2 Moment Distribution Method

Objectives:

Definition of stiffness, carry over factor, distribution factor. Analysis of continuous beams without support yielding – Analysis of continuous beams with support yielding – Analysis of portal frames – Naylor's method of cantilever moment distribution – Analysis of inclined frames – Analysis of Gable frames.

2.1 INTRODUCTION

The end moments of a redundant framed structure are determined by using the classical methods, viz. Clapeyron's theorem of three moments, strain energy method and slope deflection method. These methods of analysis require a solution of set of simultaneous equations. Solving equations is a laborious task if the unknown quantities are more than three in number. In such situations, the moment distribution method developed by Professor Hardy Cross is useful. This method is essentially balancing the moments at a joint or junction. It can be described as a method which gives solution by successive approximations of slope deflection equations.

In the moment distribution method, initially the structure is rigidly fixed at every joint or support. The fixed end moments are calculated for any loading under consideration. Subsequently, one joint at a time is then released. When the moment is released at the joint, the joint moment becomes unbalanced. The equilibrium at this joint is maintained by distributing the unbalanced moment. This joint is temporarily fixed again until all other joints have been released and restrained in the new position. This procedure of fixing the moment and releasing them is repeated several times until the desired accuracy is obtained. The experience of designers points that about five cycles of moment distribution lead to satisfactory converging results.

Basically, in the slope deflection method, the end moments are computed using the slopes and deflection at the ends. Contrarily in the moment distribution method, as a first step — the slopes at the ends are made zero. This is done by fixing the joints. Then with successive release and balancing the joint moments, the state of equilibrium is obtained. The release-balance cycles can be carried out using the following theorems.

In conclusion, when a positive moment M is applied to the hinged end of a beam a positive moment of $\left(\frac{1}{2}\right)M$ will be transferred to the fixed end.

2.2.2 Theorem 2

Consider a two span continuous beam ACB as shown in Fig. 2.2(a). A and B are fixed supports with a prop at C. A moment is applied at C and it is required to know how much moment is distributed between spans AC and CB. Let this moment M be decomposed and distributed as M_1 to CA and M_2 to CB as shown in Fig. 2.2(b).

i.e. $$M_1 + M_2 = M \tag{1}$$

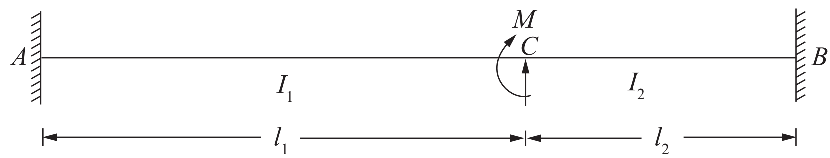


FIG. 2.2(a) Continuous beam ACB with moment M

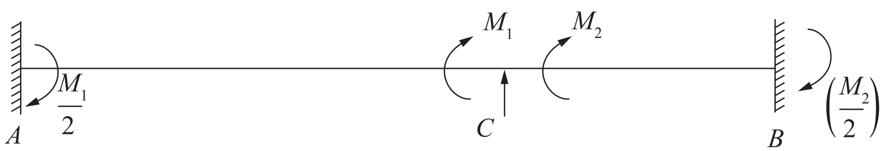


FIG. 2.2(b) Distribution of bending moments

The bending moment diagram is drawn by considering each span AC and CB respectively.

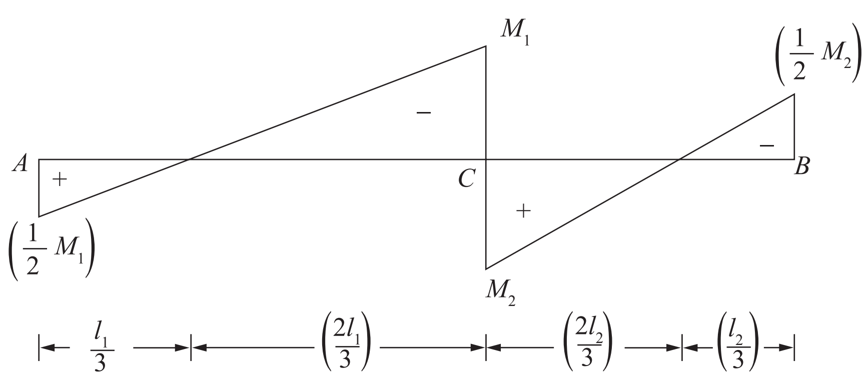


FIG. 2.2(c) Bending moment diagram

As the ends A and B are fixed; the slope between A and B is zero. That is, the area of the bending moment diagram between A and B is zero.

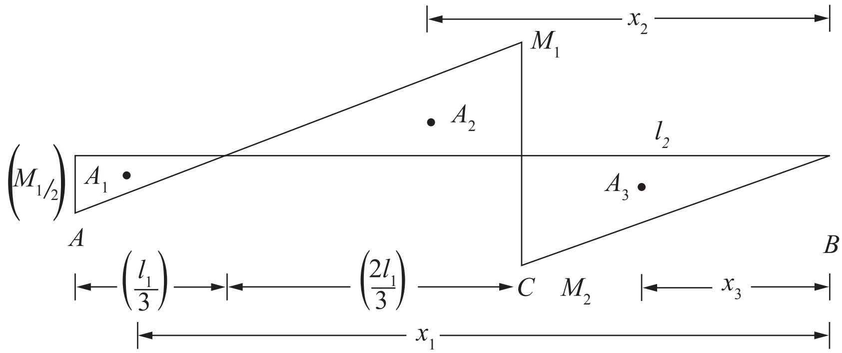


FIG. 2.4(c)

The tangent drawn at A passes through C and B. Hence, from the above figure;

$$\Sigma A_i x_i / EI_i = 0$$

i.e.

$$A_1 = (1/2)\ (l_1/3)\ (M_1/2) = \frac{M_1 l_1}{12}$$

$$x_1 = l_2 + \left(\frac{2}{3} l_1 + \frac{2}{3}\frac{l_1}{3}\right); \qquad \boxed{x_1 = l_2 + \frac{8}{9} l_1}$$

$$A_2 = (-1/2)\left(\frac{2}{3} l_1\right) M_1 = -\frac{M_1 l_1}{3}$$

$$x_2 = l_2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{l_1}{3}\right); \qquad \boxed{x_1 = l_2 + \frac{2}{9} l_1}$$

$$A_3 = (1/2)\ (l_2) M_2 = \frac{M_2 l_2}{2}$$

$$\boxed{x_3 = \frac{2}{3} l_2}$$

Thus, $\Sigma A_i x_i / EI$ gives

$$\frac{-3M_1 l_1 l_2}{12EI_1} + \frac{M_2 l_2^2}{3EI_2} = 0$$

$$\frac{M_2 l_2^2}{3EI_2} = \frac{-3M_1 l_1 l_2}{12EI_1}$$

$$\frac{l_2 M_2}{I_2} = \frac{3}{4}\frac{M_1 l_1}{I_1}$$

$$\frac{M_2}{k_2} = \frac{3}{4}\frac{M_1}{k_1}$$

i.e.,

$$\boxed{\frac{M_2}{k_2} = \frac{(3/4\,k_2)}{k_1}}$$

If one end of a member is not fixed then the "stiffness" of that member should be multiplied by (3/4).

2.2.4 Theorem 4

Consider a fixed beam AB as shown below. End B has settled by a distance δ. As the ends are fixed, there must develop a fixing moment M at the each end of the beam.

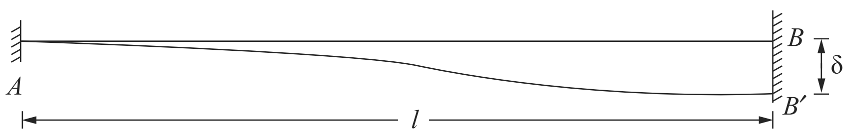


FIG. 2.5(a) Sinking of supports in a fixed beam

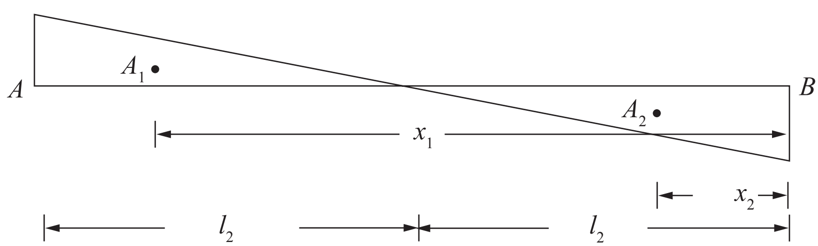


FIG. 2.5(b) Bending moment diagram

Taking moments about B and using the moment area theorem;

$$\Sigma A_i x_i / EI_i = -\delta$$

i.e.

$$A_1 = -\left(\frac{1}{2}\right)\left(\frac{l}{2}\right)M = \frac{-Ml}{4}$$

$$A_2 = \left(\frac{1}{2}\right)\left(\frac{l}{2}\right)M = \frac{Ml}{4}$$

$$x_1 = \frac{l}{2} + \frac{2}{3}\left(\frac{l}{2}\right) = \frac{5l}{6}$$

$$x_2 = \frac{1}{3}\left(\frac{l}{2}\right) = \frac{l}{6}$$

$$\frac{1}{EI}\left[-\left(\frac{Ml}{4}\right)\left(\frac{5l}{6}\right)+\left(\frac{Ml}{4}\right)\left(\frac{l}{6}\right)\right] = -\delta$$

$$\left(\frac{-5Ml^2}{24}+\frac{Ml^2}{24}\right)\frac{1}{EI} = -\delta$$

$$\frac{(-5\,+\,1)}{24}\,\frac{Ml^2}{24} = -\delta$$

$$\boxed{M = \frac{6El\delta}{l^2}}$$

i.e., when a fixed ended beam settles by an amount δ at one end, the moment required to make the ends horizontal = $6EI\delta/l^2$.

The above four theorems can be summarised as

(1) When the member is fixed at one end and a moment is applied at the other end which is simply supported or hinged, the moment induced at the fixed end is one half of the applied moment. The induced moment at the fixed end is in the same direction as the applied moment.

(2) If a moment is applied in a stiff joint of a structure, the moment is resisted by various members in proportion to their respective stiffnesses (i.e., moment of inertia divided by the length). If the stiffness of the member is more; then it resists more bending moment and it absorbs a greater proportion of the applied moment.

(3) While distributing the moments in a rigid joint, if one end of the member is not restrained then its stiffness should be multiplied by (3/4).

(4) In a fixed beam, if the support settles/subsides/sinks by an amount δ, the moment required to make the ends horizontal is $6EI\delta/l^2$.

2.3 BASIC DEFINITIONS OF TERMS IN THE MOMENT DISTRIBUTION METHOD

(a) Stiffness

Rotational stiffness can be defined as the moment required to rotate through a unit angle (radian) without translation of either end.

(b) Stiffness Factor

(i) It is the moment that must be applied at one end of a constant section member (which is unyielding supports at both ends) to produce a unit rotation of that end when the other end is fixed, i.e. $k = 4EI/l$.

(ii) It is the moment required to rotate the near end of a prismatic member through a unit angle without translation, the far end being hinged is $k = 3EI/l$.

(c) Carry Over Factor

It is the ratio of induced moment to the applied moment (Theorem 1). The carry over factor is always (1/2) for members of constant moment of inertia (prismatic section). If the end is hinged/pin connected, the carry over factor is zero. It should be mentioned here that carry over factors values differ for non-prismatic members. For non-prismatic beams (beams with variable moment of inertia); the carry over factor is not half and is different for both ends.

(d) Distribution Factors

Consider a frame with members *OA, OB*, *OC* and *OD* rigidly connected at *O* as shown in Fig. 2.6. Let *M* be the applied moment at joint *O* in the clockwise direction. Let the joint rotate through an angle θ. The members *OA*,*OB*,*OC* and *OD* also rotate by the same angle θ.

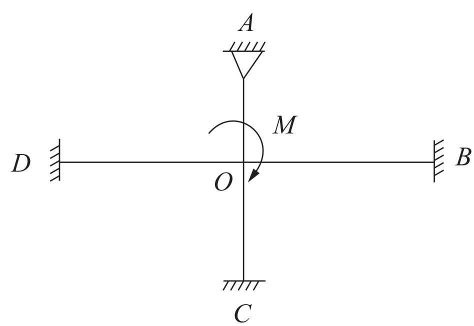


FIG. 2.6

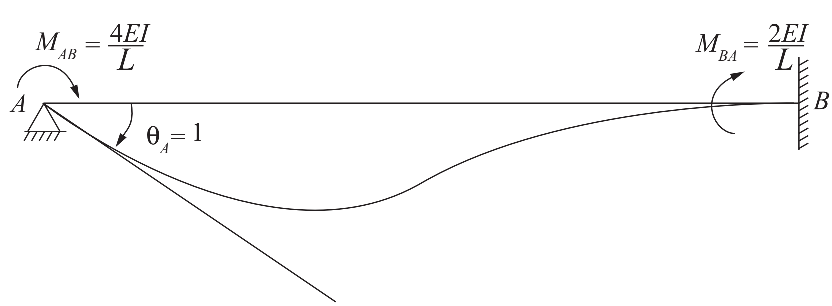
Let k_{OA}, k_{OB}, k_{OC} and k_{OD} be the stiffness values of the members *OA*, *OB*, *OC* and *OD* respectively; then

$$M_{OA} = k_{OA}\theta \qquad \text{(i)}$$

$$M_{OB} = k_{OB}\theta \qquad \text{(ii)}$$

$$M_{OC} = k_{OC}\theta \qquad \text{(iii)}$$

$$M_{OD} = k_{OD}\theta \qquad \text{(iv)}$$



(a) Beam with far end fixed

$$d_{OB} = \frac{k_{OB}}{\Sigma k} = \text{distribution factor for } OB$$

$$d_{OC} = \frac{k_{OC}}{\Sigma k} = \text{distribution factor for } OC$$

$$d_{OD} = \frac{k_{OD}}{\Sigma k} = \text{distribution factor for } OD$$

2.4 SIGN CONVENTION

Clockwise moments are considered positive and anticlockwise moments negative.

2.5 BASIC STAGES IN THE MOMENT DISTRIBUTION METHOD

The moment distribution method can be illustrated with the following example.

It is desired to draw the bending moment diagram by computing the bending moments at salient points of the given beam as shown below.

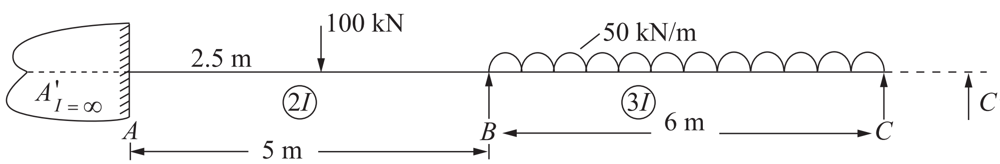


FIG. 2.8 A two span continuous beam

Step 1

Determine the distribution factor at each joint *A*,*B* and *C* respectively. The distribution factor of a member is the ratio of the stiffness of the member divided by the total stiffness of all the members meeting at that joint.

The distribution factor for the fixed support *A* is determined by assuming an imaginary span *A′A* (Fig. 2.8). The flexural rigidity of *A′A* is infinity. Hence, the stiffness is infinite. The stiffness of *AB* is $2I/5 = 0.4I$. Hence, the total stiffness is infinite. Thus, the distribution factor for *A′A* is (infinity/infinity) = 1.0 and for *AB* the distribution factor k_{AB} = ($0.4I$/infinity), i.e. zero. In essence $d_{AA'} = 1.0$ and $d_{AB} = 0.0$

The distribution factor at the support *B* is determined as follows. The stiffness of the member *BA* is (0, 4*I*). The stiffness of the member *BC* is taken as three-fourths of its stiffness (refer Theorem 3). Hence, $k_{BC} = \left(\frac{3}{4}\right)\left(\frac{3I}{6}\right) = 0.375I$. The sum of the stiffness at the joint *B* is $k = (k_{BA} + k_{BC}) = 0.775I$. Therefore, the distribution factors are $d_{BA} = k_{BA}/(k_{BA} + k_{BC})$, i.e. $d_{BA} = 0.4I/(0.4I + 0.375I) = 0.52$; similarly $d_{BC} = k_{BC}/\Sigma k$, i.e. $d_{BC} = (0.375I/0.775I) = 0.48$. The sum of the stiffnesses is $(0.52 + 0.48) = 1.0$.

The distribution factor for the simple support at *C* is determined by extending the span to *CC′*. The rigidity of *CC′* is zero and hence $k_{CC'} = 0$. On the other hand, the stiffness k_{CB} = 0.375I. The total stiffness at C is $k_{CB} + k_{CC'}$ = 0.375I. The distribution factor $d_{CB} = k_{CB}/\sum k$ = 1.0 and $d_{CC'} = k_{CC'}/\sum k$ = 0.

The above procedure is summarised in the following table for quick understanding.

Table 2.1 Distribution factor at joint B

Joint	*Members*	$k = I/l$	$\sum k$	*DF*
	BA	$\frac{2l}{5} = 0.4I$		52
B			0.775I	
	BC	$\frac{3}{4}\left(\frac{3I}{6}\right) = 0.3/5I$		0.48

Step 2

Imagine all the three joints A, B and C are rigidly fixed with horizontal tangents. Write down the fixed end moments for the beam AB as if it were built in at A and B and also for the beam BC as if it were built in B and C.

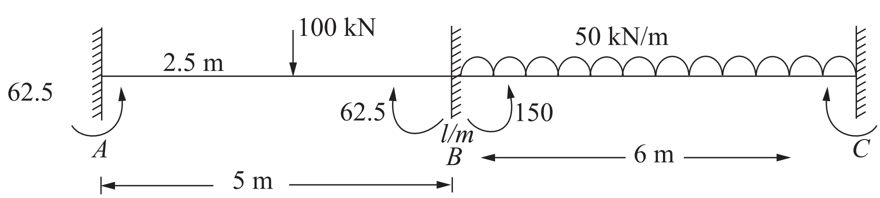


FIG. 2.9 Fixed end moments

$$M_{FAB} = -\frac{100 \times 5}{8} = -62.5\text{ kNm}$$

$$M_{FBA} = +\frac{100 \times 5}{8} = +62.5\text{ kNm}$$

$$M_{FBC} = -\frac{50 \times 6^2}{12} = -150.0\text{ kNm}$$

$$M_{FCB} = +\frac{50 \times 6^2}{12} = +150.0\text{ kNm}$$

Step 3

Each joint is released in turn and if B is released it will be out of balance. This unbalanced moment (–150.0 + 62.5 = 87.5) is shown in Fig. 2.10.

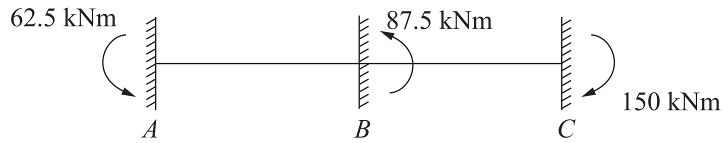


FIG. 2.10 Out of balance moments

Step 4

A moment is applied at B to balance this joint B and it will distribute itself according to the distribution factors. This is shown in the following figure.

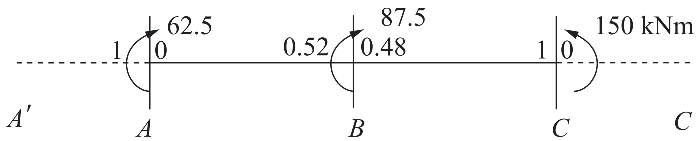


FIG. 2.11 Balancing moments

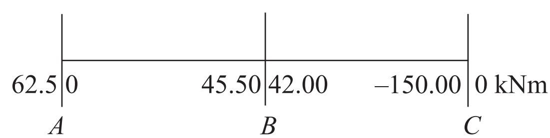


FIG. 2.12 Distributed moments

Step 5

Balance the joint C as its moment is zero (end support C is simple support). By balancing the moment –150, half of it is carried over to B. By balancing the joint B, half of the moment is carried over to joint A, i.e. half of 45.50 kNm.

Step 6

Again the joints become out of balance, and the above procedure is repeated until the moments to be distributed become negligible and can be ignored. This is illustrated in Table 2.2.

Table 2.2 Moment distribution table

Joint	*A*	*B*		*C*
Members	*AB*	*BA*	*BC*	*CB*
DF	0	0.52	0.48	1
FEMS Bal	–62.50	+62.50 +45.50	–150.00 +42.00	+150.00 –150.00
Co Bal	+22.75	 +39.00	–75.00 +36.00	
Co	+19.50			
Total	–20.25	+147.00	–147.00	0.00
Nature				*O*

In the above moment distribution table a single vertical line is drawn between the members. Double lines are drawn at the end of each joint. The pure moment diagram can be drawn using the end moments in the moment distribution table. The pure moments are the values just to the left of double line. Thus, $M_A = -20.25$ kNm, $M_B = -147.00$ kNm, $M_C = 0$.

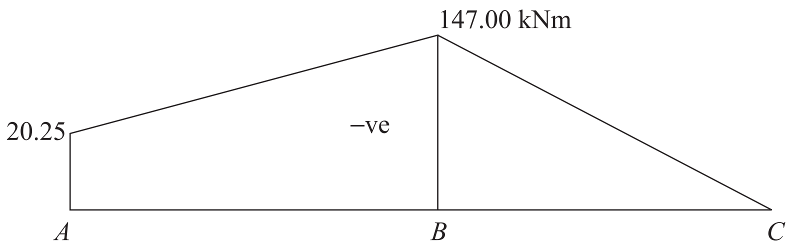


FIG. 2.13 Pure moment diagram

The simple beam moment diagram is drawn by considering each span separately. The simple beam moment diagram is always positive. While the pure moment diagram is negative. The maximum positive bending moment for span AB is $(wl/4) = 100 \times 5/4 = 125$ kNm. The maximum bending moment for a simple beam of BC is $wl^2/8 = (50 \times 6^2/8) = 225$ kNm.

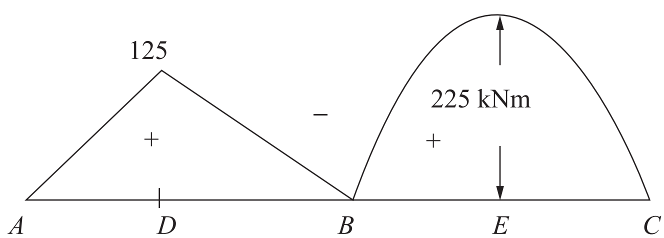


FIG. 2.14 Simple beam moment diagram

The net bending moment diagram is drawn by superimposing the pure moment diagram on the simple beam moment diagram. Thus, the net moment at D and E are

$$M_D = 125 - \left(\frac{147.00 + 20.25}{2}\right) = +41.38 \text{ kNm}$$

$$M_E = 225 - \left(\frac{147.00 + 0.00}{2}\right) = +151.5 \text{ kNm}$$

The final net bending moment diagram is as follows.

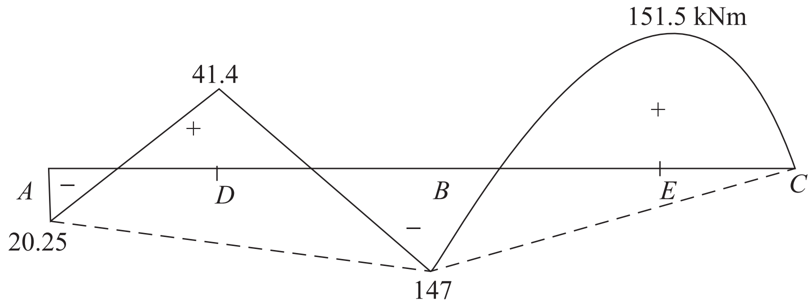


FIG. 2.15 Bending moment diagram

The net bending moment diagrams are preferable in design offices. The elastic curve is drawn using the net bending moment diagram.

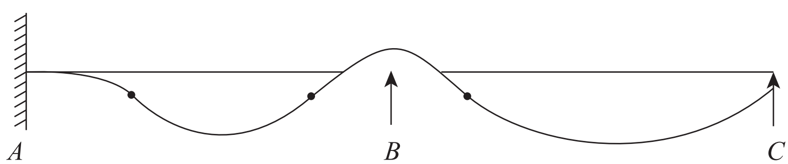


FIG. 2.16 Elastic curve

2.6 NUMERICAL EXAMPLES

2.6.1 Analysis of Continuous Beams without Support Settlement

Example 2.1 Analyse the continuous beam shown in Fig. 2.17 by the moment distribution method. Draw the bending moment diagram and shear force diagram. The beam is of uniform section.

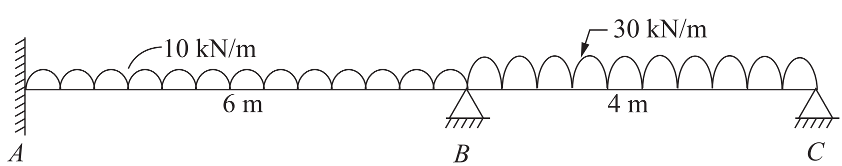


FIG. 2.17

Solution

Step 1

The distribution factors at joint B are evaluated as follows.

Table 2.3 Distribution factors

Joint	*Members*	*Relative Stiffness (k)*	*Sum Σk*	*Distribution Factors (k/Σk)*
	BA	$I/6 = 0.167I$		0.47
B			$0.3555I$	
	BC	$\frac{3}{4} \times \frac{I}{4} = 0.188I$		0.53

Step 2 Fixed End Moments

$$M_{FAB} = \frac{-10 \times 6^2}{12} = -30 \text{ kNm}, \; M_{FBA} = +30 \text{ kNm}$$

$$M_{FBC} = \frac{-30 \times 4^2}{12} = -40 \text{ kNm}, \; M_{FCB} = +40 \text{ kNm}$$

Step 3 Moment Distribution Table

As the joint *C* is a hinged end, the moment is zero. Hence, it is balanced first. Then half of this moment is carried over. Then joint *B* is balanced. From the joint *B*, the moment is carried over to *A*.

