LEXTRA LEARNING

ADVANCED LEVEL FURTHER MATHEMATICS

A2: Differential Equations – Year 13

Name: SOLUTIONS TIME ALLOWED: 1 HOUR

<u>INSTRUCTIONS TO CANDIDATES</u>

1. This examination paper contains **3** questions and all are compulsory.
2. Answer all questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT an OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

LEXTRA LEARNING

Question 1. *(7 marks)* For $x > 0$, use substitution $u = y^2$ to find the general solution to the following first-order differential equation

$$\frac{dy}{dx} - 2y = \frac{e^x}{y}$$

- **Substitution:** $u = y^2 \Rightarrow \dfrac{du}{dx} = 2y\dfrac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} - 2y &= \frac{e^x}{y}\\ \Rightarrow 2y\frac{dy}{dx} - 4y^2 &= 2e^x\\ \Rightarrow \frac{du}{dx} - 4u &= 2e^x\end{aligned}$$

- **Solving for $u(x)$:** Integrating factor $I = e^{\int -4dx} = e^{-4x}$

$$\begin{aligned}\Rightarrow \left(\frac{du}{dx} - 4u\right) I &= (2e^x)\, I\\ \Rightarrow \left(\frac{du}{dx} - 4u\right) e^{-4x} &= (2e^x)\, e^{-4x}\\ \Rightarrow e^{-4x}\frac{du}{dx} - 4ue^{-4x} &= 2e^{-3x}\\ \Rightarrow \frac{d}{dx}\left[e^{-4x}u\right] &= 2e^{-3x}\end{aligned}$$

- **Getting $y(x)$:** $e^{-4x}u = -\dfrac{2}{3}e^{-3x} + C$

$$\begin{aligned}u(x) &= -\frac{2}{3}e^x + Ce^{4x}\\ y^2 &= -\frac{2}{3}e^x + Ce^{4x}\\ \therefore y(x) &= \sqrt{Ce^{4x} - \frac{2}{3}e^x}\end{aligned}$$

Question 2. The two functions $x(t)$ and $y(t)$ obey the following two coupled simultaneous first-order differential equations

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = -x - 4y + e^{2t}$$

The boundary conditions are $x(0) = 5$ and $y(0) = 0$

(i) *(14 marks)* Eliminate y to obtain a second order differential equation for $x(t)$ and find the general solution for $x(t)$

(ii) *(3 marks)* Find the general solution for $y(t)$

(iii) *(4 marks)* Using the boundary conditions, find particular solutions to $x(t)$ and $y(t)$

(iv) *(3 marks)* Show that

$$\lim_{t\to+\infty}\left(\frac{y}{x}\right) = -\frac{1}{2}$$

(i)

- **Two to one:**

$$\frac{dx}{dt} = -x + 2y \Rightarrow y = \frac{1}{2}\left(\frac{dx}{dt} + x\right) \Rightarrow \frac{dy}{dt} = \frac{1}{2}\left(\frac{d^2x}{dt^2} + \frac{dx}{dt}\right)$$
$$\frac{dy}{dt} = -x - 4y + e^{2t}$$
$$\frac{dy}{dt} = -x - 2\left(\frac{dx}{dt} + x\right) + e^{2t}$$
$$\Rightarrow \frac{1}{2}\left(\frac{d^2x}{dt^2} + \frac{dx}{dt}\right) = -x - 2\left(\frac{dx}{dt} + x\right) + e^{2t}$$
$$\Rightarrow \frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{2t}$$

- **Complementary Function:** $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 6x = 0$

$$\lambda^2 + 5\lambda + 6 = 0$$
$$(\lambda + 2)(\lambda + 3) = 0$$
$$\Rightarrow x_{C.F}(t) = Ae^{-2t} + Be^{-3t}$$

- **Particular Integral:** Substitute $x_{P.I}(t) = Cte^{-2t}$ into the full second-order differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{2t}$$
$$\Rightarrow a = 2$$

- **General solution:** $x(t) = Ae^{-2t} + Be^{-3t} + 2te^{-2t}$

(ii)

