1.2 ALGEBRA OF EVENTS (BOOLEAN ALGEBRA)

Before talking about the assignment of probabilities to events, we introduce some operations by which new events are formed from old ones. These operations correspond to the construction of compound sentences by use of the connectives "or," "and," and "not." Let A and B be events in the same sample space. Define the *union* of A and B (denoted by $A \cup B$) as the set consisting of those points belonging to *either A or B or both.* (Unless otherwise specified, the word "or" will have, for us, the inclusive connotation. In other words, the statement "p or q" will always mean "p or q or both.") Define the *intersection* of A and B, written $A \cap B$, as the set of points that belong to *both A* and B. Define the *complement* of A, written A^c, as the set of points which do *not* belong to A.

▶ **Example 1.** Consider the experiment involving the toss of a single die, with $N =$ the result; take a sample space with six points corresponding to $N = 1, 2, 3, 4, 5, 6$. For convenience, label the points of the sample space by the integers 1 through 6.

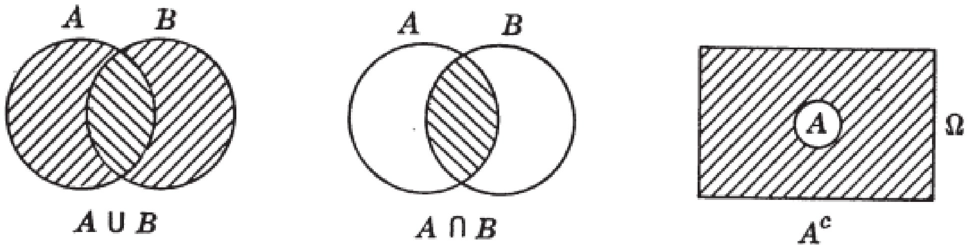


FIGURE 1.2.1 Venn Diagrams.

$$\text{Let } A = \{N \text{ is even}\} \qquad \text{and} \qquad B = \{N \geq 3\}$$

Then

$$A \cup B = \{N \text{ is even or } N \geq 3\} = \{2, 3, 4, 5, 6\}$$
$$A \cap B = \{N \text{ is even and } N \geq 3\} = \{4, 6\}$$
$$A^c = \{N \text{ is not even}\} = \{1, 3, 5\}$$
$$B^c = \{N \text{ is not } \geq 3\} = \{N < 3\} = \{1, 2\}$$ ◀

Define the *union of n events* $A_1, A_2, \ldots, A_n$ (notation: $A_1 \cup \cdots \cup A_n$, or $\bigcup_{i=1}^n A_i$) as the set consisting of those points which belong to *at least one* of the events $A_1, A_2, \ldots, A_n$. Similarly define the union of an infinite sequence of events $A_1, A_2, \ldots$ as the set of points belonging to at least one of the events $A_1, A_2, \ldots$ (notation: $A_1 \cup A_2 \cup \cdots$, or $\bigcup_{i=1}^\infty A_i$).

Define the *intersection of n events* $A_1, \ldots, A_n$ as the set of points belonging to *all* of the events $A_1, \ldots, A_n$ (notation: $A_1 \cap A_2 \cap \cdots \cap A_n$, or $\bigcap_{i=1}^n A_i$). Similarly define the intersection of an infinite sequence of events as the set of points belonging to all the events in the sequence (notation: $A_1 \cap A_2 \cap \cdots$, or $\bigcap_{i=1}^\infty A_i$). In the above example, with $A = \{N \text{ is even}\} = \{2, 4, 6\}$, $B = \{N \geq 3\} = \{3, 4, 5, 6\}$, $C = \{N = 1 \text{ or } N = 5\} = \{1, 5\}$, we have

$$A \cup B \cup C = \Omega, \qquad A \cap B \cap C = \varnothing$$
$$A \cup B^c \cup C = \{2, 4, 6\} \cup \{1, 2\} \cup \{1, 5\} = \{1, 2, 4, 5, 6\}$$
$$(A \cup C) \cap [(A \cap B)^c] = \{1, 2, 4, 5, 6\} \cap \{4, 6\}^c = \{1, 2, 5\}$$

Two events in a sample space are said to be *mutually exclusive* or *disjoint* if A and B have no points in common, that is, if it is impossible that both A and B occur during the *same* performance of the experiment. In symbols, A and B are mutually exclusive if $A \cap B = \varnothing$. In general the events $A_1, A_2, \ldots, A_n$ are said to be mutually exclusive if no two of the events have a point in common; that is, no more than one of the events can occur during

