Handout 1 - Algebra and Logic Problems

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§1 Algebra

§1.1 Expressions and Equations

Example 1.1 (Mathcounts Handbook 2018)
If a and b are real numbers such that $a + b = a - b$ and $a \neq b$, what is the value of

$$\frac{a^2b + a + b - ab^2}{a - b}?$$

Solution. a + b = a - b means a - a = b + b, 0 = 2b, $b = 0$, and a could be any number. Plug in b = 0 to $\frac{a^2b-ab^2+(a+b)}{a-b} = \frac{a}{a} = \boxed{1}$ □

Example 1.2 (Mathcounts Handbook 2018)
If the sum of a number and its reciprocal is 4, what is the absolute value of the difference of the number and its reciprocal?

Solution. Say the number is n, $n + \frac{1}{n} = 4$, multiplies both side by n and rearrange to get $n^2 - 4n + 1 = 0$. Solve using the quadratic equation. $n = \frac{4\pm\sqrt{16-4}}{2} = 2\pm\sqrt{3}$. So the absolute value of $n - \frac{1}{n} = 2+\sqrt{3} - \frac{1}{2+\sqrt{3}} = \boxed{2\sqrt{3}}$ □

Some common formulas to know are:

1. Difference of Squares:
$$a^2 - b^2 = (a + b)(a - b).$$

2. Sum and Difference of Cubes:
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2),$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

Note that the formula for sum above is the same as for the difference if we replace b y $-b$.

3. Quadratic Formula: For $ax^2 + bx + c = 0$, where $a \neq 0$,

$$x = \frac{= b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For more on solving quadratic equations, please see

https://www.mathcounts.org/resources/mini-38-solving-quadratic-equations-using-substitution-si

> **Example 1.3** (Mathcounts Handbook 2018)
> Suppose (a, b, c) is an ordered triple such that $a^2 + b^2 = c^2$. If $b = x + y\sqrt{2}$ when $a = 12 + 10\sqrt{2}$ and $c = 15 + 8\sqrt{2}$, what is the positive integer value of $x + y$?

Solution. Since b is the only variable written in terms of x and y, rewrite the equation to solve for b and plug in a, b, and c into it. $b^2 = c^2 - a^2 = x^2 + 2xy\sqrt{2} + 2y^2 = 225 + 240\sqrt{2} + 128 - 144 + 240\sqrt{2} + 200 = 9 = (x + y\sqrt{2})^2$. So $x + y\sqrt{2} = 3$. Since there is no $\sqrt{2}$ in the right half the equation, we can deduct that $y = 0$ and $x + y = x = \boxed{3}$ □

§1.2 Ratios

Definition 1.4. A ratio is the comparison of two quantities by division.

A ratio shows the relative sizes of two or more values.

> **Example 1.5** (Mathcounts Handbook 2019)
> A jar contains seven blue marbles and eight green marbles. Ming adds four yellow marbles and five blue marbles to the jar. What is the ratio of green marbles to non-green marbles in the jar? Express your answer as a common fraction.

Solution. The jar contains $7 + 5 = 12$ blue marbles, 4 yellow marbles and 8 green marbles. The total number of non-green marbles is $12 + 4 = 16$. So, the ratio of green marbles to non-green marbles is $\frac{8}{16} = \boxed{\frac{1}{2}}$. □

> **Example 1.6** (Mathcounts Handbook 2019)
> Fairy Godmother has granted wishes to Aurora, Belle and Cindi in the ratio 6:8:11. Fairy Godmother has granted at least 20 wishes each to Aurora, Belle and Cindi. What is the least possible number of wishes that she has granted to Cindi?

Solution. What's the greatest number greater than 20 that is divisible by 6? It has to be even, and have digits that sum up to a number that's divisible by 3, so we know that it's 24. That means Aurora gets 24 wishers. this means the ratio is multiplied by 4 per person, so Belle gets 32, and Cindi gets 44. Since the question asks how many she's granted to Cindy, the answer is $\boxed{44}$ □

§1.3 Mixture Problems

Mixture Problems are a type of word problem in which different quantities/values are mixed. For example liquids of different concentrations may be mixed, as seen below in example 1.4

Example 1.7 (Mathcounts Handbook 2020)

Manny's cleaning supply store receives a mixture of 80% detergent and 20% water in 15-gallon buckets. Manny would like a mixture of 60% detergent and 40% water in 5-gallon buckets. To make this, he combines some 80/20 mixture with some pure water in each 5-gallon bucket. How many gallons of pure water does Manny add to each 5-gallon bucket? Express your answer as a decimal to the nearest hundredth.

Solution. If we take w is the amount of water we put in, we must put 5 - w of 80/20 mixture into the bucket as well. Thus, we get the equation: 0.8 $\times$ (5 - w) = 0.6 $\times$ 5. This gives us 4 - 0.8 $\times$ w = 3. Solving, we get w = $\frac{1}{0.8} = \boxed{1.25}$ □

Example 1.8 (Mathcounts Handbook 2020)

To make a sand sculpture, Arthur used 2 cm^3 of red sand with a density of 4 g/cm^3, 7 cm^3 of yellow sand with a density of 5 g/cm^3 and 5 cm^3 of brown sand with a density of 6 g/cm^3. What is the average density of this sculpture in grams per cubic centimeter? Express your answer as a decimal to the nearest tenth.

