MODULAR SYSTEM

COUNTING PRINCIPLES AND PROBABILITY

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PREFACE

To the Teacher

Combinatorics, or combinatorial analysis, is perhaps one of the most interesting subjects in the study of arithmetic. This book is an introduction to two important subjects in combinatorics: counting principles and probability. Along with statistics, these two subjects have wide-ranging applications in today's world. Combinatorics is also one of the five basic subjects in the International Mathematical Olympiads.

This book is divided into two chapters. Some sections are marked as optional in the text and can be omitted if necessary to fit your program.

- *Chapter 1 has four sections. The first section introduces the basic principles of counting and enumeration. The second and third sections cover permutation and combination and different variations of these. The last section leads students to an understanding of the relationship between combination and binomial expansion, beginning with the study of Pascal's triangle.*
- *Chapter 2 is a basic introduction to probability, and is divided into five sections. The first and second sections introduce basic concepts and provide some simple examples of probability problems and applications. The third section builds on the material covered in Chapter 1 and looks at combination and probability. Sections four, five and six introduce different aspects of probability: conditional probability, the concept of dependent and independent events, and binomial probability.*

Problems in combinatorial analysis are mostly written in natural language, and understanding a question correctly is generally the first step towards finding its solution. For this reason we paid particular attention to the clarity of the text in the problems and explanations during the writing of this book. Our aim is to present the material in a student-friendly way, in order to give students the best possible chance of mastering each topic. A glossary at the end of the book also summarizes the main definitions and terms used, in clear and simple language.

This book follows a step-by-step teaching approach and explains definitions and examples in detail, just as a teacher would explain them to a class. At each stage, students' progress can be checked with regular banks of self-check questions ('Check Yourself'), plus graded exercises at the end of each section and chapter.

Mathematical applications and puzzles are one way of arousing students' interest in math. As a result, we have included a wide range of activities in this book. Some of these activities focus on real-world applications of the subject matter, while others are puzzles for students to solve using the math they have learned.

We thank you for choosing this book, and we hope that you enjoy using it.

Acknowledgements

No book of this kind could be written without the help of a large number of people. We would like to thank everyone who helped us at Zambak Publications, in particular Mustafa Kırıkçı, Ramazan Şahin, Muhammer Taşkıran and Ali Çakmak for their support and input, and Şamil Keskinoğlu for the typesetting and design. Our thanks also go to our colleagues for their valuable suggestions, to the administrators of the schools where we are working for their support and understanding while the book was being written, and to the Ankara Police Department for the provision of some of the forensic material in the book. Finally, we wish to thank our families for their patience and support during the project.

The Authors

To the Student

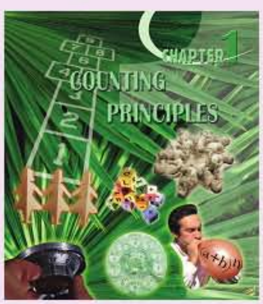
This book has been written to help you understand and use counting principles and probability. These two topics form part of the subject of combinatorics. They will be particularly useful to you if you are planning to work in any job which involves risk analysis, for example: in management, finance, insurance, politics or the military. The material in this book will give you a good, basic understanding of counting principles and probability before you begin university.

You should work through this book from the beginning, since the early material is used in later sections. Make sure that you understand the material at each step, and ask your teacher for help when you need it.

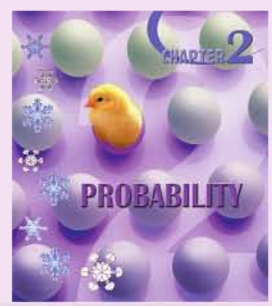
Some of the examples and problems you will study are very simple, but as you work through the book you will see how the concepts and methods you are studying could be applied to real-life situations. In addition, the chapter on probability will help you to see why a good mathematician does not gamble!

We hope that you enjoy using this book, and we wish you good luck with your studies.

Using This Book



This book is designed so that you can use it effectively. Each chapter has its own special color that you can see at the bottom of the page.



Counting Methods

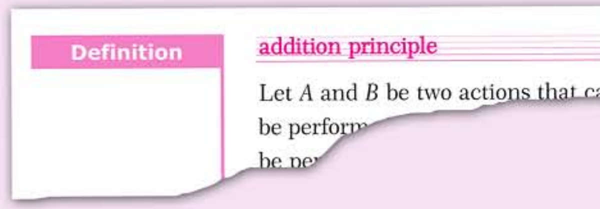
Probability

Different pieces of information in this book are useful in different ways. Look at the types of information, and how they appear in the book:

Note
Any question which can be solved using the multiplication principle.

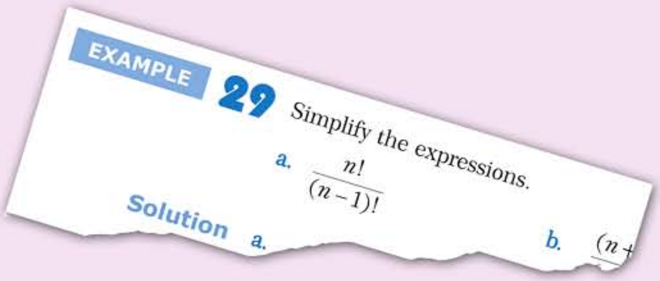
Notes help you focus on important details. When you see a note, read it twice! Make sure you understand it.

Definition boxes give formal descriptions of new concepts. The information in these boxes is very important for further understanding and for solving examples.


Definition

addition principle

Let A and B be two actions that ca
be perform
be pe

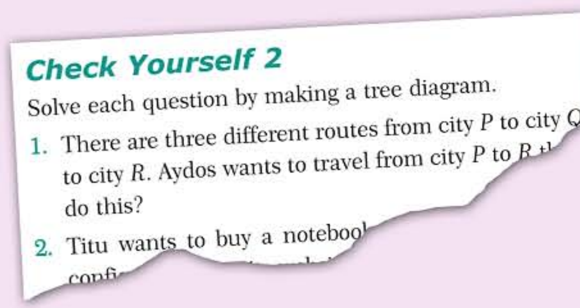

EXAMPLE 29 Simplify the expressions.

a. $\frac{n!}{(n-1)!}$

Solution a. b. $(n+$

Examples include problems related to the topic and their solutions, with explanations. The examples are numbered, so you can find them easily in the book.

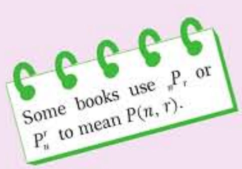
Check Yourself sections help you to check your understanding of what you have just studied. Solve the questions in these sections alone and then check your answers against the answer key provided. If your answers are correct, you can move on to the next section. If your answer is wrong, go through your working again and check back through the examples in the section.


Check Yourself 2

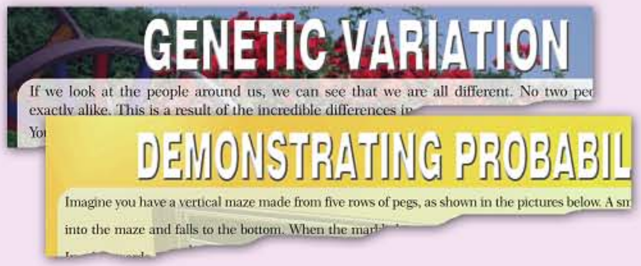
Solve each question by making a tree diagram.

1. There are three different routes from city P to city Q
to city R. Aydos wants to travel from city P to R
do this?

2. Titu wants to buy a noteboo



A small notebook in the left margin of a page reminds you of material that is related to the topic you are studying. It might help you see your mistakes, too! Notebooks are the same color as the section you are studying.


GENETIC VARIATION

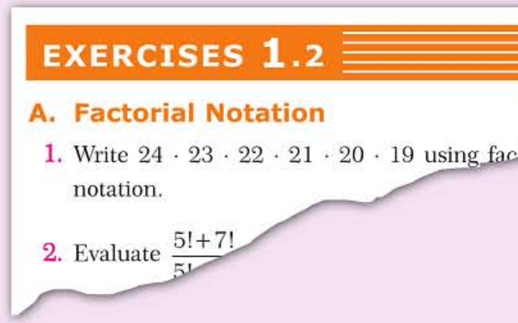
If we look at the people around us, we can see that we are all different. No two peo
exactly alike. This is a result of the incredible differences in

DEMONSTRATING PROBABIL

Imagine you have a vertical maze made from five rows of pegs, as shown in the pictures below. A sm
into the maze and falls to the bottom. When the marb

Separate pages with applications and puzzles help you to see how math is used in real life.

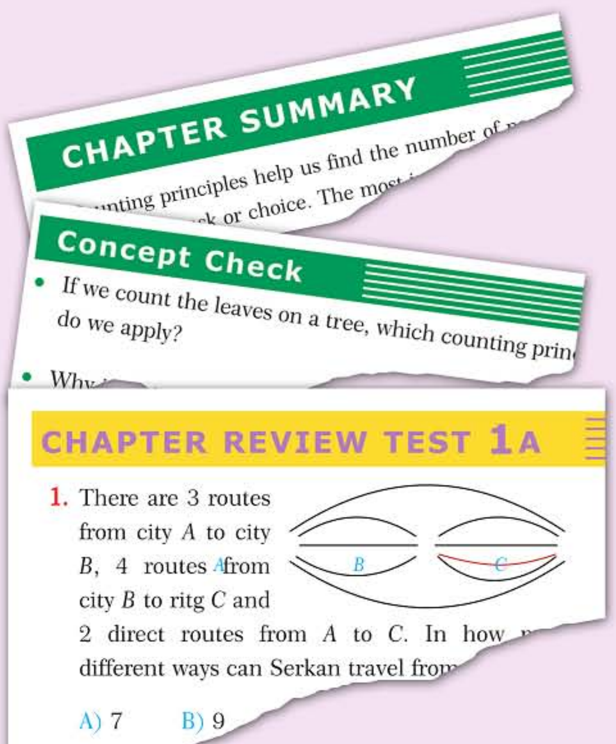
Exercises at the end of each section cover the material in the whole section. You should be able to solve all the problems which do not have a star. One star (✪) next to a question means the question is a bit more difficult. Two stars (✪✪) next to a question mean the question is for students who are looking for a challenge! The answers to the exercises are at the back of the book.


EXERCISES 1.2

A. Factorial Notation

1. Write 24 · 23 · 22 · 21 · 20 · 19 using fact
notation.

2. Evaluate $\frac{5!+7!}{5!}$


CHAPTER SUMMARY

unting principles help us find the number of
k or choice. The most

Concept Check

- If we count the leaves on a tree, which counting prin
do we apply?
- Why

CHAPTER REVIEW TEST 1A

1. There are 3 routes from city A to city B, 4 routes from city B to ritg C and 2 direct routes from A to C. In how m
different ways can Serkan travel from

A) 7 B) 9

The Chapter Summary summarizes all the important material that has been covered in the chapter. The Concept Check section contains oral questions. You do not need paper or pen to answer these questions. If you answer the Concept Check questions correctly, it means you know that topic! The answers to these questions are in the material you studied. Go back over the material if you are not sure about an answer to a Concept Check question.

Finally, Chapter Review Tests are in increasing order of difficulty and contain multiple-choice questions. The answer key for these tests is at the back of the book

CONTENTS

INTRODUCTION


The Rhind Papyrus

Combinatorics is the study of the arrangement of sets of objects or elements in groups. If you have ever wondered how to seat a group of visitors around your dinner table, or how many questions you need to try to answer in order to pass an exam, or how many bottles of soft drink you need to buy to win a prize, then you have thought about a problem in combinatorics. Probability - the study of chance - is a branch of combinatorics.

Combinatorics and Counting Principles

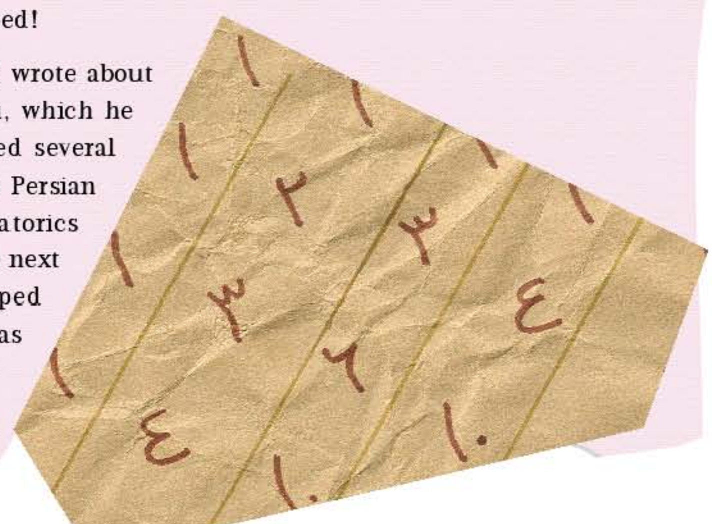
The study of combinatorics began in ancient times. The Rhind Papyrus is a very old document which dates from 1650 BC. It lists 84 mathematical problems and includes the oldest known combinatorial question, which is problem number 79 on the papyrus. The problem is as follows: There are seven houses on an estate and in each house there are seven cats. Each cat kills seven mice and each mouse has eaten seven heads of wheat. Each head of wheat would have produced seven hekats of grain. Houses, cats, mice, heads of wheat and hekats of grain: how many of these things are there in total?


