Computers and Mathematics with Applications 90 (2021) 66–72

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Guaranteed lower eigenvalue bounds for two spectral problems arising in fluid mechanics☆

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ARTICLE INFO

Dataset link: https://codeocean.com/capsule/7071558/tree/v1

Keywords:
Spectral problems in fluid mechanics
Guaranteed lower bounds
Positive semi-definite bilinear forms
Nonconforming finite elements

ABSTRACT

In this paper, we obtain guaranteed lower bounds for eigenvalues of two spectral problems arising in fluid mechanics by using the min–max principles of weak form that can be derived by the principles of operator forms. These two problems are the Laplace model for fluid–solid vibrations and the sloshing problem. We deal with the positive semi-definiteness of the associated bilinear forms by adding some constraints to the solution and finite element spaces. Numerical experiments are reported finally to validate our theoretical results.

1. Introduction

Eigenvalue problems play an important role in both natural and engineering sciences. It is of importance to obtain explicit eigenvalue bounds. As we all know, upper eigenvalue bounds are easy to compute by using conforming finite elements. Therefore, we focus on finding guaranteed lower bounds for eigenvalues.

Up to now, there have been lots of work on computing asymptotic or guaranteed lower bounds for eigenvalues by using finite element methods. In 2004, Armentano and Durán gave an identity and proved that the nonconforming Crouzeix–Raviart (CR) finite element produces asymptotic lower bounds for eigenvalues when the exact eigenfunctions are singular and the mesh sizes are small enough (see [1]). Based on the work, using nonconforming finite elements, researchers developed asymptotic lower bounds of eigenvalues in depth (e.g., see [2–10] and therein). The validity of theoretical results of asymptotic lower bounds relies on the condition that the mesh size is sufficiently small. However, there is no existing work to show how to verify the precondition. Recently, finding guaranteed lower bounds has become an attractive topic, which has no constraint of mesh size being small enough. In 2013, Liu et al. obtained guaranteed lower eigenvalue bounds for the Laplacian by using linear conforming finite element (see [11]). In 2014, Carstensen et al. obtained guaranteed lower bounds of eigenvalue for the Laplacian by using nonconforming CR finite element (see [12]) and for biharmonic equation by using nonconforming Morley finite element (see [13]). Liu et al. in [14] proposed a general framework to find guaranteed lower bounds of eigenvalue for the problem $a(u,v) = \lambda b(u,v)$, where both $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are symmetric and positive definite bilinear forms. Later, You et al. extended the framework to more general variational eigenvalue problems in which $a(\cdot,\cdot)$ is positive definite in $H^1(\Omega)$ and $b(\cdot,\cdot)$ is a positive semi-definite (see [15]). Other work on guaranteed lower bounds for eigenvalues can be found in [16–19].

In this paper, we consider two spectral problems arising in fluid mechanics. The first one is known as the Laplace model for fluid–solid vibrations and is important in nuclear engineering, which corresponds to approximating the vibrations of a bundle of tubes immersed in a fluid contained in a rigid cavity. Plentiful literature studies this problem (see e.g. [20–23] and references cited therein). The second one belongs to sloshing problems, which corresponds to the computation of the sloshing modes of fluid contained in a domain Ω with a horizontal free surface Γ_0. It has been studied in [24–27] and references cited therein. As far as we know, it has not been reported in existing literatures on finding guaranteed lower eigenvalue bounds for the two problems. Based on the arguments in [12,14,15], we successfully obtain guaranteed lower bounds for eigenvalues along with the CR and the enriched Crouzeix–Raviart (ECR) finite elements (see detail for Theorem 3.2 and 3.4). All four bilinear forms associated with the two problems are positive semi-definite in $H^1(\Omega)$. Expression $\sqrt{a(v,v)}$ is a semi-norm but not a norm in $H^1(\Omega)$. In order to deal with this problem and apply the framework of [15] (in which the min and min–max principles of weak form in the case of coercive bilinear form $a(\cdot,\cdot)$ and positive semi-definite $b(\cdot,\cdot)$ are used), we add some constraints to $H^1(\Omega)$ and finite

☆ Projects supported by the National Natural Science Foundation of China (Grant No. 11561014), the project of introduced talents for scientific research of Guizhou University of Finance and Economics, China (No. 2020YJ015), the Scientific Research Foundation of Guizhou University of Finance and Economics (No. 2020XYB10).
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https://doi.org/10.1016/j.camwa.2021.03.005
Received 25 August 2020; Received in revised form 20 January 2021; Accepted 2 March 2021
Available online 30 March 2021
0898-1221/

element space. The selection method for solution space is different from the method given in [23], which is to avoid defining and discussing the quotient space of piecewise Sobolev space.

As for the basic theory of finite element and spectral approximation, we refer to [28–31]. In this paper, C denotes a constant independent on mesh size, which may stand for different values at its different occurrences.

2. Lower eigenvalues bounds of variational eigenvalue problem

This section briefly overviews the theory from [14,15] how to compute guaranteed lower bounds of eigenvalues of an abstract eigenvalue problem given by bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ on a Hilbert space V.

Let V be a Hilbert space with inner product $a(\cdot,\cdot)$ and norm $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$, V_h be a finite dimensional space which may not be a subspace of V. Let $V(h) = V + V_h$ be a Hilbert space with inner product $a_h(\cdot,\cdot)$ and norm $\|\cdot\|_h = \sqrt{a_h(\cdot,\cdot)}$. Let us assume that bilinear form $a_h(\cdot,\cdot)$ is extension of $a(\cdot,\cdot)$ to $V(h)$ such that $a_h(u,v) = a(u,v)$ and $\|u\|_a = \|u\|_h$, for any $u, v \in V$. Let $b(\cdot,\cdot)$ be a symmetric, continuous and positive semi-definite bilinear form on $V(h)$, including the semi-norm $|\cdot|_b = \sqrt{b(\cdot,\cdot)}$. Then the following three problems are considered.

