Handout 1 - Word Problems

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§1 Introduction

§1.1 Unit Conversions

Remark 1.1. Pay attention to the units in the problem. In many Mathcounts problem, you will need to convert between units, for example from feet to inches or hours to seconds.

Some common conversions to remember are:

- 1 feet = 12 inches
- 1 yard = 3 feet
- 1 mile = 1,760 yards = 5,280 feet

§1.2 Rate of Work

Definition 1.2. If a person can do r units of work in unit time, we say that their rate of work is r.

If person a, b, c can do r_a, r_b, r_c units of work in unit time, then together they can do $r_a + r_b + r_c$ units of work in unit time. That is

$$r_{combined} = r_a + r_b + r_c.$$

Sometimes, instead of the rate of work, we are given the time t_a, t_b, t_c taken by a, b, and c to complete x units of work. Then the rate of work for each of them is $r_a = \frac{x}{t_a}, r_b = \frac{x}{t_b}$, and $r_c = \frac{x}{t_c}$,. And the combined rate of work is $r_{combined} = \frac{x}{t_{combined}}$. Plugging into the previous formula and canceling x, we get

$$\frac{1}{t_{combined}} = \frac{1}{t_a} + \frac{1}{t_b} + \frac{1}{t_c}.$$

Note 1.3. We have used 3 people above for illustration purposes; the same applies for 2 or more people.

§1.3 Speed and Distance

Definition 1.4. Speed is distance traveled divided by time taken. Average speed is the total distance traveled divided by the total time taken.

One of the implications of the above definition is that average speed is not a simple average of speeds. In particular, if the speed is v_a for t_a amount of time and v_b for t_b amount of time, then the total distance traveled is $v_a t_a + v_b t_b$ and therefore,

$$v_{average} = \frac{v_a t_a + v_b t_b}{t_a + t_b} = .$$

That is, average speed is a weighted average of the individual speeds.

§1.4 Relative Velocity or Effective Speed

You may have learned the concept of relative velocity in Physics; if you have not, you will. In some Mathcounts problem, this concept is useful to solve the problems quick. A simplified version of the concept is presented below. v_a, v_b are speed for A and B, and s_t is the distance between them at time t.

- Case 1: A and B moving towards each other. The effective speed is $v_a + v_b$ and they will meet after $\frac{s_0}{v_a+v_b}$.
- Case 2: A is moving towards B, and B is moving away from A. The effective speed is $v_a - v_b$ and they will meet after $\frac{s_0}{v_a-v_b}$, assuming $v_a > v_b$.
- Case 3: A and B moving away from each other. The effective speed is $v_a + v_b$ and the distance between them after time t will be $s_t = s_0 + (v_a + v_b) \cdot t$.

§2 Practice Problems

§2.1 Beginner Problems

Problem 2.1. *(Mathcounts)* Nan can mow $\frac{1}{5}$ of her lawn in an hour. Reagan can mow $\frac{1}{6}$ of the same lawn in an hour. If they have two mowers and work together what fractional part of the lawn can they mow in an hour?

Problem 2.2. *(Mathcounts)* Normally, the hose in Elena's garden will fill her small pool in 15 minutes. However, a leak in the hose allows $\frac{1}{3}$ of the water flowing through the hose to spill into the flower bed. How many minutes will it take to fill the pool? Express your answer as a decimal to the nearest tenth.

Problem 2.3. *(Mathcounts)* A candle that burns at a uniform rate was 11 inches tall after burning for 4 hours and 8 inches tall after burning for a total of 6 hours. How many inches tall was the candle before it was lit?

Problem 2.4. *(Mathcounts)* Ellie and Emma live 1.04 miles from each other. They decide to meet by walking toward each other, Ellie at 2.4 mi/h and Emma at 2.8 mi/h. If they both leave at 8:00 a.m., at what time in the morning will they meet?

§2.2 Intermediate Problems

Problem 2.5. *(Mandelbrot National 2009)* The tortoise and the hare are having another race. The hare hops ahead at 10 feet per second, while the tortoise advances only 1 foot every 4 seconds. The course is 500 feet long. The tortoise plods along steadily throughout the race, but the hare pauses for a half-hour nap partway through. By how many minutes does the victor win? Express your answer as a common fraction.

