RECITATION 5

Inference About a Mean Vector

Reminder

Univariate Case: Suppose a random sample of $X_1, X_2, \cdots, X_n$ is drawn from **a normal population** with mean μ and variance σ^2 (in practice σ^2 is unknown, s is used instead).

Given $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. The test statistic is:

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim t(n-1).$$

The null hypothesis is rejected if $|t|$ is large.

Multivariate Case: Let $X_1, X_2, \cdots, X_n$ be a random sample from $N_p(\mu, \Sigma)$. The hypothesis to be tested is: $H_0 : \mu = \mu_0$ versus $H1 : \mu \neq \mu_0$ where μ_0 is a $px1$ vector. The test statistic which is analog of the univariate t^2 is:

$$T^2 = n(\bar{\boldsymbol{X}} - \boldsymbol{\mu}_0)' \boldsymbol{S}^{-1} (\bar{\boldsymbol{X}} - \boldsymbol{\mu}_0) \sim \frac{(n-1)p}{n-p} F(p, n-p)$$

where

$$\bar{\boldsymbol{X}} = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{X}_i \text{ and } \boldsymbol{S} = \frac{1}{n-1} \sum_{i=1}^{n} (\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})'.$$

This test statistic is called **Hotelling's T^2 statistic.** If T^2 is "too large", i.e., $\bar{X}$ is "too far" from μ_0, then $H_0 : \mu = \mu_0$ is rejected which means μ_0 is not a plausible value for μ.

If n independent observation vectors $X_1, X_2, \cdots, X_n$ are collected, then $H_0 : \mu = \mu_0$ is rejected if $n(\bar{X} - \mu_0)' \, S^{-1} (\bar{X} - \mu_0) \;\; > c^* \;\; c^* = \frac{(n-1)p}{(n-p)} F_{\alpha\,(p, n-p)}$

Likelihood-Ratio Test

Likelihood-ratio test is a general principle for constructing test procedures. It is the ratio of the restricted likelihood function to the unrestricted likelihood function.

Let $X_1, X_2, \cdots, X_n$ be a random sample from $N_p(\mu, \Sigma)$, the likelihood function is:

$$\ell(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{\frac{np}{2}} |\boldsymbol{\Sigma}|^{\frac{n}{2}}} \exp\left[-\frac{1}{2} \sum_{i=1}^{n} (\boldsymbol{x}_i - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}_i - \boldsymbol{\mu}) \right].$$

where,

$$\hat{\mu} = \bar{x} \qquad \hat{\Sigma} = \frac{\sum_{j=1}^{n} (x_j - \bar{x})(x_j - \bar{x})'}{n} = \frac{n-1}{n} S$$

When the null hypothesis holds, there is no need of searching for μ because it is given as fixed. Hence, under $H_0 : \mu \neq \mu_0$, the restricted likelihood function is:

$$\ell(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) = \frac{1}{(2\pi)^{\frac{np}{2}} |\boldsymbol{\Sigma}_0|^{\frac{n}{2}}} \exp\left[-\frac{1}{2}np\right]$$

Therefore, the likelihood-ratio is:

$$\Lambda = \frac{\ell(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)}{\ell(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}})} = \left[\frac{|\hat{\boldsymbol{\Sigma}}|}{|\boldsymbol{\Sigma}_0|}\right]^{\frac{n}{2}}$$

$$\Rightarrow \Lambda^{\frac{2}{n}} = \frac{|\hat{\boldsymbol{\Sigma}}|}{|\boldsymbol{\Sigma}_0|}$$

This likelihood-ratio test statistic $\Lambda^{2/n}$ is called Wilks' Lambda. The null hypothesis $H_0 : \mu \neq \mu_0$ should be rejected if the value of Λ is too small, i.e., if:

$$\Lambda = \left[\frac{|\hat{\boldsymbol{\Sigma}}|}{|\boldsymbol{\Sigma}_0|}\right]^{\frac{n}{2}} < c_\alpha$$

where c_α is the lower (100α)th percentile of the distribution of Λ. But,

$$\Lambda^{\frac{2}{n}} = \left[1 + \frac{T^2}{n-1}\right]^{-1}$$

where T^2 is Hotelling T^2 statistic.