The weak form of one eigenvalue problem is to find $(\lambda, u) \in \mathbb{R}\times V$ with $|u|_b \neq 0$ such that

$$a(u,v) = \lambda b(u,v), \quad \forall\, v \in\ V. \tag{2.1}$$

The discrete form of (2.1) is to find $(\lambda_h, u_h) \in \mathbb{R}\times V_h$ with $|u_h|_b \neq 0$ such that

$$a_h(u_h, v_h) = \lambda_h b(u_h, v_h), \quad \forall\, v_h \in\ V_h. \tag{2.2}$$

The eigenvalue problem in $V(h)$ is to find $\widetilde{\lambda} \in \mathbb{R}$, $\widetilde{u} \in V(h)$ with $|\widetilde{u}|_b \neq 0$ such that

$$a_h(\widetilde{u}, v) = \widetilde{\lambda} b(\widetilde{u}, v), \quad \forall v \in V(h). \tag{2.3}$$

The following assumption is needed to guarantee well-posedness of problems (2.1)–(2.3), namely to guarantee existence of (at most) countably many positive eigenvalues of these problems.

(A1) Seminorm $|\cdot|_b$ is compact in V with respect to $\|\cdot\|_a$, i.e., every sequence of V bounded under $\|\cdot\|_a$ has a subsequence that is Cauchy under $|\cdot|_b$.

Actually, it can be shown that $|\cdot|_b$ is compact both in V_h and $V(h)$ with respect to $\|\cdot\|_h$. The former holds true because V_h is finite dimensional. The latter can be deduced by applying assumption (A1) and the fact that V_h is finite dimensional (see e.g. Theorem 1 and Remark 1 in [32]).

Let $P_h : V(h) \to V_h$ be the projection with respect to $a_h(\cdot,\cdot)$. For given $u \in V(h)$, $P_h u \in V_h$ satisfies

$$a_h(u - P_h u, v_h) = 0, \quad \forall\, v_h \in V_h.$$

An important assumption for guaranteed lower bounds is as follows:

(A2) For all $u \in V$, there exists a positive constant C_h such that

$$|u - P_h u|_b \le C_h \|u - P_h u\|_h. \tag{2.4}$$

In Theorem 2.4 of [14], Liu et al. drew guaranteed lower bounds for eigenvalues without the separation condition, which is required in [12,13]. The following lemma comes from [14,15], the difference is that we change the compactness of $|\cdot|_b$ in $V(h)$ to that in V. The lemma provides guaranteed lower bounds for eigenvalues.

Lemma 2.1. *Assume that (A1) and (A2) hold. Let λ_k and $\lambda_{k,h}$ be the kth eigenvalues of (2.1) and (2.2), respectively. Then there holds*

$$\lambda_k \ge \frac{\lambda_{k,h}}{1 + C_h^2 \lambda_{k,h}}. \tag{2.5}$$

Proof. *See the proofs of Theorem 2.1 of Ref. [14] and Theorem 2.4 of Ref. [15], in which eigenvalue problem (2.3) is used.* □

Next, based on the theoretical framework of the section, we aim to obtain guaranteed lower bounds of eigenvalues for two spectral problems arising in fluid mechanics.

3. The Laplace model for fluid–solid vibrations

We consider a bounded polygonal domain $\Omega \subset R^2$ occupied by a fluid. Symbol Γ_0 denotes outer boundary of Ω. There are $K > 0$ tubes immersed in the fluid. Cross sections of these tubes are represented as polygonal subdomains in Ω and interfaces between tubes and the fluid are denoted by Γ_i, $i = 1,\dots,K$. Each tube is modeled as a harmonic oscillator with rigidity k and mass m and the fluid is taken as perfectly incompressible with density ρ. Symbol $\mathbf{n}(x) = (n_1, n_2)$ denotes the unit outer normal vector to the boundary of Ω.

The corresponding eigenvalue problem is known as Laplace model for fluid–solid vibrations (e.g., see [23]), that is to find eigenvalue $\lambda \in \mathbb{R}$ and eigenfunction $u \neq 0$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \mathbf{n}}(x) = \lambda \left(\int_{\Gamma_i} u(y)\mathbf{n}(y)dy \right) \cdot \mathbf{n}(x) & \text{for } x \in \Gamma_i,\ i = 1,\dots,K, \end{cases} \tag{3.1}$$

where $\lambda := \rho\omega^2 / \left(k - m\omega^2\right)$, $\omega > 0$ (the vibration frequency) and u is the fluid pressure. The boundary condition on Γ_i means that the fluid touches the tube wall (the normal is oriented towards the interior of Γ_i). Eigenfunctions u of the problem corresponding to λ are determined up to an additive constant.

Let $H^1(\Omega)$ be the first order Sobolev function space that the function with the first order generalized derivative square integrable over Ω. Denote by $\|\cdot\|_{H^1(\Omega)}$ the norm of $H^1(\Omega)$. Notations $L^2(\Omega)$ and $L^2(\partial\Omega)$ denote the sets of real square integrable functions over Ω and $\partial\Omega$, respectively. The corresponding norms are $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\partial\Omega)}$, respectively.

Bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined as follows.

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \quad \text{and} \quad b(u,v) := \sum_{i=1}^{K} \left(\int_{\Gamma_i} u\mathbf{n} \right) \cdot \left(\int_{\Gamma_i} v\mathbf{n} \right). \tag{3.2}$$

Bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ have nontrivial kernels $\ker_a = \{v \in H^1(\Omega) : a(v,v) = 0\} = \mathbb{R}$ and $\ker_b = \{v \in H^1(\Omega) : \int_{\Gamma_i} v\mathbf{n} = 0, \forall i = 1,2,\dots,K\}$, respectively. The nontrivial $\ker_a$ means that $a(\cdot,\cdot)$ is not equivalent to the inner product in $H^1(\Omega)$, because it is not coercive. Therefore, a usual and natural choice for solution space of (3.1) is quotient space $H^1(\Omega)/\mathbb{R}$ (e.g., see [23]). Note that the general theory (see e.g. [15]) is well suited for bilinear forms b with nontrivial kernels.

