Handout 2 - Number Sense

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§1 Divisibility Rules

Here as some important divisibility rules to know.

- 2: The last digit is even.
- 3: The sum of the digits is divisible by 3.
- 4: The last 2 digits are divisible by 4.
- 5: The last digit is either 0 or 5.
- 6: The number is divisible by both 2 and 3
- 8: The last 3 digits are divisible by 8.
- 9: The sum of the digits is divisible by 9.
- 11: The difference of the alternating sums of digits is divisible by 11.

Example 1.1
Is 918,082 divisible by 2, 3, 4, 11?

Solution. To do this, we could use the divisibility rules by 2, 3, 4, and 11.

- 2: Last digit is even, so yes.
- 3: Sum of digits is $9 + 1 + 8 + 0 + 8 + 2 = 28$ is not divisible by 3, so no
- 4: 82 is not divisible by 4, so no.
- 11: Difference of alternating sum is $9 - 1 + 8 - 0 + 8 - 2 = 22$ is divisible by 11, so yes.

□

§2 Primes and Composites

Definition 2.1 (Prime Number)**.** A prime number is an integer greater than one that is divisible only by one and itself.

Definition 2.2 (Composite Number)**.** A composite number is a positive integer that can be formed by multiplying two smaller positive integers. Equivalently, it is a positive integer that has at least one divisor other than 1 and itself.

> **Remark 2.3.** From the above definitions, 1 is neither a prime nor a composite.

Example 2.4

Three consecutive prime numbers, each less than 100, have a sum that is a multiple of 5. What is the greatest possible sum? *(Mathcounts)*

Solution. The primes less than 100 are, in decreasing order, 97, 89, 83, 79, 73, 71, 67, 61, 59, 53, 47, 43, 41, 37, 31, 29, 23, 19, 17, 13, 11, 7, 5, 3, 2. Starting with the first triple of primes on the list, add the remainders when each prime is divided by 5 and see whether the sum is a multiple of 5, in which case the sum of the three consecutive primes is a multiple of 5: 2+4+3=9, 4+3+4=11, 3+4+3=10. Aha! This means that $83 + 79 + 73 = \boxed{235}$ is the greatest possible sum of three consecutive prime numbers, each less than 100, that have a sum that is a multiple of 5. □

Example 2.5

To determine whether a number N is prime, we must test for divisibility by every prime less than or equal to the square root of N. How many primes must we test to determine whether 2003 is prime? *(Mathcounts)*

Solution. We must test every prime less than or equal to $\sqrt{2003} < 45$. There are $\boxed{14}$ such primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, and 43. □

§3 Multiples and Divisors

§3.1 Least Common Multiple (LCM)

Example 3.1

Planets X, Y and Z take 360, 450 and 540 days, respectively, to rotate around the same sun. If the three planets are lined up in a ray having the sun as its endpoint, what is the minimum positive number of days before they are all in the exact same locations again? *(Mathcounts)*

Solution. Solution: We are asked to find the least common multiple of 360, 450 and 540. We prime factorize

$$360 = 2^3 \cdot 3^2 \cdot 5$$
$$450 = 2 \cdot 3^2 \cdot 5^2$$
$$540 = 2^2 \cdot 3^3 \cdot 5$$

and take the largest exponent for each of the primes to get a least common multiple of $2^3 \cdot 3^3 \cdot 5^2 = \boxed{5400}$. □

§4 Factors

Example 4.1

What is the largest prime factor of 999? *(Mathcounts)*

Solution. First, divide out 9 to obtain $999 = 9 \cdot 111$. Since $1 + 1 + 1 = 3$, 111 is divisible by 3. Dividing, we find $111 = 3 \cdot 37$. Therefore, $999 = 3^2 \cdot 3 \cdot 37 = 3^3 \cdot 37$ and the largest prime factor of 999 is $\boxed{37}$. □

Example 4.2

How many factors of 8000 are perfect squares? *(Mathcounts)*

Solution. Any factor of $8000 = 2^6 \cdot 5^3$ is in the form $2^a \cdot 5^b$ for $0 \le a \le 6$ and $0 \le b \le 3$. To count the number of perfect square factors, we must count the factors of $2^6 \cdot 5^3$ that have $a = 0$, 2, 4 or 6 and $b = 0$ or 2. This gives $4 \cdot 2 = \boxed{8}$ perfect square factors. □

Example 4.3

How many positive integers are less than and relatively prime to 24?

