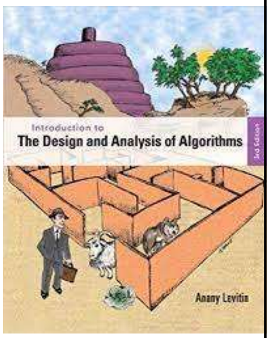
7-Space and Time Trade-Offs

A. Levitin "Introduction to the Design & Analysis of Algorithms," 3rd ed., Ch. 1 ©2012 Pearson Education, Inc. Upper Saddle River, NJ. All Rights Reserved



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Space-for-time tradeoffs

Two varieties of space-for-time algorithms:

- *input enhancement* — preprocess the input (or its part) to store some info to be used later in solving the problem
  - counting sorts (lab)
  - string searching algorithms
- *prestructuring* — preprocess the input to make accessing its elements easier
  - hashing (key, element) O(1)
  - indexing schemes (e.g., B-trees)

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Review: String searching by brute force

pattern: a string of *m* characters to search for
text: a (long) string of *n* characters to search in

D A T A
D A T A
D A T A
D A T A

Brute force algorithm

Step 1 Align pattern at beginning of text

Step 2 Moving from left to right, compare each character of pattern to the corresponding character in text until either all characters are found to match (successful search) or a mismatch is detected

Step 3 While a mismatch is detected and the text is not yet exhausted, realign pattern one position to the right and repeat Step 2

String searching by preprocessing

Several string searching algorithms are based on the input
enhancement idea of preprocessing the pattern

- Knuth-Morris-Pratt (KMP) algorithm preprocesses pattern left to right to get useful information for later searching
- Boyer -Moore algorithm preprocesses pattern right to left and store information into two tables
- Horspool's algorithm simplifies the Boyer-Moore algorithm by using just one table

Horspool's Algorithm

A simplified version of Boyer-Moore algorithm:

- preprocesses pattern to generate a shift table that determines how much to shift the pattern when a mismatch occurs
- always makes a shift based on the text's character *c* aligned with the last character in the pattern according to the shift table's entry for *c*

How far to shift?

Look at first (rightmost) character in text that was compared:

- The character is not in the pattern

```
      ......c......................... (c not in pattern)
match ⇒ BAOBAB BAOBAB (shift the pattern by its entire length)
        m-1
```

- The character is in the pattern (but not the rightmost)

```
.....O...................... (O occurs once in pattern)
BAOBAB
   BAOBAB       shift by 3.

.....A...................... (A occurs twice in pattern)
BAOBAB                    shift by 1
 BAOBAB
```

- The rightmost characters do match

```
..ABAB......................
BAOBAB          shift by 2
```

How far to shift?

Four possibilities may occur:

Case 1 If there are no c's in the pattern—e.g., c is letter S in our example—we can safely shift the pattern by its entire length (if we shift less, some character of the pattern would be aligned against the text's character c that is known not to be in the pattern):

```
s0  ...          S          ...  sn-1
                 ≠
        B A R B E R
                     B A R B E R
```

Case 2 If there are occurrences of character c in the pattern but it is not the last one there—e.g., c is letter B in our example—the shift should align the rightmost occurrence of c in the pattern with the c in the text:

```
s0  ...          B          ...  sn-1
                 ≠
        B A R B E R
            B A R B E R
```

Case 3 If c happens to be the last character in the pattern but there are no c's among its other $m-1$ characters—e.g., c is letter R in our example—the situation is similar to that of Case 1 and the pattern should be shifted by the entire pattern's length m:

```
s0  ...          M E R          ...  sn-1
                 ≠ ‖ ‖
        L E A D E R
                     L E A D E R
```

Case 4 Finally, if c happens to be the last character in the pattern and there are other c's among its first $m-1$ characters—e.g., c is letter R in our example—the situation is similar to that of Case 2 and the rightmost occurrence of c among the first $m-1$ characters in the pattern should be aligned with the text's c:

```
s0  ...          A R          ...  sn-1
                 ≠ ‖
    R E O R D E R
          R E O R D E R
```


