LEXTRA LEARNING

ADVANCED LEVEL MATHEMATICS

A2: Differentiation and Integration – Year 13

Name: SOLUTIONS

TIME ALLOWED: 1 HOUR

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **6** questions and all are compulsory.
2. Answer all questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT an OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

LEXTRA LEARNING

Question 1. The curve C has equation $x^2\tan(y) = 9$ for $0 < y < \frac{\pi}{2}$

(i) *(4 marks)* Show that

$$\frac{dy}{dx} = \frac{-18x}{x^4+81}$$

(ii) *(3 marks)* Show that the point of inflection of C lies at $x = \sqrt[4]{27}$

(i)

- **Standard Differentiation:** $\frac{d}{dx}(x^2) = 2x$
- **Formulae Booklet:** $\frac{d}{dx}\tan(x) = \sec^2(x)$
- **Implicit Differentiation:** For $y = y(x)$; $\frac{d}{dx}\tan(y) = \sec^2(y)\frac{dy}{dx}$
- **Product Rule:** $\frac{d}{dx}\left(x^2\tan(y)\right) = 2x\tan(y) + x^2\sec^2(y)\frac{dy}{dx}$
- **Equation from question:**

$$\frac{d}{dx}\left(x^2\tan(y)\right) = \frac{d}{dx}(9)$$
$$2x\tan(y) + x^2\sec^2(y)\frac{dy}{dx} = 0$$

- **Trigonometric formula:** $\tan^2(\theta) + 1 = \sec^2(\theta)$
- **Back to the equation from question (differentiated):**

$$2x\tan(y) + x^2\left[\tan^2(y) + 1\right]\frac{dy}{dx} = 0$$
$$2x\tan(y) + [x^2\tan^2(y) + x^2]\frac{dy}{dx} = 0$$

- **Original equation can be rearranged:** $\tan(y) = \dfrac{9}{x^2}$
- **Substituting back to differentiated equation:**

$$2x\left[\frac{9}{x^2}\right] + \left(x^2\left[\frac{9}{x^2}\right]^2 + x^2\right)\frac{dy}{dx} = 0$$

- **After simplifying:** $\dfrac{dy}{dx} = \dfrac{-18x}{x^4+81}$

(ii)

- **Quotient Rule:** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2}$
- **Differentiate the answer from (i):** For us $u = -18x$ and $v = x^4 + 81 \Rightarrow \dfrac{du}{dx} = -18$ and $\dfrac{dv}{dx} = 4x^3$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-18(x^4+81) - -18x(4x^3)}{(x^4+81)^2}$$

- **At point of inflection,** $\dfrac{d^2y}{dx^2} = 0$**:**

$$\Rightarrow -18(x^4+81) - -18x(4x) = 0$$
$$\Rightarrow x^4 = 27$$
$$\Rightarrow x = \sqrt[4]{27}$$

Question 2. *(7 marks)* Evaluate the following integral
HINT: USE SUBSTITUTION $\alpha = \ln(x)$, THEN INTEGRATE BY PARTS

$$I = \int \sin(\ln(x))dx$$

(ii)

- **Substitution:** $\alpha = \ln(x)$

$$\begin{aligned} d\alpha &= \frac{1}{x}dx \\ xd\alpha &= dx \\ e^{\alpha}du &= dx \end{aligned}$$

- **Replace all of x in I with α:**

$$\begin{aligned} I &= \int \sin(\ln(x))dx \\ &= \int \sin(\alpha)e^{\alpha}d\alpha \\ \therefore I &= \int e^{\alpha}\sin(\alpha)d\alpha \end{aligned}$$

- **We now have to do integration by parts:** $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$.
  We let $u = \sin(\alpha)$ and $\frac{dv}{d\alpha} = e^{\alpha} \Rightarrow \frac{du}{d\alpha} = \cos(\alpha)$ and $v = e^{\alpha}$

$$\begin{aligned} \therefore I &= e^{\alpha}\sin(\alpha) - \int e^{\alpha}\cos(\alpha)d\alpha \\ I &= e^{\alpha}\sin(\alpha) - G \end{aligned}$$

- **We now have another integration by parts equation:** $G = \int e^{\alpha} \cos(\alpha) d\alpha$.

