Option Pricing Mini-Project

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1 Monte-Carlo Simulation

Under the Black-Scholes framework, the stock prices are governed by $dS = \mu S dt + \sigma S dW$, where W is a Brownian motion. Stock price at time t is given by $S_{t+\Delta t} = S_t e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma W_t}$. Where W_t is a standard Brownian motion and μ and $\sigma > 0$ are drift and volatility parameters, which are inputs to the model. Under this formalism, we recursively simulate stock price paths for a period T and number of steps N, for an initial stock price S_0 as

$$S_{t+\Delta t} = S_t e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} Z_t}$$

Where $\Delta t = T/N$ and Z_t are independent standard normal random variables. For large enough number of steps, we can capture the exact frequecy at which stock prices are updated in the market (this varies across stocks but for most NSE stocks, millisecond frequency is common).

In the `simulate_paths` method, we simulate `n_paths` many stock trajectories with `n_steps` many steps for a time `T` and return a matrix of size `n_paths` × `n_steps`. A sample of the simulations is given in Figure 1. (Throughout the discussion code and plots, rebalance frequency refers to the time period of rebalancing in units of days)

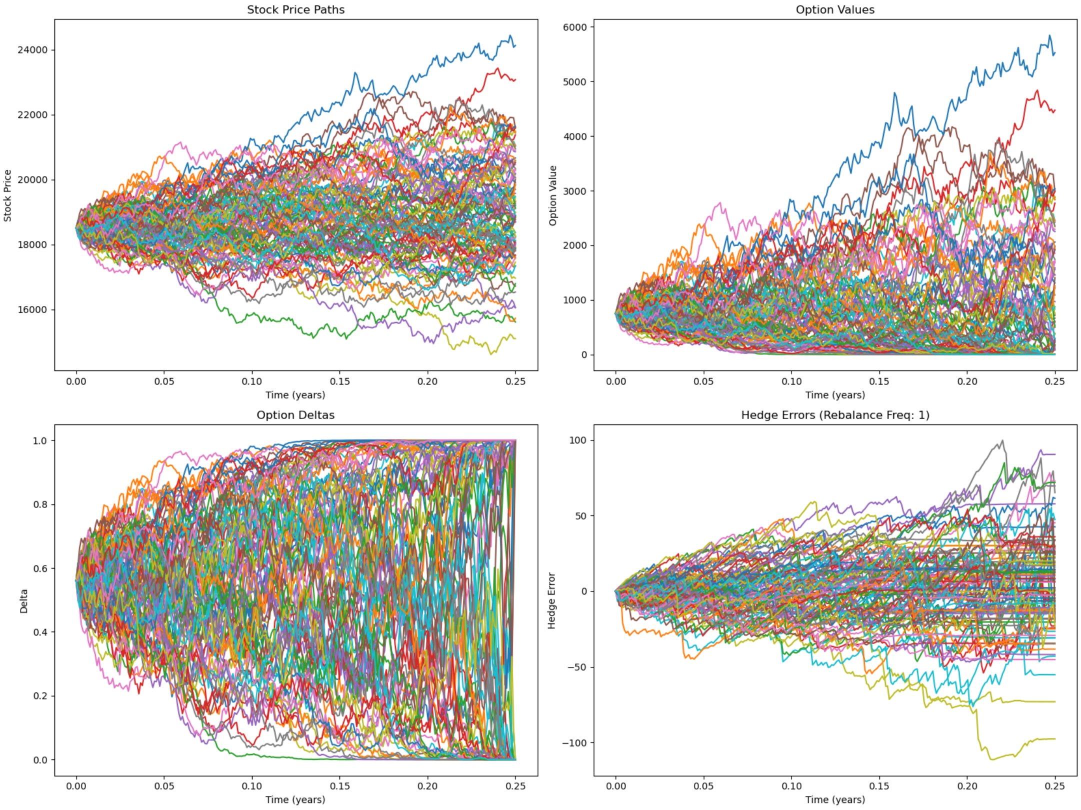

Figure 1: 1000 stock prices and related quantities simulated for one quarter

1.a Option Price and Delta

Solving the Black-Scholes PDE with the boundary conditions we obtain from the contract value at maturity time, we obtain the price of a call option as

$$C(S,t) = S_t N(d_+) - Ke^{-r(T-t)}N(d_-)$$

Where $d_\pm = \frac{\ln(S_t/K)+(r\pm\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and N is the standard normal CDF. In the code we use the notation `d_1` and `d_2` and the `price` method computes the price (for call options) according to the formula. The delta of the option is given by