- **General solution:** $x(t) = Ae^{2t} + Be^{3t} + 2te^{-2t}$ and the first differential equation from question is

$$\left[\frac{dx}{dt}\right] = -[x] + 2y$$
$$\frac{d}{dt}\left[Ae^{-2t} + Be^{-3t} + 2te^{-2t}\right] = -\left[Ae^{-2t} + Be^{-3t} + 2te^{-2t}\right] + 2y$$
$$\Rightarrow y(t) = (1-t)e^{-2t} - \frac{1}{2}Ae^{-2t} - Be^{-3t}$$

- **Boundary conditions:** For $t = 0, x = 5, y = 0$

$$\begin{aligned} A + B &= 5 \\ \frac{1}{2}A + B &= 1 \\ \Rightarrow A &= 8 \\ \Rightarrow B &= -3 \\ \therefore x(t) &= 8e^{2t} - 3e^{3t} + 2te^{-2t} \\ \therefore y(t) &= -te^{-2t} - 3e^{-2t} + 3e^{-3t} \end{aligned}$$

(iiI)

- **Limit:** $\frac{y}{x} = \frac{-t - 3 + 3e^{-t}}{2t + 8 - 3e^{-t}}$, as $t \to +\infty \Rightarrow \frac{y}{x} \to -\frac{1}{2}$

Question 3. In quantum mechanics, a particle has a wave function which encodes all the information about the quantum particle. The wave function, $\psi(x)$, for a particle with some potential energy, $U(x)$, obeys the Schrödinger equation, which is a second-order differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi = E\psi$$

where $\hbar$ is a constant known as "Reduced Plank's Constant" and m, E are the mass and energy of the particle, respectively. All these constants, except E, are assumed to be positive.
The particle is subject to the following potential, called the "Infinite Well Potential"

$$U(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & x < 0 \\ \infty & x > L \end{cases}$$

This results in the following boundary conditions, $\psi(0) = 0$ and $\psi(L) = 0$
In the region $0 \leq x \leq L$, the Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

(i) *(4 marks)* Prove why the particle cannot have negative energy $E < 0$.
HINT: Think about the boundary conditions due to the potential. Also, $\psi(x) = 0$ isn't allowed for general x for reasons that will become clear in (iv)

(ii) *(6 marks)* For $E > 0$, find the solution of $\psi(x)$ subject to the boundary condition due to Infinite Well.
HINT: $sin(\theta) = 0$ means θ is an integer multiple of $\pi \Rightarrow \theta = n\pi$

(iii) *(4 marks)* Show that the energy of the particle is given by

$$E = \frac{\hbar^2\pi^2 n^2}{2mL^2}$$

where n is a positive integer that is not zero.

(iv) *(5 marks)* Given that, $P = \int_a^b \mid \psi(x) \mid^2 dx$ represents the probability of finding the particle in the region $a \leq x \leq b$. Given that the particle has to be trapped in the Infinite Well, use the fact that

$$\int_0^L \mid \psi(x) \mid^2 dx = 1$$

to find the remaining unknown constants in $\psi(x)$

(i)

- **Schrödinger equation:** The differential equation rearranges to

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

- **If** $E < 0$ **then** $-\frac{2mE}{\hbar^2}$ **is positive:** Define $k^2 \equiv \frac{2mE}{\hbar^2} < 0$

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

- **Auxiliary equation:** Assume the form $\psi = e^{\lambda x}$, then the auxiliary equation becomes

$$\lambda^2 = -k^2 > 0$$
$$\lambda = \pm k$$

- **General solution:** General solution is then given by $\psi(x) = Ae^{kx} + Be^{-kx}$. Now subject this to the first boundary condition

$$\begin{aligned} \psi(0) &= 0 \\ Ae^{k\times 0} + Be^{-k\times 0} &= 0 \\ A + B &= 0 \\ A &= -B \end{aligned}$$

This leads us to $\psi(x) = A\left(e^{kx} - e^{-kx}\right)$. Let us subject this to the next boundary condition

$$\begin{aligned} \psi(L) &= 0 \\ A\left(e^{kL} - e^{-kL}\right) &= 0 \\ \left(e^{kL} - e^{-kL}\right) &= 0 \\ e^{kL} &= e^{-kL} \end{aligned}$$

The above boundary condition is only satisfied if $k = 0$ or $L = 0$. Clearly, $L \neq 0$ and if $k = 0$ then we are left with the trivial solution at $E = 0$ but the HINT tells us that $\psi(x) \neq 0$ for general x. Hence, no negative energy solutions exist.