Table 2.4 Moment distribution table

Joint	*A*	*B*		*C*
Members	*AB* ←	*BA*	*BC* →	*CB*
DF	0	0.47	0.53	1
FEMS	–30.00	+30.00	–40.00	+40.00
Bal	–	–	–	–40.00
Co	–	–	–20.00	
Total	–30.00	+30.00	–60.00	0.00
Bal	–	+14.10	+15.90	–
Co	+7.05	–	–	–
Final	–22.95	+44.10	–44.10	0.00

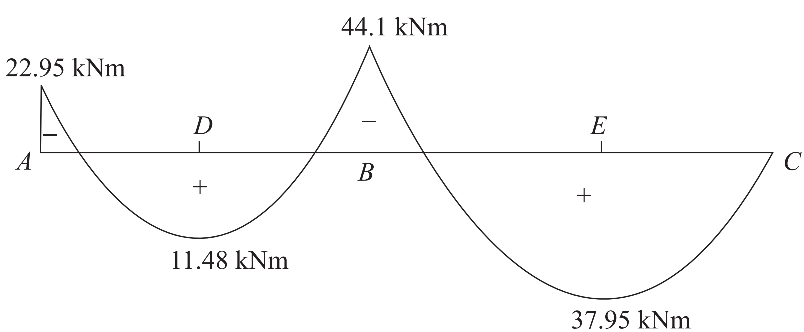


FIG. 2.18 Bending moment diagram

$$M_D = \frac{10 \times 6^2}{8} - \left(\frac{22.95 + 44.1}{2}\right) = 11.48 \text{ kNm}$$

$$M_E = \frac{30 \times 4^2}{8} - \left(\frac{44.1 + 0}{2}\right) = 37.95 \text{ kNm}$$

Step 4 Shear Force Diagrams

Equilibrium of span AB

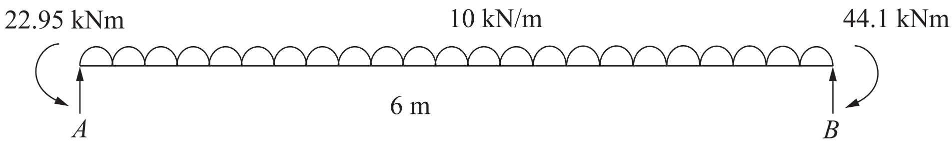


FIG. 2.19

$$\Sigma V = 0;\ V_{AB} + V_{BA} = 60 \quad (1)$$

$$\Sigma M_A = 0;\ -22.95 + 44.1 + 10 \times \frac{6^2}{2} - 6V_{BA} = 0 \quad (2)$$

$$\boxed{\begin{aligned} V_{BA} &= 33.5 \text{ kN} \\ V_{AB} &= 26.5 \text{ kN} \end{aligned}}$$

Equilibrium of span BC

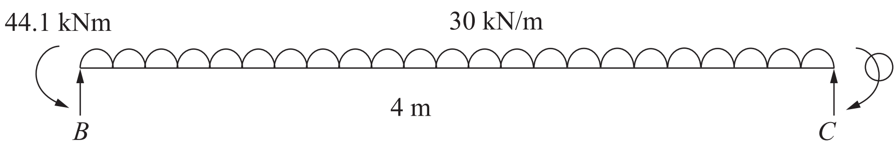


FIG. 2.20

$$\Sigma V = 0;\ V_{BC} + V_{CB} = 120 \quad (3)$$

$$\Sigma M_B = 0;\ -44.1 + 30 \times \frac{4^2}{2} - 4V_{CB} = 0$$

$$\boxed{\begin{aligned} V_{CB} &= 49 \text{ kN} \\ V_{BC} &= 71 \text{ kN} \end{aligned}}$$

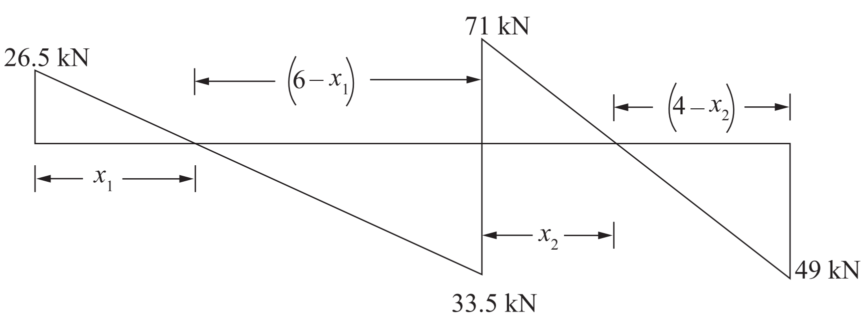


FIG. 2.21 Shear force diagram

From similar triangles

$$\frac{x_1}{(6 - x_1)} = \frac{26.5}{33.5}$$

$$33.5x_1 = 159 - 26.5x_1$$

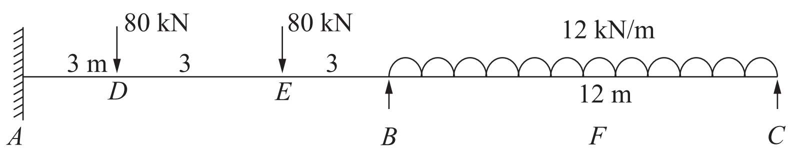
$$\therefore \qquad \boxed{x_1 = 2.65 \text{ m}}$$

$$\frac{x_2}{(4 - x_2)} = \frac{71}{49}$$

$$49x_2 = 284 - 71x_2$$

$$\therefore \qquad \boxed{x_2 = 2.37 \text{ m}}$$

Example 2.2 Analyse the continuous beam by moment distribution method. Draw the shear force diagram and bending moment diagram.



EI = Constant

FIG. 2.22

Solution

Distribution factors

The distribution factors at joint *B* are obtained as follows.

Table 2.5 Distribution factors

Joint	*Members*	*Relative Stiffness (k)*	*Sum Σk*	*Distribution Factors (k/Σk)*
	BA	$I/9 = 0.111I$		0.64
B			$0.174I$	
	BC	$\frac{3}{4} \times \frac{I}{12} = 0.063I$		0.36

Fixed end moments

$$M_{FAB} = -\frac{Wab}{l} = -\frac{80 \times 3 \times 6}{9} = -160 \text{ kNm}$$

$$M_{FBA} = +80 \times 3 \times \frac{6}{9} = +160 \text{ kNm}$$

$$M_{FBC} = -12 \times \frac{12^2}{12} = -144 \text{ kNm}$$

$$M_{FCB} = +144 \text{ kNm}$$

Table 2.6 Moment distribution table

Joint	*A*	*B*		*C*
Members	*AB*	*BA*	*BC*	*CB*
DF	0	0.64	0.36	1
FEMS Bal	–160.00	+160.00	–144.00	+144.00 –144.00
Co			–72.00	
Total Bal	–160.00	+160.00 +35.80	–216.00 +20.20	0.00
Co	+17.90			
Total	–142.10	+195.80	–195.80	0.00

Shear forces

Equilibrium of span AB

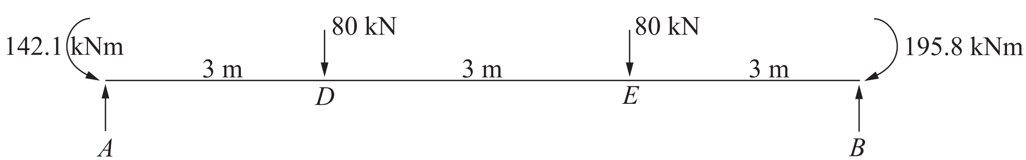


FIG. 2.23

$$\Sigma V = 0;\ V_{AB} + V_{BA} = 160 \tag{1}$$

$$\Sigma M_A = 0;\ -142.1 + 195.8 + 80 \times 3 + 80 \times 6 - 9V_{BA} = 0 \tag{2}$$

$$V_{BA} = 86 \text{ kN}$$

$$V_{AB} = 74 \text{ kN}$$

$$M_D = 74(3) - 142.1 = 80 \text{ kNm}$$

$$M_E = 86(3) - 195.8 = 62.2 \text{ kNm}$$

Equilibrium of span BC

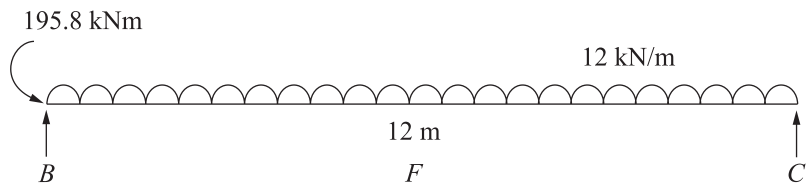


FIG. 2.24

$$\sum V = 0;\ V_{BC} + V_{CB} = 144 \tag{3}$$

$$\sum M_B = 0;\ -195.8 + 12 \times \frac{12^2}{12} - 12V_{CB} = 0 \tag{4}$$

$$V_{CB} = 55.7 \text{ kN}$$

$$V_{BC} = 88.3 \text{ kN}$$

$$M_F = 55.7 \times 6 - 12 \times \frac{6^2}{2} = 118.2 \text{ kNm}$$

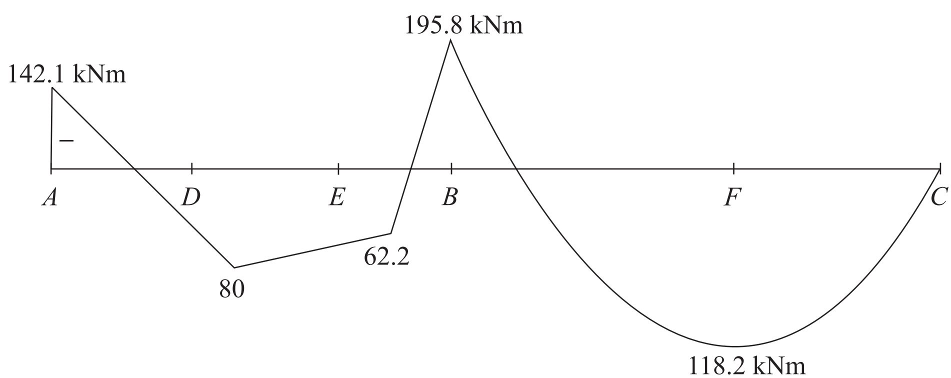


FIG. 2.25 Bending moment diagram

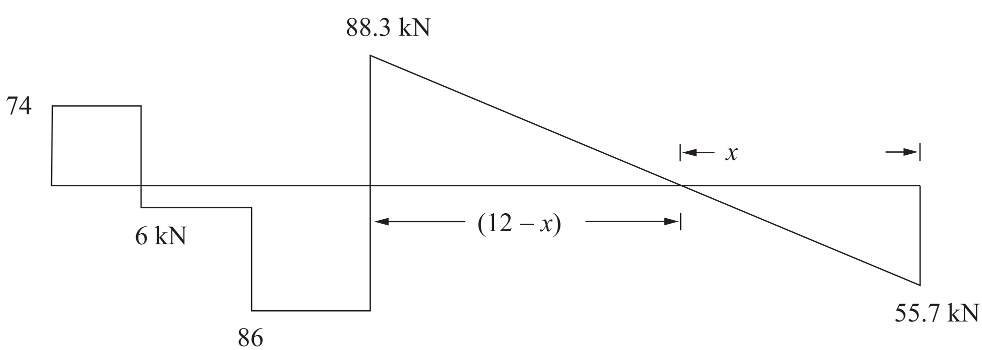


FIG. 2.26 Shear force diagram

From similar Δ's

$$\frac{x}{(12 - x)} = \frac{55.7}{88.3}$$

$$88.3x = 668.4 - 55.7x$$

$$\boxed{x = 4.64 \text{ m}}$$

Example 2.3 Analyse the continuous beam by the moment distribution method. Draw the shear force diagram and bending moment diagram.

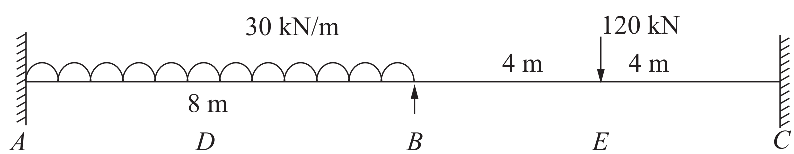


FIG. 2.27

Solution

Distribution factors

The distribution factors at joint *B* are evaluated as follows.

Table 2.7 Distribution factors

Joint	*Members*	*Relative Stiffness (k)*	*Sum Σk*	*Distribution Factors (k/Σk)*
	BA	$I/8 = 0.125I$		0.5
B			$0.25I$	
	BC	$I/8 = 0.125I$		0.5

Fixed end moments

$$M_{AB} = -30 \times \frac{8^2}{12} = -160 \text{ kNm}; \qquad M_{BC} = -\frac{120 \times 8^2}{8} = -120 \text{ kNm}$$

$$M_{BA} = +160 \text{ kNm}; \qquad M_{CB} = +120 \text{ kNm}$$

Moment Distribution

Table 2.8 Moment distribution table

Joint	*A*	*B*		*C*
Members	*AB*	*BA*	*BC*	*CB*
DF	0	0.5	0.5	0
FEMS Bal	–160 –	+160 –20	–120 –20	+120 –
Co	–10	–	–	–10
Final	–170	+140	–140	+110

Using the above end moments;

$$M_D = \frac{30 \times 8^2}{8} - \left(\frac{170 + 140}{2}\right) = 85\,\text{kNm}$$

$$M_E = 120 \times \frac{8}{4} - \left(\frac{140 + 110}{2}\right) = 115\,\text{kNm}$$

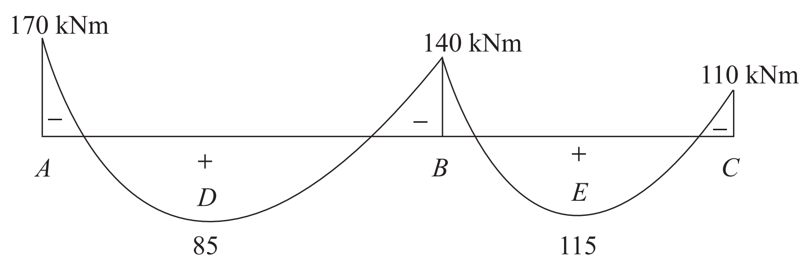


FIG. 2.28 Bending moment diagram

Shear force diagrams
Equilibrium of span AB

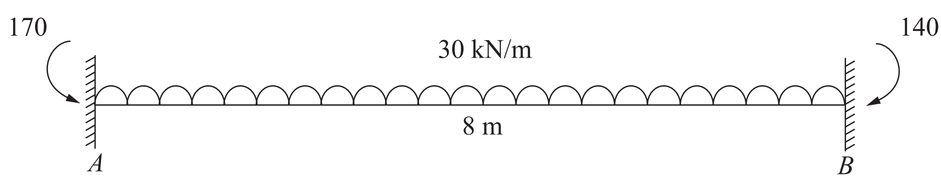


FIG. 2.29

$$\Sigma V = 0;\; V_{AB} + V_{BA} = 240 \tag{1}$$

$$\Sigma M_A = 0;\ -170 + 140 + 30 \times \frac{8^2}{2} - 8V_{BA} = 0 \tag{2}$$

$$\boxed{\begin{aligned} V_{BA} &= 116.3 \text{ kN} \\ V_{AB} &= 123.7 \text{ kN} \end{aligned}}$$

Equilibrium of span BC

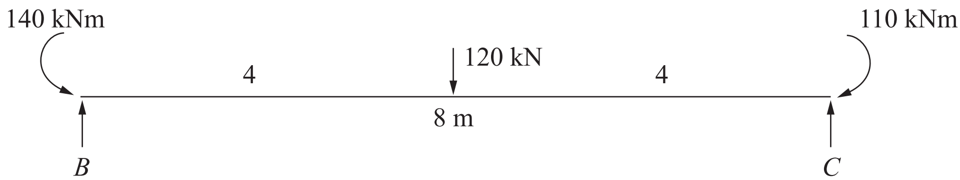


FIG. 2.30

$$\Sigma V = 0;\ V_{BC} + V_{CB} = 120 \tag{3}$$

$$\Sigma M_B = 0;\ -140 + 110 + 120(4) - 8V_{CB} = 0$$

$$\boxed{\begin{aligned} V_{CB} &= 56.3 \text{ kN} \\ V_{BC} &= 63.7 \text{ kN} \end{aligned}}$$

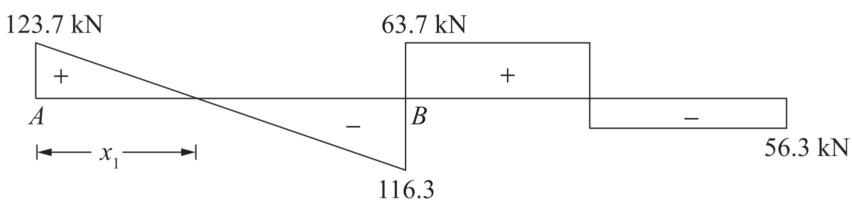


FIG. 2.31 Shear force diagram

From similar Δ's

$$\frac{x_1}{(8 - x_1)} = \frac{123.7}{116.3}$$

$$116.3x_1 = 989.6 - 123.7x_1$$

$$x_1 = 4.12 \text{ m}$$

Example 2.4 Determine the end moments for the continuous beam shown in Fig. 2.32. *EI* is constant. Use the moment distribution method.

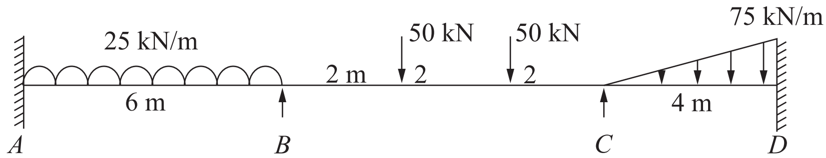


FIG. 2.32

Solution

Distribution factors

The distribution factors at joints B and C are obtained as follows.

Table 2.9 Distribution factors

Joint	*Members*	*Relative Stiffness Values (k)*	Σk	*Distribution Factors* $k/\Sigma k$
	BA	$I/6$		0.50
B			$2I/6$	
	BC	$I/6$		0.50
	CB	$I/6 = 0.167$		0.40
C			0.417	
	CD	$I/4 = 0.25$		0.60

Fixed end moments

$$M_{FAB} = -25 \times \frac{6^2}{12} = -75 \text{ kNm}$$

$$M_{FBA} = +25 \times \frac{6^2}{12} = +75 \text{ kNm}$$

$$M_{FBC} = -\frac{50 \times 2 \times 4}{6} = -66.67 \text{ kNm}$$

$$M_{FCB} = +50 \times 2 \times \frac{4}{6} = +66.67 \text{ kNm}$$

$$M_{FCD} = -75 \times 4^2/30 = -40.00 \text{ kNm}$$

$$M_{FDC} = +75 \times 4^2/20 = +60.00 \text{ kNm}$$

Table 2.10 Moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.5	0.5	0.4	0.6	0
FEMS	−75.00	+75.00	−66.67	+66.67	−40.00	+60.00
Bal		−4.17	−4.16	−10.67	+16.00	–
Co	−2.10	–	−5.34	−2.08	–	−8.00
Bal		+2.67	+2.67	+0.83	+1.25	
Co	+1.34		+0.42	+1.34		+0.63
Bal		−0.21	−0.21	−0.54	−0.80	
Co	−0.11		−0.27	−0.11		−0.40
Bal		+0.13	+0.14	+0.04	+0.07	
Co	+.07		+0.02	+0.07		+0.04
Bal		−0.01	−0.01	−0.03	−0.04	
Final	−75.8	+73.41	−73.41	+55.52	−55.52	−52.27

The end moments are

$$M_{AB} = -75.8 \text{ kNm}, \quad M_B = -73.41 \text{ kNm}$$

$$M_c = -55.52 \text{ kNm}, \; M_c = -52.27 \text{ kNm}$$

Example 2.5 Analyse and sketch the bending moment diagram for the beam shown in Fig. 2.33. The values of the second moment area of each span are indicated along the members. Modulus of elasticity is constant.

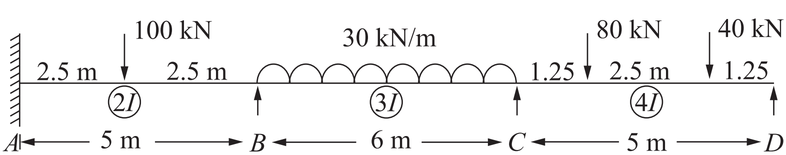


FIG. 2.33

Distribution factors

Table 2.11 Distribution factors

Joint	*Members*	*Relative Stiffness Values (k)*	Σk	*Distribution Factors* $k/\Sigma k$
	BA	$2I/5 = 0.4I$		0.44
B			$0.9I$	
	BC	$3I/6 = 0.5I$		0.56
	CB	$3I/6 = 0.5I$		0.45
C			$1.1I$	
	CD	$\frac{3}{4} \times \frac{4I}{5} = 0.6I$		0.55

Fixed end moments

$$M_{FAB} = -100 \times \frac{5}{8} = -62.5 \text{ kNm}$$

$$M_{FBA} = +100 \times \frac{5}{8} = +62.5 \text{ kNm}$$

$$M_{FBC} = -30 \times \frac{6^2}{12} = -90.0 \text{ kNm}$$

$$M_{FCB} = +30 \times \frac{6^2}{12} = +90.0 \text{ kNm}$$

$$M_{FCD} = -\frac{1.25 \times 80 \times 3.75^2}{5^2} - \frac{3.75 \times 40 \times 1.25^2}{5^2} = -65.625 \text{ kNm}$$

$$M_{FDC} = +\frac{1.25 \times 40 \times 3.75^2}{5^2} + \frac{3.75 \times 80 \times 1.25^2}{5^2} = +46.875 \text{ kNm}$$

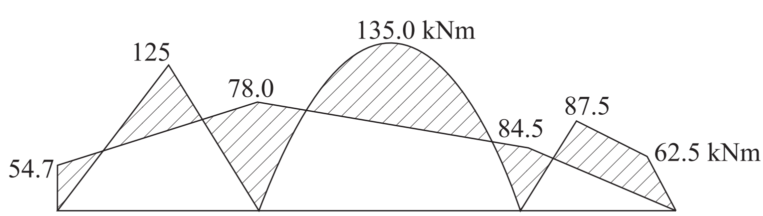


FIG. 2.34 Bending moment diagram

Table 2.12 Moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.44	0.56	0.45	0.55	1
FEMS	–62.5	+62.50	–90.00	+90.00	–65.63	+46.88
Bal		+12.10	+15.40	–10.97	–16.4	–46.88
Co	+6.05		–5.49	+7.70		
Bal		+2.42	+3.07	–3.47	–4.23	
Co	+1.21		–1.74	+1.54		
Bal		+0.77	+0.97	–0.69	–0.85	
Co	+0.39		–0.35	+0.49		
Bal		+0.15	+0.20	–0.22	–0.27	
Co	+0.08		–0.11	+0.10		
Bal		0.05	+0.06	–0.05	−0.05	
Co	+0.03		–0.03	+0.03		
Bal		+0.01	+0.02	–0.01	–0.02	
Final	–54.74	+78.0	–78.00	+84.45	–84.45	0.00

2.6.2 Analysis of Continuous Beam with Support Settlement

The settlement of supports causes the bending moment in the members. The settlement of support is mainly due to soil subsidence. In a fixed beam of *AB*, if the support *B* is lower by δ, the induced end moment is $-6EI\delta/l^2$ where δ is the settlement of supports. If the support *B* is higher by δ, the end moment is $+6EI\delta/l^2$. (Refer Basic Structural Analysis – Chapter 11) The moments are computed using the given loading and due to settlement of supports separately and summed up.

Example 2.6 Analyse the beam shown in Fig. 2.35 by the moment distribution method. Support *B* sinks by 10 mm. E = 200 kN/mm^2. $I = 4000 \times 10^4$ mm^4. Draw *BMD* and *SFD*.

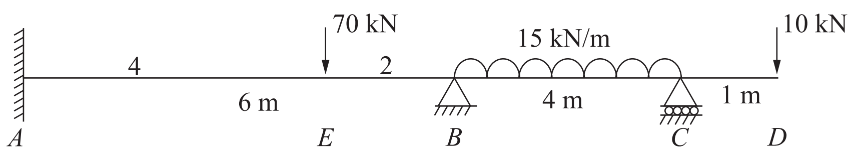


FIG. 2.35

Solution

Distribution factors

Table 2.13 Distribution factors

Joint	*Members*	*Relative Stiffness Values (k)*	Σk	*Distribution Factors* $k/\Sigma k$
	BA	$I/6 = 0.167I$		0.47
B			$0.355I$	
	BC	$\frac{3}{4}\left(\frac{I}{4}\right) = 0.188I$		0.53

Fixed moments

Due to applied loading

$$M'_{AB} = -4(70)\,\frac{2^2}{6^2} = -31.11 \text{ kNm}$$

$$M'_{BA} = +2(70)\,\frac{4^2}{6^2} = +62.22 \text{ kNm}$$

$$M'_{BC} = -15 \times 4^2/12 = -20.00 \text{ kNm}$$

$$M'_{CB} = +15 \times 4^2/12 = +20.00 \text{ kNm}$$

$$M'_{CD} = -10(1) = -10.00 \text{ kNm.}$$

Due to sinking of supports

$$M''_{AB} = M''_{BA} = \frac{-6EI\Delta_B}{l_{AB^2}} = \frac{-6 \times 8000 \times 10}{1000 \times 6^2} = -13.33 \text{ kNm}$$

$$M''_{BC} = M''_{CB} = +\frac{6EI\Delta_B}{l_{CB^2}} = \frac{6 \times 8000 \times 10}{1000 \times 4^2} = +30 \text{ kNm}$$

The actual fixed end moment, is the addition of moment due to applied loading and due to the settlement of supports.

$$M_{FAB} = -31.11 - 13.33 = -44.44 \text{ kNm}$$

$$M_{FBA} = +62.22 - 13.33 = +48.89 \text{ kNm}$$

$$M_{FBC} = -20.00 + 30.00 = +10.00 \text{ kNm}$$

$$M_{FCB} = +20.00 + 30.00 = +50.00 \text{ kNm}$$

$$M_{FCD} = -10.00 \text{ kNm}$$

Table 2.14 Moment distribution table

Joint	*A*	*B*		*C*	
Members	*AB*	*BA*	*BC*	*CB*	*CD*
DF	0	0.47	0.53	1	–
FEMS Bal	–44.44	+48.89 –27.61	+10.00 –31.22	+50.00 –40.00	–10.00
Co Bal	–13.84	 +9.40	–20.00 +10.60		
Co	+4.70				
Final	–53.58	+30.62	–30.62	+10.00	–10.00

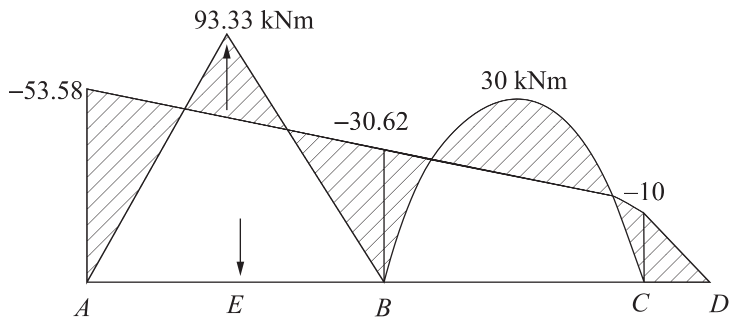


FIG. 2.36 Bending moment diagram

Shear forces

Equilibrium of span AB

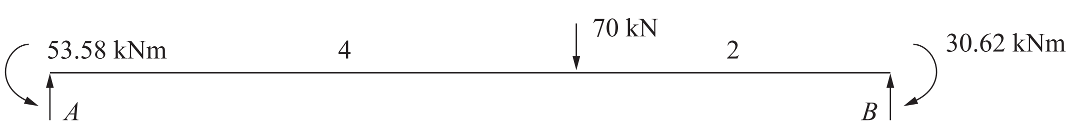


FIG. 2.37

$$\Sigma V = 0;\ V_{AB} + V_{BA} = 70 \tag{1}$$

$$\Sigma M = 0;\ -53.58 + 30.62 + 70(4) - 6V_{BA} = 0 \tag{2}$$

$$\boxed{\begin{aligned} V_{BA} &= 42.84\text{ kN} \\ V_{AB} &= 27.16\text{ kN} \end{aligned}}$$

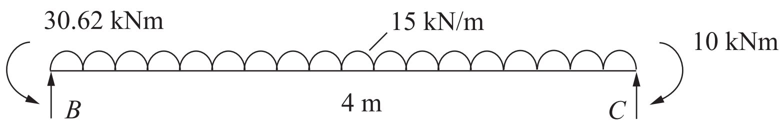


FIG. 2.38

$$\sum V = 0;\ V_{BC} + V_{CB} = 60 \tag{3}$$

$$\sum MB = 0;\ -30.62 + 10 + 15 \times \frac{4^2}{2} - 4V_{CB} = 0$$

$$\boxed{\begin{aligned} V_{CB} &= 24.85 \text{ kN} \\ V_{BC} &= 35.15 \text{ kN} \end{aligned}}$$

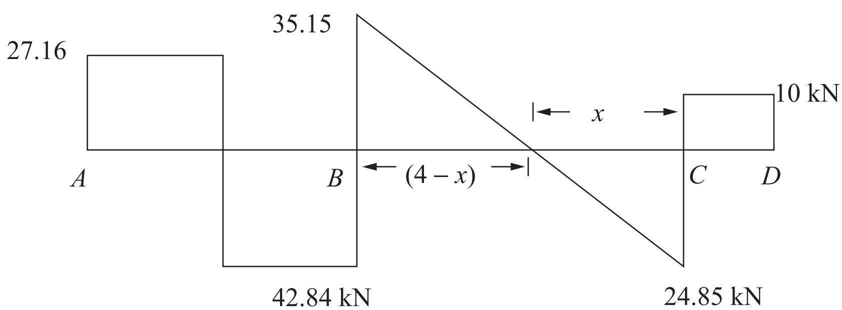


FIG. 2.39 Shear force diagram

The location of zero shear force is located from similar triangles as

$$\frac{x}{4-x} = \frac{24.85}{35.15}$$

$$35.15x = 99.4 - 24.85x$$

$$x = 1.65 \text{ m}$$

Example 2.7 Analyse the continuous beam by the moment distribution. The supports B and C settle by 8 mm and 4 mm respectively. EI = 30000 kNm^2. Sketch the SFD and BMD.