Solution. 2 cm^3 $\times$ 4 g/cm^3 + 7 cm^3 $\times$ 5 g/cm^3 + 5 cm^3 $\times$ 6 g/cm^3 = 8 + 35 + 30 g = 73 g. 2 cm^3 + 7 cm^3 + 5 cm^3 = 13 cm^3. Thus, our answer is 73/14 = $\boxed{5.2}$ □

§1.4 Functions

Example 1.9 (Mathcounts Handbook 2015)

If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is even} \\ \dfrac{x-3}{2} & \text{if } x \text{ is odd} \end{cases}$$

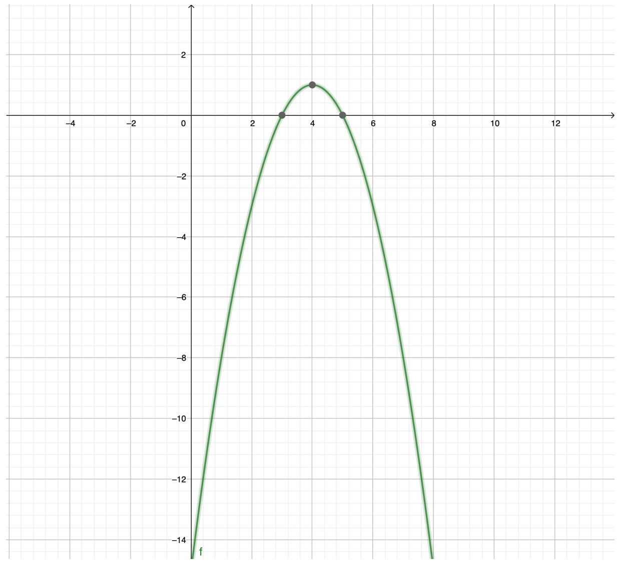
What is the value of $f(f(f(19)+1)+1)$?

Solution. f(19) + 1 = 9
f(f(19)+1)+1 = f(9) + 1 = 4
f(f(f(19)+1)+1) = f(4) = $\boxed{16}$ □

Example 1.10 (Mathcounts Handbook 2015)

The graph of $f(x) = -x^2 + 8x - 15$ contains points in how many quadrants of the Cartesian coordinate plane?

Solution. $f(x) = -(x-3)(x-5)$. For $3 < x < 5$, $f(x)$ is positive, and so in First quadrant. For $x < 3$, $f(x) < 0$, so in Third and Fourth quadrant. If $x > 5$, $f(x) < 0$, so in Fourth quadrant. Thus, we have $\boxed{3}$ solutions



□

§2 Word Problems

Please see https://www.mathcounts.org/resources/mini-19-turning-word-problems-algebra-equations.

§2.1 Measurement

Remark 2.1 (Unit Conversions)**.** Pay attention to the units in the problem. In many Mathcounts problems, you will need to convert between units, for example from feet to inches or hours to seconds. Some common conversions to remember are:

- 1 feet = 12 inches
- 1 yard = 3 feet
- 1 mile = 1,760 yards = 5,280 feet

Example 2.2 (Mathcounts Handbook 2018)

If 2 Blams equal 15 Droms and 5 Droms equal 28 Klegs, how many Klegs are in a Blam?

Solution. Rewrite algebraically, 2b = 15d, 28k = 5d, we observe the ratio of 15d (the first equation) to 3d (the second equation) is 3 to 1, so 2b = 15d = 84k. Divide everything by 2 to get 1 Blam = $\boxed{42}$ Klegs. □

Example 2.3 (Mathcounts Handbook 2018)

Vijay gives Sanjay a set of four weights of 1, 3, 8 and 26 grams. When Sanjay places weights on either side of a balance, what is the smallest positive integer number of grams that he cannot measure with this set?

Solution. Since Sanjay can place the weights on either side of the scale, a number of gram is considered to be weighed if it can be calculated by taking a sum or difference of 1, 3, 8 and 26, using each number exactly one time. Note this is because weight on left + object = weight on right. So object = weight on right - weight on left. So we can start by make a table to see if we can find a pattern:

Weight	Method
1	1
2	$3-1$
3	3
4	$1+3$
5	$8-3$
6	$8-3+1$
7	$8-1$
9	$8+1$
10	$8+3-1$
11	$8+3$
12	$8+3+1$
13	Not Possible

We are not able to write 13 as a sum or difference of given weights without repeating any weights twice. So the final answer is $\boxed{13}$ □

§2.2 Rate of Work

Definition 2.4. If a person can do r units of work in unit time, we say that their rate of work is r.

If person a, b, c can do r_a, r_b, r_c units of work in unit time, then together they can do $r_a + r_b + r_c$ units of work in unit time. That is

$$r_{combined} = r_a + r_b + r_c.$$

Sometimes, instead of the rate of work, we are given the time t_a, t_b, t_c taken by a, b, and c to complete x units of work. Then the rate of work for each of them is $r_a = \frac{x}{t_a}, r_b = \frac{x}{t_b}$, and $r_c = \frac{x}{t_c}$,. And the combined rate of work is $r_{combined} = \frac{x}{t_{combined}}$. Plugging into the previous formula and canceling x, we get

$$\frac{1}{t_{combined}} = \frac{1}{t_a} + \frac{1}{t_b} + \frac{1}{t_c}.$$

Note 2.5. We have used 3 people above for illustration purposes; the same applies for 2 or more people.

Example 2.6

Nan can mow $\frac{1}{6}$ of her lawn in an hour. Reagan can mow $\frac{1}{7}$ of the same lawn in an hour. If they have two mowers and work together what fractional part of the lawn can they mow in an hour?

Solution. When they work together, their rate of work combines and adds up to $\frac{1}{6} + \frac{1}{7} = \boxed{\frac{13}{42}}$ of lawn in an hour. □

§2.3 Speed and Distance

Definition 2.7. Speed is distance traveled divided by time taken. Average speed is the total distance traveled divided by the total time taken.