Confidence Region / Interval Approach

Once the null hypothesis $H_0 : \mu \neq \mu_0$ is rejected, then the component which is responsible for rejection has to be determined.

One at a time

$$\left(\bar{x}_j \pm t_{\alpha/2}(n-1)\sqrt{\frac{s_{jj}}{n}}\right)$$

Simultaneously

$$\left(\bar{x}_j \pm \sqrt{c^*}\sqrt{\frac{s_{jj}}{n}}\right) \text{ where } c^* = \frac{(n-1)p}{n-p} F_\alpha(p, n-p).$$

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Bonferroni: The Bonferroni confidence interval makes an adjustment on the univariate t-test critical value, not to increase type I error, by considering the total number of confidence intervals required. The (1- α)100% Bonferroni confidence interval for μ_j

$$\left(\bar{x}_j \pm t_{\alpha/2p}(n-1)\sqrt{\frac{s_{jj}}{n}}\right)$$

where p is the number of confidence intervals required.

Comparison of Several Multivariate Means

Dependent Samples

Paired Comparison : In paired comparison, two treatments or the presence and absence of a single treatment is compared by assigning both treatments to the same (e.g., persons) or identical (e.g., plots) experimental units. The paired responses are then analyzed by computing their differences.

Given p responses, 2 treatments and n experimental units. Let X_{1ij} denote the jth response of the i^{th} unit to treatment I (response before treatment) and let X_{2ij} denote the jth response of the i^{th} unit to treatment II (response after treatment).

Pre-treatment matrix				Post-treatment matrix			
Var 1	Var 2	$\cdots$	Var p	Var 1	Var 2	$\cdots$	Var p
X_{111}	X_{112}	$\cdots$	X_{11p}	X_{211}	X_{212}	$\cdots$	X_{21p}
X_{121}	X_{122}	$\cdots$	X_{12p}	X_{221}	X_{222}	$\cdots$	X_{22p}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
X_{1n1}	X_{1n2}	$\cdots$	X_{1np}	X_{2n1}	X_{2n2}	$\cdots$	X_{2np}

Taking the differences (before treatment - after treatment) of the type:

$$D_{ij} = D_{1ij} - D_{2ij};\ i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, p$$

The hypothesis of interest is: $H_0 : \mu_d = 0$ (no treatment effect for all p components) versus $H_1 : \mu_d \neq 0$. If $D_1, D_2, \cdots, D_n$ are independent random vectors distributed as $N_d(\mu_d, \Sigma_d)$, then the test statistic is:

$$T^2 = n(\bar{\boldsymbol{D}} - \boldsymbol{\mu}_d)'(\boldsymbol{S}_d)^{-1}(\bar{\boldsymbol{D}} - \boldsymbol{\mu}_d) \sim \frac{(n-1)p}{n-p}F(p, n-p)$$

where $\bar{\boldsymbol{D}} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{D}_i$ and $\boldsymbol{S}_d = \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{D}_i - \bar{\boldsymbol{D}})(\boldsymbol{D}_i - \bar{\boldsymbol{D}})'$.

Reject H_0 if $T^2 > c^*$ $\quad c^* = \frac{(n-1)p}{(n-p)}F_{\alpha\,(p,n-p)}$

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A Repeated Measures Design for Comparing Treatments: A repeated measures design is another generalization of the univariate t statistic in which q treatments are compared with respect to a single response measured from the same (identical) sampling units over time or space. Each experimental unit receives each treatment once over successive period of time. The name repeated measures stems from the fact that all treatments are administered to each unit.

Let X_{ik} be the response to the k^{th}; k = 1,2,….,q treatment on the i^{th}; i = 1,2,….,n unit.

