Black hole thermodynamics is extensive with variable Newton constant

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ABSTRACT

Inspired by the recent studies on the thermodynamics of AdS black holes in the restricted phase space formalism, we propose a similar formalism for the thermodynamics of non-AdS black holes with variable Newton constant. It is shown that, by introducing the new variables N, μ, where N is proportional to the inverse Newton constant and μ its conjugate variable, referred to as the chemical potential, the black hole thermodynamics can be formulated in a form which is consistent with the standard extensive thermodynamics for open macroscopic systems, with the first law and the Euler relation hold simultaneously. This formalism has profound implications, in particular, the mass is a homogeneous function of the first order in the extensive variables and the intensive variables are zeroth order homogeneous functions. The chemical potential is shown to be closely related to the Euclidean action evaluated at the black hole configuration.

1. Introduction

In connection with the recent studies [1,2] on the thermodynamics of AdS black holes in the reduced phase space (RPS) formalism, which indicated that the thermodynamics for AdS black holes is extensive in the standard sense, e.g. with the correct Euler relation and homogeneity behaviors, it is natural to ask what about the cases for non-AdS black holes. Since the newly introduced variables C, μ known as the central charge and the chemical potential respectively in the RPS formalism rely on the holographic interpretation [3] (see also [4–8]), the first impression may be that this formalism should work only for AdS black holes. The purpose of this work is to acknowledge that a similar formalism also works for asymptotically flat and de Sitter (dS) black holes, albeit the lack of holographic interpretation. Since there is no extended phase space formalism in these cases, it may not be appropriate to refer to the corresponding formalism as RPS thermodynamics. Therefore we simply call this new formalism as thermodynamics with variable Newton constant.

Extensivity is a much needed and long awaited feature for black hole thermodynamics which has not been realized before the works [1,2]. In the absence of extensivity, the previous formalisms for black hole thermodynamics such as the traditional [9–12] and the extended phase space [13–19] formalisms suffer from a number of issues, e.g. the black hole mass is not a ho-

mogeneous function in the extensive variables of the first order [20–22] as required in standard textbooks in thermodynamics, see, e.g. [23]; the interpretation for the total mass as enthalpy in the extended phase space formalism [13] seems to be in contradiction with the standard understanding of AD(M) mass as the conserved charge associated with the timelike Killing symmetry, *etc.* Moreover, some of the exotic thermodynamic behaviors such as intersecting isotherms and discontinuous Gibbs free energies observed in the extended phase space formalism do not appear in extensive description of thermodynamic systems. It is important to stress that, pointing out the issues in existing formalisms does not disvalue the previous works. On the contrary, there is nothing wrong in the previous works, the issues just indicate that there are some missing dimensions in the space of macro states for the black holes, which are identified to be the central charge and the chemical potential in the AdS cases. The clue for identifying the missing dimensions for the asymptotically flat and dS cases has already appeared in previous works, it only needs some time and insight to spell it out correctly, that is when and for what we come in.

2. RPS formalism for AdS and the logic behind

The RPS formalism for the AdS cases is derived from Visser's holographic thermodynamics involving an AdS black hole spacetime and a dual CFT. The basic structure of Visser's framework is captured by the following two equations,

$$\mathrm{d}E = T\mathrm{d}S - \mathcal{P}\mathrm{d}\mathcal{V} + \tilde{\Phi}\mathrm{d}\tilde{Q} + \Omega\mathrm{d}J + \mu\mathrm{d}C, \tag{1}$$

$$E = TS + \tilde{\Phi}\tilde{Q} + \Omega J + \mu C, \tag{2}$$

where E, T, S, Ω, J bear the usual sense as in literally any related references, $\tilde{\Phi}, \tilde{Q}$ are respectively the electric potential and charge with some proper rescaling, and C, μ are respectively the central charge and the chemical potential. The variables $\mathcal{P}, \mathcal{V}$ are defined such that $\mathcal{V} \sim L^{d-2}$ represents the spatial volume of the CFT and $\mathcal{P}$ is determined by the caloric equation of states $E = (d-2)\mathcal{P}\mathcal{V}$ of the CFT, where L is the AdS radius and d the dimension of the bulk spacetime. The RPS formalism arises by fixing the AdS radius L, which effectively eliminates the variables $\mathcal{P}, \mathcal{V}$, yielding

$$dE = TdS + \tilde{\Phi}d\tilde{Q} + \Omega dJ + \mu dC. \tag{3}$$

Eq. (3) represents the first law and (2) gives the corresponding Euler relation in the RPS formalism. Notice that eq. (2) is *not* the correct Euler relation associated with eq. (1), because there is a missing term $-\mathcal{P}\mathcal{V}$.

