012 - Central Limit Theorem

EPIB 607

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Statistical Concepts and Prinicples

Central Limit Theorem

Standard deviation and variance of a random variable Y

- $Y \sim \text{unknown\_distribution}(\mu, \sigma)$
- Standard Deviation σ, and Variance σ^2, of a random variable Y with mean μ.

$$\begin{aligned}
Var[Y] &= \sigma^2 = \text{mean of } (Y - \mu)^2 \\
SD[Y] &= \sigma \\
Var[Y \pm a\ constant] &= Var[Y] \\
SD[Y \pm a\ constant] &= \sigma \\
Var[Y \times a\ constant] &= constant^2 \ \times \ Var[Y] \\
SD[Y \times a\ constant] &= |constant| \times \sigma
\end{aligned}$$

Rules for Variances and SDs of sums and means of n independent random variables

Sums

$$
\begin{aligned}
Var[Y_1 + Y_2 + \cdots + Y_n] &= \sigma^2 + \sigma^2 + \cdots + \sigma^2 = n \times \sigma^2 \\
SD[Y_1 + Y_2 + \cdots + Y_n] &= \sqrt{n} \times \sigma
\end{aligned}
$$

Rules for Variances and SDs of sums and means of n independent random variables

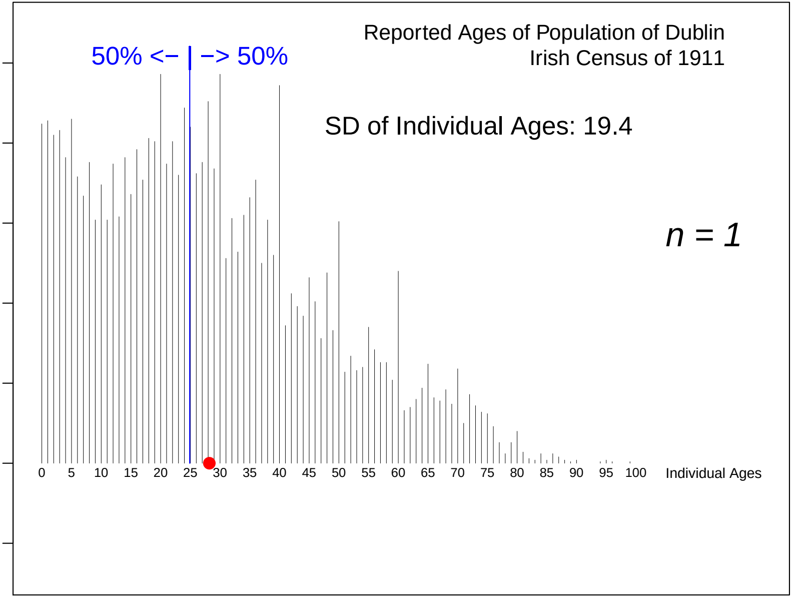
Sums

$$Var[Y_1 + Y_2 + \cdots + Y_n] = \sigma^2 + \sigma^2 + \cdots + \sigma^2 = n \times \sigma^2$$
$$SD[Y_1 + Y_2 + \cdots + Y_n] = \sqrt{n} \times \sigma$$

Means

$$Var\left[\frac{Y_1 + Y_2 + \cdots + Y_n}{n}\right] = \frac{1}{n} \times \sigma^2$$
$$SD\left[\frac{Y_1 + Y_2 + \cdots + Y_n}{n}\right] = \sqrt{\frac{1}{n}} \times \sigma$$

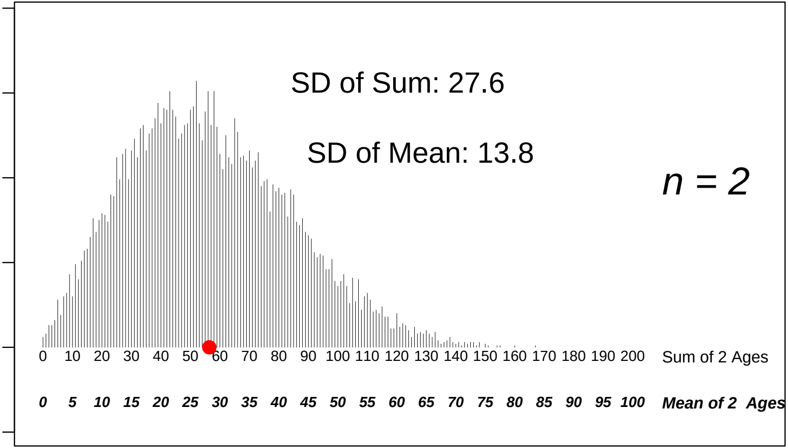
Age-distribution of the entire population of Dublin[1]