Then the weak form of (3.1) is to find $(\lambda, u) \in \mathbb{R}\times H^1(\Omega)/\mathbb{R}$ with $|u|_b \neq 0$ such that

$$a(u,v) = \lambda b(u,v), \quad \forall\, v \in\ H^1(\Omega)/\mathbb{R}. \tag{3.3}$$

The solution of problem (3.3) is given by a sequence of exactly $2K$ eigenpairs (λ_j, u_j), with positive eigenvalues that we assume to be increasingly ordered: $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{2K}$ (see Section 2.2.1 in [33]).

However the space $V(h) = H^1(\Omega)/\mathbb{R} + V_h$ will be involved in the analysis of this paper, which means the quotient space of piecewise Sobolev space needs to be defined and discussed. We have not seen any report about such a definition. Therefore, in this paper, in order to get guaranteed lower bounds for eigenvalues of (3.1) by applying the theoretical results in Section 2, we make other constraints to the elements of space $H^1(\Omega)$ and take solution space

$$V := \left\{ v \in H^1(\Omega), \sum_{i=1}^{K} \int_{\Gamma_i} v = 0 \right\}.$$

Then the weak form of (3.1) is to find $(\lambda, u) \in \mathbb{R}\times V$ with $|u|_b \neq 0$ such that

$$a(u,v) = \lambda b(u,v), \quad \forall\, v \in\ V. \tag{3.4}$$

Note that $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$ and $|\cdot|_b = \sqrt{b(\cdot,\cdot)}$ are just semi-norms in $H^1(\Omega)$, while $\|\cdot\|_a$ is a norm in V. Thanks to the norm equivalence theorem (see inequality (14.11) in [34]), we know that norms $\|\cdot\|_a$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent in V, that is

$$\|v\|_a \le \|v\|_{H^1(\Omega)} \le C\|v\|_a, \quad \text{for all } v \in V. \tag{3.5}$$

Next, we will prove that the natural space $H^1(\Omega)/\mathbb{R}$ can be replaced by the smaller space V without loosing any eigenfunction. For preparation, the following lemma is proved.

Lemma 3.1. *Assume that Ω_0 is a lipschitz domain. Let $\mathbf{n}$ denote unit inner normal vector to $\partial\Omega_0$. There holds*

$$\int_{\partial\Omega_0} \mathbf{n} = \mathbf{0}, \tag{3.6}$$

where $\mathbf{0}$ is a null vector.

Proof. The conclusion follows from the divergence theorem $\int_{\Omega_0} \operatorname{div} \mathbf{y} = \int_{\partial\Omega_0} \mathbf{y}\cdot\mathbf{n}$ with $\mathbf{y} = (1,0)$ and $\mathbf{y} = (0,1)$. This ends the proof. □

The following lemma tells us that eigenvalues of (3.3) are equal to those (3.4). For simplicity of expression, we denote $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_K$ and $|\Gamma| = \sum_{i=1}^K |\Gamma_i|$.

Lemma 3.2. *Assume $(\widetilde{\lambda}, \widetilde{u})$ is a non-zero eigenpair of (3.3). Then $(\lambda, u) = \left(\widetilde{\lambda}, \widetilde{u} - \frac{1}{|\Gamma|}\int_\Gamma \widetilde{u}\right)$ is an eigenpair of (3.4).*

Proof. From $u = \widetilde{u} - \frac{1}{|\Gamma|}\int_\Gamma \widetilde{u}$, we obtain $\int_\Gamma u = 0$. Hence, $u \in V$. For all $v \in V$, submitting $\lambda = \widetilde{\lambda}$ and $u = \widetilde{u} - \int_\Gamma \widetilde{u}$ into (3.4), we have

$$a(u,v) = \int_\Omega \nabla\widetilde{u}\cdot\nabla v = a(\widetilde{u}, v).$$

From Lemma 3.1 we know that $\int_{\Gamma_i} \mathbf{n}$ is a null vector, then

$$b(u,v) = \sum_{i=1}^K \left(\int_{\Gamma_i}\left(\widetilde{u} - \frac{1}{|\Gamma|}\int_{\Gamma_i}\widetilde{u}\right)\mathbf{n}\right)\cdot\left(\int_{\Gamma_i} v\mathbf{n}\right) = b\left(\widetilde{u}, v\right).$$

From (3.3) we know that (λ, u) is the solution of (3.4). □

Let $\pi_h = \{\kappa\}$ be a triangular subdivision of Ω such that every triangle κ has at most one edge on the boundary. The mesh diameter is defined as $h = \max\{h_\kappa\}$, where h_κ is the diameter of element κ. We denote by $|\kappa|$ the measure of κ and $|\Gamma_i|$ the measure of Γ_i. Let $\varepsilon_h(\Gamma_i) = \{e\}$ denote the set of all edges on boundary Γ_i of π_h, and $\varepsilon_h(\Gamma) = \varepsilon_h(\Gamma_1)\cup\varepsilon_h(\Gamma_2)\cup\cdots\cup\varepsilon_h(\Gamma_K)$. Let $\varepsilon_h(\Omega)$ denote the set of all interior edges of π_h, ε_h denote the set of all edges of π_h and π_h^Γ be the set of elements of π_h having an edge $e \in \varepsilon_h(\Gamma)$. Symbols $\kappa^+, \kappa^- \in \pi_h$ stand for adjacent elements having a common edge $e = \kappa^+ \cap \kappa^-$. Let $[\cdot]$ be jumps of piecewise functions over e, namely

$$[v] := v|_{\kappa^+} - v|_{\kappa^-}.$$

We take V_h in Section 2 as the following CR and ECR finite element spaces:

• The CR finite element space, proposed by Crouzeix and Raviart [35]

$$V_h = \left\{ v \in L^2(\Omega) : v|_\kappa \in P_1(\kappa), \forall\kappa\in\pi_h, \right.$$
$$\left. \int_e [v] = 0, \forall e \in \varepsilon_h(\Omega), \text{and } \sum_{i=1}^K \int_{\Gamma_i} v = 0 \right\}.$$

