Middle East Technical University
Department of Statistics
November 5, 2020, Thursday
2020-21 Fall Semester
STAT 467 Multivariate Analysis

RECITATION 2

Random Vectors and Matrices

A random vector/matrix is a vector/matrix whose elements are random variables.

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1P} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{np} \end{bmatrix}$$

Let $X = X_{ij}$ be nxp random matrix. Then, the expected value of X is

$$E(X) = \begin{bmatrix} E(X_{11}) & \cdots & E(X_{1P}) \\ \vdots & \ddots & \vdots \\ E(X_{n1}) & \cdots & E(X_{np}) \end{bmatrix}$$

Let X and Y be random matrices having same dimensions (say nxp), a,b,c be constants. A and B can be conformable matrices of constants. Then,

- $E(X+Y) = E(X) + E(Y)$
- $E(cX) = c\,E(X)$
- $E(a\,X + \,b\,Y) = a\,E(X) + \,b\,E(Y)$
- $E(\,A\,X\,B) \;=\; A\,E(X)\,B \;\neq\; ABE(X)$

Mean Vectors and Covariance Matrices

Suppose $X \;=\; [x_1, .., x_P]'$ be a $px1$ random vector. Then, each element of X is a r.v with its own marginal prob. dist. The marginal means μ_i and variances σ_i^2 are defined as $\mu_i = \; E(X_i)$ and $\sigma_i^2 = E(X_i - \mu_i)^2$ for i = 1,…p, respectively. The mean vector can be given by

$$E(X) = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_p) \end{bmatrix} = \begin{bmatrix} \mu_i \\ \vdots \\ \mu_p \end{bmatrix} = \mu_{px1}$$

Covariance matrix is defined by $Cov(X) = \Sigma = E[(X-\mu)(X-\mu)']$

$$\Sigma = \begin{bmatrix} (X_1-\mu_1)^2 & \cdots & (X_1-\mu_1)(X_p-\mu_p) \\ \vdots & \ddots & \vdots \\ (X_p-\mu_p)(X_1-\mu_1) & \cdots & (X_p-\mu_p)^2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pp} \end{bmatrix}$$

The covariance matrix is also known Var-Cov matrix since the diagonal element of this matrix represents the variance of each individual random variable and it's a pxp symmetric matrix.

Thm: If x_i and x_j are independent, then $Cov\,(x_i, x_j) \;=\; 0$. *The converse is not true in general*

users.metu.edu.tr/ozancan
e-mail: ozancan@metu.edu.tr

The population correlation matrix

$$\rho = \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} = 1 & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{1p}}\sqrt{\sigma_{1p}}} \\ \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{p1}}\sqrt{\sigma_{p1}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{pp}}} = 1 \end{bmatrix} = \begin{bmatrix} 1 & \cdots & \rho_{1p} \\ \vdots & \ddots & \vdots \\ \rho_{p1} & \cdots & 1 \end{bmatrix}$$ This is also symmetric square matrix.

Standard deviation matrix

$$V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\sigma_{pp}} \end{bmatrix} = diag\ (\sqrt{\sigma_{11}} \ldots \sqrt{\sigma_{pp}}).$$ **Thm:** $V^{1/2}\rho\, V^{1/2} = \Sigma$

Partitioning Matrices

In general, we can partition the p characteristics contained in the px1 random vector X into, for example, two group of size q and p-q respectively. Each group contains the variables being related to each other. The main purpose of this process is to reduce the dimension.

$$X = \begin{bmatrix} X_1 \\ \cdots \\ X_q \\ \cdots \\ X_{q+1} \\ \cdots \\ X_p \end{bmatrix} = \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix} \text{ where } E(X) = \begin{bmatrix} \mu_1 \\ \cdots \\ \mu_q \\ \cdots \\ \mu_{q+1} \\ \cdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} \mu^{(1)} \\ \mu^{(2)} \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Note that Σ_{11} *is qxq* Σ_{22} *is* $(p-q)x\,(p-q)$ *square matrix*

- Covariance of $X^{(1)}$ is Σ_{11}
- Covariance of $X^{(2)}$ is Σ_{22}
- Covariance between $X^{(1)}$ and $X^{(2)}$ is Σ_{12} or Σ_{21}.

