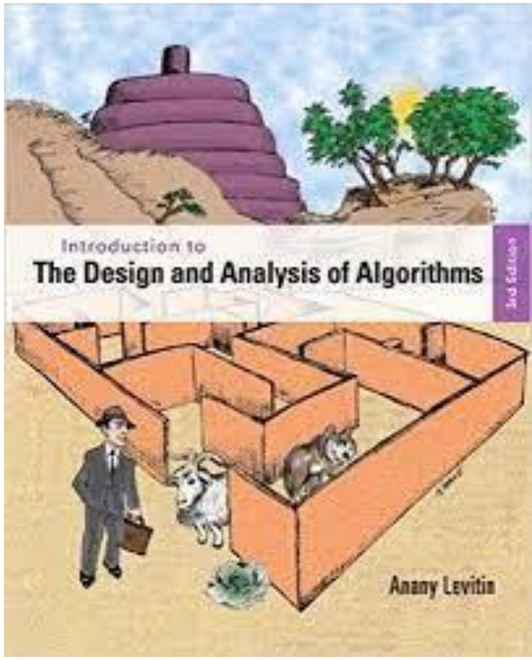
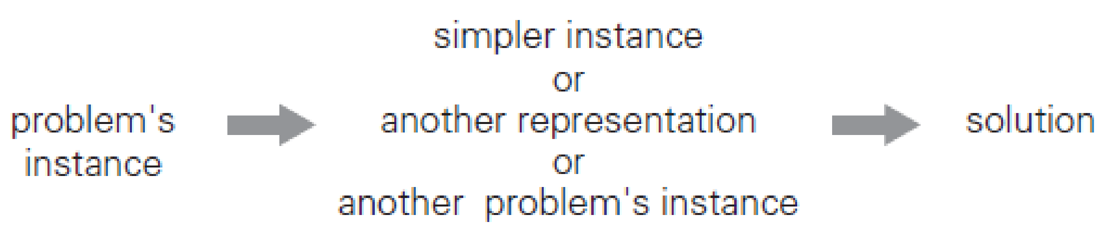
6-Transform and Conquer



Transform and Conquer

This group of techniques solves a problem by a *transformation* to

- a simpler/more convenient instance of the same problem (*instance simplification*)
- a different representation of the same instance (*representation change*)
- a different problem for which an algorithm is already available (*problem reduction*)



Instance simplification - Presorting

Solve a problem's instance by transforming it into another simpler/easier instance of the same problem

<u>Presorting</u>

Many problems involving lists are easier when list is sorted, e.g.

- searching
- computing the median (selection problem)
- checking if all elements are distinct (element uniqueness)

Also:

- Topological sorting helps solving some problems for dags.
- Presorting is used in many geometric algorithms.

How fast can we sort ?

Efficiency of algorithms involving sorting depends on efficiency of sorting.

Theorem (see Sec. 11.2): $\lceil \log_2 n! \rceil \approx n \log_2 n$ comparisons are necessary in the worst case to sort a list of size n by any comparison-based algorithm.

Note: About $n\log_2 n$ comparisons are also sufficient to sort array of size n (by mergesort).

Searching with presorting

Problem: Search for a given *K* in A[0..*n*-1]

Presorting-based algorithm:

Stage 1 Sort the array by an efficient sorting algorithm

Stage 2 Apply binary search

Efficiency: $\Theta(nlog\ n) + O(\log n) = \Theta(nlog\ n)$

Good or bad?

Why do we have our dictionaries, telephone directories, etc. sorted?

Element Uniqueness with presorting

Presorting-based algorithm

Stage 1: sort by efficient sorting algorithm (e.g. mergesort)

Stage 2: scan array to check pairs of <u>adjacent</u> elements

ALGORITHM *PresortElementUniqueness*($A[0..n-1]$)
    //Solves the element uniqueness problem by sorting the array first
    //Input: An array $A[0..n-1]$ of orderable elements
    //Output: Returns "true" if A has no equal elements, "false" otherwise
    sort the array A
    for $i \leftarrow 0$ **to** $n-2$ **do**
        if $A[i] = A[i+1]$ **return false**
    return true

Efficiency: $\Theta(n log\, n) + O(n) = \Theta(n log\, n)$

Element Uniqueness with presorting

- Brute force algorithm
  Compare all pairs of elements

  Efficiency: $O(n^2)$

- Another algorithm? Hashing

Instance simplification – Gaussian Elimination

Given: A system of *n* linear equations in *n* unknowns with an arbitrary coefficient matrix.

Transform to: An equivalent system of *n* linear equations in *n* unknowns with an upper triangular coefficient matrix.

Solve the latter by substitutions starting with the last equation and moving up to the first one.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

Gaussian Elimination (cont.)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n &= b'_1 \\
a'_{22}x_2 + \cdots + a'_{2n}x_n &= b'_2 \\
&\vdots \\
a'_{nn}x_n &= b'_n.
\end{aligned}
$$

In matrix notations, we can write this as

$$Ax = b \quad \Longrightarrow \quad A'x = b',$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \; A' = \begin{bmatrix} a'_{11} & a'_{12} & \ldots & a'_{1n} \\ 0 & a'_{22} & \ldots & a'_{2n} \\ \vdots & & & \\ 0 & 0 & \ldots & a'_{nn} \end{bmatrix}, \; b = \begin{bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{bmatrix}.$$

Gaussian Elimination (cont.)

The transformation is accomplished by a sequence of elementary operations on the system's coefficient matrix (which don't change the system's solution):

```
for i ←1 to n−1 do
   replace each of the subsequent rows
   (i.e., rows i+1, …, n) by a  difference
   between that row and an appropriate
   multiple of the i-th row to make the new
   coefficient in the i-th column of that
   row 0
```

Example of Gaussian Elimination

$$2x_1 - x_2 + x_3 = 1$$
$$4x_1 + x_2 - x_3 = 5$$
$$x_1 + x_2 + x_3 = 0.$$

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 4 & 1 & -1 & 5 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{array}{l} \\ \text{row } 2 - \frac{4}{2} \text{ row } 1 \\ \text{row } 3 - \frac{1}{2} \text{ row } 1 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{array}{l} \\ \\ \text{row } 3 - \frac{1}{2} \text{ row } 2 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

Now we can obtain the solution by back substitutions:

$$x_3 = (-2)/2 = -1, \quad x_2 = (3 - (-3)x_3)/3 = 0, \quad \text{and} \quad x_1 = (1 - x_3 - (-1)x_2)/2 = 1.$$

Pseudocode and Efficiency of Gaussian Elimination

Stage 1: Reduction to an upper-triangular matrix

for $i \leftarrow 1$ to n-1 do
    for $j \leftarrow i+1$ to n do
        for $k \leftarrow i$ to $n+1$ do
            $A[j, k] \leftarrow A[j, k] - A[i, k] * A[j, i] / A[i, i]$ //improve!

