# 20211112 HS Guts Round Problems

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## 1 Round 1

(HMMT November 2019 Guts 1,3)

For how many positive integers $a$ does the polynomial

$$x^2 - ax + 1$$

have an integer root?

Answer: $\boxed{1}$

(HMMT November 2018 Guts 1,1)

A positive integer is called *primer* if it has a prime number of distinct prime factors. Find the smallest primer number.

Answer: $\boxed{6}$

(HMMT November 2018 Guts 2,1)

Let $a, b, c$, and $n$ be positive real numbers such that $\frac{a+b}{a} = 3$, $\frac{b+c}{b} = 4$, and $\frac{c+a}{c} = n$. Find $n$.

Answer: $\boxed{\frac{7}{6}}$

## 2 Round 2

(HMMT November 2019 Guts 2,3)

Compute the sum of all positive integers $n < 2048$ such that $n$ has an even number of 1's in its binary representation

Answer: $\boxed{1048064}$

(HMMT November 2018 Guts 3,3)

Farmer James has some strange animals. His hens have 2 heads and 8 legs, his peacocks have 3 heads and 9 legs, and his zombie hens have 6 heads and 12 legs. Farmer James counts 800 heads and 2018 legs on his farm. What is the number of animals that Farmer James has on his farm?

Answer: $\boxed{203}$

(HMMT February 2018 Guts 2,1)

Compute the value of $\sqrt{105^3 - 104^3}$, given that it is a positive integer.

Answer: $\boxed{181}$

# 3 Round 3

(HMMT November 2019 Guts 3,2)

There are 36 students at the Multiples Obfuscation Program, including a singleton, a pair of identical twins, a set of identical triplets, a set of identical quadruplets, and so on, up to a set of identical octuplets. Two students look the same if and only if they are from the same identical multiple. Nithya the teaching assistant encounters a random student in the morning and a random student in the afternoon (both chosen uniformly and independently), and the two look the same. What is the probability that they are actually the same person?

Answer: $\boxed{\frac{3}{17}}$

(HMMT November 2016 Guts 3, 2)

Danielle picks a positive integer $1 \leq n \leq 2016$ uniformly at random. What is the probability that gcd(n, 2015) = 1?

Answer: $\boxed{\frac{1441}{2016}}$

(HMMT February 2019 Guts 2,2)

Alice, Bob, and Charlie roll a 4, 5, and 6-sided die, respectively. What is the probability that a number comes up exactly twice out of the three rolls?

Answer: $\boxed{\dfrac{13}{30}}$

# 4 Round 4

(HMMT November 2017 Guts 4,1)

Compute $\frac{x}{w}$ if $w \neq 0$ and $\frac{x+6y-3z}{-3x+4w} = \frac{-2y+z}{x-w} = \frac{2}{3}$

Answer: $\boxed{\dfrac{2}{3}}$

(HMMT November 2018 Guts 5,1)

Find the smallest positive integer $n$ for which

$$1! \cdot 2! \cdot ... \cdot (n-1)! > (n!)^2$$

Answer: $\boxed{8}$

(HMMT February 2019 Guts 3,3)

Let $P(x)$ be the monic polynomial with rational coefficients of minimal degree such that $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, ..., \frac{1}{\sqrt{1000}}$ are roots of $P$. What is the sum of the coefficients of $P$?

Answer: $\boxed{\dfrac{1}{16000}}$

# 5 Round 5

(HMMT November 2017 Guts 5,2)

Points $A$, $B$, $C$, and $D$ lie on a line in that order such that $\frac{AB}{BC} = \frac{DA}{CD}$. If $AC = 3$ and $BD = 4$, find $AD$.

Answer: $\boxed{6}$

(HMMT November 2017 Guts 5,3)

On a $3 \times 3$ chessboard, each square contains a knight with $\frac{1}{2}$ probability. What is the probability that there are two knights that can attack each other? (In chess, a knight can attack any piece which is two squares away from it in a particular direction and one square away in a perpendicular direction.)

Answer: $\boxed{\frac{209}{256}}$

(HMMT February 2016 Guts 4,3)

Let $R$ be the rectangle in the Cartesian plane with vertices at $(0, 0)$,$(2, 0)$,$(2, 1)$, and $(0, 1)$. R can be divided into two unit squares; the resulting figure has seven edges. Compute the number of ways to choose one or more of the seven edges such that the resulting figure is traceable without lifting a pencil. (Rotations and reflections are considered distinct.)