One of the implications of the above definition is that average speed is not a simple average of speeds. In particular, if the speed is v_a for t_a amount of time and v_b for t_b amount of time, then the total distance traveled is $v_at_a + v_bt_b$ and therefore,

$$v_{average} = \frac{v_at_a + v_bt_b}{t_a + t_b}.$$

That is, average speed is a weighted average of the individual speeds.

Example 2.8 (Mathcounts Handbook 2017)
Jack and Jill travel up a hill at a speed of 2 mi/h. They travel back down the hill at a speed of 4 mi/h. What is their average speed for the entire trip? Express your answer as a mixed number.

Solution. Let's say that Jack and Jill's hill has a distance from the base to the peak of d miles. The entire trip is 2d miles long, and d/2 + d/4 = 3d/4 hours long. 2d/(3d/4) = 8/3 = $\boxed{2\frac{2}{3}}$ □

§2.4 Relative Velocity or Effective Speed

You may have learned the concept of relative velocity in Physics; if you have not, you will. In some Mathcounts problem, this concept is useful to solve the problems quick. A simplified version of the concept is presented below. v_a, v_b are speed for A and B, and s_t is the distance between them at time t.

- Case 1: A and B moving towards each other. The effective speed is $v_a + v_b$ and they will meet after $\frac{s_0}{v_a+v_b}$.
- Case 2: A is moving towards B, and B is moving away from A. The effective speed is $v_a - v_b$ and they will meet after $\frac{s_0}{v_a-v_b}$, assuming $v_a > v_b$.
- Case 3: A and B moving away from each other. The effective speed is $v_a + v_b$ and the distance between them after time t will be $s_t = s_0 + (v_a + v_b) \cdot t$.

Example 2.9 (Mathcounts Handbook 2017)
At 2:20 p.m., Jack is at the top of the hill and starts walking down at the exact same time that Jill, who is at the bottom of the hill, starts walking up. Jill's uphill speed is 2 mi/h and Jack's downhill speed is 4 mi/h, and the distance from the bottom to the top of the hill is 1.5 miles, at what time will Jack and Jill meet?

Solution. Let's think of this question this way. There is a 1.5 mile distance between Jack and Jill that is being closed at a rate of 6 miles per hour. We can say this because they are walking towards each other, and Jill is walking at 2 mi/h, while Jack is walking at 4 mi/h. 1.5 / 6 = 1/4, so it takes 15 minutes for them to meat, so the answer is $\boxed{2:35PM}$. □

Example 2.10 (Mathcounts Handbook 2017)

Ansel left the dock in his motorboat, traveled 10 miles, and then returned to the dock along the same route. On the return trip, Ansel was traveling against the current of the river, and his average speed relative to the water was 20 mi/h. If the round-trip took Ansel 64 minutes, what is the speed of the river's current?

Solution. Let's call the speed of the current c. When Ansel goes 10 miles against the current, he goes 10/(20 - x) hours, while when he goes with the current, he goes for 10 / (20 + x) hours. Thus $\frac{10}{20-x} + \frac{10}{20+x} = \frac{64}{60}$. From here, we multiply everything by $60(20-x)(20+x)$, so we get $600(20+x) + 600(20-x) = 64(400-x^2)$. Thus 12000 + 600x + 12000 - 600x = 25600 - $64x^2$. Thus $64x^2$ = 1600, and x = $\boxed{5}$ □

§3 Logic

Some logic problems involve figuring out which statements are inconsistent to derive conclusions.

Example 3.1 (Mathcounts Handbook 2017)

A set S contains some, but not all, of the positive integers from 3 to 7. Some statements describing S are given below. The statement numbered n is true if the number n is in S and false if n is not in S. What is the product of the numbers that are in S?

3. The sum of the numbers in S is odd.
4. The sum of the numbers in S is less than 15.
5. S contains exactly one composite number.
6. S contains exactly one prime number.
7. The product of the numbers in S is odd.

Solution.

1. Statements 5 and 7 cannot both be true. The only composite numbers available are even, and any even factor will make the product even. This means we cannot have both 5 and 7 in the set.

2. If statement 3 is correct, then we can't have either 5 or 7 since two odds would make the sum even. If we left 3 out of the set and just included 5 or 7, then statement 3 would be correct, in which case 3 should be included.

3. If we try to exclude all the odd numbers, then statement 6 would be false and the set would contain only the number 4. But that would make statement 5 true, which is a contradiction.

4. All this means that 3 must be the only odd and only prime number in the set.

If 3 is the only prime, then statement 6 is correct and we include 6. Now we have to include 4 also, so that statement 5 will be false. The set must be $S = \{3, 4, 6\}$, the statements are true, true, false, true, false in that order, and the product of the numbers in S is $3 \cdot 4 \cdot 6 = \boxed{72}$. □

Venn diagrams are a useful tool for many logic problems.