Omer Khayyam

Later on, the Greek philosopher Xenocrates (396-314 BC) is said to have calculated that the letters of the Greek alphabet could form a total of 1 002 000 000 000 syllables. Xenocrates would have used his knowledge of the factorial rule to calculate his result. This rule tells us that there are $n!$ (n factorial) ways of arranging n items in order. For example, we can order three different books on a shelf in $3! = 6$ different ways. Similarly, we can order the 52 playing cards in a pack in 52! different ways. 52! is actually 80 658 175 170 943 878 571 660 636 856 403 766 975 289 505 440 883 277 824 000 000 000 000, which is a very large number indeed!


Al-Biruni

The Indian mathematician Bhascara Acharia (1114-1185) wrote about the basic theory of combinatorics in his work the Lilavati, which he published in about 1150. In his articles, Acharia discussed several different types of questions about counting methods. The Persian mathematician Abu Rayhan al-Biruni studied combinatorics problems at the beginning of the last millennium, and in the next century another mathematician, Omar Khayyam, developed the important binomial triangle that is today known as Pascal's triangle.



Probability is the study of chance. If we roll a die many times, how many times might it show the number 5? What are the chances that the lottery numbers we pick will win a prize? These are two examples of probability problems.

The concept of probability is thousands of years old, but mathematicians did not begin to study probability seriously until the middle of the seventeenth century. For this reason, probability is quite a young subject in mathematics. However, it is a very important subject for companies today. With the help of probability, life insurance companies and gambling companies can make good profits on events that are unpredictable. In fact, the annual profit of gambling companies across the United States is around 26 billion dollars! This might seem impossible until you study the probability of winning the lottery. Then you will realize that a person who purchases fifty lotto tickets each week is only likely to win the jackpot once every 5000 years. So the gambler who says 'maybe next time' might have to wait a very long time indeed.

So does studying probability make us luckier people? It may help us to make better decisions, but we should remember that the theory of probability assumes that events are always random. In fact, real world events are never totally random. The way we toss a coin may make it land in a particular way, and the speed, angle and height at which we throw a die may influence the number we roll. The gender of a baby or our chances of falling ill are determined by biological factors, not chance alone. So mathematical probability can only tell us what will probably happen in a theoretical situation. It is up to us to use this information well in the real world.

Early Roman Knucklebone

Early Egyptian Die

The earliest games of chance were played with dice and specially-shaped pieces of stone or bone called 'knucklebones'. Unfortunately, early dice were not always perfect. No two dice were exactly the same, and some dice were even weighted to show 6 or 1 more often than the other numbers. However, by the fifteenth century a few mathematicians were beginning to study probability. In 1494 the mathematician Luca Paccioli wrote the very first printed work on probability. It was called *Summa de arithmetica, geometria, proportioni e proportionalita*. In 1550 the Italian physician, mathematician and gambler Gerolamo Cardano (1501-1576) studied the Summa and wrote another book called *Liber de Ludo Aleae*, which means 'a book on games of chance'. Cardano's book was perhaps the first book to describe the basics of probability, although it was not taken seriously when it was published because of its connection with gambling.

Gerolamo Cardano

The History of Probability

A French nobleman called the Chevalier de Méré was a gambler who often gambled to win more money. One game he played involved rolling a die four times. De Méré would bet that at least one 6 would appear in these four rolls. He knew from past experience that he was often successful at this game. One day, however, de Méré decided to change the game. He bet that at least one 12 (a double 6) would appear in twenty-four rolls of two dice. Soon he realized that he won more money with his old game. De Méré asked his friend, Blaise Pascal, why his new game did not win him as much money as the old one. Pascal studied the problem and found that the probability ofwinning the new game was only 49.1 per cent, compared to a 51.8 per cent chance of winning the old game.

Pierre Fermat

Pascal continued to study other problems posed by de Méré and began exchanging letters about them with his friend Pierre Fermat. Together, Pascal and Fermat rediscovered the work of Gerolamo Cardano and worked out the basic principles of probability theory. The Dutch scientist Christiaan Huygens learned about this correspondence and soon afterwards published another book on probability theory, *De Ratiociniis in Ludo Aleae*, in 1657.

Blaise Pascal

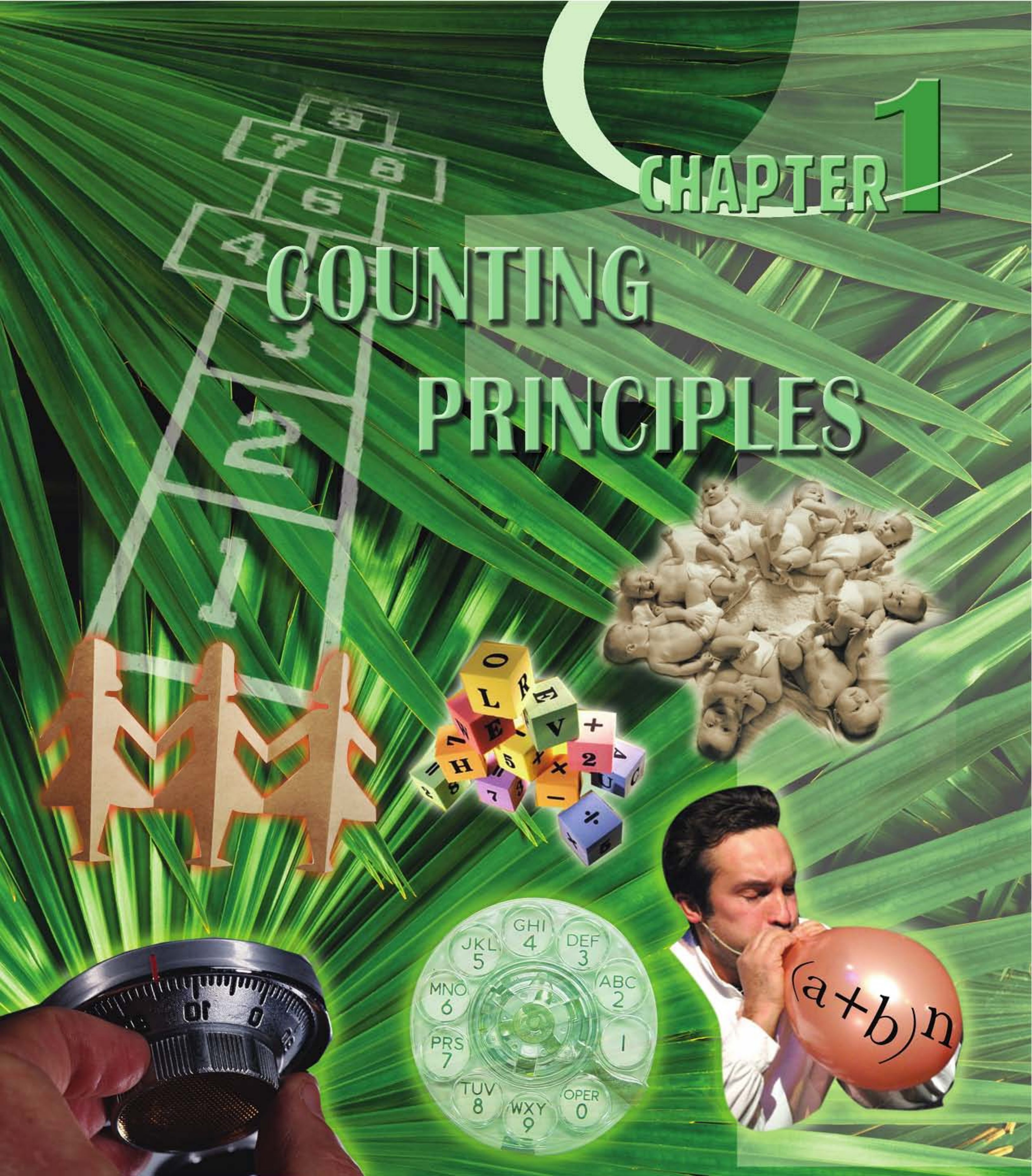
By this time, interest in games of chance was more common and probability theory soon became popular. During the seventeenth and eighteenth centuries, the most important contributors to the study of probability were Jakob Bernoulli (1654-1705) and Abraham de Moivre (1667-1754). Bernoulli's book, *The Art of Guessing*, discussed how probability could be used in government, law, economics and genetics. De Moivre developed a probability theorem that you will learn in this course.

In 1812 the French mathematician Pierre-Simon Laplace (1749-1827) restated the probability rule that Gerolamo Cardano had described over three hundred years earlier. Known as the 'Rule of Laplace', this rule can be summarized as follows: The probability of success in a game or experiment is equal to the number of successes (i.e. the number of possible ways to be successful) divided by the total number of outcomes (i.e. the total number of possible results in the game or experiment). Laplace developed this and other theories in his book, *Théorie Analytique des Probabilités*. Laplace's work was important because it changed the focus of probability theory. Before Laplace, probability theory was mostly about the mathematical analysis of games of chance. Laplace, however, showed how this theory could be applied to many scientific and practical problems.

Pierre-Simon Laplace

Today, combinatorics and the study of probability extend far beyond the subject of pure mathematics and have concrete applications in practical fields such as insurance, quality control, genetics, quantum mechanics and the kinetic theory of gases.

CHAPTER 1
COUNTING PRINCIPLES



1

COUNTING PRINCIPLES

Imagine you have a bag of marbles. Your friend asks you how many marbles there are in your bag. If you did not know the answer, you would probably count the marbles: one, two three, four, etc. Matching a word to a marble like this is one way of counting. But we can match other things, too. A prisoner might match every day he spends in his cell to a mark on the wall, or a shepherd might match every sheep he looks after to a pebble in a bag. These are different ways of counting. In these section we will look at different ways of counting.

A. THE ADDITION PRINCIPLE

Ali wants to go to the cinema to watch a movie. There are two different halls at the cinema. Three movies are showing in the first hall and four different movies are showing in the second hall. In how many ways can Ali choose a movie to watch?

Since Ali cannot be in two different halls at the same time, there are $3 + 4 = 7$ different ways for Ali to choose the movie.

Definition

addition principle

Let A and B be two actions that cannot both be performed at the same time. If action A can be performed in m ways and action B can be performed in n ways, then the action A or B can be performed in $m + n$ ways.

EXAMPLE 1 Mary has three different Barbie dolls and two different Cindy dolls. She wants to take them out to play with her friend. However, Mary's mother will only let Mary choose one doll. In how many ways can Mary choose a doll?

Solution Mary has three alternatives for her Barbie dolls and two alternatives for her Cindy dolls. So she can choose a doll in $3 + 2 = 5$ ways.

In this example, Mary had five different dolls. If we name the dolls D_1, D_2, ..., D_5 then we can list the possible results of Mary's choice as $\{D_1, D_2, ..., D_5\}$. Each element in this set is a possible outcome of Mary's choice.

B. SYSTEMATIC LISTING

In the previous examples it was easy to list the outcomes. Sometimes, however, it can be more difficult: there may be many different outcomes in a problem, or a task may be complicated. In this section we will look at different ways of listing the outcomes of a task or decision.

1. Simple Listing

If our task contains only one part, listing the possible outcomes is very easy. For example, if we roll a fair dice there are six different possible results. The list of outcomes is 1, 2, 3, 4, 5, 6.

EXAMPLE 2 David wants to buy a shirt. There are four different colored shirts in David's size. In how many different ways can David buy a shirt?

Solution Let us assume that the colors of the shirts are blue, red, yellow and green. We can list the available shirts as {blue, red, yellow, green}. Since David only chooses one shirt, each shirt in this set is a possible outcome. So the answer is four.