[1] http://www.census.nationalarchives.ie/help/about19011911census.html

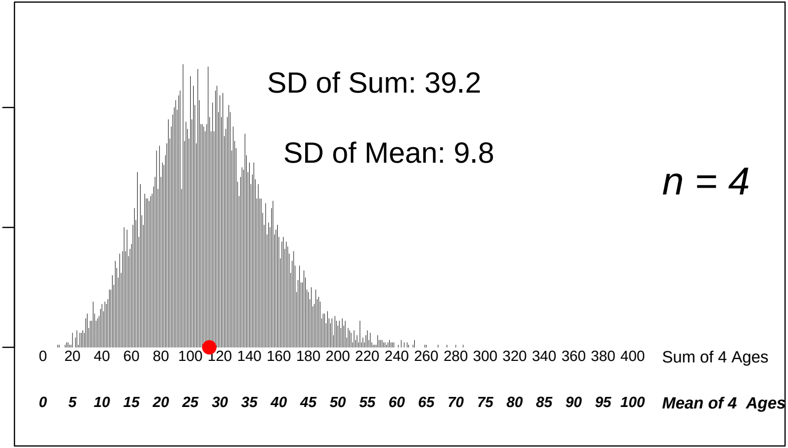
Distribution of 10000 Bootstrap samples of size 2

```
## [1] Ages of sampled persons in first 2 samples of size 2
##      [,1] [,2]
## [1,]   42   47
## [2,]   28   10
```



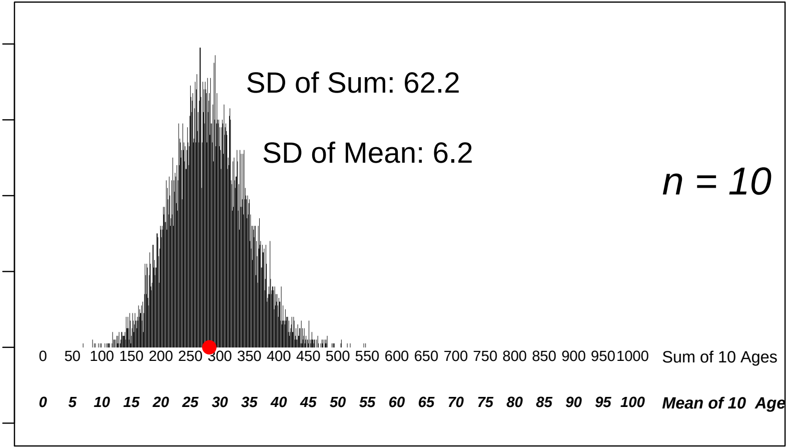
Distribution of 10000 Bootstrap samples of size 4

```
## [1] Ages of sampled persons in first 2 samples of size 4
##      [,1] [,2] [,3] [,4]
## [1,]    8   26   20    0
## [2,]   19   17   46   15
```