$$\begin{aligned}
\Delta(t,S_t) &= \frac{\partial C}{\partial S}(t,S_t) \\
&= N(d_+) + S_t f(d_+)\frac{\partial d_+}{\partial S} - Ke^{-r(T-t)}f(d_-)\frac{\partial d_-}{\partial S} \\
&= N(d_+) + S_t f(d_+)\frac{1}{S_t\sigma\sqrt{T-t}} - Ke^{-r(T-t)}f(d_-)\frac{1}{S_t\sigma\sqrt{T-t}} \\
&= N(d_+) + \frac{1}{\sigma\sqrt{T-t}}\left(f(d_+) - \frac{Ke^{-r(T-t)}}{S_t}f(d_-)\right) \\
&= N(d_+) + \frac{f(d_-)}{\sigma\sqrt{T-t}}\left(\frac{f(d_+)}{f(d_-)} - \frac{Ke^{-r(T-t)}}{S_t}\right) \\
&= N(d_+) + \frac{f(d_-)}{\sigma\sqrt{T-t}}\left(\exp\left[-\frac{1}{2}(d_+^2 - d_-^2)\right] - \frac{Ke^{-r(T-t)}}{S_t}\right) \\
&= N(d_+) + \frac{f(d_-)}{\sigma\sqrt{T-t}}\left(\exp\left[-\frac{1}{2}\frac{\left(2\ln\left(\frac{S_t}{K}\right)+2r(T-t)\right)\left(\sigma^2(T-t)\right)}{\sigma^2(T-t)}\right] - \frac{Ke^{-r(T-t)}}{S_t}\right) \\
&= N(d_+) + \frac{f(d_-)}{\sigma\sqrt{T-t}}\left(\exp\left[-\left(\ln\left(\frac{S_t}{K}\right)+r(T-t)\right)\right] - \frac{Ke^{-r(T-t)}}{S_t}\right) \\
&= N(d_+) + \frac{f(d_-)}{\sigma\sqrt{T-t}}\left(\frac{Ke^{-r(T-t)}}{S_t} - \frac{Ke^{-r(T-t)}}{S_t}\right) \\
&= N(d_+)
\end{aligned}$$

1.a.1 Put Options

We invoke the put-call parity $C - P = F = S_t - Ke^{-r(T-t)}$, where F is the value of a forward contract. We can write the price of a put option as $P(S,t) = C(S,t) - S_t + Ke^{-r(T-t)}$, this is replicated in the code as well. differentiating the put-call parity equation with respect to the stock price, we obtain $\Delta_P = \Delta_C - 1 = N(d_+) - 1$.

1.b Hedging

We use delta hedging and rebalance the hedge portfolio at a given frequency. Consider a rebalance after time Δt_r then if $t + \Delta t_r < T$, we define the single-step hedge error as the change in the hedge portfolio value. (For the sake of simplicity of analysis, we take the risk free rate to be 0)

$$HE_t = C_{t+\Delta t_r} - C_t - \Delta_t(S_{t+\Delta t_r} - S_t)$$

Note that under the limit $\Delta t_r \to 0$, the hedge error converges to zero as that is how the Black-Scholes model is constructed. So here, we make an attempt to study the distribution of the hedge errors (single-step and total) for different rebalancing frequencies. The total hedge error and root mean square hedge error are defined as

$$HE_{tot} = \sum_{i=0}^{N-1} HE_{t_0+i\Delta t_r}$$

$$HE_{rms} = \sqrt{\frac{1}{N}\sum_{i=0}^{N-1} HE^2_{t+i\Delta t_r}}$$

These quantities are of interest, studying their distribution can give us an idea of how well the discretized delta hedging strategy performs. Though finding the exact disctribution is fairly tricky but under certain approximations, we can judge the stochasticity in HE_t and make some useful comments.

1.b.1 Performance Analysis through Distributions

$$C_{t+\Delta t} - C_t \approx \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial S}(S_{t+\Delta t} - S_t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_{t+\Delta t} - S_t)^2$$

Since $(S_{t+\Delta t} - S_t)^2 \approx \sigma^2 S_t^2 \Delta t\, Z_t^2$ (keeping leading terms), we get:

$$C_{t+\Delta t} - C_t \approx \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} Z_t^2\right)\Delta t + \frac{\partial C}{\partial S}(S_{t+\Delta t} - S_t)$$

Substitute into the P&L:

$$-HE_t \approx \Delta_t(S_{t+\Delta t} - S_t) - \left[\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} Z_t^2\right)\Delta t + \frac{\partial C}{\partial S}(S_{t+\Delta t} - S_t)\right]$$