• The ECR finite element space, proposed by Lin and Hu et al. (see [8,36])

$$V_h = \left\{ v \in L^2(\Omega) : v|_\kappa \in \mathrm{ECR}(\kappa), \forall\kappa\in\pi_h, \right.$$
$$\left. \int_e [v] = 0, \forall e \in \varepsilon_h(\Omega), \text{and } \sum_{i=1}^K \int_{\Gamma_i} v = 0 \right\},$$

where

$$\mathrm{ECR}(\kappa) := P_1(\kappa) + \operatorname{span}\left\{\sum_{i=1}^2 x_i^2\right\}, \text{ for any } \kappa \in \pi_h.$$

We take $V(h) := V + V_h$. Then $a_h(\cdot,\cdot)$ is defined by

$$a_h(u_h, v_h) := \sum_{\kappa\in\pi_h}\int_\kappa \nabla u_h\cdot\nabla v_h, \quad \forall\, u_h, v_h \in V(h). \tag{3.7}$$

The discrete form of (3.4) is to find $(\lambda_h, u_h) \in \mathbb{R}\times V_h$ with $|u_h|_b \ne 0$ such that

$$a_h(u_h, v_h) = \lambda_h b(u_h, v_h), \quad \forall v_h \in V_h. \tag{3.8}$$

The theory in Section 2.2.1 of [33] allows proving that discrete problem (3.8) attains $2K$ positive eigenvalues.

The norm in $V(h)$ is denoted by $\|\cdot\|_h$, which is given by

$$\|v\|_h = \left(\sum_{\kappa\in\pi_h}\int_\kappa |\nabla v|^2\right)^{1/2}, \quad \text{for any } v \in V(h), \tag{3.9}$$

and the semi-norm $|\cdot|_b$ is given by

$$\begin{aligned} |v|_b^2 &= \sum_{i=1}^K \left(\sum_{e\in\varepsilon_h(\Gamma_i)}\int_e v\mathbf{n}\right)\cdot\left(\sum_{e\in\varepsilon_h(\Gamma_i)}\int_e v\mathbf{n}\right) \\ &= \sum_{i=1}^K\left\{\left(\sum_{e\in\varepsilon_h(\Gamma_i)}\int_e vn_1\right)^2 + \left(\sum_{e\in\varepsilon_h(\Gamma_i)}\int_e vn_2\right)^2\right\}. \end{aligned}$$

For different edge e, $\mathbf{n} = (n_1, n_2)$ may not be same.

Since $a_h(\cdot,\cdot)$ is symmetric and positive definite in $V(h)$, it is an inner product of $V(h)$. From the strengthened Cauchy–Schwarz inequality (see Theorem 1 and Remark 1 in [32]), for any Cauchy sequence $\{\tilde{v}_i\}$ in $V(h)$ which has unique splitting $\tilde{v}_i = v_i + v_{i,h}$ with $v_i \in V$ and $v_{i,h} \in V_h$, it can be shown that $\|v_{i,h}\|_h$ are bounded. Hence, there exists a subsequence $\{v_{i_k,h}\}$ of $\{v_{i,h}\}$ that converges to $v_h \in V_h$. Further, the subsequence $\{v_{i_k}\}$ of $\{v_i\}$ is also Cauchy in V and converges to $v \in V$. Then the subsequence $\{v_{i_k,h} + v_{i_k}\}$ of $\tilde{v}_i$ converges to $v + v_h \in V(h)$. It follows from the uniqueness of limit that the Cauchy sequence $\tilde{v}_i$ converges to $v + v_h \in V(h)$. Thus $V(h)$ is a Hilbert space.

Now we will prove that norm $\|v\|_{1,h} = \left(\|v\|_h^2 + \|v\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}$ is equivalent to norm $\|v\|_h$ in $V(h)$, which guarantees that $a_h(\cdot,\cdot)$ is coercive in $V(h)$.

Lemma 3.3. *For any $v \in V(h)$, the norms $\|v\|_h$ and $\|v\|_{1,h}$ are equivalent.*

Proof. According to the definitions of $\|v\|_h$ and $\|v\|_{1,h}$, it is immediate that $\|v\|_h \le \|v\|_{1,h}$. Then we prove that, for any $v \in V(h)$, there exists a constant C such that

$$\|v\|_{1,h} \le C\|v\|_h. \tag{3.10}$$

If inequality (3.10) is false, then there exists a sequence $\{v_i\} \subset V(h)$ such that

$$\|v_j\|_{1,h} = 1, \text{ for all } j \ge 1, \lim_{j\to\infty}\|v_j\|_h = 0. \tag{3.11}$$

On one hand, since the sequence $\{v_j\}$ is bounded in $V(h)$, there exists a subsequence, still denoted $\{v_j\}$ for notational convenience, that converges in $L^2(\Omega)$ (see [37]). From (3.11), $\lim_{j\to\infty}\|v_j\|_h = 0$, therefore the sequence $\{v_j\}$ converges in $V(h)$ in the sense of norm $\|\cdot\|_{1,h}$. Since $V(h)$ is complete, the limit w of $\{v_j\}$ belongs to $V(h)$ and satisfies

$$\|w\|_{1,h} = \lim_{j\to\infty}\|v_j\|_{1,h} = 1. \tag{3.12}$$

On the other hand, from (3.11), we know that $\|w\|_h = \lim_{j\to\infty}\|v_j\|_h = 0$. Then w is a constant. Noticing that $\sum_{i=1}^K \int_{\Gamma_i} w = w\sum_{i=1}^K |\Gamma_i| = 0$ in V_h, we get $w = 0$. Then $\|w\|_{1,h} = 0$, which contradicts the result of (3.12). It implies that (3.10) holds. The proof is completed. □

Next we will verify that the settings associated with problem (3.1) satisfy the conditions in previous section. Note that $a_h(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are symmetric bilinear forms in $V(h)$. The former is elliptic and continuous and the latter positive semi-definite in $V(h)$. Now we verify the continuity of $b(\cdot,\cdot)$.