Solution. A number that is relatively prime to another will have no prime factors in common. The prime factorization of 24 is $2^3 \cdot 3$, and we consider 1 to be relatively prime to all integers. Thus, we can count the integers we do not want, which are positive multiples of 2 or 3 that are 24 or below (numbers that contain a factor of 2 or 3).

There are $\frac{24}{2} = 12$ positive multiples of 2 and $\frac{24}{3} = 8$ positive multiples of 3 that are 24 or below. However, we have overcounted the numbers that are multiples of both 2 and 3 (multiples of 6), of which there are $\frac{24}{6} = 4$. So there are $12 + 8 - 4 = 16$ integers we do not want. Thus, there are $24 - 16 = \boxed{8}$ positive integers less than and relatively prime to 24. □

§5 Modular Arithmetic

Modular arithmetic is a system of integer arithmetic that enables us obtain information and draw conclusions about large quantities and calculations. It would be extremely helpful, for instance, when asked to find the units digit of 2^{2015} if we didn't really have to calculate the value of the expression to get that information. Modular arithmetic allows us to do just that!

The simplest example of modular arithmetic is commonly referred to as "clock arithmetic."

Example 5.1

It is 3 o'clock now. What time it will be in 145 hours. *(Mathcounts)*

Solution. We could count from 3 o'clock for 145 consecutive hours. We certainly wouldn't be expected to count 145 hours starting with 3 o'clock. Suppose we did counting the hours from 3 o'clock. What happens when we get to 12 o'clock? We continue counting but begin a new 12-hour cycle. Instead of counting 145 hours, we can just see how many of these 12-hour cycles we'd go through counting 145 hours. More importantly, we need to determine how many hours would remain after making it through the last full 12-hour cycle.

In this example, the value 12 is called the modulus and what is left over is called the remainder. In this case, we can determine fairly quickly that there are 12 full 12-hour cycles in 145 hours, with a remainder of 1 hour (since $12 \times 12 = 144$ and $145 - 144 = 1$).

The remainder of 1 tells us that it will be the same time 145 hours after 3 o'clock that it will be 1 hour after 3 o'clock. And that time is $\boxed{4}$ o'clock. □

Remark 5.2. The above in standard arithmetic is $145 = 12 \times 12 + 1$. In modular arithmetic we write this as

$$145 \equiv 1 \pmod{12},$$

which is read as 145 is congruent to 1 mod 12.

Example 5.3

If the current month is July, what month will it be in 152 months? *(Mathcounts)*

Solution. $152 = 12 \cdot 12 + 8$; therefore, in 152 months, we have 12 years and 8 months. Therefore, the month will be $7 + 8 = 15$. However, since there are only 12 months, this will correspond to $15 = 12 \cdot 1 + 3$, or 3rd month, that is $\boxed{\text{March}}$.

The above is equivalent to doing $(7 + 152) \pmod{12}$, which is $(7 + 8) \pmod{12}$ since $152 \equiv 8 \pmod{12}$. Therefore, our answer is $15 \equiv \boxed{3} \pmod{12}$.

□

§5.1 Modular Addition

Example 5.4

What is the remainder when $9813 + 7762 + 11252$ is divided by 10? *(Mathcounts)*

Solution.

$$\begin{aligned} 9813 + 7762 + 11252 &= (981 \times 10 + 3) + (776 \times 10 + 2) + (1125 \times 10 + 2) \\ &= (981 + 776 + 1125) \times 10 + (3 + 2 + 2) \end{aligned}$$

Since we are only interested in the remainder, we need only focus on the last part.

We see that the remainder is $3 + 2 + 2 = \boxed{7}$.

Written in modular arithmetic notation it would look like this:

$$9813 + 7761 + 11252 \equiv 3 + 2 + 2 \equiv 7 \pmod{10}.$$

□

In general, let a_1, a_2, b_1, b_2 satisfy

$$\begin{aligned} a_1 &\equiv a_2 \pmod{m} \\ b_1 &\equiv b_2 \pmod{m} \end{aligned}$$

Then addition of these integers holds as:

$$a_1 + b_1 \equiv a_2 + b_2 \pmod{m}.$$

§5.2 Modular Multiplication

Example 5.5

What is the remainder when 9813×7762 is divided by 10? *(Mathcounts)*

Solution.

$$\begin{aligned} 9813 \times 7762 &= (981 \times 10 + 3) \times (776 \times 10 + 2) \\ &= (981 \times 776 \times 102) + (981 \times 2 \times 10) + (776 \times 3 \times 10) + (3 \times 2) \end{aligned}$$

The first three terms are multiples of 10, and once again last term is the remainder $3 \times 2 = \boxed{6}$.

