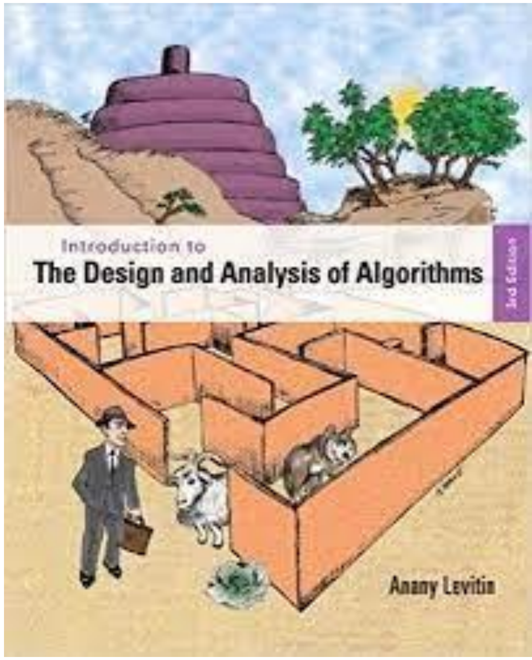
5-Divide-and-Conquer



Divide-and-Conquer

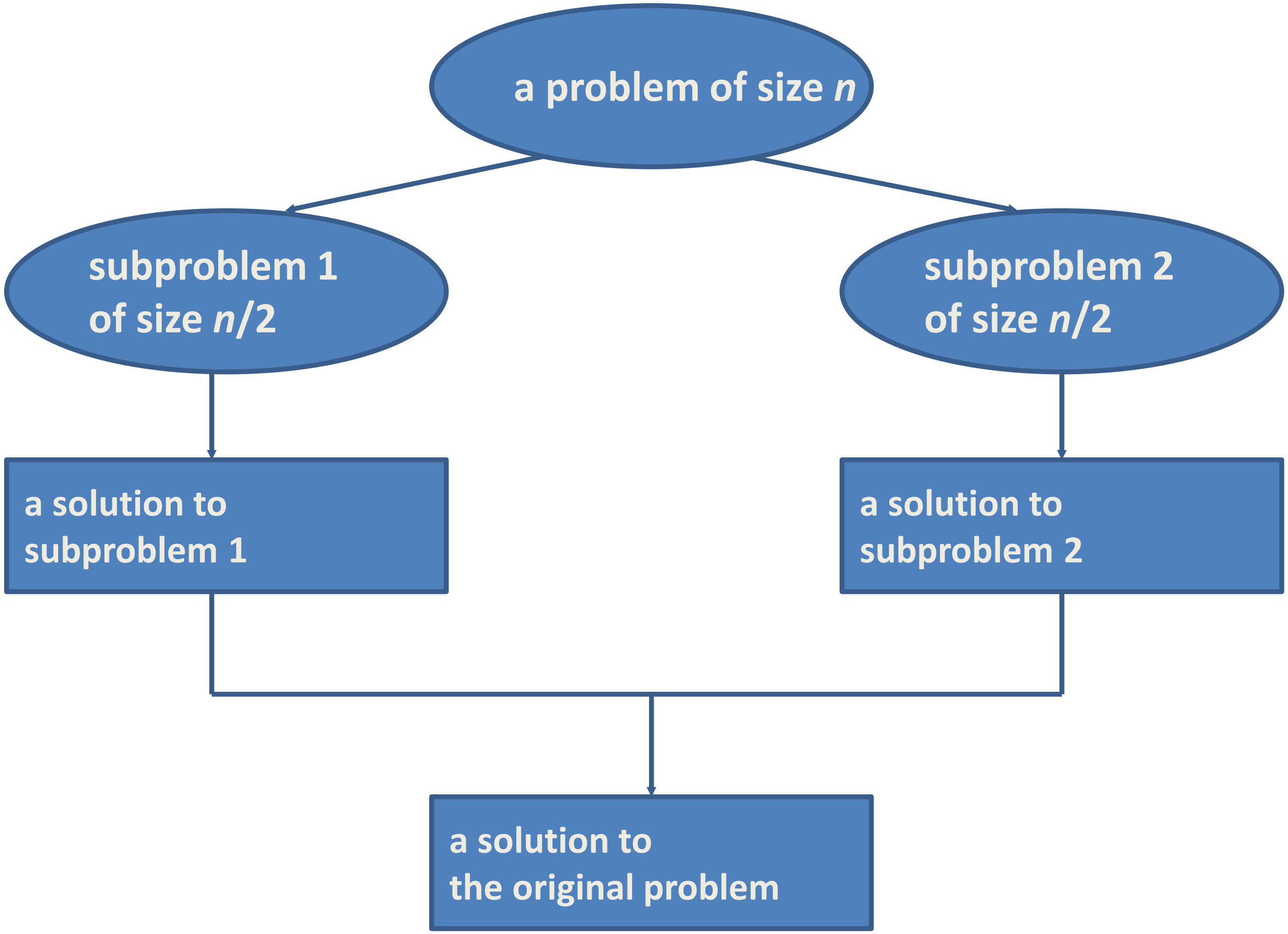
The most-well known algorithm design strategy:

