Graphical Models 114 (2021) 101100

Contents lists available at ScienceDirect

Graphical Models

journal homepage: www.elsevier.com/locate/gmod

Orthogonality of isometries in the conformal model of the 3D space

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ARTICLE INFO

Keywords:
Isometries
Euclidean model
Homogeneous model
Conformal model

ABSTRACT

Motivated by questions on orthogonality of isometries, we present a new construction of the conformal model of the 3D space using just elementary linear algebra. In addition to pictures that can help the readers to understand the conformal model, our approach allows to obtain matrix representation of isometries that can be useful, for example, in applications of computational geometry, including computer graphics, robotics, and molecular geometry.

1. Introduction

This lecture aims to introduce the conformal model of the 3D space based solely on elementary linear algebra. We center our development around orthogonality of isometries, which allows us, for example, to explain why five dimensions are required. Isometries are not even linear transformations in the Euclidean vector space $\mathbb{R}^3$ (the classical model for the 3D space), but are orthogonal in the conformal model, whose origin comes from the work of F. Wachter, a student of Gauss [19]. Our approach avoids prior knowledge of Clifford algebra as would be needed, e.g., to follow the Wikipedia article on Conformal Geometric Algebra [30].

There are many applications of Conformal Clifford Algebra (for example, in physics [5], combinatorics [28], geometric reasoning [20], image processing [3], neurocomputing [3], data science [9], computer vision [3], robotics [16], molecular geometry [13], and computational geometry [26]), but there is one with a strong connection with our approach, mentioned for the first time by Dress and Havel, through a link between the Wachter's model and Distance Geometry [8].

Distance Geometry (DG) is geometry based on the concept of distance [25], instead of points and lines [21], and the geometric meaning of the inner product in the conformal model can give new insights to DG, because the computation of distances can be greatly simplified, with new implications, for example, in 3D protein structure calculations [31], crystallography [23], and data science [18,22]. See, also, [1,2,17].

After the theoretical development (Sections 2 and 3), Section 4 presents some pictures to "illustrate" the conformal model. In Section 5, we provide an application of the conformal model, where the matrix representation of isometries is discussed. Section 6 ends the lecture with the final comments.

2. Isometries

Rigid motions play a fundamental role in geometry and applications. Mathematically, they are defined as transformations that preserve Euclidean distances, called isometries.

An *isometry* in $\mathbb{R}^n$ is a function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that, $\forall u, v \in \mathbb{R}^n$,

$$\| f(u) - f(v) \| = \| u - v \|,$$

with the norm defined by the usual inner product in $\mathbb{R}^n$, given by ($\alpha \in \mathbb{R}$ and $u, v, w \in \mathbb{R}^n$)

$$u \cdot v = v \cdot u, \tag{1}$$

$$u \cdot (v + w) = (u \cdot v) + (u \cdot w), \tag{2}$$

$$\alpha(u \cdot v) = (\alpha u) \cdot v = u \cdot (\alpha v), \tag{3}$$

and

$$u \neq 0 \Rightarrow u \cdot u > 0. \tag{4}$$

From linear algebra [4] (Theorem 11.3.2, page 204), we know that an isometry f can also be given by

$$f(u) = Au + b, \tag{5}$$

where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, such that $A^{-1} = A^T$. That is, an isometry can

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https://doi.org/10.1016/j.gmod.2021.101100
Received 15 December 2020; Received in revised form 27 January 2021; Accepted 20 February 2021
Available online 27 February 2021
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be described by an *orthogonal transformation* up to a translation.

Since translations $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ can be represented linearly in $\mathbb{R}^{n+1}$, by using the so-called *homogeneous coordinates* [11], expression (5) raises the question whether an isometry in $\mathbb{R}^n$ can be represented as an orthogonal transformation in $\mathbb{R}^{n+1}$.

We prove that the answer is no. However, if we do not insist on the positivity of the inner product (4), an isometry in $\mathbb{R}^n$ can be represented as an orthogonal transformation in $\mathbb{R}^{n+2}$. Since the results can be easily generalized, we fix $n = 3$.

3. The conformal model of the 3D space

The classical way to describe the 3*D* space is to use the *Euclidean*model, giving by the vector space $\mathbb{R}^3$ and the usual inner product. However, the 3*D* space can also be represented by the *homogeneous* or *conformal* models [10,19].