Stage 2: Back substitutions

for $j \leftarrow n$ downto 1 do
    $t \leftarrow 0$
    for $k \leftarrow j+1$ to n do
        $t \leftarrow t + A[j, k] * x[k]$
    $x[j] \leftarrow (A[j, n+1] - t) / A[j, j]$

Efficiency: $\Theta(n^3) + \Theta(n^2) = \Theta(n^3)$

Searching Problem

Problem: Given a (multi)set *S* of keys and a search key *K*, find an occurrence of *K* in *S*, if any

- Searching must be considered in the context of:
  - file size (internal vs. external)
  - dynamics of data (static vs. dynamic)

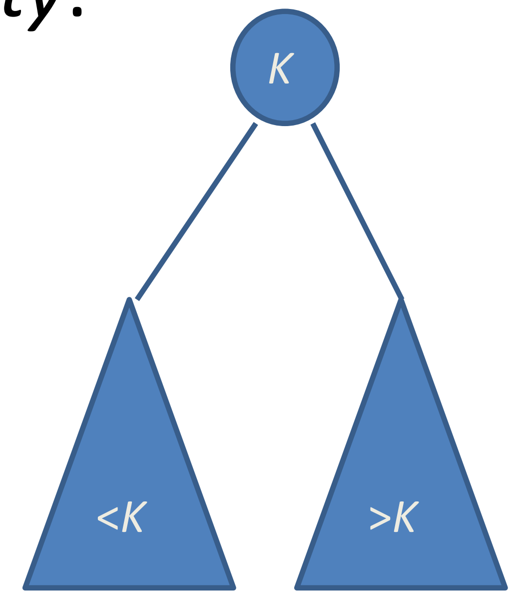
- Dictionary operations (dynamic data):
  - find (search)
  - insert
  - delete

Taxonomy of Searching Algorithms

- List searching
  - sequential search
  - binary search
  - interpolation search
- Tree searching
  - binary search tree
  - binary balanced trees: AVL trees, red-black trees
  - multiway balanced trees: 2-3 trees, 2-3-4 trees, B trees
- Hashing
  - open hashing (separate chaining)
  - closed hashing (open addressing)

Binary Search Tree

- Arrange keys in a binary tree with the *binary search tree property*:



Example: 5, 3, 1, 10, 12, 7, 9

Dictionary Operations on Binary Search Trees

- Searching – straightforward
- Insertion – search for key, insert at leaf where search terminated
- Deletion – 3 cases:
  - deleting key at a leaf
  - deleting key at node with single child
  - deleting key at node with two children

Dictionary Operations on Binary Search Trees

Efficiency depends on the tree's height:

$$\lfloor \log_2 n \rfloor \leq h \leq n - 1$$

with height average (random files) be about $3\log_2 n$

Thus, all three operations have

- worst case efficiency: $\Theta(n)$
- average case efficiency: $\Theta(\log n)$

Bonus: inorder traversal produces sorted list

Balanced Search Trees

Attractiveness of *binary search tree* is marred by the bad (linear) worst-case efficiency. Two ideas to overcome it are:

- to rebalance binary search tree when a new insertion makes the tree "too unbalanced"
  - *AVL trees*
  - *red-black trees*

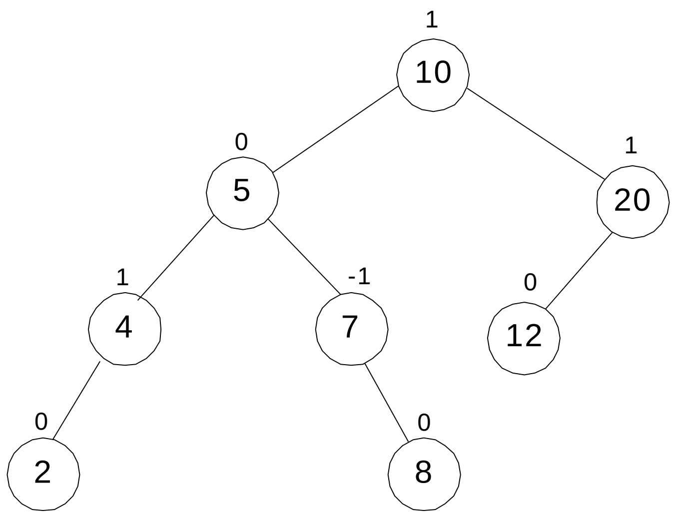
- to allow more than one key per node of a search tree
  - *2-3 trees*
  - *2-3-4 trees*
  - *B-trees*

Balanced trees: AVL trees

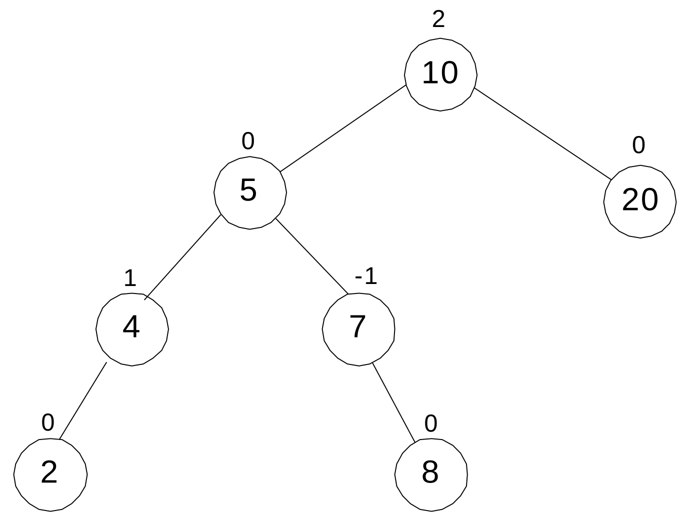
Definition

- An AVL tree is a binary search tree in which the balance factor of every node, which is defined as the difference between the heights of the node’s left and right subtrees, is either 0 or +1 or −1.
- The height of the empty tree is defined as −1. Of course, the balance factor can also be computed as the difference between the numbers of levels rather than the height difference of the node’s left and right subtrees.

Balanced trees: AVL trees



(a)



(b)

Tree (a) is an AVL tree;

tree (b) is not an AVL tree

Rotations

If a key insertion violates the balance requirement at some node, the subtree rooted at that node is transformed via one of the four *rotations.* (The rotation is always performed for a subtree rooted at an "unbalanced" node closest to the new leaf.)