1. Divide instance of problem into two or more smaller instances

2. Solve smaller instances recursively

3. Obtain solution to original (larger) instance by combining these solutions

Divide-and-Conquer Technique (cont.)



Divide-and-Conquer Examples

- Sorting: mergesort and quicksort
- Binary tree traversals
- Multiplication of large integers
- Matrix multiplication: Strassen's algorithm
- Closest-pair and convex-hull algorithms
- *Binary search: decrease-by-half (or degenerate divide&conq.)*

General Divide-and-Conquer Recurrence

$T(n) = aT(n/b) + f(n)$ where $f(n) \in \Theta(n^d)$, $d \geq 0$

Master Theorem: If $a < b^d$, $T(n) \in \Theta(n^d)$

If $a = b^d$, $T(n) \in \Theta(n^d logn)$

If $a > b^d$, $T(n) \in \Theta(n^{\log_b a})$

Note: The same results hold with O instead of Θ.

Examples: $T(n) = 4T\left(\frac{n}{2}\right) + n \rightarrow T(n) \in ?$

$T(n) = 4T\left(\frac{n}{2}\right) + n^2 \rightarrow T(n) \in ?$

$T(n) = 4T\left(\frac{n}{2}\right) + n^3 \rightarrow T(n) \in ?$

Mergesort

- Split array A[0..n-1] in two about equal halves and make copies of each half in arrays B and C
- Sort arrays B and C recursively
- Merge sorted arrays B and C into array A as follows:
  - Repeat the following until no elements remain in one of the arrays:
    - compare the first elements in the remaining unprocessed portions of the arrays
    - copy the smaller of the two into A, while incrementing the index indicating the unprocessed portion of that array
  - Once all elements in one of the arrays are processed, copy the remaining unprocessed elements from the other array into A.

Pseudocode of Mergesort

ALGORITHM *Mergesort*($A[0..n-1]$)

