Handout 6 - Mixed Practice

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§1 Beginner Problems

Problem 1.1 (Mathcounts 2021 School, Team #2)**.** A container is one-half full. After 20 ounces are poured out, the container is one-third full. How many ounces are still in the container?

Problem 1.2 (Mathcounts 2021 School, Team #10)**.** What is the remainder when 2021^{2021} is divided by 7?

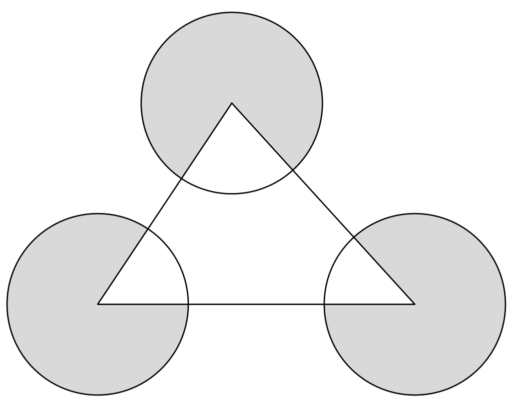
Problem 1.3 (Mathcounts)**.** Four packages are delivered to four houses, one to each house. If these packages are randomly delivered, what is the probability that exactly two of them are delivered to the correct houses? Express your answer as a common fraction.

Problem 1.4 (Mathcounts Handbook 2020-21, #236)**.** How many subsets of one or more elements of the set 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 contain the element 1 or the element 2 or both?

Problem 1.5 (Mathcounts)**.** Right $\triangle ABC$ with legs $AB = 3$ cm and $CB = 4$ cm is rotated about one of its legs. What is the greatest possible number of cubic centimeters in the volume of the resulting solid? Express your answer to the nearest whole number.

§2 Intermediate Problems

Problem 2.1 (2018 Purple Comet, HS #2)**.** A triangle with side lengths 16, 18, and 21 has a circle with radius 6 centered at each vertex. Find n so that the total area inside the three circles but outside of the triangle is $n\pi$.

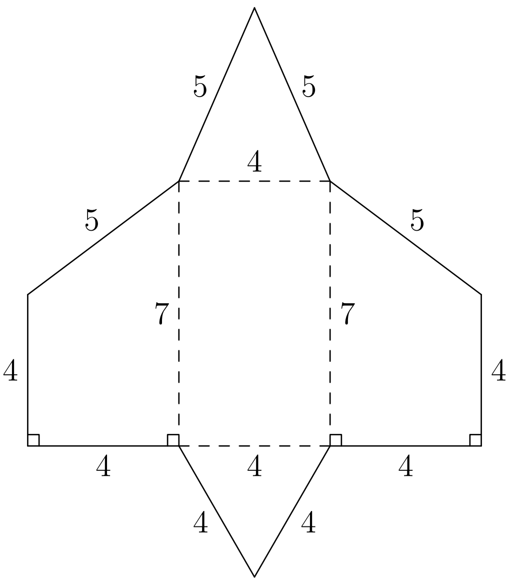


Problem 2.2 (Mathcounts 2017 State, Sprint #19)**.** A pole of radius 1 inch and height h inches is perpendicular to the ground. A snake is coiled around the pole in a spiral pattern, making a 60-degree angle with the ground. If the snake is 9π inches long and reaches from the ground to the top of the pole, then h can be written in the form $a\pi\sqrt{b}$, where a is a rational number and b is a positive integer that is not divisible by the square of any prime. What is the value of $a + b$? Express your answer as a common fraction.

Problem 2.3 (2017 Mathcounts National, Sprint #9)**.** Matt left home traveling towards the mountains to go camping. His brother Mark started along the same route 2.1 hours later traveling at a rate 35 mi/h faster trying to catch up to Matt. What was Matt's average speed if it took Mark 1.2 hours to catch up to him.