Let us fix, in $\mathbb{R}^3$, an orthonormal basis $\{e_1, e_2, e_3\}$, i.e.

$$e_i \cdot e_j = \delta_{ij},$$

where $x \in \mathbb{R}^3$ is giving by

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3,$$

with $x_1, x_2, x_3 \in \mathbb{R}$.

In the homogeneous model, the point x of the 3*D* space is represented in $\mathbb{R}^4$ by

$$X = x + e_4, \tag{6}$$

where

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3 + 0 e_4,$$

and $e_4 \in \mathbb{R}^4$ is orthogonal to $\{e_1, e_2, e_3\}$ (for simplifying the notation, the vectors e_1, e_2, e_3 and their representations in $\mathbb{R}^4$ are denoted by the same symbols).

Considering that X, Y represent $x, y \in \mathbb{R}^3$ in the homogeneous model, if there is a constant $k \in \mathbb{R}$ $(\neq 0)$ such that, $\forall x, y \in \mathbb{R}^3$,

$$X \cdot Y = k \parallel x - y \parallel^2, \tag{7}$$

then isometries in $\mathbb{R}^3$ could be coded as orthogonal transformations in the homogeneous model, since another way to define an orthogonal transformation A in $\mathbb{R}^n$ is just requiring that, $\forall u, v \in \mathbb{R}^n$,

$$(Au) \cdot (Av) = u \cdot v.$$

From (7), we obtain

$$x = y \Rightarrow X \cdot X = 0,$$

implying that the positivity of the inner product in $\mathbb{R}^4$ cannot be satisfied, if we want a "solution" for Eq. (7).

The discussion above leads to our first result.

Lemma 1. *Isometries in $\mathbb{R}^3$ cannot be represented as orthogonal transformations in $\mathbb{R}^4$.*

The next lemma extends the previous one.

Lemma 2. *Isometries in $\mathbb{R}^3$ cannot be represented as orthogonal transformations in $\mathbb{R}^4$, even if the positivity of the inner product in $\mathbb{R}^4$ is not satisfied.*

Proof. In fact, in the homogeneous model [6] (chapter 11), a point x of the 3*D* space can also be represented by

$$X = x + x_4 e_4, \tag{8}$$

where $x_4 \in \mathbb{R}$ $(x_4 \neq 0)$. Recalling that e_4 represents the vector $0 \in \mathbb{R}^3$, from now on, we will consider that $x \neq 0$. Also, without loss of generality, let us take $x_4 > 0$.

From (8),

$$\begin{aligned} X \cdot X &= 0 \\ &\Rightarrow \\ (x + x_4 e_4) \cdot (x + x_4 e_4) &= 0 \\ &\Rightarrow \\ x \cdot x + 2x_4 (x \cdot e_4) + x_4^2 (e_4 \cdot e_4) &= 0. \end{aligned}$$

Since e_4 is orthogonal to $\{e_1, e_2, e_3\}$,

$$\begin{aligned} X \cdot X &= 0 \\ &\Rightarrow \\ x \cdot x + x_4^2 (e_4 \cdot e_4) &= 0 \\ &\Rightarrow \\ e_4 \cdot e_4 &= \frac{-1}{x_4^2} \parallel x \parallel^2. \end{aligned}$$

Note that

$$x \neq 0 \Rightarrow e_4 \cdot e_4 < 0.$$

Defining the constant $c > 0$ by

$$c^2 = -(e_4 \cdot e_4),$$

we get

$$x_4 = \frac{\parallel x \parallel}{c},$$

which implies that

$$X = x + \frac{\parallel x \parallel}{c} e_4.$$

Doing the calculations, we obtain

$$\begin{aligned} X \cdot Y &= \left(x + \tfrac{\|x\|}{c} e_4\right) \cdot \left(y + \tfrac{\|y\|}{c} e_4\right) \\ &= x \cdot y + \frac{\parallel x \parallel \parallel y \parallel}{c^2} (e_4 \cdot e_4) \\ &= x \cdot y - \parallel x \parallel \parallel y \parallel . \end{aligned}$$

From (7), recalling that $k \neq 0$, we have

$$x \cdot y - \parallel x \parallel \parallel y \parallel = k \parallel x - y \parallel^2,$$

which should be valid for all $x, y \in \mathbb{R}^3$. However, considering different linearly dependent vectors x, y ($x \neq 0$ and $y \neq 0$), the Cauchy–Schwarz inequality says that

$$x \cdot y - \parallel x \parallel \parallel y \parallel = 0,$$

implying

$$k \parallel x - y \parallel^2 = 0 \Rightarrow k = 0,$$

which is a contradiction. □ Using the homogeneous model and assuming the properties (1)–(3) of the inner product, the previous lemma proves that Eq. (7) cannot be satisfied for all $x, y \in \mathbb{R}^3$. Thus, let us increase one more dimension.