Written in modular arithmetic notation would look like this:

$$9813 \times 7762 \equiv 3 \times 2 \equiv 6 \pmod{10}.$$

□

Remark 5.6. Even without knowing modular arithmetic, we know that to get the last digit when two numbers are added or multiplied we just need to consider the addition or multiplication of the last digits.

In general, let a_1, a_2, b_1, b_2 satisfy

$$\begin{aligned} a_1 &\equiv a_2 \pmod{m} \\ a_2 &\equiv b_2 \pmod{m} \end{aligned}$$

Then addition of these integers holds as:

$$a_1 \cdot b_1 \equiv a_2 \cdot b_2 \pmod{m}.$$

§5.3 More Modular Shortcuts

Example 5.7

What is the units digit of 3^{53}? *(Mathcounts)*

Solution. First notice, that the units digit of $3^0 = 1, 3^1 = 3, 3^2 = 9, 3^3 = 7$, and $3^4 = 1$. That is unit digits repeat after every 4 powers of 3. And $53 \equiv 1 \pmod 4$, so the unit digit of 3^{53} is same as the unit digit of 3^1, which is $\boxed{3}$. □

§5.4 Linear Diophantine Equations

A *Diophantine equation* is a polynomial equation with integer coefficients where only integer solutions are allowed.

A linear Diophantine equation is of the form $ax + by = c$, where a, b, c are integers and we are looking for integer solutions for x, y.

Modular arithmetic is a handy tool to solve linear Diophantine equations.

Example 5.8

Exactly one ordered pair of integers (x, y) satisfies the equation $37x + 73y = 2016$. What is the sum of $x + y$? *(Mathcounts)*

Solution. Taking modulo 37 of both sides of the equation, we get

$$\begin{aligned} -1y &\equiv 18 \pmod{37}, \\ y &\equiv (37 - 18) \pmod{37}, \\ y &\equiv 19 \pmod{37}. \end{aligned}$$

From the original equation, we also have $y \leq \frac{2016}{73} < 27$. And the only $19 \pmod{37} < 27$ is 19, so $y = 19$. Plugging this in the original equation we get $x = \frac{2016 - 73 \cdot 19}{37} = 17$. So our ordered pair is $(17, 19)$ and $x + y = 17 + 19 = \boxed{36}$.

□

§6 Base

The base 10 number system, the number system we are most familiar with, uses the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9. Numerals with these digits in the ones, tens, hundreds and higher places express specific numerical quantities. In base 10, the number 245, for example, is composed of 2 hundreds, 4 tens and 5 ones. That is, $2(10^2) + 4(10^1) + 5(10^0) = 200 + 40 + 5 = 245$.

A base b number system uses the digits $0, 1, \ldots, b - 1$. Numerical quantities are expressed with these digits in the b^0, b^1, b^2 and higher places. In base b, if $b \geq 6$, the numeral 245_b represents the number $2(b^2) + 4(b^1) + 5(b^0)$. In base 8, for example, $245_8 = 2(8^2) + 4(8^1) + 5(8^0) = 2(64) + 4(8) + 5(1) = 128 + 32 + 5 = 165$.

Bases greater than 10 use letters to represent the digits greater than 9. For example, the 12 digits used in base 12 are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A$ and B. The numeral 10 in base 12 has 1 twelve and 0 ones. That is, $10_{12} = 1(12^1) + 0(12^0) = 1(12) + 0(1) = 12 + 0 = 12$.

Example 6.1

What is the representation of each of these numbers in base 10?

1. 24_9
2. 24_8
3. 24_7

(Mathcounts)

Solution. Converting to base 10 is straightforward.

1. $24_9 = 2 \cdot (9^1) + 4 \cdot (9^0) = \boxed{22}$
2. $24_8 = 2 \cdot (8^1) + 4 \cdot (8^0) = \boxed{20}$
3. $24_7 = 2 \cdot (7^1) + 4 \cdot (7^0) = \boxed{18}$

□

Example 6.2

What is the representation of 24 in each of the following bases?