Problem 2.4 (Mathcounts 2017 State, Sprint #29)**.** Lines AB and DC are parallel, and transversals AC and BD intersect at a point X between the two lines so that $\frac{AX}{CX} = \frac{5}{7}$. Points P and Q lie on segments AB and DC, respectively. The segment PQ intersects transversals BD and AC at points M and N, respectively, so that $PM = MN = NQ$. What is the ratio $\frac{AP}{BP}$? Express your answer as a common fraction.

Problem 2.5 (Mathcounts 2015 State Target #6)**.** The shape below can be folded along the dashed lines and taped together along the edges to form a three-dimensional polyhedron. All lengths in the diagram are given in inches. What is the volume of the resulting polyhedron? Express your answer in simplest radical form.



Problem 2.6 (Mathcounts 2015 State Team #3)**.** Equilateral triangle ABC with side-length 12 cm is inscribed in a circle. What is the area of the largest equilateral triangle that can be drawn with two vertices on segment AB and the third vertex on minor arc AB of the circle? Express your answer in simplest radical form.

Problem 2.7 (Mathcounts 2018 National, Sprint #6)**.** If the three lines $3x + y = -1, x + 3y = -11$ and $ax - 2y = 3$ all intersect at one point, what is the value of a?

Problem 2.8 (Mathcounts 2018 National, Sprint #9)**.** The mean, median and unique mode of a list of seven positive integers are all equal to 10. What is the greatest possible range of the list?

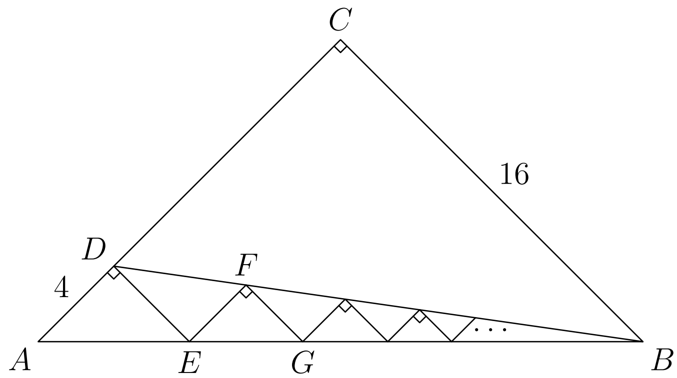
Problem 2.9 (Mathcounts 2018 National, Sprint #13)**.** What common fraction is equivalent to the sum shown?

$$\frac{1}{2018+1} + \frac{1}{2018^2+2018+1} + \frac{1}{2018^4+2(2018^3)+2(2018^2)+2018}$$

Problem 2.10 (Mathcounts 2018 National, Sprint #21)**.** What is the greatest integer that is less than or equal to $\dfrac{3^{19}+2^{19}}{3^{15}+2^{15}}$?

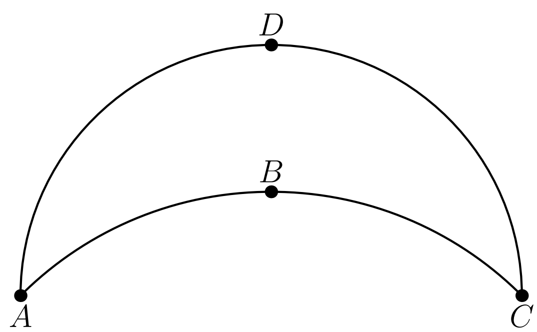
Problem 2.11 (Mathcounts 2018 National, Sprint #23)**.** Xena and Yolanda toss an unfair coin that lands heads up 75% of the time. If the coin lands heads up, Xena gets a point. Otherwise, Yolanda gets a point. The game ends when one player has a two-point lead over the other, and the player with the two-point lead is the winner. What is the probability that Xena wins the game? Express your answer as a common fraction.