The logic behind the construction is as follows. The central charge C encodes the level of conformal anomaly in the dual CFT, which takes the value

$$C = \frac{L^{d-2}}{G}, \tag{4}$$

and is related to the number of colors $\mathcal{N}$ via $C \propto \mathcal{N}^2$ or $C \propto \mathcal{N}^{3/2}$ depending on the underlying gravity model [24]. While both L and G are considered to be variable in [3], the RPS formalism keeps L fixed and only G is allowed to vary. Physically, G is proportional to $(\ell_P)^{d-2}$, where ℓ_P represents the Planck length. Therefore, C has an intuitive interpretation as the number of pieces of the size of Planck "area" that a hypersurface of radius L can be divided into, with each such piece representing a single microscopic degree of freedom. Varying G implies changing the Planck length, which in turn changes the number of pieces mentioned above. The CFT resides on the boundary of the black hole spacetime, of which the metric can be obtained using a limiting procedure. For instance, assuming that the bulk spacetime has the line element

$$ds^2_{\text{bulk}} = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + \cdots + r^2 d\Omega^2_{d-2}$$

where $A(r)$ has the asymptotic behavior

$$A(r) \sim \frac{r^2}{L^2}, \quad \text{as} \quad r \to \infty,$$

the boundary metric is obtained via

$$ds^2_{\text{bdry}} = \lim_{r \to \infty} \frac{R^2}{r^2} ds^2_{\text{bulk}} = -d\tau^2 + R^2 d\Omega^2_{d-2},$$

where R is some large constant with the dimension of L. The crucial point lies in that, setting $R = L$, the thermodynamic quantities for the bulk black hole are precisely identical to those of the dual CFT, e.g.

$$T_{\text{bulk}} = T_{\text{CFT}}, \quad S_{\text{bulk}} = S_{\text{CFT}}, \quad etc.$$

One can establish further relationships between the two theories using the AdS/CFT dictionaries, of which the most basic one is the identification of the partition functions on both sides,

$$Z_{\text{CFT}} = Z_{\text{gravity}} = \exp(-\mathcal{A}_E), \tag{5}$$

where $\mathcal{A}_E$ is the Euclidean action in the bulk taken at the black hole configuration.

An important observation is that the Gibbs free energy W of the CFT can be defined via

$$W \equiv -T \log Z_{\text{CFT}} = \mu C,$$

which, by use of the AdS/CFT dictionary (5), leads to

$$W = \mu C = T\mathcal{A}_E, \qquad \mu = \frac{T\mathcal{A}_E}{C} = \frac{GT\mathcal{A}_E}{L^{d-2}}. \tag{6}$$

Since the thermodynamic quantities for the bulk and the boundary theories are identical, μ and C are also understood to be thermodynamic quantities on the black hole side. We thus see that the chemical potential for the black hole can be defined independent of the Euler relation (2), and the Euclidean action plays an indispensable role. With $W = \mu C$ in mind, the work presented in [22] implies the correctness of the Euler relation for AdS black holes.

3. Thermodynamics with variable Newton constant

The discussion made in the last section has borrowed two concepts from the AdS/CFT correspondence, i.e. the central charge and the chemical potential. For non-AdS black holes one encounters an immediate difficulty due to the lack of the CFT description. In particular, the definition of the central charge relies on the AdS radius L, a characteristic length scale which is absent in asymptotically flat spacetimes.

However, as we will show below, the parallel line of thinking still works for non-AdS black holes. The only change is to replace L by some constant length scale, also denoted as L, with completely different meaning. For asymptotically flat black holes, L may be regarded as the radius of a containing box,[1] which corresponds to the largest possible radius which the event horizon can reach during the thermodynamic process of interests.

