

Handout 5 - Advanced Topics

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§1 Counting and Probability

§1.1 Principle of Inclusion and Exclusion (PIE)

The principle of inclusion and exclusion (PIE) is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

Memorizing a formula for PIE is not needed as long as one understands how to over-count and under-count, in the end making sure that every element is counted only once.

§1.1.1 Two Sets

Let's start with the case of 2 sets. Let A_1 and A_2 be two sets. Then to count the number of elements in the union of the 2 sets, $|A_1 \cup A_2|$, we add the number of elements in the two sets, that is $|A_1| + |A_2|$. However, we have added the elements that are in both A_1 and A_2 , that is $A_1 \cap A_2$, twice, once with A_1 and second time with A_2 ; so we need to subtract it once. Therefore,

Theorem 1.1 (PIE - Two Sets)

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Example 1.2 (2011 AMC 8)

In a town of 351 adults, every adult owns a car, motorcycle, or both. If 331 adults own cars and 45 adults own motorcycles, how many of the car owners do not own a motorcycle?

Solution.

$$|A_{\text{car}} \cap A_{\text{motorcycle}}| = |A_{\text{car}}| + |A_{\text{motorcycle}}| - |A_{\text{car}} \cup A_{\text{motorcycle}}| = 331 + 45 - 351 = 25.$$

The number of car owners who do not own a motorcycle is

$$|A_{\text{car}}| - |A_{\text{car}} \cap A_{\text{motorcycle}}| = 331 - 25 = \boxed{306}.$$

□

§1.1.2 Three Sets

We can extend the logic above to three sets. To count the number of elements in the union of the 3 sets, $|A_1 \cup A_2 \cup A_3|$, we start by adding the elements in the 3 sets, that is $|A_1| + |A_2| + |A_3|$. As before, we realize that the elements that are in both $A_1 \cap A_2$ are double counted, as are elements that are in $A_1 \cap A_3$, and $A_2 \cap A_3$. So we subtract $|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$. However, now we consider the elements that are in all 3 sets, that is $A_1 \cap A_2 \cap A_3$. These were added 3 times when we started with $|A_1| + |A_2| + |A_3|$. And, they were subtracted 3 times, when we subtracted $|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$. So these elements are not counted and we need to add $|A_1 \cap A_2 \cap A_3|$. Therefore,

Theorem 1.3 (PIE - Three Sets)

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Example 1.4 (Brilliant)

There are exactly three types of students in a school: the geeks, the wannabees, and the athletes. Each student is classified into at least one of these categories. And the total number of students in the school is 1000. Suppose that the following is given:

- The total number of students who are geeks is 310.
- The total number of students who are wannabees is 650.
- The total number of students who are athletes is 440.
- The total number of students who are both geeks and wannabees is 170.
- The total number of students who are both geeks and athletes is 150.
- The total number of students who are both wannabees and athletes is 180.

What is the total number of students who fit into all 3 categories?

Solution. Let G, W, A denote the set for geeks, wannabees, and athletes, respectively. Then by the principle of inclusion and exclusion, we have

$$|G \cup W \cup A| = |G| + |W| + |A| - |W \cap G| - |G \cap A| - |W \cap A| + |G \cap W \cap A|,$$

which gives us $1000 = 310 + 650 + 440 - 170 - 150 - 180 + |G \cap W \cap A| = 900 + |G \cap W \cap A|$. So the total number of students who fit into all 3 categories is 100. □

§1.2 Geometric Probability

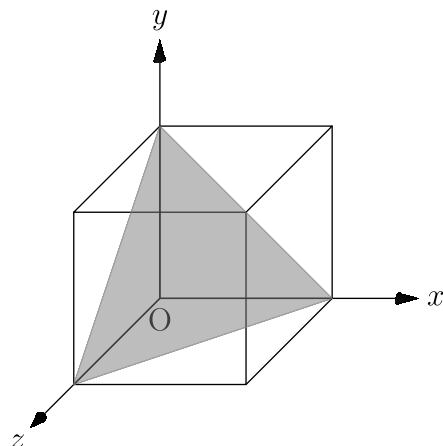
Sometimes we have to deal with continuous values instead of discrete values. In these case, we need to deal with the probability geometrically by considering either the area or the volume of the positive space vs the entire space.

Example 1.5 (Mathcounts Minis, #87)

What is the probability that the sum of three randomly drawn real numbers between 0 and 1 is less than 1?

Solution. Let's consider this geometrically. We can plot the 3 chose values in a 3 dimensional space, as x, y, z . For example, if we draw 0.5, 0.4, 0.8 in first, second, and third draw, we represent this event with the point (0.5, 0.4, 0.8) in the 3-D space.

Then the space of the universe of possibilities is a unit cube.



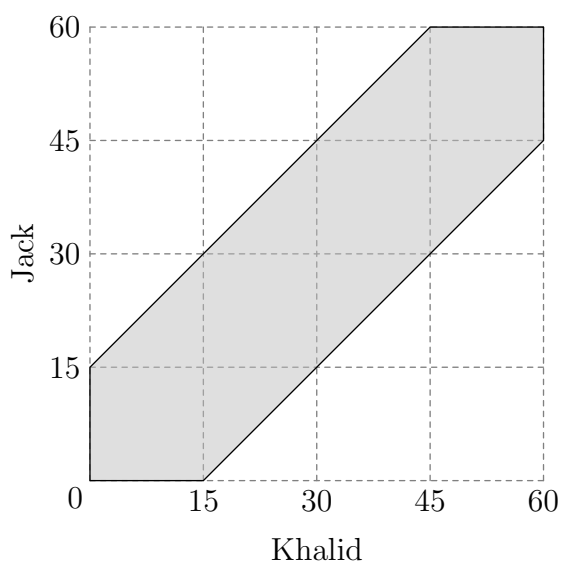
The values for $x + y + z = 1$ lie on the plane passing through $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. So the volume of good values is the tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Volume of this tetrahedron is $\frac{1}{3} \cdot 1 \cdot (\frac{1}{2} \cdot 1 \cdot 1) = \frac{1}{6}$ and therefore the desired probability is this volume divided by unit volume, that is $\boxed{\frac{1}{6}}$. \square

Remark 1.6. There is a video of this problem available at <https://artofproblemsolving.com/videos/mathcounts/mc2018/492>

Example 1.7 (Mathcounts Handbook 2014-15)

Each morning, Khalid and Jack arrive at the dining hall separately, at a random time from 8:00 to 9:00 a.m., and each remains there for exactly 15 minutes while eating. What is the probability they will see each other in the dining hall tomorrow morning? Express your answer as a common fraction.

Solution. Let us plot Khalid and Jack's arrival times on X - and Y - axis respectively.



The shaded region denotes the condition under which the two will meet. The triangle below the shaded region denotes that Khalid gets to the dining hall 15 minutes or more after Jack and the area above the shaded region denotes that Jack gets to the dining hall 15 minutes or more after Khalid.

So the needed probability is

$$\frac{\text{shaded area}}{\text{entire area}} = \frac{4^2 - 2 \cdot \frac{1}{2} \cdot 3 \cdot 3}{4^2} = \boxed{\frac{7}{16}}.$$

□

§1.3 Conditional Probability

Sometime we need to know the probability that the event B occurs given that event A has already occurred. This is called *conditional probability*. We use $P(B | A)$ to denote the probability of B given A .

Bayes theorem states that

$$P(B | A) = \frac{P(A \cap B)}{P(A)}.$$

Direct corollary of the above is

$$P(B | A) \cdot P(A) = P(A | B) \cdot P(B) = P(A \cap B).$$

Example 1.8 (Mathcounts Minis, #99)

Andy has a cube of edge length 8 cm. He paints the outside of the cube red and then divides the cube into smaller cubes, each of edge length 1 cm. Andy randomly chooses one of the unit cubes and rolls it on a table. If the cube lands so that an unpainted face is on the bottom, touching the table, what is the probability that the entire cube is unpainted?

Solution. Let A be the event that an unpainted cube is chosen and B be the event that an unpainted face is on the bottom. Then we are asked to find $P(A | B)$, that is the probability that an unpainted cube was chosen given that an unpainted face is on the bottom.

Since if an unpainted cube is chosen, it has to land with an unpainted face in the bottom, we have $P(B | A) = 1$. Therefore, from $P(A | B) \cdot P(B) = P(B | A) \cdot P(A)$, we have $P(A | B) = \frac{P(A)}{P(B)}$. Now, we just have two regular probabilities to compute.

We have $(8 - 2)^3$ unpainted cubes, and so $P(A) = \frac{6^3}{8^3}$.