Note that $\Sigma_{12} = \Sigma_{21}'$

Linear Combination of p-random variable

The linear combination $c'X = c_1X_1 + \ldots + c_pX_P$ has mean $E(c'X) = \begin{bmatrix} E(c_1X_1) \\ \vdots \\ E(c_pX_p) \end{bmatrix} = c'\mu_{px1}$ and

variance $V(c'X) = c'\Sigma\, c$ where $\Sigma = Cov\ (X)$

users.metu.edu.tr/ozancan
e-mail: ozancan@metu.edu.tr

QUESTIONS

1. The following are five measurements on the variables x_1, x_2, x_3:

x_1	9	2	6	5	8
x_2	12	8	6	4	10
x_3	3	4	0	2	1

a) Find the sample mean vector

b) Find the sample variance-covariance matrix

c) Find the sample correlation matrix

2. Let X has the following covariance matrix

$$\Sigma = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

a) Determine ρ and $V^{1/2}$

b) Multiply your matrices to check the relation $V^{1/2}\rho\, V^{1/2}$=Σ

3. You are given a random vector $X = [x_1, x_2, x_3]'$ with mean vector $\mu = [1, 2, 3]'$ and covariance matrix

$$\Sigma = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 9 \end{bmatrix}$$

Let $X^{(1)} = [x_1, x_2]'$ and

$$a = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ and\ B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

Compute the following

- **a)** $E(a'X^{(1)})$
- **b)** $E(BX^{(1)})$
- **c)** $Cov(X^{(1)})$
- **d)** $Cov(BX^{(1)})$

4. Suppose $X \sim \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}\right)$ and let $Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ where,

$z_1 = 5 + 3X_1 + 4X_2$ and $z_2 = 2 + X_1 + 5X_2$

Find $E(Z)$ and $V(Z)$

5. You are given a random vector $X' = [x_1, x_2, x_3, x_4]$ with mean vector $\mu = [4,3,2,1]'$ and variance covariance matrix.

$$\Sigma = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix}$$

Partition X as

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \dots . \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix}$$

Let $A = [1 \quad -1]$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$. Find

- **a)** $E(X^{(1)})$
- **b)** $E(AX^{(1)})$
- **c)** $Cov(X^{(1)})$
- **d)** $Cov(AX^{(1)})$
- **e)** $E(X^{(2)})$
- **f)** $E(BX^{(2)})$
- **g)** $Cov(X^{(2)})$
- **h)** $Cov(BX^{(2)})$
- **i)** $Cov(X^{(1)}, X^{(2)})$
- **j)** $Cov(AX^{(1)}, BX^{(2)})$

e-mail: ozancan@metu.edu.tr

STAT 467 / Recitation 2

Answer key

1) a) write the data in the desired form at first

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 9 & 12 & 3 \\ 2 & 8 & 4 \\ 6 & 6 & 0 \\ 5 & 4 & 2 \\ 8 & 10 & 1 \end{bmatrix} \end{array}$$

$$\bar{x}_1 = \frac{\sum_{i=1}^{n} x_{i1}}{n} = \frac{9 + \cdots + 8}{5} = \frac{30}{5} = 6$$

$$\bar{x}_2 = \frac{\sum_{i=1}^{n} x_{i2}}{n} = \frac{12 + \cdots + 10}{5} = \frac{40}{5} = 8$$

$$\bar{x}_3 = \frac{\sum_{i=1}^{n} x_{i3}}{n} = \frac{3 + \cdots + 1}{5} = \frac{10}{5} = 2$$

Therefore $\bar{x} = \begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix}$

b) $S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$ $\quad s_{ij} = \frac{1}{n} \sum_{k=1}^{n} (x_{ki} - \bar{x}_i)(x_{kj} - \bar{x}_j)$

$$s_{11} = \frac{1}{5} \sum_{k=1}^{5} (x_{k1} - \bar{x}_1)(x_{k1} - \bar{x}_1) = \frac{1}{5}\left[(9-6)^2 + (2-6)^2 + \cdots + (8-6)^2\right] = 6$$

$$s_{12} = \frac{1}{5}\left[(9-6)(12-8) + (2-6)(8-8) + \cdots + (8-6)(10-8)\right] = 4$$

$$S = \begin{bmatrix} 6 & 4 & -1.4 \\ 4 & 8 & 1.2 \\ -1.4 & 1.2 & 2 \end{bmatrix}$$

c) $r_{ik} = \dfrac{Cov(X_i, X_k)}{\sqrt{V(X_i)\, V(X_k)}}$ $\qquad r = \begin{bmatrix} 1 & 1/\sqrt{3} & -1.4/2\sqrt{3} \\ 1/\sqrt{3} & 1 & 0.3 \\ -1.4/2\sqrt{3} & 0.3 & 1 \end{bmatrix}$

$$r_{12} = \frac{Cov(X_1, X_2)}{\sqrt{V(X_1)\, V(X_2)}} = \frac{4}{\sqrt{6.8}} = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$r_{13} = \frac{Cov(X_1, X_3)}{\sqrt{V(X_1)\, V(X_3)}} = \frac{-1.4}{\sqrt{6.2}} = \frac{-1.4}{2\sqrt{3}}$$

$$r_{23} = \frac{Cov(X_2, X_3)}{\sqrt{V(X_2)\, V(X_3)}} = \frac{1.2}{\sqrt{8.2}} = 0.3$$

