force between two parallel wires carrying the currents I_1 and I_2. Magnetic field of current I_1 at wire 2 according to Biot-Savart law is $B_1 = \frac{\mu_0}{4\pi}\cdot\frac{2I_1}{b}$. The force on length Δl of wire 2 is $F_2 = I_2 \cdot \Delta l \cdot B_1$ ($\sin\alpha = 1$, since $\vec{B} \perp Id\vec{l}$). Force per unit length is



$$F_l = \frac{F}{\Delta l} = \frac{I_2 \cdot \Delta l \cdot B_1}{\Delta l} = \frac{\mu_0}{4\pi}\cdot\frac{2I_1 \cdot I_2}{b}, \tag{5.16}$$

where b is the distance between wires.

When the currents in both wires flow in the same direction, the force is attractive. When the currents flow in opposite directions, the force is repulsive.

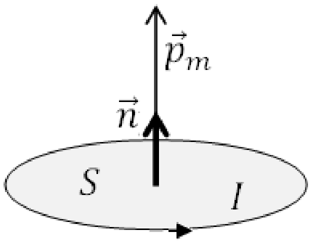
2. The force acting on current loop. Magnetic dipole moment

The net force on any arbitrarily shaped current loop placed into a magnetic field is

$$\vec{F} = I\oint\left[d\vec{l},\vec{B}\right]. \tag{5.17}$$

a) In the uniform field $\left(\vec{B} = const\right)$ the vector $\vec{B}$ can be taken outside the integral sign and $\oint d\vec{l} = 0$, hence, $\vec{F} = 0$, i.e. the force is equal to zero.

b) In the non-uniform field $\left(\vec{B} \neq const\right)$ it is necessary to calculate an integral and the force, generally speaking, is nonzero.



Let us consider a case when a contour is flat and its size is small enough (so-called ***unit loop***).

If I is the current in the loop and S is its area, the ***magnetic moment*** (***magnetic dipole moment***) is

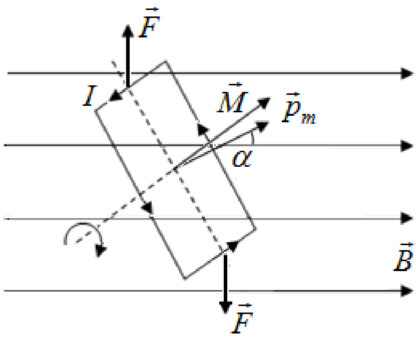
$$\vec{p}_m = I \cdot S \cdot \vec{n}, \tag{5.18}$$

where $\vec{n}$ is a normal to a loop. The direction of the magnetic moment is perpendicular to the current loop according to the right-hand rule.

Using the magnetic moment, the expression for the force is

$$\vec{F} = p_m \cdot \frac{\partial \vec{B}}{\partial n}. \tag{5.19}$$

3. The torque on a current loop. Definition of $\vec{B}$.



If an arbitrarily shaped current loop is placed into a uniform magnetic field $\vec{B}$, the total force is zero. Hence, the torque of Ampere's forces is independent on the point of the forces application.

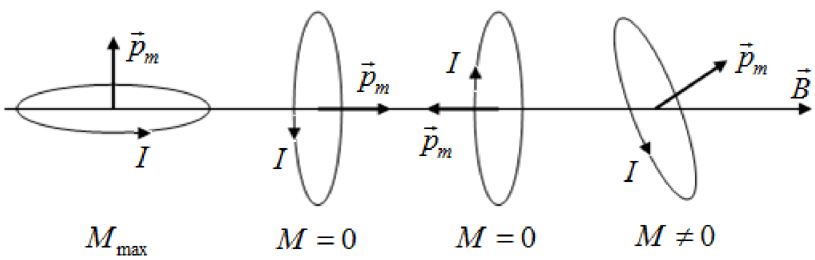
The definition of torque is

$$\vec{M} = \oint \left[\vec{r}, d\vec{F}\right], \tag{5.20}$$

where $d\vec{F} = I\left[d\vec{l}, \vec{B}\right]$.

So using the magnetic moment we may write that the ***torque on a current loop*** is

$$\vec{M} = \left[\vec{p}_m, \vec{B}\right] \tag{5.21}$$



The magnitude of torque is

$$M = p_m \cdot B \cdot \sin\alpha\,. \tag{5.22}$$

a) If vectors $\vec{p}_m$ and $\vec{B}$ are parallel, $M = 0$, i.e. the torque is zero. The loop is in the state of ***stable equilibrium***;

b) If vectors $\vec{p}_m$ and $\vec{B}$ are antiparallel, $M = 0$, the torque is zero, as well, but the loop is in a state of ***labile equilibrium***: the slightest deviation from this position leads to torque acting;

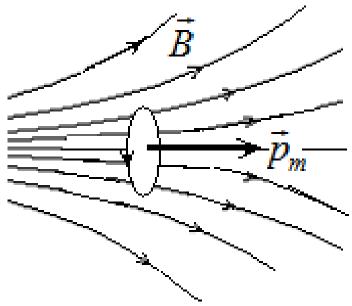
c) If $\vec{p}_m \perp \vec{B}, \quad M = M_{\max}$.