We have total $6 \cdot 8^3$ faces, of which $8 \cdot 3 + 12 \cdot 6 \cdot 2 + 6 \cdot 6 \cdot 6 = 6(4 + 24 + 36) = 6 \cdot 64$ are painted faces. So $P(\overline{B}) = \frac{6 \cdot 8^2}{6 \cdot 8^3} = \frac{1}{8}$, where \overline{B} is the complement of B , that is the probability of rolling a painted face. So $P(B) = 1 - \frac{1}{8} = \frac{7}{8}$.

Plugging these in, we have $P(A | B) = \frac{P(A)}{P(B)} = \frac{\frac{6^3}{8^3}}{\frac{7}{8}} = \frac{3^3}{8 \cdot 7} = \boxed{\frac{27}{56}}.$

□

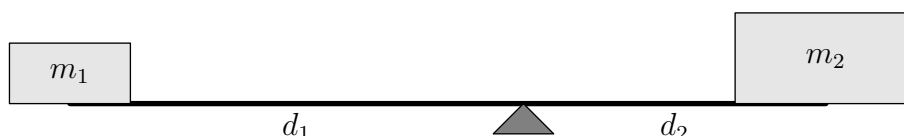
Remark 1.9. There is a video of this problem available at <https://www.mathcounts.org/resources/video-library/mathcounts-minis/mini-99-conditional-probability>.

§2 Geometry

§2.1 Mass Points

A **cevian** is a line segment that joins a vertex of a triangle with a point on the opposite side. Mass point geometry is a technique used to solve problems involving triangles and intersecting cevians by applying center of mass principles.

Mass point geometry leverages the concept of *Archimedes Principle of Levers* from Physics. The basic idea of the principle is that if there are masses at the two end of a seesaw, the seesaw can be balanced by a fulcrum such that the product of the masses on the ends and the distances from the fulcrum are the same.



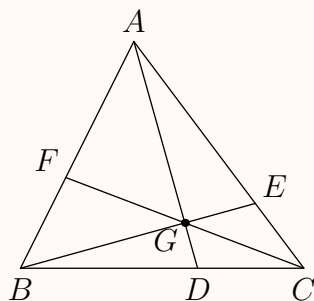
In the figure above, Archimedes Principle of Levers says

$$m_1 \cdot d_1 = m_2 \cdot d_2.$$

In geometry problems, where we are working with cevians and we have ratios associated with a cevian, we apply the mass point concept by applying masses to the vertices such that each intersection point becomes a fulcrum that balances the assigned masses.

Example 2.1 (2014-15 Mathcounts Handbook, #291-293)

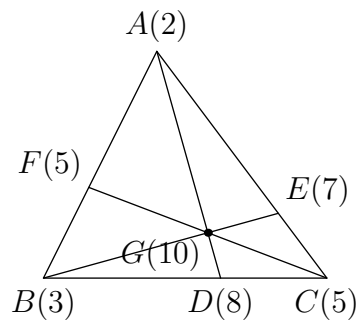
Triangle ABC , shown here, has cevians AD , BE and CF intersecting at point G , with $AF : BF = 3 : 2$ and $BD : CD = 5 : 3$.



1. What is the ratio of AE to CE ?
2. What is the ratio of BG to EG ?
3. What is the ratio of DG to AG ?

Solution. We need to place masses m_A, m_B, m_C at the vertices A, B, C , respectively, such that each intersection point D, E, F, G balances its corresponding line and masses. That is D balances BC with m_B, m_C and also AD with m_A, m_D , and so on. The masses at the intersection points m_D, m_E, m_F, m_G are the sum of masses on the corresponding segments.

Since $\frac{BD}{DC} = \frac{5}{3}$, we need $\frac{m_B}{m_C} = \frac{3}{5}$. Also, $\frac{BF}{AF} = \frac{2}{3}$, and therefore, $\frac{m_B}{m_A} = \frac{3}{2}$. Based on these ratios, we can assign $m_B = 3, m_C = 5, m_A = 2$. Then $m_D = m_B + m_C = 8$, $m_F = m_B + m_A = 5$, and $m_E = m_A + m_C = 7$. Finally, $m_G = m_A + m_D = m_B + m_E = m_C + m_F = m_A + m_B + m_C = 10$.



Now, we can use these masses to get all segment ratios:

$$1. \frac{AE}{CE} = \frac{m_C}{m_A} = \boxed{\frac{5}{2}}.$$

$$2. \frac{BG}{EG} = \frac{m_E}{m_B} = \boxed{\frac{7}{3}}.$$

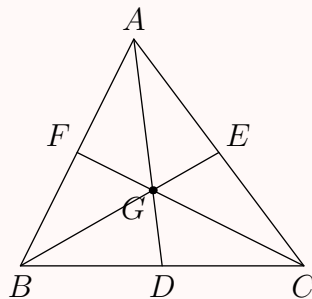
$$3. \frac{DG}{AG} = \frac{m_A}{m_D} = \frac{2}{8} = \boxed{\frac{1}{4}}.$$

□

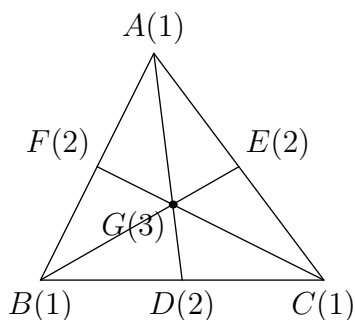
Note 2.2. In the above example, we calculated m_G while we didn't need it. Doing so helps us validate that we have a consistent set of mass assignments. We will continue doing this in subsequent mass point examples.

Example 2.3 (2014-15 Mathcounts Handbook, #294)

The medians of a triangle intersect at a point in the interior of the triangle as shown. What is the ratio of the lengths of the shorter and longer segments into which each median is divided at the point of intersection?



Solution. Since D, E, F are the midpoints of BC, AC, AB , we can simply assign $m_A = m_B = m_C = 1$, yielding $m_D = m_E = m_F = 2$ and $m_G = 3$.



Therefore, $\frac{GD}{AD} = \frac{EG}{BE} = \frac{FG}{CF} = \boxed{\frac{1}{2}}.$

□

Remark 2.4. The above argument also proves concurrency of the intersection of the 3 medians as both BE and CF intersect with AD such that $\frac{AG}{GD} = \frac{2}{1}$, and therefore at the same point.

Fact 2.5. The point of intersection of the medians is called the *centroid* of the triangle. Archimedes showed that the point where the medians are concurrent (the centroid) is the center of gravity of a triangular shape of uniform thickness and density.

§2.1.1 Split Mass with Transversals

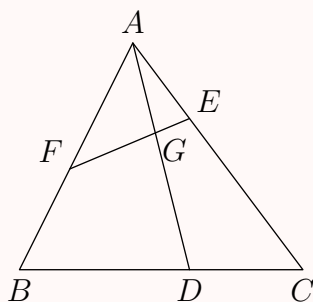
Mass point technique may also be used when we have transversals by applying a technique called *splitting masses*. The technique in summary is:

- Balance the two segments on the ends of the transversal separately and assign the common vertex two different masses (split-masses).
- Along secants from this vertex, the combined masses from the two split masses is used.

This is best illustrated with an example.

Example 2.6 (2014-15 Mathcounts Handbook, #299-300)

Triangle ABC , shown here, has cevian AD and transversal EF intersecting at G , with $AE : CE = 1 : 2$, $AF : BF = 5 : 4$ and $BD : CD = 3 : 2$.

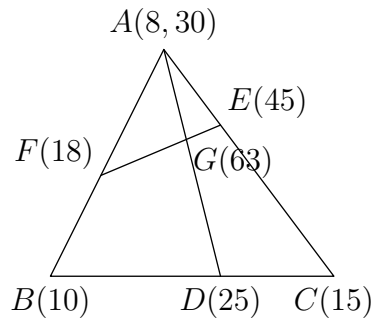


1. What is the ratio of AG to DG ?
2. What is the ratio of EG to FG ?

Solution. Since AD is a cevian, as in the prior examples, to balance BD we need to assign $\frac{m_B}{m_C} = \frac{2}{3}$.

To handle the transversal EF , we will assign A two masses, m_{AB}, m_{AC} to balance AB, AC at F, E , respectively. We need $\frac{m_{AB}}{m_B} = \frac{4}{5}$, and $\frac{m_{AC}}{m_C} = \frac{2}{1}$.

To achieve these ratios, we can assign $m_B = 10, m_C = 15, m_{AB} = 8, m_{AC} = 30$, leading to $m_F = m_{AB} + m_B = 18$, $m_E = m_{AC} + m_C = 45$, and $m_G = m_E + m_F = (m_{AB} + m_{AC}) + m_D = 63$.