Shift table

Shift sizes can be precomputed by the formula

$$t(c) = \begin{cases} \text{the pattern's length } m, \\ \text{if } c \text{ is not among the first } m-1 \text{ characters of the pattern;} \\ \\ \text{the distance from the rightmost } c \text{ among the first } m-1 \text{ characters} \\ \text{of the pattern to its last character, otherwise.} \end{cases} \qquad \textbf{(7.1)}$$

by scanning pattern before search begins and stored in a table called *shift table*

ALGORITHM *ShiftTable*$(P[0..m-1])$
//Fills the shift table used by Horspool's and Boyer-Moore algorithms
//Input: Pattern $P[0..m-1]$ and an alphabet of possible characters
//Output: $Table[0..size-1]$ indexed by the alphabet's characters and
//   filled with shift sizes computed by formula (7.1)
for $i \leftarrow 0$ **to** $size-1$ **do** $Table[i] \leftarrow m$
for $j \leftarrow 0$ **to** $m-2$ **do** $Table[P[j]] \leftarrow m-1-j$
return *Table*

Shift table

- Shift table is indexed by text and pattern alphabet
  Eg, for BAOBAB :

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	6	6	6	6	6	6	6	6	6	6	6	6	3	6	6	6	6	6	6	6	6	6	6	6

BAOBAB   BAOBAB

1    $m-1$

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Example of Horspool's alg. application

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	6	6	6	6	6	6	6	6	6	6	6	6	3	6	6	6	6	6	6	6	6	6	6	6

```
t: BARD LOVED BANANAS
p: BAOBAB          shift by 6
          BAOBAB   shift by 2
             BAOBAB   shift by 6
                      BAOBAB  (unsuccessful search)
```


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Pseudocode of Horspool's algorithm

ALGORITHM *HorspoolMatching*$(P[0..m-1], T[0..n-1])$
 //Implements Horspool's algorithm for string matching
 //Input: Pattern $P[0..m-1]$ and text $T[0..n-1]$
 //Output: The index of the left end of the first matching substring
 // or -1 if there are no matches
 ShiftTable$(P[0..m-1])$ //generate *Table* of shifts
 $i \leftarrow m-1$ //position of the pattern's right end
 while $i \leq n-1$ **do**
  $k \leftarrow 0$ //number of matched characters
  while $k \leq m-1$ **and** $P[m-1-k] = T[i-k]$ **do**
   $k \leftarrow k+1$
  if $k = m$
   return $i - m + 1$
  else $i \leftarrow i + Table[T[i]]$
 return -1

Example

BARBER

EXAMPLE As an example of a complete application of Horspool's algorithm, consider searching for the pattern BARBER in a text that comprises English letters and spaces (denoted by underscores). The shift table, as we mentioned, is filled as follows:

character c	A	B	C	D	E	F	...	R	...	Z	_
shift $t(c)$	4	2	6	6	1	6	6	3	6	6	6

The actual search in a particular text proceeds as follows:

```
J I M _ S A W _ M E _ I N _ A _ B A R B E R S H O P
B A R B E R                 B A R B E R
        B A R B E R             B A R B E R
          B A R B E R               B A R B E R
```

4 1 3

shift by 6

the worst case efficiency is $\theta(mn)$

Boyer-Moore algorithm

Based on same two ideas:

- comparing pattern characters to text from right to left
- precomputing shift sizes in two tables
  - *bad-symbol table* indicates how much to shift based on text's character causing a mismatch
  - *good-suffix table* indicates how much to shift based on matched part (suffix) of the pattern

