009 - Random Variables

EPIB 607

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Random Variables

Expected Value of a function of a RV

Variance and SD of a function of a RV

Sums, means, differences of random variables

Linear combinations of random variables (regression slopes)

Introduction

- This central chapter addresses a fundamental concept, namely the laws governing the variance of a sum of 2, or (especially) n random variables – and even more importantly – the laws governing the variance of a difference of two random variables.
- The latter is central, not just to simple contrasts involving 2 sample means or proportions, but also in the much wider world of regression, since the variance (sampling variability) of any regression slope can be viewed as the variance of a linear combination of random errors, or random deviations, or random variables.
- So, if there is one master formula to pay attention to and to own, it is the one for the variance of a linear combination of random variables. All others are special cases of this.

Objectives

- Understand the equations for expectation and variance of both a continuous and discrete random variable.
- Why it is that, when dealing with the sum of two or more independent random variables, it is not their standard deviations that sum (add), but rather their variances.
- Likewise, when dealing with the difference of two independent random variables, or some linear combination of n independent random variables involving positive and negative weights, why it is that the component variances add, and with what weights.

Discrete Random Variable (RV)

Definition 1 (Discrete Random Variable).

A random variable that assumes only a finite (or countably infinite) number of distinct values. Discrete random variables have a finite or countably infinite number of possible values, each with positive or zero probability.

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Definition 2 (Probability Mass Function (PMF)).

The probability mass function (PMF) of a discrete random variable Y provides the possible values y and their associated probabilities by $\mathrm{P}(Y = y)$. The sum of all probabilities must sum to 1, i.e. $\sum_y \mathrm{P}(Y = y) = 1$.

Continuous Random Variable

Definition 3 (Continuous Random Variable).

A random variable is continuous if both of the following apply:

1. Its set of possible values consists either of all numbers in a single interval on the number line (possibly infinite in extent, e.g., from $-\infty$ to ∞) or all numbers in a disjoint union of such intervals.
2. No possible value of the variable has positive probability, that is, $\mathrm{P}(X = c) = 0$ for any possible value c.

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Definition 4 (Probability Density Function (PDF)).

Let Y be a continuous random variable. Then a probability distribution or probability density function (pdf) of Y is a function $f(y)$ such that for any two numbers a and b with $a \leq b$,

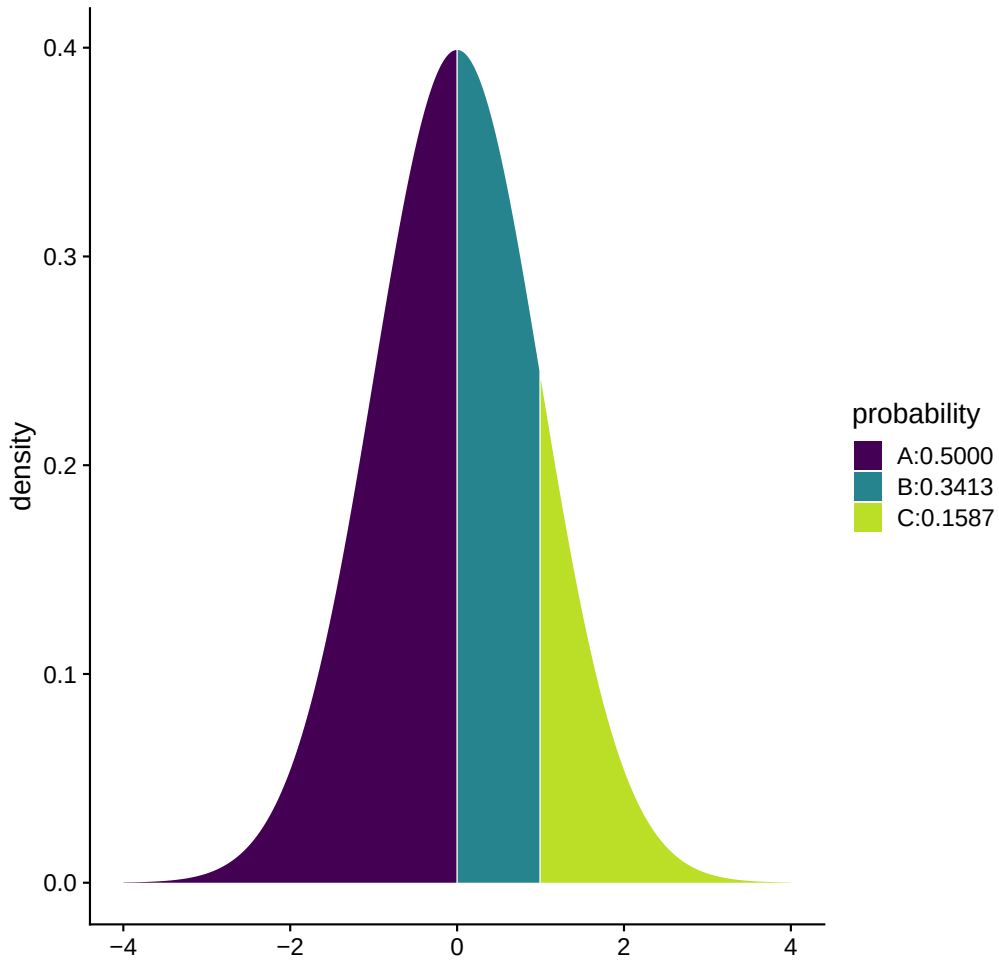
$$\mathrm{P}(a \leq Y \leq b) = \int_a^b f(y) \partial y \, .$$

That is, the probability that Y takes on a value in the interval $[a, b]$ is the area above this interval and under the graph of the density function

How Can Every Value Have Probability 0?