Now we are ready to answer all ratios:

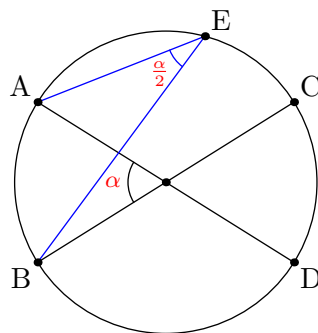
$$1. \frac{AG}{DG} = \frac{m_D}{m_A = m_{AB} + m_{AC}} = \frac{25}{8 + 30} = \boxed{\frac{25}{38}}$$

$$2. \frac{EG}{FG} = \frac{m_F}{m_E} = \frac{18}{45} = \boxed{\frac{2}{5}}$$

□

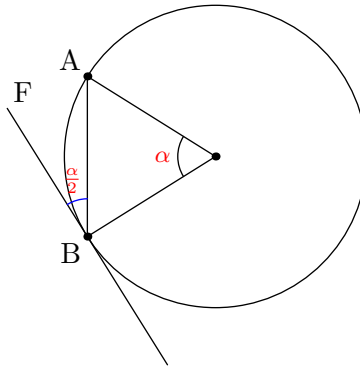
§2.2 Angles in Circles

§2.2.1 Central Angles and Inscribed Angles



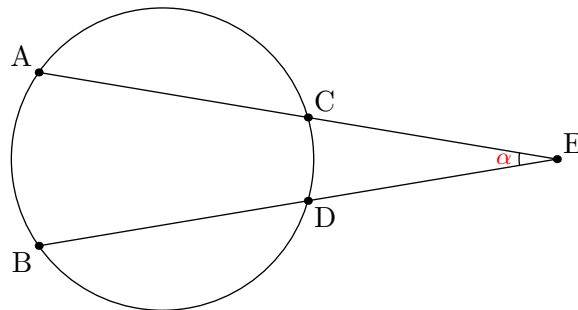
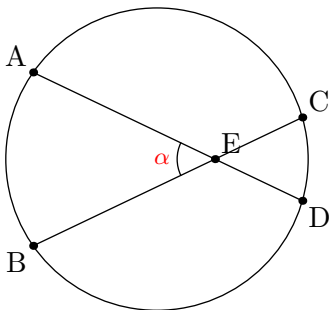
- Central angle is equal to intercepted arc; that is $m\widehat{AB} = m\widehat{CD} = \alpha$.
- Inscribed angle is half the intercepted arc that is $\angle AEB = \frac{1}{2}m\widehat{AB} = \frac{\alpha}{2}$.
- Congruent arcs have congruent central angles and vice versa.
- Congruent arcs (and central angles) have congruent chords and vice versa.

§2.2.2 Tangent Chord Angle



An angle formed by an intersecting tangent and chord has its vertex “on” the circle. In other words, the tangent chord angle is half the intercepted arc; that is $\angle ABF = \frac{1}{2}m\widehat{AB} = \frac{\alpha}{2}$.

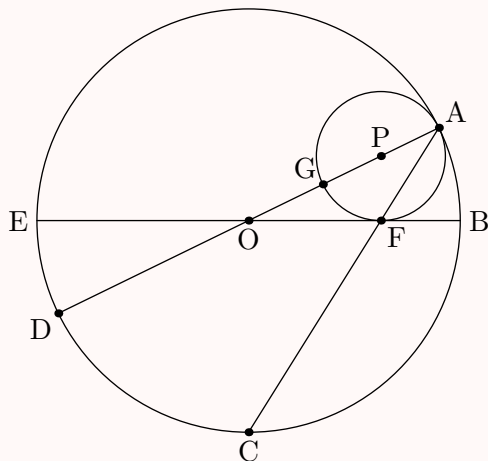
§2.2.3 Intersecting Chords/Secants



- When the chords intersect inside the circle, the angle formed is the average of the two intercepted arcs; that is $\angle AEB = \alpha = \frac{1}{2}(m\widehat{AB} + m\widehat{CD})$.
- When the secants intersect outside the circle, the angle formed is half the difference of the two intercepted arcs; that is $\angle AEB = \alpha = \frac{1}{2}(m\widehat{AB} - m\widehat{CD})$.
- The second relation applies even when one or both of the secants are **tangents**.

Example 2.7 (2016-17 Mathcounts Handbook, #240)

Circle P is internally tangent to circle O at A , as shown. AC and BE intersect at F , which is also the point of tangency between BE and circle P . AD and BE are diameters of circle O , and AG is a diameter of circle P . If $m\widehat{CD} = 50^\circ$, what is the measure of minor \widehat{BC} ?



Solution. We are given $m\widehat{CD} = m\widehat{FG} - 50^\circ$. Since AG is diameter of the circle P , we have $m\widehat{AFG} = 180^\circ$, and therefore $m\widehat{AF} = 180 - m\widehat{FG} = 130^\circ$.

Now, $\angle AOB$ is the external intersection of the two secants AG and BF for circle P . So $\angle AOB = \frac{1}{2}(m\widehat{AF} - m\widehat{GG}) = \frac{130-50}{2} = 40$. But $\angle AOB = m\widehat{DE}$.

Since EB is the diameter of the circle O , $m\widehat{DE} + m\widehat{DC} + m\widehat{CB} = 180$, and therefore, $m\widehat{BC} = 180 - 40 - 50 = 90^\circ$. \square

Remark 2.8. With some more tools in our geometry toolbox (homothety of circles), we could see that the $m\widehat{CD}$ was not needed, nor was it necessary for EB to be diameter. In the configuration above, C would be the midpoint of the arc regardless.

This is the generalized statement: Let \overline{EB} be a chord of circle O . Let P be a circle tangent to EB at F and internally tangent to O at A . Then ray AF passes through the midpoint C of \widehat{EB} not containing A .

§3 Intermediate Practice Problems

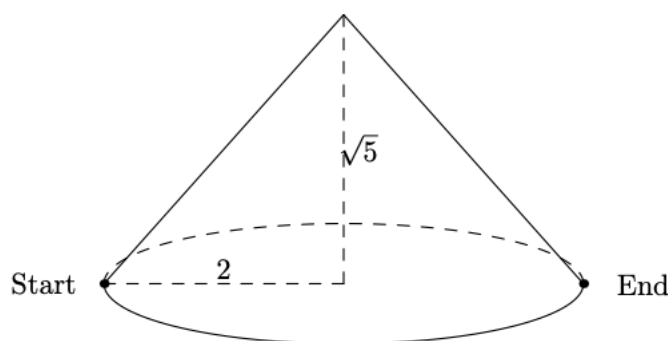
Problem 3.1 (2017 AMC 10B, #13). There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

Problem 3.2 (Mathcounts 2015 State Sprint #28). In regular pentagon $ABCDE$, point M is the midpoint of side AE , and segments AC and BM intersect at point Z . If $ZA = 3$, what is the value of AB ? Express your answer in simplest radical form.

Problem 3.3 (Mathcounts National 2018, Sprint #7). Alice and Bob play a game that begins with flipping a fair coin twice. If the coin lands heads up on both flips, Bob wins. If the coin lands heads up on only one flip, Alice wins. If the coin lands tails up on both flips, they flip two more times, and this process continues until there is a winner. What is the probability that Bob wins the game? Express your answer as a common fraction.

Problem 3.4 (Mathcounts National 2018, Sprint #10). What is the probability that a randomly chosen positive integer factor of 11,088 is not divisible by any perfect square greater than 1? Express your answer as a common fraction.

Problem 3.5 (BmMT 2019, #19). An ant sits on the circumference of the circular base of a party hat (a cone without a circular base for the ant to walk on) of radius 2 and height $\sqrt{5}$. If the ant wants to reach a point diametrically opposite of its current location on the hat, what is the minimum possible distance the ant needs to travel?