1. 9
2. 8
3. 7

(Mathcounts)

Solution. One method to compute representation of a number n in base b is to repeatedly divide n by b until we have a quotient $< b$. Then the last quotient and the remainders written in order from last to first gives us the desired representation.

$$24 = \underline{2} \cdot 9 + \underline{6}$$

Therefore, $24 = \boxed{26_9}$.

$$24 = \underline{3} \cdot 8 + \underline{0}$$

Therefore, $24 = \boxed{30_8}$.

$$24 = \underline{3} \cdot 7 + \underline{3}$$

Therefore, $24 = \boxed{33_7}$.

□

Example 6.3

What is the representation of 4991 in base 12? *(Mathcounts)*

Solution. We use the method from the previous example:

$$\begin{aligned} 4991 &= 415 \cdot 12 + \underline{11} \text{ (B)}, \\ 415 &= 34 \cdot 12 + \underline{7}, \\ 34 &= \underline{2} \cdot 12 + \underline{10} \text{ (A)}. \end{aligned}$$

Therefore, $4991 = \boxed{2A7B_{12}}$. □

Example 6.4

Which of the following is largest?

A) 10110 in base 2

B) 43 in base 5

C) 33 in base 6

(Mathcounts)

Solution. We convert all the numbers to base 10:

$10110_2 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 16 + 4 + 2 = 22.$

$43_5 = 4 \cdot 5^1 + 3 \cdot 5^0 = 20 + 3 = 23.$

$33_6 = 3 \cdot 6^1 + 3 \cdot 6^0 = 21.$

Thus we can see that the largest number is 43_5, so $\boxed{B}$ is the answer.

$\square$

§7 Beginner Practice Problems

Problem 7.1 (Mathcounts)**.** What is the sum of all two-digit multiples of three that have units digit 1?

Problem 7.2 (Mathcounts)**.** How many positive integers less than 101 are multiples of 3, 4 or 7?

Problem 7.3 (Mathcounts)**.** Given that the digits 1, 2, 3, 4, 5 and 6 are forms two three-digit numbers, what is the greatest possible positive difference that can be obtained from the two numbers?

Problem 7.4 (Mathcounts)**.** Let LCM (a, b) be the abbreviation for the least common multiple of a and b. What is LCM (LCM (8, 14), LCM (7, 12))?

Problem 7.5 (Mathcounts)**.** What is the sum of the three missing digits in the subtraction problem $5\square, 661 - \square 2, 83\square = 17, 825$?

Problem 7.6 (AHSME 1987 #3)**.** How many primes less than 100 have 7 as the ones digit? (Assume the usual base ten representation)

Problem 7.7 (AJHSME 1987 #9)**.** When finding the sum $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$, what is the least common denominator used?

§8 Intermediate Practice Problems

Problem 8.1 (Mathcounts)**.** The product of three consecutive integers is 157,410. What is their sum?

Problem 8.2 (2021 Chapter Sprint #13)**.** What is the fewest number of consecutive primes, starting with 2, that when added together produce a number divisible by 7?

Problem 8.3. Find the remainder when 23^{19} is divided by 10.

Problem 8.4. What is the value of $\dfrac{4^{26} + 8^{15}}{32^9 + 16^{11}}$?

Problem 8.5. *oplet* Natural numbers of the form $F_n = 2^{2^n} + 1$ are called Fermat numbers. In 1640, Fermat conjectured that all numbers F_n, where $n \neq 0$, are prime. (The conjecture was later shown to be false.) What is the units digit of F_{1000}?

Problem 8.6 (AMC 8 2020 #15)**.** Suppose 15% of x equals 20% of y. What percentage of x is y?

Problem 8.7 (AMC 8 2020 #17)**.** How many positive integer factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely $1, 2, 3, 4, 6,$ and 12.)

Problem 8.8 (Mathcounts 2017 State Sprint #15)**.** What is the least positive base-10 integer that can be written as a 4-digit number in base 3 and as a 3-digit number in base 4?

Problem 8.9. If $x^y = y^x$, and x and y are unequal positive integers, what is the smallest possible value of $x + y$?

Problem 8.10. Friends Megan and Heather go to different schools. Megan has math class during first period on each of the 96 days she goes to school. She will be in math class a total of 8640 minutes this year. Heather's school year also has 96 days, but her math class only meets every other day, so her class periods are longer. (One day she'll have math class, and then the next day she won't.) If Megan and Heather end up with the same number of minutes of math class each year, how long are the class periods at Heather's school?