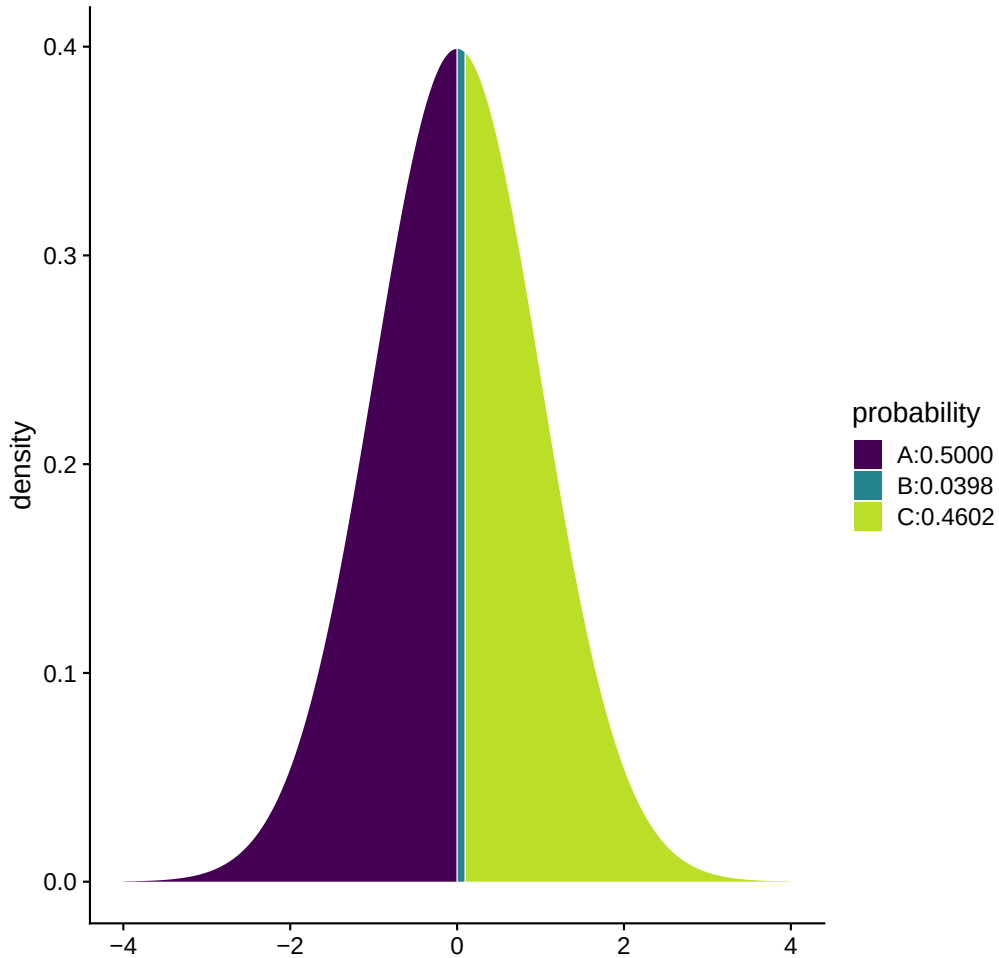
We can find a probability for any interval of z-scores. But the probability for a single z-score is zero. How can that be? Let's look at the standard Normal random variable, Z. We could find that the probability that Z lies between 0 and 1 is $P(0 \leq Z \leq 1) = 0.3413$.

```
mosaic::xpnorm(c(0,1))
```



$P(0 \leq Z \leq 0.1)$

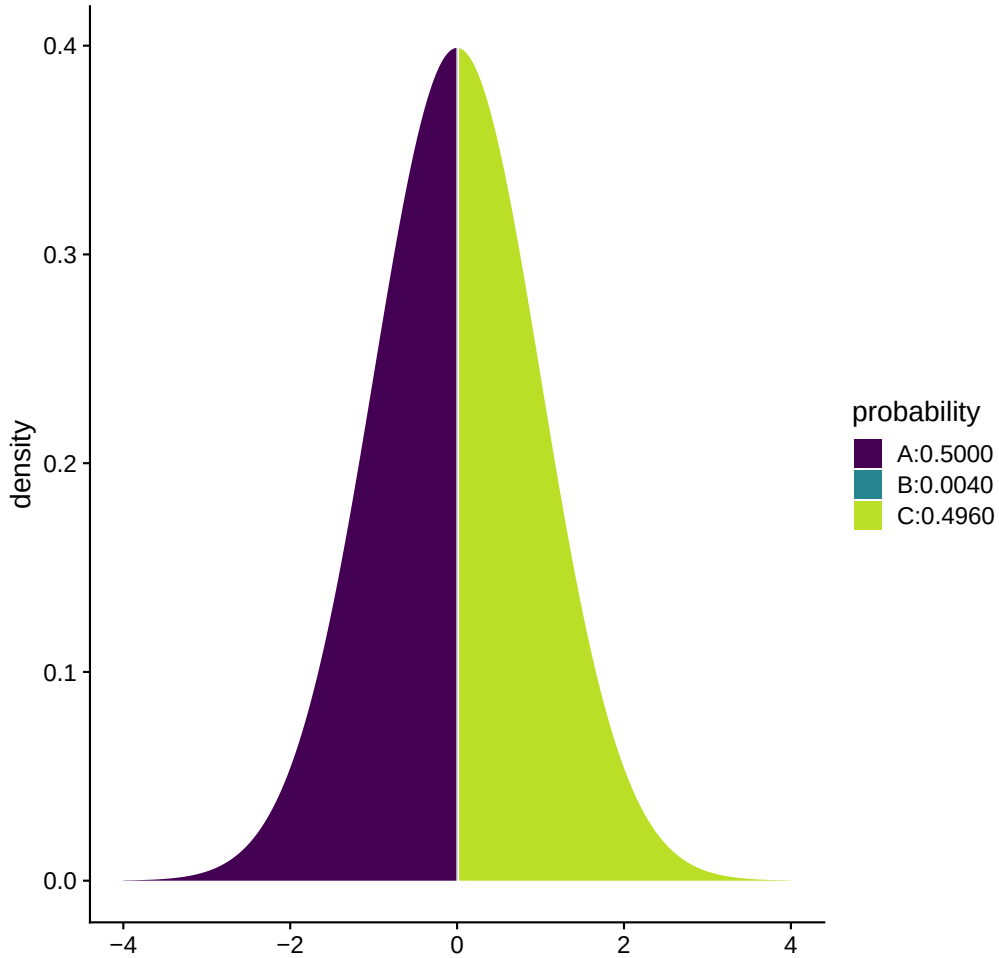
```
mosaic::xpnorm(c(0,1/10))
```




```
## [1] 0.5000000 0.5398278
```

$P(0 \leq Z \leq 0.01)$

```
mosaic::xpnorm(c(0,1/100))
```




```
## [1] 0.5000000 0.5039894
```

How Can Every Value Have Probability 0?

- So, what's the probability that Z is exactly 0? Well, there's no area under the curve right at $x = 0$, so the probability is 0.
- It's only **intervals** that have positive probability, but that's OK.
- In real life we never mean exactly 0.0000000000 or any other value. If you say "exactly 164 pounds," you might really mean between 163.5 and 164.5 pounds or even between 163.99 and 164.01 pounds, but realistically not 164.000000000 . . . pounds

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Expected Value of a function of a RV

Variance and SD of a function of a RV

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Linear combinations of random variables (regression slopes)

Expected value for a discrete RV

Definition 5.

Let Y be a discrete random variable with set of possible values $D = \{y_1, y_2, \ldots, y_k\}$ and corresponding probabilities for each value, e.g., y_1 with probability $\mathrm{P}(y_1)$, y_2 with probability $\mathrm{P}(y_2)$, y_3 with probability $\mathrm{P}(y_3)$, ..., y_k with probability $\mathrm{P}(y_k)$. Furthermore, let $g(Y)$ be some real-valued function of Y. Then the expected value of $g(Y)$ is:

$$\mathrm{E}(g(Y)) = \sum_{y \in D} g(y) \times \mathrm{P}(y) \ .$$

i.e. it is a weighted mean of the $g(y)$'s, with $\mathrm{P}(y)$'s as weights.

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Let c be a constant and Z another random variable

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Let c be a constant and Z another random variable

- $g(Y) = Y + c \rightarrow$
- $g(Y) = cY \rightarrow$
- $g(Y, Z) = Y + Z \rightarrow$

Exercise: $\mathrm{E}(g(Y)) = g(E(Y))$?