  The integration by parts for this integral is $\int \overline{u} \frac{d\overline{v}}{dx} dx = \overline{uv} - \int \overline{v} \frac{d\overline{u}}{dx} dx$.

  We will choose $\overline{u} = \cos(\alpha)$ and $\frac{d\overline{v}}{d\alpha} = e^{\alpha} \Rightarrow \frac{d\overline{u}}{d\alpha} = -\sin(\alpha)$ and $\overline{v} = e^{\alpha}$

$$
\begin{aligned}
G &= e^{\alpha} \cos(\alpha) - \int \left(e^{\alpha} \times - \sin(\alpha) \right) d\alpha \\
G &= e^{\alpha} \cos(\alpha) + \int e^{\alpha} \sin(\alpha) d\alpha \\
G &= e^{\alpha} \cos(\alpha) + I
\end{aligned}
$$

- **Substituting G back into I:**

$$
\begin{aligned}
I &= e^{\alpha} \sin(\alpha) - e^{\alpha} \cos(\alpha) - I + c \\
2I &= e^{\alpha} \left(\sin(\alpha) - \cos(\alpha) \right) + c \\
\Rightarrow I &= \frac{e^{\alpha}}{2} \left(\sin(\alpha) - \cos(\alpha) \right) + k \\
\therefore I &= \frac{x}{2} \left(\sin(\ln(x)) - \cos(\ln(x)) \right) + k
\end{aligned}
$$

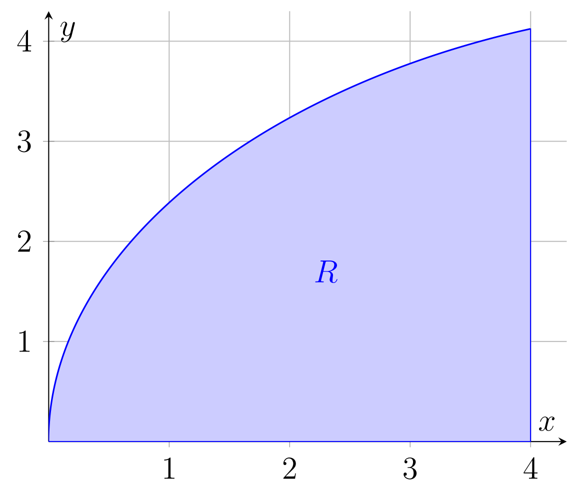

Figure 1: The curve C, with shaded region R

Question 3. For $0 \leq t \leq \dfrac{\pi}{2}$, the curve C has following parametric equations

$$\begin{aligned} x(t) &= 8\sin^2(t) \\ y(t) &= 2\sin(2t) + 3\sin(t) \end{aligned}$$

(i) *(7 marks)* Show that for region R bounded below the curve to x-axis from $x = 0$ to $x = 4$, as shown in FIGURE 1 has area given by

$$\int_0^a \big(8 - 8\cos(4t) + 48\sin^2(t)\cos(t)\big)\, dt$$

(ii) *(4 marks)* Hence, use algebraic integration to evaluate the area in region R as shown in FIGURE 1

(i)

- **Integration as area under the curve:** The area of region R is given by standard integration $R = \int_0^4 y dx$. We see here that the lowest limit is at $x = 0$ and highest one is at $x = 4$

$$\begin{aligned} &\Rightarrow x(t) = 0 \\ &\Rightarrow x(t) = 4 \end{aligned}$$

- **We need to solve for the two equations above:** The first one is trivial $x(t) = 0 \Rightarrow 8\sin^2(t) = 0 \Rightarrow t = 0$, the second one requires some work. Since we know the upper limit solution lies at $t = a$, we make this replacement

$$\begin{aligned} x(a) &= 4 \\ 8\sin^2(a) &= 4 \\ \sin^2(a) &= \frac{1}{2} \\ \sin(a) &= \frac{1}{\sqrt{2}} \\ \Rightarrow a &= \frac{\pi}{4} \end{aligned}$$