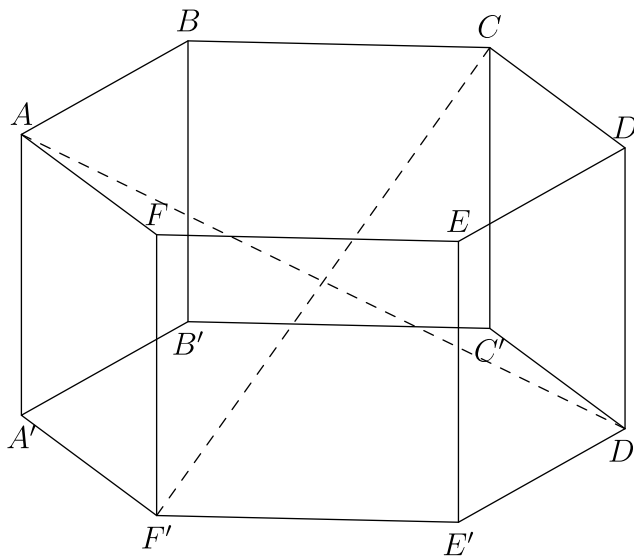


Problem 3.6 (Mathcounts National 2018, Sprint #14). Jenny rolls four standard six-sided dice, each with faces numbered one through six. What is the probability that there is at least one pair of dice whose top faces sum to 12? Express your answer as a common fraction.

Problem 3.7 (Mathcounts National 2018, Sprint #15). The faces of two cubes are painted so that the colors of the six faces on each cube are red, orange, yellow, green, blue and violet, in some random order. What is the probability that the two cubes can be rotated so that their colorings are identical? Express your answer as a common fraction.

Problem 3.8 (Mathcounts National 2018, Sprint #11). The graph of the line $2x + 3y = 8$ intersects the x -axis at A and intersects the y -axis at B . Points A , B and $O(0,0)$ form right triangle ABO . If triangle ABO is dilated about the origin by a scale factor of $-\frac{3}{4}$, what is the area of dilated triangle $A'B'O'$?

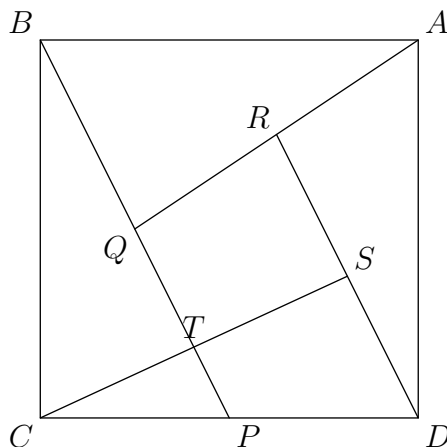
Problem 3.9 (Mathcounts National 2018, Sprint #16). The bases of the right hexagonal prism shown are regular hexagons with sides of length 4 inches. Space diagonals AD' and CF' intersect at a right angle. What is the length of edge EE' ? Express your answer in simplest radical form.



Problem 3.10 (Mathcounts National 2018, Sprint #17). In a card game, Nora draws cards at random, without replacement, from a deck of 21 cards. Twenty of the cards are numbered 1 through 20, and the other card is marked “Joker.” Nora keeps all of the cards she draws before she draws the Joker. What is the probability that the cards Nora keeps include exactly four prime-numbered cards? Express your answer as a common fraction.

Problem 3.11 (Mathcounts National 2018, Sprint #18). How many assignments of digits A, B, C and D exist such that the four-digit numbers $ABCD_6$ and $ABCD_{10}$ are both divisible by 5?

Problem 3.12 (Mathcounts National 2018, Sprint #19). In the figure, $ABCD$ is a square with area 1024 cm^2 . P is the midpoint of side CD . Q is the midpoint of segment BP . R is the midpoint of segment AQ . S is the midpoint of segment DR . T is the midpoint of segment CS . T is also the intersection of segments BP and CS . What is the area of quadrilateral $QRST$?



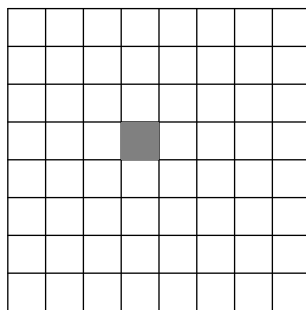
Problem 3.13 (Mathcounts National 2018, Sprint #20). Vanessa tosses a fair coin eight times. If the coin lands heads up, she steps 1 meter west. If the coin lands tails up, she steps 1 meter north. What is the probability

that Vanessa ends a distance of at most 6 meters away from her starting position? Express your answer as a common fraction.

Problem 3.14 (Mathcounts National 2018, Target #6). Micaela randomly chooses a real number w between -20 and 20 . What is the probability that the graphs of $x - |y| = w$ and $x^2 + y^2 = 50$ intersect at exactly two points? Express your answer as a decimal to the nearest hundredth.

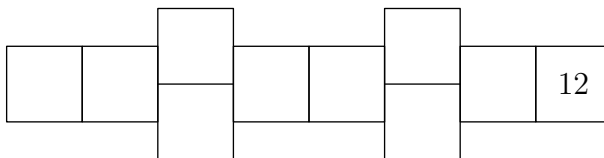
Problem 3.15. A circle with center O has radius 8 units and circle P has radius 2 units. The circles are externally tangent to each other at point Q . Segment TS is the common external tangent to circle O and circle P at points T and S , respectively. What is the length of segment OS ? Express your answer in simplest radical form.

Problem 3.16 (Mathcounts National 2017, Sprint #14). Philippa stands on the shaded square of the 8-by-8 checkerboard shown. She moves to one of the four adjacent squares sharing an edge with her starting square, with each of the four squares equally likely to be chosen. She then makes two more moves to adjacent squares in the same way. Given any square S , let $P(S)$ be the probability that Philippa lands on that square after her third move. What is the greatest possible value of $P(S)$? Express your answer as a common fraction.



Problem 3.17 (Mathcounts National 2017, Sprint #19). Sam creates a six-digit positive integer by writing the digit 7 in the hundred-thousands place, and then tossing a fair coin five times. If the coin comes up heads, he writes a 7 for the next digit; if the coin comes up tails, he writes a 0 for the next digit. What is the probability that Sam's number is divisible by 77? Express your answer as a common fraction.

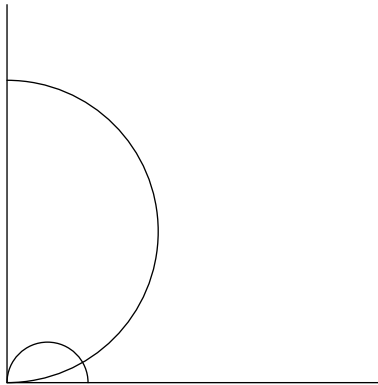
Problem 3.18 (Mathcounts National 2017, Sprint #22). How many ways are there to fill in each empty square in the diagram below with a positive integer so that no integer appears more than once in the diagram, and every integer in the diagram is less than each integer to its right?



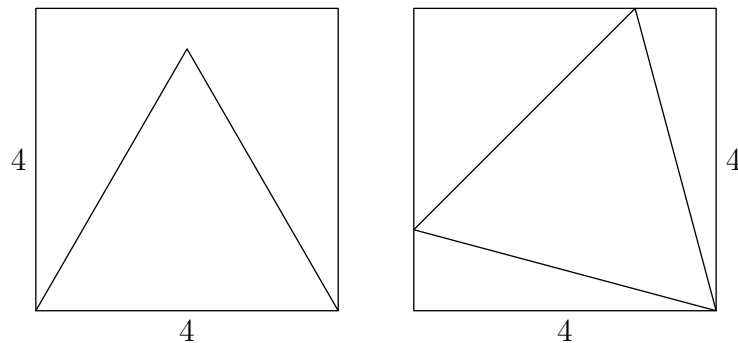
Problem 3.19 (Mathcounts National 2017, Sprint #23). What is the total surface area of the largest regular tetrahedron that can be inscribed inside of a cube of edge length 1 cm? Express your answer in simplest radical form.

Problem 3.20 (Mathcounts National 2017, Sprint #24). Penny flips three fair coins into a box with two compartments. Each compartment is equally likely to receive each of the coins. What is the probability that either of the compartments has at least two coins that landed heads? Express your answer as a common fraction.

Problem 3.21 (Mathcounts National 2017, Sprint #30). In the figure shown, two lines intersect at a right angle, and two semicircles are drawn so that each semicircle has its diameter on one line and is tangent to the other line. The larger semicircle has radius 1. The smaller semicircle intersects the larger semicircle, dividing the larger semicircular arc in the ratio 1:5. What is the radius of the smaller semicircle? Express your answer in simplest radical form.