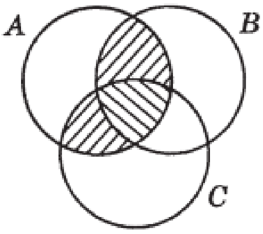


FIGURE 1.2.2 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

the same performance of the experiment. Symbolically, this condition may be written

$$A_i \cap A_j = \varnothing \qquad \text{for } i \neq j$$

Similarly, infinitely many events $A_1, A_2, \ldots$ are said to be mutually exclusive if $A_i \cap A_j = \varnothing$ for $i \neq j$.

In some ways the algebra of events is similar to the algebra of real numbers, with union corresponding to addition and intersection to multiplication. For example, the commutative and associative properties hold.

$$A \cup B = B \cup A, \qquad A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap B = B \cap A, \qquad A \cap (B \cap C) = (A \cap B) \cap C \qquad (1.2.1)$$

Furthermore, we can prove that for events A, B, and C in the same sample space we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad (1.2.2)$$

There are several ways to establish this; for example, we may verify that the sets of both the left and right sides of the equality above are represented by the area in the Venn diagram of Figure 1.2.2.

In many ways the algebra of events differs from the algebra of real numbers, as some of the identities below indicate.

$$\begin{aligned} A \cup A &= A & \quad A \cup A^c &= \Omega \\ A \cap A &= A & \quad A \cap A^c &= \varnothing \\ A \cap \Omega &= A & \quad A \cup \varnothing &= A \\ A \cup \Omega &= \Omega & \quad A \cap \varnothing &= \varnothing \end{aligned} \tag{1.2.7}$$

1.3 PROBABILITY

We now consider the assignment of probabilities to events. A technical complication arises here. It may not always be possible to regard all subsets of Ω as events. We may discard or fail to measure some of the information in the outcome corresponding to the point $\omega \in \Omega$, so that for a given subset A of Ω, it may not be possible to give a yes or no answer to the question "Is $\omega \in A$?" For example, if the experiment involves tossing a coin five times, we may record the results of only the first three tosses, so that $A = \{\text{at least four heads}\}$ will not be "measurable"; that is, membership of $\omega \in A$ cannot be determined from the given information about ω.

In a given problem there will be a particular class of subsets of Ω called the "class of events." For reasons of mathematical consistency, we require that the event class $\mathscr{F}$ form a *sigma field*, which is a collection of subsets of Ω satisfying the following three requirements.

$$\Omega \in \mathscr{F} \tag{1.3.1}$$

$$A_1, A_2, \ldots \in \mathscr{F} \quad \text{implies} \quad \bigcup_{n=1}^{\infty} A_n \in \mathscr{F} \tag{1.3.2}$$

That is, $\mathscr{F}$ is closed under finite or countable union.

$$A \in \mathscr{F} \quad \text{implies} \quad A^c \in \mathscr{F} \tag{1.3.3}$$

That is, $\mathscr{F}$ is closed under complementation.

We are now ready to talk about the assignment of probabilities to events. If $A \in \mathscr{F}$, the probability $P(A)$ should somehow reflect the long-run relative frequency of A in a large number of independent repetitions of the experiment. Thus $P(A)$ should be a number between 0 and 1, and $P(\Omega)$ should be 1.

Now if A and B are disjoint events, the number of occurrences of $A \cup B$ in n performances of the experiment is obtained by adding the number of occurrences of A to the number of occurrences of B. Thus we should have

$$P(A \cup B) = P(A) + P(B) \qquad \text{if } A \text{ and } B \text{ are disjoint}$$

and, similarly,

$$P(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) \qquad \text{if } A_1, \ldots, A_n \text{ are disjoint}$$

For mathematical convenience we require that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

when we have a *countably infinite* family of disjoint events $A_1, A_2, \ldots$.

The assumption of countable rather than simply finite additivity has not been convincingly justified physically or philosophically; however, it leads to a much richer mathematical theory.

A function that assigns a number $P(A)$ to each set A in the sigma field $\mathscr{F}$ is called a *probability measure* on $\mathscr{F}$, provided that the following conditions are satisfied.

$$P(A) \geq 0 \qquad \text{for every } A \in \mathscr{F} \tag{1.3.4}$$

$$P(\Omega) = 1 \tag{1.3.5}$$

If $A_1, A_2, \ldots$ are disjoint sets in $\mathscr{F}$, then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots \tag{1.3.6}$$

We may now give the underlying mathematical framework for probability theory.