Problem 2.12 (Mathcounts 2018 National, Sprint #24)**.** The figure shows isosceles right triangle ABC with $AC = BC = 16$ units. Triangle ABD is constructed within triangle ABC with vertex D on side AC and $AD = 4$ units. Point E is drawn on side AB to form isosceles right triangle ADE. A series of isosceles right triangles is constructed within triangle ABD, such that the hypotenuse of each lies on side AB, the 90-degree vertex of each lies on segment BD, and each shares a vertex with the preceding triangle in the series, as shown. What is the combined area of the series of isosceles right triangles within triangle ABD? Express your answer as a common fraction.

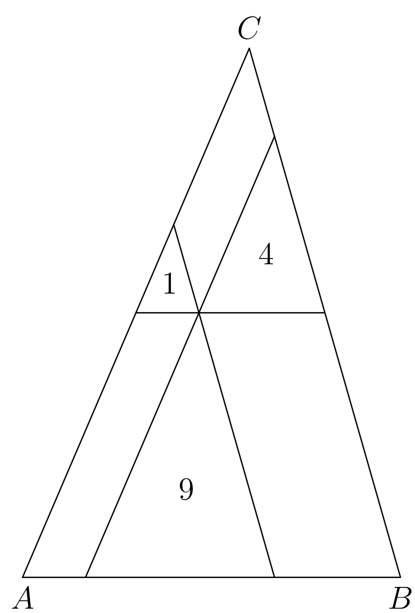


Problem 2.13 (Mathcounts)**.** A circle is tangent to the positive $x-$axis at $x = 3$. It passes through the distinct points $(6, 6)$ and (p, p) where p is not 6. What is the value of p? Express your answer as a common fraction.

Problem 2.14 (2009 Purple Comet, HS #7)**.** The figure $ABCD$ is bounded by a semicircle CDA and a quarter circle ABC. Given that the distance from A to C is 18, find the area of the figure.



Problem 2.15 (Purple Comet)**.** Three lines are drawn parallel to each of the sides of triangle ABC so that they intersect in the interior of $\triangle ABC$. The resulting three smaller triangles have areas 1, 4, and 9. Find the area of triangle ABC.



Problem 2.16 (2017 AMC 12)**.** An ice-cream novelty item consists of a cup in the shape of a 4-inch-tall frustum of a right circular cone, with a 2-inch-diameter base at the bottom and a 4-inch-diameter base at the top, packed solid with ice cream, together with a solid cone of ice cream of height 4 inches, whose base, at the bottom, is the top base of the frustum. What is the total volume of the ice cream, in cubic inches? Express your answer in terms of π.

§3 Solution Sketches

1.1 Let $6x$ be the total capacity. Then $3x - 20 = 2x$, or $x = 20$. Remaining in the container is $2c = 2 \cdot 20 = \boxed{40}$.

1.2 We can compute the remainders when the initial powers of 2021 are divided by 7.

$$\begin{aligned}
2021^1 &\equiv 5 \pmod 7,\\
2021^2 &\equiv 5 \cdot 5 \equiv 4 \pmod 7,\\
2021^3 &\equiv 4 \cdot 5 \equiv 6 \pmod 7,\\
2021^4 &\equiv 6 \cdot 5 \equiv 2 \pmod 7,\\
2021^5 &\equiv 2 \cdot 5 \equiv 3 \pmod 7,\\
2021^6 &\equiv 3 \cdot 5 \equiv 1 \pmod 7.
\end{aligned}$$

These will then repeat in cycles of 6. Since $2021 \equiv 5 \pmod 6$, we need the 5th element in the cycle, which is $\boxed{3}$.