Problem 8.11 (Mathcounts Handbook, 2012-13)**.** After Carlos accidentally spills water on his paper, he is left with the partial equation $1729^2 - 2 \times 1730^2 + 17\square\square^2 = 34\square\square$, where each $\square$ represents a smudged digit, not necessarily all the same. What is the sum of the four smudged digits? *You may use calculator for this question.*

Problem 8.12 (Mathcounts Handbook 2015-16)**.** How many whole numbers n, such that $100 \leq n \leq 1000$, have the same number of odd factors as even factors?

Problem 8.13 (Mathcounts 2009 State Target #2)**.** Two arithmetic sequences A and B both begin with 30 and have common differences of absolute value 10, with sequence A increasing and sequence B decreasing. What is the absolute value of the difference between the 51st term of sequence A and the 51st term of sequence B? *You may use calculator for this question.*

Problem 8.14 (AIME 2003, I #1)**.** Given that $\frac{((3!)!)!}{3!} = k \cdot n!$, where k and n are positive integers and n is as large as possible, find $k + n$.

Problem 8.15 (Mathcounts Handbook 2014-15)**.** To weigh an object by using a balance scale, Brady places the object on one side of the scale and places enough weights on each side to make the two sides of the scale balanced. Brady's set of weights contains the minimum number necessary to measure the whole-number weight of any object from 1 to 40 pounds, inclusive. What is the greatest weight, in pounds, of a weight in Brady's set?

§9 Challenge Practice Problems

Problem 9.1 (Mathcounts 2005, National Target #6)**.** For positive integer n such that $n < 10,000$, the number $n + 2005$ has exactly 21 positive factors. What is the sum of all the possible values of n? *You may use calculator for this.*

Problem 9.2 (Mathcounts 2017, National Sprint #8)**.** Computing in base 8, a certain two-digit base-8 number N is added to five times the sum of its digits. The sum has the same digits as N but in reverse order. What is N in base 8?

Problem 9.3. In order, the first four terms of a sequence are $2, 6, 12$ and 72, where each term, beginning with the third term, is the product of the two preceding terms. If the ninth term is $2^a 3^b$, what is the value of $a + b$?

Problem 9.4 (Folklore)**.** How many zeros are there after the last nonzero digit of 125! ?

Problem 9.5 (Pascal 2013, #24)**.** Pascal High School organized three different trips. Fifty percent of the students went on the first trip, 80% went on the second trip, and 90% went on the third trip. A total of 160 students went on all three trips, and all of the other students went on exactly two trips. How many students are at Pascal High School?

Problem 9.6 (AMC 10, 2004, #13)**.** In the United States, coins have the following thicknesses: penny, 1.55 mm; nickel, 1.95 mm; dime, 1.35 mm; quarter, 1.75 mm. If a stack of these coins is exactly 14 mm high, how many coins are in the stack?

§10 Solution Sketches

7.1 The two-digit numbers with units digit 1 are 11, 21, 31, 41, 51, 61, 71, 81 and 91. We could test each one individually, but we know a number is divisible by 3 if the digits sum to a multiple of 3. This means the first number divisible by 3 is 21 because 2 + 1 = 3. The other two numbers will be 51 and 81 because 5 + 1 = 6 and 8 + 1 = 9, and also because they are 30 or 3 x 10 away from 21. The sum of these numbers is 21 + 51 + 81 = $\boxed{153}$

7.2 The smallest positive multiple of 3 is $3 \cdot 1 = 3$ and the largest, less than 101, is $3 \cdot 33 = 99$. This means there are 33 positive multiples of 3 less than 101. Similarly, $4 \cdot 1 = 4$ is the smallest multiple of 4 and the largest we need to consider is $4 \cdot 25 = 100$. So we have 25 positive multiples of 4. Lastly, we have $7 \cdot 1 = 7$ to $7 \cdot 14 = 98$ or 14 multiples of 7. You might think that the answer will be 33 + 25 + 14 = 72, but we would be over counting some numbers. Since we are looking for multiples of 3, 4 OR 7 we have to consider all combinations of these numbers. There are 8 multiples of $3 \cdot 4 = 12$ that have been counted twice, 4 multiples of $3 \cdot 7 = 21$ that have been counted twice and 3 multiples of $4 \cdot 7 = 28$ that have been counted twice. Adjusting our count, we get 72 - 8 - 4 - 3 = 57. This still isn't our answer however. We counted the number $3 \cdot 4 \cdot 7 = 84$ three times in our original count, then we deducted it three times in adjusting for our over counting. This means it isn't represented any longer. Our final answer is 57 + 1 = $\boxed{58}$ positive integers.