- **We now look at parametric integration:** We already know the form of the integral as Cartesian integral, we just need to turn that integral from an integral of x to an integral of t

$$\begin{aligned} R &= \int_0^4 y dx \\ R &= \int_{x=0}^{x=4} y(t) dx \\ R &= \int_{x=0}^{x=4} y(t) \frac{dx}{dt} dt \\ R &= \int_{t=0}^{t=a} y(t) \frac{dx}{dt} dt \end{aligned}$$

- **Putting it all together:** We already know the value of a.
  We already know $y(t)$ as it is given to us in the question, $y(t) = 2\sin(2t) + 3\sin(t)$.
  $x(t)$ is already given to us in the question, $x(t) = 8\sin^2(t)$. There-

fore, we can solve $\frac{dx}{dt}$

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}\left(8\sin^2(t)\right)\\ &= 16\sin(t)\cos(t)\\ &= 8[2\sin(t)\cos(t)]\\ &= 8\sin(2t)\end{aligned}$$

We now return to our integral R but before we do we will require two identities from trigonometry, $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and $\cos(2\theta) = 2cos^2(\theta) - 1$.

$$\begin{aligned}R &= \int_0^{\frac{\pi}{4}} [2\sin(2t) + 3\sin(t)]\, 8\sin(2t)dt\\ &= \int_0^{\frac{\pi}{4}} \left[16\sin^2(2t) + 24\sin(t)\sin(2t)\right] dt\\ &= \int_0^{\frac{\pi}{4}} \left[16\left(1-\cos^2(2t)\right) + 24\sin(t)\times 2\sin(t)\cos(t)\right] dt\\ &= \int_0^{\frac{\pi}{4}} \left[16 - 16\cos^2(2t) + 48\sin^2(t)\cos(t)\right] dt\\ &= \int_0^{\frac{\pi}{4}} \left[16 - 16\left(\frac{\cos(4t)+1}{2}\right) + 48\sin^2(t)\cos(t)\right] dt\\ &= \int_0^{\frac{\pi}{4}} \left[16 - 8\left(\cos(4t)+1\right) + 48\sin^2(t)\cos(t)\right] dt\\ &= \int_0^{\frac{\pi}{4}} \left[8 - 8\cos(4t) + 48\sin^2(t)\cos(t)\right] dt\end{aligned}$$

(ii)

- **Split the integration into smaller, easier integration questions:**

$$
\begin{aligned}
R &= R_1 + R_2 + R_3 \\
R_1 &= \int_0^{\frac{\pi}{4}} [8] dt \\
R_2 &= \int_0^{\frac{\pi}{4}} \left[-8\cos(4t)\right] dt \\
R_3 &= \int_0^{\frac{\pi}{4}} \left[48\sin^2(t)\cos(t)\right] dt
\end{aligned}
$$

- **Evaluating R_1:** This one is rather trivial, $R_1 = [8t]_0^{\pi\backslash 4} = 2\pi$
- **Evaluating R_2:** This one is a little more involved,

$$
\begin{aligned}
R_2 &= \int_0^{\frac{\pi}{4}} \left[-8\cos(4t)\right] dt \\
&= \left[-\frac{8}{4}\sin(4t)\right]_0^{\pi\backslash 4} \\
&= 0
\end{aligned}
$$

- **Evaluating R_3:** This one requires integration by substitution, we use $u = \sin(t) \Rightarrow du = \cos(t)dt \Rightarrow dt = \dfrac{1}{\cos(t)}du$,

$$
\begin{aligned}
R_3 &= \int_{x=0}^{x=\frac{\pi}{4}} \left[48\sin^2(t)\cos(t)\right] dt \\
&= \int_{x=0}^{x=\frac{\pi}{4}} \left[48u^2\cos(t)\right] \frac{1}{\cos(t)}du \\
&= \int_{u=0}^{u=\frac{1}{\sqrt{2}}} \left[48u^2\right] du \\
&= [16u^3]_0^{1\backslash\sqrt{2}} \\
&= 4\sqrt{2}
\end{aligned}
$$

Therefore, the area in region R is given by $2\pi + 4\sqrt{2}$.