The torque magnitude depends on the magnitude of p_m. However, the ratio M/p_m at the fixed angle α at a chosen point of the field remains constant. Hence, this ratio is the characteristic of this point of the magnetic field.

Magnetic field **(*magnetic flux density*)** is a vector quantity, which magnitude is determined by expression

$$B = \frac{M_{\max}}{p_m}, \tag{5.23}$$

and the direction coincides with the direction of the normal to loop in its stable equilibrium position in the magnetic field.



The expression $\vec{M} = \left[\vec{p}, \vec{B}\right]$ is valid in every magnetic field.

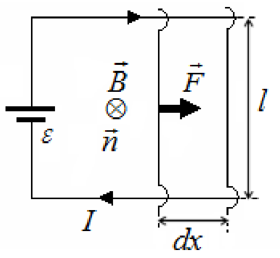
In external non-uniform magnetic field the current loop behaves analogously to, how an electric dipole behaves in an external non-uniform electric field: it turns to the position of stable equilibrium (at which vectors $\vec{p}_m$ and $\vec{B}$ are parallel) and under the action of force $\vec{F}$ the current loop is retracting into the area of a field where the magnitudes of $\vec{B}$ are larger.

4. Work done on the displacement of a current conductor in the magnetic field

The work done by Ampere’s force is

$$A = \int_1^2 \vec{F} \cdot d\vec{x},$$

where $\vec{F} = I\left[\vec{l}, \vec{B}\right]$.



Let's consider a specific case when $\vec{B} = const$, and $\vec{B}$ is normal to the plane of the figure. Then $\vec{B} \perp d\vec{l}$, and $F = I \cdot l \cdot B$.

$$A = \int_1^2 I \cdot l \cdot B \cdot dx = I\int_1^2 BdS = I\int_1^2 d\Phi = I(\Phi_2 - \Phi_1) = I\Delta\Phi,$$

$$A = I\Delta\Phi. \qquad (5.24)$$

Though this formula was obtained for a specific case, it is valid for any arbitrary displacement in any magnetic field.

IV. LORENTZ FORCE LAW

1. The magnetic force on a moving charge

The force on the current wire is $d\vec{F} = I\left[d\vec{l}, \vec{B}\right]$. The current is composed of individual charges. We want to know the magnetic force on a single charge. Recall the definition of current density, $j = \frac{I}{S}$, then $I = j \cdot S$ and $d\vec{F} = j \cdot S\left[d\vec{l}, \vec{B}\right]$. Using the expression for the drift velocity $j = n \cdot e \cdot v$, we obtain

$$d\vec{F} = I\left[d\vec{l}, \vec{B}\right] = n \cdot e \cdot v \cdot S\left[d\vec{l}, \vec{B}\right] = n \cdot e \cdot S \cdot dl\left[\vec{v}, \vec{B}\right] = e \cdot n \cdot dV\left[\vec{v}, \vec{B}\right] = dq\left[\vec{v}, \vec{B}\right].$$

The ***magnetic force acting on a single charge*** (***magnetic component of Lorentz force***) is

$$\vec{F} = q\left[\vec{v}, \vec{B}\right]. \qquad (5.25)$$

and its magnitude equals to

$$F = q \cdot v \cdot B \cdot \sin\alpha. \qquad (5.26)$$

Both the electric and magnetic force can defined from ***Lorentz force law***

$$\vec{F} = q\vec{E} + q\left[\vec{v}, \vec{B}\right] = \vec{F}_e + \vec{F}_m. \qquad (5.27)$$

The electric force is straightforward, being in the direction of the electric field if the charge q is positive, but the direction of the magnetic part of the force is given by cross product rule.

The magnitude of magnetic force is zero when 1) the angle between the velocity and the magnetic field is zero (or 180^0); 2) the velocity is zero. This implies that the magnetic force on the stationary charge or the charge moving parallel to the magnetic field is zero.

The magnetic force is perpendicular both to the velocity of the charge and to the magnetic field, therefore, no work is done. Hence magnetic forces do no work on charged particles and cannot increase their kinetic energy. If a charged particle moves through a constant magnetic field, its speed stays the same, but its direction is constantly changing.

Resolving the Lorentz force into components depends on the choice of the frame of reference as the velocity of the particle depends on the frame of reference.

Let us compare electrical and magnetic forces F_e and F_m for two equal charges moving with velocity v ($v \ll c -$ light speed).

$$\left.\begin{aligned} &F_m = q \cdot v \cdot B \\ &F_e = q \cdot E \\ &\vec{B} = \frac{\left[\vec{v}, \vec{E}\right]}{c^2} \Rightarrow \vec{B} \perp \vec{E} \Rightarrow B = \frac{v \cdot E \cdot \sin\alpha}{c^2} = \frac{v \cdot E}{c^2} \end{aligned}\right\} \frac{F_m}{F_e} = \frac{qvB}{qE} = \frac{vvE}{c^2 E} = \left(\frac{v}{c}\right)^2. \quad (5.28)$$