2. Using a Product Table

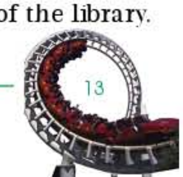
If our task contains two parts, our listing method is a little different. We can use a table called a product table to list the possible outcomes of the task. Let us look at some examples.

EXAMPLE 3 Selman needs to go into and out of the library. If the library has two doors, in how many ways can Selman go in and out?

Solution There are two tasks for Selman: going into the library and going out. Let the two library doors be A and B. For each task, Selman can choose either door.
The product table for Selman's library visit looks like this:

Selman's library visit		Go out	
		A	B
Go in	A	(A, A)	(A, B)
	B	(B, A)	(B, B)

We can see that there are four different ways for Selman to go into and out of the library.



EXAMPLE 4 How many two-letter words can be formed from the letters in the set $\{a, b, c\}$?

Solution This task contains two parts: choosing the first letter and choosing the second letter. Let us make a product table to list the possible outcomes.

Two-letter combinations		Second letter		
		a	b	c
First letter	a	aa	ab	ac
	b	ba	bb	bc
	c	ca	cb	cc

The table shows us that the outcomes are aa, ab, ac, ba, bb, bc, ca, cb and cc. So there are nine possible words.

Of course, these are not real words in English. In problems like this, 'word' means a sequence of letters, not a real English word.

EXAMPLE 5 Anton is in his first year at university. He has to take one math or science course in the first term and a different math or science course in the second term. The courses available are Algebra, Geometry, Physics, Biology and Chemistry. In how many ways can Anton choose his two courses?

Solution The required task has two parts: choosing the first term's course and choosing the second term's course. If we denote each course by its first letter, we can show the possibilities in a table as follows:

Anton's courses		Second term				
		A	G	P	B	C
First term	A		AG	AP	AB	AC
	G	GA		GP	GB	GC
	P	PA	PG		PB	PC
	B	BA	BG	BP		BC
	C	CA	CG	CP	CB	

Notice that some pairs are omitted from the table since Anton cannot take the same course twice. So there are 20 distinct possibilities.

What would happen if Anton had to take two courses together in the same term? In this case there would only be 10 possibilities. Can you see why?

Check Yourself 1

1. How many two-digit numbers can be formed from the digits 3, 5 and 9?
2. Two dice are rolled and their numbers are added. How many possibilities are there that the result is prime?

Answers

1. 9 2. 15

GENETIC VARIATION

If we look at the people around us, we can see that we are all different. No two people on Earth are exactly alike. This is a result of the incredible differences in our genes.

You inherited your basic appearance, personality and physical make-up as a combination of your parents' genes. Similarly, a flower or plant inherits genes from its parents to become one of a certain species. Scientists study how genes are inherited to learn more about how life works. The combination of the genes of two parents is called a *genetic crossing*.

Systematic listing with tables allows us to see the possible results of the genetic crossing of two parents. Some of the parents' genes have a stronger influence. They are called *dominant genes*, and written with a capital letter. Other genes have a weaker influence, and are called *recessive genes*. They are written with a lower-case letter. For example, imagine we have one gene and two types of garden pea: a yellow pea has a dominant Y gene and a green pea has a recessive y gene. Any result of the crossing of these two types of pea will inherit from two genes. Any result which has a dominant Y gene will be yellow, as this is the dominant gene. Only the result yy will be green. Crossings such as YY and yy are called *pure*, and the mixed crossings yY and Yy are called *hybrid*.

Now let us cross two pure pea types, one yellow (YY) and one green (yy):

First crossing (YY × yy)		Male gamete yy	
		y	y
Female Gamet YY	Y	Yy	Yy
	Y	Yy	Yy

We can see that the results are all yellow as they all contain the dominant Y gene. However, they are not pure.

As another experiment, let us take two seeds from this first crossing and cross them again:

Second crossing (Yy × Yy)		Male gamete Yy	
		Y	y
Female gamete Yy	Y	YY	yY
	y	Yy	yy

As a result of this crossing we have three different types of pea:

YY: pure yellow

Yy: hybrid yellow

yy: pure green.

In the examples above we looked at crossing two pea types with respect to just one genetic trait. The table opposite shows the crossing of two hybrid peas with respect to two traits, namely color and shape. The traits are as follows:

Shape: $R \to$ round (dominant), $r \to$ wrinkled

Color: $\mathrm{Y} \to$ yellow (dominant), $y \to$ grean.

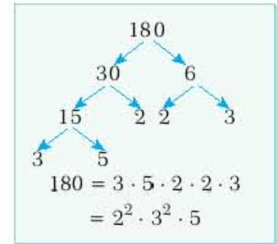
Dihybrid crossing		Male gamete			
		RY	Ry	rY	ry
Female gamete	RY	$RRYY$	$RryY$	$RrYY$	$rRyY$
	Ry	$RRYy$	$RRyy$	$rRYy$	$Rryy$
	rY	$RrYy$	$RryY$	$rrYY$	$rryY$
	ry	$RrYy$	$Rryy$	$rrYy$	$rryy$

RY : round yellow *Ry : round green*

rY : wrinkled yellow *ry : wrinkled green*

3. Using a Tree Diagram

A tree diagram is another useful way to list and count possibilities or outcomes. We use tree diagrams in several subjects. In combinatorics, they help us to cope with some complex problems that cannot be easily solved by using product tables.

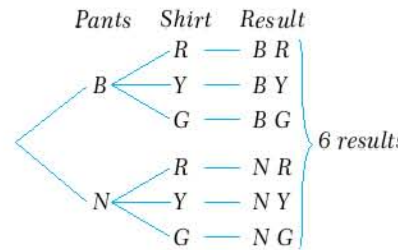


This tree diagram helps us to find the prime factorization of 180.

EXAMPLE 6 Joseph has one black and one navy pair of pants. He has also three shirts which are red, yellow and green respectively. In how many different ways can Joseph choose to wear his pants with a shirt?

Solution Let us list the choices of pants as $\{B, N\}$ and the choices of shirts as $\{R, Y, G\}$. Since it is not important whether Joseph chooses his pants or shirt first, we may assume that he chooses his pants first.

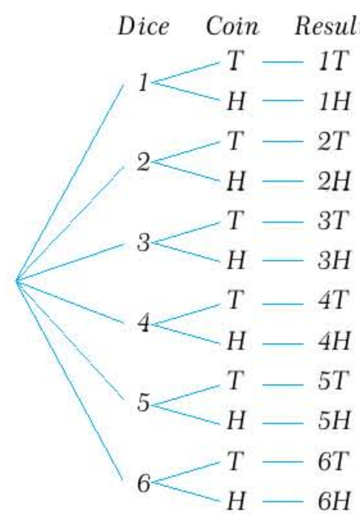
The tree diagram for this problem is opposite. We can see that Joseph has six possible choices.



EXAMPLE 7 A student rolls a dice and then tosses a coin. How many different outcomes are possible?

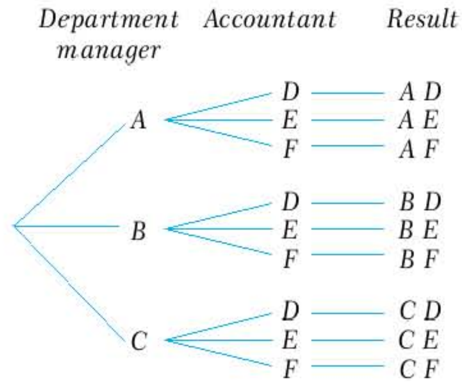
Solution This task contains two different parts which occur in an order. Constructing a tree diagram will help us to list the different possible outcomes systematically. Let us list the outcomes of the dice roll as $\{1, 2, 3, 4, 5, 6\}$ and the outcomes of tossing the coin as $\{T, H\}$.

The tree diagram shows us that there are 12 different possible outcomes.



EXAMPLE 8 In a company, three people applied for a department manager's position and three different people applied for an accountancy position. Show all the different ways of filling these two positions using a tree diagram.

Solution Let A, B and C be the people who applied for the manager's position. Similarly, let D, E and F be the people who applied for the accountancy position. Then we can construct a tree diagram as follows:

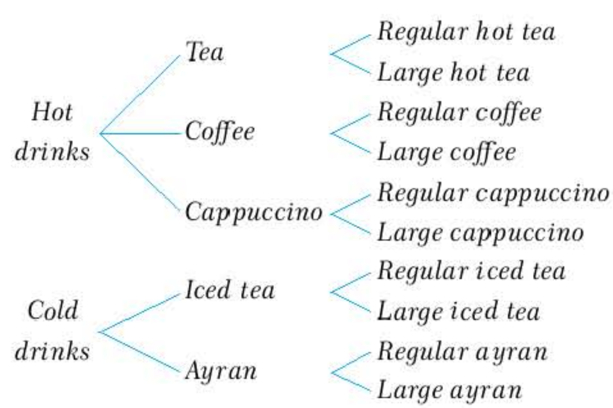


We can see that there are nine possible ways to fill the positions.

The data in the tree diagrams we have created so far could also have been shown in a product table. This is because the problems we have looked at contain at most two tasks. However, if we want to use a product table for a question that includes three or more tasks, we will have to construct a three-dimensional table. This is difficult to draw on paper. In this case, a tree diagram is the most appropriate way of listing the outcomes.

EXAMPLE 9 Merdan is at a café. The waiter offers him either a hot or a cold drink. The hot drinks are tea, coffee and cappuccino, and the cold drinks are iced tea and ayran. Each drink has two sizes: regular or large. How many choices of drink does Merdan have?

Solution This task includes three parts: deciding whether the drink will be hot or cold, then choosing the type of drink, and finally deciding on the size. Therefore our tree diagram will have three main branches.



So Merdan has ten choices of drink.

As an exercise, try solving this problem by using a product table. Can you do it?

EXAMPLE 10 Kamil, Vadim and Akira are astronauts. Two of them will be chosen to fly to either Mars or Jupiter. For this trip, one of the space shuttles Alpha-1 and Alpha-2 will be used. Any of the astronauts can fly the Alpha-2 shuttle but only Kamil can fly the Alpha-1. List all the possible ways of setting up this trip using a tree diagram.

Solution

Astronauts	*Planet*	*Shuttle*
Kamil, Vadim	*Mars*	*Alpha-1*
		Alpha-2
	Jupiter	*Alpha-1*
		Alpha-2
Vadim, Akira	*Mars*	*Alpha-2*
	Jupiter	*Alpha-2*
Akira, Kamil	*Mars*	*Alpha-1*
		Alpha-2
	Jupiter	*Alpha-1*
		Alpha-2

So there are ten ways to set up this trip.

EXAMPLE 11 Ahmet, Berk, Cemal and Deniz are in an amusement park and they want to take a roller-coaster ride. The roller-coaster car has four seats in a row. Cemal is Ahmet's younger brother and must always sit next to Ahmet. In how many ways can the four people sit in the car?

Solution We must remember that the brothers Ahmet and Cemal have to sit together as a pair. This reduces the number of possibilities. The diagram opposite shows all the possibilities. Each person is represented by his initial.

First seat	*Second seat*	*Third seat*	*Fourth seat*	
A	*C*	*D*	*B*	*ACDB*
		B	*D*	*ACBD*
B	*A*	*C*	*D*	*BACD*
	C	*A*	*D*	*BCAD*
	D	*A*	*C*	*BDAC*
		C	*A*	*BDCA*
C	*A*	*B*	*D*	*CABD*
		D	*B*	*CADB*
D	*A*	*C*	*B*	*DACB*
	B	*A*	*C*	*DBAC*
		C	*A*	*DBCA*
	C	*A*	*B*	*DCAB*

We can see that there are twelve possible arrangements.

Check Yourself 2

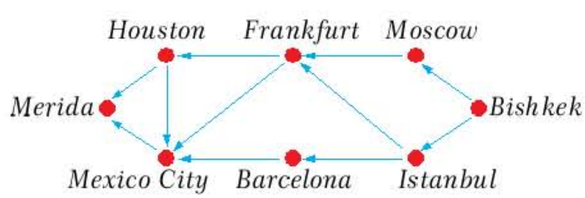
Solve each question by making a tree diagram.

1. There are three different routes from city P to city Q and four different routes from city Q to city R. Aydos wants to travel from city P to R through city Q. In how many ways can he do this?

2. Titu wants to buy a notebook computer. A computer company has different computer configurations on its website which vary according to RAM, hard disk capacity and type of optical drive. There is a choice of 256, 512 or 1024 MB RAM and a hard disk with either 80 GB or 100 GB capacity. The optical drive types are CD-ROM and DVD. A DVD drive doesn't work properly with 256 MB RAM so Titu doesn't want to have these together. How many different configurations remain for Titu to choose from?