Question 4. There exists a smooth and continuous function $y = f(x)$ which obeys the following differential equation for $0 < y < 1$

$$2\frac{dy}{dx} = y - y^2$$

(i) *(5 marks)* Using partial fractions or otherwise, find the general solution to the differential equation in the form $y = f(x)$

(ii) *(3 marks)* Evaluate the following limits

$$\lim_{x \to +\infty} f(x)$$
$$\lim_{x \to -\infty} f(x)$$

(i)

- **Turn the differential equation into an integral equation:**

$$\begin{aligned} 2\frac{dy}{dx} &= y - y^2 \\ 2\frac{dy}{dx} &= y(1-y) \\ \int \frac{1}{y(1-y)} dy &= \int 2dx \\ \int \frac{1}{y(1-y)} dy &= 2x + c \end{aligned}$$

- **We use partial fractions to split LHS:**

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$
$$\Rightarrow A = B = 1$$

- **Returning back to the integral:**

$$
\begin{aligned}
\int \frac{1}{y(1-y)} dy &= 2x + c \\
\int \left(\frac{1}{y} + \frac{1}{1-y} \right) dy &= 2x + c \\
\Rightarrow \ln(y) - \ln(1-y) &= 2x + c \\
\Rightarrow \ln\left(\frac{y}{1-y} \right) &= 2x + c \\
\Rightarrow \left(\frac{y}{1-y} \right) &= e^{2x+c} \\
&= e^{2x} e^{c} \\
\Rightarrow \left(\frac{y}{1-y} \right) &= e^{c} e^{2x}
\end{aligned}
$$

- **Constant to the power of a constant:** e^c is a constant, it is just a number. Therefore, $e^c = A$, where A is some constant.

$$
\begin{aligned}
\left(\frac{y}{1-y} \right) &= Ae^{2x} \\
\Rightarrow y &= (1-y)Ae^{2x} \\
\Rightarrow y &= Ae^{2x} - yAe^{2x} \\
\Rightarrow y + yAe^{2x} &= Ae^{2x} \\
\Rightarrow y(1 + Ae^{2x}) &= Ae^{2x} \\
\Rightarrow y = f(x) &= \frac{Ae^{2x}}{1 + Ae^{2x}}
\end{aligned}
$$

(ii)

- **Rewriting $f(x)$:**

$$f(x) = \frac{Ae^{2x}}{1 + Ae^{2x}} \times \frac{e^{-2x}}{e^{-2x}}$$
$$f(x) = \frac{A}{e^{-2x} + A}$$

- **As $x \to +\infty$:** As x increases, e^{-2x} decreases rapidly!

$$\lim_{x \to +\infty} f(x) = \frac{A}{0 + A}$$
$$= \frac{A}{A} = 1$$

- **As $x \to -\infty$:** As x increases, e^{-2x} increases rapidly! This results in $\frac{1}{e^{-\infty}}$ to decrease rapidly!

$$\lim_{x \to -\infty} f(x) = 0$$

Question 5. *(7 marks)* Boyle's Law states that when a gas is kept at a constant temperature, its pressure P, which is measured in Newton per meter Nm^{-2}, is inversely proportional to its volume V, which is measured in cubic meters m^3.

When the volume of a certain gas is $80m^3$, its pressure is $5Nm^{-2}$ and the rate at which the volume is increasing is $10m^3s^{-1}$.

Find the rate at which the pressure is decreasing at this volume.

- **A little context:** Anyone doing A-Level Physics should find this walk in the park. If you are not though, it is not that hard.