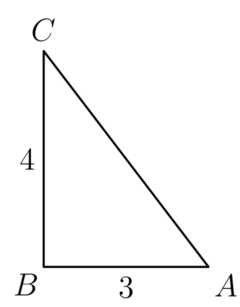
1.3 Since there are 4 houses and 4 packages, we can choose $\binom{4}{2} = 6$ pairs of houses to be the pair that will receive the correct package. In that case, the other two houses must have one another's package. The probability of this occurring for any arrangement is $\frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2}$, as the first fraction represents the probability of a given house getting the correct package, and the second fraction the subsequent probability that the other given house gets the correct package, and the final fraction the probability that the last two houses have each other's packages. So, the probability is $6 \cdot \frac{1}{2 \cdot 3 \cdot 4} = \boxed{\dfrac{1}{4}}$.

1.4 We solve this with complementary counting: $2^{10} - 1 - (2^8 - 1) = 2^8(2^2 - 1) = \boxed{768}$.

1.5 Rotating $\triangle ABC$ around leg $\overline{CB}$ produces a cone with radius 3 cm, height 4 cm, and volume

$$\frac{1}{3}\pi(3^2)(4) = 12\pi$$

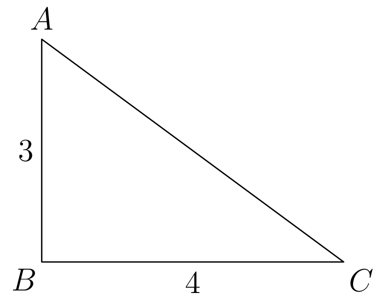
cubic cm.



Rotating $\triangle ABC$ around leg $\overline{AB}$ produces a cone with radius 4 cm, height 3 cm, and volume

$$\frac{1}{3}\pi(4^2)(3) = 16\pi$$

cubic cm.



$16\pi \approx 50.27$ cubic cm is the greater volume. To the nearest whole number, this value is $\boxed{50}$ cubic cm.

2.1 The area of the 3 circles is $3 * 6^2 * \pi = 108\pi$ We subtract $\frac{1}{2}$ a circle because the triangle has $180°$ which creates $\frac{1}{2}$ of a circle. Thus our answer is $2.5 * 6^2 * \pi = \boxed{90}\pi$

2.2 If we make a vertical cut down the poll and unroll it, we get a rectangle with width 4π and height h. The snake is making multiple 30-60-90 triangles in this rectangle, once for each time it is coiled around. If we look at a single coil, that is a single triangle, the length of snake used in that one coil is $2 \cdot 2\pi = 4\pi$; therefore, the snake makes $\frac{9\pi}{4\pi} = \frac{9}{4}$ coils. For a single coil, the height is $\frac{\sqrt{3}}{2}4\pi = 2\sqrt{3}\pi$. Then the total height is $\frac{9}{4} \cdot 2\sqrt{3}\pi = \frac{9}{2} \cdot \sqrt{3}\pi$.

Therefore, $a + b = \frac{9}{2} + 3 = \boxed{\dfrac{15}{2}}$.

2.3 Let v be Matt's average speed. Then when Mark starts, Matt is $2.1v$ miles ahead. Mark makes up this difference with a relative speed of 35 mi/h. Therefore, $\frac{2.1v}{35} = 1.2$; and, $v = \frac{1.2 \cdot 35}{2.1} = \boxed{20}$ mi/h.

2.4 We have $\frac{AB}{CD} = \frac{AX}{CX} = \frac{5}{7}$. Now, $\triangle PBM \sim \triangle QDM$, so $\frac{BP}{DQ} = \frac{PM}{QM} = \frac{1}{2}$; therefore, $DQ = 2BP$. Next, $\triangle APN \sim CQN$, so $\frac{AP}{CQ} = \frac{PN}{QN} = 2$; therefore, $CQ = \frac{1}{2}AP$.