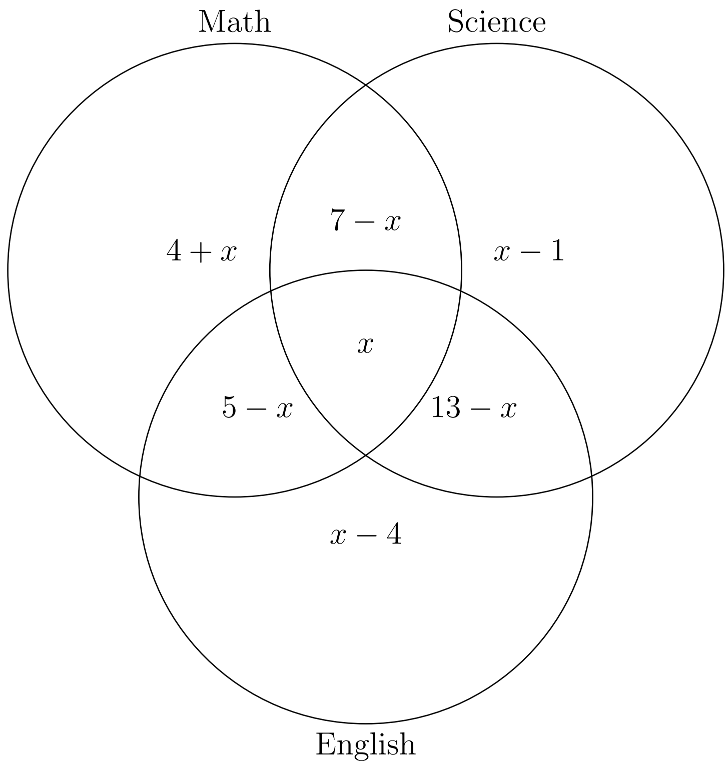
Example 3.2 (Mathcounts, Problem of the Week)

When Josh takes a survey of his 28 classmates, he gets the following results:

14 like English class
19 like Science class
16 like Math class
13 like English and Science class
7 like Science and Math class
5 like Math and English class

How many students like all English, Science and Math?

Solution. Let x be the number of students who like all 3 subjects. Then the number of students like Math and Science and Math and English, but not all 3, are $7 - x$ and $5 - x$ respectively. And the students who only like Math is $16 - (7 - x) - (5 - x) - x = 4 + x$. We can similarly fill in the other sections of the Venn diagram below.



In total, we have $4 + x + 7 - x + x - 1 + 5 - x + x + 13 - x + x - 4 = 28$. So, $24 + x = 28$, and thus x = $\boxed{4}$. □

Solution. [Solution 2, PIE.] Alternatively, students who know PIE (Principle of Inclusion and Exclusion), can directly use:

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|,\\ 28 &= 14 + 19 + 16 - 13 - 7 - 5 + x,\\ x &= 28 - (14 + 19 + 16) + (13 + 7 + 5),\\ &= \boxed{4}.\end{aligned}$$

□

§4 Beginner Practice Problems

Problem 4.1. After a brisk workout, Justin counts 32 heartbeats in 15 seconds. Based on this count, what is Justin's expected number of heartbeats in one minute?

Problem 4.2 (Mathcounts Handbook 2015)**.** Three identical boxes contain tennis balls, baseballs or both. A label is affixed to each box. The three labels correctly describe the three boxes, but none of the labels is on the correct box. Box 1 is labeled "Tennis Balls." Box 2 is labeled "Baseballs." Box 3 is labeled "Tennis Balls & Baseballs." Devon reaches into Box 3 and pulls out a baseball. Which box contains only tennis balls?

Problem 4.3 (Exeter Math Club Competition, 2016)**.** Joanna has a fully charged phone. After using it for 30 minutes, she notices that 20 percent of the battery has been consumed. Assuming a constant battery consumption rate, for how many additional minutes can she use the phone until 20 percent of the battery remains?

Problem 4.4. Dara is mixing her own paint color, using 3 parts green paint to 2 parts blue to 1 part white. Given that there are 4 quarts in a gallon, if she needs 3 gallons of her paint, how many quarts of white paint should she buy?

Problem 4.5 (Mathcounts Handbook 2017)**.** Jana begins jogging along a path and, 5 minutes later, Zhao begins riding his bicycle along the same path, which has a length of 2 miles. Zhao rides his bicycle at a speed of 10 mi/h, and Jana's jogging speed is 6 mi/h. If they both begin at one end of the path and end at the other, how many minutes after Zhao reaches the end of the path will Jana reach the end of the path?

Problem 4.6. Bobby wants to purchase animal erasers as prizes for the school fair. The animal erasers cost $11.52 per case or can be purchased individually. Bobby's parents gave him a limited allowance, and with the money he has, Bobby can either buy ten cases of animal erasers and eight individual animal erasers or he can buy 1160 animal erasers. In either situation he will not have any money remaining after the purchases. How much money does Bobby have?

§5 Intermediate Practice Problems

Problem 5.1 (Mathcounts Chapter 2021 Sprint)**.** Eli started saving money in his empty piggy bank. He inserted $1 on every Monday, $2 on every Tuesday, $3 on every Wednesday, $4 on every Thursday, $5 on every Friday, $6 on every Saturday, and $7 on every Sunday. At the end of 24 days, he opened his bank and found he had $93. On which day of the week did he start inserting money into his piggy bank?

Problem 5.2 (Mathcounts Handbook 2015)**.** Justin and Shelby board an escalator as it is moving down. Justin walks down 30 steps and reaches the bottom in 72 seconds, while Shelby walks down 40 steps and reaches the bottom in 60 seconds. If the escalator weren't moving, how many steps would Justin and Shelby each have to walk down to reach the bottom?

Problem 5.3. Eazin is traveling to visit Stefan at an average speed of 60 mi/h. Eazin realizes that if he drives at an average speed of 75 mi/h, his travel time will be reduced by 6 minutes. How many miles per hour would Eazin's travel need to average for his travel time to increase by 6 minutes?

Problem 5.4 (Mathcounts 2021 Chapter Invitational)**.** Jasmine estimates her maximum heart rate, in beats per minute, by subtracting her age from 220. Jasmine's heart rate during exercise should be between 50% and 85% of her maximum heart rate. If Jasmine is 20 years old, what is the highest heart rate, in beats per minute, that she should have during exercise? Express your answer to the nearest whole number.