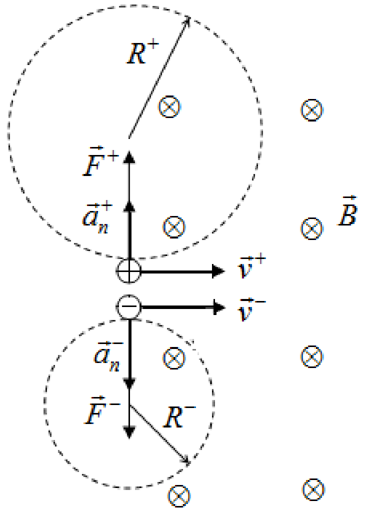
This result allows making the following conclusions:

1. The explanation of the nature of magnetic interaction is possible only on the basis of modern physics as the velocity of light $c \to \infty$ in Newtonian physics;
2. For usual velocities $v \ll c$ the magnetic force a million times less than electrical force.

But it is necessary to take it into account at least for two reasons:

1. when $v \to c$, the role of magnetic force sharply increases;
2. electrical force during the motion of electrons in wires is almost equal to zero due to the balance of the positive and negative charges inside a conductor. Then magnetic interaction is practically unique in this case.

2. Charged particles in the magnetic field



Since the magnetic force is always perpendicular to the velocity, a charged particle moves along the curvilinear path and the acceleration of the particle is the centripetal one. Firstly, let us consider the ***special case***, when $\vec{v} \perp \vec{B}$.

According to Newton's second law $ma = F \Rightarrow$ $ma_n = qvB\sin\alpha$. As $\sin\alpha = 1$, $m\dfrac{v^2}{R} = qvB$.

The ***radius*** of path is given by

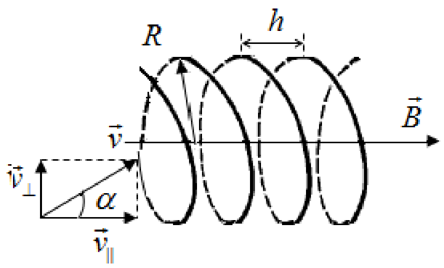
$$R = \frac{mv}{qB}. \tag{5.29}$$

All magnitudes in this expression do not change during the particle motion in uniform magnetic field, hence, the radius is constant for the given motion, and, therefore, the path of the particle is the circle.

The ***period***

$$T = \frac{2\pi R}{v} = \frac{2\pi mv}{vqB} = \frac{2\pi m}{qB}. \tag{5.30}$$

is independent on the velocity. In the given magnetic field it depends on the specific charge, i.e. the ratio q/m. It allows classifying the charged particles in the magnetic field depending on their specific charges. The mass spectrometer is an instrument which can measure the masses and the relative concentrations of atoms and molecules.



In the ***general case***, the particle takes part in two motions simultaneously: along the circular path at the velocity $\vec{v}_\perp$ and along the straight line at $\vec{v}_\parallel$ (as the charged particle moving parallel to the magnetic field does not experience a force). Therefore, the general motion is the ***helical*** motion. If the angle between $\vec{v}$ and $\vec{B}$ is α, then $v_\perp = v\sin\alpha$ and $v_\parallel = v\cos\alpha$, hence

$$R = \frac{mv_\perp}{qB}; \tag{5.31}$$

$$h = v_{\parallel} T = \frac{v_{\parallel} 2\pi m}{qB}. \qquad (5.32)$$

The property that magnetic forces compel the charged particles to change their direction of motion without changing their speed is used in high-energy particle accelerators to focus the beams of particles which eventually collide with targets to produce new particles, and in the mass spectrometers, which are used to identify elements. In these devices the beam of charged particles (ions) enters the region of the magnetic field, where they experience a force and are bent in a circular path. The amount of bending depends on the mass (and charge) of the particle, and by measuring this amount one can infer the type of particle that is present by comparing to the bending of known elements.

Chapter 6. MAGNETIC FIELD IN MATTER

I. MAGNETIZATION OF MATTER. THE MAGNETIZATION $\vec{J}$.

1. A field in a magnetic

Placing any material into the magnetic field causes changing in the field and in this material because every substance is magnetic, i.e. it can be magnetized. So magnetization is acquisition by the material of the magnetic moment.

Magnetic induces its own magnetic field $\vec{B}'$ which together with initial external magnetic field $\vec{B}_0$ creates the ***total field in magnetic***

$$\vec{B} = \vec{B}_0 + \vec{B}'. \qquad (6.1)$$

For this total field $\vec{B}$ Gauss' law is valid

$$\oint \vec{B} d\vec{S} = 0.$$

Therefore, the lines of the vector $\vec{B}$ in the presence of substance remain continuous, i.e. they form closed loops.

2. The mechanism of magnetization

The substances whose molecules have their own magnetic moments (the origin of them is tiny current loops connected with the orbital motion of electrons) are magnetized due to orientation of these moments in an external field This process induces the field $\vec{B}'$.

If molecules of a substance without external magnetic field have no magnetic moments placing of these substances into a magnetic field creates currents in molecules. Therefore, molecules and the substance as a whole gain a magnetic moment that also causes a field $\vec{B}'$ appear.

3. The magnetization $\vec{J}$

The ***magnetization*** of a material is expressed in terms of density of net magnetic dipole moments in the material as the magnetic dipole moment per unit volume, i.e.