Item	Treatment 1	Treatment 2	$\cdots$	Treatment q	
1	X_{11}	X_{12}	$\cdots$	X_{1q}	$\boldsymbol{X}_1$
2	X_{21}	X_{22}	$\cdots$	X_{1q}	$\boldsymbol{X}_2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
n	X_{n1}	X_{n2}	$\cdots$	X_{nq}	$\boldsymbol{X}_q$

The hypothesis of interest is whether $\mu_1 = \mu_2 = \ldots = \mu_q$ (no treatment effect). For comparative purposes, contrasts of the components of $\mu = E(Xi)$ are considered. These could be:

$$\underbrace{\begin{bmatrix} \mu_1-\mu_2 \\ \mu_2-\mu_3 \\ \mu_3-\mu_4 \\ \vdots \\ \mu_{q-1}-\mu_q \end{bmatrix}}_{(q-1)\times 1} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}}_{(q-1)\times q} \underbrace{\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_q \end{bmatrix}}_{q\times 1} = \boldsymbol{A\mu}$$

or

$$\underbrace{\begin{bmatrix} \mu_1-\mu_2 \\ \mu_1-\mu_3 \\ \mu_1-\mu_4 \\ \vdots \\ \mu_1-\mu_q \end{bmatrix}}_{(q-1)\times 1} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}}_{(q-1)\times q} \underbrace{\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_q \end{bmatrix}}_{q\times 1} = \boldsymbol{B\mu}$$

Since each row is a contrast and the $q-1$ rows are linearly independent, both $\boldsymbol{A}$ and $\boldsymbol{B}$ are *contrast matrices*. If $\boldsymbol{A\mu} = \boldsymbol{B\mu} = \mathbf{0}$, then $\mu_1 = \mu_2 = \cdots = \mu_q$. Hence, the hypothesis of no diff in treatments (equal treatment means) is $\boldsymbol{A\mu} = \mathbf{0}$ for any choice of contrast matrix $\boldsymbol{A}$.

Consider an $\mathcal{N}_q(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ population. If $\boldsymbol{A}$ is a contrast matrix, then $\boldsymbol{A\bar{X}} \sim \mathcal{N}(\boldsymbol{A\mu}, \boldsymbol{ASA'})$. For testing $H_0 : \boldsymbol{A\mu} = \mathbf{0}$ vs $H_1 : \boldsymbol{A\mu} \neq \mathbf{0}$, the test statistic is:

$$T^2 = n(\boldsymbol{A\bar{X}})'(\boldsymbol{ASA'})^{-1}(\boldsymbol{A\bar{X}}) \sim \frac{(n-1)(q-1)}{n-(q-1)} F[q-1, n-(q-1)].$$

Note that T^2 does not depend on the particular choice of $\boldsymbol{A}$. As usual, reject H_0 if the observed $T^2 = n(\boldsymbol{A\bar{x}})'(\boldsymbol{ASA'})^{-1}(\boldsymbol{A\bar{x}}) > c^*$.

$$\text{where } c^* = \frac{(n-1)(q-1)}{n-(q-1)} F_\alpha[q-1, n-(q-1)].$$

QUESTIONS

1. In an investigation of the Wechsler Adult Intelligence scale scores of older men and women, Doppelt & Wallace (1955) reported that the mean verbal and performance score for n=101 subjects aged to 60 to 64 were

$$\begin{bmatrix}\bar{X}_v\\ \bar{X}_p\end{bmatrix} = \begin{bmatrix}55.24\\ 34.97\end{bmatrix}$$

The sample covariance matrix of the scores was

$$S = \begin{bmatrix}210.54 & 126.99\\ 126.99 & 119.68\end{bmatrix}$$

a) Test the null hypothesis that the observations come from the mean vector $\mu_0 = \begin{bmatrix}60\\ 50\end{bmatrix}$ at $\alpha =$ 0.01 level.

b) Construct 99% simultaneous confidence intervals for the mean vector

c) Construct 99% Bonferroni confidence intervals for the mean vector

d) Construct 99% "one-at-a-time" confidence intervals for the mean vector.

2. Evaluate T^2 for testing $H_{0=}\mu \;\; = \begin{bmatrix}7\\ 11\end{bmatrix}$ using the data

$$X = \begin{bmatrix}2 & 12\\ 8 & 9\\ 6 & 9\\ 8 & 10\end{bmatrix}$$

and specify the distribution of $T^2$and test H_0 at $\alpha = 0.05$ level. Evaluate $\wedge$ and Wilk's lambda.