Problem 5.5. Jerry lives 20 miles from work. If he drives to work at 50 mi/h, how fast will he need to drive home to average 60 mi/h for the round trip? Express your answer to the nearest whole number.

Problem 5.6 (Mathcounts Handbook 2018)**.** If $x^2 - y^2 = 7, x = \frac{12}{y}$ and $y < 0$, what is the value of $x^4 + y^4$?

Problem 5.7 (Mathcounts Handbook 2019)**.** New packaging for fruit snacks contains 10% less weight than the original packaging. If the new package costs 15% more than the original package, by what fraction did the unit price increase? Express your answer as a common fraction.

Problem 5.8 (Exeter Math Club Competition, 2016)**.** In a room of government officials, two-thirds of the men are standing and 8 women are standing. There are twice as many standing men as standing women and twice as many women in total as men in total. Find the total number of government officials in the room.

Problem 5.9 (Exeter Math Competition, 2017)**.** Every day, Yannick submits 8 more problems to the EMCC problem database than he did the previous day. Every day, Vinjai submits twice as many problems to the EMCC problem database as he did the previous day. If Yannick and Vinjai initially both submit one problem to the database on a Monday, on what day of the week will the total number of Vinjai's problems first exceed the total number of Yannick's problems?

Problem 5.10 (Mathcounts 2021 State Sprint)**.** In Mr. Patterson's class, the average score among students who studied for an exam was 78. The average among students who did not study was 54. The overall class average was 70. What portion of the class did not study? Express your answer as a common fraction.

Problem 5.11 (Mathcounts 2012 State Sprint)**.** Jack and Jill drove in separate cars to their favorite hill, leaving from the same place at the same time. Jill drove 20% faster than Jack and arrived half an hour earlier. How many hours did Jack drive?

Problem 5.12 (Exeter Math Club Competition, 2015)**.** At 17:10, Totoro hopped onto a train traveling from Tianjin to Urumuqi. At 14:10 that same day, a train departed Urumuqi for Tianjin, traveling at the same speed as the 17:10 train. If the duration of a one-way trip is 13 hours, then how many hours after the two trains pass each other would Totoro reach Urumuqi?

Problem 5.13 (Exeter Math Club Competition 2015)**.** In a town, each family has either one or two children. According to a recent survey, 40% of the children in the town have a sibling. What fraction of the families in the town have two children?

Problem 5.14 (Mathcounts)**.** Otto's investment portfolio consisted of shares of internet stock and copper stock. During the year, the value of his internet shares increased 10%, but the value of his copper shares decreased from $10,000 to $9,000. During the same year, the total value of his portfolio increased by 6%. What was the dollar value of his internet shares at the end of the year?

Problem 5.15 (Mathcounts Handbook 2017)**.** In 2016 the Flying Turtles finished their baseball season with a record of 95 wins and 67 losses. The Dolphins finished the season with 84 wins and 78 losses. The Flying Turtles and Dolphins played each other 19 times during the season. If the Flying Turtles had F wins against teams other than the Dolphins, and the Dolphins had D wins against teams other than the Flying Turtles, what is the value of F + D?

§6 Challenge Practice Problems

Problem 6.1 (Mathcounts 2004 National, Team)**.** The student council sold 661 T-shirts, some at $10 and some at $12. When recording the number of T-shirts they had sold at each of the two prices, they reversed the amounts. They thought they made $378 more than they really did. How many T-shirts actually were sold at $10 per shirt?

Problem 6.2 (Mathcounts 2005 National, Team)**.** A collection of nickels, dimes and pennies has an average value of 7 cents per coin. If a nickel were replaced by five pennies, the average would drop to 6 cents per coin. What is the number of dimes in the collection?

Problem 6.3 (Mathcounts Handbook 2017)**.** Edna enters a room with 1000 bottles lined up in a row left to right. One bottle contains a tasteless magic potion. All bottles to the left of the magic potion contain tasteless water. All bottles to the right of the magic potion contain a bitter poison. Edna can drink from no more than two bottles containing poison before becoming sick and being unable to drink anything else. She can take an unlimited number of drinks from any other bottle. What is the minimum number of bottles from which Edna may need to drink to ensure she can identify the bottle containing the magic potion no matter where it is in the lineup?

Problem 6.4. In Ms. Pham's third grade class, an election with two candidates was held in which the losing candidate received 31% of the vote, expressed to the nearest whole percent. (If the percent is half-way between two integers, it is rounded up. For example, 47.5% is rounded up to 48%.) Knowing that each student cast a vote for one or the other candidate, what is the minimum number of votes that could have been cast in the election?

Problem 6.5 (1991 Mathcounts National)**.** A man is running through a train tunnel. When he is $\frac{2}{5}$ of the way through, he nears a train that is approaching the tunnel from behind him at a speed of 60 mph. Whether he runs ahead or runs back, he will reach an end of the tunnel at the same time the train reaches that end. At what rate, in miles per hour, is he running? (Assume he runs at a constant rate.)

Problem 6.6 (Mathcounts 2017 State, Team Round)**.** Eddie and Missy are swimming laps in parallel lanes of a swimming pool at different constant speeds. They start simultaneously at opposite ends of the pool. They first pass each other when Eddie has swum 72 feet. Both turn back when they reach the opposite ends, and they next pass each other when Eddie is 40 feet from Missy's starting point. What is the length of a lap?

Problem 6.7 (Mathcounts 2017 State, Target)**.** Nina's two dogs, Biter and Nipper, normally eat an entire bag of dog food kibbles in 10 days. She has fed them both for 7 days when Biter breaks a tooth and stops eating the hard food. It takes Nipper 9 more days to finish the bag. What is the ratio of the number of days it would take Biter to eat the whole bag alone to the number of days it would take Nipper to eat the whole bag alone? Express your answer as a common fraction.