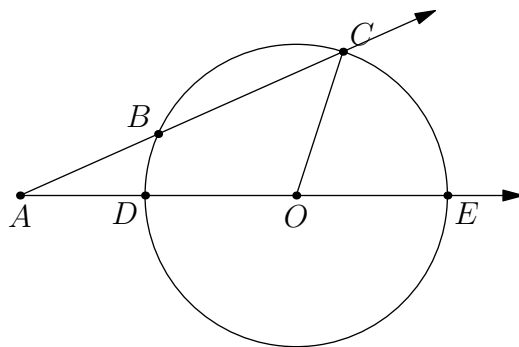


Problem 3.22 (Mathcounts National 2017, Target #6). Two congruent squares with side length 4 have equilateral triangles constructed in them as shown. In one square, one side of the equilateral triangle is a side of the square. In the other square, the equilateral triangle has one vertex at a vertex of the square and its other two vertices are on the sides of the square. The absolute difference of the areas of the two triangles can be expressed in simplest radical form as $a\sqrt{b} + c$. What is the value of $a + b + c$?



Problem 3.23 (Mathcounts National 2017, Target #3). There are a hundred competitors at the National Debating Contest, two from each of the 50 states. In how many ways can five finalists be chosen if no state may have more than one finalist?

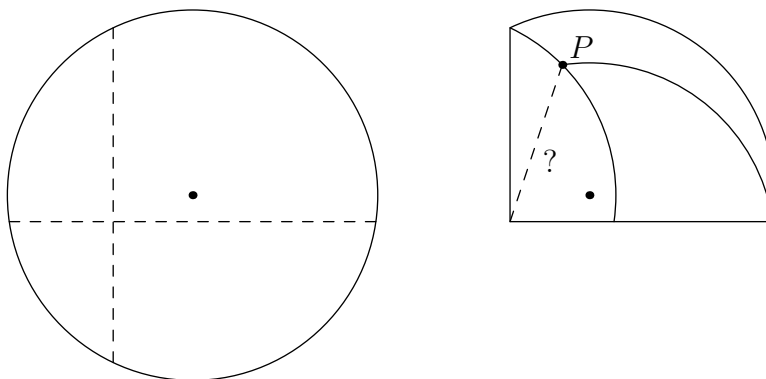
Problem 3.24 (Mathcounts National 2017, Target #4). Rays AC and AE intersect circle O at B and D , respectively. Segment DE is a diameter of circle O and $AB = \frac{1}{2}DE$. If the measure of $\angle BAD$ is 24 degrees, what is the degree measure of $\angle COE$?



Problem 3.25 (Mathcounts National 2016, Sprint #21). Manny has two red socks, two white socks, and two blue socks. He plans to choose two socks at random to wear today, two of the remaining socks to wear tomorrow, and will wear the last two remaining socks the next day. What is the probability that on all three days, Manny wears two different colored socks? Express your answer as a common fraction.

Problem 3.26 (Mathcounts National 2016, Sprint #30). In isosceles triangle ABC , with base BC of length 23 cm, points P and Q are chosen such that $BP = CQ = 9$ cm. If segments AP and AQ trisect angle BAC , what is the perimeter of $\triangle ABC$?

Problem 3.27 (Mathcounts National 2018, Target #8). Shane cuts out a paper circle of radius 7 cm. He then draws two perpendicular chords, as shown, with one a distance of 1 cm from the center and the other 3 cm from the center. Next, Shane folds the bottom of the circle up along the first chord and then folds the left side of the folded circle to the right along the second chord. If the folded arcs intersect at a point P , as shown, what is the distance from the vertex of the right angle formed by the folds to P ? Express your answer in simplest radical form.



§4 Challenge Problems

Problem 4.1 (2001 AIME I, #7). Triangle ABC has $AB = 21$, $AC = 22$ and $BC = 20$. Points D and E are located on \overline{AB} and \overline{AC} , respectively, such that \overline{DE} is parallel to \overline{BC} and contains the center of the inscribed circle of triangle ABC . Then $DE = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

Hint: The incenter is at the intersection of angle bisectors.

Problem 4.2 (1998 AIME, # 9). Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b\sqrt{c}$, where a, b , and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

Problem 4.3 (HMMT 2003). Find the smallest n such that $n!$ ends in 290 zeroes.

§5 Solution Sketches

3.1 To compute the number of students taking at least two classes we start with $|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$. However, the students taking all three classes are counted thrice in this, so need to subtract $2 \cdot |A_1 \cap A_2 \cap A_3|$. Then,

$$|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - 2 \cdot |A_1 \cap A_2 \cap A_3| = 9.$$

Now,

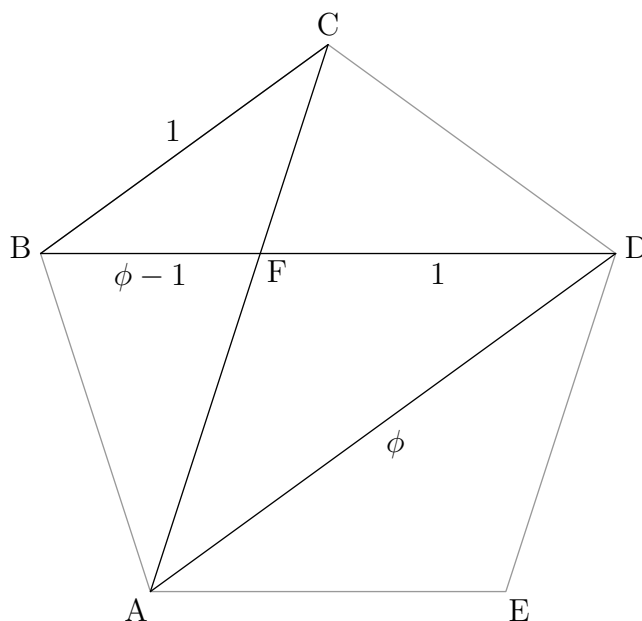
$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|,$$

$$20 = 10 + 13 + 9 - (9 + 2 \cdot |A_1 \cap A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|,$$

$$|A_1 \cap A_2 \cap A_3| = 23 - 20 = \boxed{3}.$$

3.2 We will solve this problem in the reverse direction. That is, while we are given ZA and are asked about AB , we will assume a length for AB , use it to solve for ZA and then scale back AB to make the ZA match the given value.

First, a fun fact: the diagonal of a regular pentagon and its sides are in the ratio ϕ , the golden ratio, that is $\frac{1+\sqrt{5}}{2}$. It is easy to see this using similar triangles:



In the above diagram, $DF = AE = 1$. From, similar triangles,

$$\begin{aligned} \frac{\phi}{1} &= \frac{1}{\phi - 1}, \\ \phi^2 - \phi - 1 &= 0, \\ \phi &= \frac{1 + \sqrt{5}}{2}. \end{aligned}$$

Now, we use this and mass-points to solve the problem.

3.3 Bob can win this flip with a probability $\frac{1}{4}$. And there is a $\frac{1}{4}$ chance that the game continues. If the game continues, Bob's probability of winning overall from that point is the same as from the starting point. Therefore we have

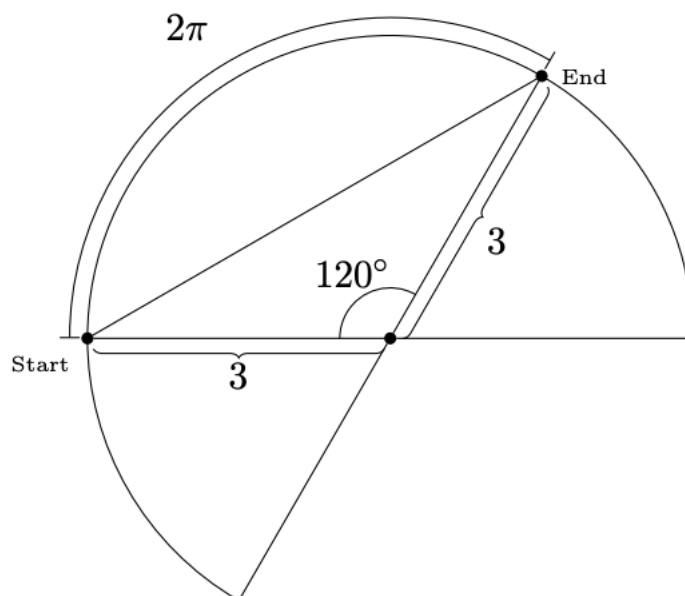
$$P(B) = \frac{1}{4} + \frac{1}{4}P(B).$$

This gives us, $P(B) = \boxed{\frac{1}{3}}$.