Theorem 3. *If the positivity of the inner product is not required, then isometries in $\mathbb{R}^3$ can be represented as orthogonal transformations in $\mathbb{R}^5$.*

Proof. In $\mathbb{R}^5$, a point x of the 3*D* space, given by

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3 + 0 e_4 + 0 e_5,$$

where $e_5 \in \mathbb{R}^5$ is orthogonal to $\{e_1, e_2, e_3, e_4\}$, which is an orthonormal set in $\mathbb{R}^4$, will be represented by

$$X = x + x_4 e_4 + x_5 e_5,$$

with $x_4, x_5 \in \mathbb{R}$.

From (7), we obtain

$$\begin{array}{rcl} X \cdot X & = & 0 \\ & \Rightarrow & \\ (x + x_4 e_4 + x_5 e_5) \cdot (x + x_4 e_4 + x_5 e_5) & = & 0 \\ & \Rightarrow & \\ x_4^2 (e_4 \cdot e_4) + x_5^2 (e_5 \cdot e_5) & = & - \| x \|^2. \end{array}$$

For $x \neq 0$, the last equality implies that $e_5 \cdot e_5$ must be negative, since $\| e_4 \| = 1$. Let us consider

$$e_5 \cdot e_5 = -1.$$

Again, from Eq. (7), the points $X \in \mathbb{R}^5$ that will represent the 3D space must satisfy

$$X \cdot X = 0. \tag{9}$$

Since vectors e_4, e_5 do not satisfy this requirement, let us replace them by other vectors that satisfy Eq. (9). For $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{array}{rcl} (\alpha e_4 + \beta e_5) \cdot (\alpha e_4 + \beta e_5) & = & 0 \\ & \Leftrightarrow & \\ \alpha^2 (e_4 \cdot e_4) + \beta^2 (e_5 \cdot e_5) & = & 0 \\ & \Leftrightarrow & \\ \alpha^2 & = & \beta^2. \end{array}$$

So, we can define a new basis for $\mathbb{R}^5$, $\{e_1, e_2, e_3, e_0, e_\infty\}$, by (see Fig. 1)

$$e_0 = e_5 - e_4,$$

$$e_\infty = e_5 + e_4, \tag{10}$$

implying that

$$e_0 \cdot e_0 = 0 \quad \text{and} \quad e_\infty \cdot e_\infty = 0.$$

To get the inner product between

$$X = x + x_0 e_0 + x_\infty e_\infty$$

and

$$Y = y + y_0 e_0 + y_\infty e_\infty,$$

for $x_0, x_\infty, y_0, y_\infty \in \mathbb{R}$, we need to calculate $e_0 \cdot e_\infty$. In order to maintain the same value commonly used in the literature [15] (page 117), we can just redefine e_0,

$$e_0 = \frac{e_5 - e_4}{2}, \tag{11}$$

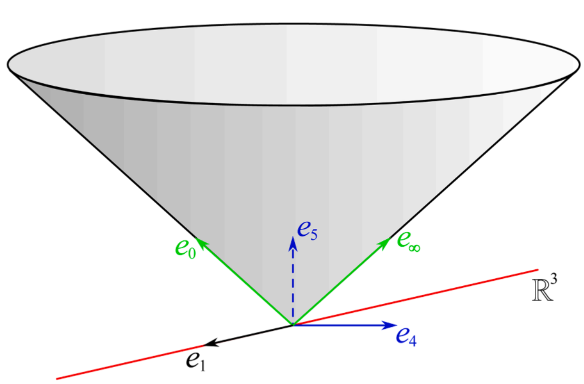

Fig. 1. The null cone.

to obtain

$$e_0 \cdot e_\infty = \left(\frac{e_5 - e_4}{2} \right) \cdot (e_5 + e_4) = -1.$$

Using this value, we have

$$\begin{aligned} X \cdot Y &= (x + x_0 e_0 + x_\infty e_\infty) \cdot (y + y_0 e_0 + y_\infty e_\infty) \\ &= x \cdot y - (x_0 y_\infty + x_\infty y_0) \end{aligned}$$