DEFINITION. A *probability space* is a triple $(\Omega, \mathscr{F}, P)$, where Ω is a set, $\mathscr{F}$ a sigma field of subsets of Ω, and P a probability measure on $\mathscr{F}$.

We shall not, at this point, embark on a general study of probability measures. However, we shall establish four facts from the definition. (All sets in the arguments to follow are assumed to belong to $\mathscr{F}$.)

1. $P(\varnothing) = 0$ (1.3.7)

2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (1.3.8)

3. If $B \subset A$, then $P(B) \leq P(A)$; in fact,

$$P(A - B) = P(A) - P(B) \tag{1.3.9}$$

where $A - B$ is the set of points that belong to A but not to B.

$$4. \quad P(A_1 \cup A_2 \cup \cdots) \leq P(A_1) + P(A_2) + \cdots \tag{1.3.10}$$

That is, the probability that at least one of a finite or countably infinite collection of events will occur is less than or equal to the sum of the probabilities; note that, for the case of two events, this follows from $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$.

3. Law of probability

3.1 The Addition Law

As we have already noted the sample space S is the set of all possible outcomes of a given experiment. Events A and B are subsets of Ω. In the previous block we defined what was meant by P (A), P (B) and their complements in the particular case in which the experiment had equally likely outcomes.

Events, like sets, can be combined to produce new events.

- $A \cup B$ denotes the event that A or B (or both) occur when the experiment is performed.

- $A \cap B$ denotes the event that both A and B occur.

In this block we obtain expressions for determining the probabilities of these combined events written $P(A \cup B)$ and $P(A \cap B)$ respectively.

Exampel 1

A bag contains 18 colored marbles: 4 are colored red, 8 are colored yellow and 6 are colored green. A marble is selected at random. What is the probability that the ball chosen is either red or green?

Assuming that any marble is as likely to be selected as any other we can say:

the probability that the chosen marble is red is 4/18
the probability that it is green is 6/18

It follows that the probability that the ball chosen is either red or green is
10/18 = 4/18 + 6/18 .

This is the case because no ball can be simultaneously red and green. We say that the events 'the ballis red' and 'the ball is green' are *mutually exclusive*.

Example 2

Consider a pack of 52 playing cards. A card is selected at random. What is the probability that the card is either a diamond or a ten?

The probability that it is a diamond is 13/52 since there are 13 diamond cards in the pack. The probability that the card is a ten is 4/52 .
There are 16 cards that fall into the category of being either a diamond or a ten: 13 of these are diamonds and there is a ten in each of the three other suits. Therefore, the probability of the card being a diamond or a ten is 16/52 not 13/52 + 4/52 = 17/52 .

We say that these events are not mutually exclusive. We must ensure in this case not to simply add the two original probabilities; this would count the ten of diamonds twice - once in each category.

In the last example P(A)= 13/52 and P(B)= 4/52 . The intersection event $A \cap B$ consists of only one member - the ten of diamonds - hence $P(A \cap B)= 1/52$.

Therefore $P(A \cup B)= 13/52 + 4/52 - 1/52 = 16/52$ as we have already argued.

3.2 Conditional Probability

Suppose a bag contains 6 balls, 3 red and 3 white. Two balls are chosen (without replacement) at random, one after the other. Consider the two events A, B:
A is event "first ball chosen is red",
B is event "second ball chosen is red"

We easily find P(A)= 3/6 = 1/2 .

However, determining the probability of B is not quite so straightforward. If the first ball chosen is red then the bag subsequently contains 2 red balls and 3 white. In this case P(B)= 2/5 .
However, if the first ball chosen is white then the bag subsequently contains 3 red balls and 2 white. In this case P(B)= 3/5 .
What this example shows is that the probability that B occurs is clearly dependent upon whether or not the event A has occurred. The probability of B occurring is conditional on the occurrence or otherwise of A.
The conditional probability of an event B occurring given that event A has occurred is written P(B|A). In this particular example P(B|A)= 2/5 and $P(B|A')= 3/5$.