Since $\Delta_t = \dfrac{\partial C}{\partial S}$ at t, the first-order terms cancel:

$$HE_t \approx \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \Gamma_t Z_t^2\right)\Delta t$$

Using the BS PDE:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \Gamma_t + rS_t\Delta_t - rC_t = 0$$

$$\implies \frac{\partial C}{\partial t} = rC_t - rS_t\Delta_t - \frac{1}{2}\sigma^2 S_t^2 \Gamma_t$$

Substitute:

$$HE_t \approx \left(rC_t - rS_t\Delta_t - \frac{1}{2}\sigma^2 S_t^2\Gamma_t + \frac{1}{2}\sigma^2 S_t^2 \Gamma_t Z_t^2\right)\Delta t$$

$$HE_t \approx (rC_t - rS_t\Delta_t)\,\Delta t + \frac{1}{2}\sigma^2 S_t^2 \Gamma_t (Z_t^2 - 1)\Delta t$$

Focusing only stochastic component (gamma effect)

$$\text{as } r = 0 HE_t \approx -\frac{1}{2}\sigma^2 S_t^2 \Gamma_t (Z_t^2 - 1)\Delta t$$

Here we have centered HE_t for ease of analysis ($r = 0$). This shows that the hedge-error is closely linked to the gamma of the option, which can be computed by differentiating the delta with respect to the stock price as $\Gamma_t = \frac{f(d_+)}{S_t\sigma\sqrt{T-t}}$. Under this, we have the centered hedge error distribution $\approx \frac{\sigma S_t f(d_+)}{2\sqrt{T-t}}\left(Z_t^2 - 1\right)\Delta t$. The standard deviation of this approximate distribution is hence $\approx \frac{\sigma S_t f(d_+)}{\sqrt{2(T-t)}}\Delta t$.

This is precisely why we would expect a linear growth in the expected hedge error with respect to the volatility.

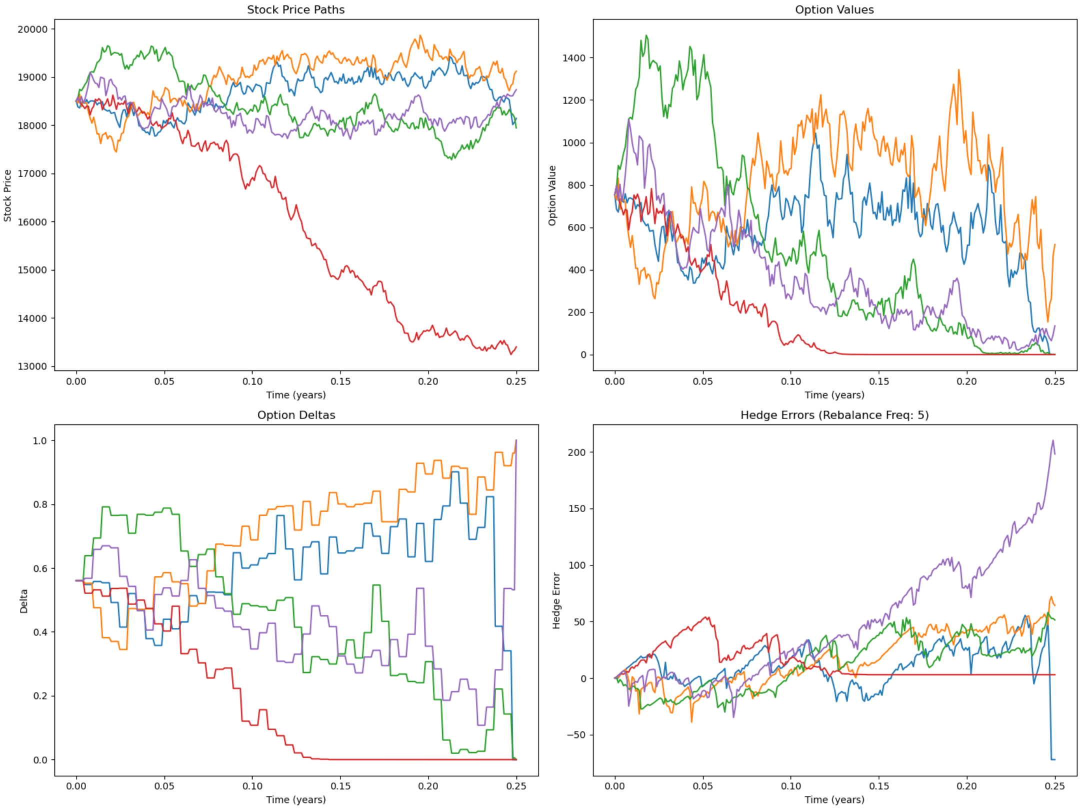

Figure 2: Hedging Metrics for different rebalancing time period of 5 days

1.b.2 Observed Metrics for Several Frequencies

The implementation simulated the delta hedging process in the `simulate_hedging` method of the `delta_hedging` class which simulates `n_paths` many paths with `n_steps` many steps and computes the option values, deltas, portfolio values, rebalancing cash flows, single step hedge errors, and total hedge errors and returns as a dictionary. The `plot_results` method aids in visualizing this simulation.