For $X = Y$,

$$X \cdot X = 0 \Rightarrow \| x \|^2 - 2 x_0 x_\infty = 0,$$

and considering $x_0 = 1$, we get

$$X = x + e_0 + \frac{1}{2} \| x \|^2 e_\infty. \tag{12}$$

Using this representation for $x, y \in \mathbb{R}^3$, we finally obtain

$$\begin{aligned} X \cdot Y &= \left(x + e_0 + \tfrac{1}{2} \| x \|^2 e_\infty \right) \cdot \left(y + e_0 + \tfrac{1}{2} \| y \|^2 e_\infty \right) \\ &= x \cdot y - \left(\frac{1}{2} \| x \|^2 + \frac{1}{2} \| y \|^2 \right) \\ &= -\frac{1}{2} \| x - y \|^2, \end{aligned}$$

which gives the solution $k = -1/2$ for Eq. (7). $\square$ We can finally define the conformal model of the 3D space.

3.1. The definition of the conformal model

The *conformal model* of the 3D space can then be defined by the $\mathbb{R}^5$, with the basis $\{e_1, e_2, e_3, e_0, e_\infty\}$, such that, for $i, j = 1, 2, 3$,

$$e_i \cdot e_j = \delta_{ij}, \tag{13}$$

$$e_0 \cdot e_i = 0, \tag{14}$$

$$e_\infty \cdot e_i = 0, \tag{15}$$

$$e_0 \cdot e_0 = e_\infty \cdot e_\infty = 0, \tag{16}$$

$$e_0 \cdot e_\infty = -1. \tag{17}$$

We emphasize that properties (1)–(3) are still satisfied for the inner product in the conformal model.

In the conformal model, the usual Euclidean metric (13) still holds for e_1, e_2, e_3, but a “strange” metric is defined for e_0, e_∞ (in fact, this is a *Minkowski metric*, considering e_4, e_5 instead of e_0, e_∞ [19]).

A natural question is to ask which points in the 3D space e_0 and e_∞ represent.

From (12), we easily obtain

$$x = 0 \Rightarrow X = e_0.$$

From the homogeneity of the conformal model, we get that

$$\frac{X}{\| x \|^2 / 2} = \frac{x}{\| x \|^2 / 2} + \frac{e_0}{\| x \|^2 / 2} + e_\infty$$

and

$$X = x + e_0 + \frac{1}{2} \| x \|^2 e_\infty$$

represent the same point of the 3D space, implying

$$\| x \|^2 \to \infty \Rightarrow \frac{X}{\| x \|^2 / 2} \to e_\infty.$$

Hence, in the conformal model, e_0 is the origin and e_∞ is "the infinity" of the 3D space, respectively. More details in Hildenbrand [12] (page 27).

4. "Visualizing" the conformal model

In order to "visualize" the conformal model, let us consider a conformal point X using the basis $\{e_1, e_4, e_5\}$, given by

$$X = x_1 e_1 + x_4 e_4 + x_5 e_5,$$

where $x_1, x_4, x_5 \in \mathbb{R}$. From Eq. (7), we get

$$\begin{aligned} X \cdot X &= 0 \\ &\Leftrightarrow \\ (x_1 e_1 + x_4 e_4 + x_5 e_5)\cdot(x_1 e_1 + x_4 e_4 + x_5 e_5) &= 0 \\ &\Leftrightarrow \\ x_1^2 + x_4^2 &= x_5^2. \end{aligned} \tag{18}$$

The last equation describes a right circular cone, with vertex at the origin and principal axis given by e_5 (see Fig. 1). This explains why the set of points satisfying Eq. (18), in the conformal model, is called the *null cone*.

From expression (12), we can see that the conformal representation of a point $x_1 \in \mathbb{R}$ is obtained first translating x_1, by the vector e_0, and then taking the corresponding point in the "parabola" given by the term $\frac{1}{2}\| x_1 \|^2 e_\infty$ (see Fig. 2).

In $\mathbb{R}^5$, the conformal points that represent the 3D space are called *horosphere* [19], given by the intersection of the null cone with the plane that passes through e_0 and is orthogonal to the plane defined by e_0, e_∞, called *Minkowski plane* [19]. In algebraic terms, these are the points satisfying equations

$$X \cdot X = 0$$

and

$$(X - e_0)\cdot e_\infty = 0.$$

There is also a beautiful relationship between the conformal model and the so-called *stereographic projection* [3,24] (pages 239 and 146, respectively). Consider a circle in the plane defined by e_1, e_4 (center and radius given by (0,0) and 1, respectively) and a point $x_1 \in \mathbb{R}$ (see Fig. 3). The stereographic projection of x_1 is its linear projection away from the point (0,1) until it reaches the circle (Fig. 3).