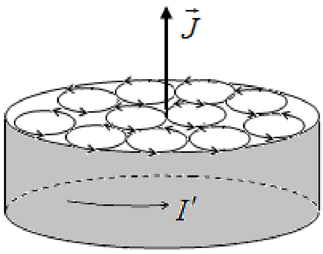
$$\vec{J} = \frac{1}{\Delta V}\sum \vec{p}_m, \qquad (6.2)$$

where ΔV a physically infinitesimal volume in a neighborhood of the given point, $\vec{p}_m$ is a magnetic moment of a molecule.

If n is the concentration of molecules and $\langle \vec{p}_m \rangle$ is an average magnetic moment of a molecule, magnetization is

$$\vec{J} = n\langle \vec{p}_m \rangle. \qquad (6.3)$$

4. Magnetization currents



The macroscopic properties of matter are a manifestation of the microscopic properties of the atoms which it is composed of. All matter is built up of atoms, and each atom consists of electrons in motion. The currents associated with this motion are termed atomic currents. Each atomic current is a tiny closed circuit of atomic dimensions, and may therefore be appropriately described as a magnetic dipole. The magnetic dipole moments of moving electrons, protons, and neutrons create the magnetic fields of bulk materials.

An elementary circular current in molecule is ***molecular current***. Orientation of molecular currents causes the appearance of macroscopic currents I', which are termed as ***magnetization currents***. Usual currents connected with transition of the carriers of current in substance are termed as ***conduction currents*** I.

To understand the appearance of magnetization currents imagine the cylinder made of the homogeneous magnetic with magnetization $\vec{J}$ directed along its axis. The planes of molecular currents in the magnetized magnetic are oriented perpendicularly to the vector $\vec{J}$. The molecular currents of adjacent molecules in places of their contact flow in opposite directions and compensate each other. Uncompensated molecular currents are exclusively on a lateral surface of the cylinder. These currents form the macroscopic surface magnetization current I' which is circulating on a lateral surface of the cylinder. The current I' induced the same macroscopic field as the molecular currents induce together.

If the magnetic is inhomogeneous there is no compensation of currents inside it, and as a result the macroscopic volume magnetization current I' appear in it. The current distribution depends not only on the configuration and properties of the magnetic but also on the field $\vec{B}$. Generally the problem of determination of field $\vec{B}$ has no simple solution. Searching the solution of this problem leads to the necessity of clearing the relationship between I' and $\vec{J}$.

5. Circulation theorem for the vector $\vec{J}$

Circulation of a magnetization $\vec{J}$ around any closed path is equal to an algebraic sum of magnetization currents I' in enclosed region

$$\oint \vec{J} d\vec{l} = I', \qquad (6.4)$$

where $I' = \int \vec{j}\,' d\vec{S}$ and $\vec{j}\,'$ is a magnetization current density.

The field of the vector $\vec{J}$ depends on ***all*** currents (both on magnetization current I' and on conduction current I). However, in some cases with a special symmetry the field of the vector $\vec{J}$ is determined only by magnetization current.

II. THE VECTOR $\vec{H}$

1. Circulation theorem for the vector $\vec{H}$

We'll formulate the circulation theorem for the vector $\vec{H}$ for magnetic fields due to direct currents.

If a magnetic is placed into an external magnetic field magnetization currents are induced in it. Now the circulation of the vector $\vec{B}$ is determined not only by conduction currents I but also by magnetization currents I', therefore,

$$\oint \vec{B}d\vec{l} = \mu_0 \left(I + I' \right).$$

It is difficult to calculate the currents I'. But it is possible to introduce an auxiliary vector whose circulation depends only on conduction currents through this closed path.

Since $\oint \vec{J}d\vec{l} = I'$ and $\oint \vec{B}d\vec{l} = \mu_0 \left(I + I' \right)$,

$$\frac{1}{\mu_0}\oint \vec{B}d\vec{l} = I = \oint \vec{J}d\vec{l}, \ \Rightarrow \ \oint \left(\frac{\vec{B}}{\mu_0} - \vec{J} \right) d\vec{l} = I .$$

Put

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{J} . \tag{6.5}$$

Then the mathematical expression for the circulation theorem is

$$\oint \vec{H}d\vec{l} = I . \tag{6.6}$$

$\left[H \right]$=A/m.

Circulation of the vector $\vec{H}$ around any closed path is equal to an algebraic sum of the conduction currents.

The ***magnetic intensity*** (or ***magnetic field strength***) $\vec{H}$ has no clear physical meaning. The only reason for its introducing is the fact that it allows calculating fields in the presence of magnetic materials without knowing the distribution of magnetization currents. However, this will be possible if we possess a constitutive relation connecting $\vec{J}$ and $\vec{H}$.

2. Dependences $\vec{J}(\vec{H})$ and $\vec{B}(\vec{H})$. Types of magnetics

As it is known the magnetization $\vec{J}$ depends on $\vec{B}$ but usually the relationship between vectors $\vec{J}$ and $\vec{H}$ is used.