(ii)

- **Schrödinger equation:** Same as before but now $k^2 > 0$

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

- **Auxiliary equation:** Assume the form $\psi = e^{\lambda x}$, then the auxiliary equation becomes

$$\begin{aligned} \lambda^2 &= -k^2 < 0 \\ \lambda &= \pm ik \end{aligned}$$

- **General solution:** General solution is then given by $\psi(x) = A\cos(kx) + B\sin(kx)$. Now subject this to the first boundary condition

$$\begin{aligned} \psi(0) &= 0 \\ A\cos(0) + B\sin(0) &= 0 \\ A &= 0 \end{aligned}$$

We are then left with $\psi(x) = B\sin(kx)$. Subject this to second boundary condition

$$\psi(L) = 0$$
$$B \sin(kL) = 0$$

We now make use of the HINT that kL is an integer multiple of $\pi \Rightarrow kL = n\pi$. n cannot be negative as we cannot allow for negative energy solutions. Similarly, n can also not be zero as this will give $\psi(x) = 0$ and would destroy the normalisation of the particle, as we are told $\psi(x) \neq 0$ for all x.

$$kL = n\pi$$
$$k = \frac{n\pi}{L}$$

Therefore, $\psi(x) = B \sin\left(\frac{n\pi x}{L}\right)$

(iii)

- **The k relation:** Remember, $k^2 \equiv \dfrac{2mE}{\hbar^2}$

$$k^2 \equiv \frac{2mE}{\hbar^2}$$
$$\left(\frac{n\pi}{L}\right)^2 = \frac{2mE}{\hbar^2}$$

This rearranges to $E = \dfrac{\hbar^2\pi^2 n^2}{2mL^2}$, for $n \in \mathbb{N} = \{1, 2, 3, 4, ...\}$. This means in quantum mechanics, when a particle in one dimension is subject to infinite well potential, it comes in chunks of energy, quanta of energy. Hence, we say that energy is quantised. For this reason it is appropriate to write E as E_n. Note that the energy can never be zero or negative. This means that the particle will always have some energy, it is never at zero energy. This is one of the key differences in certain quantum mechanical systems in comparison to classical mechanics that you study in the mechanics part of A-Level Maths.

(iv)

- **Probability integral:**

$$P = \int_0^L \mid \psi(x) \mid^2 \, dx = 1$$
$$P = \int_0^L B^2 \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

Substitute $u = \dfrac{n\pi x}{L} \Rightarrow du = \dfrac{n\pi}{L} dx \Rightarrow dx = \dfrac{L}{n\pi} du$

$$P = \int_{x=0}^{x=L} B^2 \sin^2(u) \frac{L}{n\pi} du = 1$$
$$\Rightarrow B^2 \int_{u=0}^{u=n\pi} \sin^2(u) du = \frac{n\pi}{L}$$
$$\Rightarrow B^2 \left[\frac{2u - \sin(2u)}{4}\right]_{u=0}^{u=n\pi} = \frac{n\pi}{L}$$
$$\Rightarrow B^2 \left[\frac{2(n\pi)}{4}\right] = \frac{n\pi}{L}$$
$$\therefore B = \sqrt{\frac{2}{L}}$$

Hence, $\psi(x) = \sqrt{\dfrac{2}{L}} \sin\left(\dfrac{n\pi x}{L}\right)$ is the wave function for a particle with mass m subject to infinite well potential in the region $0 \leq x \leq L$

END OF PAPER