The intersection point of the circle with the line passing through the points (c_1, c_2), $(x_1, 0)$, and (0,1) (see Fig. 3) is given by the solution of the system

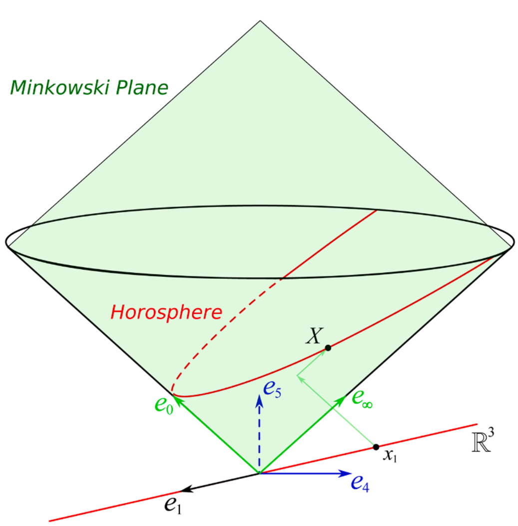

Fig. 2. Horosphere and Minkowski plane.

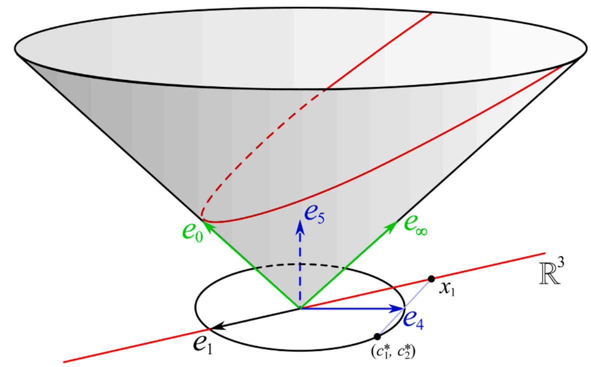

Fig. 3. The stereographic projection.

$$\begin{aligned} c_1^2 + c_2^2 &= 1, \\ \frac{c_2}{c_1 - x_1} &= \frac{-1}{x_1}, \end{aligned}$$

which is equivalent to

$$c_1^2 + \left(1 - \frac{c_1}{x_1}\right)^2 = 1 \Leftrightarrow (x_1^2 + 1)c_1^2 - 2x_1 c_1 = 0.$$

Ignoring the root $c_1 = 0$, the result of the intersection is

$$(c_1^*, c_2^*) = \left(\frac{2x_1}{x_1^2 + 1}, \frac{x_1^2 - 1}{x_1^2 + 1}\right).$$

Considering e_5 as a homogeneous coordinate, we can represent (c_1^*, c_2^*) as

$$\frac{2x_1}{x_1^2 + 1} e_1 + \frac{x_1^2 - 1}{x_1^2 + 1} e_4 + e_5. \tag{19}$$

From (10) and (11),

$$\begin{aligned} e_4 &= \frac{e_\infty - 2e_0}{2}, \\ e_5 &= \frac{e_\infty + 2e_0}{2}, \end{aligned}$$

and using these values in (19), we obtain

$$\begin{aligned} \tfrac{2x_1}{x_1^2+1} e_1 + \tfrac{x_1^2-1}{x_1^2+1} e_4 + e_5 &= \left(\tfrac{2x_1}{x_1^2+1}\right) e_1 + \left(\tfrac{x_1^2-1}{x_1^2+1}\right)\left(\tfrac{e_\infty - 2e_0}{2}\right) + \left(\tfrac{e_\infty + 2e_0}{2}\right) \\ &= \frac{2}{x_1^2 + 1}\left(x_1 e_1 + e_0 + \frac{1}{2}\| x_1 \|^2 e_\infty\right), \end{aligned}$$

which means that, using the basis $\{e_1, e_4, e_5\}$, up to a scale of factor depending on x_1, the conformal representation of a point x in the 3D space is obtained from the homogenization of the stereographic projection of x (see Fig. 4). This explains why the conformal model is also