- *single right rotation (R-rotation)*
- *single left rotation (L-rotation)*
- *double left-right rotation (LR-rotation)*
- *double right-left rotation (RL-rotation)*

Rotations

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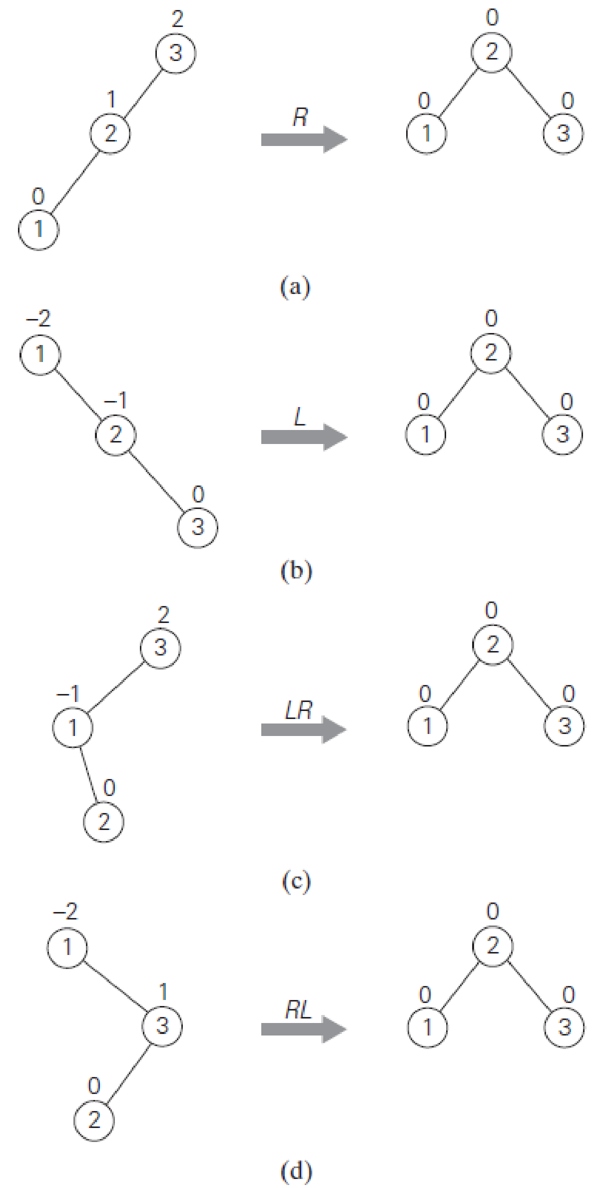
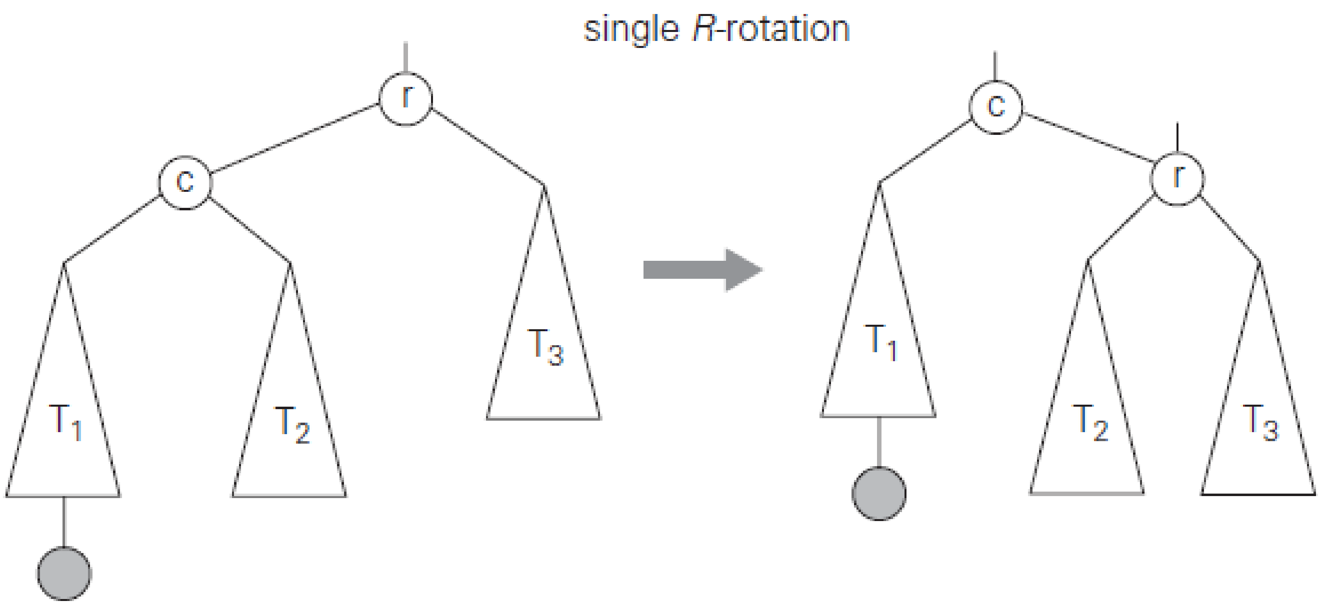
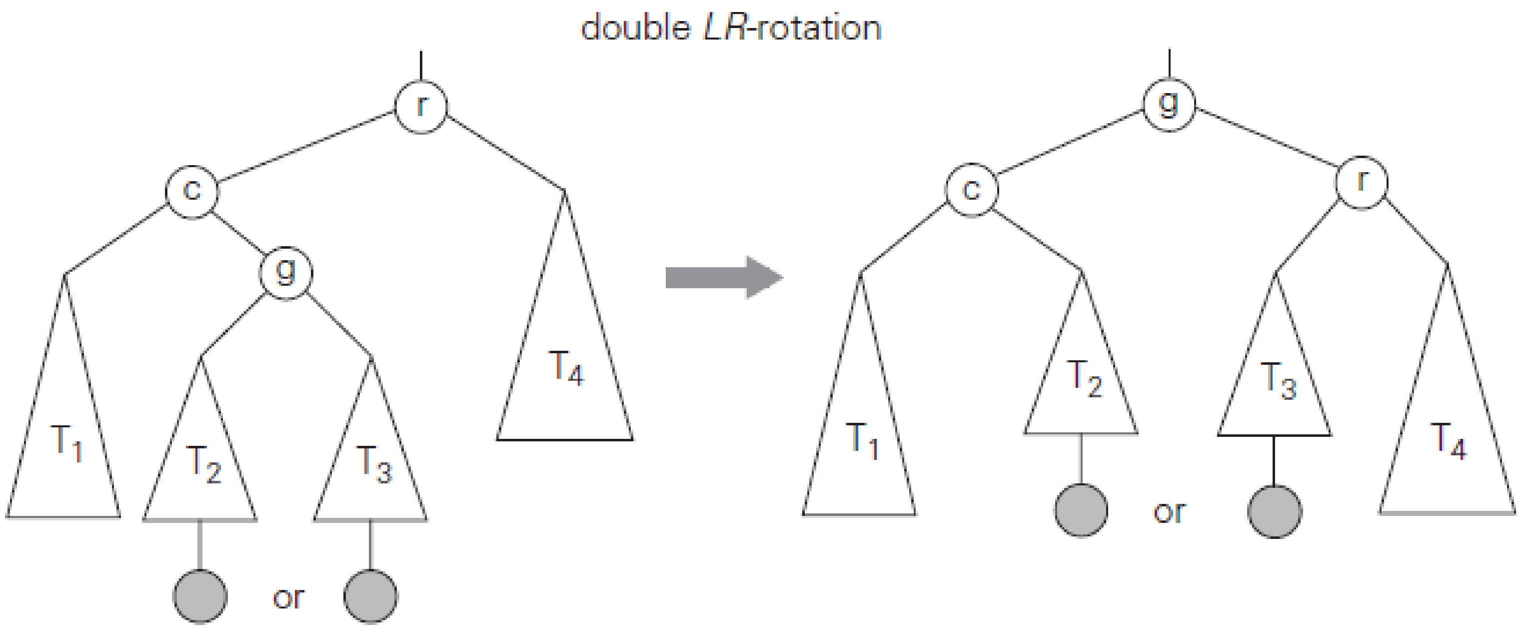