Problem 2.6. *(Mathcounts)* The tread depth on an XL-75T radial tire is $\frac{15}{16}''$. The XL-75T tire loses $\frac{1}{16}''$ of its tread every 4500 miles it is driven. Engineers have suggested that $\frac{1}{8}''$ is the minimum amount of tread depth needed for safe driving. How many miles can be driven on these tires before they are unsafe according to the engineers.

Problem 2.7. If Sally drives to her work at 40 miles per hour, she will be 15 minutes late. If she drives to her work at 60 miles per hour, she will be 15 minutes early, what speed will she arrive at her work on time? How far away is her office ?

Problem 2.8. *(Mathcounts)* Rashida is mowing a rectangular lawn that measures $100'$ x $80'$. She mows by making laps around the rectangle and her lawn mower mows a path that is $18''$ wide. What is the least number of laps necessary for Rashida to mow the lawn?

Problem 2.9. *(Mathcounts)* A firefighter uses the formula $S = 0.5N + 26$ to determine the maximum number of feet of horizontal distance (S) water will travel leaving a hose with a $\frac{3}{4}$-inch nozzle diameter. N is the nozzle pressure in pounds. If the hose has a nozzle diameter greater than $\frac{3}{4}$-inch, 5 feet is added to the distance for every $\frac{1}{8}$-inch increase in nozzle diameter. For a $1\frac{1}{8}$-inch diameter nozzle with a nozzle pressure of 80 pounds, what is the maximum horizontal distance in feet that the water will travel?

Problem 2.10. *(Mathcounts)* Daniel began painting a room at 9:00 a.m. Yeong, who can paint twice as fast as Daniel, started helping Daniel at 9:20 a.m., and they worked together until the room was fully painted at 10:00 a.m. What fraction of the room had been painted by 9:30 a.m.? Express your answer as a common fraction.

§2.3 Challenge Problems

Problem 2.11. *(1991 Mathcounts National #28)* A man is running through a train tunnel. When he is $\frac{2}{5}$ of the way through, he nears a train that is approaching the tunnel from behind him at a speed of 60 mph. Whether he runs ahead or runs back, he will reach an end of the tunnel at the same time the train reaches that end. At what rate, in miles per hour, is he running? (Assume he runs at a constant rate.)

Problem 2.12. *(Mathcounts 2017 State, Team Round, #9)* Eddie and Missy are swimming laps in parallel lanes of a swimming pool at different constant speeds. They start simultaneously at opposite ends of the pool. They first pass each other when Eddie has swum 72 feet. Both turn back when they reach the opposite ends, and they next pass each other when Eddie is 40 feet from Missy's starting point. What is the length of a lap?

Problem 2.13. *(Mathcounts 2017 State, Target #2)* Nina's two dogs, Biter and Nipper, normally eat an entire bag of dog food kibbles in 10 days. She has fed them both for 7 days when Biter breaks a tooth and stops eating the hard food. It takes Nipper 9 more days to finish the bag. What is the ratio of the number of days it would take Biter to eat the whole bag alone to the number of days it would take Nipper to eat the whole bag alone? Express your answer as a common fraction.

§3 Solution Sketches

2.1 When they work together, their rate of work combines and adds up to $\frac{1}{5} + \frac{1}{6} = \boxed{\dfrac{11}{30}}$ of lawn in an hour.

2.2 Since $\frac{1}{3}$ of the water leaks, only $\frac{2}{3}$ of the water makes it to the pool, and so the rate of filling is $\frac{2}{3}$ the regular rate and she will take $\frac{3}{2} \cdot 15 = \boxed{22.5}$ minutes.

2.3 In 2 hours the candle burnt $11 - 8 = 3$ inches, so in 4 hours it would have burnt $\frac{4}{2} \cdot 3 = 6$ inches. So it was originally, $11 + 6 = \boxed{17}$ inches tall.