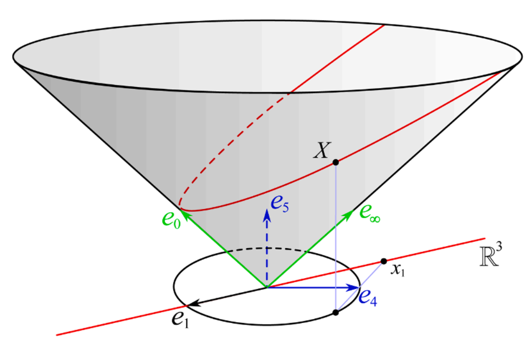

Fig. 4. The homogenization of the stereographic projection.

called *generalized homogeneous coordinates*.

5. An application: matrix representation of isometries

As an application of the conformal model, we deduce now what is the matrix form of an isometry in $\mathbb{R}^3$, represented in $\mathbb{R}^5$.

Let $f:\mathbb{R}^3\to\mathbb{R}^3$ be an isometry given by

$$f(x)=Ax+b, \tag{20}$$

where $A\in\mathbb{R}^{3\times3}$ and $b\in\mathbb{R}^3$.

From (12), $f(x)$ is given in the conformal model by

$$F(x)=(Ax+b)+e_0+\frac{1}{2}\| Ax+b \|^2 e_\infty,$$

with

$$\begin{aligned}\tfrac{1}{2}\| Ax+b \|^2 &= \tfrac{1}{2}\left[(Ax+b)^T(Ax+b)\right]\\ &=\frac{1}{2}\left[x^TA^TAx+x^TA^Tb+b^TAx+b^Tb\right]\\ &=\frac{1}{2}\left[x^TA^TAx+\left(b^TAx\right)^T+b^TAx+b^Tb\right].\end{aligned}$$

Since $A^{-1}=A^T$ and $b^TAx\in\mathbb{R}$, we obtain

$$\frac{1}{2}\| Ax+b \|^2=b^TAx+\frac{\| b \|^2}{2}+\frac{\| x \|^2}{2},$$

which allows to represent f as a linear transformation in the conformal model:

$$\begin{bmatrix} A & b & 0\\ 0 & 1 & 0\\ b^TA & \frac{\| b \|^2}{2} & 1\end{bmatrix}\begin{bmatrix} x\\ 1\\ \frac{\| x \|^2}{2}\end{bmatrix}=\begin{bmatrix} Ax+b\\ 1\\ \frac{\| Ax+b \|^2}{2}\end{bmatrix}.$$

From the homogeneous model, we already know that isometries can be represented linearly, given by

$$\begin{bmatrix} A & b\\ 0 & 1\end{bmatrix}\begin{bmatrix} x\\ 1\end{bmatrix}=\begin{bmatrix} Ax+b\\ 1\end{bmatrix},$$

but the conformal model can do more.

In the conformal model, the matrix representation of the inner product $X{\cdot}Y$ is given by

$$\begin{aligned} X{\cdot}Y &= \begin{bmatrix} x & 1 & \frac{\| x \|^2}{2}\end{bmatrix}\begin{bmatrix} I & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix} y\\ 1\\ \frac{\| y \|^2}{2}\end{bmatrix}\\ &= x{\cdot}y-\left(\frac{1}{2}\| x \|^2+\frac{1}{2}\| y \|^2\right)\\ &= \left(x+e_0+\frac{1}{2}\| x \|^2e_\infty\right)\cdot\left(y+e_0+\frac{1}{2}\| y \|^2e_\infty\right),\end{aligned}$$

where $I\in\mathbb{R}^{3\times3}$ is the identity matrix.

Recalling that the main property of an orthogonal transformation is that it can be inverted in a simple way, let us calculate

$$U^TI_cU,$$

where

$$U=\begin{bmatrix} A & b & 0\\ 0 & 1 & 0\\ b^TA & \frac{\| b \|^2}{2} & 1\end{bmatrix}\ \text{and}\ I_c=\begin{bmatrix} I & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{bmatrix},$$

resulting in

$$\begin{aligned} U^TI_cU &= \begin{bmatrix} A^T & 0 & A^Tb\\ b^T & 1 & \frac{\| b \|^2}{2}\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} I & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix} A & b & 0\\ 0 & 1 & 0\\ b^TA & \frac{\| b \|^2}{2} & 1\end{bmatrix}\\ &= \begin{bmatrix} A^T & -A^Tb & 0\\ b^T & -\frac{\| b \|^2}{2} & -1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix} A & b & 0\\ 0 & 1 & 0\\ b^TA & \frac{\| b \|^2}{2} & 1\end{bmatrix}\\ &= \begin{bmatrix} I & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{bmatrix}\\ &= I_c.\end{aligned}$$