Bad-symbol shift in Boyer-Moore algorithm

- If the rightmost character of the pattern doesn't match, BM algorithm acts as Horspool's
- If the rightmost character of the pattern does match, BM compares preceding characters right to left until either all pattern's characters match or a mismatch on text's character c is encountered after $k > 0$ matches

$$
\begin{array}{ccccccccccc}
s_0 & \dots & & c & s_{i-k+1} & \dots & s_i & \dots & s_{n-1} & \text{text} \\
 & & & \not\parallel & \parallel & & \parallel & & & \\
 & p_0 & \dots & p_{m-k-1} & p_{m-k} & \dots & p_{m-1} & & & \text{pattern}
\end{array}
$$

bad-symbol shift $d_1 = \max\{\underbrace{t_1(c)}_{\text{shift table.}} - k, 1\}$

$$
\begin{array}{ccccc}
C & A & B & A \\
\not= & = & = & = \\
B & A & B & A
\end{array}
\quad k=3
$$

(1) $t_1(c) = 6$ $6-3=3$

(2) $t_1(c) = 2$ $2-3=-1$

Bad-symbol shift in Boyer-Moore algorithm

bad-symbol shift $d_1 = \max\{t_1(c) - k, 1\}$

$t_1(c)$ is the entry in the precomputed table used by Horspool's algorithm and k is the number of matched characters

$$\begin{array}{ccccccccccl}
s_0 & \ldots & & c & s_{i-k+1} & \ldots & s_i & & \ldots & s_{n-1} & \text{text} \\
 & & & \not\parallel & \parallel & & \parallel & & & & \\
 & p_0 & \ldots & p_{m-k-1} & p_{m-k} & \ldots & p_{m-1} & & & & \text{pattern} \\
 & & & & p_0 & \ldots & & p_{m-1} & & &
\end{array}$$

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Good-suffix shift in Boyer-Moore algorithm

- Good-suffix shift d_2 is applied after $0 < k < m$ last characters were matched
- $d_2(k)$ = the distance between matched suffix of size k and its rightmost occurrence in the pattern that is not preceded by the same character as the suffix

1A

$d_2(1) = 4$

Example: CABABA $d_2(1) = 4$

k		d_2
1	CABABA	4

ABABA

It is preceded by the same character B.

I'm looking at the occurence of A where the preceeding character is different than B.

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k		d_2
1	CABABA	4
2	CABABA	6
3	CABABA	2
4	CABABA	6
5	CABABA	6

→ the preceeding char is the same (char. A). shift by the whole size of the pattern.

Good-suffix shift in Boyer-Moore algorithm

- If there is no such occurrence, match the longest part of the *k*-character suffix with corresponding prefix; if there are no such suffix-prefix matches, $d_2(k) = m$

  Example: `WOWWOW` $d_2(2) = 5$, $d_2(3) = 3$, $d_2(4) = 3$, $d_2(5) = 3$

k		d_2
1	WOWWOW	2
2	WOWWOW	5
3	WOWWOW	3
4	WOWWOW	3
5	WOWWOW	3

partial matching

k		d_2
1	BAOBAB	2
2	BAOBAB	5
3	BAOBAB	5
4	BAOBAB	5
5	BAOBAB	5

Good-suffix shift in the Boyer-Moore alg. (cont.)

After matching successfully $0 < k < m$ characters, the algorithm shifts the pattern right by

$$\boxed{d = \max\{d_1, d_2\}}$$

where $d_1 = \max\{t_1(c) - k, 1\}$ is bad-symbol shift

$d_2(k)$ is good-suffix shift

Boyer-Moore Algorithm (cont.)

Step 1 Fill in the bad-symbol shift table
Step 2 Fill in the good-suffix shift table
Step 3 Align the pattern against the beginning of the text
Step 4 Repeat until a matching substring is found or text ends:
Compare the corresponding characters right to left.
If no characters match, retrieve entry $t_1(c)$ from the bad-symbol table for the text's character c causing the mismatch and shift the pattern to the right by $t_1(c)$.
If $0 < k < m$ characters are matched, retrieve entry $t_1(c)$ from the bad-symbol table for the text's character c causing the mismatch and entry $d_2(k)$ from the good-suffix table and shift the pattern to the right by

$$\boxed{d = \max\{d_1, d_2\}}$$

where $d_1 = \max\{t_1(c) - k, 1\}$.

