Handout 5 - Advanced Topics

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§1 Counting and Probability

§1.1 Geometric Probability

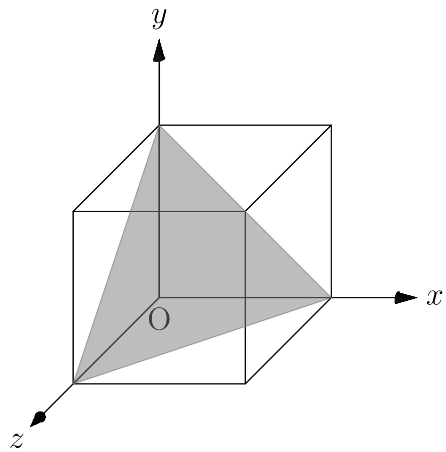
Sometimes we have to deal with continuous values instead of discrete values. In these case, we need to deal with the probability geometrically by considering either the area or the volume of the positive space vs the entire space.

Example 1.1 (Mathcounts Minis, #87)

What is the probability that the sum of three randomly drawn real numbers between 0 and 1 is less than 1?

Solution. Let's consider this geometrically. We can plot the 3 chose values in a 3 dimensional space, as x, y, z. For example, if we draw $0.5, 0.4, 0.8$ in first, second, and third draw, we represent this event with the point $(0.5, 0.4, 0.8)$ in the 3-D space.

Then the space of the universe of possibilities is a unit cube.



The values for $x + y + z = 1$ lie on the plane passing through $(1, 0, 0), (0, 1, 0), (0, 0, 0)$. So the volume of good values is the tetrahedron with vertices at $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 0)$. Volume of this tetrahedron is $\frac{1}{3} \cdot 1 \cdot (\frac{1}{2}1 \cdot 1) = \frac{1}{6}$ and therefore the desired probability is this volume divided by unit volume, that is $\boxed{\frac{1}{6}}$. □

Remark 1.2. There is a video of this problem available at https://artofproblemsolving.com/videos/mathcounts/mc2018/492

§1.2 Conditional Probability

Sometime we need to know the probability that the event B occurs given that event A has already occurred. This is called *conditional probability*. We use $P(B \mid A)$ to denote the probability of B given A.

Bayes theorem states that

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}.$$

Direct corollary of the above is

$$P(B \mid A) \cdot P(A) = P(A \mid B) \cdot P(B) = P(A \cap B).$$

Example 1.3 (Mathcounts Minis, #99)

Andy has a cube of edge length 8 cm. He paints the outside of the cube red and then divides the cube into smaller cubes, each of edge length 1 cm. Andy randomly chooses one of the unit cubes and rolls it on a table. If the cube lands so that an unpainted face is on the bottom, touching the table, what is the probability that the entire cube is unpainted?

Solution. Let A be the event that an unpainted cube is chosen and B be the event that an unpainted face is on the bottom. Then we are asked to find $P(A \mid B)$, that is the probability that an unpainted cube was chosen given that an unpainted face is on the bottom.

Since if an unpainted cube is chosen, it has to land with an unpainted face in the bottom, we have $P(B \mid A) = 1$. Therefore, from $P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A)$, we have $P(A \mid B) = \dfrac{P(A)}{P(B)}$. Now, we just have two regular probabilities to compute.

We have $(8-2)^3$ unpainted cubes, and so $P(A) = \dfrac{6^3}{8^3}$.

We have total $6 \cdot 8^3$ faces, of which $8 \cdot 3 + 12 \cdot 6 \cdot 2 + 6 \cdot 6 \cdot 6 = 6(4 + 24 + 36) = 6 \cdot 64$ are painted faces. So $P(\overline{B}) = \dfrac{6 \cdot 8^2}{6 \cdot 8^3} = \dfrac{1}{8}$, where $\overline{B}$ is the complement of B, that is the probability of rolling a painted face. So $P(B) = 1 - \dfrac{1}{8} = \dfrac{7}{8}$.

Plugging these in, we have $P(A \mid B) = \dfrac{P(A)}{P(B)} = \dfrac{\frac{6^3}{8^3}}{\frac{7}{8}} = \dfrac{3^3}{8 \cdot 7} = \boxed{\dfrac{27}{56}}$.

□

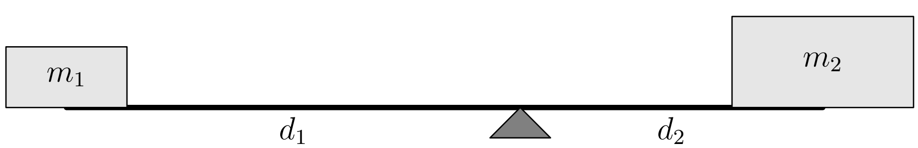
Remark 1.4. There is a video of this problem available at https://www.mathcounts.org/resources/video-library/mathcounts-minis/mini-99-conditional-probability.

§2 Geometry

§2.1 Mass Points

A ***cevian*** is a line segment that joins a vertex of a triangle with a point on the opposite side. Mass point geometry is a technique used to solve problems involving triangles and intersecting cevians by applying center of mass principles.

Mass point geometry leverages the concept of *Archimedes Principle of Levers* from Physics. The basic idea of the principle is that if there are masses at the two end of a seesaw, the seesaw can be balanced by a fulcrum such that the product of the masses on the ends and the distances from the fulcrum are the same.