3. Municipal wastewater treatment plants are required by law to monitor their discharges into rivers and streams on a regular basis. Concern about the reliability of data from one of these self-monitoring programs led to a study in which samples of effluent were divided and sent to two laboratories for testing. One-half of each sample was sent to the Wisconsin State Laboratory of Hygiene, and one-half was sent to a private commercial laboratory routinely used in the monitoring program. Measurements of biochemical oxygen demand (BOD) and suspended solids (SS) were obtained, for n=11 sample splits, from the two laboratories. The data are displayed below.

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	Commercial Lab		State Lab of Hygiene	
Sample (j)	**X_{1j1} (BOD)**	**X_{1j2} (SS)**	**X_{2j1} (BOD)**	**X_{2j2} (SS)**
1	**6**	**27**	**25**	**15**
2	**6**	**23**	**28**	**13**
3	**18**	**64**	**36**	**22**
4	**8**	**44**	**35**	**29**
5	**11**	**30**	**15**	**31**
6	**34**	**75**	**44**	**64**
7	**28**	**26**	**42**	**30**
8	**71**	**124**	**54**	**64**
9	**43**	**54**	**34**	**56**
10	**33**	**30**	**29**	**20**
11	**20**	**14**	**39**	**21**

Do the two laboratories chemical analyses agree? If differences exists what is their nature.

4. Let $X \sim N_3(\mu, \Sigma)$ where $X' = (X_1, X_2, X_3)$ and $\mu' = (\mu_1, \mu_2, \mu_3)$. The sample values and the sample statistics for a random sample are as follows:

$$X_1 = [18 \quad 4 \quad 12 \quad 10 \quad 16]$$
$$X_2 = [24 \quad 16 \quad 12 \quad 8 \quad 20]$$
$$X_3 = [6 \quad 8 \quad 0 \quad 4 \quad 2]$$

$$\bar{X} = \begin{bmatrix} 12 \\ 16 \\ 4 \end{bmatrix}, \qquad S = \begin{bmatrix} 30 & 20 & -7 \\ 20 & 40 & 6 \\ -7 & 6 & 10 \end{bmatrix}$$

Test the hypothesis $H_0 : 2\mu_1 = \mu_2 + \mu_3$ and $\mu_1 = \mu_3$, H_1 : not true at 5% significance level.

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Answer Key

1) a) $H_0 = \mu = \begin{bmatrix} 60 \\ 50 \end{bmatrix}$ $H_1 = \mu \neq \begin{bmatrix} 60 \\ 50 \end{bmatrix}$

we need to calculate test statistics T^2

$$T^2 = n \cdot (\bar{x} - \mu_0)' S^{-1} (\bar{x} - \mu_0)$$

$$= 101 \cdot \begin{bmatrix} 55.24 - 60 & 34.97 - 50 \end{bmatrix} \begin{bmatrix} 0.0132 & -0.014 \\ & 0.023 \end{bmatrix} \begin{bmatrix} 55.24 - 60 \\ 34.97 - 50 \end{bmatrix}$$

$T^2 = 357.43$

↳ compare this with test statistic

$$c^* = c^2 = \frac{(n-1)p}{(n-p)} F_{p,n-p,\alpha} = \frac{(101-1)2}{(101-2)} \cdot F_{2,99,0.01} = 2.02 \times 4.82 = 9.74$$

Since $T^2 > c^2 \rightarrow$ we reject H_0. so $\mu \neq \begin{bmatrix} 60 \\ 50 \end{bmatrix}$

b) Formula for simultaneous confidence interval

$$\bar{x}_j \pm \sqrt{c^*} \sqrt{\frac{s_{jj}}{n}}$$

$\bar{x}_1 = 55.24$ 99% CI for $\mu_1 \rightarrow \bar{x}_1 \pm \sqrt{c^*} \sqrt{\frac{s_{11}}{n}}$