Example of Boyer-Moore alg. application

t(c) shift table. BAOBAB

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	_
1	2	6	6	6	6	6	6	6	6	6	6	6	6	3	6	6	6	6	6	6	6	6	6	6	6	6

$d_1 = \max\{t_1 - k, 1\}$

```
B E S S _ K N E W _ A B O U T _ B A O B A B S
B A O B A B
```

$\boxed{d_1 = t_1(K) = 6}$

k=2

```
            B A O B A B
```

$d_1 = t_1(\_) - 2 = 4$ $= 6-2$ } max{4,5} = 5 shift by 5

$d_2(2) = 5$

k=1

```
                        B A O B A B
```

$d_1 = t_1(\_) - 1 = 5$ max{5,2}=5

$d_2(1) = 2$

```
                                    B A O B A B (success)
```

k	pattern	d_2
1	BAOBAB	2
2	BAOBAB	5
3	BAOBAB	5
4	BAOBAB	5
5	BAOBAB	5

Boyer-Moore example from their paper

Find pattern `AT_THAT` in

text `WHICH_FINALLY_HALTS.__AT_THAT`

shift table

A	T	H	_	others
1	3	2	4	7

length

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Horspool's Alg.

```
WHICH_FINALLY_HALTS.__AT_THAT
AT_THAT  shift by 7 (F)
      AT_THAT  shift by 4 (_)
          AT_THAT  shift by 3 (T)
             AT_THAT  shift by 4 (_)
                 AT_THAT  shift by 4
                     AT_THAT.
```


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Boyer_Moore

```
WHICH_FINALLY_HALTS.__AT_THAT
```

shift table

A	T	H	_	others
1	3	2	4	7

length

→ Bad symbol $d_1 = \max\{t_1(c)-k, 1\}$

good-suffix.

k		d_2
1	AT_THAT	3
2	AT_THAT	5
3	AT_THAT	5
4	AT_THAT	5
5	AT_THAT	5
6	AT_THAT	5

Boyer_Moore

```
WHICH_FINALLY_HALTS.__AT_THAT
AT_THAT shift by 7
       AT_THAT shift by 4
           AT_THAT shift by 6
                 AT_THAT shift by 5
                      AT_THAT
```

$k=1$

$d_2(1)=3$

$d_1 = t_1(L) - k$

$d_1 = 7 - 1 = 6$

$\max\{d_1, d_2\} = 6$

$k=2$

$d_2(2)=5$

$d_1 = t_1(\_) - k$

$d_1 = 4 - 2 = 2$

$\max\{d_1, d_2\} = 5$

Boyer_Moore $= 7 + 7 = 14$ comparisons

Horspool's $= 6 + 7 = 13$ comparisons.

Hashing

- A very efficient method for implementing a *dictionary,* i.e., a set with the operations:
  - find
  - insert
  - delete
- Based on representation-change and space-for-time tradeoff ideas
- Important applications:
  - symbol tables
  - databases (*extendible hashing*)

Hash tables and hash functions

The idea of *hashing* is to map keys of a given file of size *n* into a table of size *m,* called the *hash table*, by using a predefined function, called the *hash function*,

h: $K \rightarrow$ location (cell) in the hash table

key

Example: student records, key = SSN. Hash function:
$\boxed{h(K) = K \bmod m}$ where *m* is some integer (typically, prime)
If *m* = 1000, where is record with SSN= 314159265 stored?