To check this idea, let us consider the simplest example of four dimensional asymptotically flat Schwarzschild black hole with metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{7}$$

$$f(r) = 1 - \frac{2GM}{r}. \tag{8}$$

The event horizon is located at $r = r_g = 2GM$, and the temperature observed by static observers located at spatial infinity and the entropy are respectively given by

$$T = \frac{f'(r_g)}{4\pi} = \frac{1}{8\pi GM}, \quad S = \frac{A}{4G} = \frac{\pi r_g^2}{G} = 4\pi GM^2. \tag{9}$$

The traditional formalism treats G as constant and hence

$$dM = TdS, \quad M = 2TS. \tag{10}$$

This is the first law and the famous Smarr relation [25].

Now let us proceed by taking G as a variable. Assuming that the black hole is contained in a spherical box of radius L centered at $r = 0$. Due to the fact that the event horizon is a causal boundary, all meaningful degrees of freedom (whatever they are) must be distributed on the horizon surface, rather than in the volume surrounded by the horizon. Actually, there is no such notion as the volume surrounded by the horizon. Therefore, it is reasonable to assume that the number N of microscopic degrees of freedom for the black hole is proportional to L^2, a measure of the largest possible area which the event horizon can reach during the thermodynamic process of interests. In particular, we assume that

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[1] By a containing box, we refer to a hypothetical spatial hypersurface which, besides marking the largest possible scale of the event horizon, has no physical consequences and represents no physical reality. This concept should be distinguished from *confining box* or *cavity*, which represents a physical boundary which has extra consequences, e.g. in modifying the total stress energy tensor [28].

$$N = \frac{L^2}{G}. \tag{11}$$

Although N does not have the holographic interpretation as the central charge, it still represents the number of pieces of the size of Planck area that a spherical box of radius L can be divided into. We also introduce the chemical potential μ by use of the Euler relation

$$M = TS + \mu N, \tag{12}$$

which ensures that μ, N form a conjugate pair. Explicitly, we have, for Schwarzschild black hole,

$$\mu = \frac{GM}{2L^2}. \tag{13}$$

Considering S and N as functions of M and G, we can easily check that

$$dM = TdS + \mu dN. \tag{14}$$

This gives a simple example for black holes in asymptotically flat spacetime in which the first law and the Euler relation hold simultaneously with variable Newton constant. It is interesting to notice that, in the above construction, L played no essential role and hence it can have any constant value. However, by dimensional consideration it is necessary to introduce a length scale L in eq. (11), rather than simply defining N by the inverse of G. Please also note that even when G is kept fixed and the first law (14) reduces into the traditional form presented in eq. (10), the variables N and μ are still meaningful, the Euler relation (12) still holds and is distinct from the Smarr relation. One can think of our formalism with fixed G as a thermodynamic description for the black hole as a closed system, while the case with variable G as a description of the black hole as an open system.

We can go a little further by writing M as a first order homogeneous function in S and N. This is achieved by solving M from the expression for S and inserting the definition of N in the result. The final expression is given as

$$M(S, N) = \sqrt{\frac{S}{4\pi G}} = \frac{1}{2L}\sqrt{\frac{NS}{\pi}}, \tag{15}$$

which is clearly a first order homogeneous function in S and N. For comparison, in the traditional formalism with fixed Newton constant, the mass is a homogeneous function in S of order one half, thanks to the Smarr relation (10), which looks strange if M is to be taken as a thermodynamic potential as it should be.

Eq. (15) allows us to get the equations of states

$$T = \left(\frac{\partial M}{\partial S}\right)_N = \frac{1}{4L}\sqrt{\frac{N}{\pi S}}, \tag{16}$$

$$\mu = \left(\frac{\partial M}{\partial N}\right)_S = \frac{1}{4L}\sqrt{\frac{S}{\pi N}}. \tag{17}$$

Using these equations we can explicitly verify the extensivity of S by rewriting

$$S = \frac{N}{16\pi L^2 T^2} = 16\pi L^2 \mu^2 N,$$

i.e. S is proportional to N with the coefficient of proportionality depending only on the intensive variables.