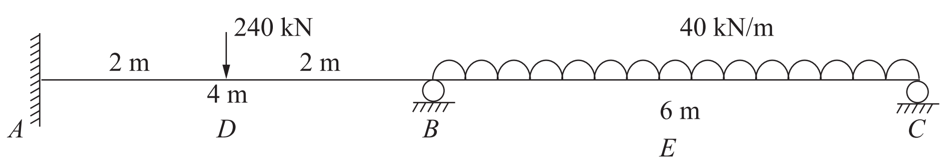


FIG. 2.40

Solution

Distribution factors

Table 2.15 Distribution factors

Joint	*Members*	*Relative Stiff Values (k)*	Σk	$k/\Sigma k$
	BA	$I/4 = 0.25I$		0.67
B			$0.375I$	
	BC	$\frac{3}{4}\times\frac{I}{6}=0.125I$		0.33

Fixed end moments

$M_{FAB} = M'_{AB} + M''_{AB}$ where M'_{AB} is the fixed end moment due to loading and M''_{AB} is the fixed end moment due to settlement of supports.

Due to applied loading

$$M'_{AB} = -240 \times \frac{4}{8} = -120 \text{ kNm}$$

$$M'_{BA} = +240 \times \frac{4}{8} = +120 \text{ kNm}$$

$$M'_{BC} = -40 \times \frac{6^2}{12} = -120 \text{ kNm}$$

$$M'_{CB} = +40 \times \frac{6^2}{12} = +120 \text{ kNm}$$

Due to settlement of supports

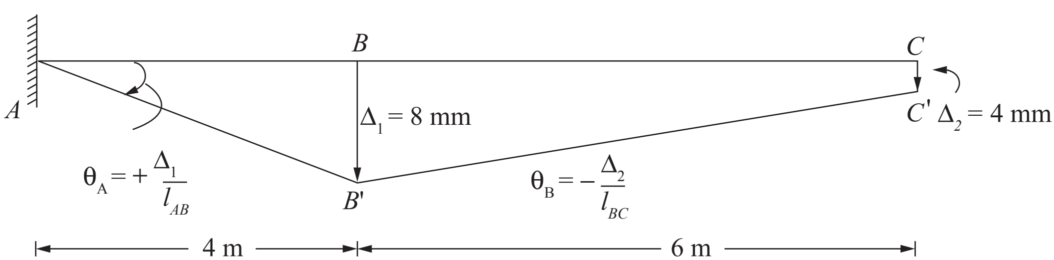


FIG. 2.41 Settlement diagram

$$M''_{AB} = M''_{BA} = \frac{-6EI\Delta_1}{l_1^2} = -6 \times 3000 \times \frac{8}{1000} \times \frac{1}{4^2} = -90 \text{ kNm}$$

$$M''_{BC} = M''_{CB} = \frac{-6EI(-\Delta_2)}{l_2^2} = +6 \times 3000 \times \frac{4}{1000} \times \frac{1}{6^2} = +20 \text{ kNm}$$

Hence, $M_{FAB} = -120 - 90 = 210$ kNm

$M_{FBA} = +120 - 90 = +30$ kNm

$M_{FBC} = -120 + 20 = -100$ kNm

$M_{FCB} = +120 + 20 = +140$ kNm

Moment distribution table

Table 2.16 Moment distribution table

Joint	*A*	*B*		*C*
Members	*AB*	*BA*	*BC*	*CB*
DF	0	0.67	0.33	1
FEMS Bal	–210.00	+30.00 +46.90	–100.00 +23.10	+140.00 –140.00
Co Bal	+23.45	– +46.90	–70.00 +23.10	–
Co	+23.45			
Total	–163.10	–123.8	–123.80	0.00

Equilibrium of span AB

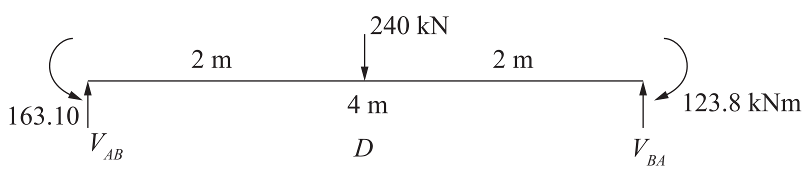


FIG. 2.42

$$\Sigma V = 0;\ V_{AB} + V_{BA} = 240 \tag{1}$$

$$\Sigma M_A = 0;\ -163.10 + 123.8 + 240(2) - 4V_{BA} = 0 \tag{2}$$

$$\boxed{\begin{aligned} V_{BA} &= 110.2 \text{ kN} \\ V_{AB} &= 129.8 \text{ kN} \end{aligned}}$$

$$M_D = -163.10 + 129.8(2) = 96.5 \text{ kNm}$$

Equilibrium of span BC

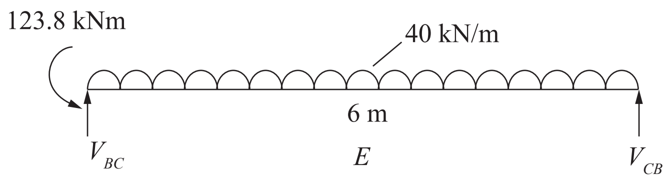


FIG. 2.43

$$\Sigma V = 0; \qquad V_{BC} + V_{CB} = 240 \tag{3}$$

$$\Sigma M_B = 0; \qquad -123.8 - 6V_{CB} + 40 \times \frac{6^2}{2} = 0$$

$$\boxed{\begin{aligned} V_{CB} &= 99.4 \text{ kN} \\ V_{BC} &= 140.6 \text{ kN} \end{aligned}}$$

$$M_E = -123.8 + 140.6 \times 3 - 40 \times \frac{3^2}{2} = 118 \text{ kNm}$$

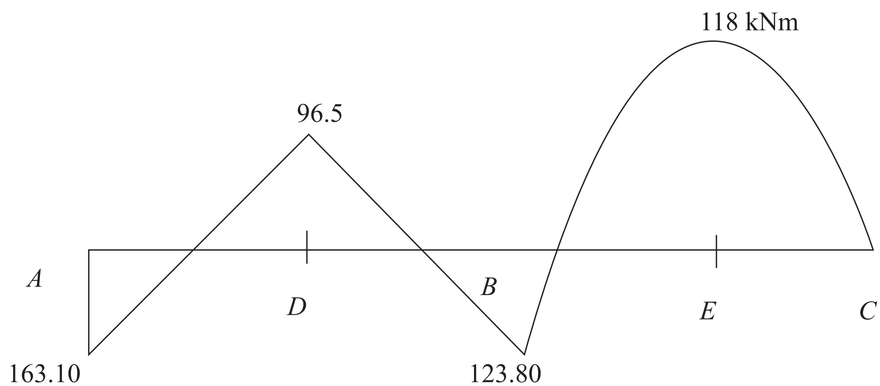


FIG. 2.44 Bending moment diagram

The reader can locate the point of contraflexure and the maximum positive bending moment using basics.

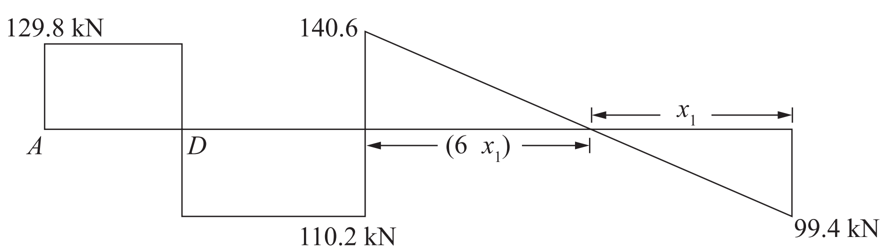


FIG. 2.45

The location of zero shear force can be obtained from similar triangles,

$$\frac{x_1}{(6-x_1)} = \frac{99.4}{140.6}$$

and hence $x_1 = 2.5$ m

Example 2.8 Analyse the continuous beam by the moment distribution method. Sketch the shear force and bending moment diagrams. Relative to the support *A*, the support *B* sinks by 1 mm and the support *C* raises by 1/2 mm, EI = 30000 kNm^2.

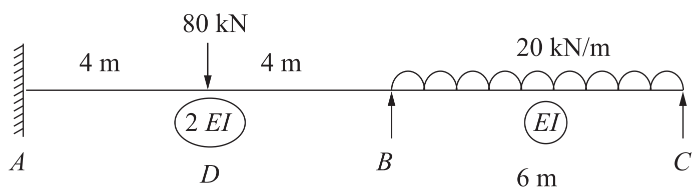


FIG. 2.46

Distribution factors

Table 2.17 Distribution factors

Joint	*Members*	*Relative Stiffness Values (k)*	Σk	$k/\Sigma k$
	BA	$\frac{2I}{8}$		0.67
B			$3I/8$	
	BC	$\frac{3}{4}\times\frac{I}{6}=\frac{I}{8}$		0.33

Fixed end moments

Due to loading

$$M'_{AB} = -80 \times \frac{8}{8} = -80 \text{ kNm}$$

$$M'_{BA} = +80 \text{ kNm}$$

$$M'_{BC} = -20 \times \frac{6^2}{12} = -60 \text{ kNm}$$

$$M'_{CB} = +60 \text{ kNm}$$

Due to settlement of supports

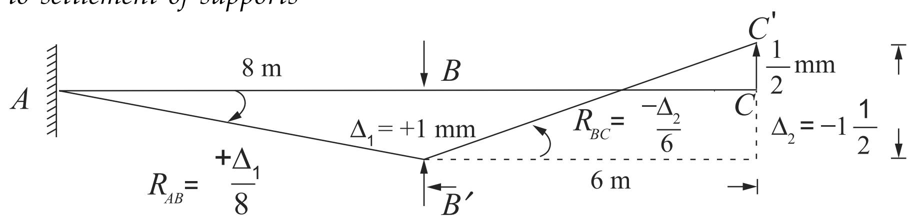


FIG. 2.47 Settlement angles diagram ($R = \Delta/l$)

$$M''_{AB} = \frac{-6E(2I)\Delta_1}{l_1^2} = -\frac{6 \times 30000 \times 1}{1000 \times 8^2} \times 2 = -5.63 \text{ kNm}$$

$$M''_{BA} = \frac{-6E(2I)\Delta_1}{l_1^2} = -5.63 \text{ kNm}$$

$$M''_{BC} = \frac{-6EI(-\Delta_2)}{l_2^2} = +\frac{6 \times 30000 \times 1.5}{1000 \times 6^2} = 7.5 \text{ kNm}$$

It should be noted that the support B settles by 1 mm. In span AB, B settles and hence AB rotates in the clockwise direction. Therefore, (Δ_1/l_1) is positive. In span BC, the support C raises by 1/2 mm from the original level AC. The total displacement is 1(1/2) mm with respect to the displaced position B'. BC displaces to $B'C'$. Hence, the rotation angle (Δ_2/l_2) is negative since the span BC rotates in the anticlockwise direction with respect to B'. The fixed end moment is the sum of the moments due to applied loading and due to the settlement of supports.

$$M_{FAB} = -80 - 5.63 = -85.63 \text{ kNm}$$

$$M_{FBA} = +80 - 5.63 = +74.37 \text{ kNm}$$

$$M_{FBC} = -60 + 7.50 = -52.50 \text{ kNm}$$

$$M_{FCB} = +60 + 7.50 = +67.50 \text{ kNm}$$

Table 2.18 Moment distribution table

Joint	*A*	*B*		*C*
Members	*AB*	*BA*	*BC*	*CB*
DF	0	0.67	0.33	1
FEMS	−85.63	+74.37	−52.50	+67.50
Bal		−14.65	−7.22	−67.50
Co	−7.33		−33.75	
Bal		+22.61	+11.14	
Co	+11.31			
Final	−81.65	+82.33	−82.33	0.00

Shear force diagrams

Equilibrium of span AB

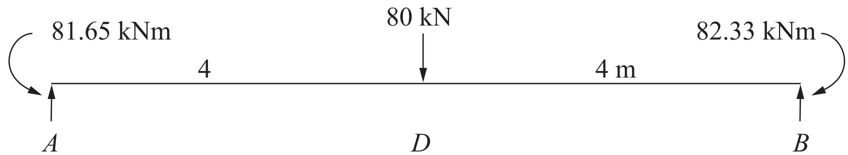


FIG. 2.48

$$\Sigma V = 0, \quad V_{AB} + V_{BA} = 80 \tag{1}$$

$$\Sigma M_A = 0; \quad -81.65 + 82.33 + 80(4) - 8V_{BA} = 0 \tag{2}$$

$$\boxed{\begin{aligned} V_{BA} &= 40.1 \text{ kN} \\ V_{AB} &= 39.9 \text{ kN} \end{aligned}}$$

$$M_D = -81.65 + 39.9 \times 4 = 77.95 \text{ kNm}$$

Equilibrium of span BC

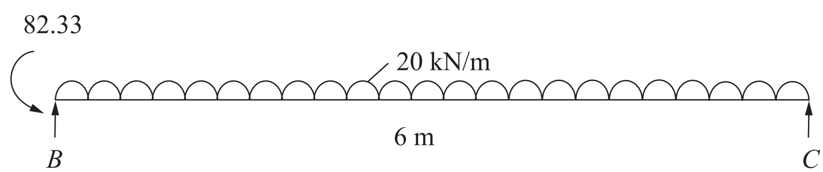


FIG. 2.49

$$\Sigma V = 0; \quad V_{BC} + V_{CB} = 120 \tag{3}$$

$$\Sigma M_B = 0; \quad -82.33 + 20 \times \frac{6^2}{2} - 6V_{CB} = 0$$

$$\boxed{\begin{aligned} V_{CB} &= 46.3 \text{ kN} \\ V_{BC} &= 73.7 \text{ kN} \end{aligned}}$$

The zero shear force location in *BC* is obtained from

$$\frac{x}{6-x} = \frac{46.3}{73.7}$$

$$73.7x = 277.8 - 46.3x$$

$$\boxed{x = 2.32 \text{ m}}$$

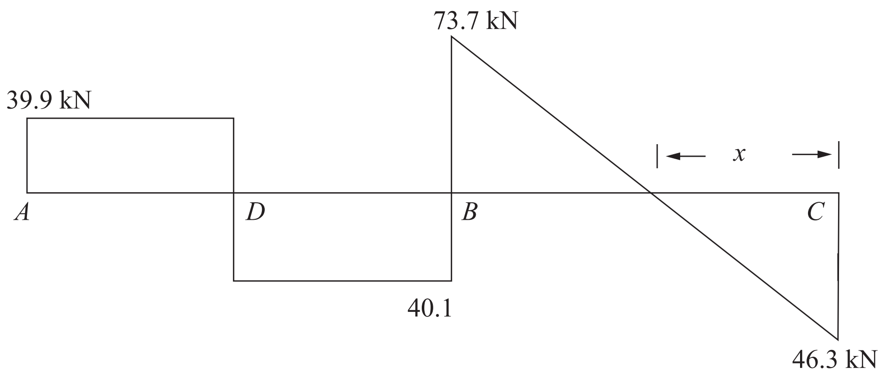


FIG. 2.50 Shear force diagram

Absolute maximum BM is $46.3 \times 2.32 - 20 \times \dfrac{2.32^2}{2} = 53.6$ kNm

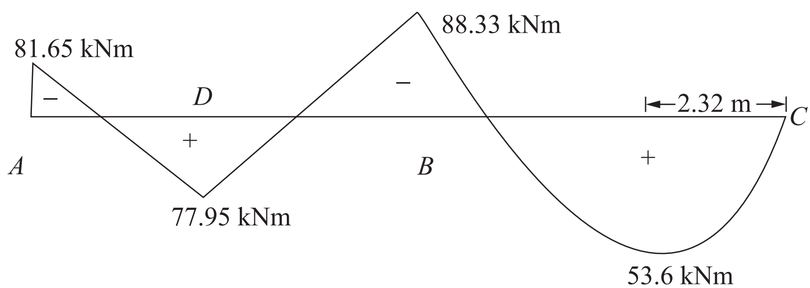


FIG. 2.51 Bending moment diagram

Example 2.9 A continuous beam is loaded as shown in Fig. 2.52. During loading the support B sinks by 10 mm. Determine the bending moments at the supports. Sketch the BMD. Given that $I = 1600(10)^4$ mm^4; $E = 200$ kN/mm^2. Use moment distribution method. Draw SFD also.

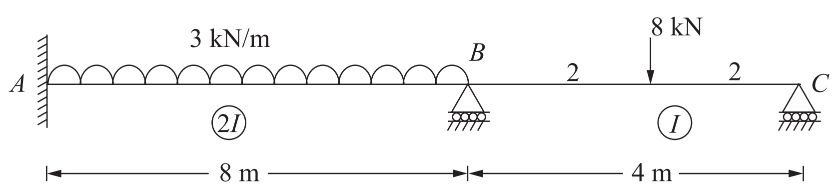


FIG. 2.52

Solution

Distribution factors

Table 2.19 Distribution factors

Joint	*Members*	*Relative Stiffness Values* (***k***)	Σk	$\frac{k}{\Sigma k}$
	BA	$\frac{2I}{8} = 0.25I$		0.57
B			$0.438I$	
	BC	$\frac{3}{4} \times \frac{I}{4} = 0.188I$		0.43

Fixed end moments

Due to loading

$$M'_{FAB} = -3 \times \frac{8^2}{12} = -16\text{ kNm}; \qquad M'_{FBA} = +16\text{ kNm}$$

$$M'_{FBC} = -8 \times \frac{4}{8} = -4\text{ kNm}; \qquad M'_{FCB} = +4\text{ kNm}$$

Due to Settlement

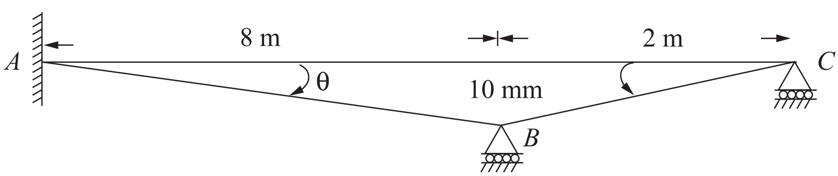


FIG. 2.53 Settlement diagram

$$M''_{FAB} = -\frac{6EI\Delta_1}{l_1^2} = -6 \times 2 \times 3200 \times \frac{10}{1000} \times \frac{1}{8^2} \; (\because \Delta_1/l_1 \text{ is positive})$$

$$M''_{FAB} = -6\text{ kNm}$$

$$\therefore \qquad M''_{FBA} = +6\text{ kNm}$$

$$M''_{FBA} = -\frac{6EI\Delta_2}{l_2^2} = -\frac{6 \times 3200 \times (-10)}{1000 \times 4^2} = +12\text{ kNm} \; (\because \Delta_2/l_2 \text{ is negative})$$

$$M''_{FCB} = +12\text{ kNm}$$

It is to be mentioned here that *B* settles by 10 mm with respect to *A*. The settlement angle is positive as *AB* rotates in clockwise direction with respect to *A*. The settlement angle for CB is negative as CB rotates in the anticlockwise direction. With respect to C. Thus, the fixed end moments at the joint is the sum of the moments due to loading and the settlement. They are as follows.

Fixed end moments

$$M_{FAB} = -16 - 6 = -22 \text{ kNm}$$

$$M_{FBA} = 16 - 6 = 10 \text{ kNm}$$

$$M_{FBC} = -4 + 12 = 8 \text{ kNm}$$

$$M_{FCB} = +4 + 12 = 16 \text{ kNm}$$

Table 2.20 Moment distribution table

Joint	*A*	*B*		*C*
Members	*AB*	*BA*	*BC*	*CB*
DF	0	0.57	0.43	1
FEMS Bal	−22.00 –	+10.00 –	+8.00 –	+16.00 −16.00
Co	–	–	−8.00	–
Total Bal	−22.00 –	+10.00 −5.70	0.00 −4.30	0.00 –
Co	−2.85	–	–	–
Final	−24.85	+4.30	−4.30	0.00

Shear forces

Free body diagram

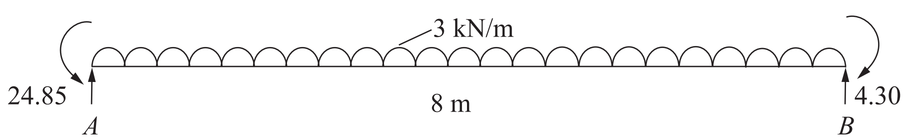


FIG. 2.54(a)

$$\Sigma V = 0; \quad V_{AB} + V_{BA} = 24 \tag{1}$$

$$\Sigma M_A = 0; \quad -24.85 + 4.30 + 3 \times \frac{8^2}{2} - 8V_{BA} = 0$$

$$\boxed{\begin{aligned} V_{BA} &= 9.4 \text{ kN} \\ V_{AB} &= 14.6 \text{ kN} \end{aligned}}$$

$$M_D = -24.85 + 14.6 \times 4 - \frac{3 \times 4^2}{2} = 9.55 \text{ kNm}$$

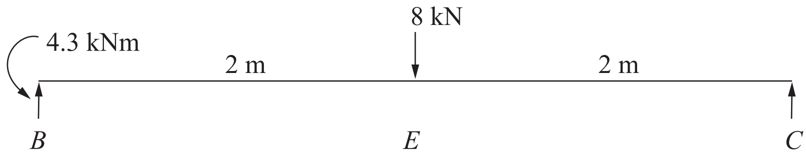


FIG. 2.54(b)

$$\sum V = 0; \quad V_{BC} + V_{CB} = 8$$

$$\sum M_B = 0; \quad -4.3 + 8(2) - 4V_{CB} = 0$$

$$\boxed{\begin{aligned} V_{CB} &= 2.9 \text{ kN} \\ V_{BC} &= 5.1 \text{ kN} \end{aligned}}$$

$$M_E = 2.9(2) = 5.8 \text{ kNm}$$

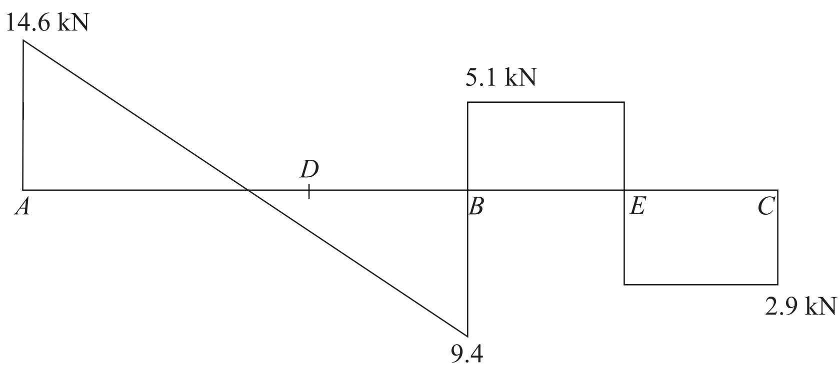


FIG. 2.55 Shear force diagram

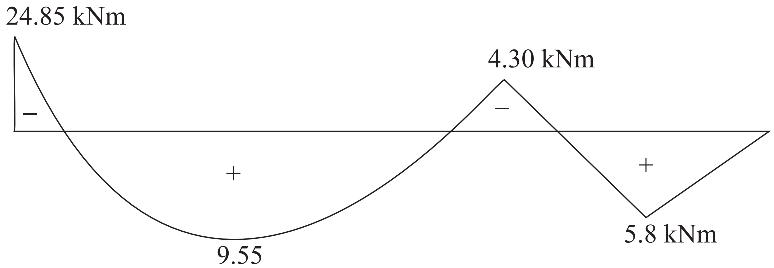


FIG. 2.56 Bending moment diagram

2.6.3 Analysis of Beams with Variable Moment of Inertia

Example 2.10 A horizontal beam *ACB*, three metres long is fixed at both ends *A* and *B*. They are at the same level. The member has a change of section at the centre of the span *C* such that the second moments of area are *I* for a distance *AC* and 2*I* for distance *CB*. A concentrated load of 500 kN acts at the midpoint *C*. Determine the fixing end moments at *A* and *B*.

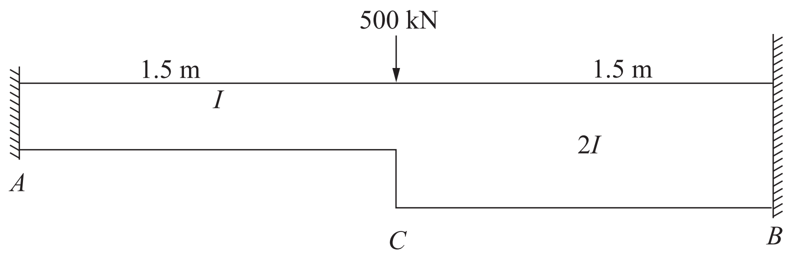


FIG. 2.57

Solution

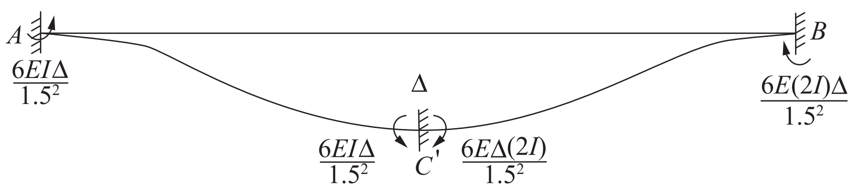


FIG. 2.58

Applying the moment distribution method,

Table 2.21 Distribution factors

Joint	*Members*	*Relative Stiffness* (*k*)	*Sum* Σk	*Distribution Factor* ($k/\Sigma k$)
	CA	$I/1.5$		0.33
C			2*I*	
	CB	$2I/1.5$		0.67

The point *C* is displaced vertically by an amount Δ to the point *C*′ keeping the section at *C* clamped (i.e., preventing any rotation). If an arbitrary end moment M_{CA} is given an arbitrary value of 100, the fixed end moment of M_{CB} will be 200.

Table 2.22 Moment distribution table

Joint	*A*	*C*		*B*
Members	*AC*	*CA*	*CB*	*BC*
DF	1	0.33	0.67	1
FEMS Bal	−100	−100 −33	+200 −67	+200
Co	−16.5			−33.5
Final	−116.5	−133	+133	+166.5

Equilibrium of span AC

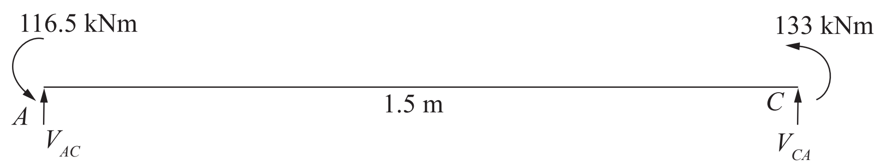


FIG. 2.59

$$\Sigma M_C = 0;$$

$$-116.5 - 133 + 1.5V_{AC} = 0$$

$$V_{AC} = 166.33 \text{ kN}$$

Equilibrium of span CB

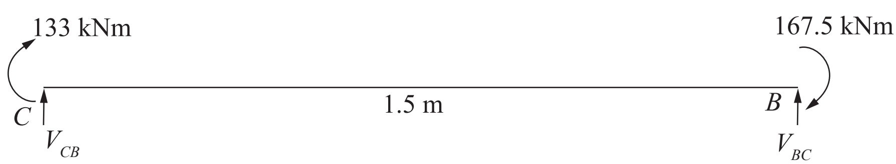


FIG. 2.60

Taking moment about *C*;

$$133 + 167.5 - 1.5V_{BC} = 0$$

$$V_{BC} = 200.33 \text{ kN}$$

Resolving the forces vertically

$$= 166.33 + 200.33 = 366.66 \text{ kN}$$

i.e., if the concentrated load applied at the centre is 366.66 kN then this will yield the moments given in the table. But the applied load is 500 kN. Hence, the moments at A and B are obtained as

$$M_A = -116.5 \times \frac{500}{366.66} = -158.9 \text{ kNm}$$

$$M_B = +167.5 \times \frac{500}{366.66} = +228.4 \text{ kNm}$$

Example 2.11 A vertical column of 8 m height is to carry a crane girder load of 50 kN applied at an eccentricity of 0.2 m. Calculate the moments at A and B due to this load assuming both ends are fixed.

Solution: Evaluate the distribution factor at C.