2) a) $V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & 0 \\ 0 & \sqrt{\sigma_{22}} & 0 \\ 0 & 0 & \sqrt{\sigma_{33}} \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \text{diag}(5, 2, 3)$

$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}$

$\rho = \left(V^{1/2}\right)^{-1} \Sigma \left(V^{1/2}\right)^{-1}$

$= \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

$= \begin{bmatrix} 5 & -2/5 & 4/5 \\ -1 & 2 & 1/2 \\ 4/3 & 1/3 & 3 \end{bmatrix} \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix}$

b) $V^{1/2} \rho V^{1/2} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 5 & -1 & 4/3 \\ -2/5 & 2 & 1/3 \\ 4/5 & 1/2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$ #

4) $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 3X_1 + 4X_2 + 5 \\ X_1 + 5X_2 + 2 \end{bmatrix} = \begin{bmatrix} 3X_1 + 4X_2 \\ X_1 + 5X_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$Z = CX + d$

$E(Z) = E(CX + d) = \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 13 \end{bmatrix}$

$V(Z) = V(CX + d) = V(CX) = C \Sigma C' = \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$

$= \begin{bmatrix} 17 & 10 \\ 13 & 22 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 139 & 127 \\ 127 & 123 \end{bmatrix}$ #

3) (a) $E(\underset{\sim}{a}'\underset{\sim}{X_1}) = \underset{\sim}{a}' E(\underset{\sim}{X_1})$

$$= (1 \quad 2)\begin{pmatrix}1\\2\end{pmatrix} = 1^2+2^2 = 5$$

(b) $E(B\underset{\sim}{X_1}) = B\,E(\underset{\sim}{X_1})$

$$= \begin{pmatrix}1 & -2\\2 & -1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-3\\0\end{pmatrix}$$

(c) $Cov(\underset{\sim}{X_1}) = \begin{pmatrix}3 & 1\\1 & 4\end{pmatrix}$

(d) $Cov(B\underset{\sim}{X_1}) = B \cdot Cov(\underset{\sim}{X_1})\, B^T$

$$= \begin{pmatrix}1 & -2\\2 & -1\end{pmatrix}\begin{pmatrix}3 & 1\\1 & 4\end{pmatrix}\begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix}$$

$$= \begin{pmatrix}1 & -7\\5 & -2\end{pmatrix}\begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix}$$

$$= \begin{pmatrix}15 & 9\\9 & 12\end{pmatrix} \qquad \#$$

5) $\mu_x = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ $\Sigma_x = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix}$

Partition X as: $X = \begin{bmatrix} X_1 \\ X_2 \\ \hline X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix}$

$A = [1 \ \ -1]$, $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

Partition of mean vector: $\begin{bmatrix} \mu^{(1)} \\ \hline \mu^{(2)} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ \hline 2 \\ 1 \end{bmatrix}$

Partition of var-cov matrix: $\Sigma_x = \left[\begin{array}{cc|cc} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ \hline 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{array}\right]$

a) $E(X^{(1)}) = \mu^{(1)} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ //

b) $E(AX^{(1)}) = A\mu^{(1)} = [1 \ \ -1]\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1$ //

c) $Cov(X^{(1)}) = \Sigma_{11} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ //

d) $Cov(AX^{(1)}) = A \cdot cov(X^{(1)}) A^T$

$= [1 \ \ -1]\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 4$ //

e) $E(X^{(2)}) = \mu^{(2)}$

$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

//

f) $E(BX^{(2)}) = B\mu^{(2)} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ //

g) $Cov(X^{(2)}) = \Sigma_{22} = \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}$

h) $Cov(BX^{(2)}) = B\,Cov(X^{(2)})B^{T}$

$$= \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 48 & -8 \\ -8 & 4 \end{bmatrix}$$ //

i) $Cov(X^{(1)}, X^{(2)}) = \Sigma_{12} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$ //

j) $Cov(AX^{(1)}, BX^{(2)}) = E\left[(AX^{(1)} - A\mu^{(1)})(BX^{(2)} - B\mu^{(2)})'\right]$

$$= E\left[A(X^{(1)} - \mu^{(1)})(X^{(2)} - \mu^{(2)})'B'\right]$$

$$= A\,Cov(X^{(1)}, X^{(2)})B'$$

$$= A\Sigma_{12}B'$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \end{bmatrix}$$ //