Four rotation types for AVL trees with three nodes. (a) Single *R*-rotation. (b) Single *L*-rotation. (c) Double *LR*-rotation. (d) Double *RL*-rotation.

General case: Single R-rotation

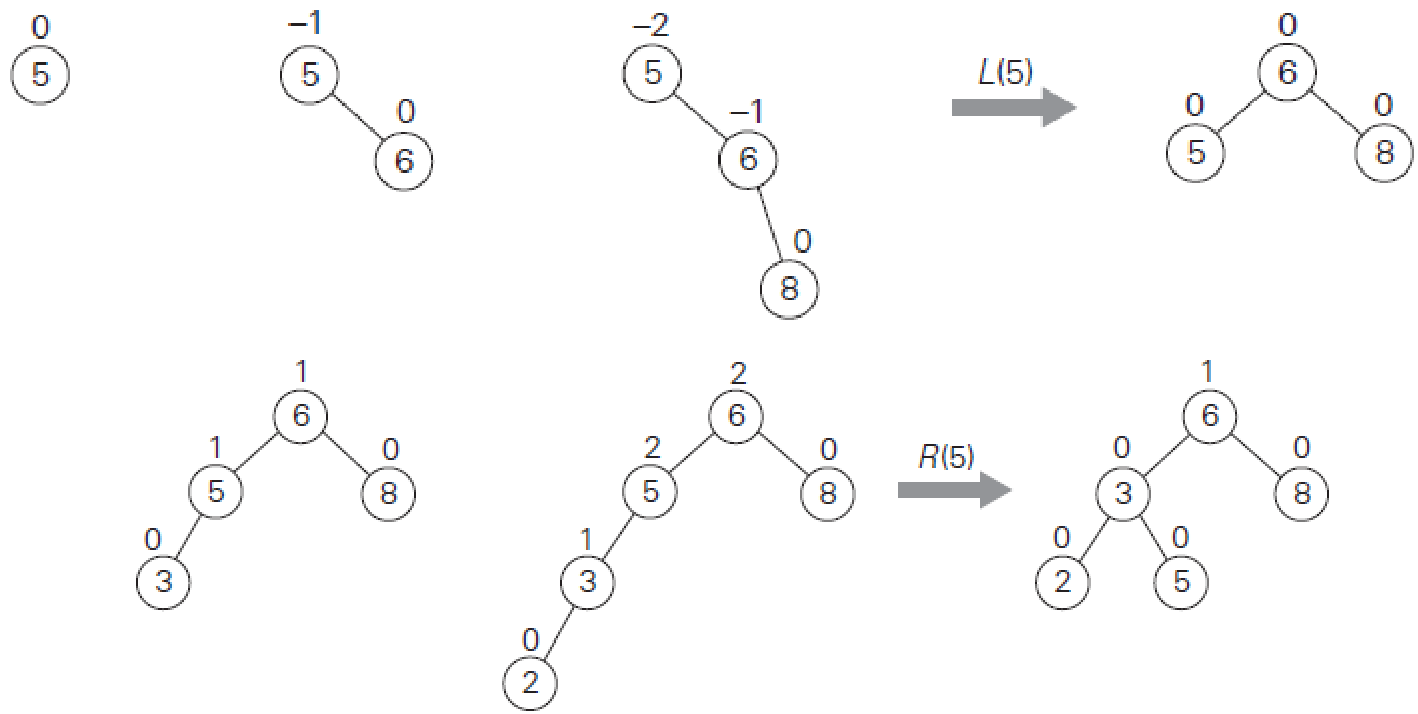


General case: Double LR-rotation



AVL tree construction - an example

Construction of an AVL tree for the list 5, 6, 8, 3, 2, 4, 7



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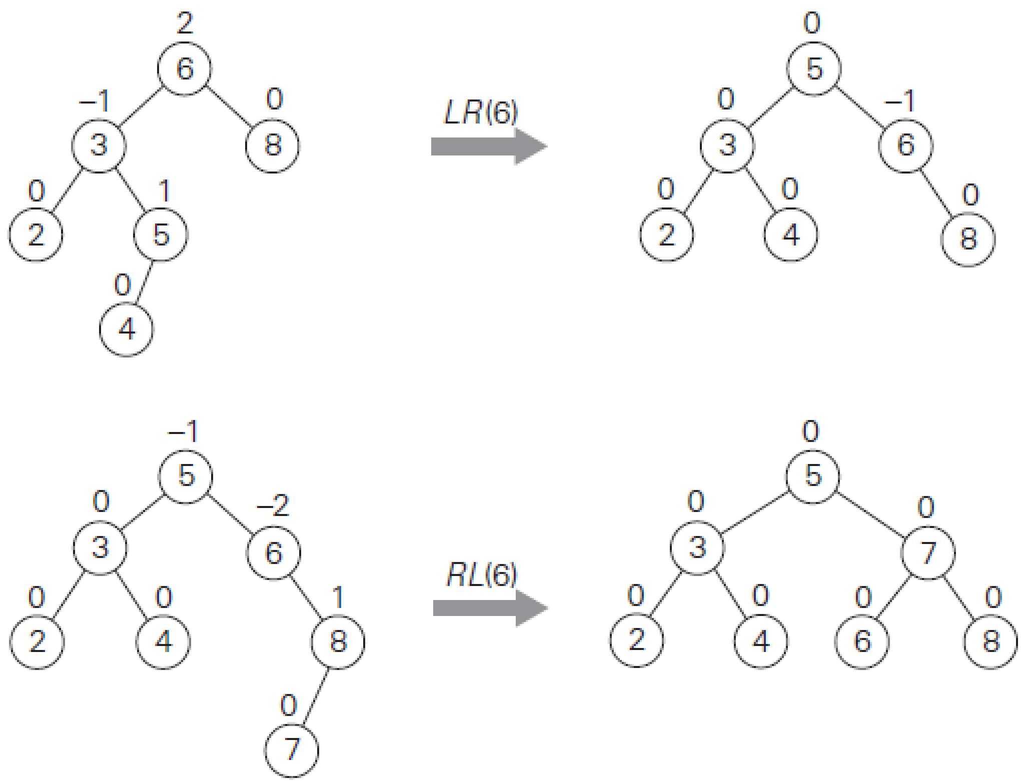


Analysis of AVL trees

- $h \le 1.4404 \log_2 (n + 2) - 1.3277$
  average height: $1.01 \log_2 n + 0.1$ for large n (found empirically)

- Search and insertion are $O(\log n)$

- Deletion is more complicated but is also $O(\log n)$

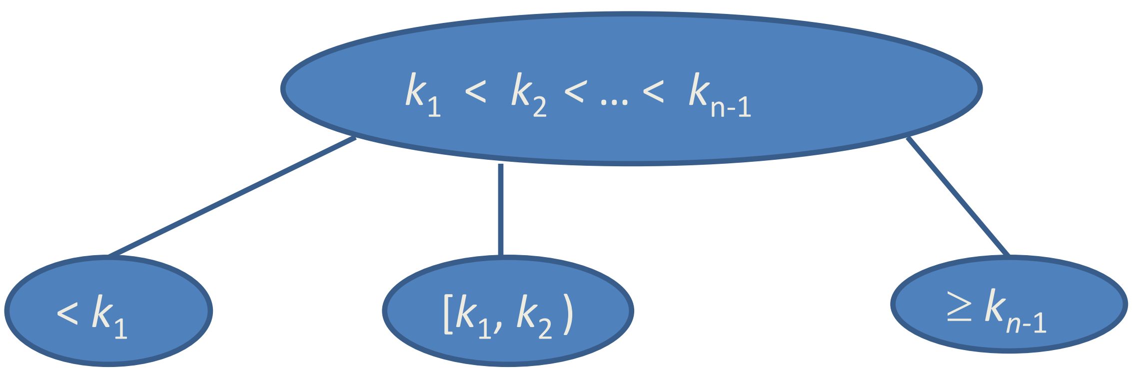