3. How many three-digit numbers can be made from the set {5, 6, 9} if
   a. any digit can be used more than once in a number?
   b. each digit can only be used once in a number?

4. An international conference is being held in Merida, Mexico. The Kyrgyz team must choose a flight route from Bishkek to Merida. The possible routes are shown in the following diagram.



Regardless of the number of flight connections, how many different possible routes are there?

Answers

1. 12   2. 10   3. a. 27   b. 6   4. 7

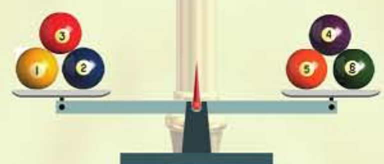
THE HEAVY BILLIARD BALL

The following puzzle is a good example of systematic listing and the use of a tree diagram.

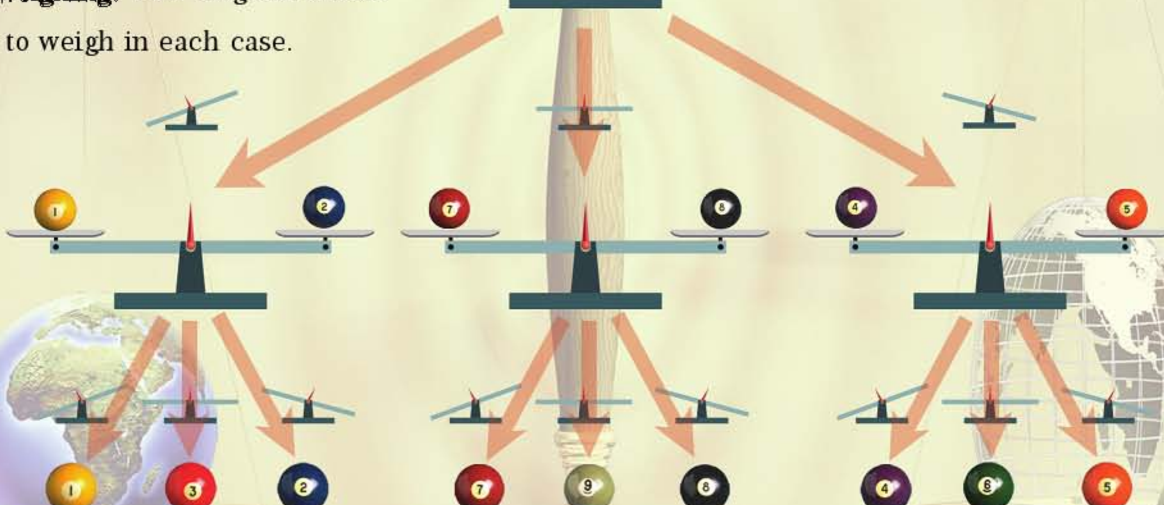
Suppose you have a set of nine billiard balls. Eight balls have equal weight and one ball is heavier than the others. You are asked to find the heavy ball, using only a simple balance scale with two pans. You can use the scale only twice and only for balancing.

The tree diagram below shows one way of solving the problem. We begin by placing three balls in each pan and noticing the position of the scales. This position tells us which balls to weigh next. The diagram uses the following symbols to show the position of the scales:

First weighing: We weigh balls 1, 2 and 3 with balls 4, 5 and 6.



Second weighing: The diagram shows the balls to weigh in each case.



The last row in the diagram tells us which ball is the heavy ball.

How many times would you need to use the scales to find the heavy ball if there were twelve balls? Could you solve the same puzzle for eight balls if you did not know whether the different ball was heavier or lighter?

A similar puzzle to this is known as the 'Counterfeit coin puzzle'. You can find out about this puzzle on the Internet.

C. THE MULTIPLICATION PRINCIPLE

In the previous section we studied different ways of listing all the possible outcomes of a particular task. However, writing out the entire list of outcomes may sometimes be very time-consuming and unnecessary. We need a different approach for problems which only ask us to calculate the number of possible outcomes.

In many of the problems we have looked at so far, the outcome of each part of a task is equally possible. We say that these tasks satisfy the uniformity criterion. However, in some problems there have been restrictions, for example: two people always have to sit together, or only one person in a team can pilot a certain type of space shuttle. These problems do not satisfy the uniformity criterion.

When we need to calculate the number of possible ways of performing a task which satisfies the uniformity criterion, we can use the multiplication principle.

Definition

multiplication principle (fundamental principle of counting)

Let a multiple-part task which satisfies the uniformity criterion consist of k parts. If the first part of the task can be performed in n_1 ways, the second part can be performed in n_2 ways and so on, then the number of ways to perform the entire task is $n_1 \cdot n_2 \cdot n_3 \cdot \ldots\ldots \cdot n_k$.

EXAMPLE 12 Nicole has four different skirts, three different blouses and two pairs of shoes which she can wear for a business meeting. In how many ways can Nicole dress for the meeting?

Solution We can use a table with three boxes, one for each part of the task. This problem satisfies the uniformity criterion because any of the three blouses can be worn with any of the skirts and shoes.

Skirts	Blouses	Shoes
4 choices	3 choices	2 choices

Using $n_1 = 4$, $n_2 = 3$ and $n_3 = 2$, by the multiplication principle Nicole can dress in $4 \cdot 3 \cdot 2 = 24$ ways.

EXAMPLE 13 How many different two-digit numbers can be formed using the digits in the set $\{1, 2, 3, 4, 5\}$?

Solution This task has two parts: choosing the tens digit and choosing the units digit. Since there are no restrictions we can start by choosing either the tens digit or the units digit, and each digit can take five different values.

Tens	Units
5 choices	5 choices

By the multiplication principle, we can form $5 \cdot 5 = 25$ different numbers.

EXAMPLE 14 In the previous example, how many numbers can be formed if a digit cannot be used twice in a number?

Solution This time we cannot use a digit that we have already used. Therefore, we can choose any of the five digits in the set for the tens but only four digits remain for the units.

This situation still satisfies the uniformity criterion because we always have five choices for the tens and four choices for the units.

Tens	Units
5	4

By the multiplication principle, the number of two-digit numbers that can be formed without using a digit twice is $5 \cdot 4 = 20$.

EXAMPLE 15 How many three-digit counting numbers are there?

Solution It is easy to think that we can use any of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 for each place value in a number. The answer therefore seems to be $10 \cdot 10 \cdot 10 = 1000$, since we can use a digit as many times as we like. However, notice that we cannot put zero in the hundreds place since this would not give us a proper three-digit number. For example, 048 is not a three-digit counting number.

Hundreds	Tens	Units
9	10	10

So the number of three-digit counting numbers is $9 \cdot 10 \cdot 10 = 900$.

EXAMPLE 16 A box contains seven different white rabbits and five different black rabbits. In how many ways can a conjurer pick out a pair of rabbits of different colors?

Solution

White rabbits	Black rabbits
7	5

There are $7 \cdot 5 = 35$ ways.

As an exercise, try to find out the format of the license plates in your city or country. How many cars can be licensed using this format?

EXAMPLE 17 How many different three-digit odd numbers can be formed using the digits in the set $\{4, 5, 6, 7, 8, 9\}$?

Solution The number formed must be an odd number. This is a restriction. In such cases we should first consider the digit(s) affected by the restriction. In this question it is the units digit that determines whether the number is odd or even. If the number is odd then the units digit must be 5, 7 or 9. So there are three ways to choose the units digit. We are free to choose the two other digits.

Hundreds	Tens	Units
6	6	3

So the answer is $6 \cdot 6 \cdot 3 = 108$ numbers.

EXAMPLE 18 In a particular country, the automobile license plates are made up of 2 letters followed by 3 digits. Any of the 26 letters of English alphabet and the digits 0-9 can be used. How many different possible license plates are there?

Solution We have 26 possibilities for each letter and 10 possibilities for each digit. So the answer is

$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^2 \cdot 10^3 = 676000.$

EXAMPLE 19 A company must form a committee comprising a manager, an assistant and a secretary from a group of nine people. Assuming that any person can do any job, in how many ways can the committee be formed?

Solution The restriction here is that no one can hold more than one position. Since the order of selection is not important, suppose that we first select the manager, then the assistant and finally the secretary. Then there are nine possible ways to select the manager, eight ways to select the assistant, and seven ways to select the secretary.

Manager	Assistant	Secretary
9	8	7

So there are $9 \cdot 8 \cdot 7 = 504$ different ways to form the committee.

EXAMPLE 20 In a particular city, all the telephone numbers are five-digit numbers which start with 4 or 5. The 2 and 9 buttons on Ulan's telephone do not work. How many phone numbers in the city can Ulan dial on his phone if

a. digits can be repeated?

b. digits cannot be repeated?

Solution a. There are ten digits on a telephone. However, the 2 and 9 buttons on Ulan's phone do not work, so there are eight available buttons. The first digit can only be 4 or 5. After this choice, if a digit can be repeated then we have eight choices for each digit.

First digit	Second digit	Third digit	Fourth digit	Fifth digit
2	8	8	8	8

So Ulan can dial $2 \cdot 8 \cdot 8 \cdot 8 \cdot 8 = 8192$ phone numbers.

b. If repetition is not allowed, we can use only one of seven different digits for the second digit since 4 or 5 has already been used. Similarly, there are only six digits left for the third place, five for the fourth place and four for the last place.

First digit	Second digit	Third digit	Fourth digit	Fifth digit
2	7	6	5	4

So Ulan will be able to dial $2 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 3360$ different phone numbers.

EXAMPLE 21 In how many different ways can five letters be delivered to three mailboxes?

Solution We are free to choose any box for the first letter, so there are three possibilities. Similarly, there are three ways to choose the box for the second letter, three for the third letter, and so on.

First letter	Second letter	Third letter	Fourth letter	Fifth letter
3	3	3	3	3

So the answer is $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5 = 243$.

Letters for the boxes or boxes for the letters?

EXAMPLE 22 We want to place eight rooks on a chessboard so that none of them can capture any of the others. In how many ways can we do this?

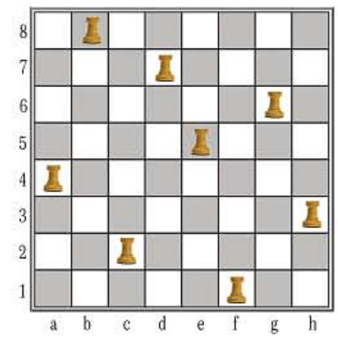
Solution A chessboard has eight rows and eight columns. If the rooks cannot capture each other, we can place only one rook in each row and column. For the first rook there are eight different possible squares in the first column. Then for the second rook we have seven possible squares in the second column (excluding the row that contains the previous rook) and so on.

In the game of chess, a the rook can capture any piece that lies in the same row or column.

So the answer is

$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320.$

One of the solutions is represented aside.



EXAMPLE 23 What is the total number of positive factors of 600?

Solution The prime factorization of 600 is $2^3 \cdot 3 \cdot 5^2$. Therefore, a number m is a positive factor of 600 as long as it is in the form $m = 2^a \cdot 3^b \cdot 5^c$ where a, b and c are natural numbers with $0 \le a \le 3$, $0 \le b \le 1$ and $0 \le c \le 2$. For example, $2^1 \cdot 3^0 \cdot 5^2 = 50$ and $2^1 \cdot 3^1 \cdot 5^1 = 60$ are two factors of 600.

So there are four different values for a, two different values for b and three different values for c.

a	b	c
4	2	3

So the number of possible factors of 600 is $4 \cdot 2 \cdot 3 = 24$.

A combination lock is a keyless lock that can be opened only when a particular sequence of digits (called the combination) is set on its dial. In a combination lock, the combination can contain repeated digits. However, the name 'combination lock' is misleading since the order of the digits is important: it is actually a permutation lock. You will learn more about permutation in section 2 of this chapter.

EXAMPLE 24 A briefcase has a five-digit combination lock. The second digit in the combination is 5. At most how many different combinations must we try if we want to open the lock?

Solution $10 \cdot 1 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$

EXAMPLE 25 A four-digit number is formed using the digits in the set $\{1, 2, 3, 4, 5, 6\}$.

a. How many different numbers can be formed?

b. How many numbers can be formed if no digit can be used more than once?

c. How many numbers greater than 4000 can be formed if no digits are repeated?

d. How many numbers less than 3000 and divisible by 5 can be formed if no digits are repeated?