§7 Solution Sketches

4.1 This is a simple unit conversion question: $\frac{32 \text{ heartbeats}}{15 \text{ seconds}} * \frac{60 \text{ seconds}}{1 \text{ minute}} = \boxed{128}$ heartbeats per minute.

4.2 Since a baseball is pulled out of Box 3, it cannot be the box that contains only tennis balls. Since none of the labels are placed on the correct box, Box 1, which is labeled "Tennis Ball", cannot be the box that contains only tennis balls either. So the correct answer is Box $\boxed{2}$.

4.3 To go from 20 percent gone (or 80 percent remaining), to 20 percent remaining, Joanna needs to use 60 percent of the battery. Since using 20 percent takes 30 minutes, using 60 percent will take $\frac{30}{20} \cdot 60 = \boxed{90}$ minutes.

4.4 First convert gallon to quarts: 3 gallons * 4 quart/gallon = 12 quarts. The 12 quarts is divided into 3 + 2 + 1 = 6 parts, and 1 out of the 6 parts is white. So $\frac{12*1}{6} = \boxed{2}$ quarts of white paint.

4.5 For this problem, we can look at each person individually and then compute the difference of their travel time. For Jana, she travels the distance 2 miles for time t_1 and speed 6 mi/h. So time is distance divided by speed, $t_1 = 2/6$ hours. Zhao rides the same distance of 2 miles with time t_2 and speed 10 mi/h. $t_2 = 5 + 2/10 = 5.25$ hours Since the answer is looking for $\boxed{3}$

4.6 Let the cost of an individual pencil be p dollars. Equating the costs of the given purchases, we see that $11.52 \cdot 10 + 8 \cdot p = 1160p$. Solving, we find that $p = 0.10$. Thus her total amount of money is $1160 \cdot 0.10 = \boxed{116}$ dollars.

5.1 Each 7-day week (Monday through Sunday), Eli adds 1 + 2 + 3 + 4 + 5 + 6 + 7 = \$28. In 3 weeks, Eli would have saved 3 × 28 = \$84. To get a total of \$93, Eli must have saved another 93 84 = \$9. So, we need to find a subset of consecutive daily amounts that have a sum of 9. We see that 2 + 3 + 4 = 9. One way to think about it is that Eli started by inserting \$2. In this case, he would have 3 full weeks of saving (Tuesday through Monday) followed by 3 days of saving (Tuesday, Wednesday, Thursday). So, the first day Eli inserted money into his piggy bank was a $\boxed{\text{Tuesday}}$.

5.2 Let x steps per second be the speed of the elevator. Then, since Justin takes 72 seconds, the elevator moves $72x$ steps in that time. Plus, he walks down additional 30, covering a total of $72x + 30$ steps. Similarly, Shelby covers a total of $60x + 40$. Since the total steps are the same, we have $72x + 30 = 60x + 40, x = \frac{10}{12}$. Then total steps are $72 \cdot \frac{10}{12} + 30 = \boxed{90}$ steps. This is the number of steps they will have to walk down if the escalator was stopped.

5.3 Distance = time · speed, the trick in this question is to pay attention to the unit and to not mess up the conversion between hour and minutes. 6 minute is 0.1 hours. D = 60 $\frac{mi}{hr}\cdot$ t = 75 $\frac{mi}{hr} \cdot (t - 0.1) = s \cdot (t + 0.1)$. $60t = 75t - 7.5$. So the original time it took him to travel is $15t = 7.5, t = 0.5$, and the distance he needs to travel is $D = 0.5 \cdot 60 = 30mi$. Finally to find the new average speed when travel time is $t + 1 = 0.6$, $30mi = 0.6hr \cdot s$, $s = \boxed{50}$ mi/h

5.4 We know that $85\% = 0.85$, so $0.85 \times (220 - 20) = 0.85 \times 200 = 85 \times 2 = \boxed{170}$ beats per minute.

5.5 Average speed = total distance / total time. The total distance is 20 miles back and forth so $20 \cdot 2 = 40$ miles. The total time is time to + time from. Time to is distance divided by speed or 0.4 hours, and time from is unknown. So 60 mi/h = $\frac{40}{0.4+t_2}, t_2 = 4/6 - 4/10 = 4/15$. Therefore, return speed is $\frac{50}{4/15} = 75$ mi/h.

Alternatively, we can use

$$\begin{aligned} v_{average} &= \frac{d_{to} + d_{back}}{t_{to} + t_{back}}, \\ &= \frac{d + d}{\frac{d}{v_{to}} + \frac{d}{v_{back}}}, \\ &= \frac{2}{\frac{1}{v_{to}} + \frac{1}{v_{back}}}, \text{ and} \\ \frac{2}{v_{average}} &= \frac{1}{v_{to}} + \frac{1}{v_{back}}. \end{aligned}$$

Therefore,

$$v_{back} = \frac{1}{\frac{2}{v_{average}} - \frac{1}{v_{to}}} = \frac{1}{\frac{2}{60} - \frac{1}{50}},$$

simplifying to $v_{to} = \boxed{75}$ mi/h.

5.6 We can start by squaring the first equation:

$$\begin{aligned} (x^2 - y^2)^2 &= 7^2, \\ x^4 + y^4 - 2x^2y^2 &= 49, \\ x^4 + y^4 &= 49 + 2(xy)^2. \end{aligned}$$

From the second given equation we have $xy = 12$, plugging this into above we get $x^4 + y^4 = 49 + 2 \cdot 144 = \boxed{337}$.