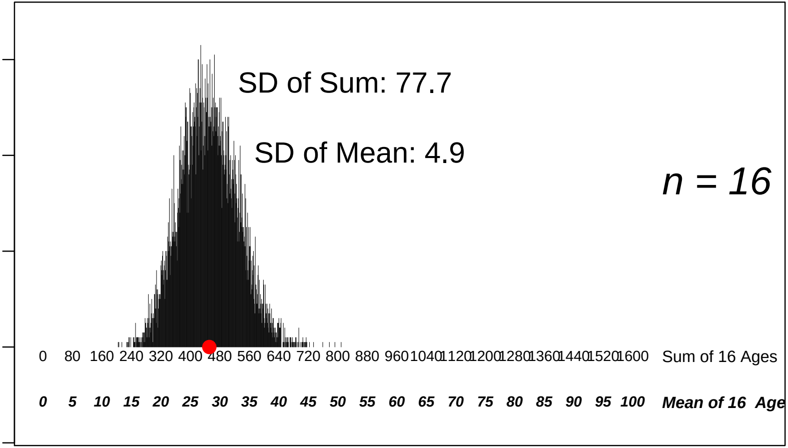


Distribution of 10000 Bootstrap samples of size 10

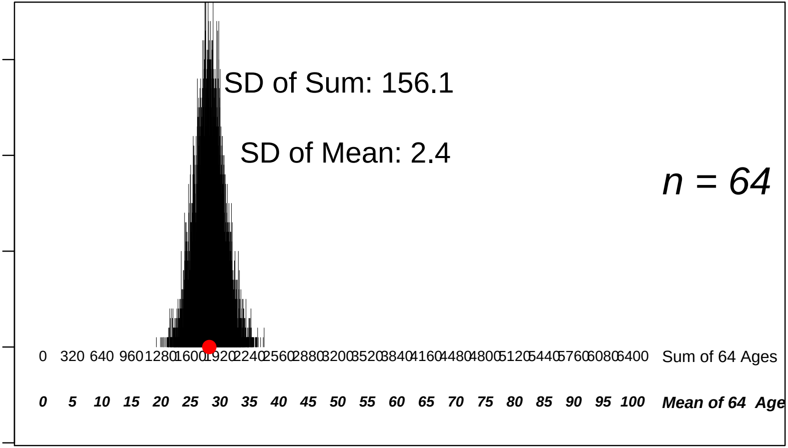
```
## [1] Ages of sampled persons in first 2 samples of size 10
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
## [1,]   15    0    6   36   24   38   16   20   50     8
## [2,]    9    6   43   19   24   13    3   45    3    21
```



Distribution of 10000 Bootstrap samples of size 16



Distribution of 10000 Bootstrap samples of size 64



Statistical Concepts and Prinicples

Central Limit Theorem

Properties of the sample mean: The Central Limit Theorem (CLT)

The sampling distribution of $\overline{Y}$ is Normal if Y is Normal. What probability distribution does the sample mean follow if Y is not Normal?

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The sampling distribution of $\overline{Y}$ is Normal if Y is Normal. What probability distribution does the sample mean follow if Y is not Normal?

As sample size increases, the distribution of $\overline{Y}$ becomes closer to a Normal distribution, no matter what the distribution of sampled variable Y!

(This is true as long as the distribution has a finite variance.)

The Central Limit Theorem (CLT)

- The sampling distribution of $\bar{y}$ is, for a large enough n, close to Gaussian in shape no matter what the shape of the distribution of individual Y values.
- This phenomenon is referred to as the CENTRAL LIMIT THEOREM
- The CLT applied also to a sample proportion, slope, correlation, or any other statistic created by aggregation of individual observations

Theorem 1 (Central Limit Theorem).

$$\textit{if } Y \sim ???(\mu_Y, \sigma_Y), \textit{ then}$$

$$\bar{y} \sim \mathcal{N}(\mu_Y, \sigma_Y/\sqrt{n})$$

Standard error (SE) of a sample statistic

- Recall: When we are talking about the variability of a **statistic**, we use the term **standard error** (not standard deviation). The standard error of the sample mean is $\sigma/\sqrt{n}$.