$s_{11} = 210.54$

$n = 101$

$$55.24 \pm \sqrt{9.74} \sqrt{\frac{210.54}{101}}$$

$(50.75, 59.73)$

$\bar{x}_2 = 34.97$

$s_{22} = 119.68$ 99% CI for $\mu_2 \rightarrow \bar{x}_2 \pm \sqrt{c^*} \sqrt{\frac{s_{11}}{n}}$

$n = 101$

$$34.97 \pm \sqrt{9.74} \sqrt{\frac{119.68}{101}}$$

$(31.57, 38.37)$

c) Formula

$$\bar{X}_J \pm t_{\alpha/2p, n-1} \sqrt{\frac{S_{JJ}}{n}}$$

For μ_1

$$55.24 \pm \underbrace{t_{\frac{0.01}{4}, 100}}_{\to 2.87} \sqrt{\underbrace{\frac{210.54}{101}}_{\to 1.44}}$$

$(51.1, 59.38)$

For μ_2 $\to$ $34.97 \pm t_{\frac{0.01}{4}, 100} \sqrt{\frac{119.68}{101}}$

$(31.85, 38.1)$

d) Formula

$$\bar{X}_J \pm t_{\alpha/2, n-1} \sqrt{\frac{S_{JJ}}{n}}$$

For $\mu_1 \to 55.24 \pm \underbrace{t_{\frac{0.01}{2}, 100}}_{\to 2.63} \sqrt{\frac{210.54}{101}}$

$(51.44, 59.04)$

For $\mu_2 \to 34.97 \pm t_{\frac{0.01}{2}, 100} \sqrt{\frac{119.68}{101}}$

$(32.11, 37.83)$

2) $\bar{X} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$ $\quad S = \begin{bmatrix} 8 & -10/3 \\ -10/3 & 2 \end{bmatrix}$ $\to S = \frac{1}{n-1}(X-\bar{X})(X-\bar{X})'$

$H_0 = \mu = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$ vs $H_1 = \mu \neq \begin{bmatrix} 7 \\ 11 \end{bmatrix}$ $\quad S^{-1} = \frac{9}{44}\begin{bmatrix} 2 & -10/3 \\ -10/3 & 8 \end{bmatrix} = \begin{bmatrix} 0.41 & 0.68 \\ 0.68 & 1.64 \end{bmatrix}$

$$T^2 = n.(\bar{X}-\mu_0)' S^{-1}(\bar{X}-\mu_0)$$

$$= 4.\begin{pmatrix} 6-7 & 10-11 \end{pmatrix} \begin{bmatrix} 0.41 & 0.68 \\ 0.68 & 1.64 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = 13.64$$

$$c^2 = \frac{(n-1)p}{n-p} F_{p,n-p,\alpha} = \frac{3.2}{2} \cdot F_{2,2,0.05} = 57.$$

Compare the test statistics with c^2

Since $T^2 < c^2$, we fail to reject H_0, that means pop mean equals to μ.

Evaluate Λ and Wilk's Lambda

$$\Lambda = \left(\frac{|\hat{\Sigma}|}{|\hat{\Sigma}_0|}\right)^{n/2} = \left(\frac{\left|\sum_{j=1}^{n} (x_j-\bar{x})(x_j-\bar{x})'\right|}{\left|\sum_{j=1}^{n} (x_j-\mu_0)(x_j-\mu_0)'\right|}\right)^{n/2}$$

$$\hat{\Sigma} = S_n = \frac{1}{4}\begin{bmatrix} 24 & -10 \\ -10 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2.5 \\ & 1.5 \end{bmatrix} \quad |\hat{\Sigma}| = (6\times 1.5) - 2.5^2 = 2.75$$

$$\hat{\Sigma}_0 = \frac{1}{4}\begin{bmatrix} 28 & -6 \\ -6 & 10 \end{bmatrix} = \begin{bmatrix} 7 & -1.5 \\ -1.5 & 2.5 \end{bmatrix} \quad |\hat{\Sigma}_0| = (7\times 2.5) - 1.5^2 = 15.25$$

$$\Lambda = \left(\frac{2.75}{15.25}\right)^{4/2} = (0.1803)^2 = 0.0325$$

Wilk's Lambda $\Lambda^{2/n} = \Lambda^{2/4} = \Lambda^{1/2} = \sqrt{0.0325} = 0.1803$