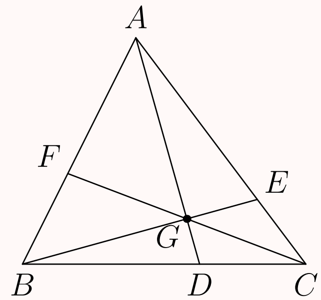
In the figure above, Archimedes Principle of Levers says

$$m_1 \cdot d_1 = m_2 \cdot d_2.$$

In geometry problems, where we are working with cevians and we have ratios associated with a cevian, we apply the mass point concept by applying masses to the vertices such that each intersection point becomes a fulcrum that balances the assigned masses.

Example 2.1 (2014-15 Mathcounts Handbook, #291-293)

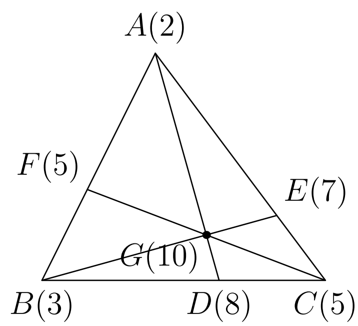
Triangle ABC, shown here, has cevians AD, BE and CF intersecting at point G, with $AF : BF = 3 : 2$ and $BD : CD = 5 : 3$.



1. What is the ratio of AE to CE?
2. What is the ratio of BG to EG?
3. What is the ratio of DG to AG?

Solution. We need to place masses m_A, m_B, m_C at the vertices A, B, C, respectively, such that each intersection point D, E, F, G balances its corresponding line and masses. That is D balances BC with m_B, m_C and also AD with m_A, m_D, and so on. The masses at the intersection points m_D, m_E, m_F, m_G are the sum of masses on the corresponding segments.

Since $\frac{BD}{DC} = \frac{5}{3}$, we need $\frac{m_B}{m_C} = \frac{3}{5}$. Also, $\frac{BF}{AF} = \frac{2}{3}$, and therefore, $\frac{m_B}{m_A} = \frac{3}{2}$. Based on these ratios, we can assign $m_B = 3, m_C = 5, m_A = 2$. Then $m_D = m_B + m_C = 8$, $m_F = m_B + m_A = 5$, and $m_E = m_A + m_C = 7$. Finally, $m_G = m_A + m_D = m_B + m_E = m_C + m_F = m_A + m_B + m_C = 10$.



Now, we can use these masses to get all segment ratios:

1. $\frac{AE}{CE} = \frac{m_C}{m_A} = \boxed{\frac{5}{2}}$.

2. $\frac{BG}{EG} = \frac{m_E}{m_B} = \boxed{\frac{7}{3}}$.

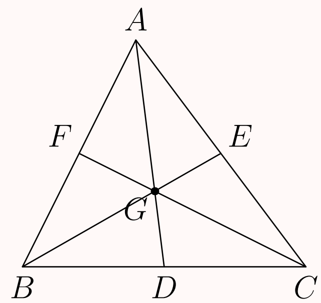
3. $\frac{DG}{AG} = \frac{m_A}{m_D} = \frac{2}{8} = \boxed{\frac{1}{4}}$.

□

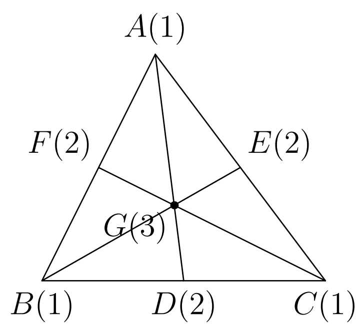
Note 2.2. In the above example, we calculated m_G while we didn't need it. Doing so helps us validate that we have a consistent set of mass assignments. We will continue doing this in subsequent mass point examples.

Example 2.3 (2014-15 Mathcounts Handbook, #294)

The medians of a triangle intersect at a point in the interior of the triangle as shown. What is the ratio of the lengths of the shorter and longer segments into which each median is divided at the point of intersection?



Solution. Since D, E, F are the midpoints of BC, AC, AB, we can simply assign $m_A = m_B = m_C = 1$, yielding $m_D = m_E = m_F = 2$ and $m_G = 3$.



Therefore, $\frac{GD}{AD} = \frac{EG}{BG} = \frac{FG}{CG} = \boxed{\frac{1}{2}}$. □

Remark 2.4. The above argument also proves concurrency of the intersection of the 3 medians as both BE and CF intersect with AD such that $\frac{AG}{GD} = \frac{2}{1}$, and therefore at the same point.

Fact 2.5. The point of intersection of the medians is called the *centroid* of the triangle. Archimedes showed that the point where the medians are concurrent (the centroid) is the center of gravity of a triangular shape of uniform thickness and density.

§2.1.1 Split Mass with Transversals

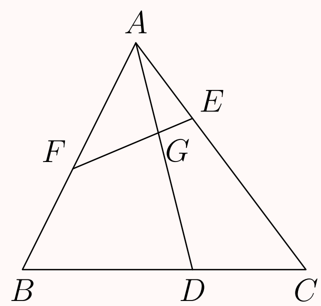
Mass point technique may also used when we have transversals by applying a technique called *splitting masses*. The technique in summary is:

- Balance the two segments on the ends of the transversal separately and assign the common vertex two different masses (split-masses).
- Along seacants from this vertex, the combined mases from the two split masses is used.