- Y = Noon Temperature (C) in Montreal on a randomly selected day of the year; $g(Y)$ = Temperature (F) = 32 + (9/5) Y

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- $g(Y_1, Y_2) = \frac{Y_1 + Y_2}{2}$
- $g(Y_i) = \frac{1}{n} \sum_{i=1}^{n} Y_i$
- Y = diameter of a randomly chosen sphere; $g(Y)$ = Volume of sphere = $\frac{\pi}{6} Y^3$

Example: $g(Y)$ = Volume of sphere = $\frac{\pi}{6} Y^3$

Example: Checking via simulation

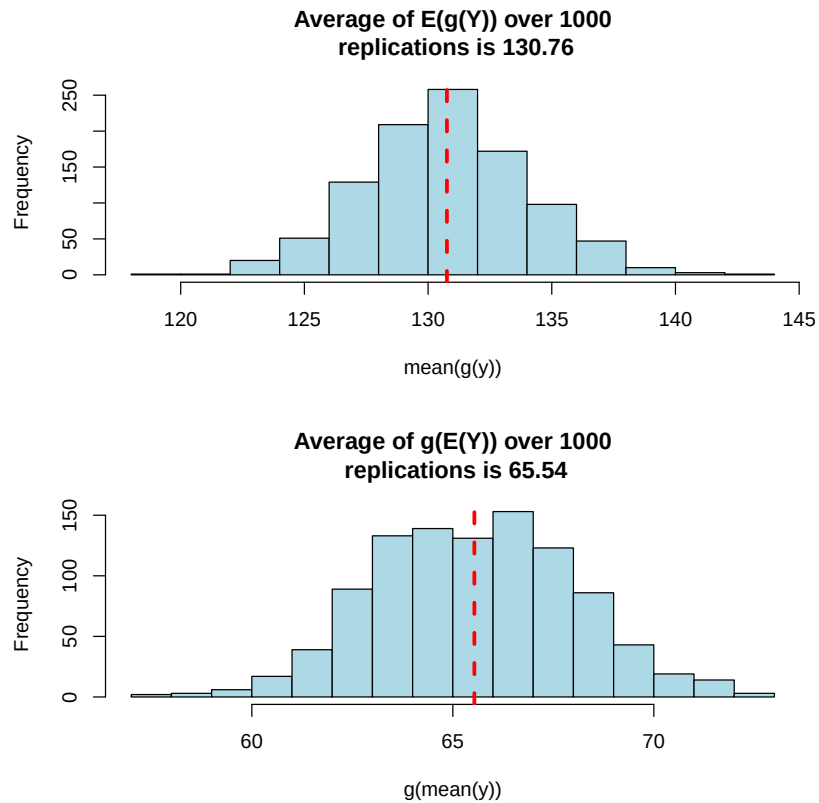
```r
g.y <- function(y) {
 (pi / 6) * y^3
}

set.seed(12)
B <- 1000; N <- 2000
E_g.y <- replicate(B, {
  diameter <- runif(N, min = 0, max = 10)
  mean(g.y(diameter)) # E(g(y))
})

g_E.y <- replicate(B, {
  diameter <- runif(N, min = 0, max = 10)
  g.y(mean(diameter)) # g(E(y))
})

par(mfrow = c(1,2))
hist(E_g.y, col = "lightblue", xlab = "mean(g(y))",
     main = sprintf("Average of E(g(Y)) over 1000\n replications
     is %0.2f",mean(E_g.y)))
abline(v = mean(E_g.y), col = "red", lty = 2, lwd = 3)

hist(g_E.y, col = "lightblue", xlab = "g(mean(y))",
     main = sprintf("Average of g(E(Y)) over 1000\n replications
     is %0.2f",mean(g_E.y)))
abline(v = mean(g_E.y), col = "red", lty = 2, lwd = 3)
```



Example: Checking the results theoretically

- We know for $Y \sim Uniform(a, b)$, the nth moment $\mathrm{E}(Y^n)$ is given by:

$$\mathrm{E}(Y^n) = \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}$$

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- Therefore, we know, theoretically, for $Y \sim Uniform(0, 10)$:

$$\mathrm{E}(Y) = \tag{1}$$

$$\mathrm{E}(Y^3) = \tag{2}$$

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- Therefore, we know, theoretically, for $Y \sim Uniform(0, 10)$:

$$\mathrm{E}(Y) = \tag{1}$$
$$\mathrm{E}(Y^3) = \tag{2}$$

- It follows that, theoretically,

$$\mathrm{E}(g(Y)) = 130.9 \tag{3}$$
$$g(\mathrm{E}(Y)) = 65.45 \tag{4}$$

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Variance of an RV

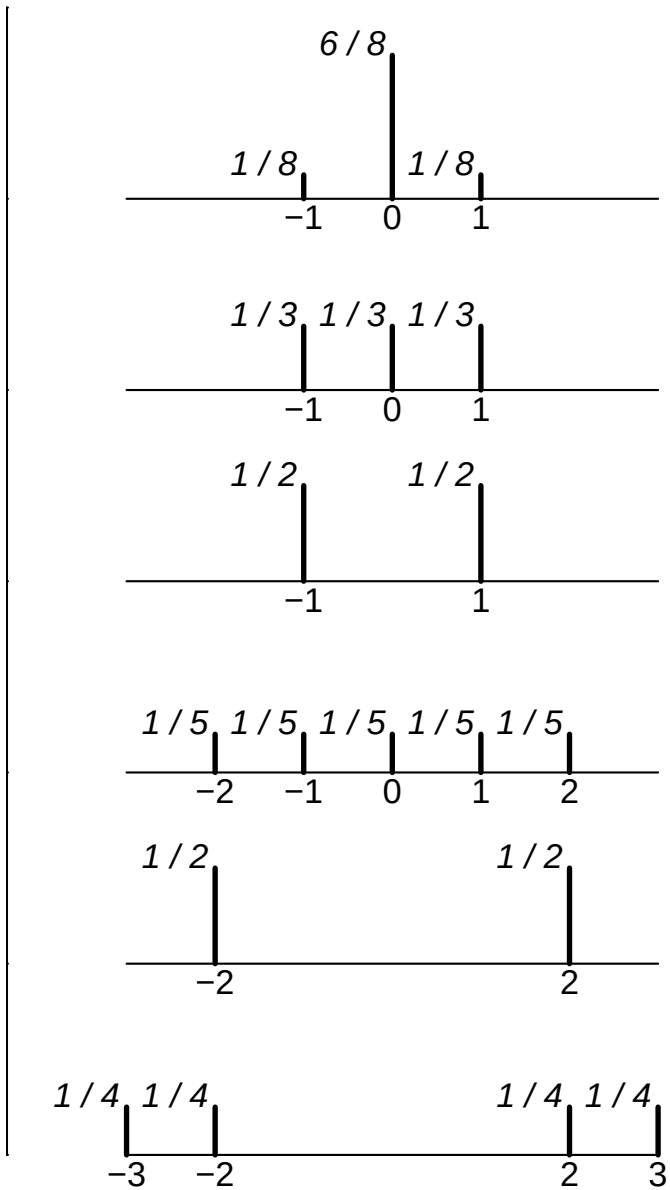
Definition 6.