- Disadvantages:
  - frequent rotations
  - complexity

- A similar idea: red-black trees (height of subtrees is allowed to differ by up to a factor of 2)

Multiway Search Trees

Definition A multiway search tree is a search tree that allows more than one key in the same node of the tree.

Definition A node of a search tree is called an n-node if it contains n-1 ordered keys (which divide the entire key range into n intervals pointed to by the node's n links to its children):

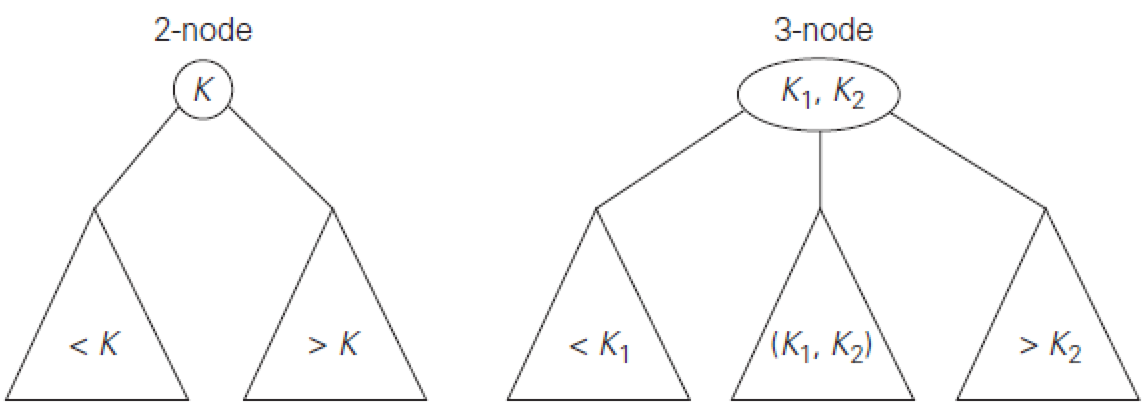


Note: Every node in a classical binary search tree is a 2-node

2-3 Tree

<u>Definition</u> A *2-3 tree* is a search tree that

- may have 2-nodes and 3-nodes
- height-balanced (all leaves are on the same level)