This is best illustrated with an example.

Example 2.6 (2014-15 Mathcounts Handbook, #299-300)

Triangle ABC, shown here, has cevian AD and transversal EF intersecting at G, with $AE : CE = 1 : 2, AF : BF = 5 : 4$ and $BD : CD = 3 : 2$.

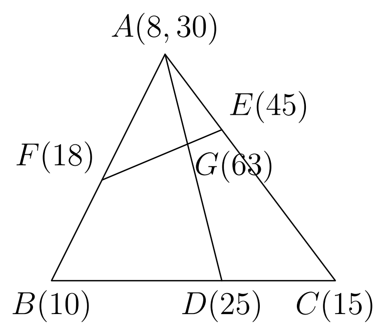


1. What is the ratio of AG to DG?
2. What is the ratio of EG to FG?

Solution. Since AD is a cevian, as in the prior examples, to balance BD we need to assign $\frac{m_B}{m_C} = \frac{2}{3}$.

To handle the transversal EF, we will assign A two masses, m_{AB}, m_{AC} to balance AB, AC at F, E, respectively. We need $\frac{m_{AB}}{m_B} = \frac{4}{5}$, and $\frac{m_{AC}}{m_C} = \frac{2}{1}$.

To achieve these ratios, we can assign $m_B = 10, m_C = 15, m_{AB} = 8, m_{AC} = 30$, leading to $m_F = m_{AB} + m_B = 18$, $m_E = m_{AC} + m_C = 45$, and $m_G = m_E + m_F = (m_{AB} + m_{AC}) + m_D = 63$.



Now we are ready to answer all ratios:

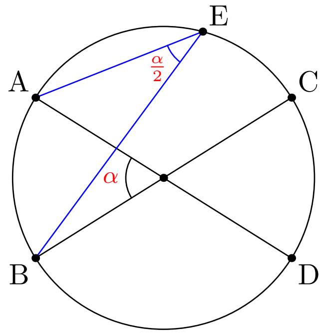
1. $\dfrac{AG}{DG} = \dfrac{m_D}{m_A = m_{AB} + m_{AC}} = \dfrac{25}{8 + 30} = \boxed{\dfrac{25}{38}}$

2. $\frac{EG}{FG} = \frac{m_F}{m_E} = \frac{18}{45} = \boxed{\frac{2}{5}}$

$\square$

§2.2 Angles in Circles

§2.2.1 Central Angles and Inscribed Angles



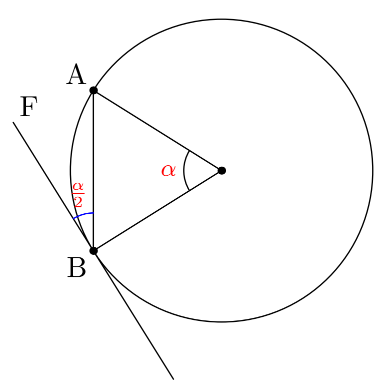
- Central angle is equal to intercepted arc; that is $m\overparen{AB} = m\overparen{CD} = \alpha$.

- Inscribed angle is half the intercepted arc that is $\angle AEB = \frac{1}{2}m\overparen{AB} = \frac{\alpha}{2}$.

- Congruent arcs have congruent central angles and vice versa.

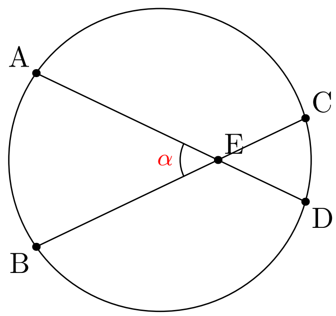
- Congruent arcs (and central angles) have congruent chords and vice versa.

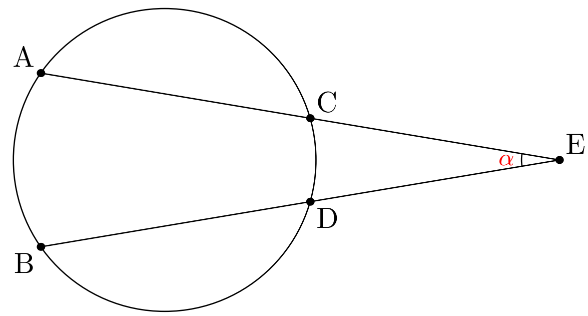
§2.2.2 Tangent Chord Angle



An angle formed by an intersecting tangent and chord has its vertex "on" the circle. In other words, the tangent chord angle is half the intercepted arc; that is $\angle ABF = \frac{1}{2}m\overparen{AB} = \frac{\alpha}{2}$.