Solution a. Since there is no restriction and there are six digits, the answer is $6 \cdot 6 \cdot 6 \cdot 6 = 1296$.

b. Since no digit can be used more than once, the answer is $6 \cdot 5 \cdot 4 \cdot 3 = 360$.

c. There are two restrictions: the thousands digit must be greater than 3 and no digit can be used more than once. Since our number must be greater than 4000, the possible values for the thousands digit are 4, 5 and 6. Therefore there are $\underbrace{3}_{4,5,6} \cdot 5 \cdot 4 \cdot 3 = 180$ numbers which satisfy the conditions.

d. This problem sets three restrictions. First, no digit can be repeated. Second, the number must be divisible by 5, so the units digit must be 5. Third, the number must be less than 3000. So there are two possible numbers for the thousands digit: 1 and 2.

Consequently, the total number of possibilities is $\underbrace{2}_{1,2} \cdot 4 \cdot 3 \cdot \underbrace{1}_{5} = 24$.

EXAMPLE 26 A three-digit number is formed using the digits $\{0, 1, 2, 3, 4, 5\}$.

a. How many different numbers can be formed?

b. How many different numbers which are greater than 300 and divisible by 5 can be formed if no digit is repeated?

c. How many even numbers greater than 200 can be formed if all the digits are different?

d. How many numbers divisible by 4 can be formed if all the digits are different?

Solution a. Provided we do not use zero in the hundreds place, all the other digits can be used without restriction.

Hundreds	Tens	Units
5	6	6

So there are $5 \cdot 6 \cdot 6 = 180$ possible numbers.

b. There are restrictions on the first and last digits. Since the digits cannot be repeated there are two cases: if a number is greater than 500 it will not end with a 5, but any other number can end with zero or 5.

First case: Consider the case in which the first digit is 5. Then for the units digit only zero is possible since the number must be divisible by 5. Since 5 and zero are used, there are four possible digits left for the tens place.

Hundreds	Tens	Units
1	4	1
only 5	*1,2,3, or 4*	*only zero*

As we can see, there are four possible numbers (they are 10, 520, 530 and 540).

Second case: Now suppose the number does not begin with 5. Then the hundreds digit will be 3 or 4. There are two possibilities for the units digit: zero and 5. Since two digits have been used for the first and last digits, there are four possible digits left for the tens.

Hundreds	Tens	Units
2	4	2
3 or 4		*zero or 5*

The answer to the question is the sum of these cases: $4 + (2 \cdot 4 \cdot 2) = 20$.

c. In this question there is a restriction on both the hundreds digit and the units digit. Because we cannot use the digit 2 twice, we need to count the numbers ending in 2 carefully. Let us consider the three possibilities for the units digit.

If the units digit is zero:

Hundreds	Tens	Units
4	4	1
2, 3, 4, 5		*zero*

If the units digit is 2:

Hundreds	Tens	Units
3	4	1
3, 4, 5		*2*

If the units digit is 4:

Hundreds	Tens	Units
3	4	1
2, 3, 5		*4*

In conclusion, we can form $(4 \cdot 4 \cdot 1) + (3 \cdot 4 \cdot 1) + (3 \cdot 4 \cdot 1) = 40$ even numbers greater than 200.

d. For a three-digit number to be divisible by 4, the last two digits must be 04, 12, 20, 24, 32, 40, or 52. This means that the last digit in any number we form must be even.

If the units digit is zero:

Hundreds	Tens	Units
4	2	1
	2 or 4	*zero*

If the units digit is 2:

Hundreds	Tens	Units
3	3	1
	1, 3 or 5	*2*

We need to consider the case in which 4 is the units digit in two parts. (Can you see why?)

4 in the units place, 2 in the tens place:

Hundreds	Tens	Units
3	1	1
1, 3, 5	*2*	*4*

4 in the units place, zero in the tens place:

Hundreds	Tens	Units
4	1	1
1, 2, 3 or 5	*zero*	*4*

The union of all these possibilities gives us the total number of three-digit numbers which are divisible by 4. So there are $(4 \cdot 2 \cdot 1) + (3 \cdot 3 \cdot 1) + (3 \cdot 1 \cdot 1) + (4 \cdot 1 \cdot 1) = 24$ numbers.

Notice that problems b, c and d in Example 26 did not satisfy the uniformity criterion directly. However, we were able to break each problem up into separate cases which satisfied the uniformity criterion and then we added he result of each case. This is a useful strategy when we are solving combinatorics problems.

Check Yourself 3

1. Kamil lives in city E. He wants to go to city G via city F. Four different bus companies travel from E to F and three more bus companies travel from F to G.
   a. In how many different ways can Kamil travel by bus from E to G?
   b. Kamil does not want to use the same companies again on his way back home. In how many different ways can he arrange his trip from E to G and back?
2. We have a list of 12 questions for the second part of this chapter. We need to choose one question as an Example, one for a Check Yourself section and one for an Exercises section. Assuming that any question can be used in any section, in how many different ways can we make our choice?
3. An astronomer wants to name 7000 celestial objects with a code made up of two letters from the English alphabet followed by a digit. Is this possible?
4. Almaz's teacher asks him to write a five-digit number whose first and last digits are even while the others are odd. How many different numbers can Almaz write if he does not want to use the same digit twice?
5. How many three-digit numbers greater than 450 can be formed from the digits in the set $\{1, 2, 3, 4, 5, 6, 7\}$?

Answers

1. a. 12 b. 72 2. 1320 3. no 4. 960 5. 168

BITS AND BYTES

The numbers we use in daily life are part of the base 10 number system. In the base 10 system, each digit in a number represents a multiple of a power of 10 and the digits are 0, 1, 2, ..., 9.

The base 2 number system (also called the binary system) has only two digits: 0 and 1. Each digit in a number is a multiple of a power of 2, and any number is written as a sequence of 0 and 1 digits. For example, the binary digit sequence 1101 represents the integer 13 in our ordinary number system:

$$1\,1\,0\,1$$

$$\begin{aligned} 1 \times 2^0 &= 1 \\ 0 \times 2^1 &= 0 \\ 1 \times 2^2 &= 4 \\ 1 \times 2^3 &= 8 \\ + \quad & \overline{13} \end{aligned}$$

Base 2 numbers are very useful in computing and in electronic circuits. An electronic circuit operates with two electrical signals: current on (=1) or current off (=0). A sequence of these signals represents a binary number, for example: 101. Each digit in the number is called a bit, so 101 is a three-bit number. Everything that happens on a computer is the result of many thousands of operations with these binary signals and numbers. Computers also use binary numbers to store information as data, for example on a hard disk.

A bit is the smallest possible amount of computer data. Because data usually comprises an incredible number of bits, we group the bits together in sequences called bytes. A byte is a sequence of eight bits. Here are three examples of bytes: (10011101), (00101111), (11111011).

The code for each character on a computer keyboard can be stored as a byte. For example, when you press the asterisk key (*) on your keyboard, the keyboard sends the byte (00101010) to your computer because the hexadecimal code for the asterisk symbol is 42, which is (00101010) in base 2.

So how many different bytes are possible? Since each byte contains eight bits and there are only two possibilities (0 or 1) for each bit, we have:

1st bit	2nd bit	3rd bit	4th bit	5th bit	6th bit	7th bit	8th bit
2	2	2	2	2	2	2	2

So by the multiplication principle there are $2^8 = 256$ different possible bytes. (00000000) is the smallest byte and (11111111) is the largest byte.

How many different possible messages could you make with two bytes? What about three bytes? One very early home computer had a memory of 1024 bytes (called one kilobyte, or 1K). What is the largest number this computer could store in full in its memory?

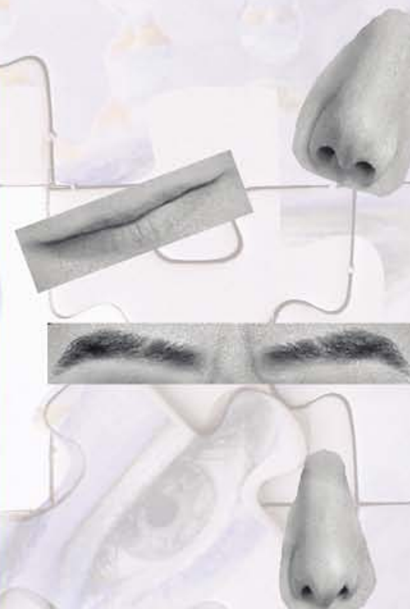
FACIAL RECONSTRUCTION

Facial identification is an important part of forensic science. When a person commits a crime, witnesses of the crime can sometimes describe the person's appearance to the police. A forensic artist works with the witnesses to make a picture of the person's face, either with pen and paper or with a computer program. This process is called *facial reconstruction*.

Facial reconstruction programs store many pictures (or *variations*) of different parts of a face: the eyes, nose, ears, mouth and hair, etc. Some of these parts, such as the eyes and nose, are very important in determining the overall appearance of a face. Sometimes the forensic artist guides the witness to keep the different parts of a face consistent and in natural proportion.

There are several different facial reconstruction programs. Newer programs allow an operator to construct a face in three dimensions. One program has the following number of variations for the different parts of a face:

Part of Face	Variations
Eyes	876
Eyebrows	834
Forehead	222
Nose	950
Face layout	927
Moustache	243
Beard	512
Glasses	162
Ears	600
Lips	873
Hair	635



The following pictures are examples of some two-dimensional faces produced by the program:

By the multiplication principle, we can calculate that this program can produce over 10^{30} possible faces using the given variations.

EXERCISES 1.1

A. The Addition Principle

1. A box contains 5 different white balls and 7 different red balls. In how many ways can we randomly pick a ball from the box?

B. Systematic Listing

2. A house has 5 windows and 2 doors. In how many ways can a burglar break in through a window and get out through a door? Show the possibilities in a product table.

3. How many two-digit prime numbers can we form using the digits 1, 2, 3 and 5?

4. In how many ways we can distribute 3 different gifts to Ömer, Ali and Cihan if each person gets one gift?

5. In how many ways can a president and a secretary be chosen from a group of 4 people if anyone can hold either position?

6. A play-off game is a contest or series of contests that are played to break a tie and determine the winner of a championship. In play-off games, the contests stop as soon as it is clear that one team will win the play-off. A play-off between 2 basketball teams has at most 3 contests. How many different play-offs are possible between the two teams?

7. A family has 3 children. List all the possible gender combinations (male or female) for these children, ordered from oldest to youngest.

8. Sheena needs to go to a school, a restaurant, a cinema and a supermarket. If she must not go to the restaurant before the supermarket, in how many different ways can her trip be arranged?

9. A power panel has 5 electric switches and the power supply depends on their positions. The power supply is on unless 2 adjacent switches are both off. How many switch settings will keep the supply on?

10. A group of tourists in a country want to visit 5 cities A, B, C, D and E once. They will fly to each city, beginning at city C and ending at city E. If there is no flight between cities B and D, in how many ways can they organize the journey? Show the possibilities in a tree diagram.

C. The Multiplication Principle

11. A dice is rolled, a coin is flipped and a card is drawn from a deck of 52 cards. How many outcomes are possible?

12. Rashid has 4 different colored pens and wants to color each letter in the word ANTARCTICA so that the same letters are the same color. In how many different ways can he do this?

13. A new scooter is available in 4 different colors with 3 types of engine and 2 types of seat. How many different configurations of this model are possible?

14. n dice are rolled together and all of them show the same number. In how many ways can this be done?

15. An ice-cream shop advertises that it can prepare 126 different varieties of ice cream. An ice-cream variety is determined by the way it is served, its flavour and its topping. It can be served in a bowl, a waffle or a cone and it can be topped with chocolate sauce or ground walnuts or hazelnuts. How many different flavours can be ordered?

16. Igor is preparing a test of 12 multiple-choice questions for his students. Each question has 4 choices of answer. How many different possible answer keys could Igor prepare?

17. Dastan is going to take the test described in the previous question. In how many ways can Dastan complete his answer sheet if

a. he is not allowed to leave any answer blank?

b. he is allowed to leave an answer blank?

18. Two teachers and 5 students will form a row to have their photos taken. In how many different ways can they be arranged if the teachers must be at the ends of the row?

19. In how many ways can 5 boys and 5 girls be seated in a row if the same genders cannot sit next to each other?

20. A word which reads the same both forward and backward is called a *palindrome*. (For example: RADAR is a palindrome.) How many palindromes of 7 letters can be formed using the 26 letters of the English alphabet?

21. The combination for a combination lock has 5 digits. We know that the second and fourth digits are the same and the last digit is odd. How many different possible combinations are there for the lock?