The `analyze_convergence` method calls the `simulate_hedging` computes the mean, etc. of the quantities, for different rebalancing frequencies and summarizes the results. The `plot_convergence` method visualizes some of these metrics. Figure 2 is a plot for a rebalancing frequency of once every 5 days with 5 paths and Figure 3 is a histogram of observed hedging errors across 1000 paths for a daily rebalancing frequency.

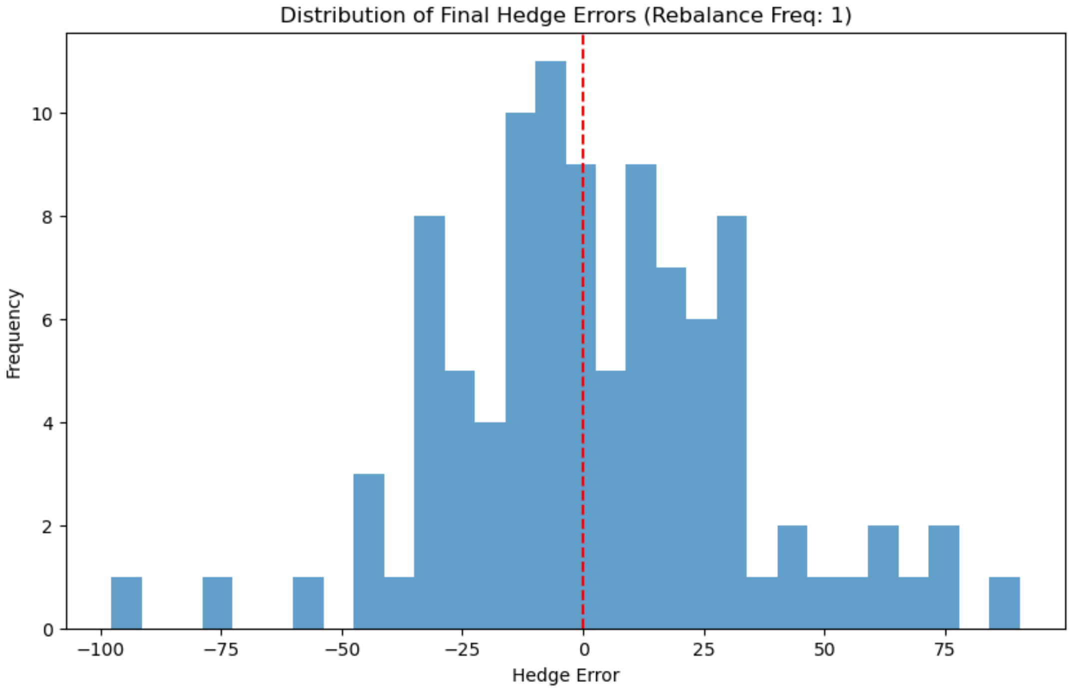

Figure 3: Histogram of observed hedge errors for daily rebalancing for 100 paths

We have noticed that there is lesser hedge error with more frequent rebalancing, as theoretically guaranteed (please refer to the histograms in the notebook `monte_carlo.ipynb`). Further we see that the hedging performance error (rms) grows almost linearly with the volatility as expected.