3.4 First, $11,088 = 2^4 \cdot 3^2 \cdot 7 \cdot 11$. The square free factors can have 0 or 1 of each of the 4 prime factors. Therefore, the desired probability is:

$$\frac{2 \cdot 2 \cdot 2 \cdot 2}{5 \cdot 3 \cdot 2 \cdot 2} = \boxed{\frac{4}{15}}.$$

3.5



"Unfold" the party hat by making a straight cut through the apex and a point on the circular base. The flattened party hat is a sector of a circle with radius 3. The total length of the arc of the sector is $2 \cdot 2\pi = 4\pi$, and the circumference of the circle is 6π , so the angle that the sector takes should be $360^\circ \cdot \frac{4\pi}{6\pi} = 240^\circ$. Similarly, the ant's desired location is $360^\circ \cdot \frac{2\pi}{6\pi} = 120^\circ$ away from its current location. Since the shortest path between these two points is the straight line between them, we draw an isosceles triangle whose two equal legs are radii (and thus of length 3) and whose vertex angle is 120° . The shortest distance to the desired location is then the length of the triangle's third side. By dropping an altitude to the third side and forming two $30 - 60 - 90$ triangles, we see that the length of the third side is $\boxed{3\sqrt{3}}$.

3.6 We do complementary counting. Probability that there is at least one pair such that ... is

$$\begin{aligned}
 & 1 - P(0) - P(1), \\
 & = 1 - \left(\frac{5}{6}\right)^4 - \binom{4}{1} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^3, \\
 & = \frac{6^4 - 5^4 - 4 \cdot 5^3}{6^4}, \\
 & = \boxed{\frac{19}{144}}.
 \end{aligned}$$

3.7 Orient the cubes so that the red face is on top the probability that they have the same color opposite the red face is $\frac{1}{5}$. Then for the remaining 4 colors there are 4 rotations that is $\frac{4}{4!}$ arrangements, so $\frac{1}{6}$ probability of getting the right colors on the side. Finally, the final probability that the 2 cubes can be rotated so that their colorings are identical is $\frac{1}{5} \cdot \frac{1}{6} = \boxed{\frac{1}{30}}$.

3.8 $A = (4, 0)$, and $B = (0, \frac{8}{3})$, so $[ABO] = \frac{1}{2} \cdot 4 \cdot \frac{8}{3} = \frac{16}{3}$. $[A'B'O'] = \left(\frac{3}{4}\right)^2 \cdot \frac{16}{3} = \boxed{3}$

3.9 Consider the crossection $ACD'F'$. Since it's a rectangle and we are given that its diagonals intersect at right angle, we know that it is square, implying $AF' = AC$.

Now if we look at the $\triangle ABC$, we have $AB = BC = s$, and $\angle ABC = 120^\circ$. From this, we can calculate $AC = \sqrt{3}s$. We could use law of cosine directly to do this. Or we can drop an altitude from C to AB , say H , and use the fact that $\triangle CFH$ is a 30-60-90 triangle to know that $CH = \frac{\sqrt{3}}{2}s$, $HF = \frac{1}{2}s$, and $AF = s + HF = \frac{3}{2}s$. Then from Pythagoras on $\triangle CHA$, we have $AC^2 = CH^2 + AH^2 = \frac{3}{4}s^2 + \frac{9}{4}s^2 = 3s^2$.

Now looking at the $\triangle AFF'$, we have from Pythagoras on it, $AF'^2 = s^2 + FF'^2$. Plugging in value of $AF' = AC$ from above, we get $FF' = \sqrt{2}s = \boxed{4\sqrt{2}}$.

3.10 We can simply ignore all cards other than the prime numbered cards and the Joker as drawing or not drawing them makes no difference. In other words we can formulate this as drawing cards from a pack of 8 prime numbered cards and 1 joker such that we draw a prime numbered card the first 4 times and Joker in the fifth. With this formulation, its easy to see that the probability is

$$\frac{8}{9} \cdot \frac{7}{8} \cdot \frac{6}{7} \cdot \frac{5}{6} \cdot \frac{1}{5} = \boxed{\frac{1}{9}}.$$

3.11 Solution 1. D must be 0 or 5, then pick any B and C between 0 and 5, inclusive. Then, there is exactly 1 choice of A from $(1, 2, 3, 4, 5)$ that makes $ABCD$ base 6 divisible by 5. Thus, we have $2 \cdot 6 \cdot 6 \cdot 1 = \boxed{72}$ possibilities.

Solution 2. First, since $ABCD_6$ is a valid number, we have

$$0 \leq A, B, C, D \leq 5, \text{ and } A \neq 0.$$

Second, $5 \mid ABCD_{10}$ tells us that

$$D \equiv 0 \pmod{5}, \text{ that is } D \in \{0, 5\}.$$

Next, from $5 \mid ABCD_6$, we have

$$\begin{aligned} A \cdot 6^3 + B \cdot 6^2 + C \cdot 6 + D &\equiv 0 \pmod{5}, \\ A \cdot 1^3 + 1 \cdot 1^2 + C \cdot 1 + 0 &\equiv 0 \pmod{5}, \\ A + B + C &\equiv 0 \pmod{5}. \end{aligned}$$

From here, we could simply do casework on A . Alternatively, we do casework on $A + B + C$:

- $A + B + C = 5$: If we have $A, B, C \geq 0$, from Stars and Bars, we have $\binom{7}{2} = 21$ ways. Of which $\binom{6}{1} = 6$ have $A = 0$. Therefore, for this case we have $21 - 6 = 15$ ways.
- $A + B + C = 10$: With $A, B, C \geq 0$, from Stars and Bars, we have $\binom{12}{2} = 66$ ways. However, in this are included the cases where
 - One of them is ≥ 6 . Let's say $A \geq 6$. Let's have $A' = A - 6 \geq 0$, and $A' + B + C = 4$. This can happen in $\binom{6}{2} = 15$ ways and since any of the three can be the one ≥ 6 we have a total of $3 \cdot 15 = 45$ ways.
 - $A = 0$. There is only 1 way, $B = 5, C = 5$.

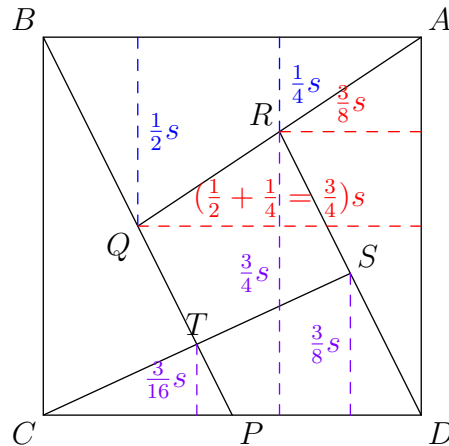
In summary, for this case we have $66 - 45 - 1 = 20$ ways.

- $A + B + C = 15$: There is only 1 way: $A = 5, B = 5, C = 5$.

So in total we have $15 + 20 + 1 = 36$ ways, and for each way D can be 0 or 5, so final answer is $2 \cdot 36 = \boxed{72}$

3.12 First,

$$[QRST] = [ABCD] - [\triangle BCP] - [\triangle ABQ] - [\triangle ADR] - [\triangle CDS] + [\triangle CPT].$$



From the above figure,

- $[\triangle BCP] = \frac{1}{2} \cdot s \cdot \frac{1}{2}s = \frac{1}{4}s^2$.
- $[\triangle ABQ] = \frac{1}{2} \cdot \frac{1}{2}s \cdot s = \frac{1}{4}s^2$.
- $[\triangle ADR] = \frac{1}{2} \cdot \frac{3}{8}s \cdot s = \frac{3}{16}s^2$.
- $[\triangle CDS] = \frac{1}{2} \cdot \frac{3}{8}s \cdot s = \frac{3}{16}s^2$.
- $[\triangle CPT] = \frac{1}{2} \cdot \frac{3}{16}s \cdot \frac{1}{2}s = \frac{3}{64}s^2$.

Putting it all together,

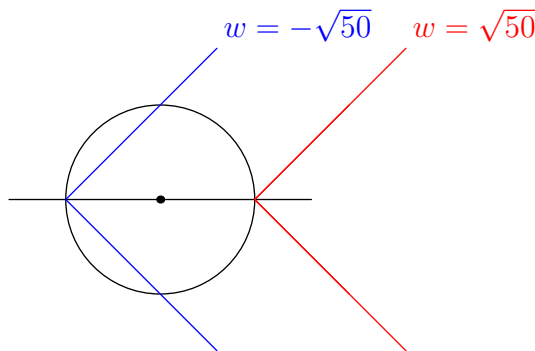
$$\begin{aligned}
 [QRST] &= s^2 - 2 \cdot \frac{1}{4}s^2 - 2 \cdot \frac{3}{16}s^2 + \frac{3}{64}s^2, \\
 &= \left(1 - \frac{1}{2} - \frac{6}{16} + \frac{3}{64}\right)s^2, \\
 &= \frac{64 - 32 - 24 + 3}{64}s^2, \\
 &= \frac{11}{64} \cdot 1024, \\
 &= \boxed{176}.
 \end{aligned}$$

3.13 First, there are $2^8 = 256$ ways she can take the steps, each equally likely. Let's count how many of these ways lands her further than 6 meters away. These are:

- $(0, 8), (-8, 0)$: 2 ways.
- $(-1, 7), (-7, 1)$: $2 \cdot \binom{8}{1} = 16$ ways.
- $(-2, 6), (-6, 2)$: $2 \cdot \binom{8}{2} = 56$ ways.