22. A briefcase is locked with two different four-digit combination locks. A thief knows that the combination for the first lock is a number from 2000 to 5999 and that the first digit of the second lock is the last digit of the first lock. At most how many different combinations must the thief test in order to open both locks?

23. In how many different ways can we pour tea into 10 cups if

a. the cups are identical?

b. the cups are different?

24. The password on Enzi's e-mail account is a number with non-repeating digits.The password is at least 2 and at most 4 digits long. How many possibilities are there for Enzi's password?

25. The auto license numbers of all the cars registered in a particular city are made up of two letters followed by three or four digits. If there are 26 possible letters, how many license numbers can end with 91?

26. An organisation wants to create a registration code format using the 26 letters A-Z and the digits 0-9. The format for all codes must have 7 characters four digits followed by three letters and there must be enough possibilities to register 200 million different people. Is this possible?

27. A man in Italy once suggested the following automobile license number format: each number should consist of two letters followed by three digits. If the first digit could not be zero and there are 21 letters in the Italian alphabet, could this format be used to register 400 000 vehicles?

28. How many two-digit numbers can we form using the elements of the set {1, 2, 3, 4, 5} if

a. repetition is allowed?

b. repetition is allowed and the number must be even?

29. In an experiment, a die is rolled 5 times and the numbers obtained are written as digits in order. How many different five-digit odd numbers greater than 40 000 can be produced in this way?

30. How many different three-digit odd numbers can be formed using the digits 0, 1, 2, 3, 4 or 5 with no repetition?

31. How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5 or 6 if no two odd digits must be next to each other?

32. How many numbers between 500 and 1000 inclusive contain repeated digits?

33. How many four-digit numbers have at least one odd digit?

34. A machine generates all the possible four-digit numbers using the digits 1, 2, 3 and 4. In how many numbers is the digit 4 on the left of (but not necessarily next to) 2 if repetition is not allowed?

35. A machine generates all the possible five-digit numbers from the digits in the set {0, 1, 2, 3, 4, 5, 6, 7}. How many of them are divisible by 25 if

a. repetition is allowed?

b. repetition is not allowed?

36. How many different numbers with nonrepeating digits from 4000 to 6000 can be produced using the digits {0,1,3,4,5,6,9} if

a. the numbers must be odd?

b. the numbers must be divisible by 9?

37. The controls on Anton's spaceship are activated with a password. Anton has forgotten the password but he knows that password is a number from 5 digit to 7 digit length which does not begin with zero. At most how many different passwords must Anton try?

38. A security password consists of 7 characters and each character must be either a digit or a lower-case letter. The password must contain at least one letter and at least one digit. If there are 26 possible letters, how many different passwords can be set?

39. Bahadır wants to set a password for his computer. The password must be between 3 and 5 letters long and the first and last letters must be vowels. If there are 26 possible letters, how many different passwords can Bahadır set?

40. A *monogram* is a symbol made up of a person's initials. Explain why in a group of 700 people at least 2 people have a monogram made from the same two-letter monogram.

PERMUTATIONS

We can define a permutation as an ordered arrangement of some or all of the elements in a given set. The way a set of books is arranged on a shelf, the seating positions of a group of people at a table or the way the players in a football team line up for a team photo are some examples of permutations since in each case, the order of the elements is important.

A. FACTORIAL NOTATION

When we are solving permutation problems, we often need to express the product of all consecutive counting numbers from 1 to a number n. Factorial notation provides an easy way to denote this product.

Definition

factorial

For any counting number n, the product of all positive integers less than or equal to n is called n factorial and denoted by $n!$:

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1.$$

For example, $3! = 3 \cdot 2 \cdot 1 = 6$ and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.
As a special case we accept that $0! = 1$.

EXAMPLE 27 Evaluate the expressions.

a. $7!$ b. $6! - 3!$ c. $(6-3)!$ d. $4! + 2!$

e. $(4+2)!$ f. $\dfrac{8!}{4!}$ g. $\left(\dfrac{8}{4}\right)!$

Solution

a. $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

b. $6! - 3! = (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) - (3 \cdot 2 \cdot 1) = 720 - 6 = 714$

c. $(6-3)! = (3)! = 3! = 3 \cdot 2 \cdot 1 = 6$

d. $4! + 2! = (4 \cdot 3 \cdot 2 \cdot 1) + (2 \cdot 1) = 24 + 2 = 26$

e. $(4+2)! = (6)! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

f. $\dfrac{8!}{4!} = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$

g. $\left(\dfrac{8}{4}\right)! = (2)! = 2 \cdot 1 = 2$

Remark

For all positive integers, $n! = n(n-1)!$

For example, $7! = 7 \cdot \underbrace{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{6!} = 7 \cdot 6!$.

As a result of this property, we can write
$n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-1)(n-2)(n-3)!$, etc.

EXAMPLE 28 Evaluate the expressions.

a. $\dfrac{9!}{8!}$ b. $\dfrac{13!}{11!}$ c. $100! - 99!$ d. $\dfrac{10!+8!}{9!-7!}$

Solution

a. $\dfrac{9!}{8!} = \dfrac{9 \cdot \cancel{8!}}{\cancel{8!}}$

b. $\dfrac{13!}{11!} = \dfrac{13 \cdot 12 \cdot \cancel{11!}}{\cancel{11!}} = 156$

c. $100! - 99! = (100 \cdot 99!) - 99! = 99!(100 - 1) = 99 \cdot 99!$

d. $\dfrac{10!+8!}{9!-7!} = \dfrac{(10 \cdot 9 \cdot 8 \cdot 7!) + (8 \cdot 7!)}{(9 \cdot 8 \cdot 7!) - 7!} = \dfrac{\cancel{7!}(10 \cdot 9 \cdot 8 + 8)}{\cancel{7!}(9 \cdot 8 - 1)} = \dfrac{728}{71}$

EXAMPLE 29 Simplify the expressions.

a. $\dfrac{n!}{(n-1)!}$ b. $\dfrac{(n+3)!}{(n-1)!} \cdot \dfrac{(n-2)!}{(n+2)!}$ c. $\dfrac{(n+1)!}{n \cdot (n-2)!}$

Solution

a. $\dfrac{n!}{(n-1)!} = \dfrac{n\,\cancel{(n-1)!}}{\cancel{(n-1)!}} = n$

b. $\dfrac{(n+3)!}{(n-1)!} \cdot \dfrac{(n-2)!}{(n+2)!} = \dfrac{(n+3) \cdot \cancel{(n+2)!}}{(n-1) \cdot \cancel{(n-2)!}} \cdot \dfrac{\cancel{(n-2)!}}{\cancel{(n+2)!}} = \dfrac{(n+3)}{(n-1)}$

c. $\dfrac{(n+1)!}{n \cdot (n-2)!} = \dfrac{(n+1) \cdot (n-1) \cdot \cancel{n} \cdot \cancel{(n-2)!}}{\cancel{n} \cdot \cancel{(n-2)!}} = (n+1) \cdot (n-1) = n^2 - 1$

EXAMPLE 30 Solve $\dfrac{(n+2)!}{(n^3-n)\cdot(n-3)!}=6n-12.$

Solution

$$\frac{(n+2)!}{(n^3-n)(n-3)!}=\frac{(n+2)\cdot(n+1)\cdot n\cdot(n-1)\cdot(n-2)\cdot(n-3)!}{n(n^2-1)(n-3)!}$$

$$=\frac{(n+2)\cdot(n-2)\cdot\cancel{n}\cdot\cancel{(n+1)}\cdot\cancel{(n-1)}\cdot\cancel{(n-3)!}}{\cancel{n}\cdot\cancel{(n+1)}\,\cancel{(n-1)}\,\cancel{(n-3)!}}$$

$$=(n+2)\cdot(n-2)=n^2-4.$$

So $n^2-4=6n-12$, which gives $n^2-6n+8=0$.

This equation has two roots: $n_1=2$ and $n_2=4$.

Since the first root makes $(n-3)!$ invalid, $n=4$.

> Remember!
> In the factorial expression $n!$, n must be a counting number.

Check Yourself 4

1. Evaluate the expressions.

   a. $\dfrac{13!}{10!3!}$ b. $\dfrac{15!-14!}{15!+14!}$ c. $\dfrac{(n+2)!(n-1)!}{(n+1)!n!}$

2. Solve for n.

   a. $\dfrac{(n+1)!}{(n-1)!}\cdot\dfrac{(n-2)!}{n!}=\dfrac{7}{5}$ b. $n(n^2-1)(n-2)!=720$

Answers

1. a. 286 b. $\dfrac{7}{8}$ c. $\dfrac{n+2}{n}$ 2. a. 6 b. 5

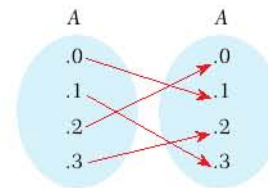
B. PERMUTATION FUNCTIONS

Definition

permutation function

Let A be a non-empty set. A one-to-one and onto function from A to A is called a permutation function in A.

For example, consider the function $f: A \to A$ with $A = \{0, 1, 2, 3\}$ and $f(0) = 1, f(1) = 3, f(2) = 0, f(3) = 2$.



f is shown in the Venn diagram opposite. We can see that it is a one-to-one and onto function, and so it is a permutation function.

$f = \{(0, 1), (1, 3), (2, 0), (3, 2)\}$.

Alternatively we can write it in the form $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \end{pmatrix}$.

This is a common way of writing a permutation function.

Note that f is not the only permutation which can be defined in A in the example above. In fact, we can define $n!$ different permutation functions in a set with n elements. So in this example we can define $4! = 24$ different permutation functions in A.

$f_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 0 & 1 \end{pmatrix}$, $f_3 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 3 \end{pmatrix}$ and $f_4 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ are three examples of such permutation functions.

EXAMPLE 31 List all the permutation functions defined in $K = \{p, q, r\}$.

Solution $f_1 = \begin{pmatrix} p & q & r \\ p & q & r \end{pmatrix}$, $f_2 = \begin{pmatrix} p & q & r \\ p & r & q \end{pmatrix}$, $f_3 = \begin{pmatrix} p & q & r \\ q & p & r \end{pmatrix}$, $f_4 = \begin{pmatrix} p & q & r \\ q & r & p \end{pmatrix}$, $f_5 = \begin{pmatrix} p & q & r \\ r & p & q \end{pmatrix}$, $f_6 = \begin{pmatrix} p & q & r \\ r & q & p \end{pmatrix}$

1. Identity Permutation Functions

Definition

identity permutation function

Let I be a permutation function defined in a set A. If $I(x) = x$ for every $x \in A$ then I is called the identity permutation function in A.

For example, if I is the identity function defined in the set $P = \{1, 2, 3, 4\}$ then $I(1) = 1$, $I(2) = 2$, $I(3) = 3$ and $I(4) = 4$.

We can write this identity permutation function as $I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

2. Composite Permutation Functions

We have already stated that a permutation function in a set A must be a one-to-one and onto function. If f and g are two permutation functions defined in A, then $f \circ g$ and $g \circ f$ are also permutations in A.

For example, suppose $f = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,1\,4\,2 \end{pmatrix}$ and $g = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,2\,4\,1 \end{pmatrix}$ are two permutation functions defined in the set $P = \{1, 2, 3, 4\}$. Then the composite function $f \circ g$ is $f \circ g(x) = f(g(x))$, so

$$\left.\begin{aligned} f \circ g(1) &= f(g(1)) = f(3) = 4 \\ f \circ g(2) &= f(g(2)) = f(2) = 1 \\ f \circ g(3) &= f(g(3)) = f(4) = 2 \\ f \circ g(4) &= f(g(4)) = f(1) = 3 \end{aligned}\right\}, \text{ i.e. } f \circ g = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,1\,2\,3 \end{pmatrix}.$$

We can visualize this as $f \circ g = f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & \boxed{4} & 2 \end{pmatrix} \circ \begin{pmatrix} \boxed{1} & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$.

EXAMPLE 32 $f = \begin{pmatrix} a & b & c & d \\ d & b & a & c \end{pmatrix}$ and $g = \begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix}$ are two permutations defined in $H = \{a, b, c, d\}$. Show that $f \circ g \neq g \circ f$.

Solution

$$\left.\begin{aligned} f \circ g &= \begin{pmatrix} a & b & c & d \\ d & b & a & c \end{pmatrix} \circ \begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix} \\ g \circ f &= \begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix} \circ \begin{pmatrix} a & b & c & d \\ d & b & a & c \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ d & c & b & d \end{pmatrix} \end{aligned}\right\} \Rightarrow f \circ g \neq g \circ f$$

Notes

'$\forall x \in A$' means 'for all elements x in A'.