§2.2.3 Intersecting Chords/Seacants

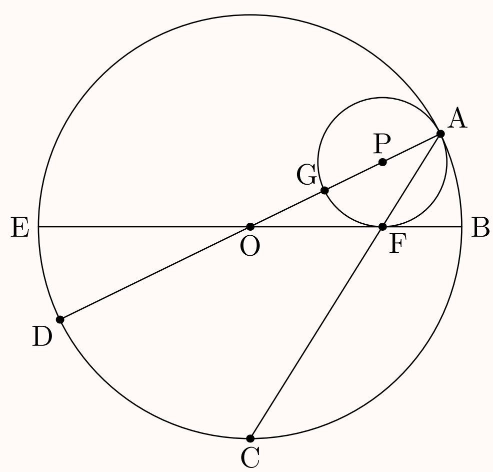




- When the chords intersect inside the circle, the angle formed is the average of the two intercepted arcs; that is $\angle AEB = \alpha = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$.
- When the seacants intersect outside the circle, the angle formed is half the difference of the two intercepted arcs; that is $\angle AEB = \alpha = \frac{1}{2}(m\overset{\frown}{AB} - m\overset{\frown}{CD})$.
- The second relation applies even when one or both of the seacants are **tangents**.

Example 2.7 (2016-17 Mathcounts Handbook, #240)

Circle P is internally tangent to circle O at A, as shown. AC and BE intersect at F, which is also the point of tangency between BE and circle P. AD and BE are diameters of circle O, and AG is a diameter of circle P. If $m\overset{\frown}{CD} = 50°$, what is the measure of minor $\overset{\frown}{BC}$?



Solution. We are given $m\overset{\frown}{CD} = m\overset{\frown}{FG} - 50°$. Since AG is diameter of the circle P, we have $m\overset{\frown}{AFG} = 180°$, and therefore $m\overset{\frown}{AF} = 180 - m\overset{\frown}{FG} = 130°$.

Now, $\angle AOB$ is the external intersection of the two seacants AG and BF for circle P. So $\angle AOB = \frac{1}{2}(m\overset{\frown}{AF} - m\overset{\frown}{GG}) = \frac{130-50}{2} = 40$. But $\angle AOB = m\overset{\frown}{DE}$.

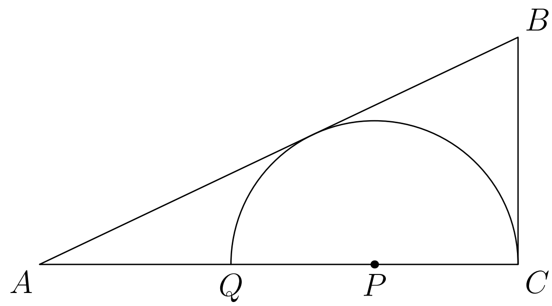
Since EB is the diameter of the circle O, $m\overset{\frown}{DE} + m\overset{\frown}{DC} + m\overset{\frown}{CB} = 180$, and therefore, $m\overset{\frown}{BC} = 180 - 40 - 50 = \boxed{90°}$. □

Remark 2.8. With some more tools in our geometry toolbox (homethety of circles), we could see that the $m\widehat{CD}$ was not needed, nor was it necessary for EB to be diameter. In the configuration above, C would be the midpoint of the arc regardless.

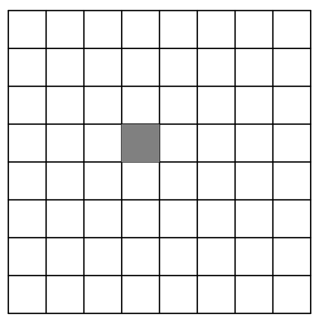
This is the generalized statement: Let $\overline{EB}$ be a chord of circle O. Let P be a circle tangent to EB at F and internally tangent to O at A. Then ray AF passes through the midpoint C of $\widehat{EB}$ not containing A.

§3 Problems

Problem 3.1 (Mathcounts National 2017, Sprint #13). In right triangle ABC with right angle at vertex C, a semicircle is constructed, as shown, with center P on leg AC, so that the semicircle is tangent to leg BC at C, tangent to the hypotenuse AB, and intersects leg AC at Q between A and C. The ratio of AQ to QC is 2:3. If BC = 12, then what is the value of AC? Express your answer in simplest radical form.

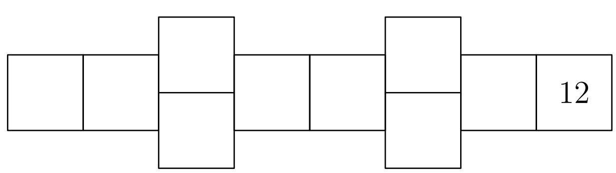


Problem 3.2 (Mathcounts National 2017, Sprint #14). Philippa stands on the shaded square of the 8-by-8 checkerboard shown. She moves to one of the four adjacent squares sharing an edge with her starting square, with each of the four squares equally likely to be chosen. She then makes two more moves to adjacent squares in the same way. Given any square S, let $P(S)$ be the probability that Philippa lands on that square after her third move. What is the greatest possible value of $P(S)$? Express your answer as a common fraction.



Problem 3.3 (Mathcounts National 2017, Sprint #19). Sam creates a six-digit positive integer by writing the digit 7 in the hundred-thousands place, and then tossing a fair coin five times. If the coin comes up heads, he writes a 7 for the next digit; if the coin comes up tails, he writes a 0 for the next digit. What is the probability that Sam's number is divisible by 77? Express your answer as a common fraction.