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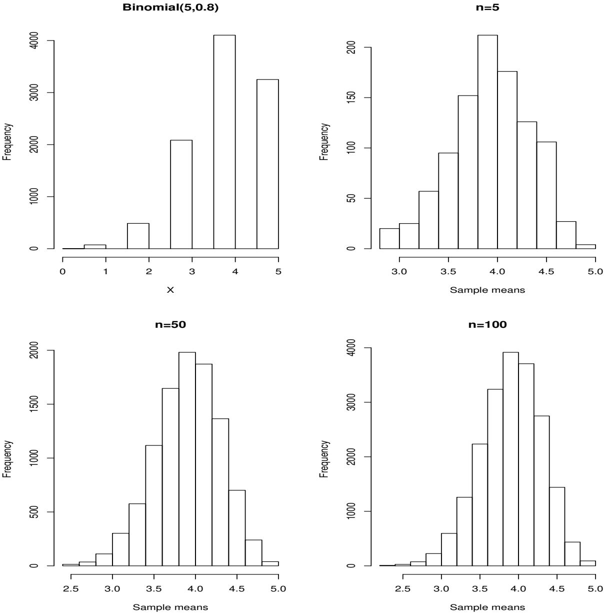
Remark 1 (SE vs. SD).

In quantifying the instability of the sample mean ($\bar{y}$) statistic, we talk of SE of the mean (SEM)

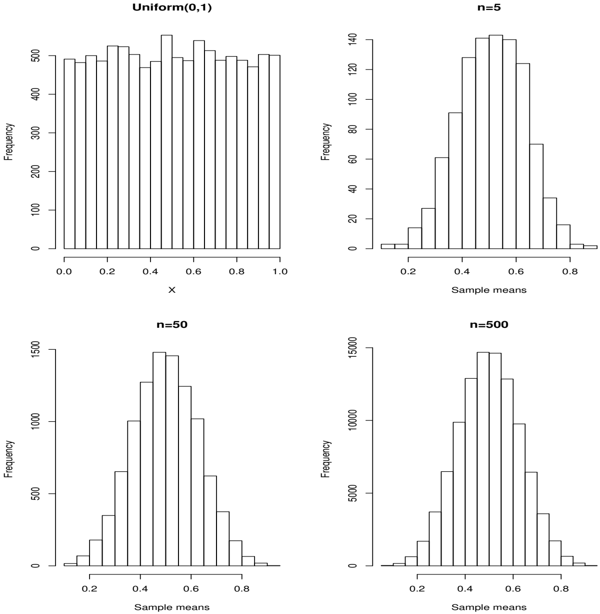
SE($\bar{y}$) describes how far $\bar{y}$ could (typically) deviate from μ;

SD(y) describes how far an individual y (typically) deviates from μ (or from $\bar{y}$).

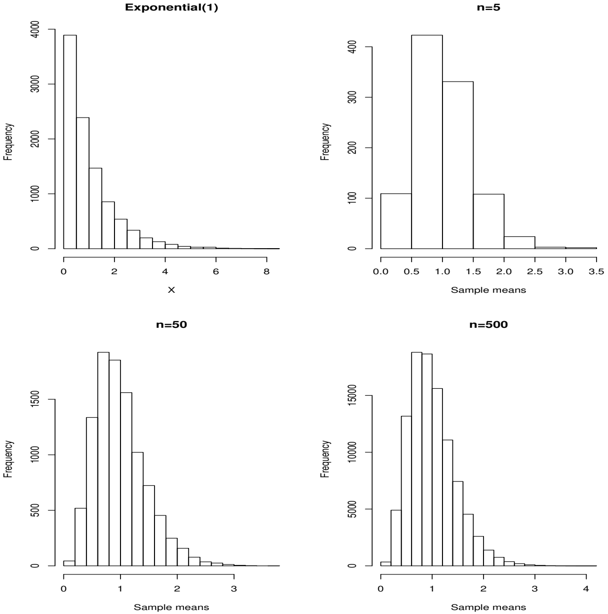
CLT in action: Binomial(n = 5,p = 0.8) distribution



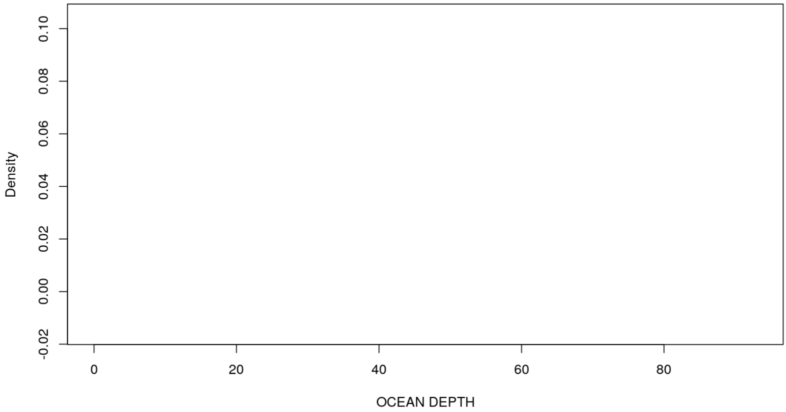
CLT in action: Uniform(a = 0, b = 1) distribution



