ANLY590 HW 2.1

Yunfei Zhang yz678

1. Compute the gradient vector for a plane in 3D space.

$z = f(x,y) = ax + by + c.$

Solution: $\nabla f(x,y) = (f_x(x,y), f_y(x,y)) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$

$= (a, b).$

2. Compute the gradient vector for a hyperplane

$z = f(X) = f(x_1, x_2, \cdots, x_N) = \sum_{i=1}^{N} a_i(x_i - b_i) + S = a_1x_1 + a_2x_2 + \cdots + a_Nx_N + d$

Solution: $\nabla f(X) = [\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_N}] \cancel{= a_1 + a_2 + \cdots + a_N}.$

$= (a_1, a_2, \cdots, a_N).$

3. Compute the partial derivative of the paraboloid function

$Z = f(x,y) = A(x-x_0)^2 + B(y-y_0)^2 + C$

Solution: $f_x(x,y) = \left(\frac{\delta f(x,y)}{\delta x}\right)_y = \cancel{2Ax + 2Ax_0} = 2A(x - x_0)$

$f_y(x,y) = \left(\frac{\delta f(x,y)}{\delta y}\right)_x = \cancel{2By + 2By_0} = 2B(y - y_0)$

4. $x = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ $y = (2\ 5\ 1)$ $A = \begin{pmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 5 \\ 5 & 2 \\ 1 & 4 \end{pmatrix}$

a. $x^T = (3\ 1\ 4)$ [1×3]

b. $y^T = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ [3×1]

c. $B^T = \begin{pmatrix} 3 & 5 & 1 \\ 5 & 2 & 4 \end{pmatrix}$ [2×3]

d. $x \cdot x$ ~~not defined~~ $= 9+1+16 = 26$ [0]

e. $x \cdot y^T$ ~~not defined~~ $= 6+5+4 = 15$ [0]

f. $x \times y = \begin{pmatrix} 6 & 15 & 3 \\ 2 & 5 & 1 \\ 8 & 20 & 4 \end{pmatrix}$ [3×3]

g. $y \times x = (2\times3+5\times1+1\times4)$ [1, 1]

$= (15)$

h. $A \times x = \begin{pmatrix} 4\times3+5\times1+2\times4 \\ 3\times3+1\times1+5\times4 \\ 6\times3+1\times4+3\times4 \end{pmatrix} = \begin{pmatrix} 25 \\ 30 \\ 34 \end{pmatrix}$ [3×1]

i. $A \times B \begin{pmatrix} 4\times3+5\times5+2\times1 & 4\times5+5\times2+2\times4 \\ 3\times3+1\times5+5\times1 & 3\times5+1\times2+5\times4 \\ 6\times3+4\times5+3\times1 & 6\times5+4\times2+3\times4 \end{pmatrix} = \begin{pmatrix} 39 & 38 \\ 19 & 37 \\ 41 & 50 \end{pmatrix}$ [3×2]

j. B.reshape(1,6) = (3, 5, 5, 2, 1, 4) [1, 6].

5. LLS - single variable SSE

Model: $y = M(x|p) = mx + b.$

$p = (p_0, p_1) = (m, b)$

Loss surface

$L(p) = L(m,b) = \sum_{i=1}^{N} (y_i - M(x_i, m, b))^2$

Solution:

To find a

~~$\frac{\delta L}{\delta m} = 0$~~

$$\frac{\delta L}{\delta b} = 0,$$

$$\sum_{i=1}^{n} \cdot (-2)(y_i - mx_i - b) = 0$$

$$\sum_{i=1}^{n} (y_i - mx_i) = n \cdot b.$$

$$b = \frac{1}{n} \sum_{i=1}^{n} (y_i - mx_i)$$

$$b = \bar{y} - m \cdot \bar{x}$$

Then, to find $\frac{\delta L}{\delta m} = 0,$

$$\sum_{i=0}^{n} (-2x_i)(y_i - b - mx_i) = 0.$$

$$\sum_{i=0}^{n} (x_i y_i - b x_i - m x_i^2) = 0.$$

$\because b = \bar{y} - m\bar{x},$

$$\therefore \sum_{i=0}^{n} (x_i y_i - \bar{y} x_i + m\bar{x} \cdot x_i - m x_i^2) = 0.$$

$$\sum_{i=0}^{n} (x_i y_i - \bar{y} x_i) = m \sum_{i=0}^{n} (x_i^2 - \bar{x} \cdot x_i)$$

$$m = \frac{\sum_{i=0}^{n} (x_i y_i - \bar{y} x_i)}{\sum_{i=0}^{n} (x_i^2 - \bar{x} \cdot x_i)}.$$

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$$m = \frac{x_i \sum_{i=0}^{n} (y_i - \bar{y})}{x_i \sum_{i=0}^{n} (x_i - \bar{x})} = \frac{\sum_{i=0}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=0}^{n} (x_i - \bar{x})(x_i - \bar{x})} = \frac{cov(X, Y)}{var(X)}.$$

$$b = \bar{y} - m \cdot \bar{x} = \bar{y} - \frac{cov(x, y)}{var(x)} \cdot \bar{x}.$$

6. LLS - multi-variable

model: ~~$y = M(X \mid P) = \theta_0 + \theta_1 x_1 + \theta_2 x_2$~~

general form: $h(x) = \vec{w}^T \cdot x.$

$$\vec{w} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{bmatrix} \quad x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}$$ where $x_0 = 1$

when $n = 2$,

$$\vec{w} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} \quad X = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$$L(X) = \sum_{i=1}^{N} (y_i - (w_0 x_0 + w_1 x_1 + w_2 x_2))^2$$

$$= (Y - X \cdot \vec{w})^T (Y - X \cdot \vec{w})$$

To find $\frac{\delta L(\vec{w})}{\delta(\vec{w})} = 0$

$$-2X^T(Y - X\vec{w}) = 0.$$

$$X^T Y = (X^T X) \cdot \vec{w}.$$

$$\vec{w} = \frac{X^T Y}{X^T X} = (X^T X)^{-1} \cdot X^T Y$$