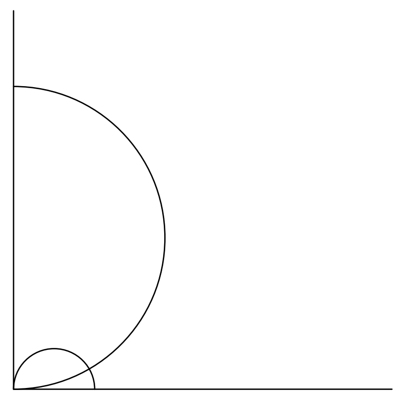
Problem 3.4 (Mathcounts National 2017, Sprint #22). How many ways are there to fill in each empty square in the diagram below with a positive integer so that no integer appears more than once in the diagram, and every integer in the diagram is less than each integer to its right?



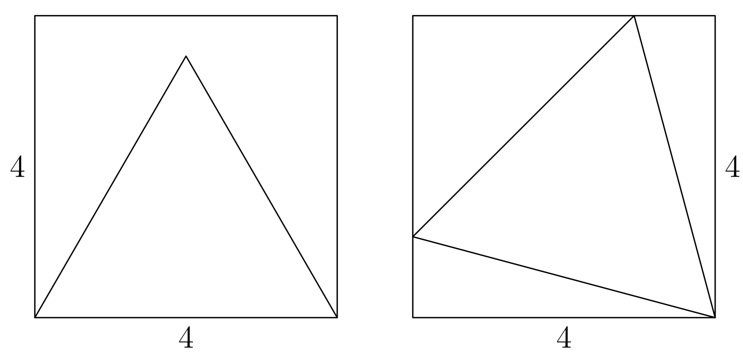
Problem 3.5 (Mathcounts National 2017, Sprint #23). What is the total surface area of the largest regular tetrahedron that can be inscribed inside of a cube of edge length 1 cm? Express your answer in simplest radical form.

Problem 3.6 (Mathcounts National 2017, Sprint #24)**.** Penny flips three fair coins into a box with two compartments. Each compartment is equally likely to receive each of the coins. What is the probability that either of the compartments has at least two coins that landed heads? Express your answer as a common fraction.

Problem 3.7 (Mathcounts National 2017, Sprint #30)**.** In the figure shown, two lines intersect at a right angle, and two semicircles are drawn so that each semicircle has its diameter on one line and is tangent to the other line. The larger semicircle has radius 1. The smaller semicircle intersects the larger semicircle, dividing the larger semicircular arc in the ratio 1:5. What is the radius of the smaller semicircle? Express your answer in simplest radical form.

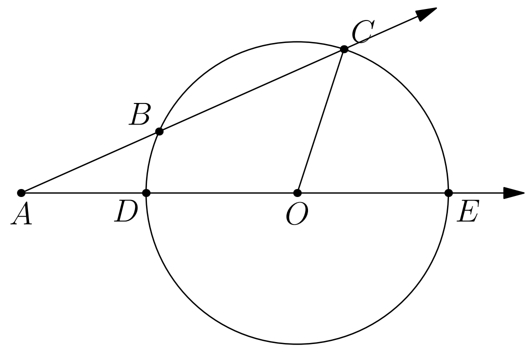


Problem 3.8 (Mathcounts National 2017, Target #6)**.** Two congruent squares with side length 4 have equilateral triangles constructed in them as shown. In one square, one side of the equilateral triangle is a side of the square. In the other square, the equilateral triangle has one vertex at a vertex of the square and its other two vertices are on the sides of the square. The absolute difference of the areas of the two triangles can be expressed in simplest radical form as $a\sqrt{b}+c$. What is the value of $a+b+c$?



Problem 3.9 (Mathcounts National 2017, Target #3)**.** There are a hundred competitors at the National Debating Contest, two from each of the 50 states. In how many ways can five finalists be chosen if no state may have more than one finalist?

Problem 3.10 (Mathcounts National 2017, Target #4)**.** Rays AC and AE intersect circle O at B and D, respectively. Segment DE is a diameter of circle O and $AB = \frac{1}{2}DE$. If the measure of $\angle BAD$ is 24 degrees, what is the degree measure of $\angle COE$?



Problem 3.11 (Mathcounts National 2016, Sprint #21)**.** Manny has two red socks, two white socks, and two blue socks. He plans to choose two socks at random to wear today, two of the remaining socks to wear tomorrow, and will wear the last two remaining socks the next day. What is the probability that on all three days, Manny wears two different colored socks? Express your answer as a common fraction.

Problem 3.12 (Mathcounts National 2016, Sprint #30)**.** In isoceles triangle ABC, with base BC of length 23 cm, points P and Q are chosen such that $BP = CQ = 9$ cm. If segments AP and AQ trisect angle BAC, what is the perimeter of $\triangle ABC$?

§4 Challenge Problems

Problem 4.1 (2001 AIME I, #7)**.** Triangle ABC has $AB = 21$, $AC = 22$ and $BC = 20$. Points D and E are located on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE}$ is parallel to $\overline{BC}$ and contains the center of the inscribed circle of triangle ABC. Then $DE = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

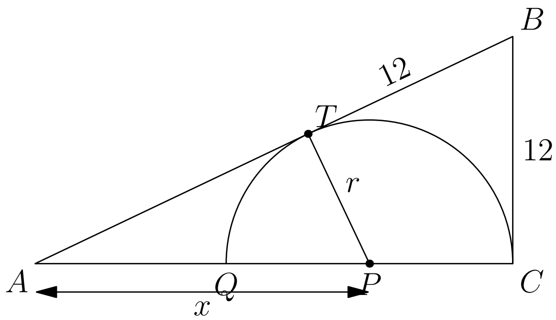
Hint: The incenter is at the intersection of angle bisectors.