Then, $-2\ln\Lambda = -2\ln(0.0325) \sim \chi^2_2 \rightarrow p + \frac{p(p+1)}{2} - \frac{p(p+1)}{2} = 2$

If $\underbrace{-2\ln(0.0325)}_{6.85} < \chi^2_{2,\alpha = 0.05} \rightarrow$ Reject Ho

$6.85 > 5.991$ so, fail to reject Ho

Alternatively, $\Lambda^{2/n} = \left(1 + \frac{T^2}{n-1}\right)^{-1} = \left(1 + \frac{13.64}{3}\right)^{-1} = \left(\frac{3}{16.64}\right) = 0.1803$

3) $H_0 = \mu_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad H_1: \mu_d \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \bar{d} = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \end{bmatrix} = \begin{bmatrix} -9.36 \\ 13.27 \end{bmatrix}$

$d_{j1} = x_{1j_1} - x_{2j_1}$ $\quad d_{j2} = x_{1j2} - x_{2j2}$

d_{j1}	d_{j2}
-19	12
-22	10
-18	42
-27	15
-4	-1
-10	11
-14	-4
17	60
9	-2
4	10
-19	-7

$$S_d = \begin{bmatrix} 199.26 & 88.38 \\ 88.38 & 418.61 \end{bmatrix}$$

$$T^2 = n.(\bar{x}_d - \mu_d)' S_d^{-1} (\bar{x}_d - \mu_d)$$

$$= 11\,(-9.36 \;\; 13.27)\begin{pmatrix} 0.0055 & -0.0012 \\ & 0.0026 \end{pmatrix}\begin{pmatrix} -9.36 \\ 13.27 \end{pmatrix}$$

$$= 13.6$$

$$c^2 = c^* = \frac{p(n-1)}{(n-p)} F_{p,n-p,\alpha} = \frac{2.10}{9} F_{2,9,0.05} = 9.47$$

Since $T^2 = 13.6 > 9.47$, we reject Ho and conclude that there is a nonzero mean difference btw the measurements of two laboratories.

4) $H_0 = \underbrace{\begin{bmatrix} 2\mu_1 - \mu_2 - \mu_3 \\ \mu_1 - \mu_3 \end{bmatrix}}_{\mu'} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mu_0}$ vs $H_1 = \mu' \neq \mu_0$

Define $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mu_y = \begin{bmatrix} \mu_{y_1} \\ \mu_{y_2} \end{bmatrix} = \begin{bmatrix} 2\mu_1 - \mu_2 - \mu_3 \\ \mu_1 - \mu_3 \end{bmatrix} = A\mu_x$ A is a matrix

So, the hypothesis can be rewritten as

$H_0 = \underbrace{\begin{bmatrix} \mu_{y_1} \\ \mu_{y_2} \end{bmatrix}}_{\mu_y} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mu_0}$ vs $H_1 = \mu_y \neq \mu_0$

Let's find A.

$\mu_y = A\mu_x$

$\begin{bmatrix} \mu_{y_1} \\ \mu_{y_2} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$

$\bar{y} = A.\bar{x}$

$= \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ 16 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$S_y = A S_x A' =$

$= \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 30 & 20 & -7 \\ 20 & 40 & 6 \\ -7 & 6 & 10 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 130 & 77 \\ 77 & 54 \end{bmatrix}$

$T^2 = n(\bar{y} - \mu_0)' S_y^{-1} (\bar{y} - \mu_0)$

$= 5.\begin{bmatrix} 4 & 8 \end{bmatrix} \begin{bmatrix} 0.05 & -0.07 \\ & 0.12 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = 19.5$

$c^2 = \frac{(n-1)p}{n-p} F_{p,n-p,\alpha} = \frac{4 \cdot 2}{3} \overbrace{F_{2,3,0.05}}^{\cancel{19} \; 9.55} = \cancel{20.6} \; \underline{25.47}$

$T^2 < c^2 \rightarrow$ Fail to reject Ho

$\boxed{\mu_y = \mu_0}$