Consider more generally, the performance of an experiment in which the outcome is a member of an event A. We can therefore say that the event A has occurred.
What is the probability that B then occurs? That is what is P(B|A)? In a sense we have a new sample space which is the event A. For B to occur some of its members must also be members of event A. So P(B|A) must be the number of outcomes in $A \cap B$ divided by the number of outcomes in A. That is

$$P(B|A)= (\text{number of outcomes in } A\cap B) / (\text{number of outcomes in } A).$$

Now if we divide both the top and bottom of this fraction by the total number of outcomes of the experiment we obtain an expression for the conditional probability of B occurring given that A has occurred:

$$P(B|A) = \frac{P(A\cap B)}{P(A)} \qquad \text{or, equivalently} \qquad P(A\cap B) = P(B|A)P(A)$$

To illustrate the use of conditional probability concepts we return to the example of the bag containing 3 black and 3 white balls in which we consider two events:
A is event "first ball is black"

B is event “second ball is black”
Let the black balls be numbered 1 to 3 and the white balls 4 to 6. If, for example, (3,5) represents the fact that the first ball is 3 (black) and the second ball is 5 (white) then we see that there are $6 \times 5 = 30$ possible outcomes to the experiment (no ball can be selected twice.) If the first ball is black then only the fifteen outcomes (1,x), (2,y), (3,z) are then possible (here $x \neq 1$, $y \neq 2$ and $z \neq 3$). Of these fifteen the six outcomes { (1,2), (1,3), (2,1), (2,3), (3,1), (3,2)} will produce the required result, i.e. the event in which both balls chosen are black, giving a probability: P(B|A)= 6/15 = 2/5 .

3.3 Independent events

If the occurrence of one event A does not affect, nor is affected by, the occurrence of another event B then we say that A and B are independent events. Clearly, if A and B are independent then P(B|A)= P(B) P(A|B)= P(A).
Then we have for independent events:

$$P(A \cap B) = P(A)P(B)$$

In Figure 1 two components a and b are connected in series.

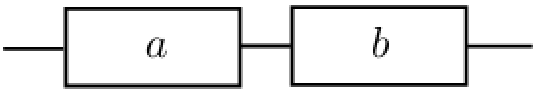


Figure 1

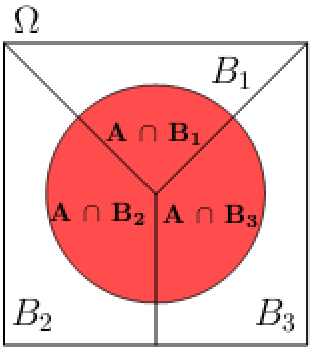
Define two events
• A is the event ‘component a is operating’
• B is the event ‘component b is operating’
P(A)=0.99, and P(B)=0.98
The circuit functions only if a and b are both operating simultaneously. The components are assumed to be independent. Then the probability that the circuit is operating is given by $P(A\cap B)= P(A)P(B)=0.99 \times 0.98 = 0.9702$
Note that this probability is smaller then either P(A) or P(B).

3.4 Law of Total Probability

Suppose the sample space is divided into 3 disjoint events B1, B2, B3 (see the figure below). Then for any event A:

$$P(A) = P(A \cap B1) + P(A \cap B2) + P(A \cap B3)$$
$$P(A) = P(A|B1)\,P(B1) + P(A|B2)\,P(B2) + P(A|B3)\,P(B3) \qquad (1)$$

The top equation says `if A is divided into 3 pieces then P(A) is the sum of the probabilities of the pieces'. The bottom equation (1) is called the law of total probability. It is just a rewriting of the top using the multiplication rule.



The sample space Ω and the event A are each divided into 3 disjoint pieces. The law holds if we divide Ω into any number of events, so long as they are disjoint and cover all of Ω. Such a division is often called a partition of Ω.

3.5 Bayes' Theorem

Bayes' theorem is a pillar of both probability and statistics and central to the rest of this course. For two events A and B Bayes' theorem (also called Bayes' rule and Bayes' formula) says

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}.$$

Comments: 1. Bayes' rule tells us how to `invert' conditional probabilities, i.e. to find P(B|A) from P(A|B).

2. In practice, P(A) is often computed using the law of total probability.

Proof of Bayes' rule

The key point is that $A \cap B$ is symmetric in A and B. So the multiplication rule says $P(B|A) * P(A) = P(A \cap B) = P(A|B) * P(B)$. Now divide through by P(A) to get Bayes' rule.

A common mistake is to confuse P(A|B) and P(B|A). They can be very different. This is illustrated in the next example.