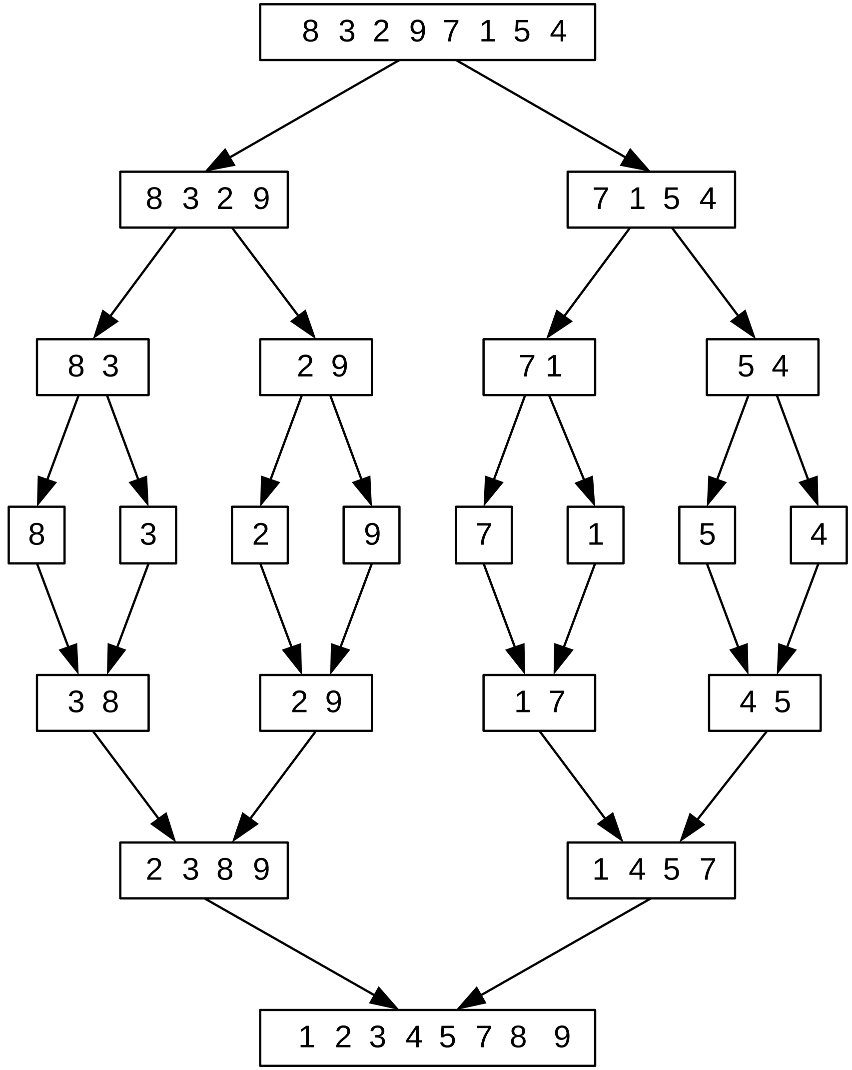
//Sorts array $A[0..n-1]$ by recursive mergesort
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $A[0..n-1]$ sorted in nondecreasing order
if $n > 1$
  copy $A[0..\lfloor n/2 \rfloor - 1]$ to $B[0..\lfloor n/2 \rfloor - 1]$
  copy $A[\lfloor n/2 \rfloor..n-1]$ to $C[0..\lceil n/2 \rceil - 1]$
  Mergesort($B[0..\lfloor n/2 \rfloor - 1]$)
  Mergesort($C[0..\lceil n/2 \rceil - 1]$)
  Merge(B, C, A) //see below

Pseudocode of Mergesort

ALGORITHM $Merge(B[0..p-1], C[0..q-1], A[0..p+q-1])$

//Merges two sorted arrays into one sorted array
//Input: Arrays $B[0..p-1]$ and $C[0..q-1]$ both sorted
//Output: Sorted array $A[0..p+q-1]$ of the elements of B and C
$i \leftarrow 0;\ \ j \leftarrow 0;\ \ k \leftarrow 0$
while $i < p$ **and** $j < q$ **do**
  if $B[i] \le C[j]$
    $A[k] \leftarrow B[i];\ \ i \leftarrow i + 1$
  else $A[k] \leftarrow C[j];\ \ j \leftarrow j + 1$
  $k \leftarrow k + 1$
if $i = p$
  copy $C[j..q-1]$ to $A[k..p+q-1]$
else copy $B[i..p-1]$ to $A[k..p+q-1]$

Mergesort Example



Analysis of Mergesort

- All cases have same efficiency: $\Theta(n \log n)$

- Number of comparisons in the worst case is close to theoretical minimum for comparison-based sorting:

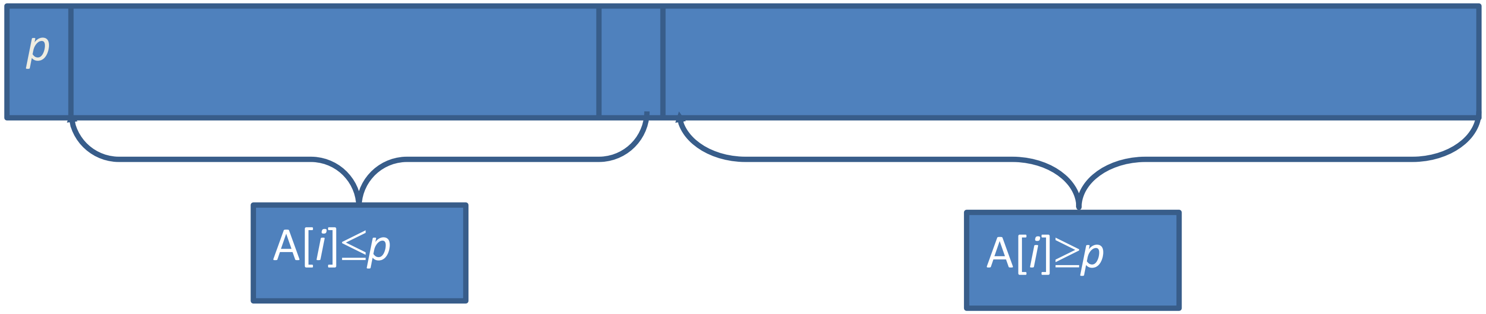
$$\lceil \log_2 n! \rceil \approx n \log_2 n - 1.44n$$

- Space requirement: $\Theta(n)$ (not in-place)

- Can be implemented without recursion (bottom-up)

Quicksort

- Select a pivot (partitioning element) – here, the first element
- Rearrange the list so that all the elements in the first s positions are smaller than or equal to the pivot and all the elements in the remaining n-s positions are larger than or equal to the pivot (see next slide for an algorithm)



- Exchange the pivot with the last element in the first (i.e., ≤) subarray — the pivot is now in its final position
- Sort the two subarrays recursively

Pseudocode of Quicksort

ALGORITHM *Quicksort*$(A[l..r])$

//Sorts a subarray by quicksort
//Input: Subarray of array $A[0..n-1]$, defined by its left and right
// indices l and r
//Output: Subarray $A[l..r]$ sorted in nondecreasing order
if $l < r$
  $s \leftarrow Partition(A[l..r])$ //s is a split position
  $Quicksort(A[l..s-1])$
  $Quicksort(A[s+1..r])$

Hoare's Partitioning Algorithm

ALGORITHM $HoarePartition(A[l..r])$

//Partitions a subarray by Hoare's algorithm, using the first element
//   as a pivot
//Input: Subarray of array $A[0..n-1]$, defined by its left and right
//   indices l and r $(l < r)$
//Output: Partition of $A[l..r]$, with the split position returned as
//   this function's value

$p \leftarrow A[l]$
$i \leftarrow l$; $j \leftarrow r + 1$
repeat
    repeat $i \leftarrow i + 1$ **until** $A[i] \geq p$
    repeat $j \leftarrow j - 1$ **until** $A[j] \leq p$
    swap($A[i]$, $A[j]$)
until $i \geq j$
swap($A[i]$, $A[j]$) //undo last swap when $i \geq j$
swap($A[l]$, $A[j]$)
return j

Quicksort Example

5 3 1 9 8 2 4 7

Analysis of Quicksort

- Best case: split in the middle — $\Theta(n \log n)$
- Worst case: sorted array! — $\Theta(n^2)$
- Average case: random arrays — $\Theta(n \log n)$