Answer: $\boxed{61}$

# 6 Round 6

(HMMT February 2017 Guts 5,1)

The game of Penta is played with teams of five players each, and there are five roles the players can play. Each of the five players chooses two of five roles they wish to play. If each player chooses their roles randomly, what is the probability that each role will have exactly two players?

Answer: $\boxed{\frac{51}{2500}}$

(HMMT February 2017 Guts 5,2)

Mrs. Toad has a class of 2017 students, with unhappiness levels 1, 2, ..., 2017 respectively. Today in class, there is a group project and Mrs. Toad wants to split the class in exactly 15 groups. The unhappiness level of a group is the average unhappiness of its members, and the unhappiness of the class is the sum of the unhappiness of all 15 groups. What's the minimum unhappiness of the class Mrs. Toad can achieve by splitting the class into 15 groups?

Answer: $\boxed{1121}$

(HMMT February 2016 Guts 5,3)

Compute $(\tan \frac{\pi}{7} \cdot \tan \frac{2\pi}{7} \cdot \tan \frac{3\pi}{7})^2$

Answer: $\boxed{7}$

## 7 Round 7

(HMMT November 2018 Guts 7,1)

Let $A$ be the number of unordered pairs of ordered pairs of integers between 1 and 6 inclusive, and let $B$ be the number of ordered pairs of unordered pairs of integers between 1 and 6 inclusive. (Repetitions are allowed in both ordered and unordered pairs.) Find $A - B$.

Answer: $\boxed{225}$

(HMMT February 2017 Guts 6,3)

Let $ABCD$ be a quadrilateral with side lengths $AB = 2$, $BC = 3$, $CD = 5$, and $DA = 4$. The maximum possible radius of a circle inscribed in quadrilateral ABCD can be expressed as $\frac{a\sqrt{b}}{c}$, where $a$, $b$, and $c$ are integers, $a$ and $c$ are relatively prime, and $b$ is not divisible by the square of any integer. What is $a+b+c$?

Answer: $\boxed{39}$

(HMMT February 2014 Guts 7,1)

Let $ABCD$ be a trapezoid with $AB \parallel CD$. The bisectors of $\angle CDA$ and $\angle DAB$ meet at $E$, the bisectors of $\angle ABC$ and $\angle BCD$ meet at $F$, the bisectors of $\angle BCD$ and $\angle CDA$ meet at $G$, and the bisectors of $\angle DAB$ and $\angle ABC$ meet at $H$. Quadrilaterals $EABF$ and $EDCF$ have areas 24 and 36, respectively, and triangle $ABH$ has area 25. Find the area of triangle $CDG$.

Answer: $\boxed{\frac{256}{7}}$

## 8 Round 8

(HMMT February 2017 Guts 7,3)

Let $P$ and $A$ denote the perimeter and area, respectively, of a right triangle with relatively prime integer side-lengths. Find the largest possible integral value of $\frac{P^2}{A}$

Answer: $\boxed{45}$

(HMMT February 2016 Guts 8,1)

On the Cartesian plane $\mathbb{R}^2$, a circle is said to be nice if its center is at the origin $(0, 0)$ and it passes through at least one lattice point (i.e. a point with integer coordinates). Define the points $A = (20, 15)$ and $B = (20, 16)$. How many nice circles intersect the open segment $AB$?

For reference, the numbers 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691 are the only prime numbers between 600 and 700.

Answer: $\boxed{10}$

(HMMT February 2014 Guts 7,3)

Compute the number of ordered quintuples of nonnegative integers $(a_1, a_2, a_3, a_4, a_5)$ such that $0 \leq a_1, a_2, a_3, a_4, a_5 \leq 7$ and 5 divides $2^{a_1} + 2^{a_2} + 2^{a_3} + 2^{a_4} + 2^{a_5}$.

Answer: $\boxed{6528}$

# 9 Round 9

(HMMT February 2014 Guts 8,3)

Let $A = a_1, a_2, ..., a_7$ be a set of distinct positive integers such that the mean of the elements of any nonempty subset of $A$ is an integer. Find the smallest possible value of the sum of the elements in $A$.

Answer: $\boxed{1267}$

(HMMT February 2014 Guts 8,2)

Let $S = 100, 99, 98, \ . \ . \ . \ , 99, 100$. Choose a 50-element subset $T$ of $S$ at random. Find the expected number of elements of the set $\{|x| : x \epsilon T\}$.