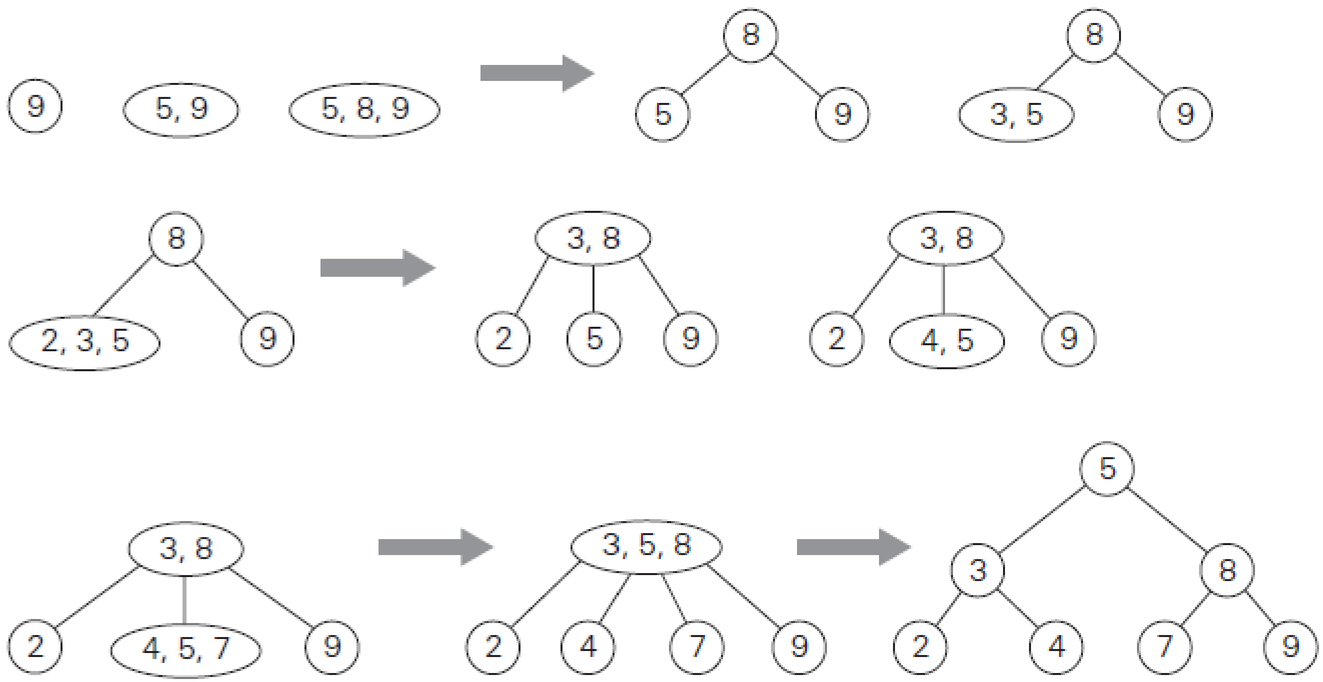
2-3 Tree

- A 2-3 tree is constructed by successive insertions of keys given, with a new key always inserted into a leaf of the tree.

- If the leaf is a 3-node, it's split into two with the middle key promoted to the parent.

2-3 tree construction – an example

Construct a 2-3 tree the list 9, 5, 8, 3, 2, 4, 7



Analysis of 2-3 trees

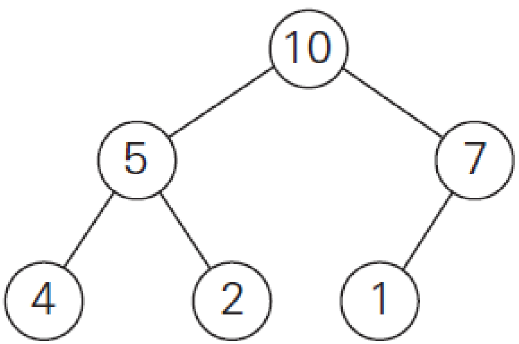
- $\log_3(n + 1) - 1 \leq h \leq \log_2(n + 1) - 1$
- Search, insertion, and deletion are in $\Theta(\log n)$
- The idea of 2-3 tree can be generalized by allowing more keys per node
  - 2-3-4 trees
  - B-trees

Heaps and Heapsort

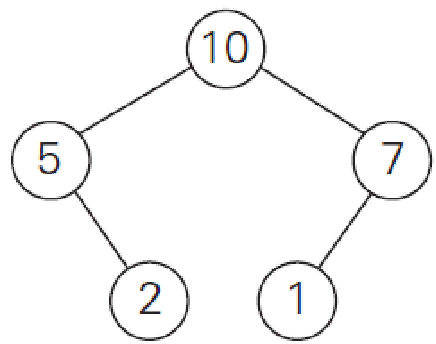
Definition A heap can be defined as a binary tree with keys assigned to its nodes, one key per node, provided the following two conditions are met:

1. The shape property—the binary tree is essentially complete (or simply complete), i.e., all its levels are full except possibly the last level, where only some rightmost leaves may be missing.
2. The parental dominance or heap property—the key in each node is greater than or equal to the keys in its children

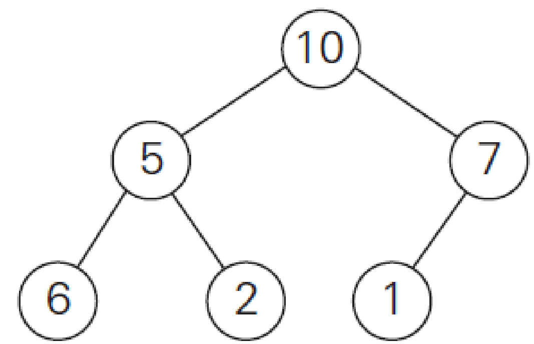
Heaps



a heap



not a heap



not a heap

Note: Heap's elements are ordered top down (along any path down from its root), but they are not ordered left to right

Some Important Properties of a Heap

- Given *n*, there exists a unique binary tree with *n* nodes that is essentially complete, with $h = \lfloor \log_2 n \rfloor$

- The root contains the largest key

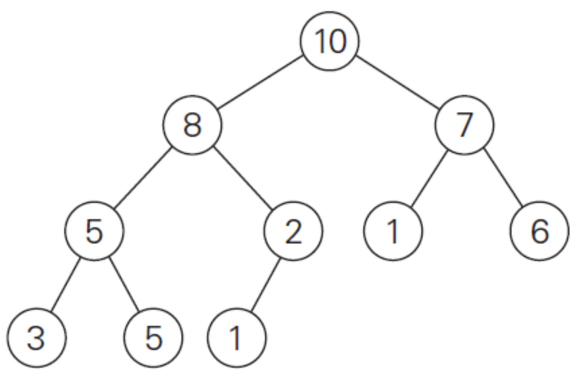
- The subtree rooted at any node of a heap is also a heap

- A heap can be represented as an array

Heap’s Array Representation

Store heap’s elements in an array (whose elements indexed, for convenience, 1 to n) in top-down left-to-right order

Example:



the array representation

index	0	1	2	3	4	5	6	7	8	9	10
value		10	8	7	5	2	1	6	3	5	1

parents | leaves

- Left child of node j is at $2j$
- Right child of node j is at $2j+1$
- Parent of node j is at $\lfloor j/2 \rfloor$
- Parental nodes are represented in the first $\lfloor n/2 \rfloor$ locations

Heap Construction (bottom-up)

