

## LEXTRA LEARNING

### ADVANCED LEVEL FURTHER MATHEMATICS

#### A2: Differential Equations – Year 13

Name:

TIME ALLOWED: 1 HOUR

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#### INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **3** questions and all are compulsory.
2. Answer all questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT an OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

## LEXTRA LEARNING

A2: Differential Equations

**Question 1.** (*7 marks*) For  $x > 0$ , use substitution  $u = y^2$  to find the general solution to the following first-order differential equation

$$\frac{dy}{dx} - 2y = \frac{e^x}{y}$$

## A2: Differential Equations

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**Question 2.** The two functions  $x(t)$  and  $y(t)$  obey the following two coupled simultaneous first-order differential equations

$$\begin{aligned}\frac{dx}{dt} &= -x + 2y \\ \frac{dy}{dt} &= -x - 4y + e^{2t}\end{aligned}$$

The boundary conditions are  $x(0) = 5$  and  $y(0) = 0$

- (i) *(14 marks)* Eliminate  $y$  to obtain a second order differential equation for  $x(t)$  and find the general solution for  $x(t)$
- (ii) *(3 marks)* Find the general solution for  $y(t)$
- (iii) *(4 marks)* Using the boundary conditions, find particular solutions to  $x(t)$  and  $y(t)$
- (iv) *(3 marks)* Show that

$$\lim_{t \rightarrow +\infty} \left( \frac{y}{x} \right) = -\frac{1}{2}$$

## A2: Differential Equations

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**Question 3.** In quantum mechanics, a particle has a wave function which encodes all the information about the quantum particle. The wave function,  $\psi(x)$ , for a particle with some potential energy,  $U(x)$ , obeys the Schrödinger equation, which is a second-order differential equation

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + U(x)\psi = E\psi$$

where  $\hbar$  is a constant known as "Reduced Plank's Constant" and  $m$ ,  $E$  are the mass and energy of the particle, respectively. All these constants, except  $E$ , are assumed to be positive.

The particle is subject to the following potential, called the "Infinite Well Potential"

$$U(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & x < 0 \\ \infty & x > L \end{cases}$$

This results in the following boundary conditions,  $\psi(0) = 0$  and  $\psi(L) = 0$ . In the region  $0 \leq x \leq L$ , the Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi$$

- (i) (4 marks) Prove why the particle cannot have negative energy  $E < 0$ .  
HINT: THINK ABOUT THE BOUNDARY CONDITIONS DUE TO THE POTENTIAL. ALSO,  $\psi(x) = 0$  ISN'T ALLOWED FOR GENERAL  $x$  FOR REASONS THAT WILL BECOME CLEAR IN (IV)
- (ii) (6 marks) For  $E > 0$ , find the solution of  $\psi(x)$  subject to the boundary condition due to Infinite Well.  
HINT:  $\sin(\theta) = 0$  MEANS  $\theta$  IS AN INTEGER MULTIPLE OF  $\pi \Rightarrow \theta = n\pi$
- (iii) (4 marks) Show that the energy of the particle is given by

$$E = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$$

where  $n$  is a positive integer that is not zero.

- (iv) (5 marks) Given that,  $P = \int_a^b |\psi(x)|^2 dx$  represents the probability of finding the particle in the region  $a \leq x \leq b$ . Given that the particle

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has to be trapped in the Infinite Well, use the fact that

$$\int_0^L |\psi(x)|^2 dx = 1$$

to find the remaining unknown constants in  $\psi(x)$

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**END OF PAPER**