5.7 The unit price is the ratio cost/unit weight. Let the original unit price be c/w. With a 10% reduction in weight and a 15% increase in cost, the new unit price is 1.15c/0.9w = 1.15/0.9 ×c/w = 115/90 × c/w = 23/18 × c/w. The increase in unit price is 23/181 = 23/1818/18 = $\boxed{5/18}$.

5.8 There are $8 \cdot 2 = 16$ men standing, and thus $\frac{16}{2/3} = 24$ men in the room. Then we deduce there are $24 \cdot 2 = 48$ women, so the room contains $24 + 48 = \boxed{72}$ government officials in total.

5.9 Let the Monday when both Yannick and Vinjai submit 1 problem be day 1, then: On day 2, Yannick submits 9 problems, Vinjai submits 2 problems. On day 3, Yannick submits 17 problems, Vinjai submits 4 problems. On day 4, Yannick submits 25 problems, Vinjai submits 8 problems. On day 5, Yannick submits 33 problems, Vinjai submits 16 problems. On day 6, Yannick submits 41 problems, Vinjai submits 32 problems. On day 7, Yannick submits 49 problems, Vinjai submits 64 problems. At this point Yannick have submitted $((1 + 49)7)/2 = 175$ problems, and Vinjai have submitted $27 - 1 = 127$ problems. On day 8, Yannick submits 57 problems, Vinjai submits 128 problems. At this point Yannick have submitted 175 + 57 = 232 problems, and Vinjai have submited 127 + 128 = 255 problems. Therefore, day 8 is the first day that Vinjai have submitted more problems than Yannick in total. Since day 1 is Monday, day 8 is $\boxed{\text{Monday}}$ as well.

5.10 The value 70 is $\frac{70-54}{78-54} = \frac{2}{3}$ of the way from 54 to 78. Thus, $\frac{2}{3}$ of the students are in the group whose average score was 78, and the group whose average score was 54 has the remaining $\boxed{\frac{1}{3}}$ of the students who did not study.

5.11 Let x denote the time that Jack drove in hours, and s be the distance. Then Jack's speed is $\dfrac{s}{x}$. Now, Jill's speed is 20% faster, so we get

$$\begin{aligned}
\frac{s}{1.2\frac{s}{x}} &= x - 0.5, \\
\frac{x}{1.2} &= x - 0.5, \\
x &= 1.2x - 0.6, \\
0.2x &= 0.6, \\
x &= \boxed{3} \text{ hours.}
\end{aligned}$$

5.12 Let t be the number of hours that Totoro has traveled when the two trains pass each other. Since the train from Urumuqi departs 3 hours earlier, it has been traveling for $t + 3$ hours. We are also given that the one way trip time is 13 hours, so $t + (t + 3) = 13 \implies t = 5$. Thus Totoro has $13\ \ 5 = \boxed{8}$ hours left on his train.

5.13 Suppose that there are n children in the town. Then $0.4n$ children are from families with two children and $0.6n$ children are from families with one child. Therefore, there are $\frac{0.4n}{2} = 0.2n$ families with two children and $0.6n$ families with one child, for a total $0.2n + 0.6n = 0.8n$ families. Thus the desired fraction is $\frac{0.2n}{0.8n} = \boxed{\dfrac{1}{4}}$

5.14 Let I be the initial value of Otto's internet stock, in dollars. At the beginning of the year, the total value of his portfolio is $I + 10{,}000$ dollars. At the end of the year, the total value of his portfolio is $1.1I + 9{,}000$ dollars. Since increasing by 6% is equivalent to multiplying by 1.06, we have

$$1.06(I + 10{,}000) = 1.1I + 9{,}000.$$

Distributing and collecting terms, we find $I = 40{,}000$. At the end of the year, his internet stock is worth $40{,}000 \cdot 1.1 = \boxed{44{,}000}$ dollars.

5.15 If we add the wins of both Flying Turtles and Dolphins, we have to then subtract the 19 wins for the games they played against each other regardless of who won those games. The value of F + D, then, is 95 + 84 – 19 = $\boxed{160}$.

6.1 Let n be the number of T-shirts that were sold at \$10 each. Since $661 - n$ T-shirts were sold at twelve dollars each, they made $10n + 12(661 - n)$ dollars. Switching the numbers of ten and twelve dollar T-shirts sold, they thought they made $12n + 10(661 - n)$ dollars. We solve

$$\begin{aligned}
12n + 10(661 - n) &= 10n + 12(661 - n) + 378 \\
2n + 10 \cdot 661 &= 12 \cdot 661 - 2n + 378 \\
4n &= 2 \cdot 661 + 378 \\
n &= \boxed{425}.
\end{aligned}$$

6.2 Let M be the total number of cents in the collection, and let N be the total number of coins in the collection. Since the average is 7 cents per coin, $\frac{M}{N} = 7$, or $M = 7N$. If a nickel is replaced by five pennies, the amount of money doesn't change, but the number of coins increases by 4. So $\frac{M}{N+4} = 6$. Substituting $M = 7N$ we get $\frac{7N}{N+4} = 6$, and solving gives $N = 24$. Then $M = 7(24) = 168$.

Since we have a total of 168 cents in pennies, nickels, and dimes, the number of pennies must be 3, 8, 13, 18, or so on (since nickels and dimes contribute multiples of 5 cents). Suppose we have 8 or more pennies. We have a total of 24 coins, so there are at most 16 nickels and dimes. At least one of these is a nickel, since the problem says we can exchange one nickel for five pennies. So, the most money we could have is 8 pennies, 1 nickel, and 15 dimes, a total of $8 + 5 + 15(10) = 163$ cents. But we know we have more than this: we have 168 cents. This is a contradiction, so we must have less than 8 pennies. Thus, we have 3 pennies. This means we have total of 21 nickels and dimes, and these nickels and dimes are worth 165 cents. Letting n and d be the number of nickels and dimes we have, respectively, we get $n + d = 21$ and $5n + 10d = 165$. Dividing the second equation by 5, we have $n + 2d = 33$, and subtracting the first equation from this gives $d = \boxed{12}$.