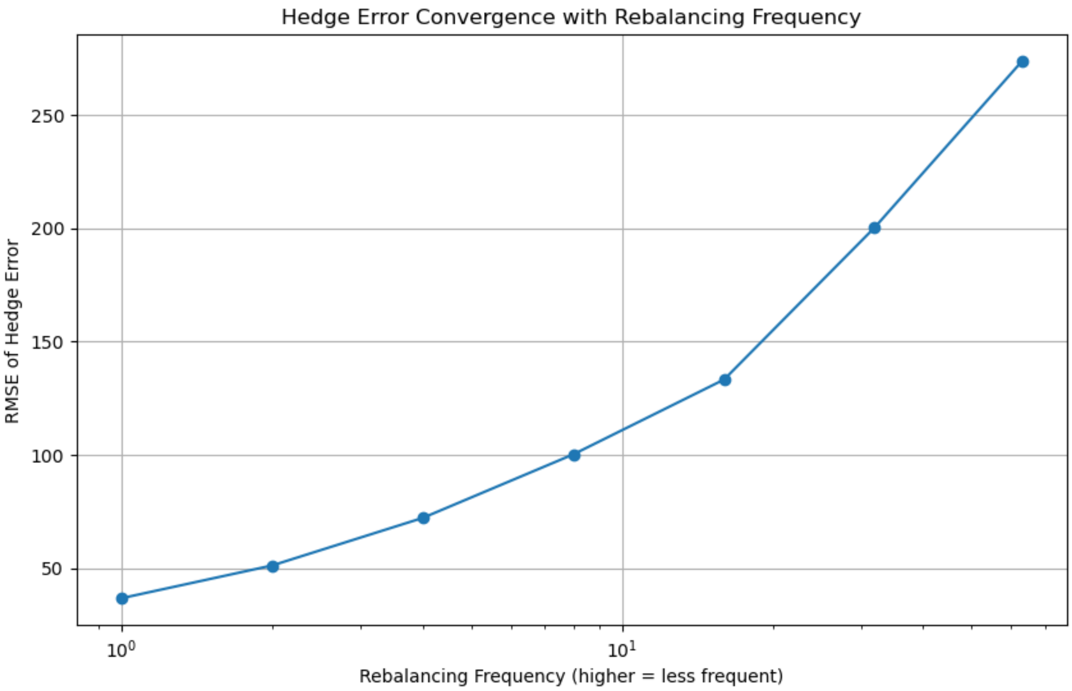

Figure 4: RMS (Note RMSE) hedge error with frequency

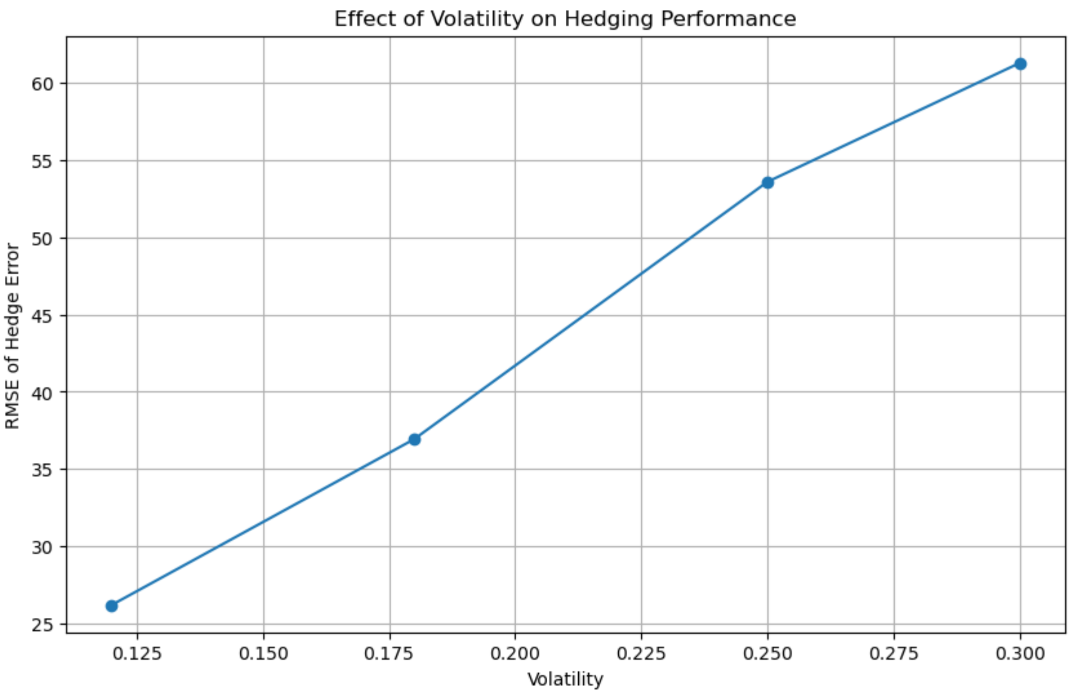

Figure 5: RMS (Not RMSE) hedge error with volatility

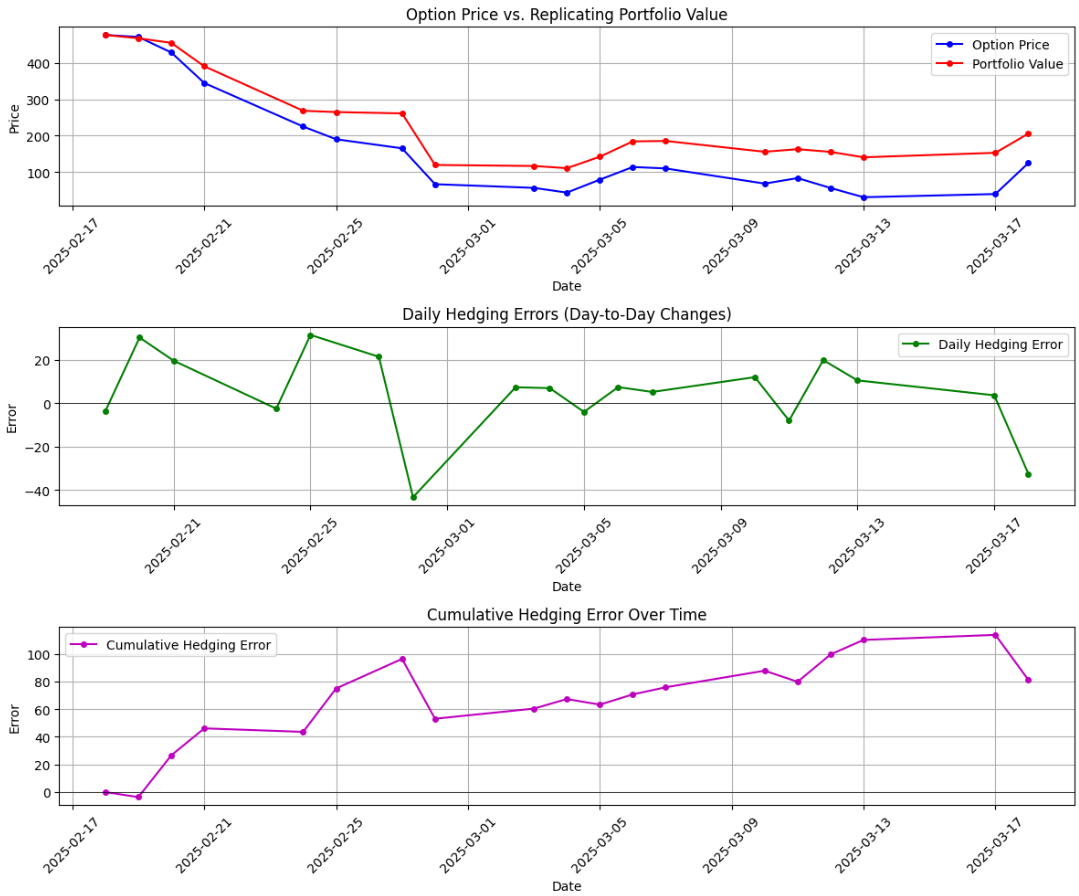

Figure 6: Graphs for delta hedging with newton Raphson IV, part 1

2 Emperical Delta Hedging

We price the option by assuming a constant volatility 0.1 (inspired from data of other NIFTY options with similar strike and time to maturity) for a NIFTY-50 option. For hedging, we can either simply substitute the observed stock price S_t in $N(d_+)$, or we can use option and stock from NSE data, either assuming the same volatility or updating at each step using root-finding methods. We observed that the methods of updating σ at each step are rather robust to the change in initial guess and perform slighlty better than assuming a fixed volatility. In our work we have not explored the numerical computation of delta using option and stock data from NSE.

The `jupyter` notebooks contain the code for the all the above mentioned methods and have the necessary comments and explanations in markdown (inspired from the theory in the revious section and computations are adapted for the non-zero risk-free rate case) hence we omit the theory here. The plots found are attached confirming out intuitions made in the theory section, explanations of the deviations are done in the notebook.