From Schwarz's inequality, for any $f \in V(h)$, we have

$$
\begin{aligned}
|f|_b^2 \le \sum_{i=1}^{K} &\left(\left(\sum_{e\in\varepsilon_h(\Gamma_i)} |n_1|\sqrt{|e|}\left(\int_e f^2\right)^{\frac{1}{2}}\right)^2\right.\\
&\left.+\left(\sum_{e\in\varepsilon_h(\Gamma_i)} |n_2|\sqrt{|e|}\left(\int_e f^2\right)^{\frac{1}{2}}\right)^2\right)\\
\le& \max_{i=1,2,\ldots,K}\left\{|\Gamma_i|\right\}\sum_{e\in\varepsilon_h(\Gamma)}\int_e f^2 = \max_{i=1,2,\ldots,K}\left\{|\Gamma_i|\right\}\sum_{e\in\varepsilon_h(\Gamma)}\|f\|^2_{L^2(e)}.
\end{aligned}
\tag{3.13}
$$

Then by Schwarz's inequality and trace inequality, we deduce

$$
\begin{aligned}
|b(u,v)| =& \left|\sum_{i=1}^{K}\left(\left(\sum_{e\in\varepsilon_h(\Gamma_i)}\int_e un_1\right)\left(\sum_{e\in\varepsilon_h(\Gamma_i)}\int_e vn_1\right)\right.\right.\\
&\left.\left.+\left(\sum_{e\in\varepsilon_h(\Gamma_i)}\int_e un_2\right)\left(\sum_{e\in\varepsilon_h(\Gamma_i)}\int_e vn_2\right)\right)\right|\\
\le& |u|_b|v|_b \le C\|u\|_h\|v\|_h, \qquad \forall u,v\in V(h),
\end{aligned}
$$

thus $b(\cdot,\cdot)$ is continuous.

Consider a sequence in V bounded in $\|\cdot\|_a$. By (3.5), it is bounded in $\|\cdot\|_{H^1(\Omega)}$ as well. Theorem 10.2.2 in [38] says that every sequence in V bounded in $\|\cdot\|_{H^1(\Omega)}$ has a subsequence that is Cauchy in $\|\cdot\|_{L^2(\partial\Omega)}$. By (3.13) this subsequence is Cauchy in $|\cdot|_b$. That is Assumption (A1) is satisfied by the above settings.

The construction of space V_h is complex. In practical computation, we can simplify it. Define common CR and ECR finite element spaces as follows.

• The common CR finite element space:

$$
\widetilde{V}_h = \left\{ v\in L^2(\Omega):\ v|_\kappa \in P_1(\kappa), \forall\kappa\in\pi_h, \int_e [v]=0, \forall e\in\varepsilon_h(\Omega)\right\}.
$$

• The common ECR finite element space:

$$
\widetilde{V}_h = \left\{ v\in L^2(\Omega):\ v|_\kappa \in \mathrm{ECR}(\kappa), \forall\kappa\in\pi_h, \int_e [v]=0, \forall e\in\varepsilon_h(\Omega)\right\}.
$$

Define the eigenvalue problem in $\widetilde{V}_h$ as follows: Find $\left(\widetilde{\lambda}_h, \widetilde{u}_h\right) \in \mathbb{R}\times\widetilde{V}_h$ such that

$$
a_h\left(\widetilde{u}_h, v_h\right) = \widetilde{\lambda}_h b\left(\widetilde{u}_h, v_h\right), \quad \forall v_h\in\widetilde{V}_h. \tag{3.14}
$$

In contrast to (3.8) problem (3.14) has a zero eigenvalue corresponding to the constant eigenfunction.

Using similar proof to Lemma 3.2, we can conclude the following lemma.

Lemma 3.4. *Let $(\widetilde{\lambda}_h,\widetilde{u}_h)$ be a non-zero eigenpair of (3.14). Then $\left(\lambda_h,u_h\right) = \left(\widetilde{\lambda}_h, \widetilde{u}_h - \frac{1}{|\Gamma|}\int_\Gamma \widetilde{u}_h\right)$ is an eigenpair of (3.8).*

Remark 3.1. According to Lemma 3.4, we know that the construction of space V_h can be simplified as the construction of space $\widetilde{V}_h$ in practical computation. That is to say, we just need to compute eigenvalue problem (3.14) rather than eigenvalue problem (3.8).

Referring to the definition of projection operator P_h provided in Section 2, we define the projection operator $P_h : V(h) \to V_h$ with respect to $a_h(\cdot,\cdot)$.

Next, our goal is to evaluate the constant C_h that satisfies (2.4). For preparation of evaluating C_h, we introduce interpolation operators $I_h : V(h) \to V_h$ as follows:

• For CR finite element, the interpolation operator I_h is defined by

$$
\int_e I_h u = \int_e u; \quad \forall e\in\varepsilon_h.
$$

• For ECR finite element, the interpolation operator I_h is defined by

$$
\int_e I_h u = \int_e u; \int_\kappa I_h u = \int_\kappa u, \quad \forall e\in\varepsilon_h, \kappa\in\pi_h.
$$

From Section 5 in [36] and the equality (7.4) in [8], for each element $\kappa\in\pi_h$, there holds

$$
\int_\kappa \nabla\left(u - I_h u\right)\cdot\nabla v_h = 0, \qquad \forall v_h\in V_h. \tag{3.15}
$$

In addition, we give error estimates for I_h, which will be used to obtain explicit value for C_h in (2.4).

Denote the altitude of triangle κ with respect to edge e by H_κ. From Theorem 3.2 in [15] and Theorem 3 in [19], we have the following lemma.

Lemma 3.5. *For a given element κ, the following error estimate holds for any $u\in H^1(\kappa)$:*

$$
\|u - I_h u\|_{0,e} \le \beta\frac{h_\kappa}{\sqrt{H_\kappa}}\|u - I_h u\|_{h,\kappa}, \tag{3.16}
$$

where

• $\beta = 0.6711$, *for CR finite element;*
• $\beta = 0.5852$, *for ECR finite element.*

Theorem 3.1. *Given $u\in V$, there holds the following error estimate:*

$$
|u - I_h u|_b \le \beta\sqrt{\max_{i=1,2,\ldots,K}\left\{|\Gamma_i|\right\}}\cdot\max_{\kappa\in\pi_h^\Gamma}\left\{\frac{h_\kappa}{\sqrt{H_\kappa}}\right\}\|u - I_h u\|_h, \tag{3.17}
$$

where H_κ stands for the altitude of κ with respect to the edge lying on $\Gamma_i (i = 1,2,\ldots,K)$.