We can also introduce the heat capacity for Schwarzschild black hole unambiguously,

$$C_N = T\left(\frac{\partial S}{\partial T}\right)_N = -\frac{N}{8\pi L^2 T^2} = -2S.$$

As expected, C_N is always negative, which signifies the thermodynamic instability of the Schwarzschild black hole.

The surprising validity for the first law and Euler relation in the Schwarzschild case with variable G forces us to consider also the validity of the same formalism for de Sitter black holes. To this end, let us again consider a simple example, i.e. Schwarzschild-de Sitter case. The metric is still written in the form (7), but with $f(r)$ replaced by

$$f(r) = 1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}, \tag{18}$$

where the cosmological constant $\Lambda > 0$ corresponds to de Sitter case and $\Lambda < 0$ corresponds to AdS case. We assume that $9\Lambda G^2 M^2 < 1$, so that $f(r)$ has two distinct real zeros r_h and r_c which correspond respectively to the black hole event horizon and the cosmological horizon.

Comparing to the cases of asymptotically flat and AdS black holes, the thermodynamics for de Sitter black holes is more subtle due to the presence of two horizons. The existence of the cosmological horizon makes it impossible to normalize the timelike Killing vector at spatial infinity, which brings in some ambiguity in determining the values of surface gravity and hence the temperatures. A reasonable way out is to consider thermodynamic relations on only one of the horizons, fixing the other as a boundary [26]. Then the temperature can be determined unambiguously using the Euclidean period approach. In the following, we shall follow this line of thinking and consider only the thermodynamics associated with the event horizon. Similar constructions associated with the cosmological horizon should also work, but we simply omit it for simplicity and brevity.

Now M can be represented in terms of the radius of the event horizon r_h by use of the equation $f(r_h) = 0$ which determines the horizon radius, i.e.

$$M = \frac{r_h(3 - \Lambda r_h^2)}{6G}. \tag{19}$$

The associated temperature and entropy are given by

$$T = \frac{f'(r_h)}{4\pi} = \frac{2GM}{r_h^2} - \frac{2\Lambda r_h}{3} = \frac{1 - \Lambda r_h^2}{4\pi r_h}, \tag{20}$$

$$S = \frac{A}{4G} = \frac{\pi r_h^2}{G}. \tag{21}$$

Notice that both M and S are inversely proportional to G if they are considered as functions of (r_h, G). This feature will be useful in later analysis.

The number N of microscopic degrees of freedom for the black hole is still assumed to take the form (11), however please be aware that L is not related in any way to the cosmological constant. Using the Euler relation $M = TS + \mu N$, we can calculate the value of μ, yielding

$$\mu = \frac{r_h(3 + \Lambda r_h^2)}{12L^2}. \tag{22}$$

Finally, it is straightforward to check that the first law (14) indeed holds in this case, assuming that G is an independent variable.

From the values for N and S we can get

$$G = \frac{L^2}{N}, \quad r_h = L\sqrt{\frac{S}{\pi N}}.$$

Inserting these relations into eqs. (19)-(22) yields

$$M = \frac{3\pi N - \Lambda L^2 S}{6\pi^{3/2} L}\sqrt{\frac{S}{N}}, \tag{23}$$

$$T = \frac{\pi N - \Lambda L^2 S}{4\pi^{3/2} LS}\sqrt{\frac{S}{N}}, \tag{24}$$

$$\mu = \frac{3\pi N + \Lambda L^2 S}{12\pi^{3/2} L N} \sqrt{\frac{S}{N}}. \tag{25}$$

The first order homogeneity of M and zeroth order homogeneity of T, μ in S, N are crystal clear from the above equations.

Although eqs. (23)-(25) are derived specifically for the de Sitter case, they actually hold also for asymptotically flat and AdS cases. In particular, for $\Lambda = 0$, eqs. (23)-(25) fall back to eqs. (15)-(17); for $\Lambda < 0$, L can be chosen to be independent of the AdS radius, which is a feature not mentioned in our previous work [1].