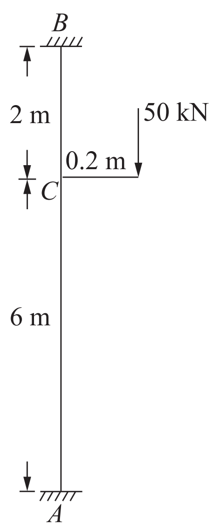


FIG. 2.61

Table 2.23 Distribution factors

Joint	*Members*	*Relative Stiffness (k)*	Σk	$(k/\Sigma k)$
	CA	$I/6$		0.25
C			$0.667I$	
	CB	$I/2$		0.75

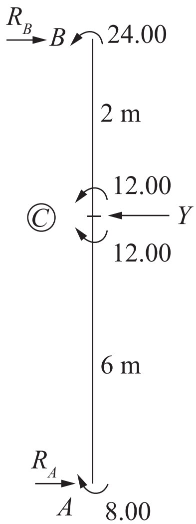


FIG. 2.64

Taking moment about C for the portion AC,

$$12 + 8 - 6R_A = 0$$

$$R_A = 3.33 \text{ kN}$$

Resolving all the forces horizontally

$$R_A + R_B - Y = 0$$

$$3.33 + 18 - Y = 0$$

$$\boxed{Y = 21.33 \text{ kN}}$$

The sway moments corresponding to the sway force 5 kN is obtained by multiplying the moments in the moment distribution table as

$$M''_{AC} = \left(\frac{5}{21.33}\right)8 = 1.88 \text{ kNm}$$

$$M''_{CA} = \left(\frac{5}{21.33}\right)12 = 2.81 \text{ kNm}$$

$$M''_{CB} = \left(\frac{5}{21.33}\right) \times -12 = -2.81 \text{ kNm}$$

$$M''_{BC} = \left(\frac{5}{21.33}\right) \times -24 = -5.63 \text{ kNm}$$

Then the final moments is the sum of sway and nonsway moments as

$$M_{AC} = M'_{AC} + M''_{AC} = 1.25 + 1.88 = 3.13 \text{ kNm}$$

$$M_{BC} = M'_{BC} + M''_{BC} = 3.75 - 5.63 = -1.88 \text{ kNm}$$

2.7 ANALYSIS OF RECTILINEAR FRAMES

Example 2.12 Analyse the frame shown in Fig. 2.65 by moment distribution method. Draw the bending moment diagram.

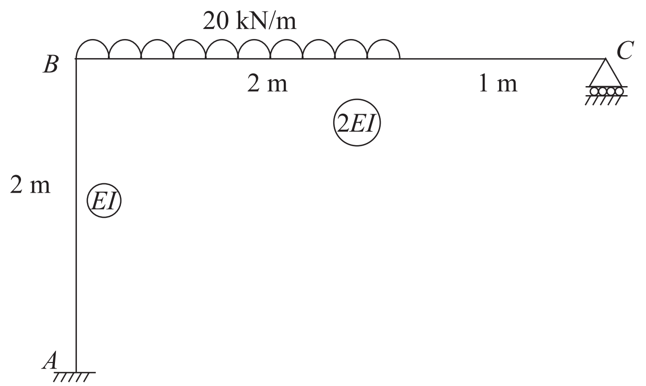


FIG. 2.65

Solution

Distribution factors

Table 2.25 Distribution factors

Joint	*Members*	*Relative Stiffness Values (I/l)*	Σk	$k/\Sigma k$
	BA	$I/2$		0.5
B			I	
	BC	$\frac{3}{4}\left(\frac{2I}{3}\right) = I/2$		0.5

Fixed end moments

$$M_{BC} = \int_0^2 \frac{xwdx(l-x)^2}{l^2} = \int_0^2 \frac{x(20dx)\,(3-x)^2}{3^2} = 13.33 \text{ kNm}$$

$$M_{CB} = \int_0^2 (I - x)\, wdx\, \frac{x^2}{l^2} = \int_0^2 (3-x)\, 20dx\, \frac{x^2}{9} = 8.89 \text{ kNm}$$

Nonsway Analysis

Table 2.26 Moment distribution table

Joint	*A*	*B*		*C*
Members	*AB*	*BA*	*BC*	*CB*
DF	0	0.5	0.5	1.0
FEMS			−13.33	+8.89
Bal		+6.66	+6.67	−8.89
Co	+3.33		−4.44	
Bal		+2.22	+2.22	
Co	+1.11			
	+4.44	+8.88	−8.88	0.00

$$M_{BC} = -8.88 + 50(0.107) = -3.530 \text{ kNm}$$

$$M_{CB} = 0$$

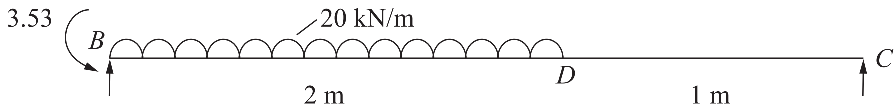


FIG. 2.67

$$\Sigma M_B = 0$$

$$3V_C + 3.53 = \frac{20 \times 2^2}{2}$$

$$\boxed{V_C = 12.157 \text{ kN}}$$

$$\boxed{M_D = 12.157(1) = 12.157 \text{ kNm}}$$

$$\Sigma V = 0$$

$$V_B + V_C = 2(20)$$

$$\boxed{V_B = 40 - 12.157 = 27.843 \text{ kN}}$$

$$(SF)_X = 27.843 - 20x = 0$$

$$\boxed{x = 1.392 \text{ m}}$$

$$\text{Maximum +ve BM} = 27.843(1.392) - \frac{20(1.392)^2}{2} - 3.53$$

$$M_E = 15.851 \text{ kNm}$$

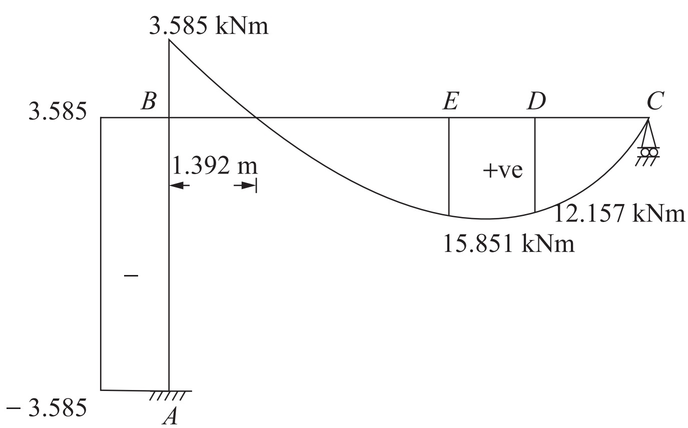


FIG. 2.68 Bending moment diagram

Example 2.13 Analyse the frame shown in Fig. 2.69 by the moment distribution method. Draw the bending moment diagram.

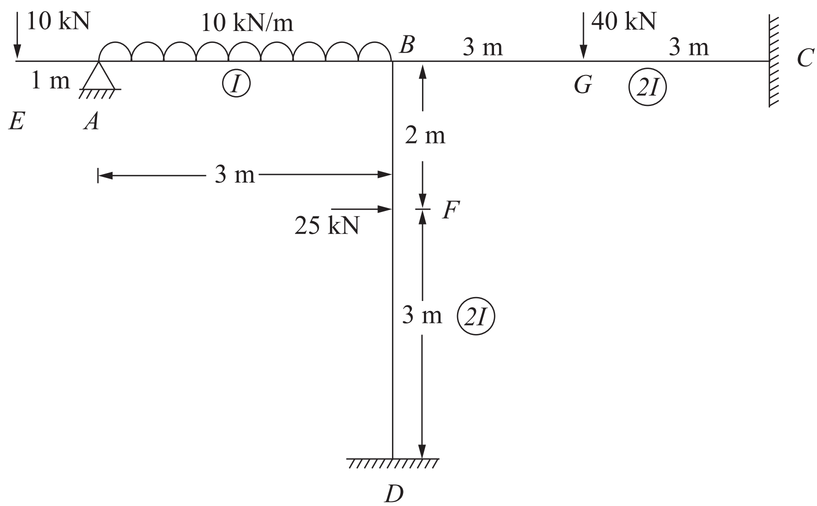


FIG. 2.69

Solution

Distribution factors

Table 2.28 Distribution factors

Joint	*Members*	*Relative Stiffness Values (I/l)*	Σk	$(k/\Sigma k)$
	BA	$\frac{3}{4}\times\frac{I}{3}=\frac{I}{4}$		0.26
B	*BD*	$(2I/5)$	0.98	0.41
	BC	$2I/6$		0.33

Fixed end moments

$$M_{FAE} = +10 \times 1 = 10 \text{ kNm}$$

$$M_{FAB} = -10 \times \frac{3^2}{12} = -7.5 \text{ kNm}$$

$$M_{FBA} = +10 \times \frac{3^2}{12} = +7.5 \text{ kNm}$$

$$M_{FBC} = -40 \times \frac{6}{8} = -30.0 \text{ kNm}$$

$$M_{FCB} = +40 \times \frac{6}{8} = +30.0 \text{ kNm}$$

$$M_{FBD} = +2 \times 25 \times 3^2/5^2 = +18.0 \text{ kNm}$$

$$M_{FBE} = -3 \times 25 \times 2^2/5^2 = -12.0 \text{ kNm}$$

Table 2.29 Moment distribution table

Joint	*A*			*B*		*C*	*D*
Members	*AE*	*AB*	*BA*	*BD*	*BC*	*CB*	*DB*
DF	–	1.00	0.26	0.41	0.33	0.00	0.00
FEMS	+10.00	–7.50	+7.50	+18.00	–30.00	+30.00	–12.00
Bal		–2.50	+1.17	+1.85	+1.48		
Co			–1.25			+0.74	+0.93
Bal			+0.33	+0.51	+0.41		
Co						0.21	
Total	+10.00	–10.00	+7.75	+20.36	–28.11	+30.95	–11.07

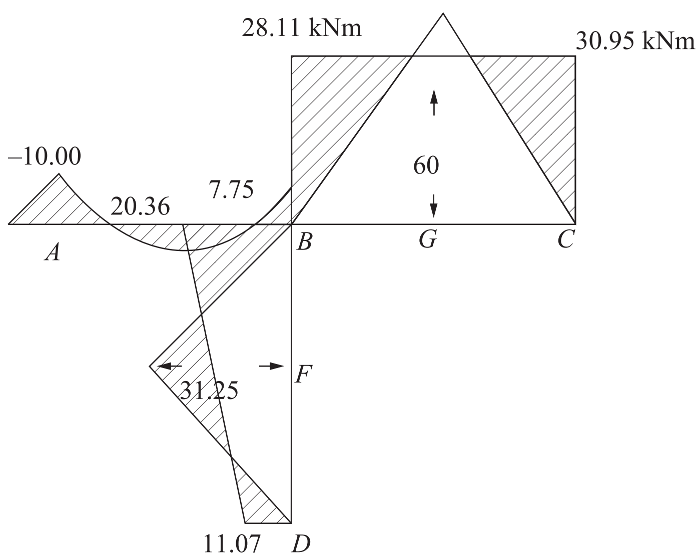


FIG. 2.70 Bending moment diagram

Example 2.14 Analyse the frame by the moment distribution method. Draw the bending moment diagram.

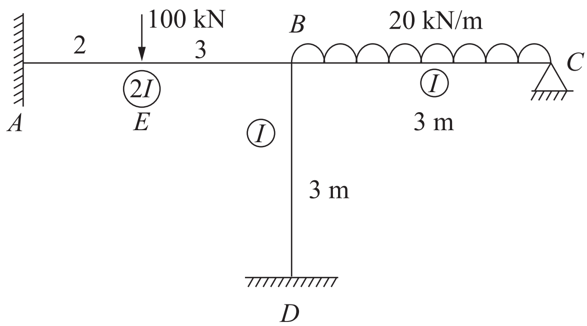


FIG. 2.71

Solution

Distribution factors

Table 2.30 Distribution factors

Joint	*Members*	*Relative Stiffness (k)*	*Σk*	*DF = k/Σk*
	BA	$2I/5 = 0.4I$		0.41
B	*BD*	$I/3 = 0.33I$	$0.98I$	0.34
	BC	$\frac{3}{4}(I/3) = 0.25I$		0.25

Fixed end moments

$$M_{FAB} = -2 \times 100 \times \frac{3^2}{5^2} = -72 \text{ kNm}$$

$$M_{FBA} = +3 \times 100 \times \frac{2^2}{5^2} = +48 \text{ kNm}$$

$$M_{FBC} = -20 \times \frac{3^2}{12} = -15 \text{ kNm}$$

$$M_{FCB} = +20 \times \frac{3^2}{12} = +15 \text{ kNm}$$

Table 2.31 Moment distribution table

Joint	*A*		*B*		*C*	*D*
Members	*AB*	*BA*	*BD*	*BC*	*CB*	*DB*
DF	0	0.41	0.34	0.25	1	0
FEMS	–72.00	+48.00	–	–15.00	+15.00	
Bal	–	–13.53	–11.22	–8.25	–15.00	
Co	–6.77			–7.50		–5.61
Bal		+3.08	+2.55	+1.88		
Co	+1.54					+1.28
Total	–77.23	+37.55	–8.67	–28.87	0.00	–4.33

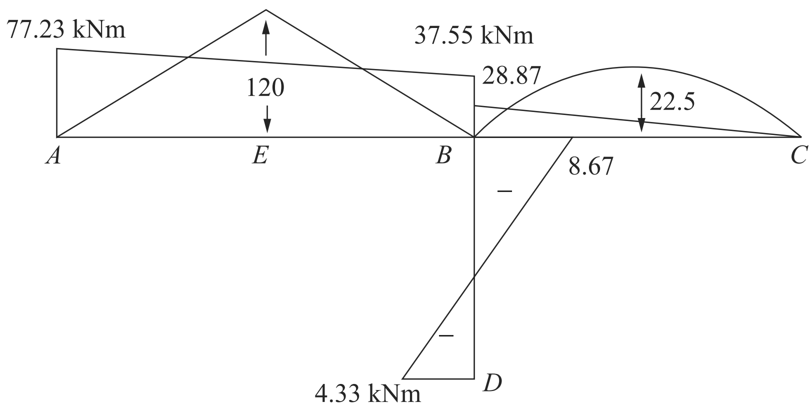


FIG. 2.72 Bending moment diagram

Example 2.15 Analyse the frame by the moment distribution method. Draw the bending moment diagram.

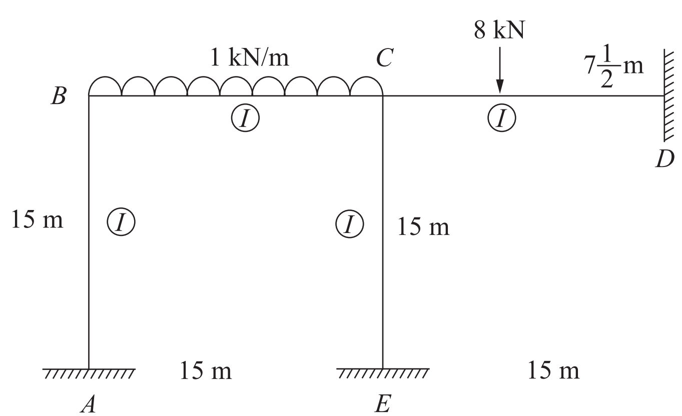


FIG. 2.73

Solution

Distribution factors

Table 2.32 Distribution factors

Joint	*Members*	*Relative Stiffness (k)*	*Σk*	*k/Σk*
B	*BA*	*I*/15	2*I*/15	0.5
	BC	*I*/15		0.5
C	*CB*	*I*/15	3*I*/15	0.33
	CE	*I*/15		0.33
	CD	*I*/15		0.33

Fixed end moments

$$M_{FBC} = -1 \times 15^2/12 = -18.75 \text{ kNm}$$

$$M_{FCB} = +1 \times 15^2/12 = +18.75 \text{ kNm}$$

$$M_{FCD} = -8 \times 15/8 = -15.00 \text{ kNm}$$

$$M_{FDC} = +8 \times 15/8 = +15.00 \text{ kNm}$$

Table 2.33 Moment distribution table

Joint	*A*		*B*			*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CE*	*CD*	*DC*	*EC*
Distribution Factors	0	0.5	0.5	0.33	0.33	0.33	0	0
FEMS			−18.75	+18.75		−15.00	+15.00	
Balance		+9.38	+9.37	−1.25	−1.25	−1.25		
Co	+4.69		−0.63	+4.69			−0.63	−0.63
Bal		+0.32	+0.31	−1.56	−1.57	−1.56		
Co	+0.16		−0.78	+0.16			−0.78	−0.79
Bal		+0.39	+0.39	−0.05	−0.05	−0.06		
Co	+0.20		−0.03	+0.20			−0.03	−0.03
Bal		+0.02	+0.01	−0.07	−0.06	−0.06		
Final								
Moments	+5.05	+10.11	−10.11	+20.87	−2.93	−17.93	+13.56	−1.45

Table 2.34 Distribution factors for symmetrical case

Joint	*Members*	*Relative Stiffness*	Σk	$DF = k/\Sigma k$
	BA	$\frac{I}{5}$		0.8
B			$0.25I$	
	BC	$\frac{1}{2}\left(\frac{I}{10}\right) = 0.05I$		0.2

Fixed end moment

$$M_{FBC} = -\frac{Wl}{8} = -\frac{W(10)}{8} = -1.25W$$

Table 2.35 Moment distribution table

Joint	*A*	*B*	
Members	*AB*	*BA*	*BC*
DF	0	0.8	0.2
FEM Bal Co	 $+0.5W$	 $+1.00W$ 	$-1.25W$ $+0.25W$
Final	$+0.5W$	$+1.00W$	$-1.00W$

Thus, from the above moment distribution table, joint moment

$$1.00W = 300$$

$\therefore$ $W = 300$ kN

Hence, $M_{AB} = 0.5 \times 300 = 150$ kNm

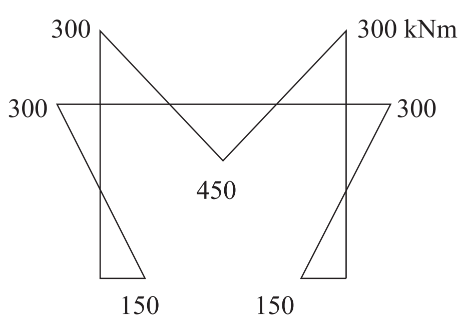


FIG 2.76 Bending moment diagram

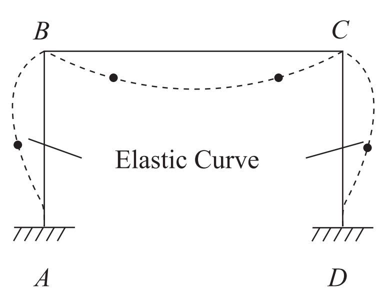


FIG 2.77

Table 2.36 Moment distribution table

Joint	*A*	*B*	
Members	*AB*	*BA*	*BC*
DF	0.0	0.5	0.5
FEMS Balance		 +15.00	–30.00 +15.00
Co	+7.50		
Final Moments	+7.50	+15.00	–15.00

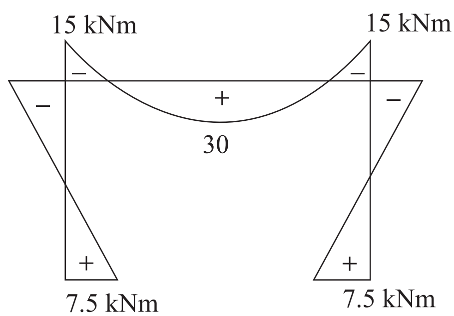


FIG. 2.80 Bending moment diagram

Example 2.18 The culvert shown in Fig. 2.81 is of constant section throughout and the top beam is subjected to a central concentrated load of 25 kN. Assume the base pressure is uniform throughout and analyse the box culvert. Draw the bending moment diagram.

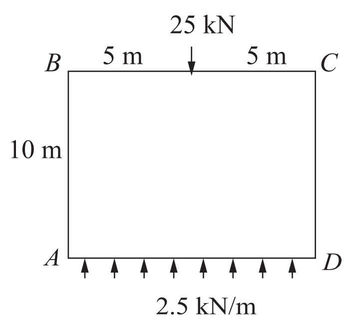


FIG. 2.81

Solution: The above frame is symmetrical and symmetrically loaded. This can be analysed using moment distribution in a straightforward and in a simpler manner by taking advantage of symmetry. The distribution factors are worked out by taking the central member BC as half of its value of stiffness. The process of moment distribution is carried out by considering only half of the frame.

Distribution factors

Table 2.37

Joint	*Members*	*Relative Stiffness Values*	Σk	$k/\Sigma k$
	BA	$\frac{I}{10} = 0.1I$		0.67
B			$0.15I$	
	BC	$\frac{1}{2}\left(\frac{I}{10}\right) = 0.05I$		0.33
	AB	$\frac{I}{10} = 0.1I$		0.67
A			$0.15I$	
	AD	$\frac{1}{2}\left(\frac{I}{10}\right) = 0.05I$		0.33

Fixed end moments

$$M_{FBC} = -\frac{Wl}{8} = -25 \times \frac{10}{8} = -31.25 \text{ kNm}$$

$$M_{FAD} = +\frac{wl^2}{12} = +2.5 \times \frac{10^2}{12} = +20.83 \text{ kNm}$$

Moment Distribution

Due to symmetry analyse half of the frame. The joints A, B, C and D are rigid. This frame is recognised as a continuous frame. It implies that when joint moments are balanced, it is being carried over to the neighbouring joints. This carry over moment is again balanced, the process is continued and all the joints are balanced.

Hence, the joint moments are

$$M_A = M_D = 9.06 \text{ kNm}$$

$$M_B = M_C = 16.95 \text{ kNm}$$

Table 2.38 Moment distribution table

Joint	*A*		*B*	
Members	*AD*	*AB* ⟷	*BA*	*BC*
DF	0.33	0.67	0.67	0.33
FEMS	+20.83			−31.25
Bal	−6.94	−13.89	+20.83	+10.42
Co		+10.42	−6.95	
Bal	−3.47	−6.95	+4.63	+2.32
Co		+2.32	−3.48	
Bal	−0.77	−1.55	+2.32	+1.16
Co		+1.16	−0.78	
Bal	−0.38	−0.78	+0.52	+0.26
Co		+0.26	−0.39	
Bal	−0.17	−0.08	+0.26	+0.13
Co		+0.13	−0.04	
Bal	−0.04	−0.09	+0.03	+.01
Final	9.06	−9.06	16.95	−16.95

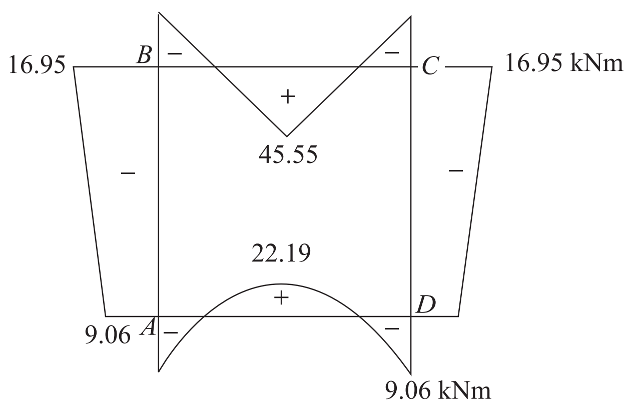


FIG. 2.82 Bending moment diagram

2.9 ANALYSIS OF UNSYMMETRICAL FRAMES

In the previous article, the frame was symmetrical and was symmetrically loaded. Half frame technique was used by taking beam stiffness as half of its original stiffness. It indicates that sways do not occur in the following cases (i) a structure which is held against lateral movement (Ref. Example 2.15) (ii) where both geometry of the frame and the loading are symmetrical (Ref. Examples 2.16 - 2.18).

In reality, the frames are unsymmetrical and the loading could also be unsymmetrical. In such frames, there will be rotation of joints as well as the lateral displacement joints. The lateral displacement causes the frame to sway either to the right or to the left with respect to the centre line of the frame. The sway (side sway) is caused by

(i) Portal frames with unequal columns (Fig. 2.83a)
(ii) Portal frames whose columns are of different moment of inertia (Fig. 2.83b)
(iii) Unsymmetrical loading on the beams (Fig. 2.83c)
(iv) Lateral loading (Fig. 2.83d)

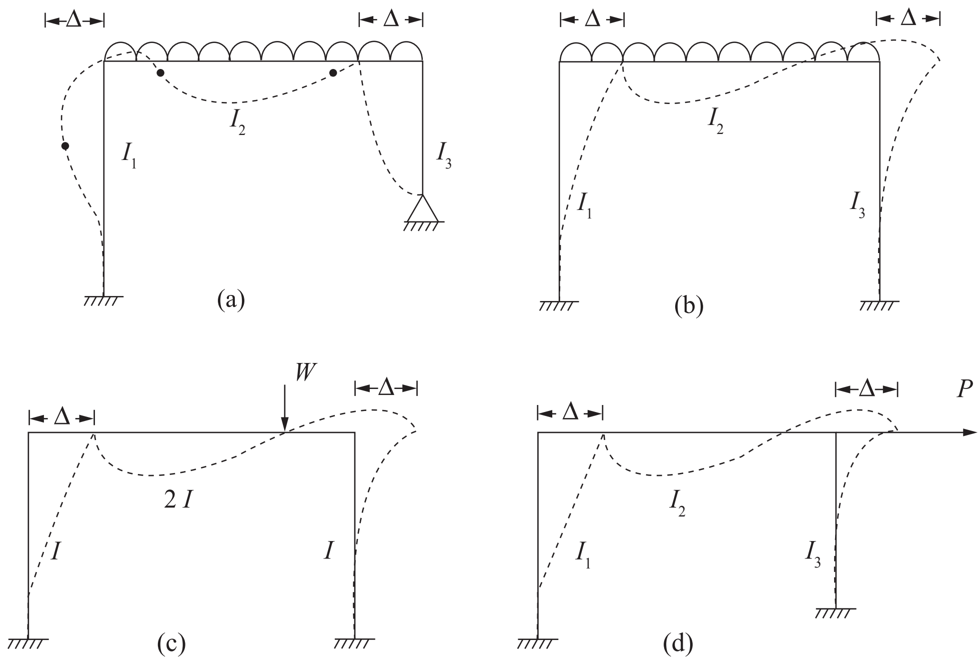


FIG. 2.83 Examples of sway frames

Analysis of sway frames is done in the following way.

(1) The moment distribution was carried out by assuming that the joints do not get displaced. The moments obtained from the moment distribution table are called nonsway moments.
(2) The horizontal reactions at the base of the columns are found out which helps in finding the net out of balance force. A prop is assumed to act opposite to the direction of the above force.
(3) Allow the frame to sway in the direction of sway force which is equal and opposite to the prop force (which acts along the axis of the beam level). Let the actual prop force be X.
(4) Perform the sway moment distribution, by assuming arbitrary moments as per the joint moment ratios.

(5) Calculate the displacement force (Y) from the sway moment distribution.

(6) The correction factor/sway factor $k = X/Y$. The correction factor gives the direction of sway of the frame. If the value of k is positive, then the frame sways in the direction of the sway solution. If the value of k is negative, then the frame sways in the direction opposite to that of assumed sway.

(7) The final end moments are obtained as

Final end moments = Nonsway moments +'Correction factor' × Sway moments

Expressing mathematically;

$$M_{AB} = M'_{AB} + k\,M''_{AB}$$

where M_{AB} is the end moment of the member AB, M'_{AB} is the moment obtained from nonsway moment distribution, M''_{AB} is the computed moment obtained from sway moment distribution k is the correction factor.

2.9.1 Joint Moment Ratios

The various moment ratios for rectangular frames are given below.