In a large class of substances the magnetization vector $\vec{J}$ is proportional to the magnetic strength $\vec{H}$:

$$\vec{J} = \chi\vec{H}, \tag{6.7}$$

where χ is called the ***magnetic susceptibility*** (***magnetizability***) of substance.

The magnetic susceptibility χ can be both positive and negative, in contrast to the electrical susceptibility κ which is always positive.

If $\vec{J} = \chi\vec{H}$, then $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{J} = \frac{\vec{B}}{\mu_0} - \chi\vec{H}$. Therefore, $\vec{H}(1+\chi) = \frac{\vec{B}}{\mu_0}$. Put

$$\mu = 1+\chi. \tag{6.8}$$

Then

$$\vec{B} = \mu\mu_0\vec{H}. \tag{6.9}$$

Dimensionless quantity μ is called ***magnetic permeability*** of substance, $\mu_r = \mu\mu_0$ is called ***relative magnetic permeability***, a quantity which measures the ratio of the internal magnetization to the applied magnetic field.

If the material does not respond to the magnetic field by magnetizing, then the field in the material will be just the applied field and the relative permeability $\mu_r = 1$. A positive relative permeability greater than 1 implies that the material magnetizes in response to the applied magnetic field. As $\chi = \mu - 1$, then the magnetic susceptibility is zero if the material does not respond to any magnetization. So both quantities give the same information, and both are dimensionless quantities.

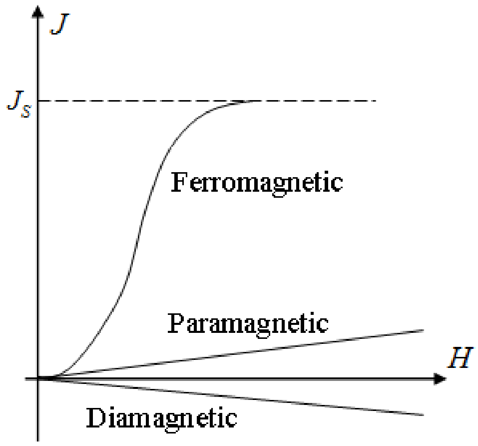
For ordinary solids and liquids at room temperature, the relative permeability μ_r is typically in the range from 1.00001 to 1.003. We recognize this weak magnetic character of common materials by saying "they are not magnetic", which recognizes their great contrast to the magnetic response of ferromagnetic materials. More precisely, they are either paramagnetic or diamagnetic, but that represents a very small magnetic response compared to ferromagnets.

Type of magnetic material	Magnetic susceptibility, χ	Magnetic permeability, μ	Dependence $\vec{J}\left(\vec{H}\right)$
Diamagnetic	$\chi < 0$	$\mu < 1$	linear
Paramagnetic	$\chi > 0$	$\mu > 1$	linear
Ferromagnetic	$\chi > 0 \, (\chi \gg 0)$	$\mu > 1 \, (\mu \gg 1)$	nonlinear

3. Diamagnetic and paramagnetic materials

		Paramagnetic materials	Diamagnetic materials
Examples		Na, Mg, K, Ca, Al, Pt, O_2, solutions of salts	Bi, Cu, Ag, Au, Hg, Be, Ce, inertial gases
Magnetic susceptibility		$\chi > 0$	$\chi < 0$
Magnetic permeability		$\mu > 1$	$\mu < 1$
Magnetic dipole moment p_m		Atoms have their own magnetic dipole moment	Atomic magnetic moment is absent
Behavior in external magnetic field			Induced magnetic moment appears
	Uniform field	Magnetic moments are oriented along the field $\vec{J} \uparrow\uparrow \vec{H}$	Magnetic moments are oriented against the field $\vec{J} \uparrow\downarrow \vec{H}$
	Nonuniform field	Are retracted into a field	Are pushed out from a field

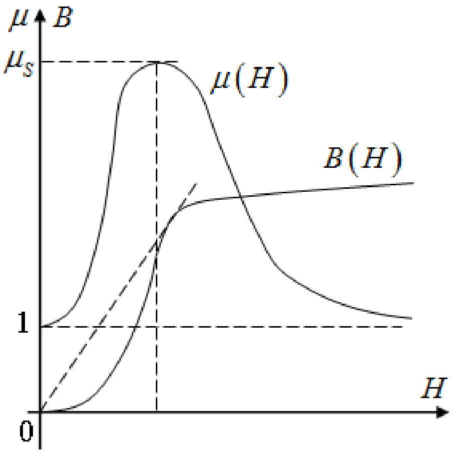
III. THE FERROMAGNETISM



Iron, nickel, cobalt and some of the rare earths (gadolinium, dysprosium) exhibit a unique magnetic behavior which is called ***ferromagnetism*** because iron (ferric) is the most common and most dramatic example. Ferromagnets will tend to stay magnetized to some extent after being subjected to an external magnetic field.