Let us now proceed with some discussions for generic choice of Λ. In order to ensure $T \geq 0$, we need

$$\pi N - \Lambda L^2 S \geq 0. \tag{26}$$

The heat capacity evaluated using the above equations of states reads

$$C_N = -2S \left(\frac{\pi N - \Lambda L^2 S}{\pi N + \Lambda L^2 S} \right).$$

For $\Lambda \to 0$, we have $C_N \to -2S$, which recovers the Schwarzschild result. For $\Lambda > 0$, C_N is always non-positive, thanks to the bound (26), which indicates that the Schwarzschild-de Sitter black hole is also thermodynamically unstable. For $\Lambda < 0$, the bound (26) imposes no restriction on the N and S values, thus for sufficiently large S which makes $\pi N + \Lambda L^2 S < 0$ (with $\Lambda < 0$), C_N can become positive. Therefore, the Schwarzschild-AdS black hole can be thermodynamically stable.

Another point to be noticed is that, according to eq. (25), the chemical potential is strictly positive if $\Lambda \geq 0$, which implies that the microscopic degrees of freedom are repulsive, which might help to understand the thermodynamic instability in these cases. On the contrary, if $\Lambda < 0$, μ becomes zero at $N = -\frac{\Lambda L^2 S}{3\pi}$ and negative for $N < -\frac{\Lambda L^2 S}{3\pi}$. The former condition reconfirms that the Hawking-Page transition [27] appears only for AdS but not for asymptotically flat and de Sitter black holes, while the latter signifies that the microscopic degrees of freedom are attractive for sufficiently large S, and hence the large AdS black hole becomes thermodynamically stable.

The proceeding two simple examples make us confident that the same formalism involving variable G should work for generic black holes, be it AdS or not. The correct Euler relation is a key ingredient in this formalism, which guarantees the extensivity. Indeed, we have verified the validity of eq. (3) for Kerr-Newman black holes in both asymptotically flat and de Sitter spacetimes, where C is given by eq. (4) and μ is defined via the Euler relation (2).[2] There is no necessary relationship between L and Λ, the only requirement is that both of them are constant, and that G is treated as a variable which defines C. That the Euler relation and the first law together guarantees extensivity of the thermodynamics can be explained very easily. We can simply rewrite eq. (2) in the form (we use M in place of E and N in place of C)

$$M = \left(\frac{\partial M}{\partial S} \right)_{\tilde{Q}, J, N} S + \left(\frac{\partial M}{\partial \tilde{Q}} \right)_{S, J, N} \tilde{Q} + \left(\frac{\partial M}{\partial J} \right)_{S, \tilde{Q}, N} J$$
$$+ \left(\frac{\partial M}{\partial N} \right)_{S, \tilde{Q}, J} N. \tag{27}$$

Then it is clear that M scales as $M \to \lambda M$ if $S, \tilde{Q}, J, N$ all scales in the same way, i.e. $S \to \lambda S$, etc. Then it is evident that the partial

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derivatives, which are actually the intensive variables $T, \hat{\Phi}, \Omega, \mu$, will not get rescaled. What remains is a proper understanding about μ, because now it is defined using the Euler relation, rather than defined independently as in the AdS cases.

4. Understanding the chemical potential

In the variable G formalism discussed in the last section, the chemical potential μ is defined via the Euler relation. This is somewhat puzzling and needs further understanding, because the first law and the Euler relation are commonly regarded as independent relations.

To solve the puzzle, let us recall that, for Kerr-Newman black hole, the Euclidean action $\mathcal{A}_E$ is verified to satisfy the relation [20]

$$W = T\mathcal{A}_E = M - TS - \Phi Q - \Omega J.$$

This is a purely algebraic relation which does not rely on whether G is variable or not. The above relation reduces to

$$T\mathcal{A}_E = M - TS \tag{28}$$

in the case of static neutral black holes. Meanwhile, for fixed G, the traditional formalism of black hole thermodynamics admits the first law

$$\tilde{d}M = T\tilde{d}S + \Phi\tilde{d}Q + \Omega\tilde{d}J,$$

which reduces to

$$\tilde{d}M = T\tilde{d}S$$

for static neutral black holes, where $\tilde{d}$ represents the total differential taken when G is considered to be a constant.