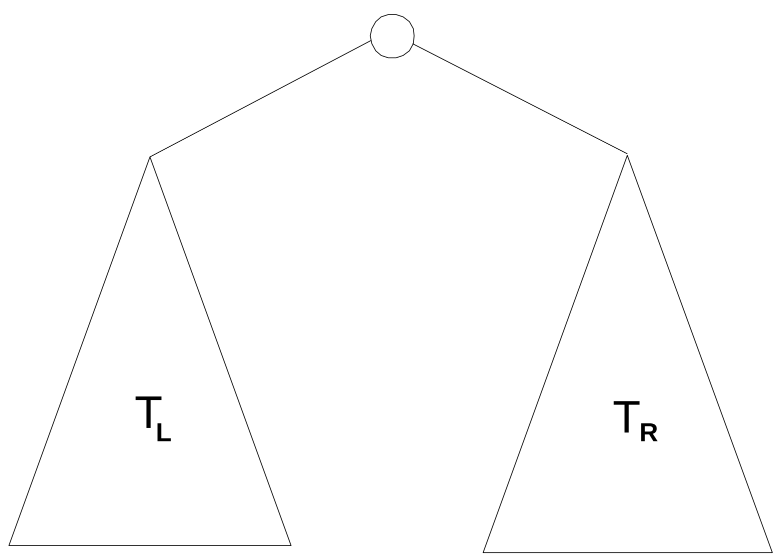
- Improvements:
  - better pivot selection: median of three partitioning
  - switch to insertion sort on small subfiles
  - elimination of recursion

  These combine to 20-25% improvement

- Considered the method of choice for internal sorting of large files ($n \geq 10000$)

Binary Tree Algorithms

Binary tree is a divide-and-conquer ready structure!



Binary Tree Algorithms

Ex. 1: Classic traversals (preorder, inorder, postorder)

- In the **preorder** traversal, the root is visited before the left and right subtrees are visited (in that order).
- In the **inorder** traversal, the root is visited after visiting its left subtree but before visiting the right subtree.
- In the **postorder** traversal, the root is visited after visiting the left and right subtrees (in that order).

- Efficiency: $\Theta(n)$

Binary Tree Algorithms (cont.)

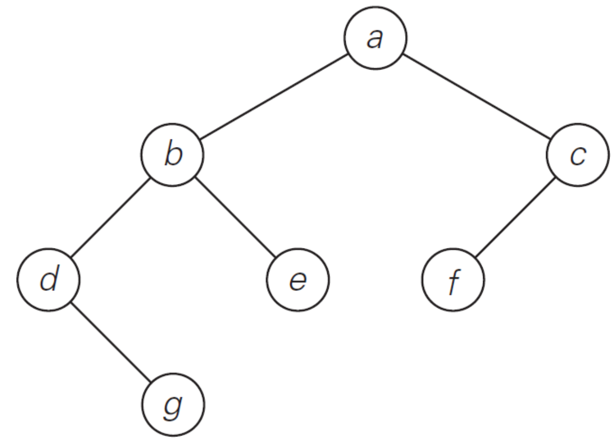
Algorithm *Inorder*(T)

if $T \neq \varnothing$

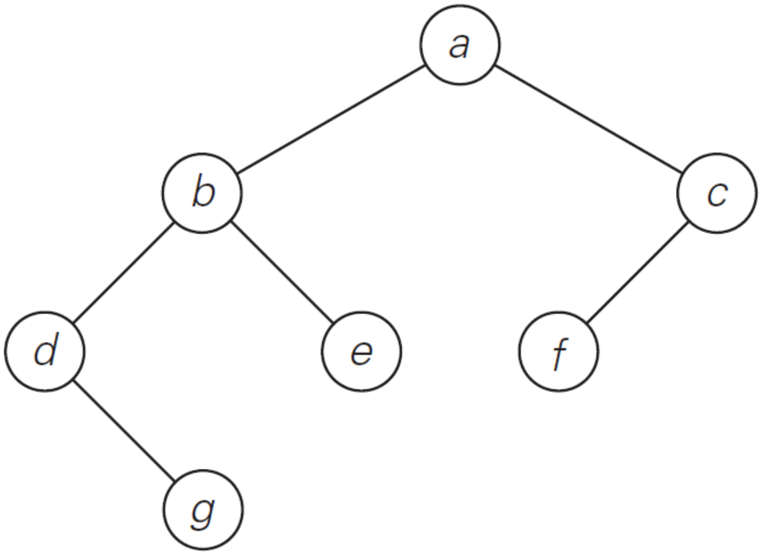
  Inorder(T_{left})

  print(root of T)

  Inorder(T_{right})



Binary Tree Algorithms (cont.)