2.4 The two are moving towards each other at a combined speed of $2.4 + 2.8 = 5.2$ mi/h, and therefore will take $\frac{1.04}{5.2} = \frac{1}{5}$ h, that is 12 minutes. So they will meet at $\boxed{8:12}$ a.m.

2.5 The tortoise walks 1 foot every 4 seconds, so he will need 2000 seconds to finish the 500 foot course. The hare will only take $500/10 = 50$ seconds to cover the same distance, but we must add in $60 \cdot 30 = 1800$ seconds for her nap. She still wins though, by $2000 \quad 1850 = 150$ seconds, which translates to $\boxed{\dfrac{5}{2}}$ minutes.

2.6 $\boxed{58500}$ miles.

2.7 Let t be the time it would take her to reach on time. Set up the given conditions in terms of t, that is $(t - \frac{1}{4}) \cdot 60 = (t + \frac{1}{4}) \cdot 40$ and solve for t. Alternatively, realize that the average speed is the harmonic mean. Answer: $\boxed{60}$ miles and $\boxed{48}$ mph.

2.8 Two things to keep in mind: a. $1' = 12''$, and b. a lap means going to the end and coming back. Therefore our answer is $\frac{80 \cdot 12}{18 \cdot 2} = \boxed{27}$ laps.

2.9 First, for a regular nozzle the distance given by the formula is $S = 40 + 26 = 66$. Next, the diameter is $\frac{9}{8} - \frac{6}{8} = \frac{3}{8} = 3 \cdot \frac{1}{8}$ larger and so it adds $3 \cdot 5 = 15$. So total distance is $66 + 15 = \boxed{81}$ feet.

2.10 Let x be Yeong's rate of work. Then the total work done by 10:00 a.m. is $x \cdot 20 + 3x \cdot 40$ and the work done till 9:30 is $x \cdot 20 + 3x \cdot 10$. So the fraction of the room painted at 9:30 is $\frac{x \cdot 20 + 3x \cdot 40}{x \cdot 20 + 3x \cdot 10} = \boxed{\dfrac{5}{14}}$

2.11 When the man went back $\frac{2}{5}$ of the length of the tunnel, the train is at the beginning of the tunnel. Same thing happened if the men went ahead $\frac{2}{5}$ of the length of the tunnel. The time it took the train to run through the length of the tunnel, the man could only run $\frac{1}{5}$ of the same distance. So the speed of the man is $\frac{1}{5}$ of the train's speed, which is $\frac{1}{5} \cdot 60 = \boxed{12}$ mph.

2.12 Let L be the length of the pool. When they first cross Eddie has swam 72 feet and Missy has swam $L - 72$. When they cross the second time, Eddy has swam a lap plus 40 feet so $L + 40$, where as Missy is short of completing second lap by 40 so $2L - 40$. Since they are swimming at constant speeds, ratio of the distances are the same, therefore:

$$\begin{aligned} \frac{72}{L - 72} &= \frac{L + 40}{2L - 40}, \\ 72(2L - 40) &= (L + 40)(L - 72), \\ 144L - 40 \cdot 72 &= L^2 - 32L - 40 \cdot 72, \\ L^2 - 176L &= 0, \\ L(L - 176) &= 0. \end{aligned}$$

Since, $L = 0$ doesn't make sense, $L = \boxed{176}$ feet.

2.13 Let b and n be the rate at which Biter and Nipper eat the kibbles (in bags per day) respectively. Then it takes Biter $\frac{1}{b}$ days to eat the whole bag alone and it takses Nipper $\frac{1}{n}$. Therefore, the asked for ratio is $\frac{\frac{1}{b}}{\frac{1}{n}} = \dfrac{n}{b}$.

Now, when they are working together (rates added) they eat 1 bag in 10 days, so

$$1 = 10(b + n).$$

Also, when they work together for 7 days and Nibber works alone for 9 days, they eat one bag, so

$$1 = 7(n + b) + 9n.$$

Equating the two equations, we get

$$\begin{aligned} 10n + 10b &= 7n + 7b + 9n, \\ 3b &= 6n, \\ \frac{n}{b} &= \frac{3}{6} = \boxed{\frac{1}{2}}. \end{aligned}$$