The variance of the random variable Y is given by

$$\mathrm{Var}(Y) = \mathrm{E}[(Y-\mu)^2].$$

- Discrete RV: $\mathrm{Var}(Y) = \sum_y (y - \mathrm{E}(Y))^2 \times f(y)$
- Continous RV: $\mathrm{Var}(Y) = \int_y (y - \mathrm{E}(Y))^2 \times f(y) \partial y$

Graphical representation of variance of a RV



	Mean Absolute Deviation	Mean Squared Deviation (Variance)	Root Mean Squared Deviation (SD; écart type)
	0.25	0.25	0.5
	0.67	0.67	0.82
	1	1	1
	1.2	2	1.41
	2	4	2
	2.5	6.5	2.55

Variance of a function of a RV

- Y = Noon Temperature (C) in Mtl on a randomly selected day of the year; $g(Y)$ = Temperature (F) = $32 + (9/5)\ Y$

Variance of a function of a RV

- Y = Noon Temperature (C) in Mtl on a randomly selected day of the year; $g(Y)$ = Temperature (F) = 32 + (9/5) Y

- Y = Years of publication of all the books in the McGill Library, with Years measured from 1439 AD; $W = Y - 1439$ vs. $W = 2020 - Y$

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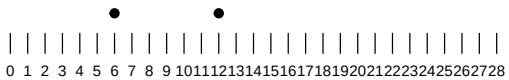
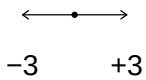
A sum of 2 random variables

PROBABILITY: $\frac{1}{2}$ $\frac{1}{2}$

RV_1

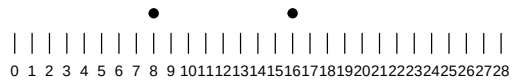


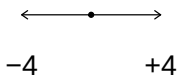


VARIANCE $= \frac{1}{2}(-3)^2 + \frac{1}{2}(+3)^2 = 3^2$

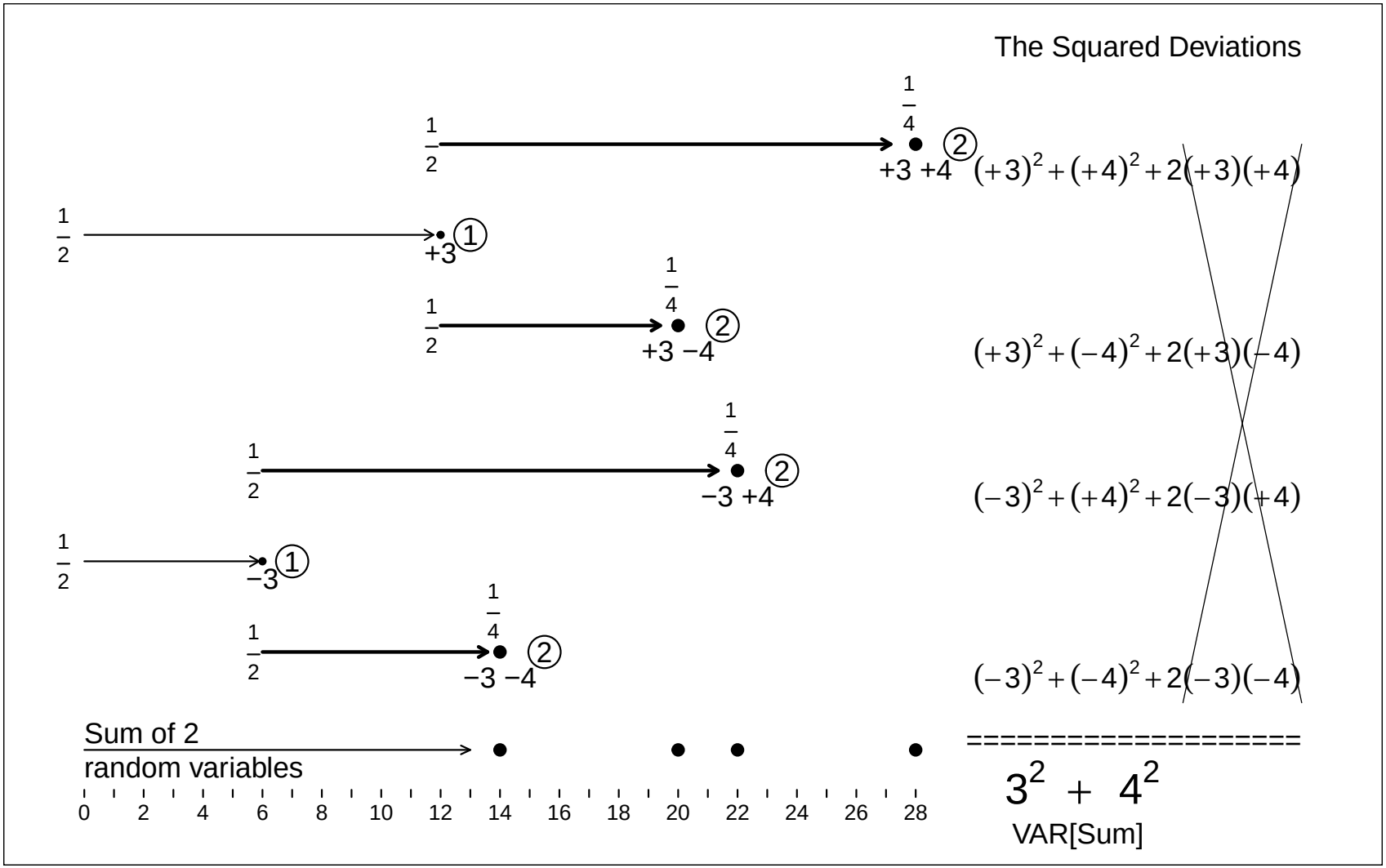
PROBABILITY: $\frac{1}{2}$ $\frac{1}{2}$