This gives, $CD = DQ + CQ = \frac{1}{2}AP + 2BP$. But, $CD = \frac{7}{5}AB = \frac{7}{5}(AP + BP)$. Therefore,

$$\begin{aligned}
\frac{7}{5}(AP + BP) &= \frac{1}{2}AP + 2BP, \\
(\frac{7}{5} - \frac{1}{2})AP &= (2 - \frac{7}{5})BP, \\
\frac{9}{10}AP &= \frac{3}{5}BP, \\
3AP &= 2BP, \\
\frac{AP}{BP} &= \boxed{\frac{2}{3}}.
\end{aligned}$$

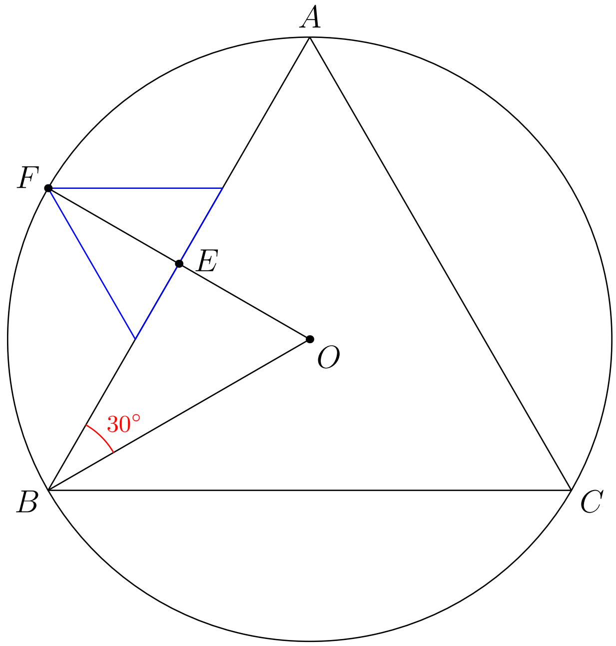
2.5 The figure formed consists of a triangular prism, with $4 - 4 - 4$ triangle as its base and height 4, and a rectangular pyramid with base $(7 - 4 =)3 \times 4$ and a triangular face of $5 - 5 - 4$.

The volume of the triangular prism is $B \cdot h$, where B is the area of the triangular base, which is $\frac{\sqrt{3}}{4} \cdot 4^2$. Therefore, the volume of the triangular prism is $\frac{\sqrt{3}}{4} \cdot 4^2 \cdot 4 = 16\sqrt{3}$.

The volume of a rectangular pyramid is $\frac{1}{3}B \cdot h$. In this case, $B = 3 \cdot 4 = 12$. Height is the same as the height of the $4 - 4 - 4$ triangle, which is $\frac{\sqrt{3}}{2} \cdot 4 = 2\sqrt{3}$. Therefore, the volume of the rectangular pyramid is $\frac{1}{3} \cdot 12 \cdot 2\sqrt{3} = 8\sqrt{3}$.

Therefore, the total volume is $16\sqrt{3} + 8\sqrt{3} = \boxed{24\sqrt{3}}$ in^3.

2.6 Note that finding the largest equilateral triangle is same as finding where the triangle can have the largest height. The largest height will happen at the perpendicular bisector of AB, which is the same as the radius of the circumcircle of $\triangle ABC$ passing through the midpoint of AB.



From the $30-60-90\triangle OBE$, $OB = \frac{\sqrt{3}}{2}BE = \frac{\sqrt{3}}{\sqrt{3}}\frac{1}{2}AB = \frac{1}{\sqrt{3}}12 = 4\sqrt{3}$. And, $OE = \frac{1}{2}OB = 2\sqrt{3}$.

Next, $FE = OF - OE = OB - OE = 4\sqrt{3} - 2\sqrt{3} = 2\sqrt{3}$. Then the side of the equilateral triangle $s = \frac{2}{\sqrt{3}}h = \frac{2}{\sqrt{3}}2\sqrt{3} = 4$.

Finally, the area of the equilateral triangle is $\frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}4^2 = \boxed{4\sqrt{3}}\ \text{cm}^2$.