Portal frames with both ends fixed

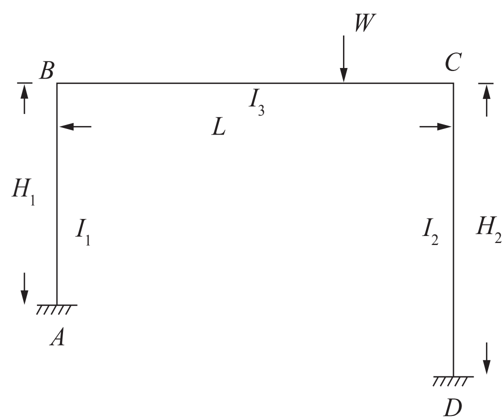


FIG. 2.84(a)

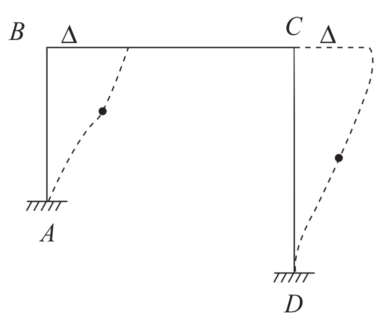


FIG. 2.84(b)

$$M_{AB} = M_{BA} = -\frac{6EI_1\Delta}{H_1^2}$$

and
$$M_{CD} = M_{DC} = -\frac{6EI_2\Delta}{H_2^2}$$

$\therefore$
$$\boxed{\frac{M_{AB}}{M_{DC}} = \frac{I_1H_2^2}{I_2H_1^2}}$$

Portal frame with end fixed and other hinged

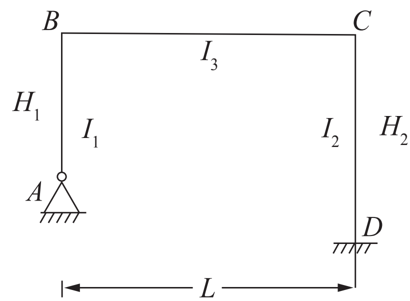


FIG. 2.85(a)

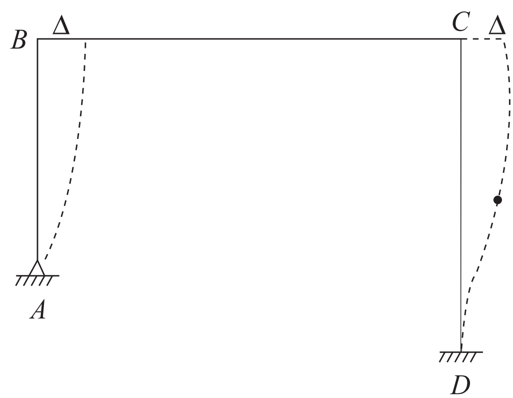


FIG. 2.85(b)

$$M_{AB} = 0 \quad (\because \text{ Hinged})$$

$$M_{BA} = -\frac{3EI_1\Delta}{H_1^2}$$

$$M_{DC} = M_{CD} = -\frac{6EI_2\Delta}{H_2^2}$$

Nonsway Analysis

Table 2.40 Moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.45	0.55	0.55	0.45	0
FEMS			−28.80	+19.20		
Bal		+12.96	15.84	−10.56	−8.64	
Co	6.48		−5.28	+7.92		−4.32
Bal		+2.38	+2.90	−4.36	−3.56	
Co	1.19		−2.18	+1.45		−1.78
Bal		+0.98	+1.20	−0.80	−0.65	
Co	0.49		−0.40	+0.60		−0.33
Bal		+0.18	+0.22	−0.33	−0.27	
Co	0.09		−0.17	+0.11		−0.14
Bal		+0.08	+0.09	−0.06	−0.05	
Co	0.04		−0.03	+0.05		−0.03
Bal		+0.01	+0.02	−0.03	−0.02	
Total	8.29	16.59	−16.59	13.19	−13.19	−6.60
	M'_{AB}	M'_{BA}	M'_{BC}	M'_{CB}	M'_{CD}	M'_{DC}

Sway Analysis

The assumed sway moments are in the joint ratio

$$\frac{M_{BA}}{M_{CD}} = \frac{-6EI\Delta/3^2}{-6EI\Delta/3^2}$$

i.e. $$M_{BA} : M_{CD} = -1 : -1$$

The moments are arbitrarily assumed in the columns as −100 kNm.

Final moments

These are obtained by adding the nonsway and sway moments as

$$M_{AB} = 8.29 - 82.33k$$

$$M_{BA} = 16.59 - 64.73k$$

$$M_{BC} = -16.59 + 64.73k$$

$$M_{CB} = 13.19 + 64.73k$$

$$M_{CD} = -13.19 - 64.73k$$

$$M_{DC} = -6.60 - 82.33k$$

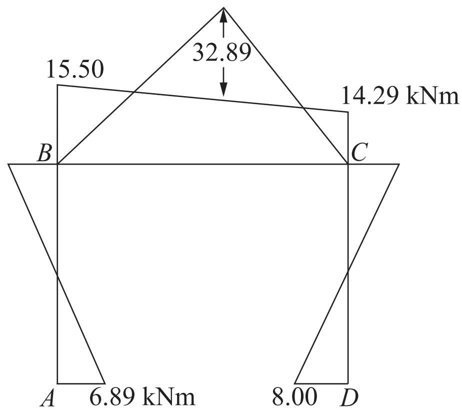


FIG. 2.88 Bending moment diagram

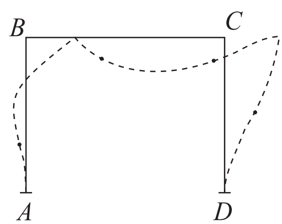


FIG. 2.89 Elastic curve

Example 2.20 Analyse the given frame and draw the bending moment diagram

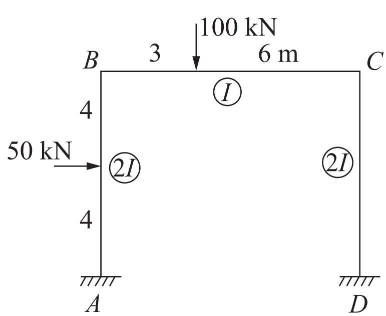


FIG. 2.90

Solution

Distribution factors

Table 2.42

Joint	*Members*	*Relative Stiff Values (I/l)*	Σk	$k/\Sigma k$
	BA	$2I/8 = 0.25I$		0.7
B			$0.36I$	
	BC	$I/9 = 0.11I$		0.3
	CB	$I/9 = 0.11I$		0.3
C			$0.36I$	
	CD	$2I/8 = 0.25I$		0.7

Fixed end moments

$$M_{FAB} = -\frac{Wl}{8} = -50 \times \frac{8}{8} = -50\,\text{kNm}$$

$$M_{FBA} = +\frac{Wl}{8} = +50 \times \frac{8}{8} = +50\,\text{kNm}$$

$$M_{FBC} = -\frac{Wab^2}{l^2} = -100 \times 3 \times \frac{6^2}{9^2} = -133.33\,\text{kNm}$$

$$M_{FCB} = +\frac{Wa^2b}{l^2} = +100 \times 3^2 \times \frac{6}{9^2} = +66.67\,\text{kNm}$$

Nonsway Analysis

Table 2.43 Moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB* ←	*BA*	*BC* ↔	*CB*	*CD* →	*DC*
DF	0	0.70	0.30	0.30	0.70	0
FEMS	−50.00	+50.00	−133.33	+66.67		
Bal		+58.33	+25.00	−20.00	−46.67	
Co	+29.20		−10.00	+12.50		−23.34
Bal		+7.00	+3.00	−3.75	−8.75	
Co	+3.50		−1.88	+1.50		−4.38
Bal		+1.32	+0.56	−0.45	−1.05	
Co	+0.66		−0.23	+0.28		−0.53
Bal		+0.16	+0.07	−0.08	−0.20	
Co	+0.08		−0.04	+0.04		−0.10
Bal		+0.03	+0.01	−0.01	−0.03	
	−16.56	+116.84	−116.84	56.70	−56.70	−28.35
	M'_{AB}	M'_{BA}	M'_{BC}	M'_{CB}	M'_{CD}	M'_{DC}

Sway Analysis

Table 2.44 Moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.70	0.30	0.30	0.70	0
FEMS	+100.00	+100.00			+100.00	+100.00
Bal		–70.00	–30.00	–30.00	–70.00	
CO	–35.00		–15.00	–15.00		–35.00
Bal		+10.50	+4.50	+4.50	+10.50	
Co	+5.25		+2.25	+2.25		+5.25
Bal		–1.57	–0.68	–0.68	–1.57	
CO	–0.79		–0.34	–0.34		–0.79
Bal		+0.24	+0.10	+0.10	+0.24	
CO	+0.12		+0.05	+0.05		+0.12
Bal		–0.04	–0.01	–0.02	–0.03	
Final	69.58	39.13	–39.13	–39.13	+39.13	+69.58
	M''_{AB}	M''_{BA}	M''_{BC}	M''_{CB}	M''_{CD}	M''_{DC}

The final moments are the addition of nonsway moments and a constant times sway moments; for example, $M_{AB} = M'_{AB} + k\, M''_{AB}$. Therefore,

End moments

$$M_{AB} = -16.56 + 69.58\, k$$

$$M_{BA} = +116.84 + 39.13\, k$$

$$M_{BC} = -116.84 - 39.13\, k$$

$$M_{CB} = +56.70 - 39.13\, k$$

$$M_{CD} = -56.70 + 39.13\, k$$

$$M_{DC} = -28.35 + 69.58\, k$$

The value of k is determined from the column shear condition as follows.

Column shear condition

Considering the free body diagram of column AB and the value of H_A is determined as

$$23.096 - 27.18\,k = 50$$

$$\boxed{k = -0.99}$$

Hence,

Final moments

$$M_{AB} = -16.56 + 69.58(-0.99) = -85.4 \text{ kNm}$$

$$M_{BA} = +116.84 + 39.13(-0.99) = +78.1 \text{ kNm}$$

$$M_{BC} = -116.84 - 39.13(-0.99) = -78.1 \text{ kNm}$$

$$M_{CB} = +56.70 - 39.13(-0.99) = +95.4 \text{ kNm}$$

$$M_{CD} = -56.70 + 39.13(-0.99) = -95.4 \text{ kNm}$$

$$M_{DC} = -28.35 + 69.58(-0.99) = -97.2 \text{ kNm}$$

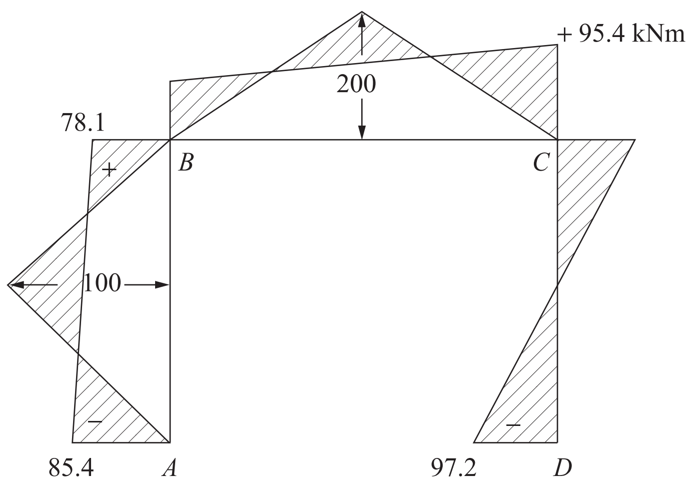


FIG. 2.93 Bending moment diagram

Example 2.21 Analyse the portal frame shown in Fig. 2.94 by the moment distribution method. Sketch the bending moment diagram.

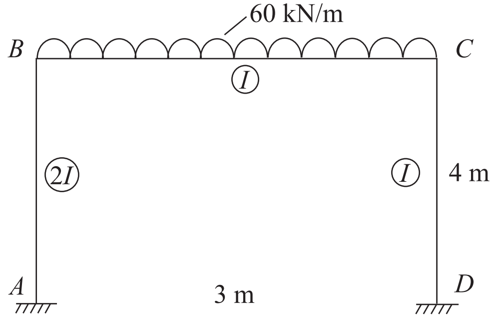


FIG. 2.94

Solution

Distribution factors

Table 2.45

Joint	*Members*	*Relative Stiffness Values*	Σk	$k/\Sigma k$
B	*BA*	$2I/4$	$0.83I$	0.60
	BC	$I/3$		0.40
C	*CB*	$I/3$	$0.58I$	0.57
	CD	$I/4$		0.33

Fixed end moments

$$M_{FBC} = -60 \times \frac{3^2}{12} = -45 \text{ kNm}$$

$$M_{FCB} = +60 \times \frac{3^2}{12} = +45 \text{ kNm}$$

Though the boundary conditions and the loading are symmetrical, the frame sways as the columns *AB* and *CD* are having different moments of inertia of the cross–section.

Hence, we have to do nonsway and sway analysis to determine the joint moments.

Nonsway Analysis

Table 2.46 Moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.60	0.40	0.57	0.33	0
FEMS			–45.00	+45.00		
Bal		+27.00	+18.00	–25.65	–19.35	
Co	+13.50		–12.83	+9.00		–9.68
Bal		+7.70	+5.13	–5.13	–3.87	
Co	+3.85		–2.56	+2.56		–1.94
Bal		+1.54	+1.02	–1.43	–1.13	
Co	+0.77		–0.72	+0.51		–0.57
Bal		+0.43	+0.29	–0.29	–0.22	
Co	+0.22		–0.15	+0.15		–0.11
Bal		+0.09	+0.06	–0.08	–0.06	
Co	+0.05		–0.04	+0.03		–0.03
Bal		+0.02	+0.02	–0.02	–0.01	
Final	+18.39	+36.78	–36.78	+24.65	–24.65	–12.33
	M'_{AB}	M'_{BA}	M'_{BC}	M'_{CB}	M'_{CD}	M'_{DC}

Column shear condition

In the given portal frame, the columns are fixed and there is no horizontal load acting externally. Hence, the horizontal equilibrium gives

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} = 0$$

$$(18.39 + 36.78 - 24.65 - 12.33) - (14.57 + 9.09 + 7.26 + 8.63)\,k = 0$$

$$18.19 - 39.55\,k = 0$$

$$\boxed{k = 0.46}$$

Substituting this value of k in the above equations;

End moments

$$M_{AB} = 18.39 - 14.57(0.46) = 11.7 \text{ kNm}$$

$$M_{BA} = 36.78 - 9.09(0.46) = 32.6 \text{ kNm}$$

$$M_{BC} = -36.78 + 9.09(0.46) = -32.6 \text{ kNm}$$

$$M_{CB} = 24.65 + 7.26(0.46) = +28.0 \text{ kNm}$$

$$M_{CD} = -24.65 - 7.26(0.46) = -28.0 \text{ kNm}$$

$$M_{DC} = -12.33 - 8.63(0.46) = +16.3 \text{ kNm}$$

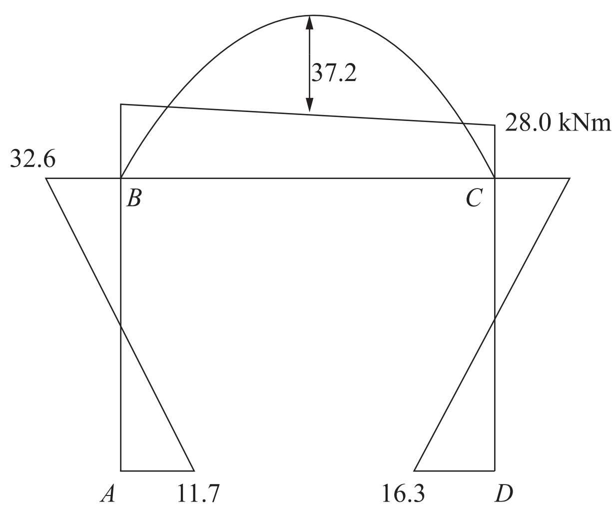


FIG. 2.95 Bending moment diagram

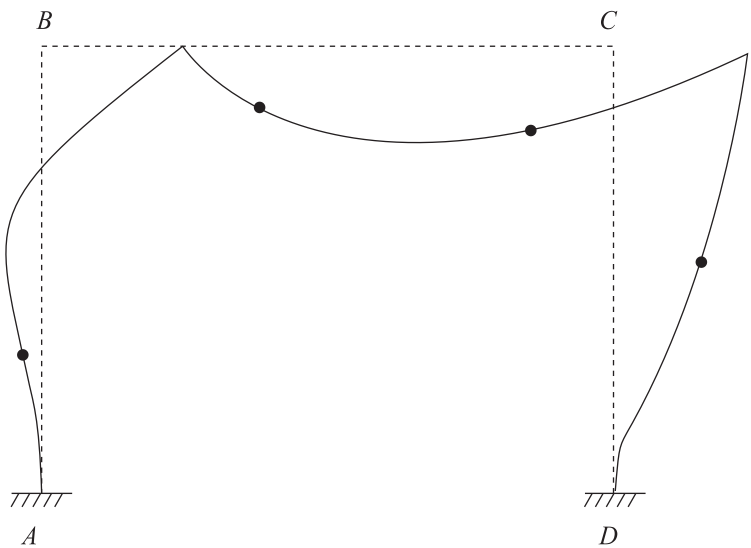


FIG. 2.96 Elastic curve

Example 2.22 Analyse the portal frame shown in Fig. 2.97 by the moment distribution method. Draw the bending moment diagram.

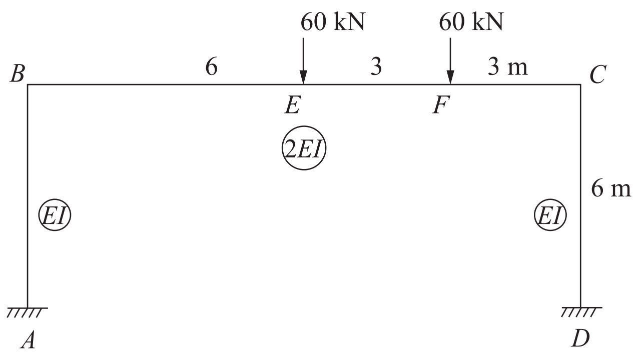


FIG. 2.97

Solution: The frame is subjected to asymmetrical loading. Hence, nonsway and sway analyses were carried out separately and the results are summed up.

Distribution factors

Table 2.48

Joint	*Members*	*Relative Stiff Values (k)*	Σk	*DF*
B	*BA*	$I/6$	$I/3$	0.5
	BC	$2I/12 = I/6$		0.5
C	*CB*	$2I/12 = I/6$	$I/3$	0.5
	CD	$I/6$		0.5

Fixed end moments

$$M_{FBC} = -\frac{Wl}{8} - \frac{Wab^2}{l^2} = -60 \times \frac{12}{8} - \frac{60 \times 9 \times 3^2}{12^2} = -123.75 \text{ kNm}$$

$$M_{FCB} = +\frac{Wl}{8} + \frac{Wa^2b}{l^2} = 60 \times \frac{12}{8} + \frac{60 \times 3 \times 9^2}{12^2} = -191.25 \text{ kNm}$$

Nonsway Analysis

Table 2.49 Moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB* ←	*BA*	*BC* ↔	*CB*	*CD* →	*DC*
DF	0.0	0.5	0.5	0.5	0.5	0.0
FEMS			–123.75	+191.25		
Bal		+61.88	+61.87	–95.63	–95.62	
Co	+30.94		–47.82	+30.94		–47.81
Bal		+23.91	+23.91	–15.47	–15.47	
Co	+11.96		–7.74	+11.96		–7.74
Bal		+3.87	+3.87	–5.98	–5.98	
Co	+1.94		2.99	+1.94		–2.99
Bal		+1.50	+1.50	–0.97	–0.97	
Co	+0.75		–0.49	+0.75		–0.49
Bal		+0.24	+0.25	–0.37	–0.38	
Co	+0.12		–0.19	+0.13		–0.19
Bal		+0.10	+0.09	–0.06	–0.07	
Final	45.71	91.50	–91.50	118.49	–118.49	–59.22
	M'_{AB}	M'_{BA}	M'_{BC}	M'_{CB}	M'_{CD}	M'_{DC}

Sway moments

The sway moments are assumed in the joint ratio of

$$\frac{M_{BA}}{M_{DC}} = \frac{-6EI\Delta/6^2}{-6EI\Delta/6^2} = \frac{-1}{-1}$$

$\therefore$ $M_{BA} : M_{DC} = -100.00 : -100.00$

Example 2.23 Analyse the given frame by the moment distribution method. Draw the bending moment diagram and shear force diagram.

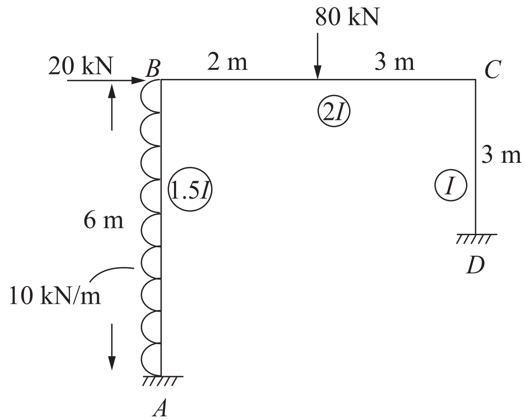


FIG. 2.100

Distribution factors

Table 2.51

Joint	*Members*	*Relative Stiffness*	Σk	$k/\Sigma k$
	BA	$\frac{1.5I}{6} = 0.25I$		0.38
B			$0.65I$	
	BC	$\frac{2I}{5} = 0.4I$		0.62
	CB	$\frac{2I}{5} = 0.4I$		0.55
C			$0.73I$	
	CD	$\frac{I}{3} = 0.33I$		0.45

Fixed end moments

$$M_{FAB} = -10 \times \frac{6^2}{12} = -30 \text{ kNm}$$

$$M_{FBA} = +10 \times \frac{6^2}{12} = +30 \text{ kNm}$$

$$M_{FBC} = -\frac{2(90)3^2}{5^2} = -64.8 \text{ kNm}$$

$$M_{FCB} = +3(90)\,\frac{3^2}{5^2} = +43.2\ \text{kNm}$$

Nonsway Analysis

Table 2.52 Moment distribution table

Joint	A	B		C		D
Members	AB	BA	BC	CB	CD	DC
DF	0	0.38	0.62	0.55	0.45	0
FEMS	–30.00	+30.00	–64.80	+43.20	–	–
Bal		+13.22	+21.58	–23.76	–19.44	
Co	+6.61		–11.88	+10.79		–9.72
Bal		+4.51	+7.37	–5.93	–4.86	
Co	+2.26		–2.97	3.69		–2.43
Bal		+1.13	+1.84	–2.03	–1.66	
Co	+0.57		–1.02	+0.92		–0.83
Bal		+0.39	+0.63	–0.51	–0.41	
Co	+0.20		–0.26	0.34		–0.21
Bal		+0.10	+0.16	–0.19	–0.15	
Co	+0.05		–0.10	+0.08		–0.08
Bal		+0.04	+0.06	–0.04	–0.04	
Final	–20.31	+49.39	–49.39	+26.56	–26.56	–12.44
	M'_{AB}	M'_{BA}	M'_{BC}	M'_{CB}	M'_{CD}	M'_{DC}

Sway Analysis

$$M_{BA} : M_{CD} = -6E(1.5I)\,\frac{\Delta}{6^2} : -6E(I)\,\frac{\Delta}{3^2}$$

$$M_{BA} : M_{CD} = -15.00 : -40.00\ \text{kNm}$$

Correction factor

$$M_{AB} = -20.31 - 14.15\,k$$

$$M_{BA} = +49.39 - 13.34\,k$$

$$M_{BC} = -49.39 + 13.34\,k$$

$$M_{CB} = +26.56 + 22.63\,k$$

$$M_{CD} = -26.56 - 22.63\,k$$

$$M_{DC} = -12.44 - 31.30\,k$$

Table 2.53 Moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.38	0.62	0.55	0.45	0
FEMS	–15.00	–15.00	–	–	–40.00	–40.00
Bal		+5.70	+9.30	+22.00	+18.00	
Co	+2.85		+11.00	+4.65		+9.00
Bal		–4.18	–6.82	–2.56	–2.09	
Co	–2.09		–1.28	–3.41		–1.05
Bal		+0.49	+0.79	+1.88	+1.53	
Co	+0.25		+0.94	+0.40		+0.77
Bal		–0.36	–0.58	–0.22	–0.18	
Co	–0.18		–0.11	–0.29		–0.09
Bal		+0.04	+0.07	+0.16	+0.13	
Co	+0.02		+0.08	+0.04		+0.07
Bal		–0.03	–0.05	–0.02	–0.02	
Final	–14.15	–13.34	+13.34	+22.63	–22.63	–31.30
	M''_{AB}	M''_{BA}	M''_{BC}	M''_{CB}	M''_{CD}	M''_{DC}

The value of k is determined from the column shear condition as

Column shear condition

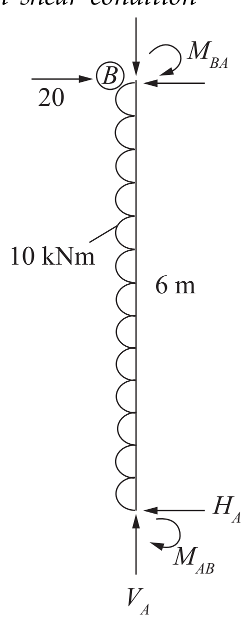


FIG. 2.101

$$\Sigma M_B = 0$$

$$M_{AB} + M_{BA} + 6H_A = 10 \times \frac{6^2}{2}$$

and hence

$$\boxed{H_A = 30 - \frac{(M_{AB} + M_{BA})}{6}}$$

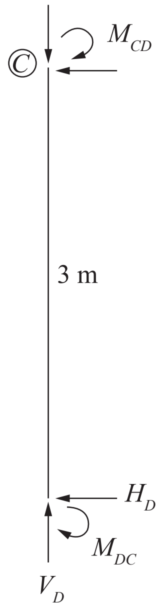


FIG. 2.102

$$\Sigma M_C = 0$$

$$M_{CD} + M_{DC} + 3H_D = 0$$

$$\boxed{H_D = -\frac{(M_{CD} + M_{DC})}{3}}$$

$$H_A + H_D = 6(10) + 20$$

$$30 - \frac{(M_{AB} + M_{BA})}{6} - \frac{(M_{CD} + M_{DC})}{3} = 80$$

Substituting the values of M_{AB}, M_{BA}, M_{CD} and M_{DC} from the above equations; and solving

$$\boxed{k = 1.843}$$

Final moments

$$M_{AB} = -20.31 - 14.15(1.843) = -46.4$$

$$M_{BA} = +49.39 - 13.34(1.843) = +24.8$$

$$M_{BC} = -49.39 + 13.34(1.843) = -24.8$$

$$M_{CB} = +26.56 + 22.63(1.843) = +68.3$$

$$M_{CD} = -26.56 - 22.63(1.843) = -68.3$$

$$M_{DC} = -12.44 - 31.30(1.843) = -70.1$$

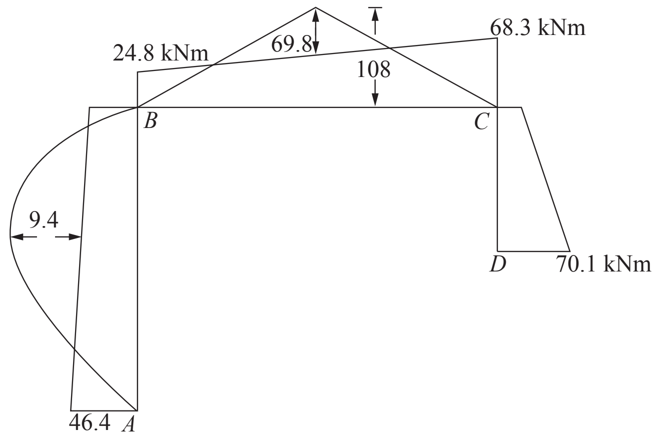


FIG. 2.103 Bending moment diagram

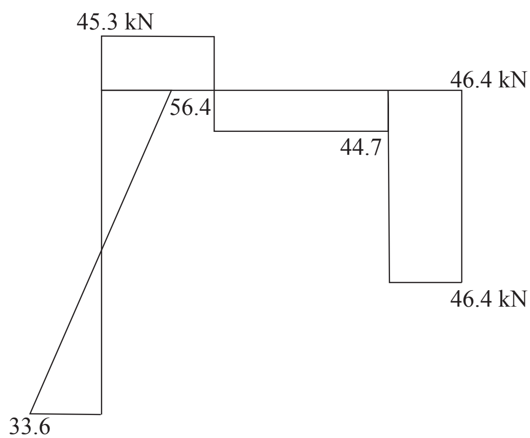


FIG. 2.104 Shear force diagram

Example 2.24 Analyse the rigid portal frame by moment distribution method and hence draw the bending moment diagram (Fig. 2.105).