So the number of ways, she ends up within a distance of 6 is $256 - 2 - 16 - 56 = 182$, and the desired probability is $\frac{182}{256} = \boxed{\frac{91}{128}}$

3.14 As seen in the diagram below, $-\sqrt{50} \leq w < \sqrt{50}$ satisfy the criteria. Therefore, the probability is $\frac{2 \cdot \sqrt{50}}{40} = \boxed{0.35}$.



3.15 This is easy to visualize if we make the two circles tangent to a horizontal line (say x -axis). The distance $TS = \sqrt{(8+2)^2 - (8-2)^2} = 8$. Then, $OS = \sqrt{8^2 + 8^2} = \boxed{8\sqrt{2}}$.

3.16 At any step, Phillipa can move L, R, U, D each with probability $\frac{1}{4}$. Also, L and R cancel each other as well as U and D cancel each other. Next, $x + y$ of the final position has to be odd and less than 3. So the 3 (considering symmetry) choices are:

- $(0, 1), (1, 0), (0, -1), (-1, 0)$: Let's consider $(0, 1)$. We can get there with some combination of U, L, R or U, D, U . We have $3! = 6$ ways for U, L, R and $\binom{3}{1} = 3$ ways for U, D, U , for a total of 9 ways. This gives probability of these cells to be $\frac{6+3}{4^3} = \boxed{\frac{9}{64}}$.

- $(1, 2), (2, 1), (2, -1), (1, -2), \dots$: Let's consider $(1, 2)$. We get there with a combination of U, R, R , which is in $\binom{3}{1} = 3$ ways, for a probability of $\frac{3}{64}$.
- $(0, 3), (3, 0), \dots$: Let's consider $(0, 3)$. We get there only one way, U, U, U . So probability of $\frac{1}{64}$.

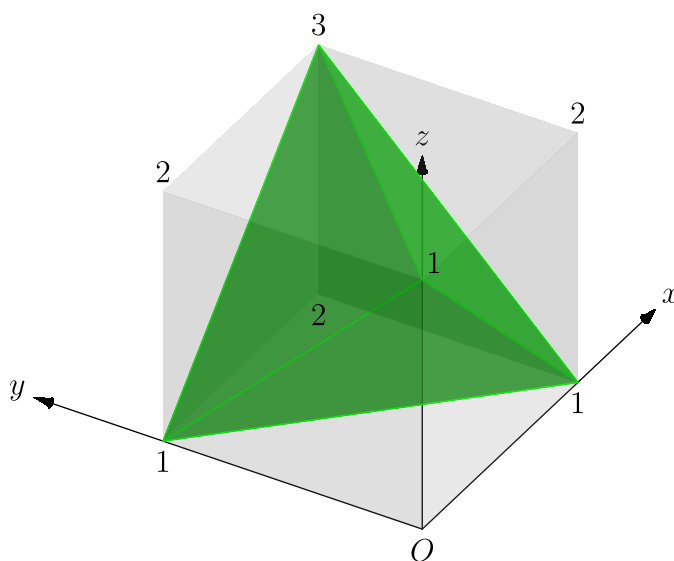
3.17 Since the digits are only 7 or 0, it will always be divisible by 7. So we only need to worry about divisibility by 11. We use divisibility by 11 rule and do case work on total number of 7s:

- 2 7s: The additional 7 can go in one of the three odd places - 3 ways
- 4 7s: 2 additional 7s should go in one of the three odd places, and one should go in one of the 2 even places - $\binom{3}{2} \cdot \binom{2}{1} = 6$ ways
- 6 7's: 1 way

Therefore, we can get the desired number in $3 + 6 + 1 = 10$ ways and there are 2^5 total numbers. Hence, the desired probability is $\frac{10}{2^5} = \boxed{\frac{5}{16}}$.

3.18 Once we have chosen the 9 numbers to use, we have only 2 choices at points where we have vertically stacked cells. Therefore we can do this in $\binom{11}{9} \cdot 2 \cdot 2 = \boxed{220}$ ways.

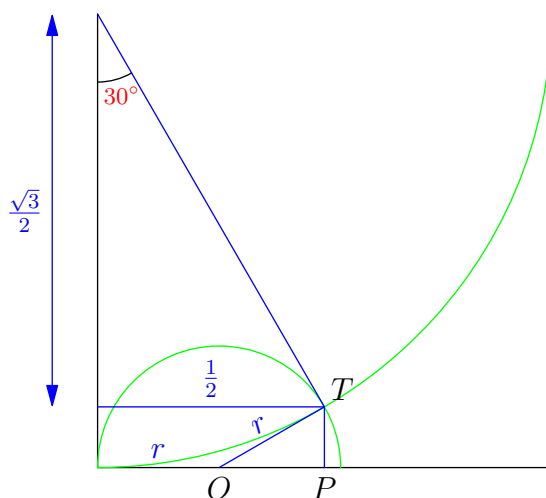
3.19 The largest tetrahedron can be fit in a cube by putting one vertex of the tetrahedron on a vertex of the cube and the the 3 vertices on the vertices that share a face with the first vertex and are diagonally opposite to it. In this configuration the side of the tetrahedron $a = \sqrt{2}s = \sqrt{2}$. And surface area of a regular tetrahedron is $\sqrt{3}a^2 = \boxed{2\sqrt{3}}$.



3.20 We can calculate $P(\text{at least two heads})$

$$\begin{aligned}
 &= P(\text{at least two heads} \mid \text{three in box}) \cdot P(\text{three in box}) \\
 &\quad + P(\text{two heads} \mid \text{two in box}) \cdot P(\text{two in box}), \\
 &= \left(3 \cdot \frac{1}{2^3} + 1 \cdot \frac{1}{2^3}\right) \cdot \frac{2}{2^3} + \frac{1}{2^2} \cdot \left(1 - \frac{2}{2^3}\right), \\
 &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4}, \\
 &= \boxed{\frac{5}{16}}.
 \end{aligned}$$

3.21 First, note that the smaller portion of the cut arc is $\frac{1}{6} \cdot 180 = 30^\circ$. We know that $\sin 30 = \frac{1}{2}$ and $\cos 30 = \frac{\sqrt{3}}{2}$.



Applying Pythagoras on $\triangle OTP$, we have

$$\begin{aligned}
 r^2 &= \left(\frac{1}{2} - r\right)^2 + \left(1 - \frac{\sqrt{3}}{2}\right)^2, \\
 r &= \boxed{2 - \sqrt{3}}.
 \end{aligned}$$

3.22 The first equilateral triangle is one with side length 4, so its area is $\frac{1}{2} \frac{\sqrt{3}}{2} s^2 = 4\sqrt{3}$.

In the other square, we see that the big right triangle is a 45-45-90 triangle. So one of the angles of the small right triangles is $180 - 60 - 45 = 75^\circ$, and so it's a 75-15-90 triangle, and its side length is $s = \frac{4}{\cos 15^\circ}$. If one knows that $\cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$, they can plug it in to get $s = 4(\sqrt{6} - \sqrt{2})$.

If not, this can be solved by Pythagoras theorem. Length of a leg of the 45-45-90 triangle be x , then by Pythagoras on that triangle we have $s^2 = 2x^2$, and by Pythagoras on the 75-15-90 triangle, we have $s^2 = (4 - x)^2 + 4^2$. Setting the two equal and solving the quadratic we get $x = 4(\sqrt{3} - 1)$, and therefore, $s = 4(\sqrt{6} - \sqrt{2})$.

Remark 5.1. This configuration is one of the ways to derive the value for $\cos 15^\circ$.

The difference in the areas between two equilateral triangles is $\frac{\sqrt{3}}{4}((\sqrt{6} - \sqrt{2})^2 - 1^2) = 28\sqrt{3} - 48$, and so the desired answer is $28 + 3 - 48 = \boxed{-17}$.

3.23 We have to choose 5 states from 50 states, and then for each selected state, we can select from one of the two contestants. We can do this in $\binom{50}{5} \cdot 2^5 = \boxed{67800320}$ ways.