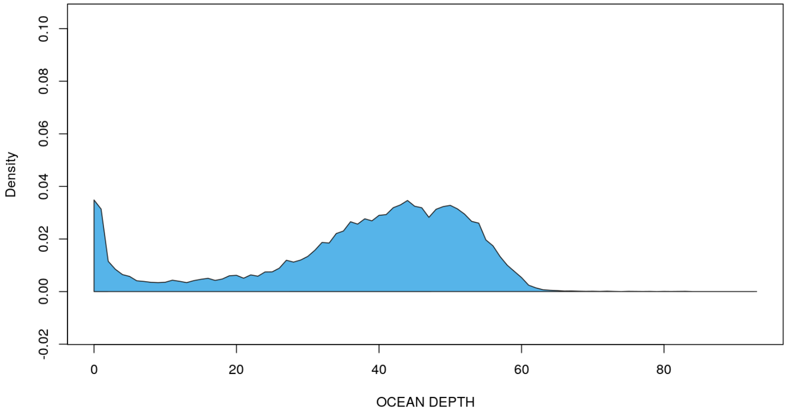
CLT in action: Exponential($\lambda = 1$) distribution



CLT in action: Depths of the ocean



CLT in action: Depths of the ocean



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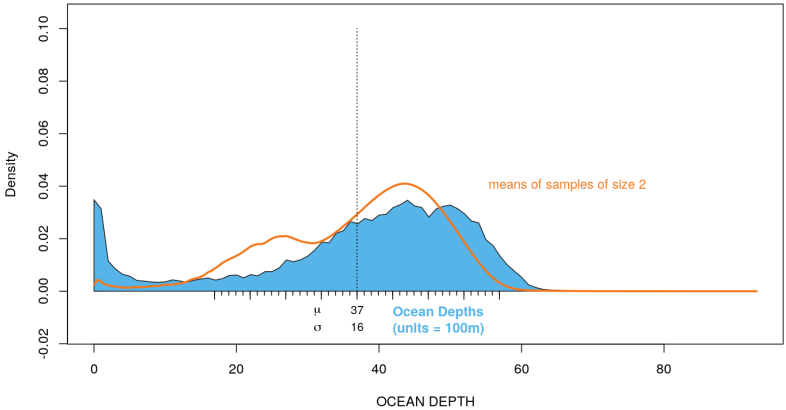
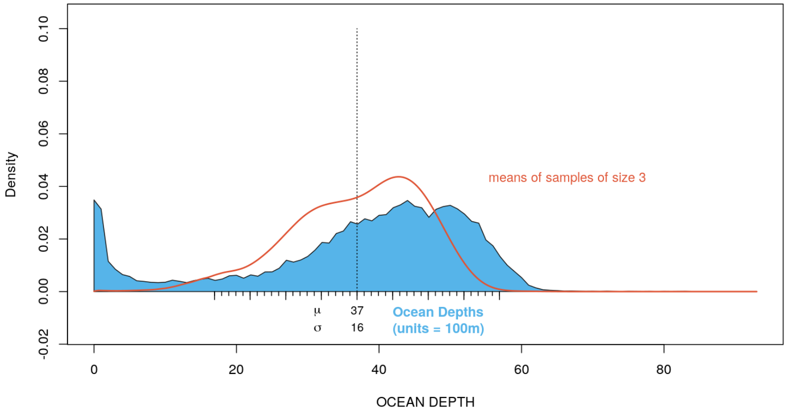
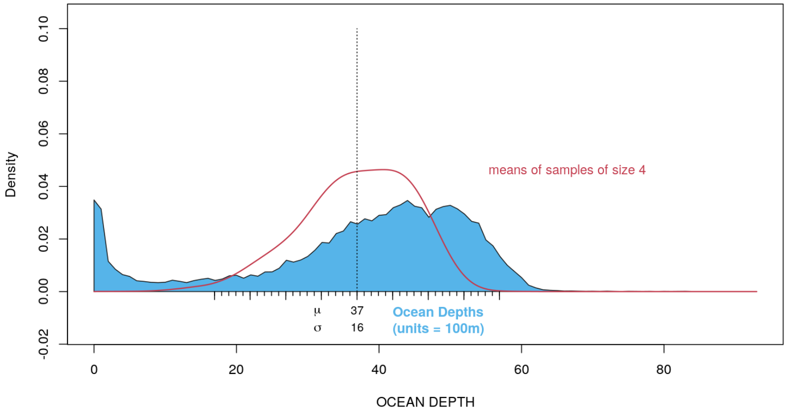