Solution

Distribution factors

Table 2.54

Joint	*Members*	*Relative Stiffness (I/l)*	Σk	$k/\Sigma k$
B	*BA*	$2I/5$	$0.9I$	0.44
	BC	$2I/4$		0.56
C	*CB*	$2I/4$	$0.875I$	0.57
	CD	$\frac{3}{4} \times \frac{1.5I}{3}$		0.43

Column shear equation

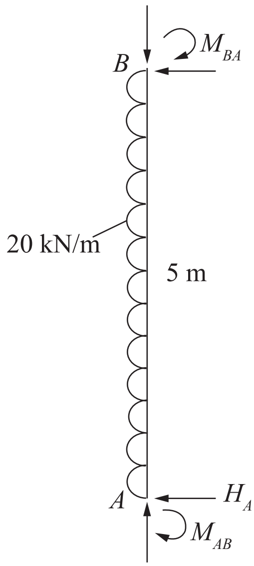


FIG. 2.106

$$\Sigma M_B = 0$$

$$M_{AB} + M_{BA} + 5H_A - 20 \times \frac{5^2}{2} = 0$$

$$\boxed{\therefore \quad H_A = 50 - 0.2(M_{AB} + M_{BA})}$$

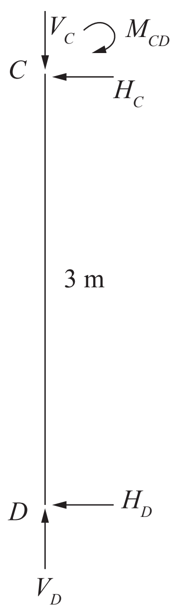


FIG. 2.107

$$\Sigma M_C = 0$$

$$M_{CD} + 3H_D = 0$$

$$\boxed{\therefore \quad H_D = -0.33M_{CD}}$$

$$\Sigma H = 0; \quad H_A + H_D = 20(5)$$

$$\therefore \qquad 50 - 0.2(M_{AB} + M_{BA}) - 0.33\,M_{CD} = 100 \qquad (1)$$

The end moments are obtained using the sway and nonsway moment distribution table as

$$M_{AB} = -45.49 - 86.00\ k$$
$$M_{BA} = +34.05 - 64.09\ k$$
$$M_{BC} = -34.05 + 64.09\ k$$
$$M_{CB} = 6.50 + 44.11\ k$$
$$M_{CD} = -6.50 - 44.11\ k$$
$$M_{DC} = 0.00$$

Substituting the above end moments in Eq. (1);

$$\boxed{k = 1.022}$$

Hence, the end moments are obtained by back substitution as

$$M_{AB} = -133.4 \text{ kNm};\ M_{BA} = -31.5 \text{ kNm};\ M_{BC} = +31.5 \text{ kNm};$$
$$M_{CB} = +51.6 \text{ kNm};\ M_{CD} = -51.6 \text{ kNm};\ M_{DC} = 0$$

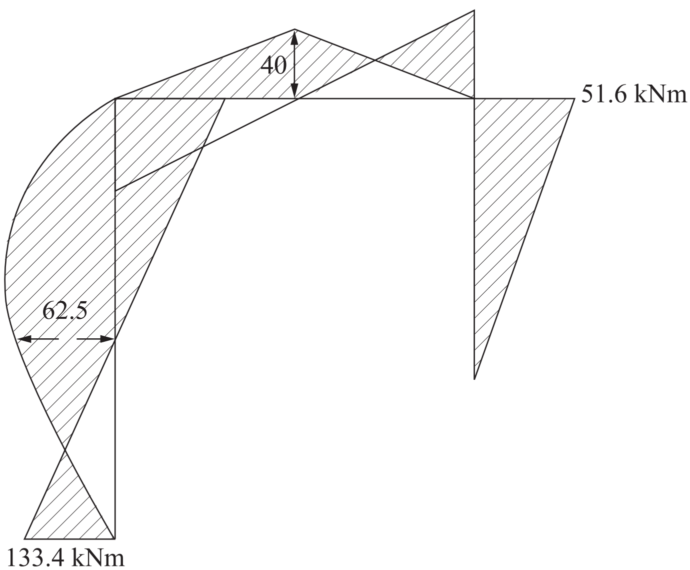


FIG. 2.108 Bending moment diagram

Example 2.25 The frame shown in Fig. 2.109 is hinged at A. The end D is fixed and the joints B and C are rigid. Column CD is subjected to horizontal loading of 30 kN/m. A concentrated load of 90 kN acts on BC at 1 m from B. Analyse the frame and sketch the BMD?

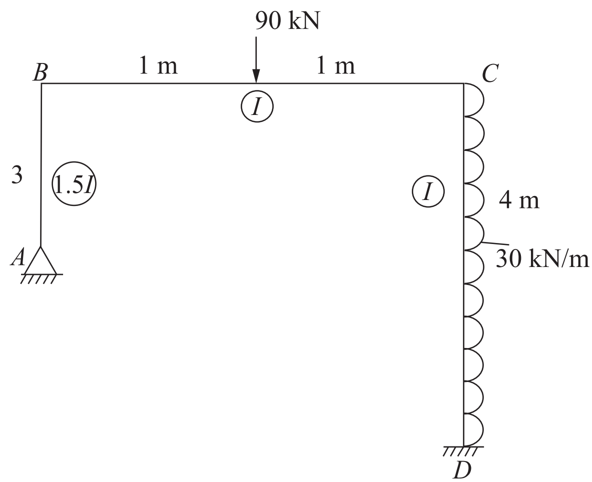


FIG. 2.109

Solution

Distribution factors

Table 2.57

Joint	*Members*	*Relative stiffness k*	Σk	$k/\Sigma k$
B	*BA*	$\frac{3}{4} \times \frac{1.5I}{3} = 0.375I$	$0.875I$	0.43
	BC	$I/2 = 0.5I$		0.57
C	*CB*	$I/2 = 0.5I$	$0.75I$	0.67
	CD	$I/4 = 0.25I$		0.33

Fixed end moments

$$M_{FBC} = -90 \times \frac{2}{8} = -22.5 \text{ kNm}$$

$$M_{FCB} = +90 \times \frac{2}{8} = +22.5 \text{ kNm}$$

$$M_{FCD} = -30 \times \frac{4^2}{12} = -40 \text{ kNm}$$

$$M_{FDC} = +30 \times \frac{4^2}{12} = +40 \text{ kNm}$$

Nonsway Analysis

Table 2.58 Moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB* ←	*BA*	*BC*	*CB*	*CD* →	*DC*
DF	1	0.43	0.57	0.67	0.33	0
FEMS			–22.50	+22.50	–40.00	+40.00
Bal		+9.68	+12.82	+11.73	+5.77	
Co			+5.87	+6.41		+2.89
Bal		–2.52	–3.35	–4.30	–2.11	
Co			–2.15	–1.68		–1.06
Bal		+0.92	+1.23	+1.12	+0.56	
Co			+0.56	+0.62		+0.28
Bal		–0.24	–0.32	–0.42	–0.20	
Co			–0.21	–0.16		–0.10
Bal		+0.09	+0.12	+0.11	+0.05	
Co			+0.06	+0.06		+0.03
Bal		–0.03	–0.03	–0.04	–0.02	
Final	0	+7.90	–7.90	+35.95	–35.95	+42.04
Members	M'_{AB}	M'_{BA}	M'_{BC}	M'_{CB}	M'_{CD}	M'_{DC}

Sway Analysis

The sway moments are assumed in the following ratios

$$\frac{M_{BA}}{M_{CD}} = \frac{3E(1.5I)\Delta/3^2}{6E(I)\Delta/4^2}$$

$$\frac{M_{BA}}{M_{CD}} = \frac{4}{3}$$

$$\therefore \qquad M_{BA} : M_{CD} = +40 : +30 \text{ kNm}$$

Final moments

$$M_{AB} = 0$$

$$M_{BA} = 7.90 + 25.80\,k$$

$$M_{BC} = -7.90 - 25.80\,k$$

$$M_{CB} = +35.95 - 23.22\,k$$

$$M_{CD} = -35.95 + 23.22\,k$$

$$M_{DC} = +42.04 + 26.58\,k$$

Equilibrium of column CD

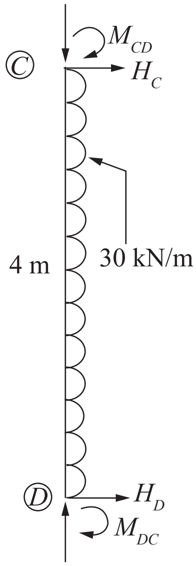


FIG. 2.111

$\Sigma M_C = 0;$

$$M_{CD} + M_{DC} + 30 \times \frac{4^2}{2} - 4H_D = 0$$

$$\boxed{H_D = 60 + \frac{1}{4}(M_{CD} + M_{DC})}$$

Considering the horizontal equilibrium of the whole structure;

$$H_A + H_D = 4(30)$$

i.e.

$$\frac{M_{BA}}{3} + 60 + \frac{(M_{CD} + M_{DC})}{4} = 120$$

Substituting the values of the moments from the final moment equations;

$$\boxed{k = 2.65}$$

The end moments were calculated as

$$M_{AB} = 0$$

$$M_{BA} = 7.90 + 25.80 \times 2.65 = 76.3 \text{ kNm}$$

$$M_{BC} = -7.90 - 25.80 \times 2.65 = -76.3 \text{ kNm}$$

$$M_{CB} = +35.95 - 23.22 \times 2.65 = -25.6 \text{ kNm}$$

$$M_{CD} = -35.95 + 23.22 \times 2.65 = +25.6 \text{ kNm}$$

$$M_{DC} = +42.04 + 26.58 \times 2.65 = +112.5 \text{ kNm}$$

Sway Analysis

Table 2.62 Moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.5	0.5	0.5	0.5	0
FEMS	–100.00	–100.00			–100.00	–100.00
Bal		+50.00	+50.00	+50.00	+50.00	
Co	+25.00		+25.00	+25.00		+25.00
Bal		–12.50	–12.50	–12.50	–12.50	
Co	–6.25		–6.25	–6.25		–6.25
Bal		+3.12	+3.13	+3.13	+3.12	
Co	+1.56		+1.56	+1.57		+1.56
Bal		–0.78	–0.78	–0.79	–0.78	
Co	–0.39		–0.40	–0.39		–0.69
Bal		+0.20	+0.20	+0.20	+0.20	
Co	+0.10		+0.10	+0.10		+0.10
Bal		–0.05	–0.05	–0.05	–0.05	
Final	–79.98	–60.01	+60.01	60.02	–60.02	–79.98
	M''_{AB}	M''_{BA}	M''_{BC}	M''_{CB}	M''_{CD}	M''_{DC}

Column shear condition

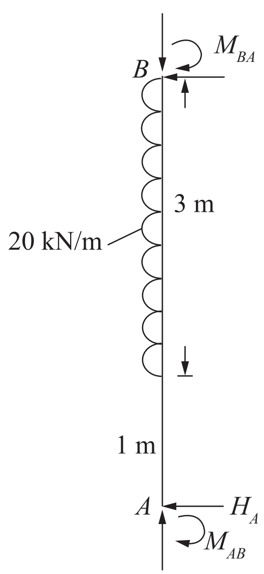


FIG. 2.114

Equilibrium of column AB

$$\Sigma M_B = 0;$$

$$M_{AB} + M_{BA} - 20 \times \frac{3^2}{2} + 4H_A = 0$$

$$\boxed{H_A = \frac{90 - (M_{AB} + M_{BA})}{4}}$$

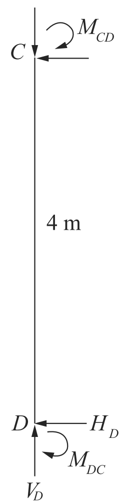


FIG. 2.115

$$\Sigma M_C = 0;$$

$$M_{CD} + M_{DC} + 4H_D = 0$$

$$\boxed{H_D = \frac{-(M_{CD} + M_{DC})}{4}}$$

Considering the horizontal equilibrium
Substituting the values in terms of moments and simplifying

$$H_A + H_D = 20 \times 3$$

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} = -150$$

$$(-26.41 + 11.81 + 3.38 + 1.68) - (79.98 + 60.01 + 60.02 + 79.98)k = -150$$

$$\boxed{k = 0.502}$$

End moments

$$M_{AB} = -26.41 - 79.98(0.502) = 66.6 \text{ kNm}$$
$$M_{BA} = 11.81 - 60.01(0.502) = -18.3 \text{ kNm}$$
$$M_{BC} = -11.81 + 60.01(0.502) = +18.3 \text{ kNm}$$
$$M_{CB} = -3.38 + 60.02(0.502) = +26.8 \text{ kNm}$$
$$M_{CD} = +3.38 - 60.02(0.502) = -26.8 \text{ kNm}$$
$$M_{DC} = 1.68 - 79.98(0.502) = -38.5 \text{ kNm}$$

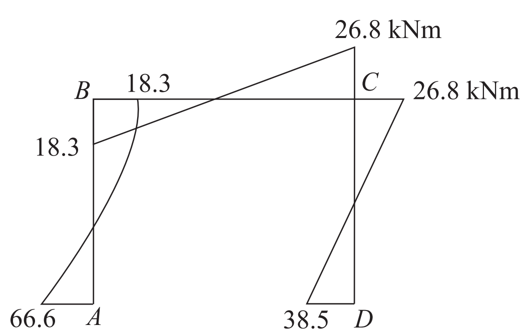


FIG. 2.116 Bending moment diagram

Example 2.27 Analyse the rigid frame shown in Fig. 2.117 by the moment distribution method. Draw the bending moment diagram.

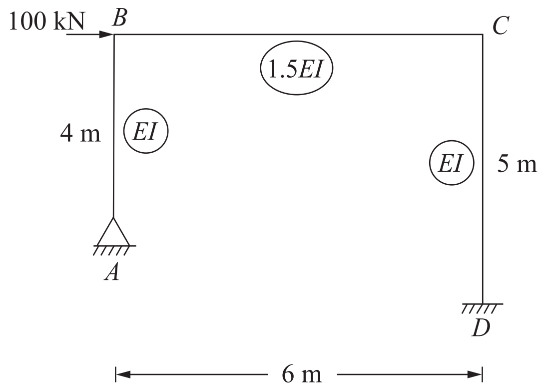


FIG. 2.117

Solution: In the above problem, a lack of symmetry makes the frame to sway to the right. In the sway analysis, it is assumed that the joints B and C have no movement of rotation but there is only lateral translation.

In the first instant, an external force necessary to prevent the lateral translation is assumed. The moments at the supports and joints are computed for the above external force. This is followed by a redistribution of moments allowing for the side sway by removing the assumed external force.

The frame is analysed by considering the sway moments only.

Distribution factors

Table 2.63

Joint	*B*		*C*	
Members	*BA*	*BC*	*CB*	*CD*
k	$\frac{3}{4}\left(\frac{I}{4}\right) = 0.188I$	$\frac{1.5I}{6} = 0.25I$	$\frac{1.5I}{6} = 0.25I$	$\frac{I}{5} = 0.20I$
Σk	$0.438I$		$0.45I$	
$DF = k/\Sigma k$	0.43	0.57	0.56	0.44

Sway moments

The sway moments are assumed in the following ratio

$$\frac{M_{BA}}{M_{CD}} = \frac{-3EI\Delta/4^2}{-6EI\Delta/5^2}$$

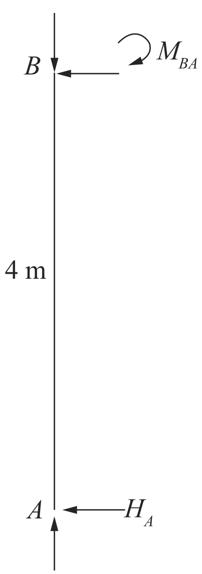
$\therefore$ $$M_{BA} : M_{CD} = -25.00 : -32.00 \text{ kNm}$$

Moment distribution table

Table 2.64

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.43	0.57	0.56	0.44 1	
FEMS	0	−25.00			−32.00	−32.00
Bal		+10.75	+14.25	+17.92	+14.08	
Co			+8.96	+7.13		+7.04
Bal		−3.85	−5.11	−4.00	−3.13	
Co			−2.00	−2.56		−1.57
Bal		+0.86	+1.14	+1.43	+1.13	
Co			+0.72	+0.72		+0.57
Bal		−0.31	−0.41	−0.40	−0.32	
Co			−0.20	−0.21		−0.16
Bal		+0.08	+0.12	+0.12	+0.09	
Co			+0.06	+0.06		+0.05
Bal		−0.03	−0.03	−0.03	−0.03	
Final		−17.50	+17.50	+20.18	−20.18	−26.07
Moment	0					

Column shear condition

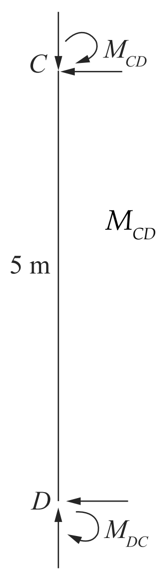


$$\Sigma M_B = 0$$

$$4H_A + M_{BA} = 0$$

$$\boxed{H_A = \frac{-M_{BA}}{4}}$$

FIG. 2.118



$$\Sigma M_C = 0$$

$$M_{CD} + M_{DC} + 5H_D = 0$$

$$\boxed{H_D = \frac{-(M_{CD} + M_{DC})}{5}}$$

FIG. 2.119

Considering the horizontal equilibrium, substituting the values of moments, H_A and H_D are calculated as

$$H_A = \frac{-(-17.50)}{4} = 4.38\,\text{kN}$$

$$H_D = \frac{-1}{5}(-20.18 - 26.07) = 9.25\,\text{kN}$$

The addition of H_A and H_D gives the lateral load which produces the assumed moments. Thus, $H_A + H_D$ gives 13.63 kN but the applied load is 100 kN. Hence, the end moments in the table are to be multiplied by the ratio (100/13.63 = 7.34).

The final moments are

$$M_{AB} = 0 \times 7.34 = 0$$

$$M_{BA} = -17.50 \times 7.34 = -128.5\ \text{kNm}$$

$$M_{BC} = +17.50 \times 7.34 = +128.5\ \text{kNm}$$

$$M_{CB} = +20.18 \times 7.34 = +148.1\ \text{kNm}$$

$$M_{CD} = -20.18 \times 7.34 = -148.1\ \text{kNm}$$

$$M_{DC} = -26.07 \times 7.34 = -191.4\ \text{kNm}$$

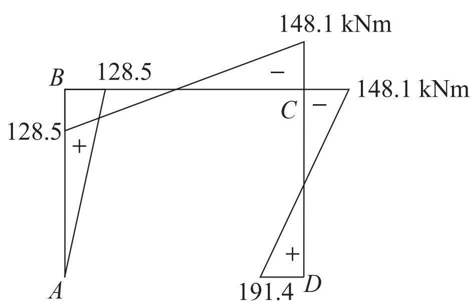


FIG. 2.120 Bending moment diagram

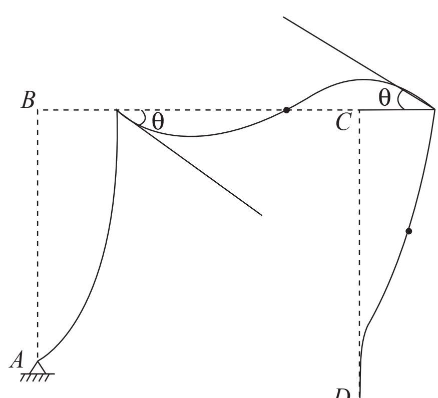


FIG. 2.121 Elastic curve

Example 2.28 Analyse the frame by moment distribution method and draw the shear force diagram and bending moment diagram.

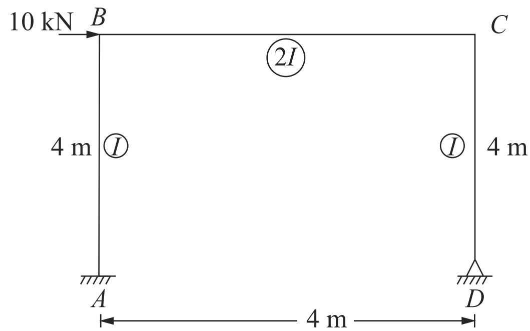


FIG. 2.122

Solution

Distribution factors

Table 2.65

Joint	*Members*	*Relative Stiff (I/l)*	*Σk*	*k/Σk*
B	BA	$I/4$	$3I/4$	0.33
	BC	$2I/4$		0.67
C	CB	$2I/4$	$0.688I$	0.73
	CD	$\frac{3}{4}\left(\frac{I}{4}\right) = 0.188I$		0.27

Sway moments

The sway moments are assumed in the following ratio:

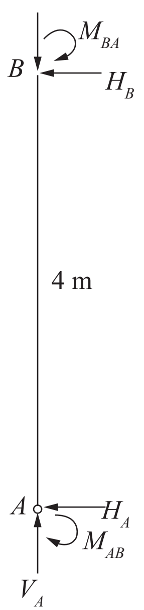
$$\frac{M_{BA}}{M_{CD}} = \frac{-6EI\Delta/4^2}{-3EI\Delta/4^2} = \frac{-2}{-1}$$

Hence, $\quad M_{BA} : M_{CD} = -20 : -10$ kNm

Table 2.66 Sway moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	1	0.33	0.67	0.73	0.27	0
FEMS	−20.00	−20.00			−10.00	
Bal		+6.60	+13.40	+7.30	+2.70	
Co	+3.30		+3.65	+6.70		
Bal		−1.21	−2.44	−4.89	−1.81	
Co	−0.61		−2.45	−1.22		
Bal		+0.81	+1.64	+0.89	+0.33	
Co	+0.41		+0.45	+0.82		
Bal		−0.15	−0.30	−0.60	−0.22	
Co	−0.08		−0.30	−0.15		
Bal		+0.10	+0.20	+0.11	+0.04	
Co	+0.05		+0.06	+0.10		
Bal		−0.02	−0.04	−0.07	−0.03	
	−16.93	−13.87	+13.87	8.99	−8.99	0.00
	M'_{AB}	M'_{BA}	M'_{BC}	M'_{CB}	M'_{CD}	M'_{DC}

Column shear equation

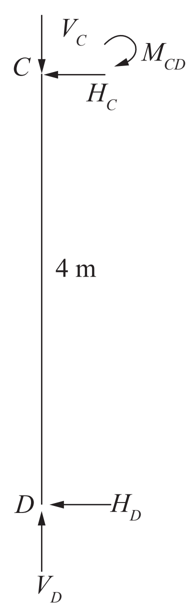


$$\Sigma M_B = 0$$

$$M_{AB} + M_{BA} + 4H_A = 0$$

$$\boxed{H_A = -\frac{1}{4}(M_{AB} + M_{BA})}$$

FIG. 2.123



$$\Sigma M_C = 0$$

$$M_{CD} + 4H_D = 0$$

$$\boxed{H_D = \frac{M_{CD}}{4}}$$

FIG. 2.124

Example 2.29 Analyse the frame shown in Fig. 2.127 by the moment distribution method. Draw the bending moment diagram.

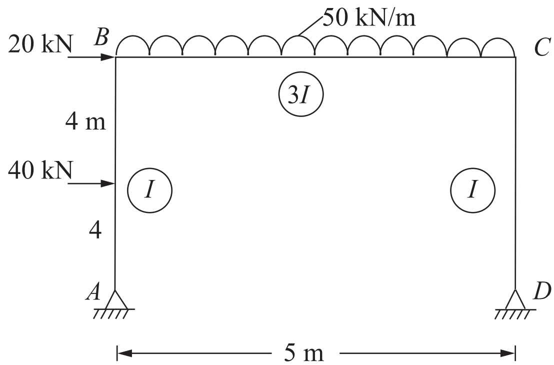


FIG. 2.127

Solution

Distribution factors

Table 2.67

Joint	*Members*	*Relative Stiffness (I/l)*	Σk	$k/\Sigma k$
B	BA	$\frac{3}{4}\left(\frac{1}{8}\right) = .094$	0.694I	0.14
	BC	$\frac{3I}{5} = 0.6I$		0.86
C	CB	$\frac{3I}{5} = 0.6I$	0.694I	0.86
	CD	$\frac{3}{4}\left(\frac{I}{8}\right) = .094I$		0.14

Fixed end moments

$$M_{FAB} = -40 \times \frac{8}{8} = -40.0 \text{ kNm}$$

$$M_{FBA} = +40.0 \text{ kNm}$$

$$M_{FBC} = \frac{-50 \times 5^2}{12} = -104.17 \text{ kNm}$$

$$M_{FCB} = +50 \times \frac{5^2}{12} = +104.17 \text{ kNm}$$

End moments

$$M_{AB} = 0$$

$$M_{BA} = 75.18 - 90.27\ k$$

$$M_{BC} = -75.18 + 90.27\ k$$

$$M_{CB} = +21.07 + 90.27\ k$$

$$M_{CD} = -21.07 - 90.27\ k$$

$$M_{DC} = 0$$

Nonsway Analysis

Table 2.68

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	1	0.14	0.86	0.86	0.14	1
FEMS Bal	−40.00 +40.00	+40.00 +8.98	−104.17 +55.19	+104.17 −89.58	0 −14.59	0
Co Bal		+20.00 +3.47	−44.79 +21.32	+27.60 −23.74	−3.86	
Co Bal		+1.66	−11.87 +10.21	+10.66 −9.16	−1.50	
Co Bal	0	+0.64	−4.58 +3.94	+5.11 −4.40	−0.71	0
Co Bal		+0.31	−2.20 +1.89	+1.97 −1.69	−0.28	
Co Bal		+0.12	−0.85 +0.73	+0.95 −0.82	−0.13	
		75.18	−75.18	+21.07	−21.07	
	M'_{AB}	M'_{BA}	M'_{BC}	M'_{CB}	M'_{CD}	M'_{DC}

Table 2.69 Moment distribution table

Joint	A	B		C		D
Members	AB	BA	BC	CB	CD	DC
DF	1	0.14	0.86	0.86	0.14	1
FEMS	0	–100.00	0	0	–100.00	0
Bal		+14.00	+86.00	+86.00	+14.00	
Co			+43.00	+43.00		
Bal		–6.02	–36.98	–36.98	–6.02	
Co			–18.49	–18.49		
Bal		+2.59	+15.90	+15.90	+2.59	
Co			+7.95	+7.95		
Bal		–1.11	–6.84	–6.84	–1.11	
Co			–3.42	–3.42		
Bal		+0.48	+2.94	+2.94	+0.48	
Co			+1.47	+1.47		
Bal		–0.21	–1.26	–1.26	–0.21	
	0	–90.27	+90.27	+90.27	–90.27	0
	M''_{AB}	M''_{BA}	M''_{BC}	M''_{CB}	M''_{CD}	M''_{DC}

Hence, $M_{BA} : M_{CD} = -100.00 : -100.00$

$$20 - 0.125M_{BA} - 0.125M_{CD} = 60$$

$$20 - 0.125(75.18 - 90.27k) - 0.125(-21.07 - 90.27k) = 60$$

Solving for $\boxed{k = 2.07}$

Substituting the values of k by back substitution:

$$M_{AB} = 0,\ M_{BA} = -111.86,\ M_{BC} = +111.86,$$

$$M_{CB} = 207.9,\ M_{CD} = -207.9 \text{ kNm},\ M_{DC} = 0$$

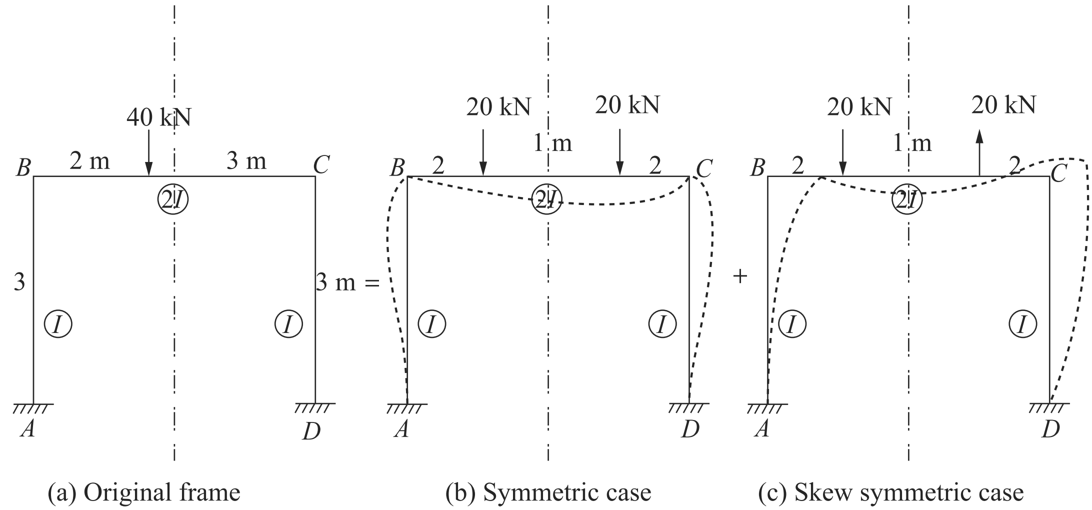


FIG. 2.133

Decomposition of the structure taking advantage of symmetry.