2.7 Let's find where the first two lines meet. Multiplying the first equation by 3 and subtracting the second from it, we get $8x = 8$, that is $x = 1$ and $y = -4$.

Plugging this point into the third line's equation we get $a + 8 = 3$, that is $a = \boxed{-5}$.

2.8 From the given conditions, we have $\underline{?}\ \underline{?}\ \underline{?}\ \underline{10}\ \underline{?}\ \underline{?}\ \underline{?}$. Our goal is to put as small values in the spaces other than the last space. This means that the two other spaces to the immediate right of the center, we need to put 10. This gives $\underline{?}\ \underline{?}\ \underline{?}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{?}$.

Now, we need to have the smallest values in the first 3 spaces, but to preserve unique mode we can not repeat an element 3 times. This leads to: $\underline{1}\ \underline{1}\ \underline{2}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{?}$.

To preserve the mean as 10, the last digit is $10 + 2\cdot(10-1) + (10-2) = 36$, and the range is $36 - 1 = \boxed{35}$

2.9 First, note that $x^2 + x + 1 = \frac{x^3-1}{x-1}$. Next, factorize $x^4 + 2x^3 + 2x^2 + x$,

$$\begin{aligned}
&= x^4 + x^3 + x^2 + x^3 + x^2 + x, \\
&= x^2(x^2 + x + 1) + x(x^2 + x + 1), \\
&= (x^2 + x)(x^2 + x + 1), \\
&= x(x+1)(x^2 + x + 1).
\end{aligned}$$

So the given fraction (with 2018 replaced by x) can be written as,

$$\begin{aligned}
&\frac{1}{x+1}+\frac{x-1}{x^3-1}+\frac{x-1}{x(x+1)(x^3-1)},\\
&=\frac{x(x^3-1)+x(x+1)(x-1)+(x-1)}{x(x+1)(x^3-1)},\\
&=\frac{x(x^3-1)+x^3-x+x-1}{x(x+1)(x^3-1)},\\
&=\frac{(x^3-1)(x+1)}{x(x+1)(x^3-1)},\\
&=\frac{1}{x}.
\end{aligned}$$

And, so our answer is $\boxed{\dfrac{1}{2018}}$

2.10

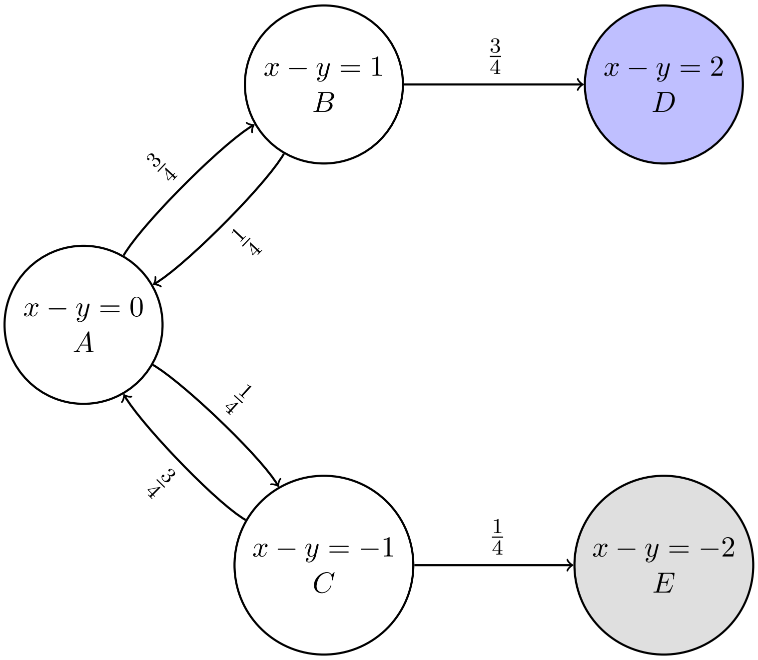
$$\begin{aligned}
\frac{3^{19}+2^{19}}{3^{15}+2^{15}} &= \frac{3^4\cdot 3^{15}+3^4\cdot 2^{15}-3^4\cdot 2^{15}+2^{19}}{3^{15}+2^{15}},\\
&=3^4-\frac{3^4\cdot 2^{15}-2^{19}}{3^{15}+2^{15}},\\
&=3^4-\frac{2^{15}\cdot 65}{3^{15}+2^{15}},\\
&=3^4-\frac{2^{15}\cdot(2^6+1)}{3^{15}+2^{15}},\\
&=3^4-\frac{2^{21}+2^{15}}{3^{15}+2^{15}}.
\end{aligned}$$