Proof. Using (3.13) with $f = u - I_h u$, we conclude

$$
|u - I_h u|_b^2 \le \max_{i=1,2,\ldots,K}\left\{|\Gamma_i|\right\}\sum_{e\in\varepsilon_h(\Gamma)}\|u - I_h u\|^2_{0,e}.
$$

Noticing that all elements κ of π_h have at most one edge on the boundary and using (3.16), we deduce

$$
|u - I_h u|_b^2 \le \max_{i=1,2,\ldots,K}\left\{|\Gamma_i|\right\}\sum_{\kappa\in\pi_h^\Gamma}\beta^2\frac{h_\kappa^2}{H_\kappa}\|u - I_h u\|^2_{h,\kappa}.
$$

Then (3.17) is obtained. □

The following theorem provides guaranteed lower bounds for eigenvalue λ_k of problem (3.1).

Theorem 3.2. *Let λ_k and $\lambda_{k,h}$ be the kth eigenvalues of (3.4) and (3.8), respectively. Then we have the following lower bound for eigenvalue:*

$$
\lambda_k \ge \frac{\lambda_{k,h}}{1 + C_h^2\lambda_{k,h}}, \tag{3.18}
$$

where $C_h = \beta\sqrt{\max_{i=1,2,\ldots,K}\left\{|\Gamma_i|\right\}}\cdot\max_{\kappa\in\pi_h^\Gamma}\left\{\frac{h_\kappa}{\sqrt{H_\kappa}}\right\}$.

Proof. From (3.15) and the definition of P_h, in fact the interpolation operator I_h is a projection operator. Thus, we take $P_h = I_h$ and from Theorem 3.1 we find the constant $C_h = \beta\sqrt{\max_{i=1,2,\ldots,K}\left\{|\Gamma_i|\right\}}\cdot\max_{\kappa\in\pi_h^\Gamma}\left\{\frac{h_\kappa}{\sqrt{H_\kappa}}\right\}$ satisfying

$$
|u - P_h u|_b \le C_h\|u - P_h u\|_h, \quad \text{for all } u\in V.
$$

Then conclusion (3.18) follows from Lemma 2.1. The proof is completed. □

4. Sloshing mode of fluid

In this subsection, the following eigenvalue problem is considered. Find eigenvalue $\lambda \in \mathbb{R}$ and eigenfunction $u \neq 0$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_1, \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^2$ is the bounded polygonal domain. Symbols Γ_0 and Γ_1 stand for disjoint open subsets of $\partial\Omega = \Gamma$ such that $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$ and $|\Gamma_0| \neq 0$.

For applying the results in Section 2, we take Hilbert space

$$V := \left\{ v \in H^1(\Omega), \int_{\Gamma_0} v = 0 \right\}.$$

Define

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \quad \text{and} \quad b(u, v) := \int_{\Gamma_0} uv. \tag{4.2}$$

Then the weak form of (4.1) is to find $(\lambda, u) \in \mathbb{R} \times V$ with $|u|_b \neq 0$ such that

$$a(u, v) = \lambda b(u, v), \forall v \in V. \tag{4.3}$$

For the sloshing problem, we take nonconforming finite element space V_h as follows:

• The CR finite element space

$$V_h = \left\{ v \in L^2(\Omega) : v|_\kappa \in P_1(\kappa), \forall \kappa \in \pi_h, \right.$$
$$\left. \int_e [v] = 0, \forall e \in \varepsilon_h(\Omega), \text{ and } \int_{\Gamma_0} v = 0 \right\}.$$

• The ECR finite element space

$$V_h = \left\{ v \in L^2(\Omega) : v|_\kappa \in \mathrm{ECR}(\kappa), \forall \kappa \in \pi_h, \right.$$
$$\left. \int_e [v] = 0, \forall e \in \varepsilon_h(\Omega), \text{ and } \int_{\Gamma_0} v = 0 \right\}.$$

As in the previous section, we set $V(h) = V + V_h$ and define bilinear form $a_h(\cdot, \cdot)$ and the norm $\|\cdot\|_h$ by (3.7) and (3.9).

Then the discrete form of (4.3) is to find $(\lambda_h, u_h) \in \mathbb{R} \times V_h$ with $|u_h|_b \neq 0$ such that

$$a_h(u_h, v_h) = \lambda_h b(u_h, v_h), \quad \forall v_h \in V_h. \tag{4.4}$$

Using almost the same arguments as in the previous section, we know that the settings associated with the eigenvalue problem satisfy assumption (A1).

Remark 4.1. We can simplify the construction of space V_h by computing the following eigenvalue problem over $\widetilde{V}_h$ that is defined in previous subsection. Find $(\widetilde{\lambda}_h, \widetilde{u}_h) \in \mathbb{R} \times \widetilde{V}_h$ such that

$$a_h(\widetilde{u}_h, v_h) = \widetilde{\lambda}_h b(\widetilde{u}_h, v_h), \quad \forall v_h \in \widetilde{V}_h. \tag{4.5}$$

It is obvious that eigenpairs of (4.4) are also eigenpairs of (4.5). Now, we prove the opposite, namely that nonzero eigenpairs of (4.5) are eigenpairs of (4.4). Since that (4.5) is valid for any $v_h \in \widetilde{V}_h$, we select v_h as a nonzero constant C. Then we deduce that

$$0 = \widetilde{\lambda}_h C \int_{\Gamma_0} \widetilde{u}_h.$$

Hence, we have $\int_{\Gamma_0} \widetilde{u}_h = 0$, that is $\widetilde{u}_h \in V_h$.

Using similar arguments to Theorem 3.1 and 3.2 in Section 3, we prove the following theorems.

