J integral

SIF is calculated using two methods as shown in the *Figure 1*. The attached program evaluates the **area J-integral**.

1, line integral around the crack tip.

$$J^k = \int_\Gamma \left(W n_k - \sigma_{ij} \frac{\partial u_i}{\partial x_k} n_i \right) d\Gamma \quad ,(i,j,k=1,2)$$

$$J_M^{\ k} = \int_\Gamma \left(W_M n_k - \sigma_{Mij} \frac{\partial u_{Mi}}{\partial x_k} n_i \right) d\Gamma \quad ,(i,j,k=1,2)$$

Where,

$$W_M = \frac{1}{2}\left(\sigma_{M11}\varepsilon_{M11} + \sigma_{M22}\varepsilon_{M22} + 2\sigma_{M12}\varepsilon_{M12} \right)$$

$$\varepsilon_{Mij} = \frac{\partial u_{Mi}}{\partial x_j}, \ (i,j=1,2)$$

$$\sigma_{Mim} = E_{ij}\varepsilon_{Mij}, \ (i,j=1,2)$$

2. Area integral around the crack tip.

$$J_M^{\ k} = \int_\Gamma \left(\sigma_{Mij} \frac{\partial u_{Mi}}{\partial x_1} W_M \frac{\partial q}{\partial x_j} - W_M \frac{\partial q}{\partial x_1} \right) dA \quad ,(i,j,M=1,2)$$

The function q is any smooth function which has a value of zero at the outer boundary and unity at the inner boundary. A particular area is chosen around the crack tip as in *Figure 1*. Selected domain could have any shape such as rectangular or circle. Figure 2also shows the distribution of the *q*.

$$q = \begin{cases} 1, & \textit{at the inner boundary} \\ 0, & \textit{at the outer boundary} \end{cases}$$

Stress-displacement fields for plane stress, mode 1 are given,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\phi}{2}\right) \begin{Bmatrix} 1 - \sin\left(\frac{\phi}{2}\right)\sin\left(\frac{3\phi}{2}\right) \\ 1 + \sin\left(\frac{\phi}{2}\right)\sin\left(\frac{3\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right)\cos\left(\frac{3\phi}{2}\right) \end{Bmatrix}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{K_I}{2G}\sqrt{\frac{r}{2\pi}}(k - \cos\phi)\begin{Bmatrix} \cos\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) \end{Bmatrix}$$
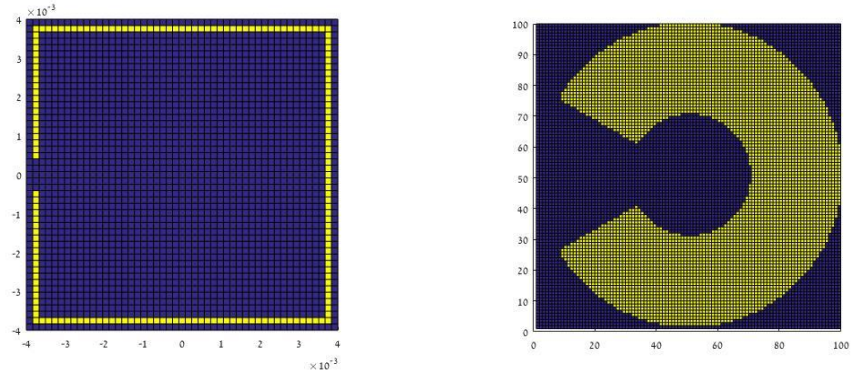

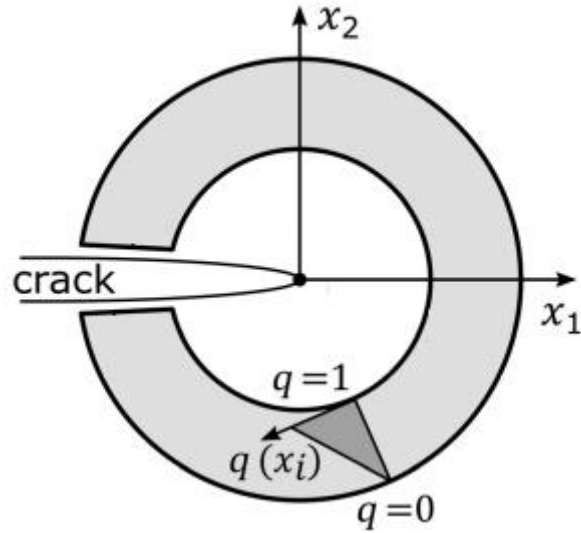*Figure 1 Two types of J integral around the crack tip. a) Line integral b) Area integral.*

Figure 2 q function in 2D area integral.