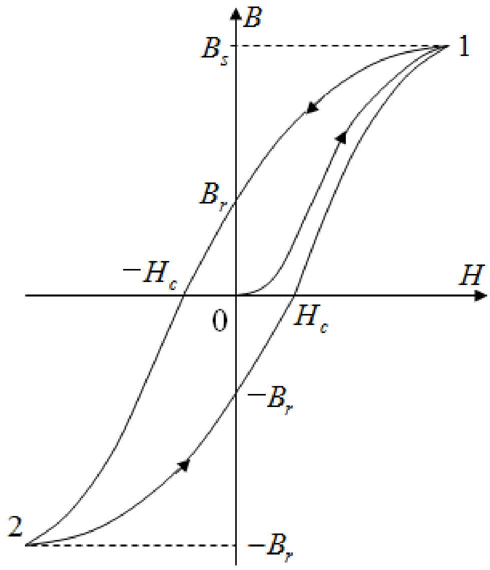
The magnetization of ferromagnets in a huge amount of times (~ 10^{16}) exceeds the magnetization of paramagnetic materials and diamagnets.

1. The basic magnetization curve



The magnetic permeability μ for ferromagnets is not constant. It depends on a magnetic field strength and the dependence $\mu(H)$ is nonlinear. Magnitudes of the maximum magnetic permeability vary over a wide range depending on the material. For example, $\mu \approx 5000$ (iron) and $\mu \approx 10^6$ (superpermalloy).

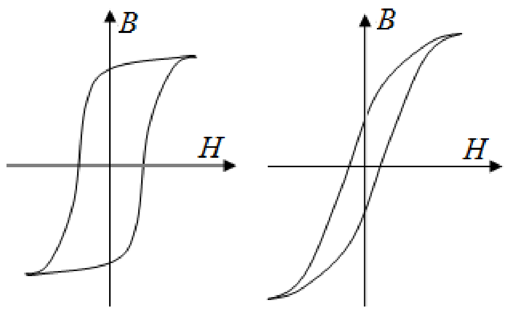
2. Magnetic hysteresis



The ferromagnet follows a non-linear magnetization curve when magnetized from a zero field value. The dependence $B(H)$ for ferromagnets shows that the magnetization of a ferromagnetic substance depends on the history of the substance as well as on the strength of the field. Plot B versus H is called the ***magnetization curve***.

If an alternating magnetic field is applied to the material, its magnetization will trace out a loop called a ***hysteresis loop***. The lack of retraceability of the magnetization curve is the property called ***hysteresis*** (hysteresis means ``to lag behind").

When a ferromagnetic material is magnetized in one direction, it will not relax back to zero magnetization when the imposed magnetizing field is removed. The amount of magnetization it retains at zero driving field is called its ***remanence*** (B_r). It must be driven back to zero by a field in the opposite direction; the amount of reverse driving field required to demagnetize it is called its ***coercivity*** (H_c). The materials with high remanence and high coercivity from which permanent magnets are made are sometimes said to be "***magnetically hard***" (or high-coercitivity, or retentive) materials to contrast them with the "***magnetically soft***" (or low-coercitivity, or nonretentive) materials from which transformer cores and coils for electronics are made.

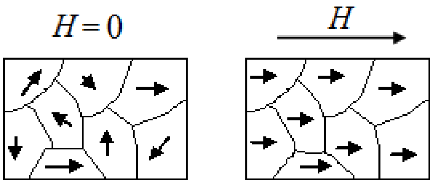


magnetically hard material magnetically soft material

The points 1 and 2 match the ***saturation magnetization***. The area of the hysteresis loop is related to the amount of energy dissipation upon reversal of field.

3. The Curie temperature. The physical nature of ferromagnetism

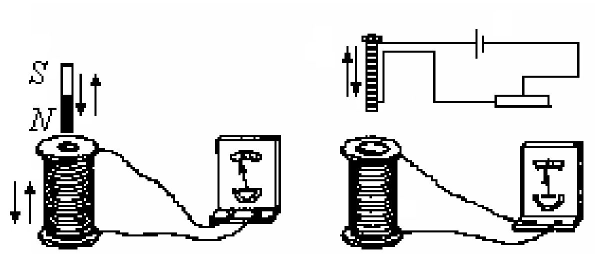
Increasing temperature courses decrease the ability of a ferromagnet to be magnetized as saturation magnetization falls down. For a given ferromagnetic material its ferromagnetic properties abruptly disappears at certain temperature which is called the ***Curie temperature*** for the material. The Curie temperature of iron is about 1043 K.



The physical nature of ferromagnetism can be understood only by means of quantum mechanics. In this material, there are domains in which the magnetic fields of the individual atoms align. The long range order which creates magnetic domains arises from quantum mechanical interaction at atomic level. This interaction is remarkable in that it locks the magnetic moments of neighboring atoms into rigid parallel order over a large number of atoms in spite of the thermal agitation which tends to randomize any atomic level order. The sizes of domains range from 0.1 μm to a few mm. The main implication of the domains is that there is already a high degree of magnetization in ferromagnetic materials within individual domains, but in the absence of external magnetic fields those domains are randomly oriented. A modest applied magnetic field can cause a larger degree of alignment of the magnetic moments with the external field, giving a large multiplication of the applied field. The microscopic evidence about magnetization indicates that the net magnetization of ferromagnetic materials in response to an external magnetic field may actually occur more by the growth of the domains parallel to the applied field at the expense of other domains rather than the reorientation of the domains themselves.