- **Boyle's Law:** We are told that Bolye's law is $P \propto \frac{1}{V}$. However, at $V = 80$ we are told $P = 5$

$$\begin{aligned} P &\propto \frac{1}{V} \\ P &= \frac{k}{V} \\ \Rightarrow PV &= k \\ \Rightarrow 5 \times 80 &= k \\ \Rightarrow 400 &= k \end{aligned}$$

Also keep in mind, $P = \frac{400}{V} = 400V^{-1}$
$\Rightarrow \frac{dP}{dV} = -400V^{-2} = -\frac{400}{V^2}$.
We are also told $\left.\frac{dV}{dt}\right|_{V=80} = 10$

- **Chain Rule:** We know from chain rule $\frac{dP}{dt} = \frac{dP}{dV} \times \frac{dV}{dt}$. However, we need this to be evaluated at the point $V = 80$.

$$\begin{aligned}
\left.\frac{dP}{dt}\right|_{V=80} &= \left.\frac{dP}{dV}\right|_{V=80} \times \left.\frac{dV}{dt}\right|_{V=80} \\
\Rightarrow \left.\frac{dP}{dt}\right|_{V=80} &= -\left.\frac{400}{V^2}\right|_{V=80} \times 10 \\
\Rightarrow \left.\frac{dP}{dt}\right|_{V=80} &= -\frac{400}{80^2} \times 10 \\
\Rightarrow \left.\frac{dP}{dt}\right|_{V=80} &= -0.625 Nm^{-2}s^{-1}
\end{aligned}$$

Question 6. A curve has equation $y = \dfrac{xe^{2x}}{x+k}$, where k is a constant and $k \neq 0$.

(i) *(5 marks)* Show that

$$\frac{dy}{dx} = \frac{e^{2x}\left(2x^2 + 2kx + k\right)}{(x+k)^2}$$

(ii) *(5 marks)* Given that y only has one stationary point, find the value of k and determine the nature of this stationary point.

(i)

- **Quotient Rule:** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2}$

  For us $u = xe^{2x}$ and $v = x + k$. Therefore, $\dfrac{du}{dx} = e^{2x} + 2xe^{2x}$ and $\dfrac{dv}{dx} = 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(e^{2x} + 2xe^{2x}\right)(x+k) - xe^{2x}}{(x+k)^2} \\ &= e^{2x}\frac{(1+2x)(x+k) - x}{(x+k)^2} \\ &= e^{2x}\frac{x + k + 2x^2 + 2kx - x}{(x+k)^2} \\ &= \frac{e^{2x}\left(2x^2 + 2kx + k\right)}{(x+k)^2}\end{aligned}$$

(ii)

- **Stationary point:** This means first derivative is zero. That is only possible if $e^{2x}\left(2x^2+2kx+k\right)=0$

$$\begin{aligned} e^{2x}\left(2x^2+2kx+k\right) &= 0 \\ \left(2x^2+2kx+k\right) &= 0 \end{aligned}$$

- **Only one stationary point:** The above quadratic has one solution if the discriminant is zero.
  For a general quadratic $ax^2+bx+c=0$, the discriminant is b^2-4ac.

$$\begin{aligned} \Rightarrow (2k)^2-4(2\times k) &= 0 \\ 4k^2-8k &= 0 \\ k(4k-8) &= 0 \end{aligned}$$

  Given $k\neq 0$, we deduce that $k=2$.

- **Nature of stationary point:** We could keep going and differentiate the first derivative to determine the point but we can be more clever!
  Let us first find the stationary point, remember above we said the stationary point satisfies the following quadratic $2x^2+4x+2=0$.

$$\begin{aligned} \left(2x^2+4x+2\right) &= 0 \\ \left(x^2+2x+1\right) &= 0 \\ (x+1)^2 &= 0 \end{aligned}$$

  Hence, the stationary point is at $x=-1$.
  Now we attempt to be clever, though not the most rigorous.
  If this stationary point is minimum, the y value at this point will be some number, but the y value slightly beyond this point will be higher than what it was before. The same analysis applies to the maximum point too, with it being lower than before! At $x=-1$, $y=-e^{-2}\approx -0.135335$

At $x = -0.8$, $y = -\dfrac{2}{3e^{8\backslash\ 5}} \approx -0.134598$
We see that the value of y has decreased as we went slightly above the stationary point. Therefore, this is a maximum point!

END OF PAPER