Since $2^3<3^2$, we have $2^{21}<3^{14}$. Therefore,

$$0<\frac{2^{21}+2^{15}}{3^{15}+2^{15}}<1.$$

And our answer is $\boxed{80}$

2.11 Let x be Xena's points and y be Yolanda's points. Then we can think of the problem in terms of $x-y$ and the different values for $x-y$ and its transitions are shown in the diagram below:



Let P_A, P_B, P_C denote the chances of Xena winning given that we are currently in state A, B, C, respectively. Then from the above diagram, we can see that:

$$\begin{aligned}
P_A &= \frac{3}{4}P_B + \frac{1}{4}P_C,\\
P_B &= \frac{3}{4} + \frac{1}{4}P_A,\\
P_C &= \frac{3}{4}P_A.
\end{aligned}$$

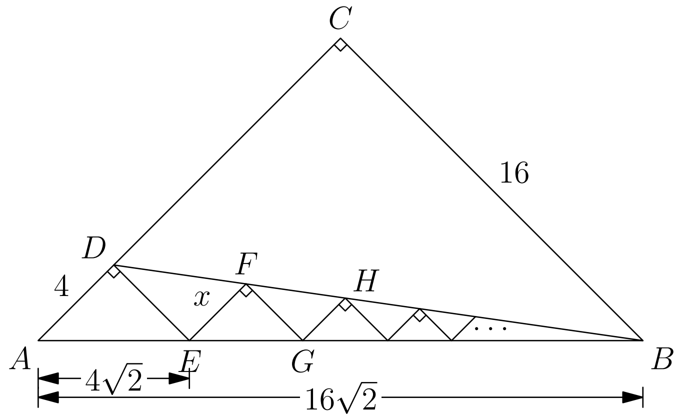
We can solve the above equations to get $P_B = \frac{39}{40}$, and

$$P_A = \frac{12}{13} \cdot \frac{39}{40} = \boxed{\frac{9}{10}}.$$

2.12 Since $\triangle DEF \sim \triangle FGH$, $\frac{EF}{DE} = \frac{GH}{FG}$, and because the triangles are isoceles this means $\frac{EF}{AD} = \frac{GH}{EF}$. That is each successive triangle is a dilation of the previous one by a constant factor, say k. Therefore, the total area of these triangles is

$$\begin{aligned}
&(1 + k^2 + k^4 + k^{16} + \dots)[\triangle ADE],\\
&= \frac{1}{1-k^2} \cdot \frac{1}{2} \cdot 4 \cdot 4,\\
&= \frac{1}{1-k^2} \cdot 8.
\end{aligned}$$

We need to find k.