Theorem 4.1. *Given $u \in V$, there holds the following error estimate:*

$$|u - I_h u|_b \leq \beta \max_{\kappa \in \pi_h^{\Gamma_0}} \left\{ \frac{h_\kappa}{\sqrt{H_\kappa}} \right\} \|u - I_h u\|_h. \tag{4.6}$$

Proof. Noticing that all elements κ of π_h have at most one edge on the boundary and using (3.16), we deduce

$$|u - I_h u|_b^2 = \sum_{e \in \varepsilon_h(\Gamma_0)} \|u - I_h u\|_{0,e}^2 \leq \sum_{\kappa \in \pi_h^{\Gamma_0}} \beta^2 \frac{h_\kappa^2}{H_\kappa} \|u - I_h u\|_{h,\kappa}^2.$$

Then (4.6) is immediately obtained. □

Theorem 4.2. *Let λ_k and $\lambda_{k,h}$ be the kth eigenvalues of (4.3) and (4.4), respectively. Then we have the following lower bound for eigenvalue:*

$$\lambda_k \geq \frac{\lambda_{k,h}}{1 + C_h^2 \lambda_{k,h}}, \tag{4.7}$$

where $C_h = \beta \max_{\kappa \in \pi_h^{\Gamma_0}} \left\{ \frac{h_\kappa}{\sqrt{H_\kappa}} \right\}$.

Proof. Due to (3.15) and the definition of P_h, the interpolate operator I_h is a projection operator. Thus, we take $P_h = I_h$ and from Theorem 4.1 we find the constant $C_h = \beta \max_{\kappa \in \pi_h^{\Gamma_0}} \left\{ \frac{h_\kappa}{\sqrt{H_\kappa}} \right\}$ satisfying

$$|u - P_h u|_b \leq C_h \|u - P_h u\|_h, \text{for all } u \in V.$$

The conclusion follows from Lemma 2.1. The proof is completed. □

Remark 4.2. The common Steklov type eigenvalue problem: find eigenvalue $\lambda \in \mathbb{R}$ and eigenfunction $u \neq 0$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u & \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

is a special case of (4.1). Thus, all results derived in this section also hold true for (4.8).

5. Numerical experiments

In this section, numerical results illustrating the theoretical analysis are presented. For the Laplace model for fluid–solid vibrations, guaranteed lower eigenvalue bounds are obtained by applying Theorem 3.2. For the sloshing mode of fluid, they are obtained by Theorem 4.2. The discrete eigenvalue problems are solved in MATLAB 2018b on an Lenovo IdeaPad PC with 1.8 GHZ CPU and 8 GB RAM. Our program is compiled under the package of iFEM [39]. The following notations are adopted in tables.

h : The mesh diameter.

C_h : The interpolation constant.

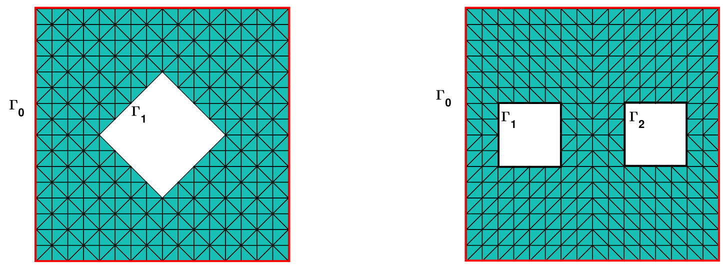
λ_j : The jth exact eigenvalue.

$\lambda_{j,h}$: The jth discrete eigenvalue.

$\lambda_{j,h}^G$: The guaranteed lower bounds of λ_j.

5.1. Results on the Laplace model for fluid–solid vibrations

For the Laplace model for fluid–solid vibrations, we consider two examples. For the first one, we consider a rhomboidal tube with side length $2\sqrt{2}$ centered in a square cavity of side length 8 (as shown in Fig. 1(a)). In this case, the problem has one eigenvalue with multiplicity two, which we list in tables below. The numerical results by CR and ECR finite elements are listed in Tables 1 and 2, respectively. The reference value $\lambda_1 = 0.07896$ is provided in Section 6 of [23].



(a) Square cavity with a rhomboidal tube (b) Square cavity with two square tubes

Fig. 1. Domains and meshes with diameter $h = \frac{\sqrt{2}}{2}$.

Table 1
Guaranteed lower eigenvalue bounds on the square cavity with a rhomboidal tube: CR finite element.

h	C_h	$\lambda_{1,h}$	$\lambda^G_{1,h}$
$\frac{\sqrt{2}}{4}$	1.8982	0.07625	0.05982
$\frac{\sqrt{2}}{8}$	1.3422	0.07786	0.06828
$\frac{\sqrt{2}}{16}$	0.9491	0.07852	0.07333
$\frac{\sqrt{2}}{32}$	0.6711	0.07878	0.07608

Table 2
Guaranteed lower eigenvalue bounds on the square cavity with a rhomboidal tube: ECR finite element.

h	C_h	$\lambda_{1,h}$	$\lambda^G_{1,h}$
$\frac{\sqrt{2}}{4}$	1.6552	0.07625	0.06307
$\frac{\sqrt{2}}{8}$	1.1704	0.07786	0.07036
$\frac{\sqrt{2}}{16}$	0.8276	0.07852	0.07451
$\frac{\sqrt{2}}{32}$	0.5852	0.07878	0.07671

From convergence order of eigenvalue, we know that the eigenfunction corresponding to λ_1 is singular.