RV_2

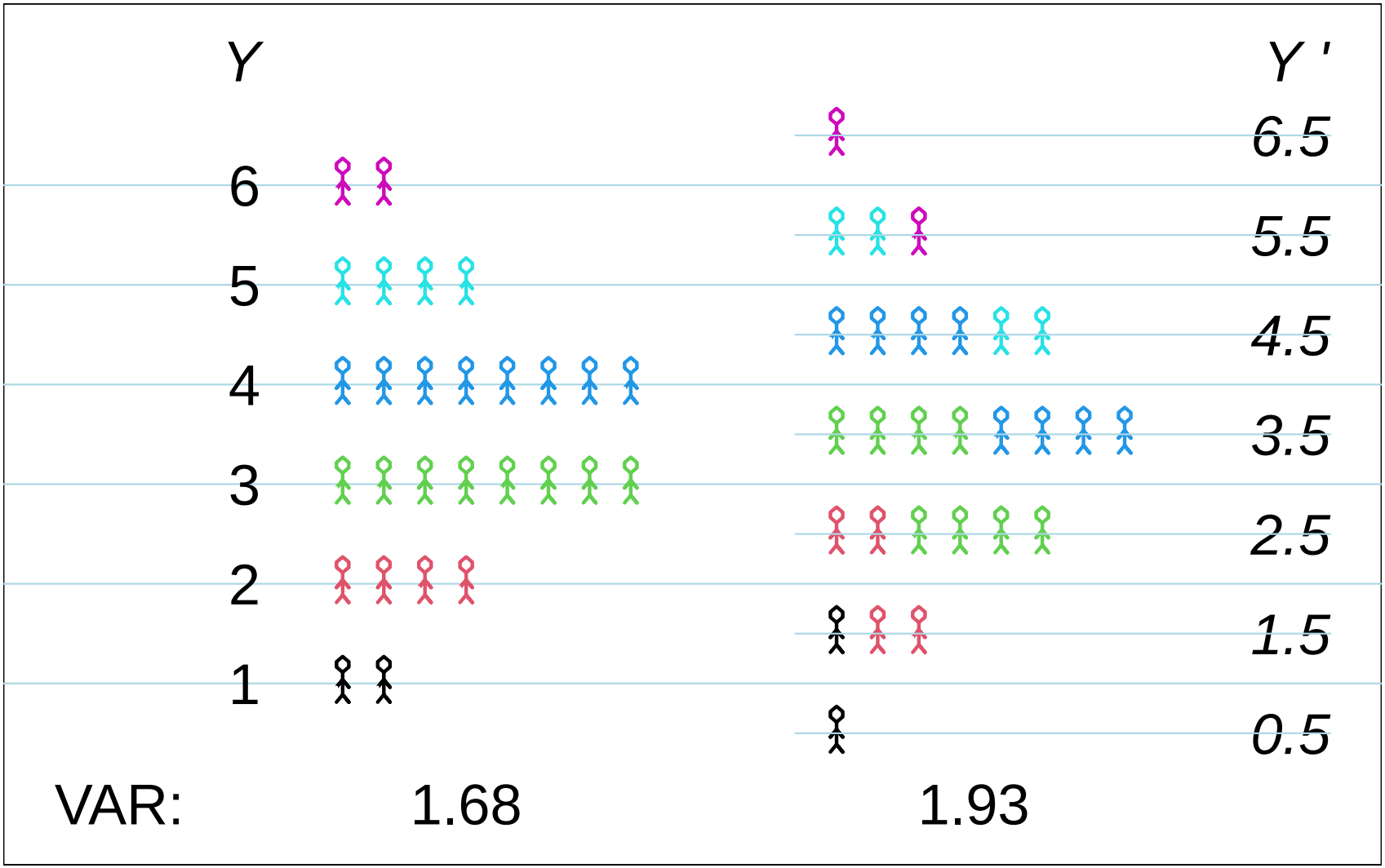




VARIANCE $= \frac{1}{2}(-4)^2 + \frac{1}{2}(+4)^2 = 4^2$


The Squared Deviations
1/2
1/4
+3 +4 ②
(+3)² + (+4)² + 2(+3)(+4)
1/2
+3 ①
1/2
1/4
+3 −4 ②
(+3)² + (−4)² + 2(+3)(−4)
1/2
1/4
−3 +4 ②
(−3)² + (+4)² + 2(−3)(+4)
1/2
−3 ①
1/2
1/4
−3 −4 ②
(−3)² + (−4)² + 2(−3)(−4)
Sum of 2
random variables
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28
==========
3² + 4²
VAR[Sum]

Application: Measurement errors



A sum of n random variables

- Up to now, to keep things general, we used n non-identical – but independent – random variables. If we consider the Variance and the sum of n **identical** – and independent – random variables, so the n Variances (each abbreviated to Var) are all equal, the laws simplify:

- First, since the variances add, we have that

$$\mathrm{Var}(RV_1 + RV_2 + \cdots + RV_n) = \mathrm{Var}_1 + \mathrm{Var}_2 + \cdots + \mathrm{Var}_n = n \times \textit{each Var}\,.$$

- Taking square roots,

$$SD(\ RV_1 + RV_2 + \cdots + RV_n\ ) = \sqrt{n \times \text{ each Var}} = \sqrt{n}\ \times \text{ each } SD$$