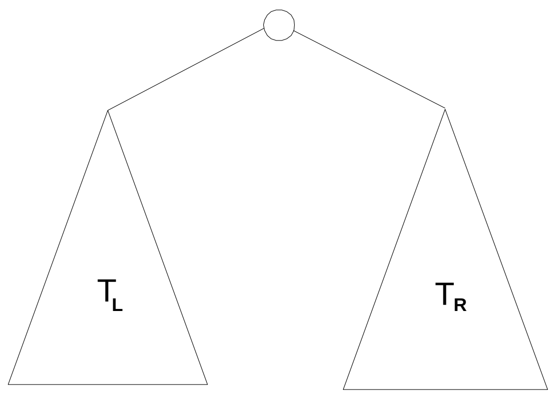
preorder (Root, Left, Right)

inorder (Left, Root, Right)

Postorder (Left, Right, Root)

Binary Tree Algorithms (cont.)

Ex. 2: Computing the height of a binary tree



$$h(T) = \max\{h(T_L), h(T_R)\} + 1 \ \ if\ T \neq \emptyset \ \ and \ \ h(\emptyset) = -1$$

Efficiency: $\Theta(n)$

Binary Tree Algorithms (cont.)

ALGORITHM $Height(T)$

//Computes recursively the height of a binary tree
//Input: A binary tree T
//Output: The height of T
if $T = \varnothing$ **return** -1
else return $\max\{Height(T_{left}), Height(T_{right})\} + 1$

Multiplication of Large Integers

Consider the problem of multiplying two (large) n-digit integers represented by arrays of their digits such as:

A = 12345678901357986429   B = 87654321284820912836

The grade-school algorithm:

$$
\begin{array}{r}
a_1\ a_2 \dots\ a_n \\
\underline{b_1\ b_2 \dots\ b_n} \\
(d_{10})\ d_{11} d_{12} \dots d_{1n} \\
(d_{20})\ d_{21} d_{22} \dots d_{2n} \qquad\quad \\
\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \qquad\quad \\
(d_{n0})\ d_{n1} d_{n2} \dots d_{nn} \qquad\qquad\quad \\
\hline
\end{array}
$$

Efficiency: n^2 one-digit multiplications

First Divide-and-Conquer Algorithm

A small example: $A * B$ where $A = 2135$ and $B = 4014$
$A = (21 \cdot 10^2 + 35)$, $B = (40 \cdot 10^2 + 14)$
So, $A * B = (21 \cdot 10^2 + 35) * (40 \cdot 10^2 + 14)$
$= 21 * 40 \cdot 10^4 + (21 * 14 + 35 * 40) \cdot 10^2 + 35 * 14$

In general, if $A = A_1A_2$ and $B = B_1B_2$ (where A and B are *n*-digit, A_1, A_2, B_1, B_2 are $n/2$-digit numbers),
$A * B = A_1 * B_1 \cdot 10^n + (A_1 * B_2 + A_2 * B_1) \cdot 10^{n/2} + A_2 * B_2$

Recurrence for the number of one-digit multiplications $M(n)$:
$$M(n) = 4M(n/2), \quad M(1) = 1$$
Solution: $M(n) = n^2$

Second Divide-and-Conquer Algorithm

$A * B = A_1 * B_1 \cdot 10^n + (A_1 * B_2 + A_2 * B_1) \cdot 10^{n/2} + A_2 * B_2$

The idea is to decrease the number of multiplications from 4 to 3:

$$(A_1 + A_2) * (B_1 + B_2) = A_1 * B_1 + (A_1 * B_2 + A_2 * B_1) + A_2 * B_2,$$

I.e., $(A_1 * B_2 + A_2 * B_1) = (A_1 + A_2) * (B_1 + B_2) - A_1 * B_1 - A_2 * B_2,$
which requires only 3 multiplications at the expense of (4-1) extra add/sub.