CLT in action: Depths of the ocean



CLT in action: Depths of the ocean



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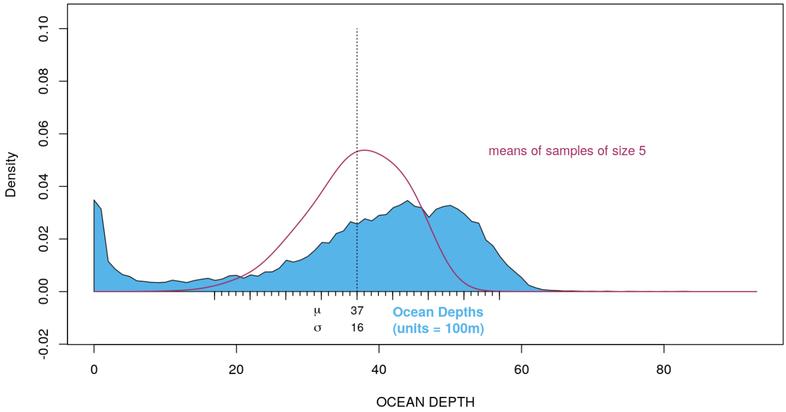

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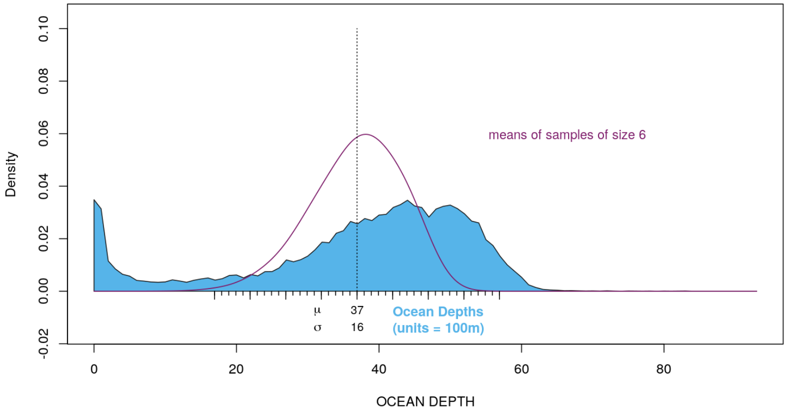
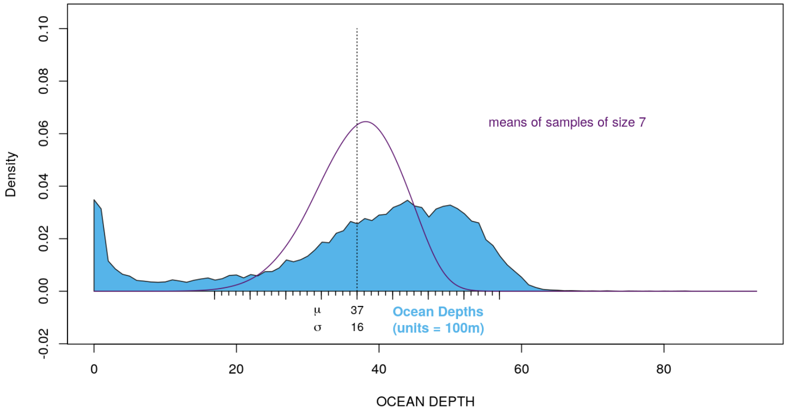
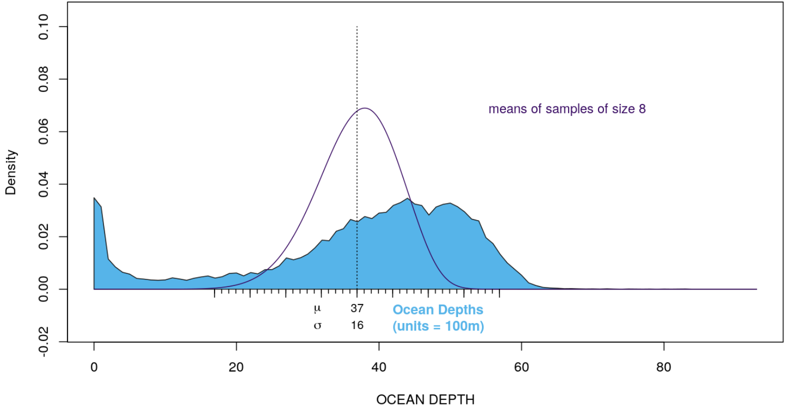