In order to get the orthogonality of U, we also need to calculate

$$\begin{aligned} (I_c)^2 &= \begin{bmatrix} I & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix} I & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{bmatrix}\\ &= \begin{bmatrix} I & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\end{aligned}$$

which implies that (X,Y are the conformal representations of $x,y\in\mathbb{R}^3$)

$$\begin{aligned} UX &= Y\\ &\Leftrightarrow\\ \left(U^TI_c\right)UX &= \left(U^TI_c\right)Y\\ &\Leftrightarrow\\ \left(U^TI_cU\right)X &= \left(U^TI_c\right)Y\\ &\Leftrightarrow\\ I_cX &= \left(U^TI_c\right)Y\\ &\Leftrightarrow\\ X &= \left(I_cU^TI_c\right)Y.\end{aligned}$$

This means that, in the conformal model, isometries are not merely linear transformations (as in the homogeneous model), but also orthogonal ones. Of course, the orthogonality is defined in terms of the new metric given by (13)–(17), whose matrix representation is

$$U^{-1}=I_cU^TI_c.$$

5.1. A concrete example

In computer graphics, one of the celebrated advantage of the homogeneous model, compared to the Euclidean model of the 3D space, is that composition of isometries can be given by multiplying matrices, which can be useful because of representational simplicity and computational efficiency. In the conformal model, this property is preserved and, in addition to that, the inversion of the composition is also encoded as matrix products.

For example, consider two rigid motions $f_1:\mathbb{R}^3\to\mathbb{R}^3$ and $f_2:\mathbb{R}^3\to\mathbb{R}^3$, given by

$$f_1(x)=A_1x+b_1$$

and

$$f_2(x)=A_2x+b_2,$$

where $A_1,A_2\in\mathbb{R}^{3\times3}$ and $b_1,b_2\in\mathbb{R}^3$.

In the conformal model, if f_1 and f_2 are represented respectively by

$$U_1 = \begin{bmatrix} A_1 & b_1 & 0 \\ 0 & 1 & 0 \\ b_1^T A_1 & \frac{\| b_1\|^2}{2} & 1 \end{bmatrix}$$

and

$$U_2 = \begin{bmatrix} A_2 & b_2 & 0 \\ 0 & 1 & 0 \\ b_2^T A_2 & \frac{\| b_2\|^2}{2} & 1 \end{bmatrix},$$

the composition $f_2 \circ f_1$ would be given by

$$U_2 U_1,$$

instead of

$$(f_2 \circ f_1)(x) = f_2(f_1(x)) = A_2(A_1 x + b_1) + b_2 = A_2 A_1 x + A_2 b_1 + b_2,$$

and the inversion $(f_2 \circ f_1)^{-1}$ would be given by

$$(U_2 U_1)^{-1} = I_c (U_2 U_1)^T I_c,$$

instead of

$$\begin{aligned} (f_2 \circ f_1)^{-1}(y) &= f_1^{-1}(f_2^{-1}(y)) \\ &= f_1^{-1}\left(A_2^T (y - b_2)\right) \\ &= f_1^{-1}\left(A_2^T y - A_2^T b_2\right) \\ &= A_1^T\left(\left(A_2^T y - A_2^T b_2\right) - b_1\right) \\ &= A_1^T A_2^T y - A_1^T A_2^T b_2 - A_1^T b_1. \end{aligned}$$

Just to illustrate numerically, let f_1 be a rotation around axis e_1 defined by $\theta_1 = \frac{\pi}{6}$ radians, followed by a translation given by vector $b_1 = [0.1, -0.2, 0.3]^T$, and f_2 be a rotation around axis e_3 defined by $\theta_2 = \frac{\pi}{3}$ radians, followed by a translation given by vector $b_2 = [0.3, -0.2, 0.1]^T$. Doing the calculations, we obtain

$$U_2 U_1 = \begin{bmatrix} 0.50000 & -0.75000 & 0.43301 & 0.52321 & 0.00000 \\ 0.86603 & 0.43301 & -0.25000 & -0.21340 & 0.00000 \\ 0.00000 & 0.50000 & 0.86603 & 0.40000 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0.00000 \\ 0.07679 & -0.28481 & 0.62631 & 0.23964 & 1.00000 \end{bmatrix}$$

and

$$(U_2 U_1)^{-1} = \begin{bmatrix} 0.50000 & 0.86603 & 0.00000 & -0.07679 & 0.00000 \\ -0.75000 & 0.43301 & 0.50000 & 0.28481 & 0.00000 \\ 0.43301 & -0.25000 & 0.86603 & -0.62631 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0.00000 \\ -0.52321 & 0.21340 & -0.40000 & 0.23964 & 1.00000 \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. Final comments