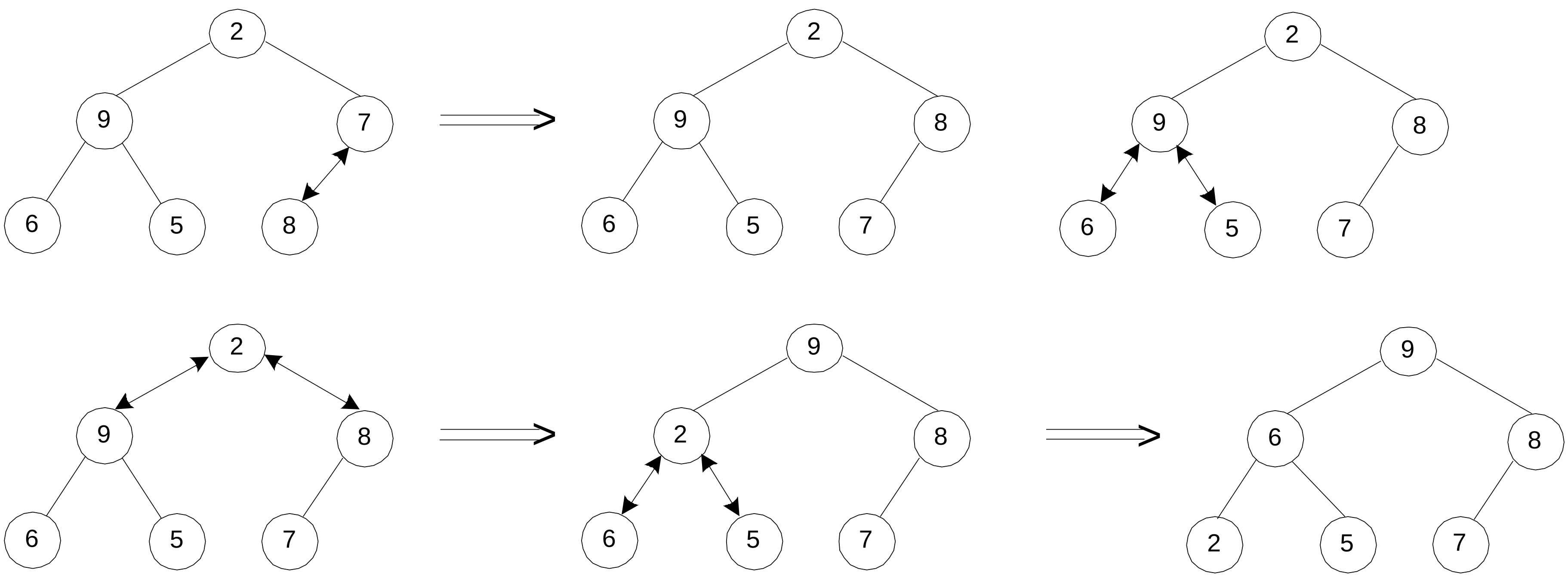
Step 0: Initialize the structure with keys in the order given

Step 1: Starting with the last (rightmost) parental node, fix the heap rooted at it, if it doesn't satisfy the heap condition: keep exchanging it with its largest child until the heap condition holds

Step 2: Repeat Step 1 for the preceding parental node

Example of Heap Construction

Construct a heap for the list 2, 9, 7, 6, 5, 8



Pseudopodia of bottom-up heap construction

ALGORITHM $HeapBottomUp(H[1..n])$

//Constructs a heap from elements of a given array
// by the bottom-up algorithm
//Input: An array $H[1..n]$ of orderable items
//Output: A heap $H[1..n]$
for $i \leftarrow \lfloor n/2 \rfloor$ **downto** 1 **do**
    $k \leftarrow i; \quad v \leftarrow H[k]$
    $heap \leftarrow$ **false**
    while not $heap$ **and** $2 * k \le n$ **do**
        $j \leftarrow 2 * k$
        if $j < n$ //there are two children
            if $H[j] < H[j+1]$ $j \leftarrow j + 1$
        if $v \ge H[j]$
            $heap \leftarrow$ **true**
        else $H[k] \leftarrow H[j]; \quad k \leftarrow j$
    $H[k] \leftarrow v$

Heapsort

Stage 1: Construct a heap for a given list of *n* keys

Stage 2: Repeat operation of root removal *n*-1 times:

- Exchange keys in the root and in the last (rightmost) leaf
- Decrease heap size by 1
- If necessary, swap new root with larger child until the heap condition holds

Example of Sorting by Heapsort

Sort the list 2, 9, 7, 6, 5, 8 by heapsort

Stage 1 (heap construction)

1 9 $\underline{7}$ 6 5 8

2 $\underline{9}$ 8 6 5 7

$\underline{2}$ 9 8 6 5 7

9 $\underline{2}$ 8 6 5 7

9 6 8 2 5 7

Stage 2 (root/max removal)

$\underline{9}$ 6 8 2 5 7

7 6 8 2 5 | 9

$\underline{8}$ 6 7 2 5 | 9

5 6 7 2 | 8 9

$\underline{7}$ 6 5 2 | 8 9

2 6 5 | 7 8 9

$\underline{6}$ 2 5 | 7 8 9

5 2 | 6 7 8 9

$\underline{5}$ 2 | 6 7 8 9

2 | 5 6 7 8 9

Analysis of Heapsort

Stage 1: Build heap for a given list of *n* keys
worst-case

$$C(n) \; = \sum_{i=0}^{h-1} 2(h-i)2^i = 2(n - \log_2(n+1)) \in \Theta(n)$$

nodes at
level *i*

Stage 2: Repeat operation of root removal *n*-1 times (fix heap)
worst-case

$$C(n) \; = \sum_{i=1}^{n-1} 2\log_2 i \in \Theta(nlogn)$$

Both worst-case and average-case efficiency: $\Theta(nlogn)$
In-place: yes
Stability: no

Priority Queue

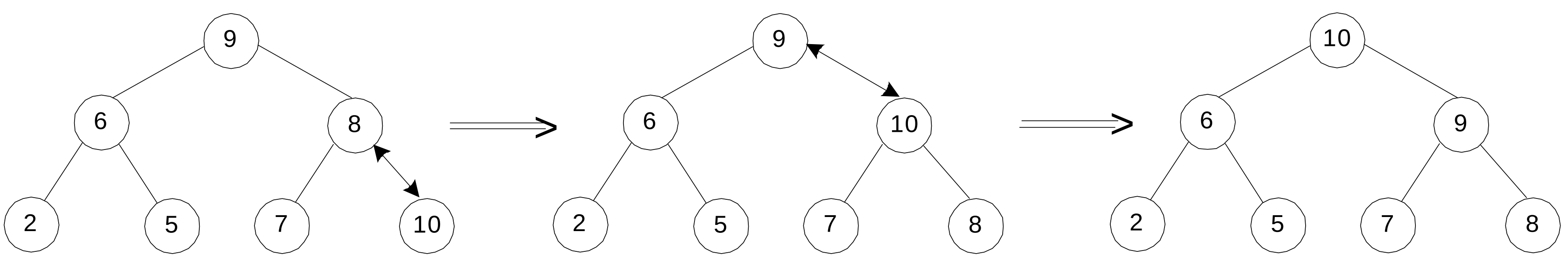
A *priority queue* is the ADT of a set of elements with numerical priorities with the following operations:

- find element with highest priority
- delete element with highest priority
- insert element with assigned priority (see below)

- Heap is a very efficient way for implementing priority queues
- Two ways to handle priority queue in which highest priority = smallest number

Insertion of a New Element into a Heap

- Insert the new element at last position in heap.
- Compare it with its parent and, if it violates heap condition, exchange them
- Continue comparing the new element with nodes up the tree until the heap condition is satisfied
- Example: Insert key 10



Efficiency: $O(\log n)$

Horner's Rule For Polynomial Evaluation

Given a polynomial of degree n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

and a specific value of x, find the value of p at that point.

Two brute-force algorithms:

$p \leftarrow 0$
for $i \leftarrow n$ downto 0 do
    $power \leftarrow 1$
    for $j \leftarrow 1$ to i do
        $power \leftarrow power * x$
    $p \leftarrow p + a_i * power$
return p

$p \leftarrow a_0$; $power \leftarrow 1$
for $i \leftarrow 1$ to n do
    $power \leftarrow power * x$
    $p \leftarrow p + a_i * power$
return p

Horner's Rule

$$\begin{aligned} p(x) &= 2x^4 - x^3 + 3x^2 + x - 5 \\ &= x(2x^3 - x^2 + 3x + 1) - 5 \\ &= x(x(2x^2 - x + 3) + 1) - 5 \\ &= x(x(x(2x - 1) + 3) + 1) - 5. \end{aligned}$$

Substitution into the last formula leads to a faster algorithm

Same sequence of computations are obtained by simply arranging the coefficient in a table and proceeding as follows:

coefficients	2	-1	3	1	-5
x=3					

Horner's Rule pseudocode

ALGORITHM $Horner(P[0..n], x)$

//Evaluates a polynomial at a given point by Horner's rule
//Input: An array $P[0..n]$ of coefficients of a polynomial of degree n,
//   stored from the lowest to the highest and a number x
//Output: The value of the polynomial at x
$p \leftarrow P[n]$
for $i \leftarrow n - 1$ **downto** 0 **do**
    $p \leftarrow x * p + P[i]$
return p

Efficiency of Horner's Rule: # multiplications = # additions = *n*

Synthetic Division

- S*ynthetic division* of of $p(x)$ by $(x\text{-}x_0)$
  Example: Let $p(x) = 2x^4 - x^3 + 3x^2 + x - 5$. Find $p(x){:}(x\text{-}3)$

Binary Exponentiation

<u>Left-to-right binary exponentiation</u>
Initialize product accumulator by 1.

Scan *n*'s binary expansion from left to right and do the following:

If the current binary digit is 0, square the accumulator (S);
if the binary digit is 1, square the accumulator and multiply it by *a* (SM).

ALGORITHM *LeftRightBinaryExponentiation*$(a, b(n))$
//Computes a^n by the left-to-right binary exponentiation algorithm
//Input: A number a and a list $b(n)$ of binary digits $b_I, \ldots, b_0$
// in the binary expansion of a positive integer n
//Output: The value of a^n
$product \leftarrow a$
for $i \leftarrow I - 1$ **downto** 0 **do**
  $product \leftarrow product * product$
  if $b_i = 1$ $product \leftarrow product * a$
return $product$

Binary Exponentiation

Example: Compute a^{13}. Here, $n = 13 = 1101_2$

binary digits of n	1	1	0	1
product accumulator	a	$a^2 \cdot a = a^3$	$(a^3)^2 = a^6$	$(a^6)^2 \cdot a = a^{13}$

Efficiency: $b \leq M(n) \leq 2b$ where $b = \lfloor \log_2 n \rfloor + 1$

Binary Exponentiation

Right-to-left binary exponentiation

Scan *n*'s binary expansion from right to left and compute a^n as the product of terms a^{2^i} corresponding to 1's in this expansion.

ALGORITHM *RightLeftBinaryExponentiation*$(a, b(n))$

//Computes a^n by the right-to-left binary exponentiation algorithm
//Input: A number a and a list $b(n)$ of binary digits $b_I, \ldots, b_0$
//   in the binary expansion of a nonnegative integer n
//Output: The value of a^n

$term \leftarrow a$ //initializes a^{2^i}
if $b_0 = 1$ $product \leftarrow a$
else $product \leftarrow 1$
for $i \leftarrow 1$ **to** I **do**
    $term \leftarrow term * term$
    if $b_i = 1$ $product \leftarrow product * term$
return $product$

Binary Exponentiation

Example: Compute a^{13}. Here, $n = 13 = 1101_2$

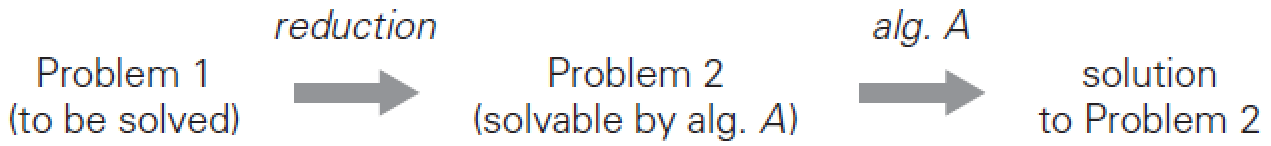
1	1	0	1	binary digits of n
a^8	a^4	a^2	a	terms a^{2^i}
$a^5 \cdot a^8 = a^{13}$	$a \cdot a^4 = a^5$		a	product accumulator

Efficiency: same as that of left-to-right binary exponentiation

Problem Reduction

This variation of transform-and-conquer solves a problem by a transforming it into different problem for which an algorithm is already available.

To be of practical value, the combined time of the transformation and solving the other problem should be smaller than solving the problem as given by another method.



Examples of Solving Problems by Reduction

- computing lcm(m, n) via computing gcd(m, n)
- counting number of paths of length n in a graph by raising the graph's adjacency matrix to the n-th power
- transforming a maximization problem to a minimization problem and vice versa (also, min-heap construction)
- linear programming
- reduction to graph problems (e.g., solving puzzles via state-space graphs)