1. The composition of permutation functions is not commutative: $f \circ g \neq g \circ f$.
2. The composition of permutation functions is associative: $(f \circ g) \circ h = f \circ (g \circ h)$ because $(f \circ g) \circ h = f(g(h(x))) = f \circ (g \circ h)$.
3. For any permutation f and identity permutation I in a set A, $f \circ I = I \circ f = f$ since $\forall x \in A,\ f \circ I(x) = f(I(x)) = f(x)$ and $I \circ f(x) = I(f(x)) = f(x)$.

3. The Inverse of a Permutation Function

Since a permutation f in a set A is both one-to-one and onto, by reversing the ordered pairs of f we get the inverse permutation function of f, denoted by f^{-1}.

For example, if $P = \{0, 1, 2, 3\}$ is a set and

$f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 1 & 0 & 2 \end{pmatrix}$ is a permutation in P then $f^{-1} = \begin{pmatrix} 3 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$, i.e. $f^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix}$.

Note that $f \circ f^{-1} = f^{-1} \circ f = I$.

EXAMPLE 33 In a set $K = \{\bigcirc, \blacklozenge, \star, \square, \triangle\}$, the permutation function $g = \begin{pmatrix} \Delta & \square & \bigcirc & \star & \blacklozenge \\ \square & \blacklozenge & \Delta & \star & \bigcirc \end{pmatrix}$ is defined. Show that $g \circ g^{-1} = I$.

Solution If $g = \begin{pmatrix} \Delta & \square & \bigcirc & \star & \blacklozenge \\ \square & \blacklozenge & \Delta & \star & \bigcirc \end{pmatrix}$ then $g^{-1} = \begin{pmatrix} \Delta & \square & \bigcirc & \star & \blacklozenge \\ \bigcirc & \Delta & \blacklozenge & \star & \square \end{pmatrix}$.

So $g \circ g^{-1} = \begin{pmatrix} \Delta & \square & \bigcirc & \star & \blacklozenge \\ \square & \blacklozenge & \Delta & \star & \bigcirc \end{pmatrix} \circ \begin{pmatrix} \Delta & \square & \bigcirc & \star & \blacklozenge \\ \bigcirc & \Delta & \blacklozenge & \star & \square \end{pmatrix} = \begin{pmatrix} \Delta & \square & \bigcirc & \star & \blacklozenge \\ \Delta & \square & \bigcirc & \star & \blacklozenge \end{pmatrix} = I.$

EXAMPLE 34 $f = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 3 & 7 & 9 & 5 & 1 \end{pmatrix}$ and g are two permutations defined in the set $K = \{1, 3, 5, 7, 9\}$.

$g \circ f = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 7 & 5 & 3 & 9 & 1 \end{pmatrix}$ is given. Find g.

Solution To find g, we have to eliminate f from $g \circ f$.

We can achieve this by composing $g \circ f$ with the inverse of f, since

$(g \circ f) \circ f^{-1} = g \circ (f \circ f^{-1}) = g \circ I = g.$

So we must find f^{-1}. If $f = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 3 & 7 & 9 & 5 & 1 \end{pmatrix}$ then $f^{-1} = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 9 & 1 & 7 & 3 & 5 \end{pmatrix}$.

So $g = (g \circ f) \circ f^{-1} = \underbrace{\begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 7 & 5 & 3 & 9 & 1 \end{pmatrix}}_{given} \circ \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 9 & 1 & 7 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 1 & 7 & 9 & 5 & 3 \end{pmatrix}$.

Check Yourself 5

1. $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix}$ is defined in $K = \{0, 1, 2, 3\}$. Find f^{-1}.

2. $g = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 6 & 2 & 8 & 4 \end{pmatrix}$ is defined in $P = \{2, 4, 6, 8\}$. Find $g \circ g$.

3. $g = \begin{pmatrix} \triangle & \square & \bigcirc & \star \\ \square & \star & \bigcirc & \triangle \end{pmatrix}$ and f are defined in $H = \{\bigcirc, \star, \square, \triangle\}$. $g \circ f = \begin{pmatrix} \triangle & \square & \bigcirc & \star \\ \square & \bigcirc & \triangle & \star \end{pmatrix}$ is given. Find f.

Answers

1. $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix}$ 2. $\begin{pmatrix} 2 & 4 & 6 & 8 \\ 8 & 6 & 4 & 2 \end{pmatrix}$ 3. $\begin{pmatrix} \triangle & \square & \bigcirc & \star \\ \triangle & \bigcirc & \star & \square \end{pmatrix}$

C. PERMUTATIONS OF *n* ELEMENTS

We have defined a permutation as ordered arrangement of a set of elements or items. In a permutation, the order of the items is important. We considered some permutation problems in our study of the multiplication principle.

Here is another example of a permutation problem: in how many different ways can the three students Faruk, Oleg and Evgeny be seated at a desk?

Definition

permutation

An ordered arrangement of some or all of the elements of a given set is called a permutation.

The number of permutations of all of the n distinct elements in a set is denoted by $P(n, n)$, where $P(n, n) = n(n - 1) \cdot (n - 2) \cdot ... \cdot 2 \cdot 1 = n!$.

In our library seating problem we can see that there are six ways for three students to sit at a desk. Using the permutation notation described above, we can write $P(3, 3) = 6$.

EXAMPLE 35 What is the number of permutations of 5 different math books piled on a table?

Solution By the definition above, the answer is $P(5, 5) = 5! = 120$ permutations because there are five distinct books.

We can check this answer using the counting technique we learned when we studied the multiplication principle:

First book	Second book	Third book	Fourth book	Fifth book
5	4	3	2	1

Again we find that there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to put the books in a pile.

EXAMPLE 36 How many different eight-letter permutations are there of the letters in the word ISTANBUL?

Solution Notice that there are eight letters and each letter is distinct. So we can use the formula for $P(n, n)$ using $n = 8$: $P(8, 8) = 8! = 40320$ permutations.

EXAMPLE 37 A football league has 18 teams. How many different rankings from first to last are possible in the end-of-season league table, assuming that there are no ties?

Solution Since each football team is distinct we can use the permutation formula. Therefore the answer is $P(18, 18) = 18!$ possible rankings.

EXAMPLE 38 Murat has 5 different math books, 3 different biology books and 4 different physics books. In how many different ways can Murat arrange his books

a. on a book shelf?

b. in three different file holders, if each holder is for a different subject?

Solution a. There is no restriction on the order of the books on the shelf so we do not need to consider the subjects. Since there are twelve books, the answer is $P(12, 12) = 12!$ different ways.

b. In this case we need to consider the subjects separately.

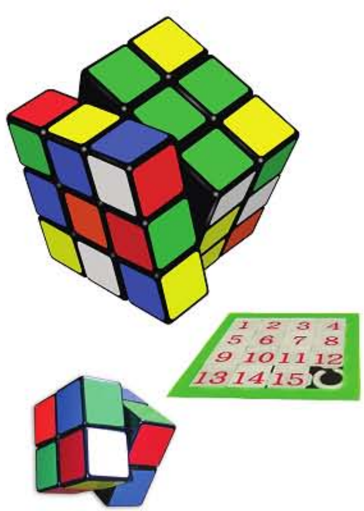
$$\underbrace{P(5,\ 5)}_{math} \cdot \underbrace{P(3,\ 3)}_{biology} \cdot \underbrace{P(4,\ 4)}_{physics} \cdot 3! = 5! \cdot 3! \cdot 4! \cdot 3! = 120 \cdot 6 \cdot 24 \cdot 6 = 103\,680 \text{ arrangements.}$$

Check Yourself 6

1. In how many different ways can 5 students form a queue?
2. Rasim, Togrul and Elnur are going to establish a company whose name will be a combination of their initials. How many company names are possible?
3. There are 10 desks in a classroom and each desk has two seats. In how many different ways can 20 students sit in the classroom?

Answers

1. 120 2. 6 3. 20!



These toys are some interesting samples for permutation puzzles.

D. PERMUTATIONS OF *r* ELEMENTS SELECTED FROM *n* ELEMENTS

Many permutation problems ask us to consider arrangements of r things chosen from n things $(0 \le r \le n)$, i.e. permutations of r elements chosen from a set of n elements.

EXAMPLE 39 How many different two-letter combinations can we form from the letters of the word KANO if a letter cannot be used more than once?

Solution The order of the letters is important and a letter cannot be used more than once. By the multiplication principle, the number of combinations is: $\underline{4} \cdot \underline{3} = 12$. These combinations are

KA	AK	NK	OK
KN	AN	NA	OA
KO	AO	NO	ON

.

In this section we will use a new formula to solve problems of this type.

Definition permutation of r elements selected from n elements

The number of permutations of r elements selected from a set of n elements is

$$P(n, r) = \frac{n!}{(n-r)!} \quad (n, r \in \mathbb{N} \text{ and } 0 \le r \le n).$$

Some books use ${}_nP_r$ or P_n^r to mean $P(n, r)$.

If we apply this formula to Example 39, we can write the answer as

$$P(4,\ 2) = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 12.$$

Note

Any question which can be solved using this permutation formula can also be solved using the multiplication principle.

EXAMPLE 40 Calculate $P(5,\ 3) \cdot P(7,\ 2)$.

Solution $$P(5,\ 3) \cdot P(7,\ 2) = \frac{5!}{(5-3)!} \cdot \frac{7!}{(7-2)!} = \frac{\cancel{5!}}{2!} \cdot \frac{7!}{\cancel{5!}} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$$

EXAMPLE 41 Evaluate the expressions.

a. $P(7, 3)$ b. $P(n, n)$ c. $P(n, 0)$

Solution a. $P(7,3) = \dfrac{7!}{(7-3)!} = \dfrac{7!}{4!} = \dfrac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}} = 210$

b. $P(n,n) = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = \dfrac{n!}{1} = n!$

c. $P(n,0) = \dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = \dfrac{n!}{n!} = 1$

EXAMPLE 42 $P(n, 3) \cdot 5 = P(n, 4)$ is given. Find n.

Solution $P(n, 3) \cdot 5 = P(n, 4)$

$$\frac{\cancel{n!}}{(n-3)!} \cdot 5 = \frac{\cancel{n!}}{(n-4)!}$$

$$\frac{5}{(n-3) \cdot \cancel{(n-4)!}} = \frac{1}{\cancel{(n-4)!}}$$

$$5 = n-3$$

$$n = 8$$

EXAMPLE 43 How many three-digit numbers can be formed from the digits in the number 13567 if a digit cannot be used more than once?

Solution We are choosing three digits from five digits. So there are

$P(5,3) = \dfrac{5!}{(5-3)!} = \dfrac{5 \cdot 4 \cdot 3 \cdot \cancel{2!}}{\cancel{2!}} = 60$ different three-digit numbers.

Notice that we could have solved the same question using the multiplication principle: $5 \cdot 4 \cdot 3 = 60$.

EXAMPLE 44 Three raffle tickets will be selected in order from a box containing 30 tickets. The person holding the first ticket will win a car, the person with the second ticket will win a computer, and the person with the third ticket will win a CD player. In how many different ways can these prizes be awarded?

Solution Since the question is about an ordered arrangement of three tickets selected from thirty tickets, we can use the formula:

$P(30,3) = \dfrac{30!}{(30-3)!} = \dfrac{30!}{27!} = \dfrac{30 \cdot 29 \cdot 28 \cdot \cancel{27!}}{\cancel{27!}} = 24360.$

Remark

The number of permutations of r elements selected from n elements is the product of r successive numbers less than or equal to n:

$$P(n,\ r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{r\ factors}.$$

For example,

$P(5,\ 4) = \underbrace{5 \cdot 4 \cdot 3 \cdot 2}_{4\ factors} = 120$, $P(10,\ 3) = \underbrace{10 \cdot 9 \cdot 8}_{3\ factors} = 720$ and $P(20,\ 1) = \underbrace{20}_{1\ factor} = 20$.

EXAMPLE 45 A fighter plane has seats for a pilot and a copilot. In how many different ways can these be selected from a squadron of 18 soldiers?

Solution $P(18,\ 2) = \underbrace{18 \cdot 17}_{2\ factors} = 306$

EXAMPLE 46 How many different combinations of at least 3 letters can be formed from the letters in the word MATHS if no letter can be used more than once?

Solution 'At least 3 letters' means the combination can have 3 letters, 4 letters or 5 letters. So we need to consider three mutually exclusive cases: combinations of 3 letters, 4 letters and 5 letters.