Problem 4.2 (1998 AIME, # 9)**.** Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

Problem 4.3 (HMMT 2003)**.** Find the smallest n such that $n!$ ends in 290 zeroes.

§5 Solution Sketches

3.1 Using the given ratio $\frac{AQ}{CQ} = \frac{x-r}{2r} = \frac{2}{3}$, we get $\frac{x}{r} = \frac{7}{3}$. Next, since $\triangle PTA \sim \triangle BCA$, $\frac{x}{r} = \frac{12+AT}{12} = \frac{7}{3}$. Therefore, $AT = 16$.



From Pythagoas on $\triangle ABC$, $AC^2 = 28^2 - 12^2 = 640$, therefore, $AC = \boxed{8\sqrt{10}}$.

3.2 At any step, Phillipa can move L, R, U, D each with probability $\frac{1}{4}$. Also, L and R cancel each other as well as U and D cancel each other. Next, $x + y$ of the final position has to be odd and less than 3. So the 3 (considering symmetry) choices are:

- (0, 1), (1, 0), (0 - 1), (-1, 0): Let's consider (0, 1). We can get there with some combination of U, L, R or U, D, U. We have $3! = 6$ ways for U, L, R and $\binom{3}{1} = 3$ ways for U, D, U, for a total of 9 ways. This gives probability of these cells to be $\frac{6+3}{4^3} = \boxed{\frac{9}{64}}$.
- (1, 2), (2, 1), (2, -1), (1, -2), ...: Let's consider (1, 2). We get there with a combination of U, R, R, which is in $\binom{3}{1} = 3$ ways, for a probability of $\frac{3}{64}$.
- (0, 3), (3, 0), ...: Let's consider (0, 3). We get there only one way, U, U, U. So probability of $\frac{1}{64}$.

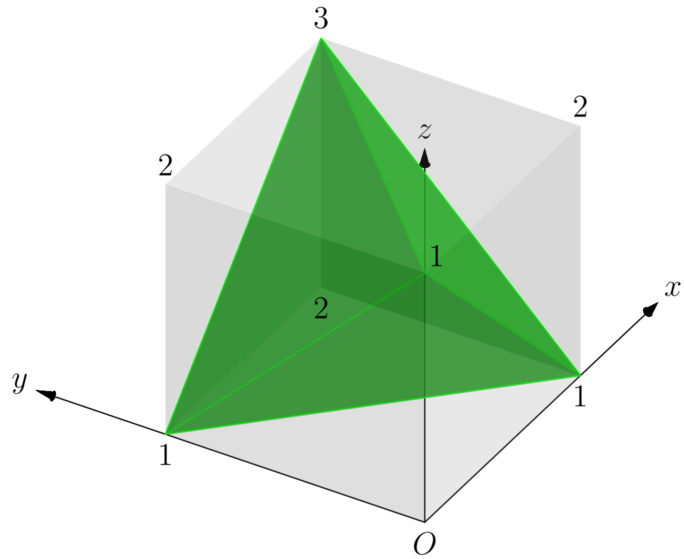
3.3 Since the digits are only 7 or 0, it will always be divisible by 7. So we only need to worry about divisibility by 11. We use divisibility by 11 rule and do case work on total number of 7s:

- 2 7s: The additional 7 can go in one of the three odd places - 3 ways
- 4 7s: 2 additional 7s should go in one of the three odd places, and one should go in one of the 2 even places - $\binom{3}{2} \cdot \binom{2}{1} = 6$ ways
- 6 7's: 1 way

Therefore, we can get the desired number in $3 + 6 + 1 = 10$ ways and there are 2^5 total numbers. Hence, the desired probability is $\frac{10}{2^5} = \boxed{\frac{5}{16}}$.

3.4 Once we have chosen the 9 numbers to use, we have only 2 choices at points where we have vertically stacked cells. Therefore we can do this in $\binom{11}{9} \cdot 2 \cdot 2 = \boxed{220}$ ways.

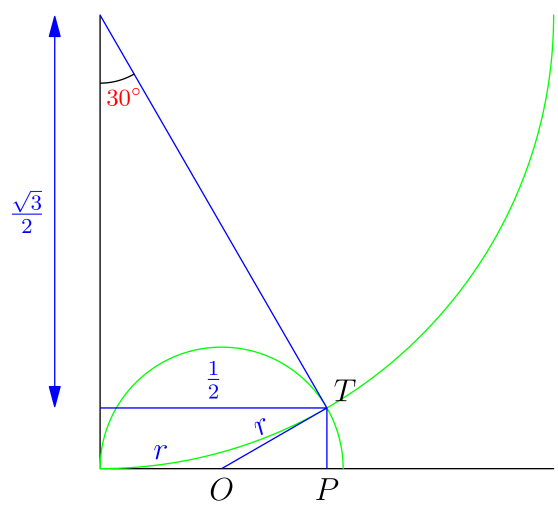
3.5 The largest tetrahedron can be fit in a cube by putting one vertex of the tetrahedron on a vertex of the cube and the the 3 vertices on the vertices that share a face with the first vertex and are diagnonally opposite to it. In this configuration the side of the tetrahedron $a = \sqrt{2}s = \sqrt{2}$. And surface area of a regular tetrahedron is $\sqrt{3}a^2 = \boxed{2\sqrt{3}}$.