6.3 If Edna starts with the first bottle on the left and samples every bottle, she will eventually taste a bitter poison and know that the previous bottle contains the magic potion. If Edna samples every even-numbered bottle, she will have to check the odd-numbered bottle right before the first bitter poison she gets. If the odd bottle is tasteless, then it contains the magic potion; if it's bitter, then she will become sick and unable to drink anything else, but she will have succeeded in being able to identify the bottle that contains the magic potion. Likewise, if Edna samples every third bottle, she can go back after she gets the first bitter bottle, but she has to go back to check the bottle after the last tasteless bottle. If we continue with this line of reasoning, we realize that Edna can skip-count by any number she wants. As soon as she gets to a bitter bottle, she has to go back to the bottle that comes right after the last tasteless bottle and check every bottle until she gets to another bitter one. At this point, some students will arrive at the idea of using the square root of 1000, which is a little more than 31. Edna could set out to check multiples of 31 until she comes to a bitter bottle. Edna would check bottles 31, 62, 93, ..., up until 992, which is 32×31. In the worst-case scenario, bottle 991 would contain the magic potion. Edna would find bottle 992 to be the first bitter one she samples and she would have to go back to check bottles 962 through 991, which would all be tasteless. She would then know that bottle 991 contains the magic potion, but she would have sampled $32 + 30 = 62$ bottles. (The total is the same if she counts by 32s instead of 31s.) This is pretty good, but it turns out that Edna can do better. She can take bigger steps at the beginning and reduce the size of her jump as she goes. If she reduces the jump by 1 each time, then what we want is a triangular number that is close to 1000. The 45th triangular number is $4562 = 1035$, so Edna should first sample bottle 45, then bottle $45 + 44 = 89$, then bottle $89 + 43 = 132$, etc. At any point, when she samples a bitter bottle, she will have to go back and check the bottles in between, and the total number of bottles sampled is always $\boxed{45}$ bottles..

6.4 Solution: Let n be the total number of votes, and let k be the number of votes in favor of the losing candidate. So the loser won $\frac{k}{n}$ of the vote, or $100\left(\frac{k}{n}\right) = \frac{100k}{n}$ percent of the vote. Since $\frac{100k}{n}$ rounded to the nearest whole number is 31, we must have

$$30.5 \le \frac{100k}{n} < 31.5 \Rightarrow \frac{61}{2} < \frac{100k}{n} \le \frac{63}{2}.$$

Taking the reciprocal of everything (in doing this we have to flip the inequality), and then multiplying by $100k$, we get

$$\frac{200}{63}k \le n < \frac{200}{61}k.$$

So we want the smallest integer n such that $\frac{200}{63}k \le n < \frac{200}{61}k$ for some positive integer k. If $k = 1$, this becomes $3.17\ldots \le n < 3.28\ldots$, which no integer n satisfies. If $k = 2$, this becomes $6.34\ldots \le n < 6.56\ldots$, which again no integer n satisfies. We can check that $k = 3$ doesn't work either. But for $k = 4$, we have $12.69\ldots \le n < 13.11\ldots$, which $n = 13$ satisfies. So $n = \boxed{13}$ is the minimum number of votes that could be cast.

6.5 When the man went back $\frac{2}{5}$ of the length of the tunnel, the train is at the beginning of the tunnel. Same thing happened if the men went ahead $\frac{2}{5}$ of the length of the tunnel. The time it took the train to run through the length of the tunnel, the man could only run $\frac{1}{5}$ of the same distance. So the speed of the man is $\frac{1}{5}$ of the train's speed, which is $\frac{1}{5} \cdot 60 = \boxed{12}$ mph.

6.6 Let L be the length of the pool. When they first cross Eddie has swam 72 feet and Missy has swam $L - 72$. When they cross the second time, Eddy has swam a lap plus 40 feet so $L + 40$, where as Missy is short of completing second lap by 40 so $2L - 40$. Since they are swimming at constant speeds, ratio of the distances are the same, therefore:

$$\begin{aligned}
\frac{72}{L-72} &= \frac{L+40}{2L-40}, \\
72(2L-40) &= (L+40)(L-72), \\
144L - 40 \cdot 72 &= L^2 - 32L - 40 \cdot 72, \\
L^2 - 176L &= 0, \\
L(L-176) &= 0.
\end{aligned}$$

Since, $L = 0$ doesn't make sense, $L = \boxed{176}$ feet.

6.7 Let b and n be the rate at which Biter and Nipper eat the kibbles (in bags per day) respectively. Then it takes Biter $\frac{1}{b}$ days to eat the whole bag alone and it takses Nipper $\frac{1}{n}$. Therefore, the asked for ratio is $\frac{\frac{1}{b}}{\frac{1}{n}} = \frac{n}{b}$.

Now, when they are working together (rates added) they eat 1 bag in 10 days, so

$$1 = 10(b+n).$$

Also, when they work together for 7 days and Nibber works alone for 9 days, they eat one bag, so

$$1 = 7(n+b) + 9n.$$

Equating the two equations, we get

$$\begin{aligned}
10n + 10b &= 7n + 7b + 9n, \\
3b &= 6n, \\
\frac{n}{b} &= \frac{3}{6} = \boxed{\frac{1}{2}}.
\end{aligned}$$