Chapter 7. ELECTROMAGNETIC INDUCTION

I. FARADAY'S LAW OF INDUCTION. LENZ'S LAW



After Ampere and others had investigated the magnetic effect of a current, Faraday tried to find its opposite. He tried to produce a current by means of a magnetic field. He began working on the problem in 1825 but didn't succeed until 1831. The apparatus with which he worked consisted of coils and galvanometer. Any change in the magnetic environment of a coil of wire will cause a voltage (emf) to be "induced" in the coil. No matter how the change is produced, the voltage will be generated. The change could be produced by changing the magnetic field strength, moving a magnet toward or away from the coil, moving the coil into or out of the magnetic field, rotating the coil relative to the magnet, etc.

Faraday's Law of induction: The line integral of the electric field-strength $\vec{E}$ around a closed loop is equal to the negative of the rate of change of the magnetic flux through the area enclosed by the loop.

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}. \quad (7.1)$$

The left part of this equation is electromotive force (emf), therefore, another form of Faraday's law of induction is

$$\mathrm{E}_i = -\frac{d\Phi}{dt}. \qquad (7.2)$$

This induced emf (E_i) is called ***motional emf***.

Generated (or induced) voltage in any loop depends on the rate of changing the magnetic flux linking the circuit. The current stipulated by this emf is ***induced current***.

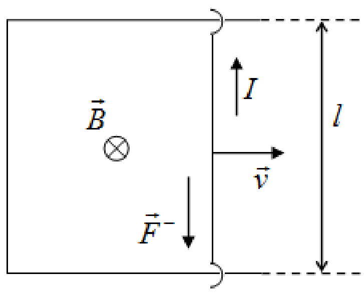
The minus sign in this formula denotes ***Lenz's law***. When an emf is generated by a change in magnetic flux according to Faraday's law, the polarity of the induced emf is such that it produces a current whose magnetic field is opposite to the magnetic field change which produces it. The induced magnetic field inside any loop of wire always acts to keep the magnetic flux in the loop constant. If the external magnetic field is increasing, the induced field acts in opposition to it. If it is decreasing, the induced field acts in the direction of the applied field to try to keep it constant. Another variant of Lenz's law is: the induced current flows in a direction such as to fight the change in a field.

If the wire has N turns, the total magnetic flux (***flux linkage***) is $\Phi = N \cdot \Phi_1$, where Φ_1 is the flux through one loop. Then Faraday's law is

$$\mathrm{E}_i = -N\frac{d\Phi}{dt}. \qquad (7.3)$$

II. THE NATURE OF THE ELECTROMAGNETIC INDUCTION

1. A wire movement in a stationary magnetic field



Consider a wire of length l moving at a velocity $\vec{v}$ in a stationary magnetic field $\vec{B}$ whose direction is perpendicular to the plane of the figure. All electrons in the have the same velocities $\vec{v}$. This movement of charged particles in a magnetic field makes the charges in the wire feel a downward magnetic force (Lorentz force)

$$\vec{F} = -|e| \cdot \left[\vec{v}, \vec{B}\right].$$

Negative charges are accumulated at the bottom of the wire. The excess of positive charges is at the top of the wire. This creates a downward electric field in the wire. The electric field is

$$\vec{E}^* = \frac{\vec{F}}{|e|} = \frac{-|e| \cdot \left[\vec{v}, \vec{B}\right]}{|e|} = -\left[\vec{v}, \vec{B}\right].$$

The emf can be found as circulation of the vector E$_I$ along the length of wire

$$\mathrm{E}_i = -\int \vec{E}^* \cdot dl = -v \cdot B \cdot l, \qquad (7.4)$$

The product $v \cdot l$ is an increment of area per unit time $\frac{dS}{dt}$, therefore, $v \cdot l = B\frac{dS}{dt} = \frac{d\Phi}{dt}$. Hence,

$$\mathrm{E}_i = -\frac{d\Phi}{dt}.$$

This law is valid for any wire or a loop of wire moving in a stationary magnetic field.

2. A wire loop in a changing magnetic field

The appearance of an induction current in the wire in the case mentioned above testifies that the magnetic field varying in time causes the appearance of the extraneous forces in it. These forces aren't magnetic (not proportional to $\left[\vec{v}, \vec{B}\right]$) and, therefore, can't make the motionless charges move. But other forces, except for $q\vec{E}$ and $\left[\vec{v}, \vec{B}\right]$, aren't. So, it is necessary to assume that the induction current is stipulated by the electric field which has been induced in a wire. Just this field is responsible for appearance of emf in a motionless loop when a magnetic field is changing.

Maxwell has assumed that just the magnetic field varying in time induces the appearance of an electric field in space irrespectively a conductive loop presence. The presence of a loop allows detecting the existence of this electric field.

Thus, according to Maxwell's theory, the varying magnetic field induces an electric field. This electric field is not potential (its circulation around a closed loop is not equal to zero). This is a rotational field.

Faraday's law of induction is valid when the magnetic flux through a loop varies both due to a loop motion and due to changing a magnetic field (or when both processes take place).