Now if G allowed to vary, then

$$dM = \tilde{d}M + \left(\frac{\partial M}{\partial G} \right) dG = \tilde{d}M - M\frac{dG}{G}, \tag{29}$$

where we have used the fact that M is inversely proportional to G. Similarly we have

$$dS = \tilde{d}S - S\frac{dG}{G}.$$

Using the above relations we have

$$dM = T\left[dS + S\frac{dG}{G} \right] - M\frac{dG}{G} = TdS - (M - TS)\frac{dG}{G}$$
$$= TdS + \frac{M - TS}{N}dN = TdS + \mu dN. \tag{30}$$

We recognize that this is precisely the first law in the formalism with variable G, in which μ is given by use of the Euler relation (12). The same procedure also works for more general black hole solutions in generic dimension d. However, please be reminded that, for charged black holes, M is not inversely proportional to G, because there is a G factor in front of Q^2 which appears in the expression for M. In order to make sure that the first law still holds with variable G, a G-dependent rescaling of Q and Φ maintaining the product ΦQ is needed, as did in [1].

Comparing the Euler relation (12) with the Euclidean action formula (28), we get

$$\mu = \frac{T\mathcal{A}_E}{N} = \frac{GT\mathcal{A}_E}{L^2}.$$

This reproduces the result of eq. (6) for $d = 4$, but now for non-AdS black holes without holographic interpretation. The fact that the chemical potential is connected with the Euclidean action is natural. Let us remark that, since the Newton constant is an overall

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[2] To keep the presentation of the present manuscript as clean as possible, we omit the details for the Kerr-Newman and Kerr-Newman-de Sitter cases. The readers are encouraged to reproduce the explicit procedures by themselves, which is a little involved but can still be carried out using simple algebra and calculus.

factor in the Einstein-Hilbert action (denoted as $\mathcal{A}$), the expression $G\mathcal{A}/L^2$ is the *mechanical* conjugate to $N = L^2/G$. The thermodynamic conjugate differs from the mechanical conjugate in that the temperature T enters in the expression for μ, and that the action $\mathcal{A}$ is replaced by its Euclidean counterpart $\mathcal{A}_E$.

Before closing this section, let us mention that the Euclidean action for RN-AdS black hole together with the $\Lambda \to 0$ limit was considered in [28]. The connection between Euclidean action approach and the extended phase space formalism was discussed in [29]. In either cases the Euclidean action is not associated with the chemical potential and the formalisms are thus not extensive.

5. Concluding remarks

The work presented in this paper indicates that the thermodynamics for black holes fall in the standard framework of extensive thermodynamics, provided the Newton constant is allowed to vary. This is true for all kinds of asymptotics, including asymptotically flat and de Sitter cases. Although the latter cases do not admit holographic interpretations, the relation $\mu = \frac{T\mathcal{A}_E}{N}$ between the chemical potential μ and the Euclidean action $\mathcal{A}_E$ is universal. We hope the new formalism will be useful in further exploring and understanding the physics of black holes from the macroscopic perspective.

While finishing this manuscript, we noticed that the idea of including variable Newton constant in the formulation of black hole thermodynamics has already been used in [30]. However, the motivation and the major points of interests are different from ours. In particular, extensivity of the formalism, including the Euler relation and the related homogeneity behaviors are not considered in [30].

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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References

[1] Z. Gao, L. Zhao, Restricted phase space thermodynamics for AdS black holes via holography, arXiv:2112.02386.

[2] Z. Gao, X. Kong, L. Zhao, Thermodynamics of Kerr-AdS black holes in the restricted phase space, arXiv:2112.08672.

[3] M.R. Visser, Holographic thermodynamics requires a chemical potential for color, arXiv:2101.04145.

[4] D. Kastor, S. Ray, J. Traschen, Chemical potential in the first law for holographic entanglement entropy, J. High Energy Phys. (2014) 120, arXiv:1409.3521.

[5] J.-L. Zhang, R.G. Cai, H. Yu, Phase transition and thermodynamical geometry of Reissner-Nordström-AdS black holes in extended phase space, Phys. Rev. D 91 (2015) 044028, arXiv:1502.01428.

[6] A. Karch, B. Robinson, Holographic black hole chemistry, J. High Energy Phys. (2015) 1–15, arXiv:1510.02472.