CLT in action: Depths of the ocean



CLT in action: Depths of the ocean



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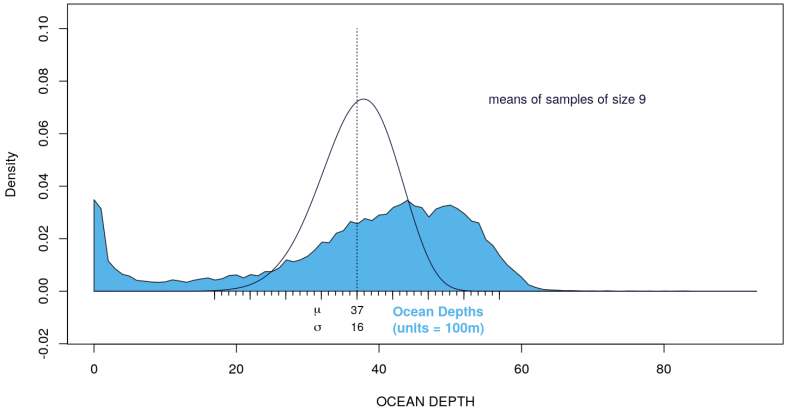

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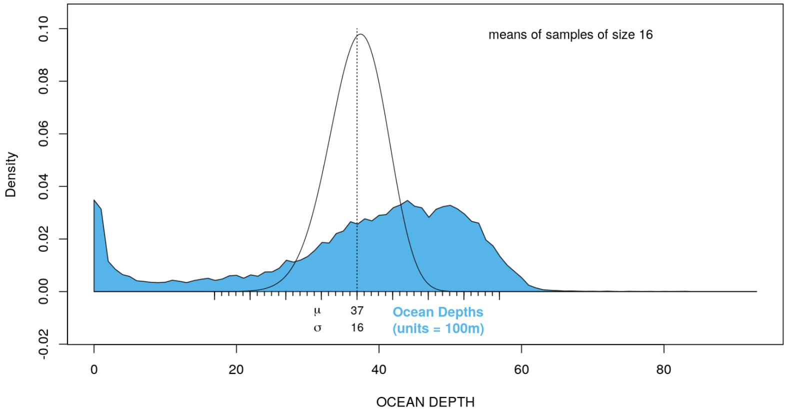
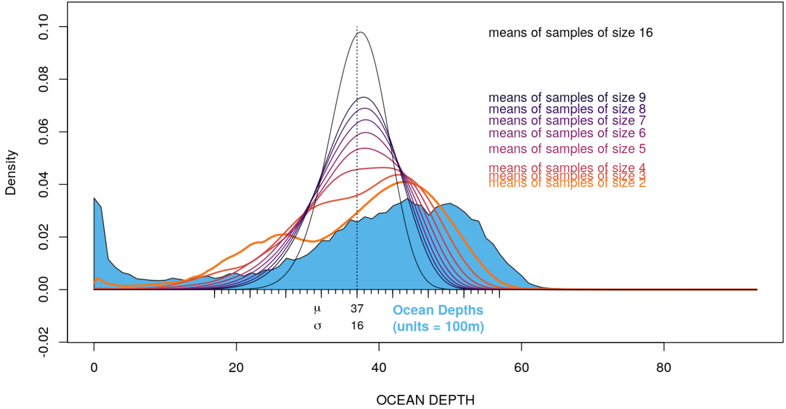
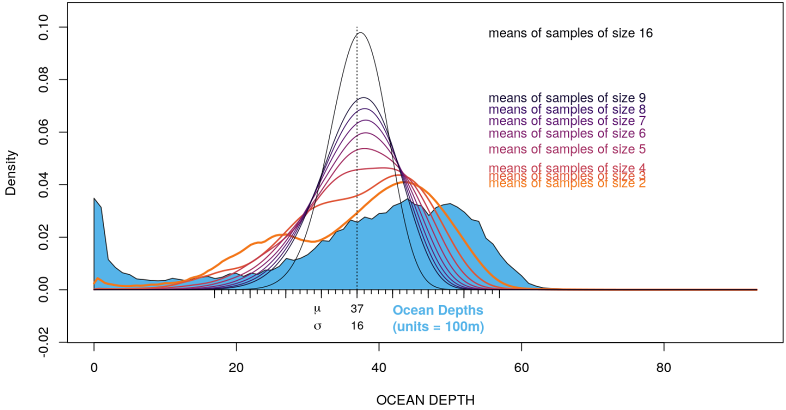


CLT in action: Depths of the ocean



How long does it take for the CLT to ’kick in’?

- How *fast* or slowly the CLT will kick in is a function of how symmetric, or how asymmetric and CLT-unfriendly, the distribution of Y (the depths of the ocean) is



Quadruple the work, half the benefit

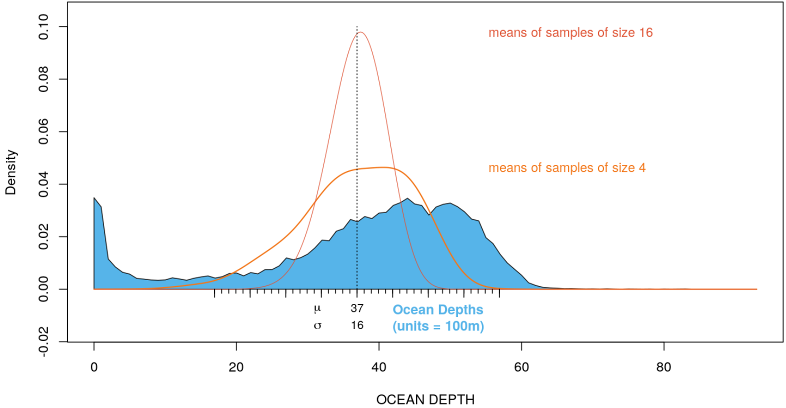


Figure: When the sample size increases from 4 to 16, the spread of the sampling distribution for the mean is reduced by a half, i.e., the range is cut in half. This is known as the curse of the $\sqrt{n}$

Session Info

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Platform: x86_64-pc-linux-gnu (64-bit)
Running under: Pop!_OS 21.04

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LAPACK: /usr/lib/x86_64-linux-gnu/openblas-pthread/libopenblasp-r0.3.13.so

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[8] base     

other attached packages:
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 [9] socviz_1.2        gapminder_0.3.0   here_0.1          NCStats_0.4.7    
[13] FSA_0.8.30        forcats_0.5.1     stringr_1.4.0     dplyr_1.0.7      
[17] purrr_0.3.4       readr_1.4.0       tidyr_1.1.3       tibble_3.1.3     
[21] ggplot2_3.3.5     tidyverse_1.3.0   knitr_1.33       

loaded via a namespace (and not attached):
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