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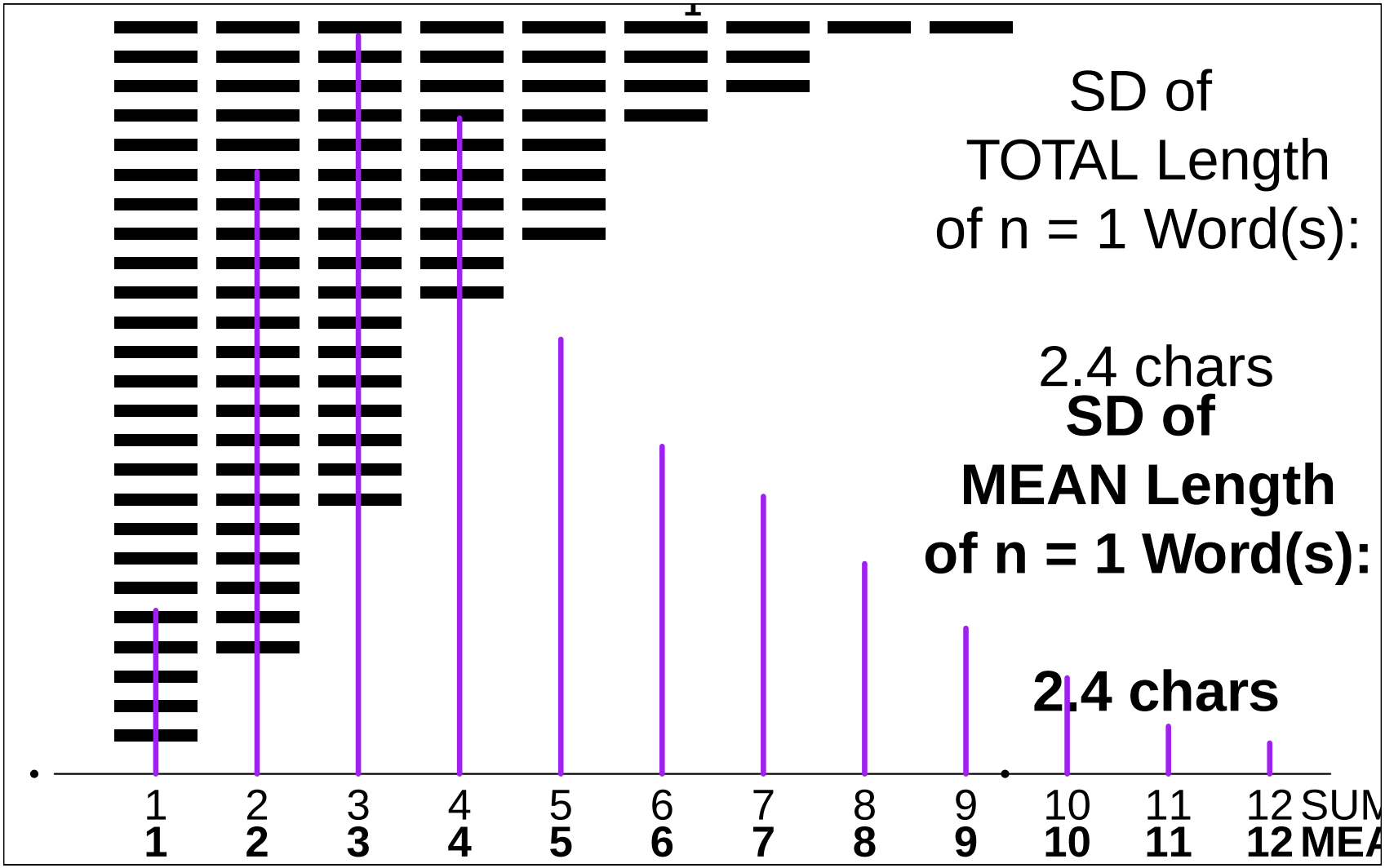
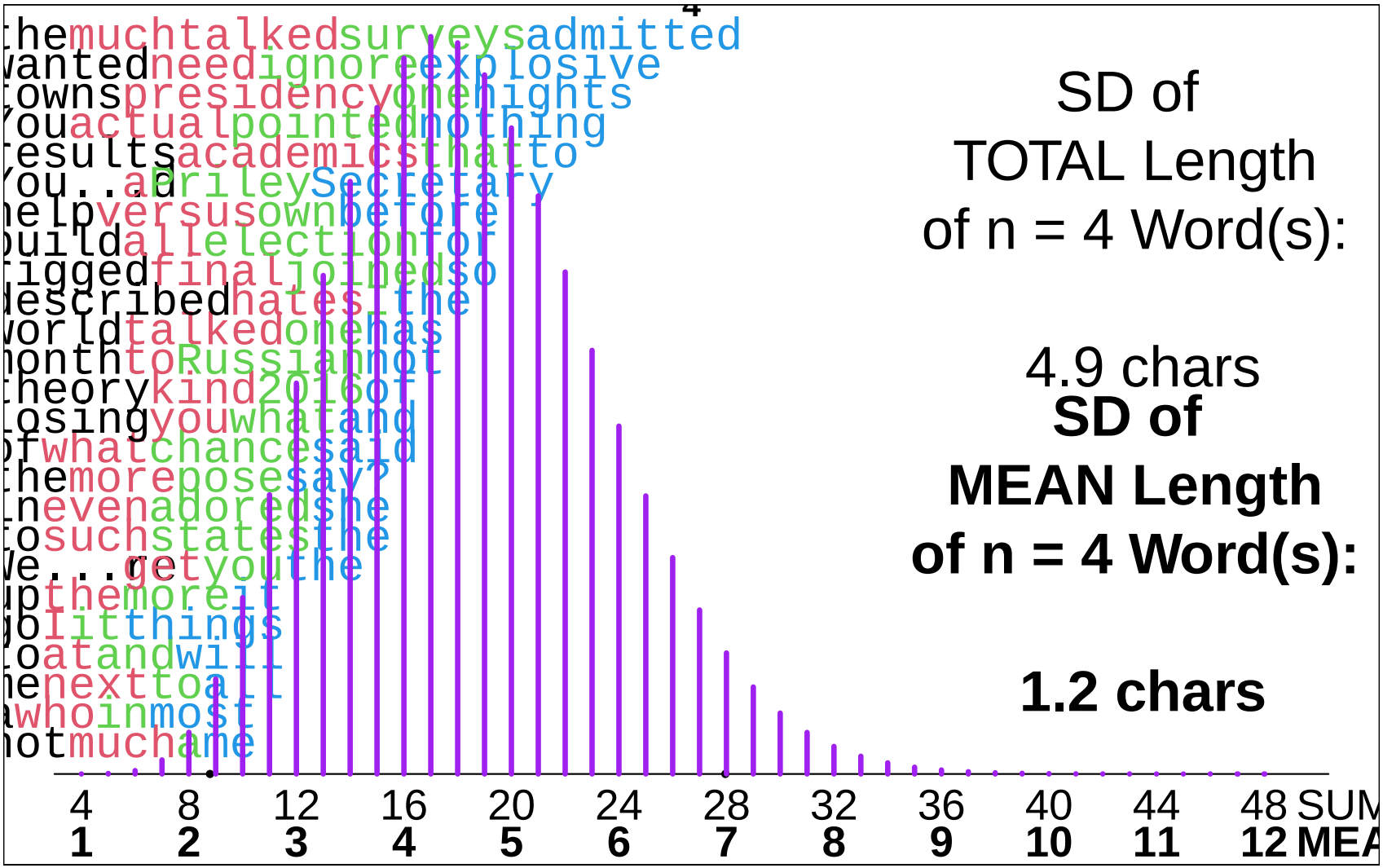
- $$SD\left(\frac{RV_1 + RV_2 + \cdots + RV_n}{n}\right) = \frac{\sqrt{n}\ \times \text{ each } SD}{n} = \frac{\text{common } SD}{\sqrt{n}}$$

- $$\text{Var}\left(\frac{RV_1 + RV_2 + \cdots + RV_n}{n}\right) = \frac{\text{common Var}}{n}$$

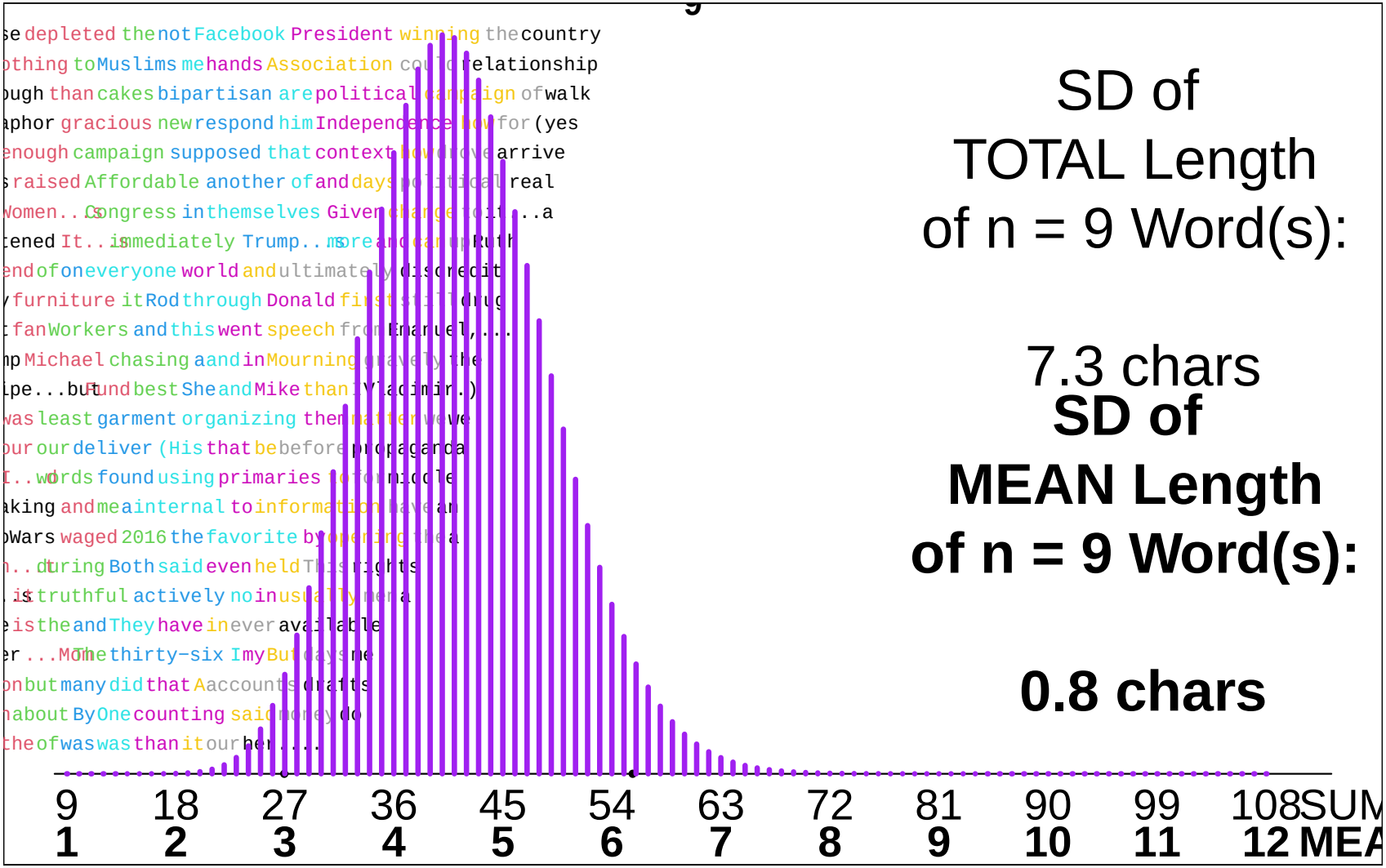
Example: Length of Words in a Book - $n = 1$ word