Recurrence for the number of multiplications M(n):

$$M(n) = 3M(n/2), \quad M(1) = 1$$

Solution: $M(n) = 3^{\log_2 n} = n^{\log_2 3} \approx n^{1.585}$

Example of Large-Integer Multiplication

$2135 * 4014$

Strassen's Matrix Multiplication

Strassen observed [1969] that the product of two matrices can be computed as follows:

$$\begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} * \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}$$

$$= \begin{pmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 + M_3 - M_2 + M_6 \end{pmatrix}$$

Formulas for Strassen's Algorithm

$M_1 = (A_{00} + A_{11}) * (B_{00} + B_{11})$

$M_2 = (A_{10} + A_{11}) * B_{00}$

$M_3 = A_{00} * (B_{01} - B_{11})$

$M_4 = A_{11} * (B_{10} - B_{00})$

$M_5 = (A_{00} + A_{01}) * B_{11}$

$M_6 = (A_{10} - A_{00}) * (B_{00} + B_{01})$

$M_7 = (A_{01} - A_{11}) * (B_{10} + B_{11})$

Analysis of Strassen's Algorithm

If n is not a power of 2, matrices can be padded with zeros.

Number of multiplications:

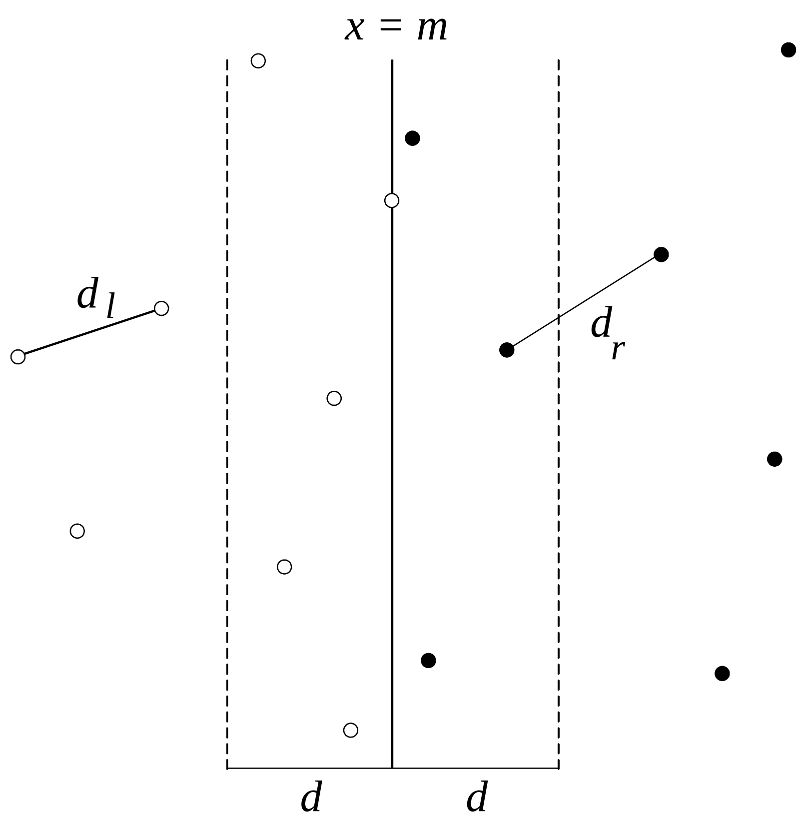
$$M(n) = 7M(n/2), \quad M(1) = 1$$

Solution: $M(n) = 7^{\log_2 n} = n^{\log_2 7} \approx n^{2.807}$ vs. n^3 of brute-force alg.