3.6 We can calculate $P(\text{at least two heads})$

$$\begin{aligned}
&= P(\text{at least two heads} \mid \text{three in box}) \cdot P(\text{three in box}) \\
&\quad + P(\text{two heads} \mid \text{two in box}) \cdot P(\text{two in box}), \\
&= (3 \cdot \frac{1}{2^3} + 1 \cdot \frac{1}{2^3}) \cdot \frac{2}{2^3} + \frac{1}{2^2} \cdot (1 - \frac{2}{2^3}), \\
&= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4}, \\
&= \boxed{\frac{5}{16}}.
\end{aligned}$$

3.7 First, note that the smaller portion of the cut arc is $\frac{1}{6} \cdot 180 = 30°$. We know that $\sin 30 = \frac{1}{2}$ and $\cos 30 = \frac{\sqrt{3}}{2}$.



Applying Pythagoras on $\triangle OTP$, we have

$$\begin{aligned}
r^2 &= (\frac{1}{2} - r)^2 + (1 - \frac{\sqrt{3}}{2})^2, \\
r &= \boxed{2 - \sqrt{3}}.
\end{aligned}$$

3.8 The first equilateral triangle is one with side length 4, so its area is $\frac{1}{2}\frac{\sqrt{3}}{2}s^2 = 4\sqrt{3}$.

In the other square, we see that the big right triangle is a 45-45-90 triangle. So one of the angles of the small right triangles is $180 - 60 - 45 = 75°$, and so it's a 75-15-90 triangle, and its side length is $s = \frac{4}{\cos 15°}$. If one knows that $\cos 15° = \frac{\sqrt{6}+\sqrt{2}}{4}$, they can plug it in to get $s = 4(\sqrt{6} - \sqrt{2})$.

If not, this can be solved by Pythagoras theorem. Length of a leg of the 45-45-90 triangle be x, then by Pythagoras on that triangle we have $s^2 = 2x^2$, and by Pythagoras on the 75-15-90 triangle, we have $s^2 = (4 - x)^2 + 4^2$. Setting the two equal and solving the quadratic we get $x = 4(\sqrt{3} - 1)$, and therefore, $s = 4(\sqrt{6} - \sqrt{2})$.

> **Remark 5.1.** This configuration is one of the ways to derive the value for $\cos 15°$.

The difference in the areas between two equilateral triangles is $\frac{\sqrt{3}}{4}((\sqrt{6} - \sqrt{2})^2 - 1^2) = 28\sqrt{3} - 48$, and so the desired answer is $28 + 3 - 48 = \boxed{-17}$.

3.9 We have to choose 5 states from 50 states, and then for each selected state, we can select from one of the two contestants. We can do this in $\binom{50}{5} \cdot 2^5 = \boxed{67800320}$ ways.

3.10 First, $AB = \frac{1}{2}DE$ means that AB is same size as radius and therefore, $\triangle ABO$ is isoceles and $\angle BOA = \angle BAO = 24°$.

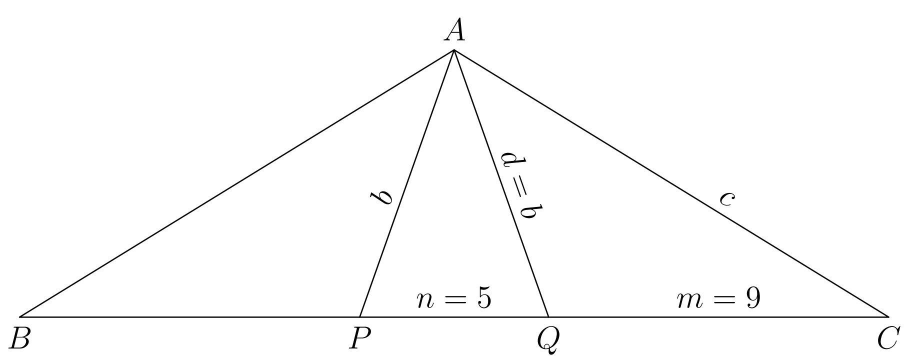
Next, from the formula for two seacants intersecting outside the circle, we have $= \frac{1}{2}(\widehat{CE} - \widehat{BD}) = \frac{1}{2}(\angle COE - \angle BOA)$. From the angle equivalence above, we get $\angle BAO = \frac{1}{2}(\angle COE - \angle BAO)$, or $= 3 \cdot \angle BAO = 3 \cdot 24 = \boxed{72}$ degrees.

3.11 First, the total number of ways to select socks is $\binom{6}{2}$ on the first day and $\binom{4}{2}$ ways on the second day, for a total of $\binom{6}{2}\binom{4}{2} = 90$ ways.