$n = 4$ words


thehemuchtalkedsurveysadmitted
wantedneedignoreexplosive
townspresidencyonenights
youactualpointednothing
resultsacademicsthatto
you...aBrileySecretary
helpversusownbefore
buildallelectionfor
riggedfinaljoinedso
describedhatesthe
worldtalkedonehas
monthtoRussiannot
theorykind2016of
losingyouwhatand
ofwhatchancesaid
themoreposesay?
nevenadoredshe
tosuchstatesthe
We...getyouthe
upthemoreit
golittlethings
toatandwill
henexttoall
awhoinmost
notmuchame
4 8 12 16 20 24 28 32 36 40 44 48 SUM
1 2 3 4 5 6 7 8 9 10 11 12 MEA
SD of
TOTAL Length
of n = 4 Word(s):
4.9 chars
SD of
MEAN Length
of n = 4 Word(s):
1.2 chars

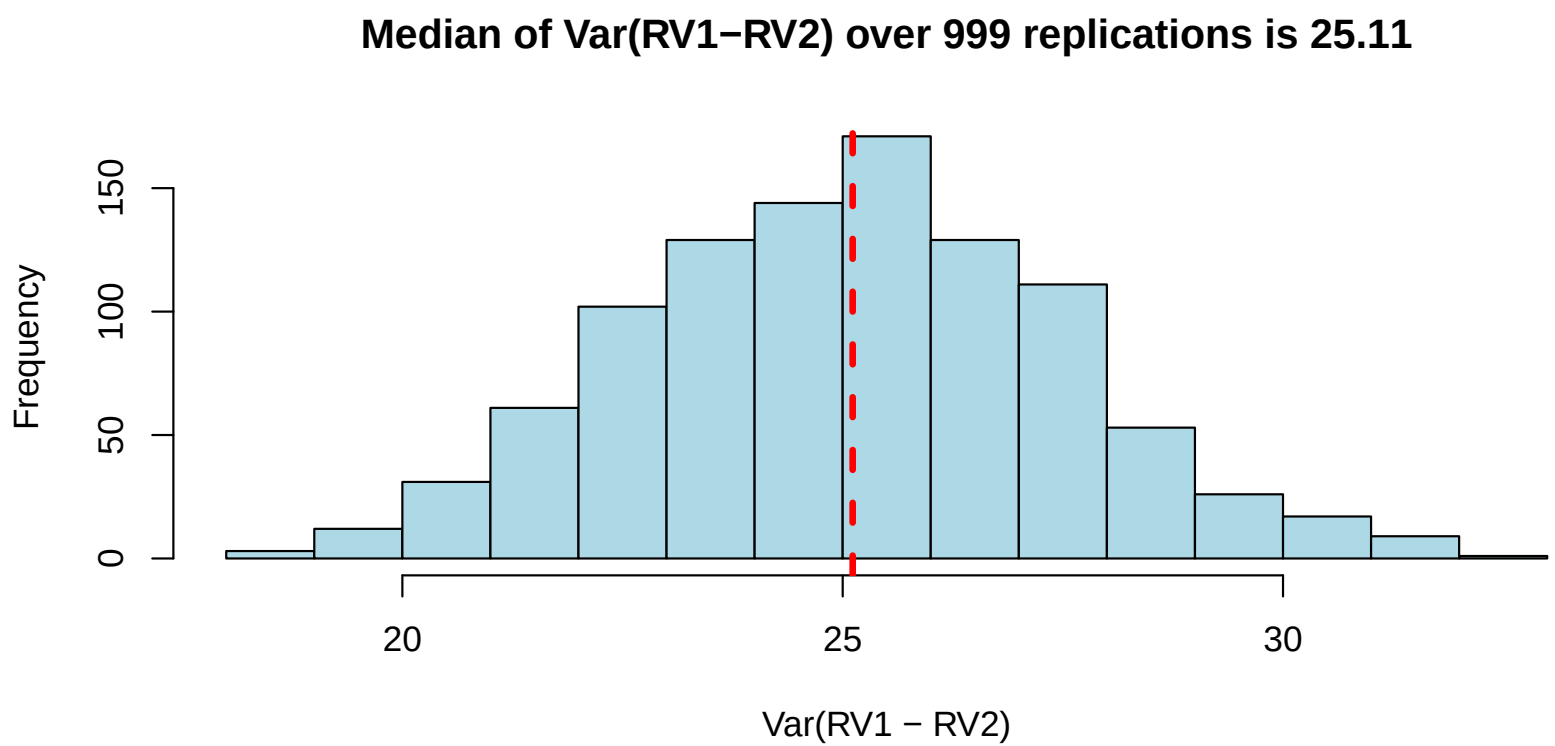
$n = 9$ words


SD of
TOTAL Length
of n = 9 Word(s):
7.3 chars
SD of
MEAN Length
of n = 9 Word(s):
0.8 chars
9 18 27 36 45 54 63 72 81 90 99 108 SUM
1 2 3 4 5 6 7 8 9 10 11 12 MEA

Difference of 2 Random Variables via Simulation

```r
set.seed(12)
B <- 999; N <- 200
  var_diff <- replicate(B, {
  RV1 <- rnorm(N, mean = 2, sd = 3)
  RV2 <- rnorm(N, mean = 4, sd = 4)
  var(RV1 - RV2)
})

hist(var_diff, col = "lightblue", xlab = "Var(RV1 - RV2)",
    main = sprintf("Median of Var(RV1-RV2) over 999 replications is %0.2f",median(var_diff)))
abline(v = median(var_diff), col = "red", lty = 2, lwd = 3)
```