Algorithms with better asymptotic efficiency are known but they are even more complex.

Closest-Pair Problem by Divide-and-Conquer

Step 1 Divide the points given into two subsets P_l and P_r by a vertical line $x = m$ so that half the points lie to the left or on the line and half the points lie to the right or on the line.



Closest Pair by Divide-and-Conquer (cont.)

Step 2 Find recursively the closest pairs for the left and right subsets.

Step 3 Set $d = \min\{d_l, d_r\}$

We can limit our attention to the points in the symmetric vertical strip *S* of width 2*d* as possible closest pair. (The points are stored and processed in increasing order of their *y* coordinates.)

Step 4 Scan the points in the vertical strip *S* from the lowest up. For every point $p(x,y)$ in the strip, inspect points in in the strip that may be closer to *p* than *d*. There can be no more than 5 such points following *p* on the strip list!

Efficiency of the Closest-Pair Algorithm

Running time of the algorithm is described by

$$T(n) = 2T(n/2) + M(n), \text{ where } M(n) \in O(n)$$

By the Master Theorem (with $a = 2$, $b = 2$, $d = 1$)

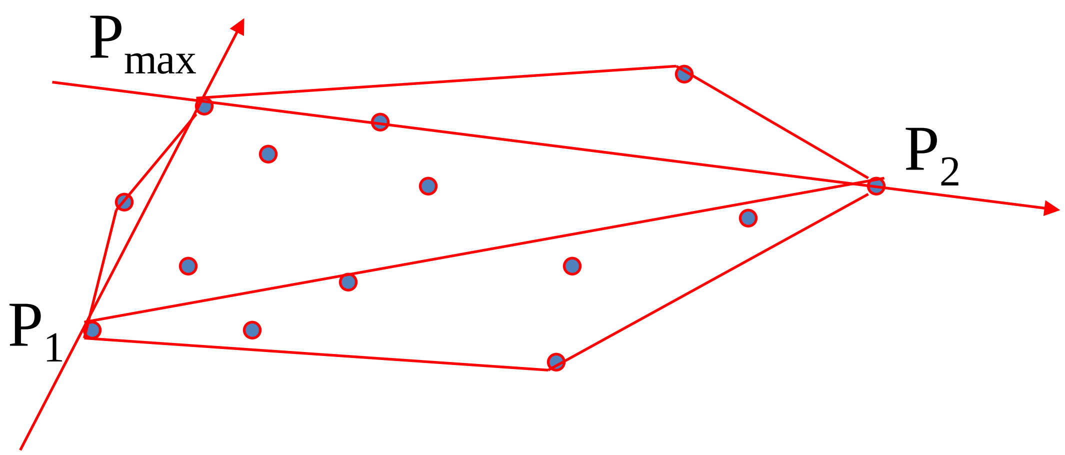
$$T(n) \in O(n \log n)$$

Quickhull Algorithm

Convex hull: smallest convex set that includes given points

- Assume points are sorted by *x*-coordinate values
- Identify *extreme points* P_1 and P_2 (leftmost and rightmost)
- Compute *upper hull* recursively:
  - find point P_{max} that is farthest away from line P_1P_2
  - compute the upper hull of the points to the left of line P_1P_{max}
  - compute the upper hull of the points to the left of line $P_{max}P_2$
- Compute *lower hull* in a similar manner

Quickhull Algorithm



Efficiency of Quickhull Algorithm

- Finding point farthest away from line P_1P_2 can be done in linear time
- Time efficiency:
  - worst case: $\Theta(n^2)$ (as quicksort)
  - average case: $\Theta(n)$ (under reasonable assumptions about distribution of points given)

- If points are not initially sorted by *x*-coordinate value, this can be accomplished in $O(n \log n)$ time

- Several $O(n \log n)$ algorithms for convex hull are known