LEXTRA LEARNING

ADVANCED LEVEL FURTHER MATHEMATICS

A2F: Differential Equations – Year 13

Name: TIME ALLOWED: 1 HOUR

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **3** questions and all are compulsory.
2. Answer all questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT an OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

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Question 1. *(7 marks)* For $x > 0$, use substitution $u = y^2$ to find the general solution to the following first-order differential equation

$$\frac{dy}{dx} - 2y = \frac{e^x}{y}$$

Question 2. The two functions $x(t)$ and $y(t)$ obey the following two coupled simultaneous first-order differential equations

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = -x - 4y + e^{2t}$$

The boundary conditions are $x(0) = 5$ and $y(0) = 0$

(i) *(14 marks)* Eliminate y to obtain a second order differential equation for $x(t)$ and find the general solution for $x(t)$

(ii) *(3 marks)* Find the general solution for $y(t)$

(iii) *(4 marks)* Using the boundary conditions, find particular solutions to $x(t)$ and $y(t)$

(iv) *(3 marks)* Show that

$$\lim_{t\to+\infty}\left(\frac{y}{x}\right) = -\frac{1}{2}$$

Question 3. In quantum mechanics, a particle has a wave function which encodes all the information about the quantum particle. The wave function, $\psi(x)$, for a particle with some potential energy, $U(x)$, obeys the Schrödinger equation, which is a second-order differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi = E\psi$$

where $\hbar$ is a constant known as "Reduced Plank's Constant" and m, E are the mass and energy of the particle, respectively. All these constants, except E, are assumed to be positive.
The particle is subject to the following potential, called the "Infinite Well Potential"

$$U(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & x < 0 \\ \infty & x > L \end{cases}$$

This results in the following boundary conditions, $\psi(0) = 0$ and $\psi(L) = 0$
In the region $0 \leq x \leq L$, the Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

(i) *(4 marks)* Prove why the particle cannot have negative energy $E < 0$.
HINT: Think about the boundary conditions due to the potential. Also, $\psi(x) = 0$ isn't allowed for general x for reasons that will become clear in (iv)

(ii) *(6 marks)* For $E > 0$, find the solution of $\psi(x)$ subject to the boundary condition due to Infinite Well.
HINT: $sin(\theta) = 0$ means θ is an integer multiple of $\pi \Rightarrow \theta = n\pi$

(iii) *(4 marks)* Show that the energy of the particle is given by

$$E = \frac{\hbar^2\pi^2 n^2}{2mL^2}$$

where n is a positive integer that is not zero.

(iv) *(5 marks)* Given that in position representation,

$$P(a \leq x \leq b) = \int_a^b \mid \psi(x) \mid^2 dx$$

represents the probability of finding the particle in the region $a \leq x \leq b$. Given that the particle has to be trapped in the Infinite Well, use the fact that

$$\int_0^L \mid \psi(x) \mid^2 dx = 1$$

to find the remaining unknown constants in $\psi(x)$

END OF PAPER