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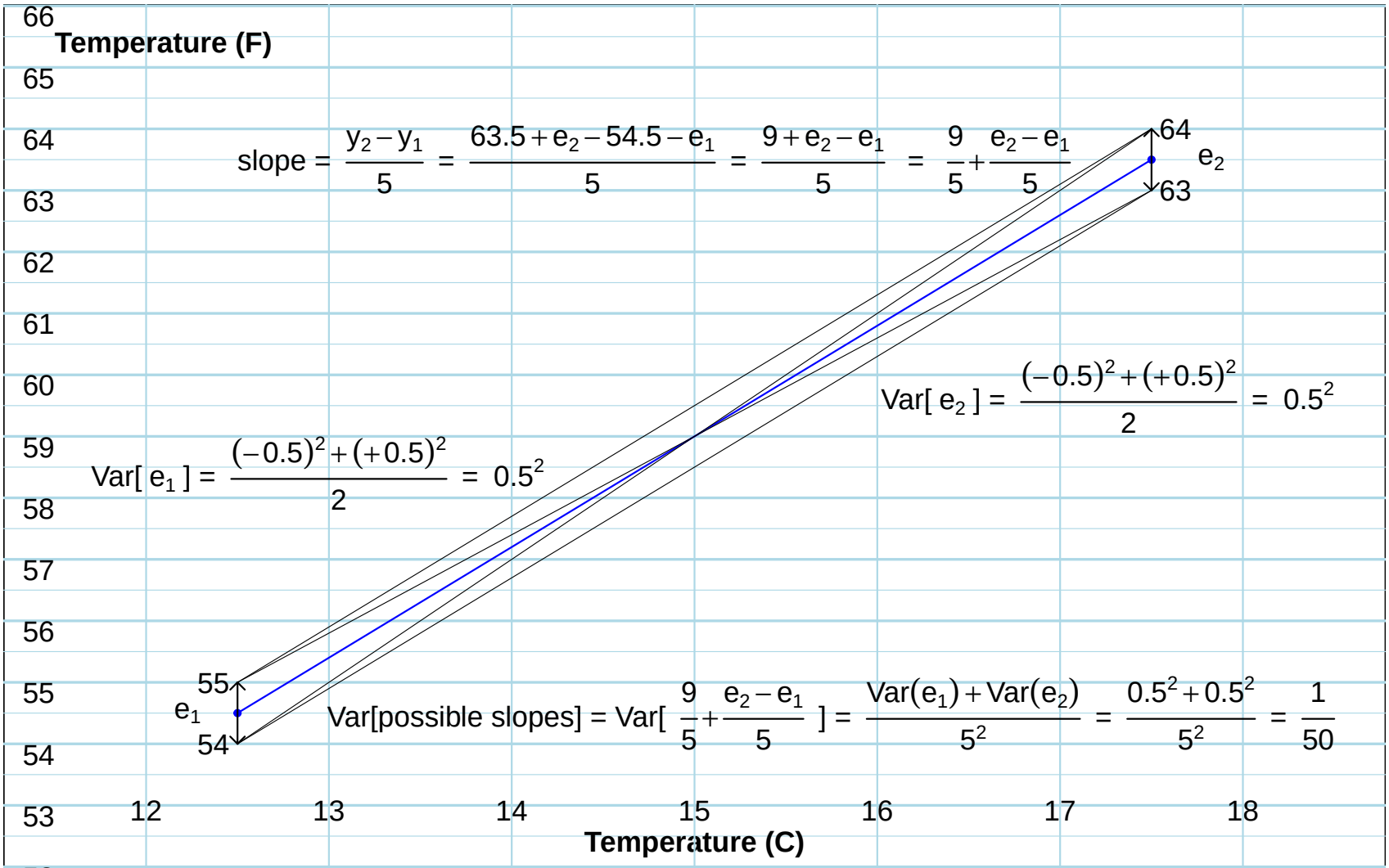
Linear Combinations of random variables



Session Info

```
R version 4.0.4 (2021-02-15)
Platform: x86_64-pc-linux-gnu (64-bit)
Running under: Pop!_OS 21.04

Matrix products: default
BLAS:   /usr/lib/x86_64-linux-gnu/openblas-pthread/libblas.so.3
LAPACK: /usr/lib/x86_64-linux-gnu/openblas-pthread/libopenblasp-r0.3.13.so

attached base packages:
[1] tools     stats     graphics  grDevices utils     datasets  methods
[8] base

other attached packages:
 [1] DT_0.16           mosaic_1.7.0      Matrix_1.3-2      mosaicData_0.20.1
 [5] ggformula_0.9.4   ggstance_0.3.4    lattice_0.20-41   kableExtra_1.2.1
 [9] socviz_1.2        gapminder_0.3.0   here_0.1          NCStats_0.4.7
[13] FSA_0.8.30        forcats_0.5.1     stringr_1.4.0     dplyr_1.0.7
[17] purrr_0.3.4       readr_1.4.0       tidyr_1.1.3       tibble_3.1.3
[21] ggplot2_3.3.5     tidyverse_1.3.0   knitr_1.33

loaded via a namespace (and not attached):
 [1] fs_1.5.0           lubridate_1.7.9    webshot_0.5.2      httr_1.4.2
 [5] rprojroot_2.0.2    backports_1.2.1    utf8_1.2.2         R6_2.5.1
 [9] DBI_1.1.1          colorspace_2.0-2   withr_2.4.2        tidyselect_1.1.1
[13] gridExtra_2.3      leaflet_2.0.3      curl_4.3.2         compiler_4.0.4
[17] cli_3.0.1          rvest_1.0.0        pacman_0.5.1       xml2_1.3.2
[21] ggdendro_0.1.22    labeling_0.4.2     mosaicCore_0.8.0   scales_1.1.1
[25] digest_0.6.27      foreign_0.8-81     rmarkdown_2.9.7    rio_0.5.16
[29] pkgconfig_2.0.3    htmltools_0.5.2    highr_0.9          dbplyr_1.4.4
[33] fastmap_1.1.0      htmlwidgets_1.5.3  rlang_0.4.11       readxl_1.3.1
[37] rstudioapi_0.13    farver_2.1.0       generics_0.1.0     jsonlite_1.7.2
[41] crosstalk_1.1.1    zip_2.2.0          car_3.0-9          magrittr_2.0.1
[45] Rcpp_1.0.7         munsell_0.5.0      fansi_0.5.0        abind_1.4-5
[49] lifecycle_1.0.0    stringi_1.7.3      carData_3.0-4      MASS_7.3-53.1
[53] plyr_1.8.6         grid_4.0.4         blob_1.2.1         ggrepel_0.8.2
[57] crayon_1.4.1       cowplot_1.1.0      haven_2.3.1        splines_4.0.4
[61] hms_1.0.0          pillar_1.6.2       reprex_0.3.0       glue_1.4.2
[65] evaluate_0.14      data.table_1.14.0  modelr_0.1.8       vctrs_0.3.8
[69] tweenr_1.0.1       cellranger_1.1.0   gtable_0.3.0       polyclip_1.10-0
[73] assertthat_0.2.1   TeachingDemos_2.12 xfun_0.25          ggforce_0.3.2
[77] openxlsx_4.1.5     broom_0.7.2        viridisLite_0.4.0  ellipsis_0.3.2
```