$h(k) = 265$

Generally, a hash function should:
- be easy to compute
- distribute keys about evenly throughout the hash table

Collisions

- If h(K1) = h(K2), there is a collision
- Good hash functions result in fewer collisions, but some collisions should be expected (birthday paradox)
- Two principal hashing schemes handle collisions differently:
  - Open hashing
    - each cell is a header of linked list of all keys hashed to it
  - Closed hashing
    - one key per cell
    - in case of collision, finds another cell by
      - linear probing: use next free bucket
      - double hashing: use second hash function to compute increment

Open hashing (Separate chaining)

A → 1
B → 2
C → 3
D → 4
E → 5
F → 6
N → 14

Keys are stored in linked lists outside a hash table whose elements serve as the lists' headers.

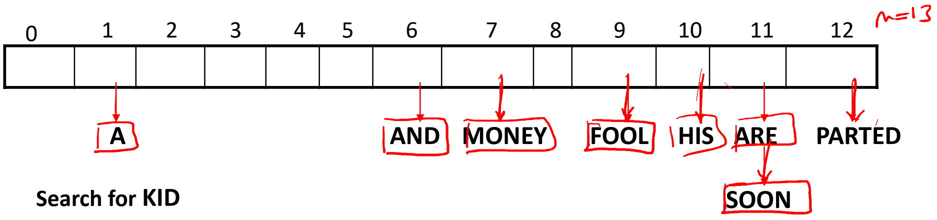
Example: A, FOOL, AND, HIS, MONEY, ARE, SOON, PARTED

h(K) = sum of K 's letters' positions in the alphabet MOD 13

1 mod 13   1 14 4

Key	A	FOOL	AND	HIS	MONEY	ARE	SOON	PARTED
h(*K*)	1	9	6	10	7	11	11	12

m=13



Search for KID

Open hashing (cont.)

- If the hash function distributes n keys among m cells of the hash table about evenly, each list will be about n/m keys long. This ratio ($\alpha = n/m$) is called load factor.
- Average number of probes in successful, S, and unsuccessful searches, U:

$$S \approx 1 + \alpha/2, \quad U = \alpha$$

- Load α is typically kept small (ideally, about 1)
- Open hashing still works if n > m

Closed hashing (Open addressing)

linear probing

- Keys are stored inside a hash table. (next available slot.)

keys	A	FOOL	AND	HIS	MONEY	ARE	SOON	PARTED
hash addresses	1	9	6	10	7	11	11	12

12

0	1	2	3	4	5	6	7	8	9	10	11	12
	A											
	A								FOOL			
	A					AND			FOOL			
	A					AND			FOOL	HIS		
	A					AND	MONEY		FOOL	HIS		
	A					AND	MONEY		FOOL	HIS	ARE	
	A					AND	MONEY		FOOL	HIS	ARE	SOON
PARTED	A					AND	MONEY		FOOL	HIS	ARE	SOON

Closed hashing (cont.)

- Does not work if n > m
- Avoids pointers
- Deletions are not straightforward
- Number of probes to find/insert/delete a key depends on load factor α = n/m (hash table density) and collision resolution strategy. For linear probing:
- S = (½) (1+ 1/(1- α)) and U = (½) (1+ 1/(1- α)²)
- As the table gets filled (α approaches 1), number of probes in linear probing increases dramatically:

α	$\frac{1}{2}(1+\frac{1}{1-\alpha})$	$\frac{1}{2}(1+\frac{1}{(1-\alpha)^2})$
50%	1.5	2.5
75%	2.5	8.5
90%	5.5	50.5

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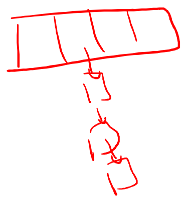
linear probing.

$h'(a_k, i) = (h(a_k) + i) \bmod m$



m : size

i : # of collisions.

wit Hashing : searching, insertion, deletion } Θ(?) Θ(1)

average

(n) unlikely worst case } → Θ(?) Θ(n)

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