Now, let's count the ways he can select so that on any day he wears different colors. First, he can choose the days to wear red in $\binom{3}{2}$ ways. Then he can choose the days to wear blue also in $\binom{3}{2}$ ways; however, if he chooses blue on the same days as red, then white will end up being used together. Correcting for this, the number of valid ways to choose the days to wear blue is $\binom{3}{2} - 1$. However, we are not done with the numerator. When we first counted the total number of possible ways Manny could pair his socks, we assumed that each sock was distinct, namely that the two red socks were different, and so on. This led to an overcount on the total number of pairings (For example, on the first day, there are only 6 possibilities if socks were not distinct, as opposed to the 15 we calculated). To compensate, we multiply our previous result of $\binom{3}{2} \cdot (\binom{3}{2} - 1)$ by 2^3 (2 ways for each pair of socks since we are assuming each sock is distinct, as we did when finding the total pairings).

Therefore the desired probability is $\frac{6 \cdot 8}{90} = \boxed{\frac{8}{15}}$.

3.12 We can use the angle bisector theorem to get the ratio of the cevian, b in the diagram below, to one of the sides, c in the diagram, as $\frac{b}{n} = \frac{c}{m}$, that is $\frac{b}{5} = \frac{c}{9}$, or $b = \frac{5c}{9}$.

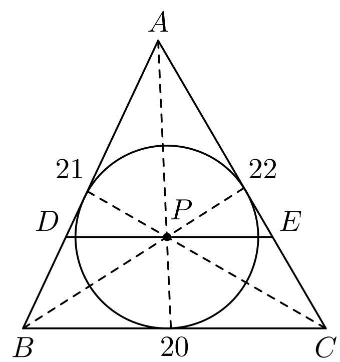


Then we can solve this by either doing Pythagoras theorem twice, or by using Stewart's theorem. Using the Stewart's theorem, we get

$$\begin{aligned}
(m+n)(mn+d^2) &= b^2 \cdot m + c^2 \cdot n, \\
14(45+b^2) &= 9b^2 + 5c^2, \\
14 \cdot 45 &= 5c^2 - 5b^2, \\
c^2 - b^2 &= 14 \cdot 9, \\
c^2 - \frac{81}{25}c^2 &= 14 \cdot 9, \\
c^2 &= \frac{9 \cdot 81}{4}, \\
c &= \frac{27}{2}.
\end{aligned}$$

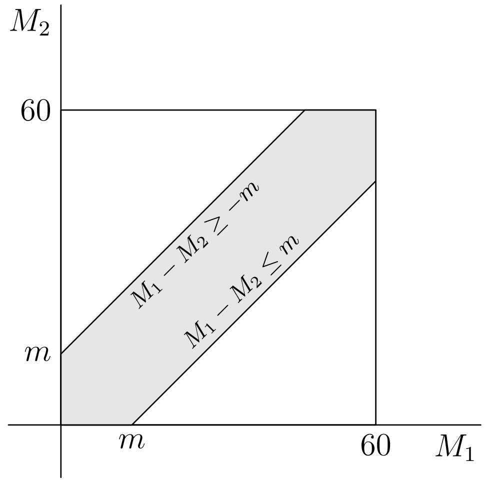
Therefore, the perimeter is $2 \cdot \frac{27}{2} + 23 = \boxed{50}$.

4.1 Let P be the incenter; then it is be the intersection of all three angle bisectors. Draw the bisector AP to where it intersects BC, and name the intersection F.



Using the angle bisector theorem, we know the ratio $BF : CF$ is $21 : 22$, thus we shall assign a weight of 22 to point B and a weight of 21 to point C, giving F a weight of 43. In the same manner, using another bisector, we find that A has a weight of 20. So, now we know P has a weight of 63, and the ratio of $FP : PA$ is $20 : 43$. Therefore, the smaller similar triangle ADE is $43/63$ the height of the original triangle ABC. So, DE is $43/63$ the size of BC. Multiplying this ratio by the length of BC, we find DE is $860/63 = m/n$. Therefore, $m + n = \boxed{923}$.

4.2 Let the two mathematicians be M_1 and M_2. Consider plotting the times that they are on break on a coordinate plane with one axis being the time M_1 arrives and the second axis being the time M_2 arrives (in minutes past 9 a.m.). The two mathematicians meet each other when $|M_1 - M_2| \leq m$. Also because the mathematicians arrive between 9 and 10, $0 \leq M_1, M_2 \leq 60$. Therefore, 60×60 square represents the possible arrival times of the mathematicians, while the shaded region represents the arrival times where they meet.



It's easier to compute the area of the unshaded region over the area of the total region, which is the probability that the mathematicians do not meet:
$\frac{(60-m)^2}{60^2} = 0.6 \Rightarrow (60-m)^2 = 36 \cdot 60 \Rightarrow 60 - m = 12\sqrt{15} \Rightarrow m = 60 - 12\sqrt{15}$
So the answer is $60 + 12 + 15 = \boxed{87}$.

4.3 Since there are more 2's than 5's in $n!$, we need to find the smallest value of n such that $n!$ has 290 5's. That is,

$$290 = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots .$$

We can approximate the sum by losing the floors and treating it as a geometric series. Therefore,

$$290 \approx \frac{\frac{n}{5}}{1 - \frac{1}{5}},$$

yielding $b \approx 1160$. When we plug 1160 into the original formula, instead of 290 we get 288. So we need two more 5s, that is go up by 10, and our desired $n = \boxed{1170}$.