7.3 We need to start with creating the largest difference in the hundreds column, so we can start by placing a 6 and 1 in each of those boxes. Then we look at the tens column and place the remaining largest and smallest numbers in each of those respectively. Finally, we place the remaining two in the ones column. This gives us 654 as the first number and 123 as the second number. The difference between those is $\boxed{531}$.

7.4 The prime factorization of the four integers are $8 = 2^3$, $14 = 2 \cdot 7$, $7 = 7$ and $12 = 2^2 \cdot 3$. This means LCM (8, 14) $= 2^3 \cdot 7$ and LCM (7, 12) $= 2^2 \cdot 3 \cdot 7$. So the LCM (LCM (8, 14), LCM (7, 12)) $= 2^3 \cdot 3 \cdot 7 = \boxed{168}$. Note: you could skip a step by considering the problem as LCM (8, 14, 7, 12).

7.5 Starting with the ones column, we are taking something away from 1 and getting 5, so we will have to carry the 1 from the tens column to the ones column to get 11 - _ = 5, so the ones blank must be 6. Our remaining information is 5_,661 - _ 2,836 = 17,825. In the hundreds column, a difference of 8 requires you to have carried from the thousands column. In the thousands column, _ − 2 = 7 suggests the digit should be 0 to then become a 9 when the digit is carried. Thus, in the difference 50,661 - _ 2,836 17,825 = the remaining digit must be 3. Adding the digits 3, 0 and 6 gives a sum of $\boxed{9}$.

7.6 List out all numbers that have 7 as the ones digit less than 100: $7, 17, 27, 37, 47, 57, 67, 77, 87, 97$. Only $7, 17, 37, 47, 67$, and 97 are prime. Thus, it is $\boxed{6}$.

7.7 We want the least common multiple of $2, 3, 4, 5, 6, 7$, which is $\boxed{420}$.

8.1 The first step is to factor 157,410:

- 157,410 = 10 × 15,741 the number ends in 0, 10 is a factor
- 15,741 = 9 × 1749 the digit sum is 1 + 5 + 7 + 4 + 1 = 18, 9 is a factor
- 1749 = 11 × 159 the alternating ± digits sum is 1 7 + 4 9 = 11, 11 is a factor
- 159 = 3 × 53 the digit sum is 1 + 5 + 9 = 15, 3 is a factor

So the prime factorization of 157,410 is $2 \times 3^3 \times 5 \times 11 \times 53$. We know 53 is one of the consecutive integers and can then see that $5 \times 11 = 55$ and $2 \times 3^3 = 54$ are the other two. Their sum is $53 + 54 + 55 = \boxed{162}$.

8.2 Since the consecutive primes we consider must begin with 2, let's just add consecutive primes until we reach the first multiple of 7.

- 2 primes: $2 + 3 = 5$, not a multiple of 7
- 3 primes: $2 + 3 + 5 = 10$, not a multiple of 7
- 4 primes: $2 + 3 + 5 + 7 = 17$, not a multiple of 7
- 5 primes: $2 + 3 + 5 + 7 + 11 = 28$, a multiple of 7
- Therefore, for the fewest consecutive primes beginning with 2 that sum to a multiple of 7 is $\boxed{5}$ primes.

8.3 First, since we are interested in only the units digit, we may as well just think of 3^{19}. Now, for 3^x, the unit digits repeat in groups of 4 and they are $3, 9, 7, 1$. Since $19 = 4 \cdot 4 + 3$, our desired digit is the 3rd in the group, that is $\boxed{7}$.

8.4

$$\begin{aligned}\frac{4^{26} + 8^{15}}{32^9 + 16^{11}} &= \frac{2^{2\cdot 26} + 2^{3\cdot 15}}{2^{5\cdot 9} + 2^{4\cdot 11}},\\ &= \frac{2^{52} + 2^{45}}{2^{45} + 2^{44}},\\ &= \frac{2^{45}(2^7 + 1)}{2^{44}(2 + 1)},\\ &= \frac{2 \cdot 129}{3},\\ &= \boxed{86}.\end{aligned}$$

8.5 Let's first try to find the cycle of units digits of 2^n, starting with $n = 1$: $2, 4, 8, 6, 2, 4, 8, 6, \ldots$. The cycle of units digits of 2^n is 4 digits long: 2, 4, 8, 6. To find the units digit of 2^n, for any positive integer n, we simply need to find the remainder, R, when n is divided by 4 ($R = 1$ corresponds to the units digit 2, $R = 2$ corresponds to the units digit 4, etc.) Since $2^{1000} \div 4 = 2^{998}$ without remainder, the units digit of $2^{2^{1000}}$ is 6. Therefore, the units digit of $F_n = 2^{2^{1000}} + 1$ is $6 + 1 = \boxed{7}$.