Answer: $\boxed{\frac{8825}{201}}$

(HMMT February 2011 Guts 8,3)

In how many ways may thirteen beads be placed on a circular necklace if each bead is either blue or yellow and no two yellow beads may be placed in adjacent positions? (Beads of the same color are considered to be identical, and two arrangements are considered to be the same if and only if each can be obtained from the other by rotation).

Answer: $\boxed{41}$

## 10 Round 10

(HMMT February 2019 Guts 10,1)

Let $\triangle ABC$ be a triangle inscribed in a unit circle with center $O$. Let $I$ be the incenter of $\triangle ABC$, and let $D$ be the intersection of $BC$ and the angle bisector of $\angle BAC$. Suppose that the circumcircle of $\triangle ADO$ intersects $BC$ again at a point $E$ such that $E$ lies on IO. If $\cos A = \frac{12}{13}$, find the area of $\triangle ABC$.

Answer: $\boxed{\dfrac{15}{169}}$

(HMMT February 2011 Guts 9,2)

In how many ways can 13 bishops be placed on an 8×8 chessboard such that (i) a bishop is placed on the second square in the second row, (ii) at most one bishop is placed on each square, (iii) no bishop is placed on the same diagonal as another bishop, and (iv) every diagonal contains a bishop? (For the purposes of this problem, consider all diagonals of the chessboard to be diagonals, not just the main diagonals).

Answer: $\boxed{1152}$

(HMMT February 2015 Guts 10,1)

Let $w$, $x$, $y$, and $z$ be positive real numbers such that

$$\begin{aligned}
0 &\neq \cos w \cdot \cos x \cdot \cos y \cdot \cos z \\
2 &= w + x + y + z \\
3 \tan w &= k(1 + \sec w) \\
4 \tan x &= k(1 + \sec x) \\
5 \tan y &= k(1 + \sec y) \\
6 \tan z &= k(1 + \sec z)
\end{aligned}$$

(Here $\sec t$ denotes $\frac{1}{\cos t}$ when $\cos t \neq 0$) Find $k^2$.

Answer: $\boxed{19}$

## 11 Round 11

(HMMT February 2016 Guts 10,3)

Determine the number of triples $0 \leq k, m, n \leq 100$ of integers such that

$$2^m n - 2^n m = 2^k$$

Answer: $\boxed{10}$

(HMMT February 2016 Guts 11,2)

How many equilateral hexagons of side length $\sqrt{13}$ have one vertex at (0, 0) and the other five vertices at lattice points?

Answer: $\boxed{216}$

(HMMT February 2011 Guts 12,2)

Let $w = w_1, w_2, ..., w_6$ be a permutation of the integers $\{1,\ 2,\ .\ \ .\ \ .\ ,\ 6\}$. If there do not exist indices $i < j < k$ such that $w_i < w_j < w_k$ or indices $i < j < k < l$ such that $w_i > w_j > w_k > w_l$, then $w$ is said to be exquisite. Find the number of exquisite permutations.

Answer: $\boxed{25}$

## 12 Round 12

(HMMT February 2016 Guts 11,1)

The Lucas numbers are defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for every $n \geq 0$. There are $N$ integers $1 \leq n \leq 2016$ such that $L_n$ contains the digit 1. Estimate $N$.

An estimate of $E$ earns $\lfloor 20 - 2|N - E| \rfloor$ or 0 points, whichever is greater.

(HMMT February 2015 Guts 12,3)

A prime number $p$ is twin if at least one of $p + 2$ or $p\ \ 2$ is prime and sexy if at least one of $p + 6$ and $p\ \ 6$ is prime. How many sexy twin primes (i.e. primes that are both twin and sexy) are there less than 109? Express your answer as a positive integer $N$ in decimal notation; for example, 521495223. If your answer is in this form, your score for this problem will be $\max\{0, 25 - \lfloor \frac{1}{10000}|A - N|\}$, where $A$ is the actual answer to this problem. Otherwise, your score will be zero.

Answer: $\boxed{1462105}$

(HMMT February 2020 Guts 12,3)

A collection $S$ of 10000 points is formed by picking each point uniformly at random inside a circle of radius 1. Let $N$ be the expected number of points of $S$

which are vertices of the convex hull of the $S$. (The convex hull is the smallest convex polygon containing every point of $S$.) Estimate $N$.

An estimate of $E > 0$ will earn $\max(\lfloor 22 - |E - N| \rfloor, 0)$ points.

Answer: $\boxed{\approx 72.8}$