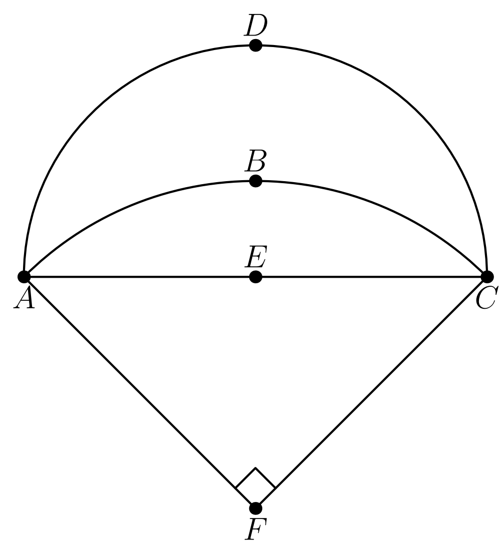
We have

$$k = \frac{x}{4} = \frac{16\sqrt{2} - 4\sqrt{2}}{16\sqrt{2}} = \frac{3}{4}.$$

Plugging this in our previous expression, we get $\dfrac{1}{1 - \frac{9}{16}} \cdot 8 = \boxed{\dfrac{128}{7}}$

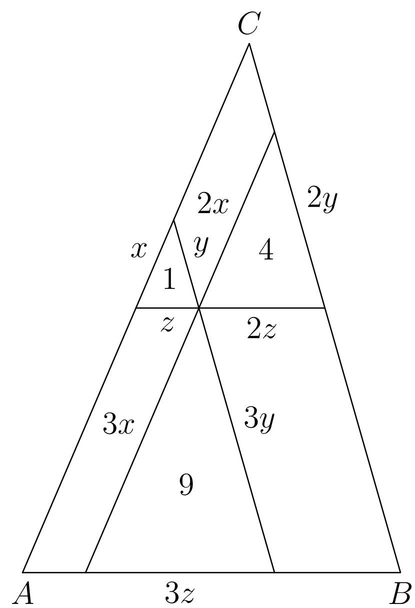
2.13 Since the circle is tangent to the point (3,0), we know the center of the circle is at some point (3,y). If we let the radius of the circle be R, we can write the equation of the circle to be $(x-3)^2 + (y-R)^2 = R^2$, since the Y component is R above the x-axis. Plugging in (6,6), we find that R is $\frac{15}{4}$. Replacing R with $\frac{15}{4}$ and plugging in (p,p) as another point, we arrive with the equation $(p-3)^2 + (y - \frac{15}{4})^2 = (\frac{15}{4})^2$, we get that p = $\boxed{\dfrac{3}{4}}$.

2.14 Let E be the center of the semicircle and F be the center of the quarter circle.



Since $AC = 18$, the radius of the semicircle is $AE = 9$. Since triangle AFC is a 45-45-90 triangle we have $AF = CF = \frac{18}{\sqrt{2}} = 9\sqrt{2}$. The area of the quarter circle is therefore $\frac{(9\sqrt{2})^2\pi}{4} = \frac{81\pi}{2}$. It follows the area of the curvy $AECB$ region is $\frac{81\pi}{2} - [AFC] = \frac{81\pi}{2} - 81$. The area of semicircle is $\frac{9^2\pi}{2} = \frac{81\pi}{2}$. The answer is $\frac{81\pi}{2} - \left(\frac{81\pi}{2} - 81\right) = \boxed{81}$.

2.15 By using the AA postulate, we find that triangle ABC is similar to all 3 triangles, so all angles in each triangle are congruent. Since the ratio of area of similar triangles is square of the ratio of the sides of the similar triangle, the sides of the 3 triangles are in the ration 1:2:3. As shown in the diagram, we can then workout that the sides of $\triangle ABC$ are 6 times the sides of the smallest triangle. Therefore, its area is $6^2 = \boxed{36}$.



2.16 The top cone has radius 2 and height 4 so it has volume $\dfrac{1}{3}\pi(2)^2 \times 4$.

The frustum is made up by taking away a small cone of radius 1, height 4 from a large cone of radius 2, height 8, so it has volume $\dfrac{1}{3}\pi(2)^2 \times 8 - \dfrac{1}{3}\pi(1)^2 \times 4$.

Adding, we get $\dfrac{1}{3}\pi(16 + 32 - 4) = \boxed{\dfrac{44\pi}{3}}$.