8.6 Letting $x = 100$, the condition becomes $0.15 \cdot 100 = 0.2 \cdot y \Rightarrow 15 = \frac{y}{5} \Rightarrow y = 75$. Clearly, it follows that y is 75% of x, so the answer is $\boxed{75}$.

8.7 Since $2020 = 2^2 \cdot 5 \cdot 101$, we can simply list its factors:

$$1, 2, 4, 5, 10, 20, 101, 202, 404, 505, 1010, 2020.$$

There are 12 of these; only $1, 2, 4, 5, 101$ (i.e. 5 of them) don't have over 3 factors, so the remaining $12 - 5 = \boxed{7}$ factors have more than 3 factors.

8.8 The least positive 4-digit value in base 3 is $1000_3 = 3^3 = 27_{10}$; the least positive 3-digit value in base 4 is $100_4 = 4^2 = 16_{10}$. The least value satisfying both is the greater of the two, namely $\boxed{27}$. That decimal value is equal to 1000_3 and 123_4, confirming satisfaction of the stated criteria.

8.9 $2^4 = 4^2$, therefore smallest $x + y = 2 + 4 = \boxed{6}$.

8.10 Let's start with Megan. Megan has 96 days of school, so 96 days of math class, so she takes 96 days of math class. However, even though Heather also has 180 days of school, she has $\frac{96}{2} = 48$ days of math class. So Heather has to complete 8640 minutes of math in 48 days, giving her a math period of $\frac{8640}{48} = \boxed{180}$ minutes of math class each time math class meets.

8.11 Let A, B, C, D represent the missing digits. First, we know that $3400 \leq 34CD < 3500$. Therefore, $3400 \leq 1729^2 - 2 \times 1730^2 + 17AB^2 < 3500$. We have $1729^2 - 2 \times 1730^2 = -2,996,359$. Which means that $3400 + 2,996,359 \leq 17AB^2 < 3500 + 2,996,359$. Using calculator to take square roots, this gives us $1731.98 < 17AB < 1732.01$. Therefore $17AB = 1732$. Plugging this in we get the right side of the original equation as 3465. Therefore $A = 3, B = 2, C = 6, D = 5$ and $A + B + C + D = \boxed{16}$.

8.12 2 is the only even prime number. Consider the prime factorization $2^a p^b q^c$ where p and q are prime numbers. All of the positive factors of this number are of the form $2^x p^y q^z$, where x is an integer 0 to a, inclusive; y is drawn from 0 to b, and z is chosen from 0 to c.

If $x = 0$, the factor is odd. If $x \geq 1$, the factor is even. So for the number to have an equal number of odd and even factors, we must have $a = 1$ so that half the factors use $x = 0$ and the other half $x = 1$. This applies to other numbers with a number of odd prime factors besides 2.

So we wish to count the even numbers from 100 to 1000, inclusive, that are not divisible by 4. The even numbers are $100, 102, 104, \ldots, 1000$, of which there are 451. Every other one, starting at 100, is a multiple of 4, so there are 226 multiples of 4 and $\boxed{225}$ numbers divisible by 2 but not 4.

8.13 The 51st term of sequence A is $30 + (10 \cdot 50) = 30 + 500 = 530$.The 51st term of sequence B is $30–(10 \cdot 50) = 30–500 = –470$. The difference is $530–(–470) = 530 + 470 = \boxed{1000}$.

8.14 We use the definition of a factorial to get $\frac{((3!)!)!}{3!} = \frac{(6!)!}{3!} = \frac{720!}{3!} = \frac{720!}{6} = \frac{720 \cdot 719!}{6} = 120 \cdot 719! = k \cdot n!$ We certainly can't make n any larger if k is going to stay an integer, so the answer is $k + n = 120 + 719 = \boxed{839}$.