[7] R. Maity, P. Roy, T. Sarkar, Black hole phase transitions and the chemical potential, Phys. Lett. B 765 (2017) 386–394, arXiv:1512.05541.

[8] S.W. Wei, B. Liang, Y.X. Liu, Critical phenomena and chemical potential of a charged AdS black hole, Phys. Rev. D 96 (2017) 124018, arXiv:1705.08596.

[9] J.D. Bekenstein, Black holes and the second law, Lett. Nuovo Cimento 4 (1972) 737–740.

[10] J.D. Bekenstein, Black holes and entropy, Phys. Rev. D 7 (1973) 2333–2346.

[11] J.M. Bardeen, B. Carter, S.W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. 31 (1973) 161–170.

[12] S.W. Hawking, Particle creation by black holes, Commun. Math. Phys. 43 (1975) 199–220; Commun. Math. Phys. 46 (1976) 206 (Erratum).

[13] D. Kastor, S. Ray, J. Traschen, Enthalpy and the mechanics of AdS black holes, Class. Quantum Gravity 26 (2009) 195011, arXiv:0904.2765.

[14] B.P. Dolan, The cosmological constant and the black hole equation of state, Class. Quantum Gravity 28 (2011) 125020, arXiv:1008.5023.

[15] B.P. Dolan, Pressure and volume in the first law of black hole thermodynamics, Class. Quantum Gravity 28 (2011) 235017, arXiv:1106.6260.

[16] Dolan, Compressibility of rotating black holes, Phys. Rev. D 84 (2011) 127503, arXiv:1109.0198.

[17] D. Kubiznak, R.B. Mann, P-v criticality of charged AdS black holes, J. High Energy Phys. 1207 (2012) 033, arXiv:1205.0559.

[18] R.-G. Cai, L.-M. Cao, L. Li, R.-Q. Yang, P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space, J. High Energy Phys. (2013) 005, arXiv:1306.6233.

[19] D. Kubiznak, R.B. Mann, M. Teo, Black hole chemistry: thermodynamics with lambda, Class. Quantum Gravity 34 (2017) 063001, arXiv:1608.06147.

[20] G.W. Gibbons, S.W. Hawking, Action integrals and partition functions in quantum gravity, Phys. Rev. D 15 (1977) 2752.

[21] J.W. York Jr., Black-hole thermodynamics and the Euclidean Einstein action, Phys. Rev. D 33 (1987) 2092.

[22] G.W. Gibbons, M.J. Perry, C.N. Pope, The first law of thermodynamics for Kerr-anti-de Sitter black holes, Class. Quantum Gravity 22 (2005) 1503, arXiv:hep-th/0408217.

[23] H.B. Callen, Thermodynamics and an Introduction to Thermostatistics, 2nd ed., John Willy & Sons, 1985.

[24] M. Rafiee, S.A.H. Mansoori, S.W. Wei, R.B. Mann, Universal criticality of thermodynamic geometry for boundary conformal field theories in gauge/gravity duality, arXiv:2107.08883.

[25] L. Smarr, Mass formula for Kerr black holes, Phys. Rev. Lett. 30 (1973) 71–73; Phys. Rev. Lett. 30 (1973) 521 (Erratum).

[26] A. Gomberoff, C. Teitelboim, de Sitter black holes with either of the two horizons as a boundary, Phys. Rev. D 67 (2003) 104024, arXiv:hep-th/0302204.

[27] S.W. Hawking, D.N. Page, Thermodynamics of black holes in anti-de Sitter space, Commun. Math. Phys. 87 (1983) 577.

[28] C.S. Peça, J.P.S. Lemos, Thermodynamics of Reissner-Nordstrom-anti-de Sitter black holes in the grand canonical ensemble, Phys. Rev. D 59 (1999) 124007, arXiv:gr-qc/9805004.

[29] J.P.S. Lemos, O.B. Zaslavskii, Black hole thermodynamics with the cosmological constant as independent variable: bridge between the enthalpy and the Euclidean path integral approaches, Phys. Lett. B 786 (2018) 296, arXiv:1806.07910.

[30] G. Volovik, Varying Newton constant and black hole to white hole quantum tunneling, Universe 6 (9) (2020) 133, arXiv:2003.10331.