Fixed end moments in symmetrical case

$$M_{FBC} = -20 \times 2 \times \frac{3}{5} = -24\ \text{kNm}$$

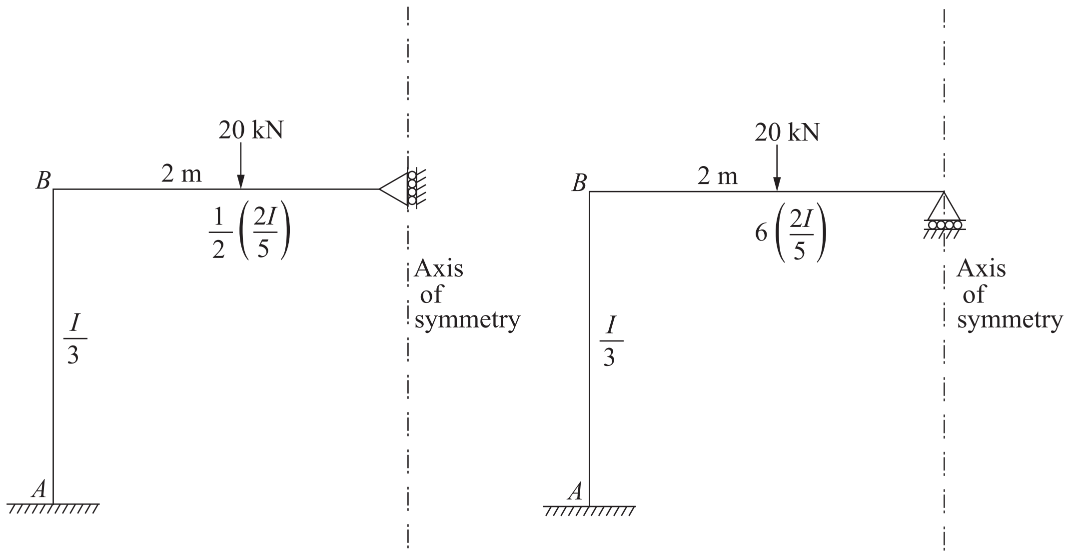
$$M_{FCB} = +20 \times 2 \times \frac{3}{5} = +24\ \text{kNm}$$

Table 2.71 Moment distribution for symmetric case

Joint	*A*	*B*	
Members	*AB*	*BA*	*BC*
DF	0	0.62	0.38
FEM Bal		 +14.88	−24.00 +9.12
Co	7.44		
Final	7.44	14.88	−14.88

Skew–symmetric case

The cantilever stiffness for the column *AB* is taken as $(I/3)$. While the skew stiffness of the beam *BC* is taken as $6(2I/5)$. The derivation of the cantilever stiffness and skew stiffness are available in standard references.



(a) Symmetric case (b) Skew symmetric case

FIG. 2.134

Table 2.72 Distribution factors for skews symmetric case

Joint	*Members*	*Relative Stiffness*	Σk	*DF* = $k/\Sigma k$
	BA	$I/3 = 0.33I$		0.12
B			$2.73I$	
	BC	$6(2I)/5 = 2.4I$		0.88

Fixed end moments in skew symmetric case

$$M_{FBC} = -\frac{2 \times 20 \times 3^2}{5^2} + \frac{3 \times 20 \times 2^2}{5^2} = -4.8 \text{ kNm}$$

$$M_{FCB} = -4.8 \text{ kNm}$$

Table 2.73 Moment distribution for skew symmetric case

Joint	*A*	*B*	
Members	*AB*	*BA*	*BC*
	0	0.12	0.88
FEMS			−4.80
Bal		+0.58	+4.22
Co	−0.58		
Final	−0.58	+0.5	−0.58

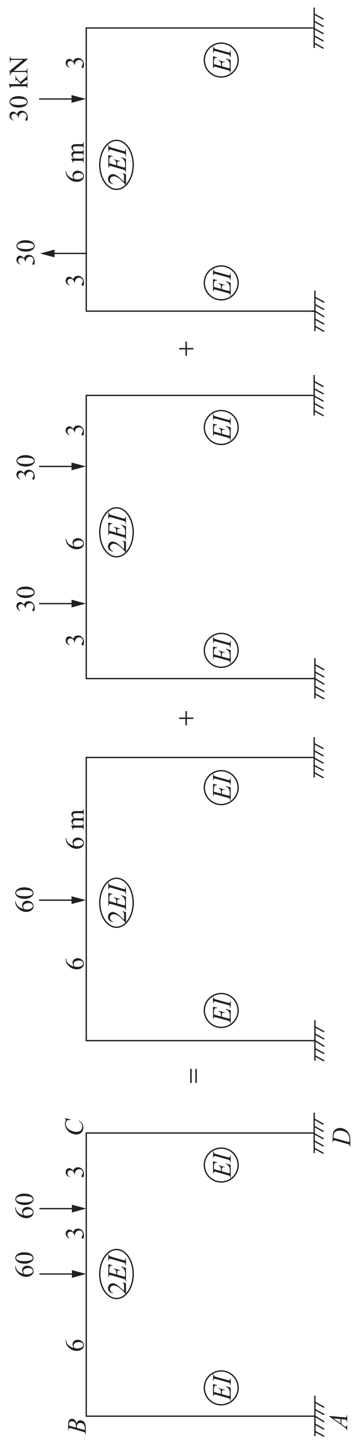


FIG. 2.137 Decomposition of given load system to symmetric and skew symmetric loading systems

Table 2.76

Joint	*A*	*B*	
Members	*AB*	*BA*	*BC*
DF	0.0	0.67	0.33
FEMS Bal		 +45.0	–67.5 +22.5
CO	+22.5		
Total	22.5	45.0	–45.0

Case (iii) Skew symmetric case

The distribution factors at joint B are calculated by considering the cantilever stiffness and taking the skew stiffness of the beam. $M_{FBC} = (3 \times 30 \times 9^2/12^2 - 9 \times 30 \times 3^2/12^2) = +33.75$ kNm. The cantilever stiffness is taken as $(I/6)$ and for the beam $(6 \times 2I/12)$. Hence,

$$d_{BA} : d_{BC} = (I/6) : (6 \times 2I/12), \quad \text{i.e. } d_{BA} : d_{BC} = 1 : 6$$

i.e. $$d_{BA} = (1/7) \text{ and } d_{BC} = (6/7).$$

Table 2.77

Joint	*A*	*B*	
Members	*AB*	*BA*	*BC*
DF	0	0.14	0.86
FEMS Bal		 –4.72	+33.75 –29.03
CO	+4.72		
Total	+4.72	–4.72	+4.72

The final moments are obtained by adding:

Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
Case (i)	30.00	60.00	–60.00	+60.00	–60.00	–30.00
Case (ii)	22.50	45.00	–45.00	+45.00	–45.00	–22.50
Case (iii)	4.72	–4.72	+4.72	+4.72	–4.72	+4.72
Total	57.22	100.28	–100.28	109.72	–109.72	–47.78

The above moments tally with the moments obtained by classical moment distribution (Ref. Ex. 2.22). The Naylor's method is very much useful if the reader identifies the application of the same to specific problems.

The distribution factors are obtained as follows.

$$\begin{array}{llll} & & B & \\ & BA & & BC \\ k & \frac{I}{4} & & \frac{I}{4} \\ \Sigma k & & \frac{I}{2} & \\ DF = \frac{k}{\Sigma k} & 0.5 & & 0.5 \end{array}$$

Table 2.78 Member end moments for symmetrical loading

Members	*AB*	*BA*	*BC*
DF	0	0.5	0.5
FEMS	−26.67	+26.67	
Bal		−13.33	−13.34
CO	−6.67		
Bal	0.00	0.00	0.00
Total	−33.34	13.34	−13.34

The analysis for antisymmetrical distribution is carried out using cantilever moment distribution. The stiffness for the column is taken as $(EI/4)$ and for the beam it is taken as six times the original stiffness. Hence, $6E(2I)/4 = EI$. The fixed end moments for the columns are taken as

$$M_{FAB} = -\frac{wl^2}{12} - \frac{wl}{2}\left(\frac{l}{2}\right) = -\frac{wl^2}{3} = -20 \times \frac{4^2}{3} = -106.67 \text{ kNm}$$

$$M_{FBA} = +\frac{wl^2}{12} - \frac{wl}{2}\left(\frac{l}{2}\right) = -\frac{wl^2}{6} = -\frac{20 \times 4^2}{6} = -53.33 \text{ kNm}$$

The carryover factor to the column is (−1) in antisymmetrical loading. The distribution factors for the antisymmetrical loading is

Joint	*B*	
Member	*BA*	*BC*
k	$\frac{I}{4}$	$\frac{6(2I)}{4}$
k	1	12
$\sum k$	13	
$\frac{k}{\Sigma k}$	0.077	0.923

Table 2.79 Member end moments for symmetrical loading

Members	*AB*	*BA*	*BC*
DF	0	0.77	0.923
FEMS Bal	–106.67	–53.33 +4.17	 +49.16
CO	4.17		
Bal	0.00	0.00	0.00
Final	–110.84	–49.16	+49.16

The end moments of the members were obtained by adding the moments due to symmetric loading and antisymmetric loading.

Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
Symmetric Loading	–33.34	+13.34	–13.34	+13.34	–13.34	+33.34
Antisymmetric Loading	–110.84	–49.16	+49.16	+49.16	–49.16	–110.84
Total	–144.18	–35.82	+35.82	+62.5	–62.50	–77.50

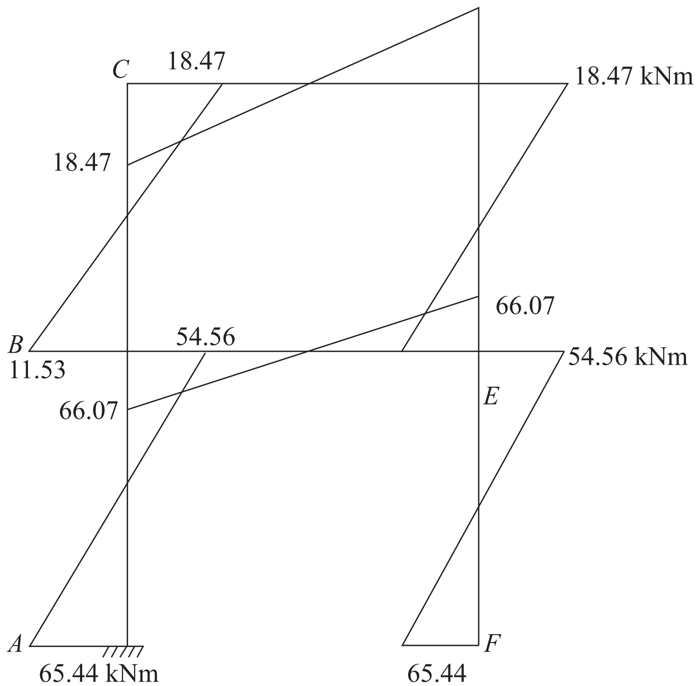


FIG. 2.142 Bending moment diagram

2.11 ANALYSIS OF FRAMES WITH INCLINED LEGS

Example 2.34 Analyse the rigid frame shown in figure by the moment distribution method and draw the BM diagram Support A is hinged and support D is fixed support.

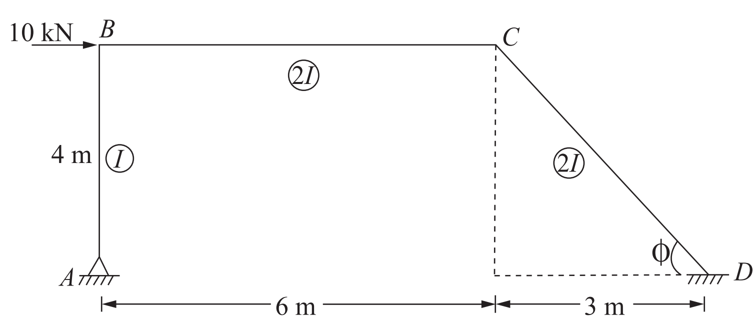


FIG. 2.143

Solution

Determination of sway (Δ) of members

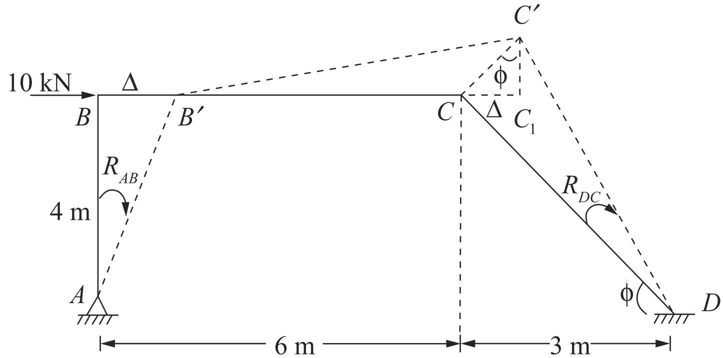


FIG. 2.144 Sway displacement diagram

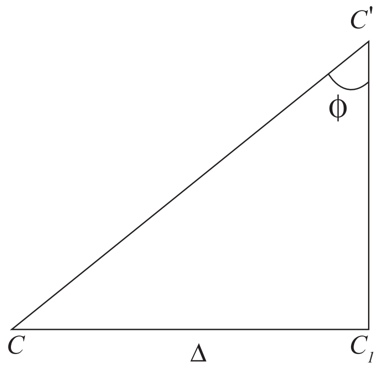


FIG. 2.145

Consider the above triangle,

$$C_1C' = \Delta \cot\phi = 0.75\Delta$$

$$CC' = \Delta \operatorname{cosec}\phi = 1.25\Delta$$

$$\text{Sway of } AB = \Delta_1 = \Delta_{AB} = BB' = +\Delta$$

$$\text{Sway of } BC = \Delta_2 = \Delta_{BC} = -C_1C' = -\Delta \cot\phi = -0.75\Delta$$

$$\text{Sway of } CD = \Delta_3 = \Delta_{CD} = +CC' = \Delta \operatorname{cosec}\phi = +1.25\Delta$$

Sway fixed end moments

$$M_{FAB} = 0$$

$$M_{FAB} = -3EI_1\Delta_1/l_1^2 = -3EI\Delta/4^2 = -\frac{3}{16}EI\Delta$$

$$M_{FBC} = M_{FCB} = -6EI_2\Delta_2/l_2^2 = -6E(2I)(-0.75\Delta)/6^2 = \frac{EI\Delta}{4}$$

$$M_{FCD} = M_{FDC} = -6EI_3\Delta_3/l_3^2 = -6E(2I)(-0.25\Delta)/5^2 = \frac{3EI\Delta}{5}$$

$$\therefore \qquad M_{FBA} : M_{FBC} : M_{FCD} = -\frac{3}{16} : \frac{1}{4} : -\frac{3}{5}$$

$$-15 : +20 : -48$$

Relative Stiffness k Values and Distribution Factors

Table 2.82 Distribution factors

Joint	*Members*	*Relative Stiffness Values I/l*	Σk	$DF = k/\Sigma k$
	BA	$\frac{3}{4}\left(\frac{I}{4}\right) = \frac{3}{16}I$		0.36
B		$= 0.1875I$	$= 0.5175I$	
	BC	$\frac{2I}{6} = 0.33I$		0.64
	CB	$\frac{2I}{6} = 0.33I$		0.45
C			$0.73I$	
	CD	$\frac{2I}{5} = 0.4I$		0.55

Table 2.83 Sway moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC* ⟷	*CB*	*CD* ⟶	*DC*
DF	1	0.36	0.64	0.45	0.55	0
FEMS		−15.00	+20.00	+20.00	−48.00	−48.00
Bal		−1.80	−3.20	+12.60	+15.40	
CO			+6.30	−1.60		+7.70
Bal		−2.27	−4.03	+0.72	+0.88	
CO			+0.36	−2.02		+0.44
Bal		+0.13	−0.23	+0.91	+1.11	
CO			+0.46	−0.12		+0.56
Bal		−0.17	−0.29	+0.05	+0.07	
CO			+0.03	−0.15		+0.04
Bal		−0.01	−0.02	+0.07	+0.08	
CO			+0.04	−0.01		+0.04
Bal		−0.01	−0.03	+0.005	+0.005	
		−19.39	19.39	30.46	−30.46	−39.22

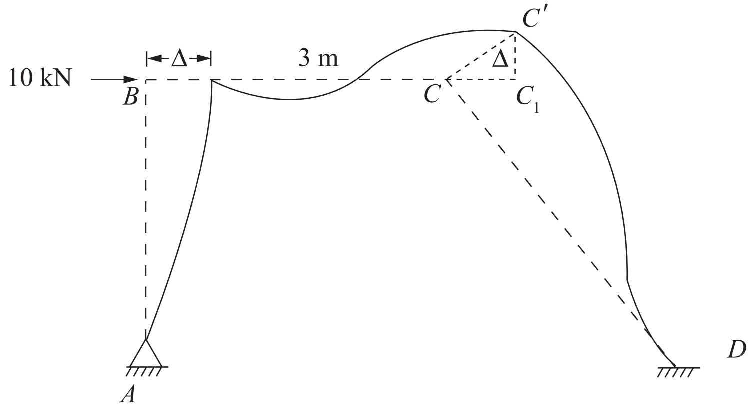


FIG. 2.148 Elastic curve

Example 2.35 Determine the end moments and draw the bending moment diagram.

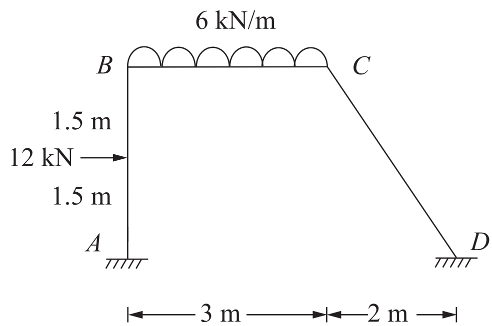


FIG. 2.149

Solution: In inclined frames, the sway of the members is to be determined from geometry. Let the column *AB* sway by an amount Δ. The beam member *BC* moves horizontally to *B′C′* as shown in Fig. 2.150. The displacement *CC′* is related to sway Δ and the angle of the inclined member *CD*. Thus, using the geometry of the deflected shape the sway of the members are related.

Fixed end moments due to loading

$$M_{FAB} = -12 \times \frac{3}{8} = -4.5 \text{ kNm};$$

$$M_{FBA} = +12 \times 310 = +4.5 \text{ kNm}$$

$$M_{FBC} = -6 \times \frac{3^2}{12} = -4.5 \text{ kNm}$$

$$M_{FCB} = +6 \times \frac{3^2}{12} = +4.5 \text{ kNm}$$

Determination of sway (Δ) of the member

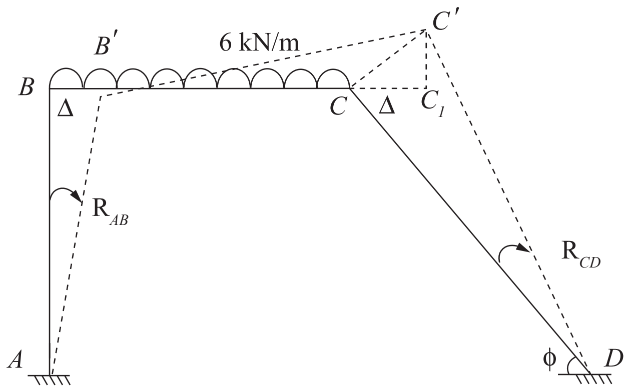


FIG. 2.150 Sway displacement diagram

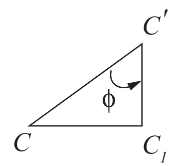


FIG. 2.151

From $\Delta CC_1C'$; $\quad \cot\phi = \dfrac{C_1C'}{\Delta}$

$$\boxed{C_1C' = \Delta \cot\phi}$$

$$\operatorname{cosec}\phi = \frac{CC'}{\Delta}$$

$$\boxed{CC' = \Delta \operatorname{cosec}\phi}$$

Referring to the above given frame

$$\cot\phi = 2/3 = 0.67$$

$$\operatorname{cosec}\phi = \sqrt{13}/3 = 1.20$$

Sway of $AB = \Delta$

Sway of $BC = -0.67\Delta$

Sway of $CD = +1.20\Delta$

Sway moment distribution

The sway moments are assumed as

$$M_{FAB} = M_{FBA} = -\frac{6EI_1}{l_1^2} = -\frac{6EI\Delta}{3^2} = -0.667EI\Delta$$

$$M_{FBC} = M_{FCB} = -\frac{6EI_2\Delta_2}{l_2^2} = \frac{6EI(-0.7\Delta)}{3^2} = +0.447EI\Delta$$

$$M_{FCD} = M_{FDC} = -\frac{6EI_3\Delta_3}{l_3^2} = -\frac{6EI(1.2\Delta)}{13} = -0.554EI\Delta$$

$$\therefore \quad M_{FBA} : M_{FBC} : M_{FCD} = -0.667EI\Delta : +0.447EI\Delta : -0.554EI\Delta$$

$$= -66.70 : +44.70 - 55.40$$

Table 2.84 Moment distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB* ←	*BA*	*BC* ↔	*CB*	*CD* →	*DC*
	0	0.5	0.5	0.55	0.45	0
FEMS	−66.70	−66.70	+44.70	+44.70	−55.40	−55.40
Bal		+11.00	+11.00	+5.89	+4.82	
CO	+5.50		+2.95	+5.50		+2.41
Bal		−1.48	−1.48	−3.03	−2.47	
CO	−0.74		−1.52	−0.74		−1.24
Bal		+0.76	+0.76	+0.41	+0.33	
CO	+0.38		+0.21	+0.38		+0.17
Bal		−0.10	−0.11	−0.21	−0.17	
CO	−0.05		−0.11	−0.06		−0.09
Bal		+0.06	+0.05	+0.03	+0.03	
CO	+0.03		+0.02	+0.03		+0.02
Bal		−0.01	−0.01	−0.02	−0.01	
Total	−61.58	−56.47	+56.47	+52.88	−52.88	−54.13
	M''_{AB}	M''_{BA}	M''_{BC}	M''_{CB}	M''_{CD}	M''_{DC}

Distribution factors

Table 2.85

Joint	*Members*	*Relative Stiff Values (k)*	Σk	$k/\Sigma k$
B	*BA*	$I/3$	$2I/3$	0.5
	BC	$I/3$		0.5
C	*CB*	$I/3$	$2I/3$	0.55
	CD	$I/\sqrt{3}$		0.45

Table 2.86 Nonsway member distribution table

Joint	*A*	*B*		*C*		*D*
Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.5	0.5	0.55	0.45	0
FEMS	−4.50	+4.50	−4.50	+4.50	0	0
Bal		0	0	−2.48	−2.02	
CO	0		−1.24	0		−1.01
Bal		+0.62	+0.62	0	0	
CO	+0.31		0	+0.31		0
Bal		0	0	−0.17	−0.14	
CO	0		−0.09	0		−0.07
Bal		+0.05	+0.04	0	0	
CO	+0.03		0	+0.02		0
Bal		0	0	−0.01	−0.01	
Total	−4.16 M'_{AB}	+5.17 M'_{BA}	−5.17 M'_{BC}	2.17 M'_{CB}	−2.17 M'_{CD}	−108 M'_{DC}
	−61.58 M''_{AB}	−56.47 M''_{BA}	+56.47 M''_{BC}	52.88 M''_{CB}	−52.88 M''_{CD}	−54.13 M''_{DC}

End moments

The end moments are the sum of the moments of the nonsway and sway moments.

$$M_{AB} = -4.16 - 61.58\ k$$

$$M_{BA} = +5.17 - 56.47\ k$$

$$M_{BC} = -5.17 + 56.47\ k$$

$$M_{CB} = +2.17 + 52.88\ k$$

$$M_{CD} = -2.17 - 52.88\ k$$

$$M_{DC} = -1.08 - 54.13\ k$$

Column shear equation

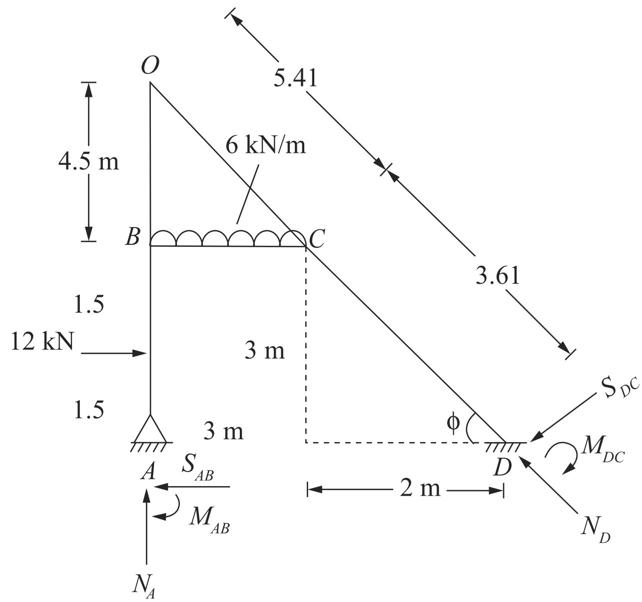
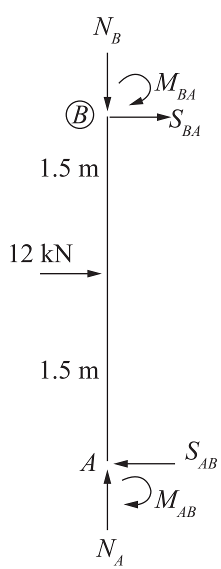


FIG. 2.152(a)



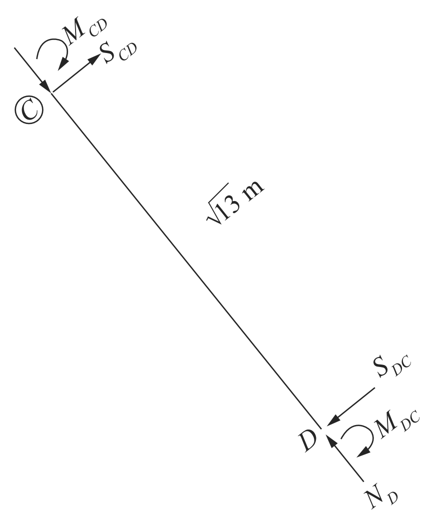


FIG. 2.152(b)

FIG. 2.152(c)

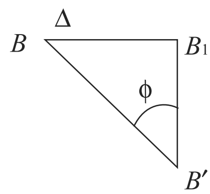


FIG. 2.156

$$\operatorname{cosec}\phi = 5/4 = 1.25$$

$$\cot\phi = 3/4 = 0.75$$

From the displacement diagram

$$\cot\phi = \frac{B_1B'}{\Delta}$$

$$\boxed{B_1B' = \Delta \cot\phi}$$

$$\operatorname{cosec}\phi = \frac{BB'}{BB_1}$$

$$BB' = BB_1 \operatorname{cosec}\phi$$

$$\boxed{BB' = 1.25\Delta}$$

$$\therefore \qquad \boxed{CC' = 1.25\Delta}$$

Vertical displacement between B' and C'

$$= B'B_1 + C_1C' = 2\Delta \cot\phi = 1.5\Delta$$

$$\text{Sway of } AB = \Delta_{AB} = BB' = \Delta \operatorname{cosec}\phi = +1.25\Delta$$