8.15 We can put weights on both sides of the scale, so if we have a set of weights that can balance any weight from 1 to N pounds, they will also balance any weight from -1 to $-N$ pounds. So, the next weight you would want to add to the scale would weigh $2N + 1$ pounds, so we can balance any weight from 1 to $3N + 1$ pounds. So if we start with a 1 pound weight, we get $2(1) + 1 = 3$ pounds, and for those 2, they can balance $1 + 3 = 4$ pounds. Now using the 3 pound weight $2(4) + 1 = 9$ pounds for the next weight, which will give us a total of 13 pounds. Similarly, $2(13) + 1 = 27$, so the last weight is 27 pounds, giving us a total of 40. So the answer is $\boxed{27}$. (You can also think of this as a base 3 representation problem.)

9.1 First, since $n + 2005$ has 21 positive factors, it is of the form $p^6 q^2$ as $(1 + 6) \cdot (1 + 2) = 21$. Next, we have $2005 < p^6 q^2 < 1,205$, which gives $45 \leq p^3 q < 109$. If we try $p = 2$, $7 \leq q \leq 13$ fits the constraint. If we try $p = 3$, $q = 2$ is the only one that fits the constraint. No $p \geq 5$ fits the constraint. Therefore, the sum of the 4 possibilities is

$$\left(2^6 \cdot \left(7^2 + 11^2 + 13^2\right) + 3^6 \cdot 2^2\right) - 4 \cdot 2005 = \boxed{16,592}.$$

9.2 Let $N = \overline{AB}_8$. Then the given condition translates to

$$8A + B + 5(A + B) = 8B + A.$$

Simplifying, we get $6A = B$, and the only valid digits in base 8 satisfying this are $A = 1, B = 6$. Therefore, $N = \boxed{16}$ base 8.

9.3 Let us write out the prime factorizations of each term of the sequence. We have:

$$2 = 2^1 \cdot 3^0$$
$$6 = 2^1 \cdot 3^1$$
$$12 = 2^2 \cdot 3^1$$
$$72 = 2^3 \cdot 3^2.$$

Notice that the exponent of the power of 2 on each term is equal to the sum of the exponents of the power of 2 on the previous two terms, and similarly for 3. Indeed, if two consecutive terms of the sequence are given by $2^{e_2} \cdot 3^{e_3}$ and $2^{f_2} \cdot 3^{f_3}$, then their product is $2^{e_2+f_2} \cdot 3^{e_3+f_3}$. It follows that the exponents of the 2 and 3 satisfy the Fibonacci sequence. After inspection, the ninth term is equal to $2^{F_9} \cdot 3^{F_8}$, where we calculate $F_9 = 34, F_8 = 21$. Thus, the answer is $34 + 21 = \boxed{55}$.

9.4 Trailing zero comes from power of 10, or power of 2×5, since 125! has less divisors which are multiple of 5 than 2, we count the multiples of 5^1, 5^2, and 5^3. For every 5's there are in 125!, there is a 2's to match with it to form a 10 and thus a trailing zero. The number of 5's can be found by $\lfloor \frac{125}{5^1} \rfloor + \lfloor \frac{125}{5^2} \rfloor + \lfloor \frac{125}{5^3} \rfloor = 25 + 5 + 1 = \boxed{31}$. Where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

9.5 Let x be the total number of students at Pascal H.S.
Let a be the total number of students who went on both the first trip and the second trip, but did not go on the third trip.
Let b be the total number of students who went on both the first trip and the third trip, but did not go on the second trip.
Let c be the total number of students who went on both the second trip and the third trip, but did not go on the first trip.
We note that no student went on one trip only, and that 160 students went on all three trips. Since the total number of students is x, x = a + b + c + 160 We also know

0.5x = a + b + 160,
0.8x = a + c + 160, and
0.9x = b + c + 160.

Combining this information:

x = a + b + c + 160
2x = 2a + 2b + 2c + 320
2x = 0.5x + 0.8x + 0.9x - 160
2x = 2.2x - 160
x = 800 So, there are $\boxed{800}$ students at Pascal High School.

9.6 Let p, n, d, and q be the number of pennies, nickels, dimes, and quarters used in the stack.
From the conditions above, we get the following equation:

$$155p + 195n + 135d + 175q = 1400.$$

Then we divide each side by five to get

$$31p + 39n + 27d + 35q = 280.$$

Writing both sides in terms of mod 4, we have $-p - n - d - q \equiv 0 \pmod 4$.

This means that the sum $p + n + d + q$ is divisible by 4. However, $\frac{14}{1.95} \leq p + n + d + q \leq \frac{14}{1.35}$, giving us $7 < p + n + d + q < 10$. Therefore $p + n + d + q = \boxed{8}$.