Example. Toss a coin 5 times. Let H_1 = "first toss is heads" and let H_A = "all 5 tosses are heads". Then $P(H_1|H_A) = 1$ but $P(H_A|H_1) = 1/16$:

For practice, let's check this with Bayes' theorem. The terms are $P(H_A|H_1) = 1/16$,

$P(H_1) = 1/2$, $P(H_A) = 1/32$. So, $P(H_1|H_A) = \frac{P(H_A|H_1)P(H_1)}{P(H_A)} = \frac{(1/16)\cdot(1/2)}{1/32} = 1.$

3.6 Bernoulli Trials Probability

Imagine some experiment (for example, tossing a coin) that only has two possible outcomes. Such an experiment is called **Bernoulli trial**.
Now image a series of such experiments. What is the probability that tossing a fair coin 5 times we will get exactly 2 heads (and hence 3 tails)?
Let *A*— "There will be 2 heads in 5 trials". Then, the probability is given by:

$$P(A)=C_5^2 \frac{1}{2}^2 \frac{1}{2}^3 =10*1/32=0.3125$$

$$\text{Generally: } P(A)=C_n^k p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k}, \quad 0 \le k \le n$$

Where
n is the number of trials;
k is the number of successes;
p the probability for a success;
q the probability for a failure;

3.7 Properties of the Bernoulli Trials Probability

1. The recurrence formula $P(n,k)$ is $P(n,k+1)=\frac{n-k}{k+1}*\frac{p}{q}*P(n,k)$;

2. The number k_0, which corresponds to the maximum probability $P(n,k_0)$ is called the most probable number of occurrences of the event A:
$$np-q\le k_0\le np+p;$$

3. Probability $P(n,k_1\le k\le k_2)$ that in n Bernoulli Trials scheme event A will appear from k_1 to k_2 times is equal to
$$P(n,k_1\le k\le k_2)=\sum_{k=k_1}^{k_2}P(n,k)=\sum_{k=k_1}^{k_2}C_n^k p^k q^{n-k};$$

4. Probability $P(n,1\le k\le k_2)$ that in n trials event A will appear at least once is equal to
$$P(n,1\le k\le k_2)=1-P(n,0)=1-q^n.$$

Let n independent experiments is carried out. Each of them has $r(r\ge 2)$ pairwise disjoint and unique outcomes $A_1,A_2...A_r$ with probabilities $p_1=P(A_1),p_2=P(A_2),...,p_r=P(A_r)$. It is required to determine that probability that from series of n independent trials outcomes A_1 will appear k_1 times, $A_2-k_2,...,A_r-k_r(k_1+k_2+...+k_3=n)$ then $P(n,k_1,...,k_r)=\frac{(k_1+k_2+...+k_r)!}{k_1!k_2!...k_r!}p_1^{k_1}p_2^{k_2}...p_r^{k_r}$.

Calculation of the probabilities $P(n,k)$ for large values of n using formula of Bernoulli Trials is problematic. That’s why calculation of the corresponding probabilities is carried out using following approximate formulas.

If number of trials is large $n\to\infty$ and probability of event is small $p\to 0$, so that $np\to a\ \ 0\le a\le\infty$ and $p\ll\frac{1}{\sqrt{n}}$ then using **Poisson’s Formula**
$$P(n,k)\approx\frac{a^k}{k!}e^{-a},k=\overline{0,n}.$$
If number of trials is large and probabilities p and q are not small, so that next conditions are fullfield $0<np-3\sqrt{npq}$ and $np+3\sqrt{npq}<n$, then we have to use approximate **Moivre-Laplace’s Formula:**

- local $P(n,k) \approx \dfrac{\varphi(x)}{\sqrt{npq}}$, where $\varphi(x) = \dfrac{1}{\sqrt{2\Pi}} \exp(-\dfrac{x^2}{2})$, $x = \dfrac{k-np}{\sqrt{npq}}$

- integral $P(n, k_1 \le k \le k_2) \approx \Phi(x_2) - \Phi(x_1)$, where $x_1 = \dfrac{k_1 - np}{\sqrt{npq}}, x_2 = \dfrac{k_2 - np}{\sqrt{npq}}$, and

Laplace's Function $\Phi(x) = \dfrac{1}{2\Pi} \int_0^x \exp(-\dfrac{x^2}{2}) dx$.

Functions $\varphi(x)$ and $\Phi(x)$ are tabulated. When we are using the table we have to remember that $\varphi(x)$ is even $\varphi(-x) = \varphi(x)$ and Laplace's Function is odd $\Phi(-x) = -\Phi(x)$.