3.24 First, $AB = \frac{1}{2}DE$ means that AB is same size as radius and therefore, $\triangle ABO$ is isocles and $\angle BOA = \angle BAO = 24^\circ$.

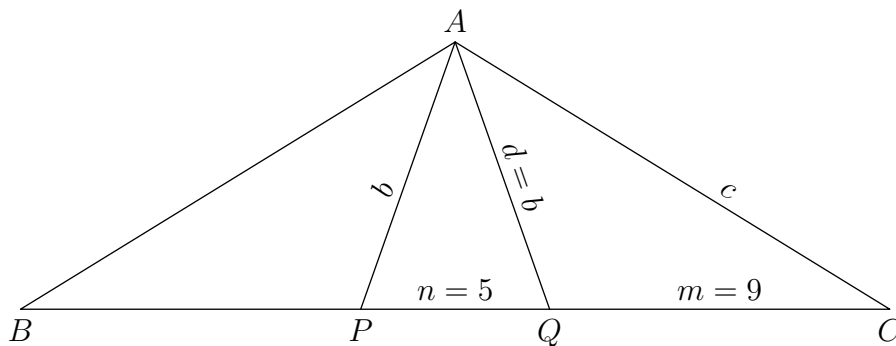
Next, from the formula for two seacants intersecting outside the circle, we have $= \frac{1}{2}(\widehat{CE} - \widehat{BD}) = \frac{1}{2}(\angle COE - \angle BOA)$. From the angle equivalence above, we get $\angle BAO = \frac{1}{2}(\angle COE - \angle BAO)$, or $= 3 \cdot \angle BAO = 3 \cdot 24 = \boxed{72}$ degrees.

3.25 First, the total number of ways to select socks is $\binom{6}{2}$ on the first day and $\binom{4}{2}$ ways on the second day, for a total of $\binom{6}{2}\binom{4}{2} = 90$ ways.

Now, let's count the ways he can select so that on any day he wears different colors. First, he can choose the days to wear red in $\binom{3}{2}$ ways. Then he can choose the days to wear blue also in $\binom{3}{2}$ ways; however, if he chooses blue on the same days as red, then white will end up being used together. Correcting for this, the number of valid ways to choose the days to wear blue is $\binom{3}{2} - 1$. However, we are not done with the numerator. When we first counted the total number of possible ways Manny could pair his socks, we assumed that each sock was distinct, namely that the two red socks were different, and so on. This led to an overcount on the total number of pairings (For example, on the first day, there are only 6 possibilities if socks were not distinct, as opposed to the 15 we calculated). To compensate, we multiply our previous result of $\binom{3}{2} \cdot (\binom{3}{2} - 1)$ by 2^3 (2 ways for each pair of socks since we are assuming each sock is distinct, as we did when finding the total pairings).

Therefore the desired probability is $\frac{6 \cdot 8}{90} = \boxed{\frac{8}{15}}$.

3.26 We can use the angle bisector theorem to get the ratio of the cevian, b in the diagram below, to one of the sides, c in the diagram, as $\frac{b}{n} = \frac{c}{m}$, that is $\frac{b}{5} = \frac{c}{9}$, or $b = \frac{5c}{9}$.



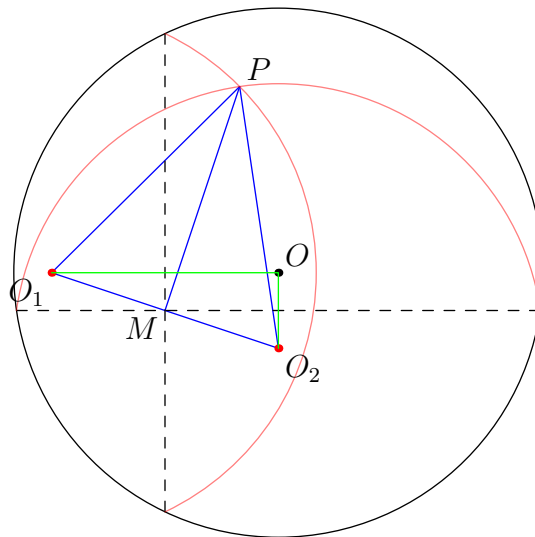
Then we can solve this by either doing Pythagoras theorem twice, or by using Stewart's theorem. Using the

Stewart's theorem, we get

$$\begin{aligned}(m+n)(mn+d^2) &= b^2 \cdot m + c^2 \cdot n, \\ 14(45+b^2) &= 9b^2 + 5c^2, \\ 14 \cdot 45 &= 5c^2 - 5b^2, \\ c^2 - b^2 &= 14 \cdot 9, \\ c^2 - \frac{81}{25}c^2 &= 14 \cdot 9, \\ c^2 &= \frac{9 \cdot 81}{4}, \\ c &= \frac{27}{2}.\end{aligned}$$

Therefore, the perimeter is $2 \cdot \frac{27}{2} + 23 = \boxed{50}$.

3.27 From the $\triangle O_1OO_2$, we have M is the midpoint of O_1O_2 because the two lines of folding are the midlines of those sides.



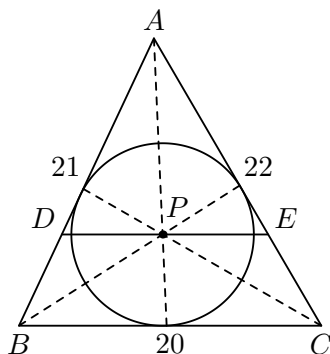
Therefore, $\triangle PO_1O_2$ is an isosceles triangle with PM as its median. We have

$$O_2M^2 = 3^2 + 1^2 = 10.$$

Then

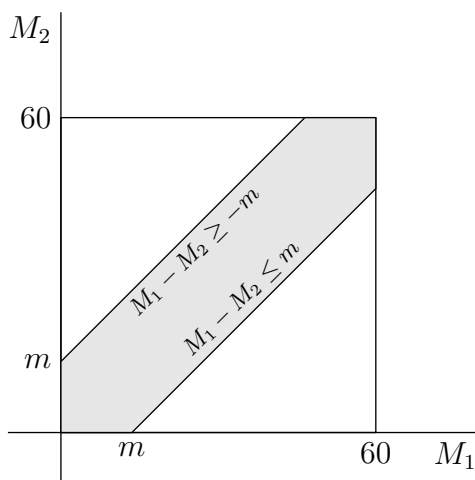
$$PM = \sqrt{PO_2^2 - O_2M^2} = \sqrt{7^2 - 10} = \boxed{\sqrt{39}}.$$

4.1 Let P be the incenter; then it is the intersection of all three angle bisectors. Draw the bisector AP to where it intersects BC , and name the intersection F .



Using the angle bisector theorem, we know the ratio $BF : CF$ is $21 : 22$, thus we shall assign a weight of 22 to point B and a weight of 21 to point C , giving F a weight of 43. In the same manner, using another bisector, we find that A has a weight of 20. So, now we know P has a weight of 63, and the ratio of $FP : PA$ is $20 : 43$. Therefore, the smaller similar triangle ADE is $43/63$ the height of the original triangle ABC . So, DE is $43/63$ the size of BC . Multiplying this ratio by the length of BC , we find DE is $860/63 = m/n$. Therefore, $m + n = \boxed{923}$.

4.2 Let the two mathematicians be M_1 and M_2 . Consider plotting the times that they are on break on a coordinate plane with one axis being the time M_1 arrives and the second axis being the time M_2 arrives (in minutes past 9 a.m.). The two mathematicians meet each other when $|M_1 - M_2| \leq m$. Also because the mathematicians arrive between 9 and 10, $0 \leq M_1, M_2 \leq 60$. Therefore, 60×60 square represents the possible arrival times of the mathematicians, while the shaded region represents the arrival times where they meet.



It's easier to compute the area of the unshaded region over the area of the total region, which is the probability that the mathematicians do not meet:

$$\frac{(60-m)^2}{60^2} = 0.6 \Rightarrow (60-m)^2 = 36 \cdot 60 \Rightarrow 60-m = 12\sqrt{15} \Rightarrow m = 60 - 12\sqrt{15}$$

So the answer is $60 + 12 + 15 = \boxed{87}$.

4.3 Since there are more 2's than 5's in $n!$, we need to find the smallest value of n such that $n!$ has 290 5's. That is,

$$290 = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots$$

We can approximate the sum by losing the floors and treating it as a geometric series. Therefore,

$$290 \approx \frac{\frac{n}{5}}{1 - \frac{1}{5}},$$

yielding $b \approx 1160$. When we plug 1160 into the original formula, instead of 290 we get 288. So we need two more 5s, that is go up by 10, and our desired $n = \boxed{1170}$.