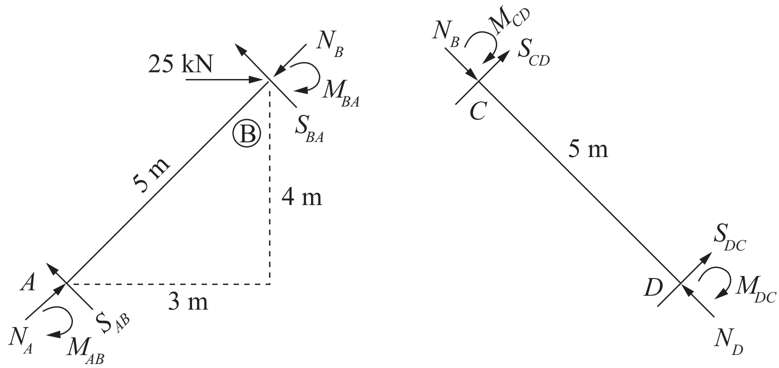
$$\text{Sway of } BC = \Delta_{BC} = -1.5\Delta$$

$$\text{Sway of } CD = \Delta_{CD} = \Delta \operatorname{cosec}\phi = +1.25\Delta$$

Nonsway Analysis

$$M_{FBC} = -100 \times \frac{8}{8} = -100\,\text{kNm}$$

$$M_{FCB} = +100 \times \frac{8}{8} = +100\,\text{kNm}$$



(a) (b)

FIG. 2.157

Considering the above free body diagram,

$$\Sigma M_B = 0;$$

$$M_{AB} + M_{BA} + 5\ S_{AB} = 0$$

i.e.
$$\boxed{S_{AB} = -\frac{(M_{AB} + M_{BA})}{5}} \tag{1}$$

$$\Sigma M_C = 0;$$

$$M_{CD} + M_{DC} - 5\ S_{DC} = 0$$

$$\boxed{S_{DC} = -\frac{(M_{CD} + M_{DC})}{5}} \tag{2}$$

On simplifying

$$(M_{AB} + M_{DC}) + 1.75(M_{BA} + M_{CD}) + 100 = 0$$

The values of the moments are obtained from the above nonsway moment distribution and sway moment distribution case as

$$\{(38.27 - 26.39k) - (38.27 + 24.78k)\} + 1.75\{(76.55 - 22.74k) + (-76.55 - 24.78k)\} + 100 = 0$$

$$-51.17\,k - 83.16\,k + 100 = 0$$

$$\boxed{k = 0.74}$$

Final moments

$$M_{AB} = 38.27 - 26.39\,k(0.74) = 18.74 \text{ kNm}$$

$$M_{BA} = 76.55 - 22.74\,k(0.74) = 59.72 \text{ kNm}$$

$$M_{BC} = -76.55 + 22.74(0.74) = -59.72 \text{ kNm}$$

$$M_{CB} = +76.55 + 24.78(0.74) = 94.88 \text{ kNm}$$

$$M_{CD} = -76.55 - 24.78(0.74) = -94.88 \text{ kNm}$$

$$M_{DC} = -38.27 - 27.40(0.74) = -58.55 \text{ kNm}$$

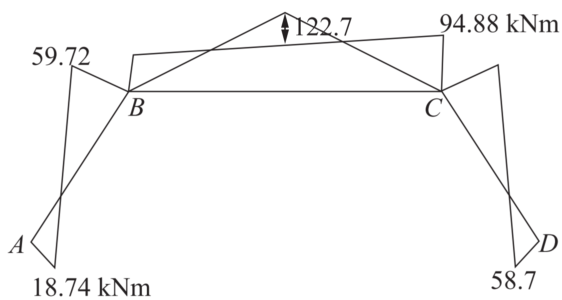


FIG. 2.159 Bending moment diagram

$$M_E = 100 \times \frac{8}{4} - \frac{1}{2}(59.72 + 94.88) = 122.7 \text{ kNm}$$

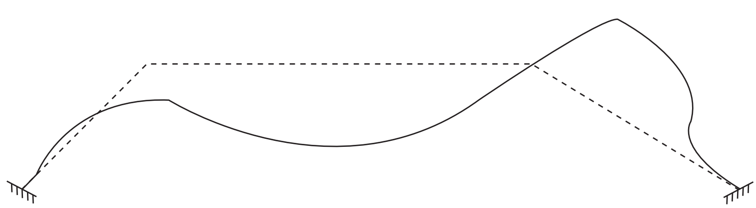


FIG. 2.160 Elastic curve

2.12 ANALYSIS OF GABLE FRAMES

Example 2.37 Analyse the rigid frame shown in Fig. 2.161 by the moment distribution method and sketch the bending moment diagram.

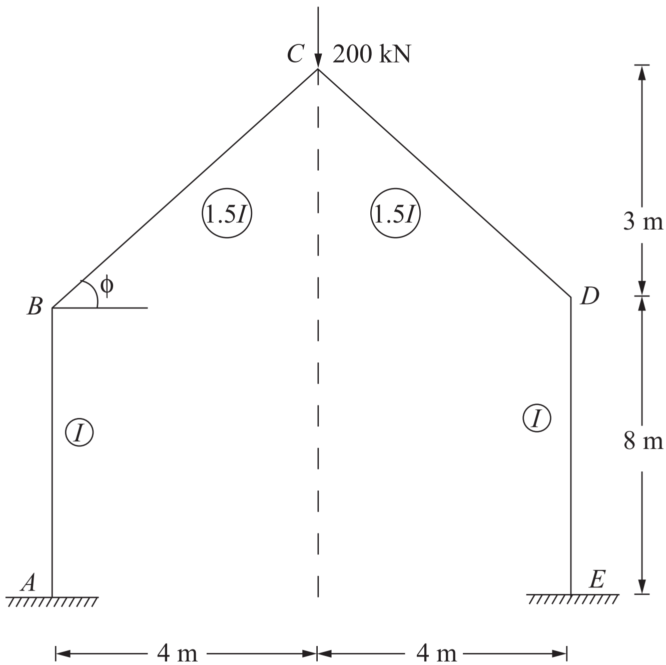


FIG. 2.161

Solution

Sway (Δ) values of the members

The Gable frame sways equally on both sides (either side) at B and D by virtue of symmetry on the frame and the loading on YY axis.

Sway of $AB = \Delta_{AB} = -BB' = -\Delta$ (negative sign because AB sways in the anticlockwise direction with respect to AB)

Sway of $BC = \Delta_{BC} = +C_1C' = +\Delta \operatorname{cosec}\phi$

$$\therefore \quad \Delta_{BC} = +\frac{5}{3}\Delta$$

Sway of $CD = \Delta_{CD} = -C_2C' = -\Delta \operatorname{cosec}\phi$

$$\therefore \quad \Delta_{CD} = -\frac{5}{3}\Delta$$

Sway of $DE = \Delta_{DE} = DD'$

$$\therefore \quad \Delta_{DE} = +\Delta$$

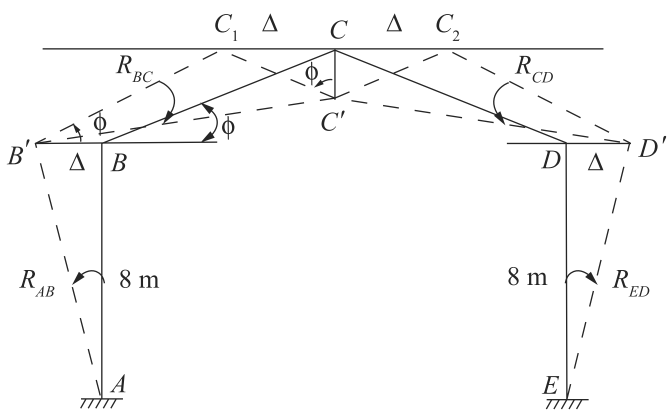


FIG. 2.162 Sway diagram

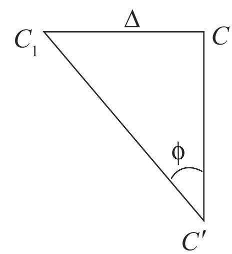


FIG. 2.163

Sway moment values

(Fixed end moments due to sway)

$$M_{FAB} = M_{FBA} = -\frac{EI_1 \;\; {}_1}{{}_2} = -\frac{-6EI\Delta}{8^2} = -\frac{6EI(-\Delta)}{64} = +\frac{EI}{64}$$

$$M_{FBC} = M_{FCB} = -\frac{6EI_2\Delta_2}{l_2^2} = -\frac{-6E(1.5\Delta)\,(5/3\Delta)}{5^2} = -\frac{15EI\Delta}{25}$$

$$M_{FCD} = M_{FDC} = -\frac{6EI_3\Delta_3}{l_3^2} = -\frac{-6E(1.5Z)\,(-5/3\Delta)}{5^2} = +\frac{15EI\Delta}{25}$$

$$M_{FDE} = M_{FED} = \frac{6EI_4\Delta_4}{l_4^2} = -\frac{-6EI\Delta}{8^2} = -\frac{6EI\Delta}{64}$$

$$\therefore \qquad \frac{M_{FAB}}{M_{FBC}} = \frac{+6EI\Delta/64}{-15EI\Delta/25} = \frac{+10}{-64}$$

Assume the sway moments in the above proportion as follows:

$$M_{FAB} = M_{FBA} = +10.00 \text{ kNm}$$

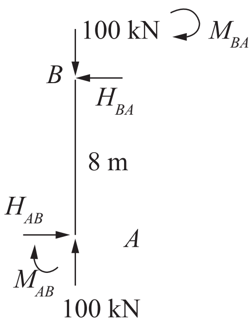
$$M_{FBC} = M_{FCB} = -64.00 \text{ kNm}$$

$$M_{FCD} = M_{FDC} = +64.00 \text{ kNm}$$

$$M_{FDE} = M_{FED} = -10.00 \text{ kNm}$$

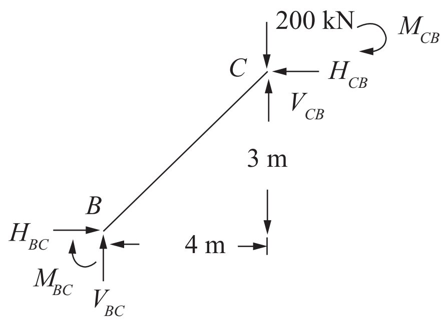
Distribution factors

Due to symmetry, consider joints B and C only.



Column AB

FIG. 2.164(b)



Member BC

FIG. 2.164(c)

In Fig. 2.164(b);

$$\Sigma M_B = 0;$$

$$-8H_{AB} + M_{BA} + M_{AB} = 0$$

$\therefore$
$$\boxed{H_{AB} = \frac{(M_{AB} + M_{BA})}{8}}$$

$$H_{BC} = H_{BA} = H_{AB} = (M_{AB} + M_{BA})/8$$

$\therefore$
$$V_{AB} = V_{BC} = V_{BA} = 100 \text{ kN}$$

Referring to Fig. 2.164(c) and taking moment about (c),

$$4\ V_{BC} - 3\ H_{BC} + M_{BC} + M_{CB} = 0$$

Substituting the value of H_{BC} in terms of moments,

$$4(100) - 3\frac{(M_{AB} + M_{BA})}{8} + (M_{BC} + M_{CB}) = 0$$

$$3200 - 3(M_{AB} + M_{BA}) + 8(M_{BC} + M_{CB}) = 0$$

Substituting the end moments from moment distribution table with a multiplier 'k';

$$3200 - 3(18.1k + 26.2k) + 8(-26.2k - 45.1k) = 0$$

$\therefore$
$$\boxed{k = 4.55}$$

Solution: The Gable frame sways equally on either side *B* and *D* by virtue of symmetry of the frame and loading.

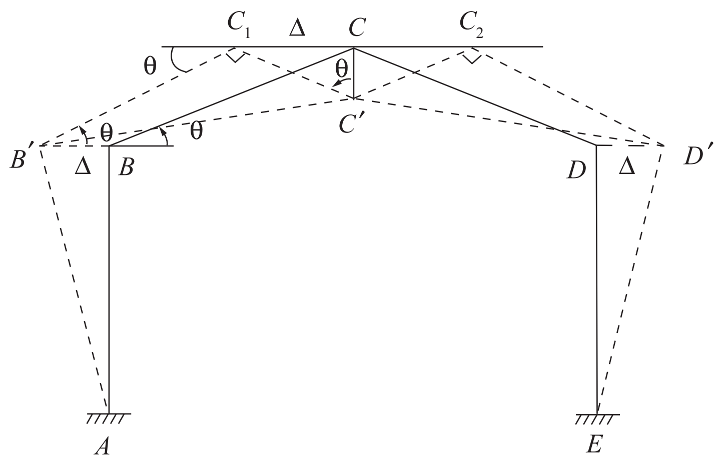


FIG. 2.167 Sway diagram

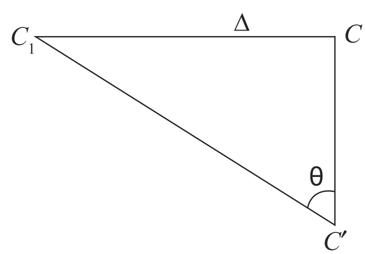


FIG. 2.168

In Fig. 2.166 length $BC = \sqrt{6^2 + 3^2} = 6.71$ m

$$\operatorname{cosec}\theta = 6.71/3 = 2.237$$

In Fig 2.167 $\operatorname{cosec}\theta = \dfrac{C_1C'}{\Delta}$

$$C_1C' = \Delta \operatorname{cosec}\theta = 2.237\Delta$$

In Fig. 2.167 Sway of $AB = BB' = -\Delta$

Sway of $BC = C_1C' = +2.237\Delta$

Sway of $CD = C_2C' = -2.237\Delta$

Sway of $ED = DD' = +\Delta$

Fixed end moments

Due to loading

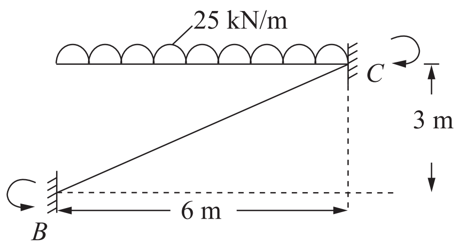


FIG. 2.169

$$M_{FAB} = M_{FBA} = M_{FDE} = M_{FED} = 0$$

$$M_{FBC} = -25 \times \frac{6^2}{12} = -75 \text{ kNm} = M_{FCD}$$

$$M_{FCB} = +25 \times \frac{6^2}{12} = +75 \text{ kNm} = M_{FDC}$$

Sway moment values

$$M_{FAB} = M_{FBA} = -\frac{EI_1 \quad 1}{2} = -\frac{6E(2I)(-\Delta)}{6^2} = \frac{12EI}{36}$$

$$M_{FBC} = M_{FCB} = -\frac{6EI_2\Delta_2}{l_2^2} = -\frac{6E(3I)(2.237\Delta)}{6.71^2} = \frac{-40.27EI\Delta}{45}$$

$$\therefore \qquad \frac{M_{FAB}}{M_{FBC}} = \frac{+15}{-40.27}$$

Assume the sway moments in the above proportion as

$$M_{FAB} = M_{FBA} = +15.00 \text{ kNm}$$

$$M_{FBC} = M_{FCB} = -40.27 \text{ kNm}$$

Distribution factors

Due to symmetry, consider joints B and C only.

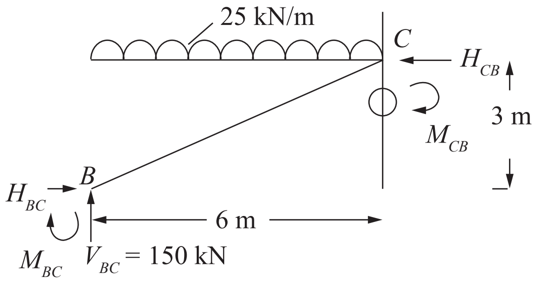


FIG. 2.171

$$\sum M_C = 0$$

$$150(6) + M_{BC} + M_{CB} - 3H_{BC} - 25 \times \frac{6^2}{2} = 0$$

As $H_{BC} = H_{BA} = H_{AB}$ and replacing in terms of moments as above.

$$150 \times 6 + M_{BC} + M_{CB} - \frac{3}{6}(M_{AB} + M_{BA}) - 450 = 0$$

$$900 + (64.13 - 58.94k) - 0.5(48.38 + 46.3k) - 450 = 0$$

$$489.94 - 82.09k = 0$$

$$\boxed{k = 5.97}$$

End moments

The end moments are obtained by substituting the value of k in the above equations as

$$M_{AB} = 16.13 + 20.43(5.97) = 138.1 \text{ kNm}$$

$$M_{BA} = 32.25 + 25.87(5.97) = 186.7 \text{ kNm}$$

$$M_{BC} = -32.25 - 25.87(5.97) = -186.7 \text{ kNm}$$

$$M_{CB} = +96.38 - 33.07(5.97) = -101.0 \text{ kNm}$$

$$H_{AB} = \frac{M_{AB} + M_{BA}}{6} = 54.13 \text{ kN}$$

From geometry; $\cos\theta = 6/6.71 = 0.894$ and

$$\sin\theta = 3/6.71 = 0.447$$

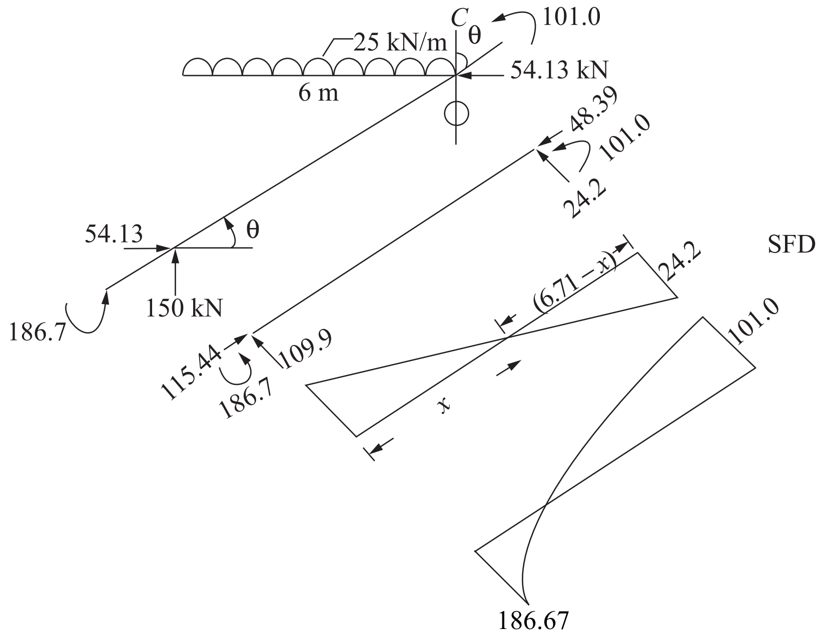
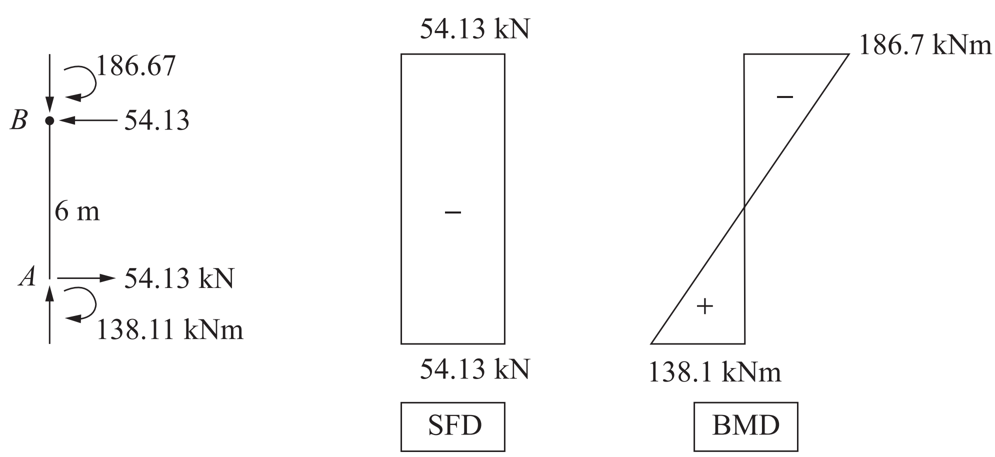


FIG. 2.172 Bending moment diagram (BMD)



(a) (b) (c)

FIG. 2.173

Referring to shear force diagram,

$$\frac{x}{6.71 - x} = \frac{109.9}{24.2}$$

$$\boxed{x = 5.5 \text{ m}}$$

$$M_{FCD} = M_{FDC} = -\frac{6EI_2\Delta_2}{l_2^2} = -\frac{6EI\Delta}{4^2} = \frac{3}{8}EI\Delta$$

$$M_{EF} = M_{FE} = -\frac{6EI_3\Delta_3}{l_3^2} = -\frac{6EI\Delta}{3^2} = \frac{6EI\Delta}{9}$$

$$M_{FBC} = M_{FCB} = 0$$

$$M_{FCE} = M_{FEC} = 0$$

$$\therefore \quad M_{FBA} : M_{FCD} : M_{FEF} = -\frac{12EI\Delta}{25} : -\frac{3EI\Delta}{8} : -\frac{6EI\Delta}{9}$$

$$M_{FAB} : M_{FCD} : M_{FEF} = -0.48\Delta : -0.375\Delta : -0.67\Delta$$

Hence,

$$M_{FAB} : M_{FCD} : M_{FEF} = -48.00 : -37.50 : -66.7 \text{ kNm}$$

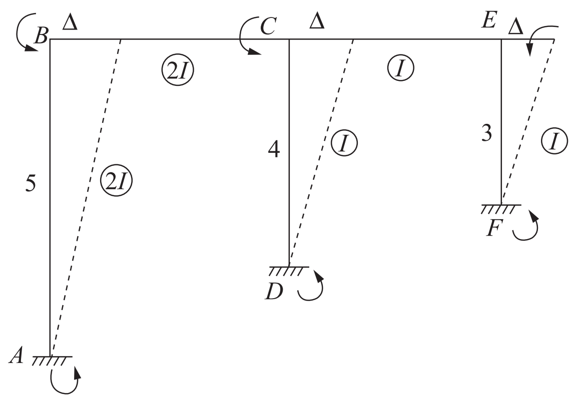


FIG. 2.177

Distribution factors

Table 2.93

Joint	*Members*	*Relative Stiffness Values (k)*	$\Sigma I/l$	$k/\Sigma k$
B	*BA*	$2I/5$	$4I/5$	0.50
	BC	$2I/5$		0.50
C	*CB*	$2I/5$	$0.9I$	0.44
	CD	$I/4$		0.28
	CE	$I/4$		0.28
E	*EC*	$I/4$	$0.58I$	0.43
	EF	$I/3$		0.57

Column shear equation

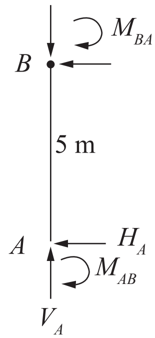


FIG. 2.178

$$\Sigma M_B = 0$$

$$M_{AB} + M_{BA} + 5H_A = 0$$

$$\boxed{H_A = -0.2(M_{AB} + M_{BA})}$$

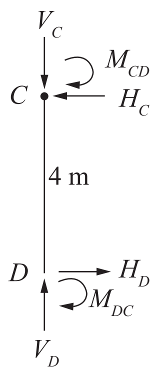


FIG. 2.179

$$\Sigma M_C = 0$$

$$M_{CD} + M_{DC} + 4H_D = 0$$

$$\boxed{H_D = -0.25(M_{CD} + M_{DC})}$$

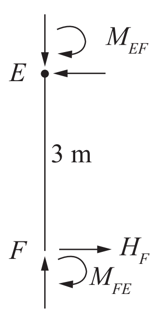


FIG. 2.180

$$\Sigma M_E = 0$$

$$M_{EF} + M_{FE} + 3H_F = 0$$

$$\boxed{H_F = -0.33(M_{EF} + M_{FE})}$$

Considering the horizontal equilibrium

$$80 - H_B - H_D - H_F = 0$$

$$80 + 0.2(M_{AB} + M_{BA}) + 0.25(M_{CD} + M_{DC}) + 0.33(M_{EF} + M_{FE}) = 0$$

$$80 + \{0.2(-36.66 - 25.36) + 0.25(-34.09 - 35.77) + 0.33(-29.68 - 48.18)\}k = 0$$

$$\boxed{k = 1.44}$$

Final moments

The moments obtained at the end of moment distribution table is multiplied by k and hence

$$M_{AB} = -52.8 \text{ kNm},\ M_{BA} = -36.5 \text{ kNm},\ M_{BC} = -36.5 \text{ kNm}$$

$$M_{CB} = +24.1 \text{ kNm},\ M_{CD} = -49.1 \text{ kNm},\ M_{DC} = -51.5 \text{ kNm}$$

$$M_{CE} = +35.4 \text{ kNm},\ M_{EC} = +42.7 \text{ kNm},\ M_{EF} = -42.7 \text{ kNm}$$

$$M_{FE} = -69.4 \text{ kNm}$$

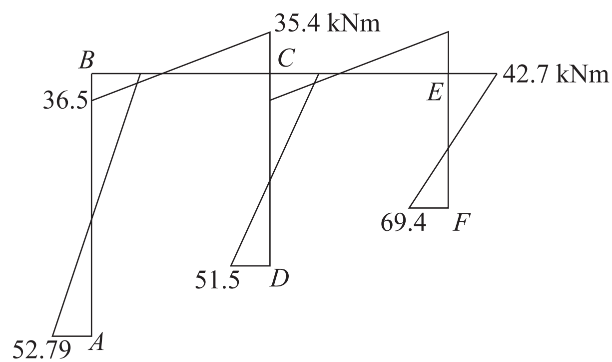


FIG. 2.181 Bending moment diagram

REVIEW QUESTIONS

Remembrance

2.1. Who has developed the moment distribution method?

2.2. Is the moment distribution a stiffness method or flexibility method?

2.3. List the advantages of the moment distribution method?

2.4. List the important steps in the moment distribution method?

2.5. Does the axial deformation considered in the development of the moment distribution method?

2.6. Define rotational stiffness?

2.7. Explain distribution factor?

2.8. Define carry over factor?

2.9. Which moment distribution is preferable for symmetric frames subjected to lateral loads as storey heights?

2.10. What are the other names for cantilever moment distribution method?

2.11. What are the advantages of Naylor's method?

2.12. What is the rotational stiffness of a cantilever?

2.13. What is the magnitude of a stiffness of a member under antisymmetric bending in Naylor's method?

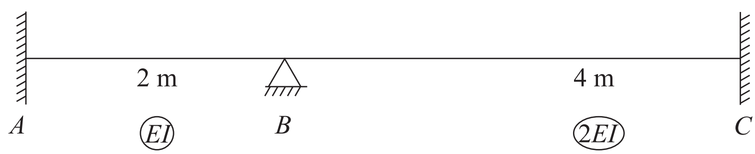
2.14. List the values of symmetric stiffness factor, skew symmetric stiffness factor and cantilever stiffness factor?

Understanding

2.1. In a member *AB*, if a moment of 10 kNm is applied at *A*, what is the moment carried over to the fixed end *B*?

2.2. What is sinking of supports? What is its effect on the end moments of the member?

2.3. Calculate the M_{FBA} and M_{FBC} for the beam shown in figure below due to sinking of support by 2 mm. $E = 6000 \text{ N/mm}^2$ and $I = 1.6(10)^8 \text{ mm}^4$.



2.4. Calculate the M_{FBC} and M_{FCB} for the beam shown in figure below $EI = 10 \times 10^{11} \text{ Nmm}^2$. Support *B* and *C* sinks by 2 mm and 3 mm respectively.

2.5. Is it possible to determine the beam deflections in a continuous beam by moment distribution method?

2.6. List the reasons for sidesway of the portal frame.

2.7. What is balancing at a joint?

2.8. While applying the moment distribution method, a designer remembers that "nothing comes back from the fixed end". Justify.

2.9. What we can understand from the computed sway correction factor, i.e., if it is positive what does it indicate as well as if it is negative what does it point out?

2.10. Why sway correction is required when we analyse unsymmetrical frames?

2.11. Does the carry over factor is half for non-prismatic members?

2.12. When a symmetrical structure is subjected to symmetrical loading, it does not sway. Does this statement applicable to Gable frames? If not, why?