Task 1.

A bag contains 20 marbles, 3 are coloured red, 6 are coloured green, 4are coloured blue, 2 are coloured white and 5 are coloured yellow. One ball is selected at random. Find the probabilities of the following events.
(a) the ball is either red or green
(b) the ball is not blue
(c) the ball is either red or white or blue. (Hint: consider the complementary event.)

Answer

Note that a ball can only have one colour, which are designated by the letters R, G, B, W, Y.

(a) $P(R \cup G) = P(R) + P(G) = \frac{3}{20} + \frac{6}{20} = \frac{9}{20}$.

(b) $P(B') = 1 - P(B) = 1 - \frac{4}{20} = \frac{16}{20} = \frac{4}{5}$.

(c) The complementary event is $G \cup Y$ and $P(G \cup Y) = \frac{6}{20} + \frac{5}{20} = \frac{11}{20}$.

Hence $P(R \cup W \cup B) = 1 - \frac{11}{20} = \frac{9}{20}$

In the last example (part (c)) we could alternatively have used an obvious extension of the law of addition for mutually exclusive events:

$P(R \cup W \cup B) = P(R) + P(W) + P(B) = 3/20 + 2/20 + 4/20 = 9/20$.

Task 2.

Figure 1 shows a simplified circuit in which two components a and b are connected in parallel.

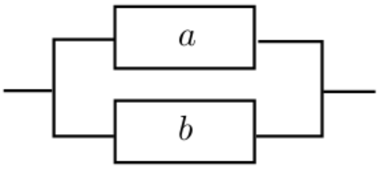


Figure 1

The circuit functions if either or both of the components are operational. It is known that if A is the event 'component a is operating' and B is the event 'component b is operating' then $P(A) = 0.99$, $P(B) = 0.98$ and $P(A \cap B) = 0.9702$. Find the probability that the circuit is functioning.

Answer

The probability that the circuit is functioning is $P(A \cup B)$. In words: either a or b must be functioning if the circuit is to function. Using the keypoint:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.99 + 0.98 - 0.9702 = 0.9998 \end{aligned}$$

Not surprisingly the probability that the circuit functions is greater than the probability that either of the individual components functions.

1. The following people are in a room: 5 men over 21, 4 men under 21, 6 women over 21, and 3 women under 21. One person is chosen at random. The following events are defined: $A = \{\text{the person is over 21}\}$; $B = \{\text{the person is under 21}\}$; $C = \{\text{the person is male}\}$; $D = \{\text{ the person is female}\}$. Evalute the following:

   (a) $P(B \cap D)$

   (b) $P(A' \cup C')$ Express the meaning of these quantities in words.

2. A lot consists of 10 good articles, 4 with minor defects, and 2 with major defects. One article is chosen at random. Find the probability that:

   (a) it has no defects

   (b) it has no major defects

   (c) it is either good or has major defects

3. A card is drawn at random from a deck of 52 playing cards. What is the probability that it is an ace or a face card?

4. In a single throw of two dice, what is the probability that neither a double nor a 9 will appear?

1. (a) $P(B \cup D) = P(B) + P(D) - P(B \cap D) \qquad p(B) = \frac{7}{18}, \quad p(D) = \frac{9}{18} = \frac{1}{2}$

   $\therefore \qquad P(B \cap D) = \frac{3}{18} = \frac{1}{6} \qquad \therefore \quad P(B \cup D) = \frac{7}{18} + \frac{9}{18} - \frac{3}{18} = \frac{13}{18}$

   (b) $P(A' \cap C') \qquad A' = \{\text{people under 21}\} \quad C' = \{\text{people who are female}\}$

   $\therefore \qquad P(A' \cap C') = \frac{3}{18} = \frac{1}{6}$

2. $G = \{\text{article is good}\}, \ M_j = \{\text{major defect}\} \quad M_n = \{\text{minor defect}\}$

   (a) $P(G) = \frac{10}{16} = \frac{5}{8}$

   (b) $P(G \cup M_n) = P(G) + P(M_n) - P(G \cap M_n) = \frac{5}{8} + \frac{4}{16} - 0 = \frac{7}{8}$

   (c) $P(G \cup M_j) = P(G) + P(M_j) - P(G \cap M_j) = \frac{5}{8} + \frac{2}{16} + 0 = \frac{6}{8} = \frac{3}{4}$

3. $F = \{\text{face card}\} \quad A = \{\text{card is ace}\} \quad P(F) = \frac{12}{52}, \quad P(A) = \frac{4}{52}$

   $\therefore \qquad P(F \cup A) = P(F) + P(A) - P(F \cap A) = \frac{12}{52} + \frac{4}{52} - 0 = \frac{16}{52}$

4. $D = \{\text{double is thrown}\}$ $\quad N = \{\text{sum is } 9\}$

$P(D) = \frac{6}{36}$ (36 possible outcomes in an experiment in which all the outcomes are equally probable).

