LEXTRA LEARNING

ADVANCED LEVEL FURTHER MATHEMATICS

A2: Differential Equations – Year 13

Name: TIME ALLOWED: 1 HOUR

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **3** questions and all are compulsory.
2. Answer all questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT an OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

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Question 1. *(7 marks)* For $x > 0$, use substitution $u = y^2$ to find the general solution to the following first-order differential equation

$$\frac{dy}{dx} - 2y = \frac{e^x}{y}$$

A2: Differential Equations

Question 2. The two functions $x(t)$ and $y(t)$ obey the following two coupled simultaneous first-order differential equations

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = -x - 4y + e^{2t}$$

The boundary conditions are $x(0) = 5$ and $y(0) = 0$

(i) *(14 marks)* Eliminate y to obtain a second order differential equation for $x(t)$ and find the general solution for $x(t)$

(ii) *(3 marks)* Find the general solution for $y(t)$

(iii) *(4 marks)* Using the boundary conditions, find particular solutions

(iv) *(3 marks)* Show that

$$\lim_{t \to +\infty} \left(\frac{y}{x} \right) = -\frac{1}{2}$$

A2: Differential Equations

Question 3. In quantum mechanics, a particle has a wave function which encodes all the information about the quantum particle. The wave function, $\psi(x)$, for a particle with some potential energy, $U(x)$, obeys the Schrödinger equation, which is a second-order differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi = E\psi$$

where $\hbar$ is a constant known as "Reduced Plank's Constant" and m, E are the mass and energy of the particle, respectively.
The particle is subject to the following potential, called the "Infinite Well Potential"

$$U(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & x < 0 \\ \infty & x > L \end{cases}$$

This results in the following boundary conditions, $\psi(0) = 0$ and $\psi(L) = 0$
In the region $0 \leq x \leq L$, the Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

(i) *(4 marks)* Prove why the particle cannot have negative energy $E < 0$.
HINT: Think about the boundary conditions due to the potential

(ii) *(6 marks)* For $E > 0$, find the solution of $\psi(x)$ subject to the boundary condition due to Infinite Well.
HINT: $sin(\theta) = 0$ means θ is an integer multiple of π

(iii) *(4 marks)* Show that the energy of the particle is given by

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$$

where n is a positive integer that is not zero.

(iv) *(5 marks)* Given that, $P = \int_a^b \mid \psi(x) \mid^2 dx$ represents the probability of finding the particle in the region $a \leq x \leq b$. Given that the particle has to be trapped in the Infinite Well, use the fact that $P = \int_0^L \mid \psi(x) \mid^2 dx$ $= 1$ to find the remaining unknown constants in $\psi(x)$

END OF PAPER