An even richer structure (a *Clifford algebra* [14,27,29]) can be obtained from the conformal model if a more general product, called *geometric product*, is defined [7] (page 439). This product is associative and distributive (but not commutative in general), and has the following properties, for $i, j = 1, 2, 3$:

$$e_i e_j + e_j e_i = 2\delta_{ij}, \tag{21}$$

$$e_0 e_i = -e_i e_0, \tag{22}$$

$$e_\infty e_i = -e_i e_\infty, \tag{23}$$

$$e_0^2 = e_\infty^2 = 0, \tag{24}$$

$$e_0 e_\infty + e_\infty e_0 = -2. \tag{25}$$

An important consequence from the properties above is that we can define the inner product in the conformal model as

$$e_i \cdot e_j = \frac{1}{2}\left(e_i e_j + e_j e_i\right), \tag{26}$$

for $i, j = 0, 1, 2, 3, \infty$, implying that properties (13)–(17) are obtained from (21) to (25).

More interestingly, also because the geometric product is not commutative, we can define another product, called *outer product* [7] (page 441), given by

$$e_i \wedge e_j = \frac{1}{2}\left(e_i e_j - e_j e_i\right), \tag{27}$$

for $i, j = 0, 1, 2, 3, \infty$ (from (27), notice that $e_i \wedge e_j = 0$ implies that e_i and e_j are parallel vectors, for $i, j = 1, 2, 3$, which means that both equations, $e_i \cdot e_j = 0$ and $e_i \wedge e_j = 0$, have geometric meaning).

From (21) and (27), and considering $i \neq j$, for $i, j = 1, 2, 3$, we obtain

$$\begin{aligned} (e_i \wedge e_j)^2 &= \left(\tfrac{1}{2}\left(e_i e_j - e_j e_i\right)\right)\left(\tfrac{1}{2}\left(e_i e_j - e_j e_i\right)\right) \\ &= \frac{1}{4}\left(e_i e_j e_i e_j - e_i e_j e_j e_i - e_j e_i e_i e_j - e_i e_j e_j e_i\right) \\ &= \frac{1}{4}(-1 - 1 - 1 - 1) \\ &= -1, \end{aligned}$$

which implies that $e_i \wedge e_j$ is not a real number. Since the norm of a vector v in conformal geometric algebra (CGA) is defined by $\| v \|^2 = v^2$ [7] (page 439), $e_i \wedge e_j$ can neither be a vector, because we would get

$$(e_i \wedge e_j)^2 \geq 0.$$

As a consequence, $e_i \wedge e_j$, for $i \neq j$ and $i, j = 1, 2, 3$, is a new object, called a *bivector* (or a 2-vector). In a similar way, 3-vectors, 4-vectors, and 5-vectors can be defined in CGA. Scalars and vectors are also called 0-vectors and 1-vectors, respectively. The linear combination of all these k-vectors, $k = 0, 1, \ldots, 5$, called *multivectors*, generates a 32D space and offers a very rich mathematical structure with many applications, as said in the Introduction. Much more details can be obtained in the Wikipedia article on CGA [30] and in the books cited here.

We hope that our distinct approach to CGA, compared to the classical literature, introducing first the conformal model of the 3D space, with a new inner product in $\mathbb{R}^5$, and giving intuition about the extra two dimensions (e_0 and e_∞), encourages the readers to get a step further studying CGA, which can be seen as an ideal framework for computational Euclidean geometry.

The conformal model is covered by the US Patent 6853964 (D. Hestenes, H. Li, A. Rockwood, System for encoding and manipulating models of objects, granted 8 February 2005), although there are no restrictions for academic research.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We are grateful to the Brazilian research agencies CNPq (424507/2018-8) and FAPESP (2019/20047-8) for their financial support and to the reviewer comments.

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