$P(N) = P\{(6 \cap 3) \cup (5 \cap 4)(4 \cap 5) \cup (3 \cap 6)\} = \frac{4}{36}$

$P(D \cup N) = p(D) + P(N) - P(D \cap N) = \frac{6}{36} + \frac{4}{36} - 0 = \frac{10}{36}$

$P(\overline{D \cup N}) = 1 - P(D \cup N) = 1 - \frac{10}{36} = \frac{13}{18}$

Task

A bag contains 10 balls, 3 red and 7 white. Let Aand B be the events as previously. Calculate the probability that the second ball is white.

Answer

The two mutually exclusive events are $A \cap B'$ and $A' \cap B'$. However the event that the second ball chosen is white is B' and $B' = (A \cap B') \cup (A' \cap B')$

$P(A \cap B') = P(B'|A)P(A) = \frac{7}{9} \times \frac{3}{10} = \frac{21}{90}$

$P(A' \cap B') = P(B'|A')P(A') = \frac{6}{9} \times \frac{7}{10} = \frac{42}{90}$

Therefore the required probability is $\frac{21}{90} + \frac{42}{90} = \frac{63}{90} = \frac{7}{10}$.

More exercises for you to try

1. A box contains 4 bad and 6 good tubes. Two are drawn out together. One of them is tested and found to be good. What is the probability that the other one is also good?

2. In the above problem the tubes are checked by drawing a tube at random, testing it and repeating the process until all 4 bad tubes are located. What is the probability that the fourth bad tube will be located

   (a) on the fifth test? $\quad$ (b) on the tenth test?

3. A man owns a house in town and a cottage in the country. In any one year the probability of the house being burgled is 0.01 and the probability of the cottage being burgled is 0.05. In any one year what is the probability that:

   (a) both will be burgled? $\quad$ (b) one or the other (but not both) will be burgled ?

4. In a Series, teams A and B play until one team has won 4 games. If team A has probability $2/3$ of winning against B in a single game, what is the probability that the Series will end only after 7 games are played?

5. The probability that a single aircraft engine will fail during flight is q. A plane makes a successful flight if at least half its engines run. Assuming that the engines are independent, find the values of q for which a two-engine plane is to be preferred to a four-engined one.

6. Current flows through a relay only if it is closed. The probability of any relay being closed is 0.95. Calculate the probability that a current will flow through a circuit composed of 3 relays in parallel. What assumption must be made?

<u>*Answer*</u>

1. Let $G_i = \{i^{th} \text{ tube is good}\}$ $\quad B_i = \{i^{th} \text{ tube is bad}\}$

$P(G_2|G_1) = \frac{5}{9}$ (only 5 good tubes left out of 9).

2. Same events as in question 1.

(a) This will occur if event
$\{B_1 \cap B_2 \cap B_3 \cap G_4 \cap B_5\} \cup \{\ldots \cap B_5\} \cup \{\ldots B_5\} \cup \{\ldots B_5\}$ occurs.

Here we have a number of events in which B_5 *must* appear in the last position and there must be just three appearances of the B symbol in the first 4 slots. Now the number of ways of arranging 3 from 4 is ${}^4C_3$ (see following Section). Thus the probability of the required event occurring is ${}^4C_3 P\{B_1 \cap B_2 \cap B_3 \cap G_4 \cap B_5\}$.