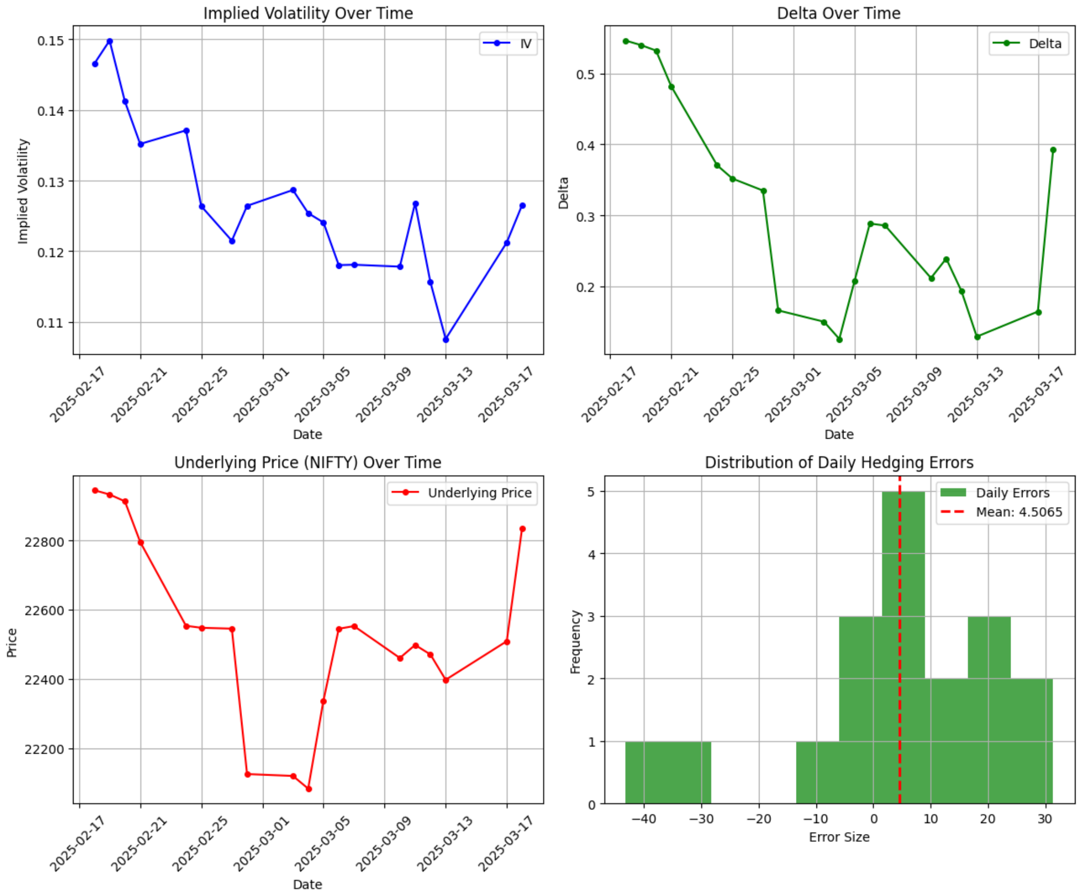

Figure 7: Graphs for delta hedging with newton Raphson IV, part 2

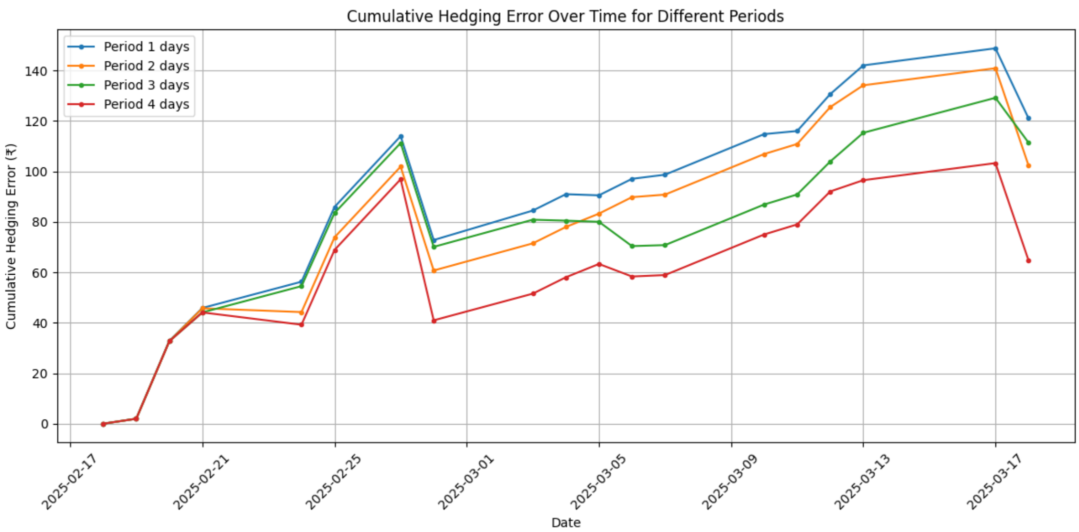

Figure 8: Hedging Metrics for different rebalancing time, part 1

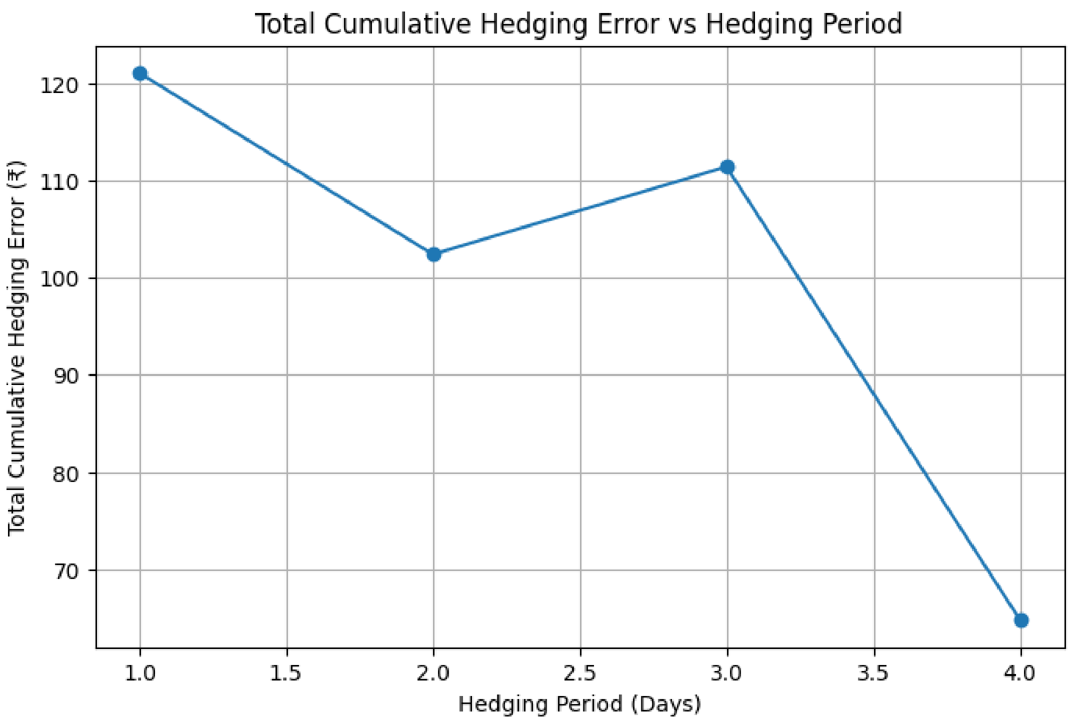

Figure 9: Hedging Metrics for different rebalancing time period, part 2

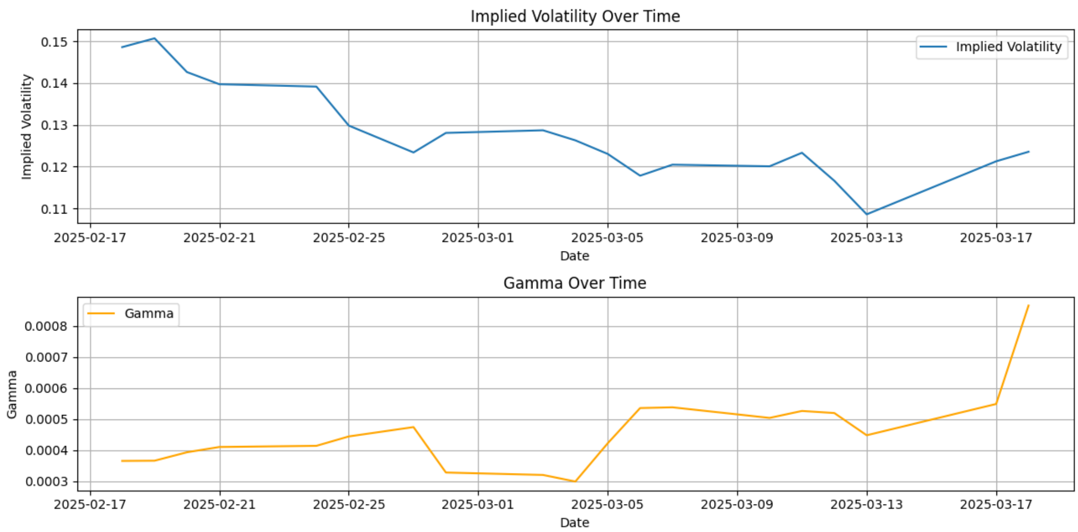

Figure 10: Gamma calculation for NSE options data

3 Contributions

Monte-Carlo and analytics (first part of assignment) and Theory: Ishaq Hamza (22187)
Emperical Delta Hedging and analytics for constant and Newton-Raphson IV: Rohit Jorige (22166)
Emperical Delta Hedging for different frequencies and different methods for IV: Balaji R