For the second problem, we consider two square tubes immersed in a fluid occupying a square cavity (as shown in Fig. 1(b)). In this case, the problem has four positive eigenvalues. Only the first two numerical eigenvalues and corresponding guaranteed lower bounds are listed in Table 3 (for CR finite element) and Table 4 (for ECR finite element). With the aim of reducing computational cost, some techniques from subsection 5.1 in [23] are used. According to the convergence order, we know that the eigenfunctions corresponding to λ_1 and λ_2 are singular. From Tables 1–4, we see that the guaranteed lower bounds of eigenvalues are obtained whatever the mesh diameters are. Comparing Table 1 with Table 2 (resp. Table 3 with Table 4), we see that although the lower eigenvalue bounds obtained by the ECR finite element are slightly more accurate than the ones obtained by the CR finite element with the same mesh diameters, they have same number of significant digits. We should also note that the ECR finite element method is more costly because the ECR finite element space contains more degrees of freedom. For example, in Tables 1 and 2, when $h = \frac{\sqrt{2}}{32}$, the number of degree of freedom for the ECR space is 287 359, which is larger than the number of 172 671 for the CR finite element space. It can be seen from the numerical results that the guaranteed lower eigenvalue bound obtained by (3.18) is not optimal when compared with the approximate eigenvalues themselves, which also occurs for the classical Steklov eigenvalue problem (see [15]).

Remark 5.1. It is worth noting that, for problem (3.1), C_h can be 0 since $|u - I_h u|_b = 0$, which means the approximate eigenvalue $\lambda_{j,h}$ obtained by both CR and ECR finite elements is optimal guaranteed lower eigenvalue bounds.

5.2. Results on sloshing mode of a two dimensional fluid contained in Ω

In this subsection, we report a numerical example on problem (4.1) on the unit square $\Omega = [0,1]^2$ with Γ_0 and Γ_1 as shown in Fig. 2. The problem corresponds to the computation of the sloshing modes of a two-dimensional fluid contained in Ω with a horizontal free surface Γ_0. The exact solutions of this problem are $\lambda_n = n\pi \tanh(n\pi)$, where $n \in \mathbb{N}$ (for comparison, we select the reference eigenvalues as $\lambda_1 \approx 3.129881$ and $\lambda_4 \approx 12.566371$, respectively). We only list two numerical eigenvalues $\lambda_{1,h}$, $\lambda_{4,h}$ and corresponding lower bounds $\lambda^G_{1,h}$, $\lambda^G_{4,h}$ in Table 5 (for CR finite element) and Table 6 (for ECR finite element). According to the convergence order, we know that the function corresponding to λ_1 is smooth and the eigenfunction corresponding to λ_4 is singular. From Table 5, on one hand, we see that the asymptotic lower bounds for λ_1 and λ_4 can be obtained as the mesh diameters are sufficiently small, while $\lambda_{4,h}$ is larger than the exact value as mesh diameter is $\frac{\sqrt{2}}{4}$. On the other hand, we see that $\lambda^G_{4,h}$ are lower bounds of the exact eigenvalue whatever the mesh diameters are. Comparing Tables 5 and 6, we obtain same conclusion as the previous problem.

Table 3
Guaranteed lower eigenvalue bounds on the square cavity with two square tubes: CR finite element.

h	C_h	$\lambda_{1,h}$	$\lambda^G_{1,h}$	$\lambda_{2,h}$	$\lambda^G_{2,h}$
$\frac{\sqrt{2}}{4}$	1.3422	0.19730	0.14556	0.17183	0.13121
$\frac{\sqrt{2}}{8}$	0.9491	0.20366	0.17209	0.17756	0.15307
$\frac{\sqrt{2}}{16}$	0.6711	0.20629	0.18875	0.17992	0.16643
$\frac{\sqrt{2}}{32}$	0.4745	0.20735	0.19810	0.18087	0.17379

Table 4
Guaranteed lower eigenvalue bounds on the square cavity with two square tubes: ECR finite element.

h	C_h	$\lambda_{1,h}$	$\lambda^G_{1,h}$	$\lambda_{2,h}$	$\lambda^G_{2,h}$
$\frac{\sqrt{2}}{4}$	1.1704	0.19730	0.15532	0.17183	0.13909
$\frac{\sqrt{2}}{8}$	0.8276	0.20366	0.17873	0.17756	0.15830
$\frac{\sqrt{2}}{16}$	0.5852	0.20629	0.19268	0.17992	0.16948
$\frac{\sqrt{2}}{32}$	0.4138	0.20735	0.20024	0.18087	0.17544

Link to the reproducible capsule

The permanent code ocean link is https://codeocean.com/capsule/7071558/tree/v1.

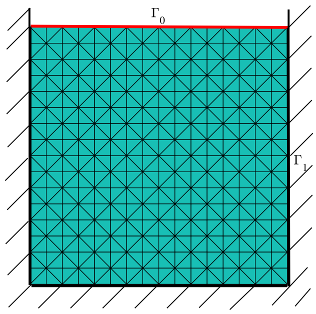


Fig. 2. Domain and meshes with diameter $h = \frac{\sqrt{2}}{16}$.

Table 5
Guaranteed lower eigenvalue bounds on the unit square: CR finite element.

h	C_h	$\lambda_{1,h}$	$\lambda_{1,h}^G$	$\lambda_{4,h}$	$\lambda_{4,h}^G$
$\frac{\sqrt{2}}{4}$	0.4745	2.89619	1.75295	24.00000	3.74736
$\frac{\sqrt{2}}{8}$	0.3356	3.03785	2.26360	8.67824	4.38934
$\frac{\sqrt{2}}{16}$	0.2373	3.10344	2.64187	11.19333	6.86644
$\frac{\sqrt{2}}{32}$	0.1678	3.12288	2.87055	12.15078	9.05406
$\frac{\sqrt{2}}{64}$	0.1186	3.12808	2.99618	12.45341	10.59620

Table 6
Guaranteed lower eigenvalue bounds on the unit square: ECR finite element.

h	C_h	$\lambda_{1,h}$	$\lambda_{1,h}^G$	$\lambda_{4,h}$	$\lambda_{4,h}^G$
$\frac{\sqrt{2}}{8}$	0.2069	3.03785	2.68826	8.67824	6.32759
$\frac{\sqrt{2}}{16}$	0.1463	3.10344	2.91014	11.19333	9.02995
$\frac{\sqrt{2}}{32}$	0.1034	3.12288	3.02189	12.15078	10.75256
$\frac{\sqrt{2}}{64}$	0.0732	3.12808	3.07659	12.45341	11.67540

Acknowledgments

The authors cordially thank the editor and the referees for their valuable comments and suggestions that lead to the improvement of this paper.

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