STAT467 – Multivariate Analysis

Homework I

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1) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

a) A matrix is positive definite if $x'Ax > 0$ for all possible $x \neq 0$.

Let's say that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $x \neq 0$

$$x'Ax = \underbrace{\underset{1\times 2}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \underset{1\times 2}{\begin{bmatrix} 3x_1 + x_2 & x_1 + 3x_2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$$= 3x_1^2 + x_1x_2 + x_1x_2 + 3x_2^2 = 3x_1^2 + 2x_1x_2 + 3x_2^2$$

$$= 3\left(x_1^2 + \frac{2}{3}x_1x_2 + x_2^2\right)$$

$$= \underbrace{3\left(\left(x_1 + \frac{1}{3}x_2\right)^2 + \frac{8x_2^2}{9}\right)}_{>0 \text{ for all } x \neq 0}$$

⇒ thus, A is positive definite ✓

b) $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$

$$= (3-\lambda)^2 - 1 = 9 - 6\lambda + \lambda^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2)$$

$$\Rightarrow \boxed{\lambda_1 = 4, \ \lambda_2 = 2}$$ Eigen values of A

Normalized eigen vectors of A: (e1 & e2 are eigen vectors)

→ $\lambda_1 = 4$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} = 4 \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} \quad \left(\xrightarrow{\text{(where)}} e_1 = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} \right)$$

$3e_{11} + e_{12} = 4e_{11} \rightarrow e_{11} = e_{12}$ ✓

$e_{11} + 3e_{12} = 4e_{12} \rightarrow e_{12} = e_{11}$ ✓

→ There is infinitively many solution.
→ So, we can assign any value.

→ say, $e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

→ $\lambda_2 = 2$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} = 2 \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} \quad \left(\text{where } e_2 = \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} \right)$$

$3e_{21} + e_{22} = 2e_{21} \rightarrow -e_{21} = e_{22}$ ✓

$e_{21} + 3e_{22} = 2e_{22} \rightarrow -e_{22} = e_{21}$ ✓

Again, infinitely many soln. exists.

→ say, $e_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Then normalize the eigenvectors:

$$e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\text{Normalize}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$e_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \xrightarrow{\text{Normalize}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

c) $A = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2'$ } spectral decomposition of matrix A.

$$A = 4 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + 2 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

d) $A^{1/2} = \Gamma \Lambda^{\frac{1}{2}} \Gamma^{-1}$ where $\Lambda^{1/2} = \begin{pmatrix} \sqrt{\lambda_1} & 0 & & \\ 0 & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

In our case, $\Lambda^{1/2} = \begin{pmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$, $\Gamma = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$A^{1/2} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1.707106 & 0.29289 \\ 0.29289 & 1.707106 \end{pmatrix}$$

2) $X \sim N_3(\mu, \Sigma)$ $\mu = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

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a) $\frac{X_1+X_2}{2}$ and X_3

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \frac{X_1+X_2}{2} \\ X_3 \end{bmatrix} = \overset{c}{\underset{\parallel}{}} c\,X = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \frac{X_1+X_2}{2} \\ X_3 \end{bmatrix}$$

$$V(z) = c\,\Sigma\,c' = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V(z) = \begin{array}{cc} & \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{array}{c} z_1 \\ z_2 \end{array} & \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \end{array}$$

$\sigma_{12} = 0$, $\frac{X_1+X_2}{2}$ & X_3 are independent.

b) Dist. of $X_2 - \frac{5}{2}X_1 - X_3 \longrightarrow X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

$$z = cX = -\frac{5}{2}X_1 + X_2 - X_3 = [c_1 \;\; c_2 \;\; c_3] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$\rightsquigarrow$ Thus, $c = [-5/2 \;\; 1 \;\; -1]$

$$\mu_z = c\,\mu_X = [-5/2 \;\; 1 \;\; -1] \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} = \frac{15}{2} + 1 - 4 = \frac{+9}{2} = +4.5$$

$$\sigma_z = c\,\Sigma\,c' = [-5/2 \;\; 1 \;\; -1] \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} -5/2 \\ 1 \\ -1 \end{pmatrix} = 23{,}25$$

Then, $z = -\frac{5}{2}X_1 + X_2 - X_3 \sim N(+4.5, 23.25)$

c) Dist. of $X_2 \mid X_1 = x_1, X_3 = x_3$

$$x' = \left[\begin{array}{c} X_2 \\ \hline X_1 \\ X_3 \end{array}\right] \begin{array}{l} \to x^{(1)} \\ \\ \to x^{(2)} \end{array}, \quad \mu' = \left[\begin{array}{c} 1 \\ \hline -3 \\ 4 \end{array}\right] \begin{array}{l} \to \mu^{(1)} \\ \\ \to \mu^{(2)} \end{array}, \quad \Sigma = \begin{array}{c|c} & \begin{array}{ccc} 2 & 1 & 3 \end{array} \\ \begin{array}{c} 2 \\ 1 \\ 3 \end{array} & \left[\begin{array}{c|cc} 5 & -2 & 0 \\ \hline -2 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right] \end{array}$$

Σ_{11}, Σ_{12}, Σ_{21}, Σ_{22}

$$\text{mean} = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(X^{(2)} - \mu^{(2)})$$

$$= 1 + [-2 \;\; 0] \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \left(\begin{bmatrix} x_1 + 3 \\ x_3 - 4 \end{bmatrix} \right)$$

$$= 1 + \underset{1\times 2}{[-2 \;\; 0]} \underset{2\times 1}{\begin{bmatrix} x_1 + 3 \\ x_3 - 4 \end{bmatrix}} = 1 - 2x_1 - 6 = -2x_1 - 5$$

$$\text{cov} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$= 5 - [-2 \;\; 0] \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = 5 - 4 = 1$$

$$\boxed{X_2 \mid X_1 = x_1, X_3 = x_3 \sim N(-2x_1 - 5, 1)}$$

3.
a)
i. False, an observation should be taken as an "outlier" if $|Z_{ij}|>3$ (where $Z_{ij} \frac{= Y_{ij} - \bar{Y}}{s_i} \sim$ N(0,1))
ii. True
iii. False, "not normalized eigen vector". A matrix can be written as A=λv (where v is the eigenvector matrix)
Thus, A=$\lambda_1 v_1$ +$\lambda_2 v_2$ + $\lambda_3 v_3$ + … + $\lambda_n v_n$
iv. True

b)

- Transformation
- Discart the outliers
- Applying nonparametric methods
- Increasing the sample size, n if applicable.
- Generate bootsrap samples.