To offer an explanation it was necessary to use different phenomena: for a moving loop – a magnetic force action, and for a field varying in time – the representation about rotational electric field due to changing magnetic field. Both of these phenomena are independent of each other, but in both cases a motional emf in a loop is always equal to a magnetic flux changing rate.

III. SELF-INDUCTION AND MUTIAL INDUCTION

The induction phenomenon is observed in all cases when the magnetic flux through a contour varies. The modification of a current in a contour leads to origin of motional emf in this contour. This appearance is termed as a self-induction.

1. Inductance

The phenomenon called ***self-induction*** was discovered by Joseph Henry in 1832.

When current flows through a coil, it sets up a magnetic field. And that field threads the coil which produces it. If the current I through the coil is changed (for example, by means of variable resistance), the flux linked with the turns of the coil changes. An emf is therefore induced in the coil. By Lenz's law the direction of the induced emf will be such as to oppose the change of current.

When an emf is induced in a circuit by a change in the current through that circuit, the emf induced is called ***a back-emf***.

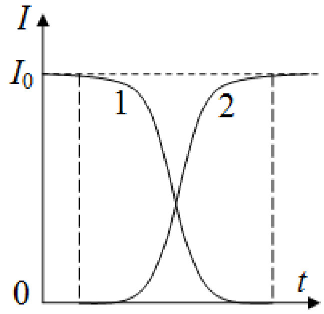
To discuss the effects of self-induction more fully, note that the total flux Φ is proportional to the current in the circuit. The coefficient L in this linear dependence ($L > 0$) is ***inductance*** (or ***self-inductance***)

$$\Phi = L \cdot I . \tag{7.5}$$

So proceeding from Faraday's law, the back-emf generated to oppose a given change in current is

$$\mathrm{E}_s = -L\frac{dI}{dt} . \tag{7.6}$$

Inductance depends on the shape and sizes of a circuit and also on magnetic properties of the surrounding medium. If the circuit is rigid and ferromagnetic medium is absent, the inductance is a constant for a given circuit (independent of a current I).



$$[L] = \frac{\text{Wb}}{\text{m}} = \text{Henry} = \text{H}.$$

Let's calculate the inductance of the long solenoid. If V is a volume of the solenoid, n is a number of turns of wire per the unit length, μ is a magnetic permeability of substance inside the solenoid, B is the magnetic field in solenoid, then

$$B = \mu\mu_0 nI \Rightarrow \Phi_1 = BS = \mu\mu_0 nIS \Rightarrow \Phi = N\Phi_1 = nlBS = nl\mu\mu_0 nIS = \mu\mu_0 n^2 VI,$$

$$L = \Phi/I = \mu\mu_0 n^2 V. \tag{7.7}$$

The examples of self-induction phenomenon are the processes of switching on and switching off in electrical circuits. Appearance of the current when switching on and disappearance of the current when switching off do not occur instantly. These processes take the more time the more inductance is.

2. Mutual induction

Consider two circuits located close to each other. The flux through the first circuit with current I_1 is equal to $\Phi_1 = L_{21} I_1$. Similarly, for the second circuit: $\Phi_2 = L_{12} I_2$. Coefficients L_{12} as well as L_{21} are called ***the mutual inductance*** of circuits. It is possible to show that $L_{12} = L_{21}$ (this property is ***reciprocity theorem***). The fact that a change in the current in one circuit affects the current and emf in the second circuit is quantified in the property called ***mutual induction***.

The inductive coupling between circuits is that emf is produced in one circuit because of the change in current in a coupled circuit.

The emf is described by Faraday's law and its direction is always opposite to the change in the magnetic field produced in it by the coupled circuit. The induced emf in circuits due to self-induction is

$$\mathrm{E}_{s1} = -\frac{d\Phi_1}{dt} = L_{12}\frac{dI_2}{dt}, \quad \mathrm{E}_{s2} = -\frac{d\Phi_2}{dt} = L_{21}\frac{dI_1}{dt}. \tag{7.8}$$

The mutual inductance can be defined as the coefficient of proportionality between the emf generated in circuit 1 to the change in current in circuit 2 which produced it.

IV. MAGNETIC FIELD ENERGY

The circuit with inductance L and without ferromagnets has energy storage

$$W = \frac{LI^2}{2} = \frac{I\Phi}{2} = \frac{\Phi^2}{2L}. \qquad (7.9)$$

These formulas express energy using characteristics of circuit. Let’s express magnetic field energy through the field characteristics. We’ll find the energy for solenoid.

Inductance of the solenoid is $L = \mu\mu_0 n^2 V$. Taking into account that $nI = H = B/\mu\mu_0$, we obtain

$$W = \frac{LI^2}{2} = \frac{\mu\mu_0 n^2 I^2 V}{2} = \frac{B^2 V}{2\mu\mu_0} = \frac{\left(\vec{B},\vec{H}\right)}{2}V. \qquad (7.10)$$

This formula is valid for a uniform magnetic field in any volume V.

In general theory it is shown that the energy can be expressed by means of vectors $\vec{B}$ and $\vec{H}$ in any case (in the absence of ferromagnetic media) by formula

$$W = \int \frac{\left(\vec{B},\vec{H}\right)}{2} dV. \qquad (7.11)$$