$$\begin{aligned}
\therefore \qquad P(\text{event occuring}) &= {}^4C_3 P\{B_1 \cap B_2 \cap B_3 \cap G_4 \cap B_5\} \\
&= {}^4C_3 P(B_5|(B_1 \cap B_2 \cap B_3 \cap G_4))P(B_1 \cap B_2 \cap B_3 \cap G_4) \\
&= {}^4C_3 P(B_5|(B_1 \cap B_2 \cap B_3 \cap G_4))P\ (G_4|(B_1 \cap B_2 \cap B_3)) \\
&\cdot \qquad\qquad \ldots\ldots P(B_3|(B_1 \cap B_2))P(B_2|B_1)P(B_1) \\
&= {}^4C_3 \frac{1}{6} \cdot \frac{6}{7} \cdot \frac{2}{8} \cdot \frac{3}{9} \cdot \frac{4}{10} = \frac{4}{210} = \frac{2}{105}
\end{aligned}$$

(b) Same idea as in (a)

$$\begin{aligned}
\text{Req'd probability} &= {}^9C_3 p\{B_{10}|B_1 \cap B_2 \ldots \cap G_4 \cap \ldots \cap G_9\} \cdot P\{B_1 \cap \ldots \cap G_9\} \\
&= {}^9C_3 \tfrac{1}{1} \cdot \tfrac{1}{2} \cdot \tfrac{2}{3} \cdot \tfrac{3}{4} \cdot \tfrac{4}{5} \cdot \tfrac{5}{6} \cdot \tfrac{6}{7} \cdot \tfrac{2}{8} \cdot \tfrac{3}{9} \cdot \tfrac{4}{10} = \tfrac{2}{5}
\end{aligned}$$

3. $H = \{\text{house is burgled}\}$ $C = \{\text{cottage is burgled}\}$

$P(H \cap C) = P(H)P(C) = (0.01)(0.05) = 0.0005$ since events independent

$$\begin{aligned} P(\text{one or the other (but not both)}) &= P((H \cap C') \cup (H' \cap C)) = P(H \cap C') + P(H' \cap C) \\ &= P(H)P(C') + P(H')P(C) \\ &= (0.01)(0.95) + (0.99)(0.05) = 0.059. \end{aligned}$$

4. Let A_i be event $\{A \text{ wins the } i^{th} \text{ game}\}$

req'd event is $\underbrace{\{A_1 \cap A_2 \cap A_3 \cap A_4' \cap A_5' \cap A_6')(\ldots)(\ldots)}_{\text{no. of ways of arranging 3 in 6 i.e. } {}^6C_3}$

$P(\text{req'd event}) = {}^6C_3 P(A_1 \cap A_2 \cap A_3 \cap A_4' \cap A_5' \cap A_6') = {}^6C_3 [P(A_1)]^3 P[A_1']^3 = \frac{160}{729}$

5. Let E_i be event $\{i^{th} \text{ engine success}\}$

2 engine plane: flight success if $\{(E_1 \cap E_2) \cup (E'_1 \cap E_2)(E_1 \cap E'_2)\}$ occurs

$$\begin{aligned} P(\text{req'd event}) &= P(E_1)P(E_2) + P(E_1')P(E_2) + P(E_1)P(E_2') \\ &= (1-q)^2 + 2q(1-q) = 1 - q^2 \end{aligned}$$

4 engine plane: success if following event occurs

$$\underbrace{\{E_1 \cap E_2 \cap E_3' \cap E_4'\}}_{{}^4C_2 \text{ ways}} \cap \underbrace{\{E_1 \cap E_2 \cap E_3 \cap E_4'\}}_{{}^4C_1 \text{ ways}} \cap \underbrace{\{E_1 \cap E_2 \cap E_3 \cap E_4\}}_{{}^4C_0 \text{ ways}}$$

req'd probability $= 6(1-q)^2q^2 + 4(1-q)^3q + (1-q)^4 = 3q^4 - 4q^3 + 1$

Two engine plane is preferred if

$$1 - q^2 > 3q^4 - 4q^3 + 1 \qquad \text{i.e. if} \quad 0 > q^2(3q-1)(q-1)$$

Let $y = (3q-1)(q-1)$. By drawing a graph of this quadratic you will quickly see that a two engine plane is preferred if $\frac{1}{3} < q < 1$

6. Let A be event $\{\text{relay } A \text{ is closed}\}$: Similarly for B, C

req'd event is $\{A \cap B \cap C\} \cup \underbrace{\{A' \cap B \cap C\}}_{{}^3C_1} \cap \underbrace{\{A' \cap B' \cap C\}}_{{}^3C_2}$

$P(\text{req'd event}) = (0.95)^3 + 3(0.95)^2(0.05) + 3(0.95)(0.05)^2 = 0.999875$

(or $1 - p(\text{all relays open}) = 1 - (0.05)^3 = 0.999875$.)