Mutually exclusive cases are cases which cannot happen at the same time.

Then we add the number of permutations in each case:

$$\underbrace{P(5,\ 3)}_{3\text{-letter words}} + \underbrace{P(5,\ 4)}_{4\text{-letter words}} + \underbrace{P(5,\ 5)}_{5\text{-letter words}} = (5 \cdot 4 \cdot 3) + (5 \cdot 4 \cdot 3 \cdot 2) + (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 60 + 120 + 120 = 300.$$

So there are 300 possible combinations.

EXAMPLE 47 Kemal's bookcase has three shelves. Kemal has 5 different math books, 6 different biology books and 7 different physics books. He wants to arrange 3 math books, 4 biology books and 5 physics books on the shelves so that each shelf is for one subject only. In how many different ways can Kemal arrange his books?

Solution There are $P(5,\ 3)$ possible ways to arrange the math books. There are also $P(6,\ 4)$ and $P(7,\ 5)$ different possible ways to order the biology and physics books respectively.

However, Kemal can choose the shelves for the subjects in $3!$ ways. As a result there are

$$\underbrace{P(5,\ 3)}_{math} \cdot \underbrace{P(6,\ 4)}_{biology} \cdot \underbrace{P(7,\ 5)}_{physics} \cdot \underbrace{3!}_{shelves} = 60 \cdot 360 \cdot 840 \cdot 6$$

$= 108\ 864\ 000$ ways for Kemal to arrange his books.

EXAMPLE 48 A three-digit number is formed by choosing elements from the set {1, 3, 4, 5, 7, 8, 9} without repetition.

a. How many numbers do not contain the digit 5?

b. How many numbers contain the digit 5?

c. How many numbers contain 1 or 7 or both 1 and 7?

Solution

a. The problem is the same as finding the number of three-digit permutations of the set {1, 3, 4, 7, 8, 9} (5 excluded): $P(6, 3) = 6 \cdot 5 \cdot 4 = 120$.

b. The total number of three-digit permutations of the set {1, 3, 4, 5, 7, 8, 9} is $P(7, 3) = 7 \cdot 6 \cdot 5 = 210$. From part a, 120 of these permutations do not contain the digit 5. So there are $210 - 120 = 90$ three-digit numbers which contain the digit 5.

c. We begin by calculating the number of three-digit permutations in which 1 and 7 are not used: $P(5, 3) = 5 \cdot 4 \cdot 3 = 60$ permutations. So there are $210 - 60 = 150$ three-digit numbers which contain 1 or 7 or both 1 and 7.

Check Yourself 7

1. There are 7 different pieces of fruit on a tray. We will choose 3 of them and arrange them in a row on a plate. How many different arrangements are possible?

2. The students in a class are photographed in pairs such that each student is photographed with every other student. If there are 90 photos, how many students are there in the class?

3. A machine generates all the possible two-letter combinations of the letters *ABCDE*, without using a letter twice. What percentage of the combinations do not contain a consonant?

4. How many of the four-digit numbers formed from the digits of the number 12345 without repetition do not begin with the digit 2?

5. A group *A* contains 6 students and a group *B* contains 8 students. In a class photo, two students who are to sit in the front will be from *A* and three students who are to stand at the back will be from *B*. How many arrangements are possible?

Answers

1. $P(7, 3) = 210$ 2. 10 3. 10% 4. $P(5, 4) - P(4, 3) = 96$ 5. $P(6, 2) \cdot P(8, 3) = 10080$

E. PERMUTATIONS WITH RESTRICTIONS

1. Permutations with Grouped Elements

Sometimes a permutation problem states that we should not separate two or more elements in the set. In this case we count the elements as a single element. Then we apply the ordinary permutation rules. However, we need to consider the number of arrangements within the group of combined elements. By the multiplication principle, we multiply the results to get the answer.

EXAMPLE 49 How many five-letter words can we form using all the letters in the word MERAK if A and K must be next to each other?

Solution First we count the group of letters A and K as a single element. Then the problem is to find the number of permutations of four elements, namely M, E, R and (A, K). However, for each permutation of these four elements there are two arrangements within the group (A, K). Therefore the answer is

$$(\underbrace{3}_{M,\,E,\,R}+\underbrace{1}_{\substack{\text{group of}\\ A \text{ and } K}})!\cdot\underbrace{2!}_{A,\,K}=4!\cdot 2!=24\cdot 2=48 \text{ words.}$$

EXAMPLE 50 Solve the roller-coaster problem in Example 11 by using grouping and permutation.

Solution Since Ahmet and Cemal must sit together, we count them as single element.

$$(\underbrace{2}_{\text{Berk and Deniz}}+\underbrace{1}_{\substack{\text{group of}\\ \text{Ahmet and Cemal}}})!\cdot\underbrace{2!}_{\text{Ahmet, Cemal}}=3!\cdot 2!$$

$= 12$. This is the answer we found in Example 11.

EXAMPLE 51 In how many ways can the children Anar, Maksat, Sasha, Dilshat, Catalin and Mehmet sit in a row if Maksat and Catalin must not sit together?

Solution The total number of possible arrangements of six children is $6! = 720$.

From these, the number of permutations in which Maksat and Catalin sit together will be

$$(4+1)!\cdot\underbrace{2!}_{\text{Maksat, Catalin}}=5!\cdot 2!=240.$$

So the number of permutations in which Maksat and Catalin are not together is

$720 - 240 = 480$.

EXAMPLE 52 In Example 38, Murat had five different math books, three different biology books and four different physics books. In how many ways can Murat arrange his books on a shelf if

a. the math books must be kept together?

b. the biology and physics books must be kept together in two different groups?

c. all the books on the same subject must be kept together?

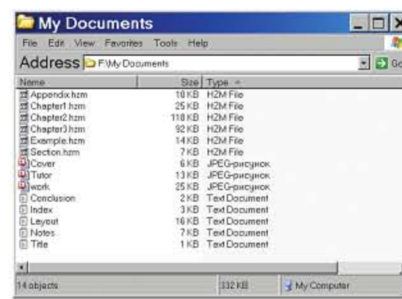
Solution a. Since the math books must be together we consider them as a single book. So the answer is

$$(\underbrace{1}_{\substack{\text{group of}\\ \text{math books}}} + \underbrace{3}_{\substack{\text{biology}\\ \text{books}}} + \underbrace{4}_{\substack{\text{physics}\\ \text{books}}})! \cdot \underbrace{5!}_{\substack{\text{math}\\ \text{books}}} = 8! \cdot 5!.$$

b. $$(\underbrace{1}_{\substack{\text{group of}\\ \text{biology}\\ \text{books}}} + \underbrace{5}_{\substack{\text{math}\\ \text{books}}} + \underbrace{1}_{\substack{\text{group of}\\ \text{physics}\\ \text{books}}})! \cdot \underbrace{3!}_{\substack{\text{biology}\\ \text{books}}} \cdot \underbrace{4!}_{\substack{\text{physics}\\ \text{books}}} = 7! \cdot 3! \cdot 4!$$

c. $$(\underbrace{1}_{\substack{\text{group of}\\ \text{biology}\\ \text{books}}} + \underbrace{1}_{\substack{\text{group of}\\ \text{math books}}} + \underbrace{1}_{\substack{\text{group of}\\ \text{physics}\\ \text{books}}})! \cdot \underbrace{3!}_{\substack{\text{biology}\\ \text{books}}} \cdot \underbrace{4!}_{\substack{\text{physics}\\ \text{books}}} \cdot \underbrace{5!}_{\substack{\text{math}\\ \text{books}}} = 3! \cdot 3! \cdot 4! \cdot 5!$$

EXAMPLE 53 In the *My Documents* folder on my computer there are 3 files with the extension *jpg*, 5 files with the extension *txt*, and 6 files with the extension *hzm*. In how many ways can the files be listed if the files are ordered by type, ignoring alphabetical order?



Solution $$\left(\underbrace{1}_{\substack{\text{group of}\\ \text{hzm files}}} + \underbrace{1}_{\substack{\text{group of}\\ \text{jpg files}}} + \underbrace{1}_{\substack{\text{group of}\\ \text{txt files}}}\right)! \cdot \underbrace{6!}_{\text{hzm files}} \cdot \underbrace{3!}_{\text{jpg files}} \cdot \underbrace{5!}_{\text{txt files}} = 3! \cdot 6! \cdot 3! \cdot 5!$$

Check Yourself 8

1. A company owns 3 different green cars, 2 different red cars, one blue car and one yellow car. In how many different ways can they be parked in the company's parking lot so that cars of the same color are parked together?

2. Five different countries each send 3 people to an international meeting. A photographer wants to photograph all the people in a row such that people from the same country stand together. How many different photographs are possible?

Answers

1. $4! \cdot 3! \cdot 2!$ 2. $5! \cdot 3! \cdot 3! \cdot 3! \cdot 3! \cdot 3!$

2. Permutations with Identical Elements

Remember that order is important in a permutation: for three objects A, B and C, the permutations (A, C, B) and (B, A, C) are different.

Now suppose that we are asked to find the number of permutations of the letters A, A, B, C, D. The number of permutations of five objects is $P(5, 5) = 5! = 120$. However, some of these of these permutations will be the same because there are two A's in the list. For example, let A_1 and A_2 be the two A's. Then the permutations (A_1, B, A_2, D, C) and (A_2, B, A_1, D, C) are indistinguishable. In order to find the number of distinguishable permutations we can use the following formula:

Theorem

distinguishable permutations of a set with identical elements

Let A be a set of n elements which has n_1 of one kind of element, n_2 of a second kind, n_3 of a third kind, and so on, where $n_1 + n_2 + n_3 + \ldots + n_r = n$.

Then the number of distinguishable permutations in A is $\dfrac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \ldots \cdot n_{r-1}! \cdot n_r!}$.

EXAMPLE 54 What is the number of permutations of the digits in the number 5711?

Solution By the formula above, there are $\dfrac{4!}{2! \cdot 1! \cdot 1!} = 12$ different permutations.

To check our answer let us list the possible permutations. To avoid confusion between the two 1 digits we will name them 1_1 and 1_2.

571_11_2	51_171_2	51_11_27	751_11_2	71_151_2	71_11_25	1_1571_2	1_151_27	1_1751_2	1_171_25	1_11_257	1_11_275
571_21_1	51_271_1	51_21_17	751_21_1	71_251_1	71_21_15	1_2571_1	1_251_17	1_2751_1	1_271_15	1_21_157	1_21_175

We can see that the permutations in the bottom row are the same as those in the top row. In the bottom row, we have simply swapped the positions of the two identical digits. So there are indeed 12 different permutations.

EXAMPLE 55 How many distinguishable permutations are there of the letters in the word NAHCIVAN?

Solution The letters A and N each occur twice.

By the formula, the answer is $\dfrac{8!}{2! \cdot 2!} = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot 3 \cdot 2!}{\cancel{2!} \cdot \cancel{2} \cdot 1} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 3 \cdot 2 = 10080$.

EXAMPLE 56 In a kitchen there are 3 identical porcelain dishes, 2 identical metal dishes and 4 identical glass dishes. In how many ways can they be piled up?

Solution $\dfrac{n!_{total}}{n!_{porcelain} \cdot n!_{metal} \cdot n!_{glass}} = \dfrac{9!}{3! \cdot 2! \cdot 4!} = 1260$

EXAMPLE 57 A teacher has 5 identical math books, 3 identical biology books and 4 identical physics books. In how many different ways can the teacher arrange her books on a shelf if books about the same subject must be together?

Solution The books are identical.

$$(\underbrace{1}_{\substack{\text{group of}\\\text{biology}\\\text{books}}} + \underbrace{1}_{\substack{\text{group of}\\\text{math books}}} + \underbrace{1}_{\substack{\text{group of}\\\text{physics}\\\text{books}}})! \cdot \underbrace{\frac{3!}{3!}}_{\substack{\text{biology}\\\text{books}}} \cdot \underbrace{\frac{4!}{4!}}_{\substack{\text{physics}\\\text{books}}} \cdot \underbrace{\frac{5!}{5!}}_{\substack{\text{math}\\\text{books}}} = 3! \cdot 1 \cdot 1 \cdot 1$$

$= 6$ different ways.

As an extension to this example, try calculating the number of possible arrangements if books on the same subject do not have to be kept together.

EXAMPLE 58 How many three-letter words can be formed from the letters in the word NARIN if each word must contain both N's?

Solution We have three possible sets of letters: $\{A, N, N\}$, $\{R, N, N\}$ and $\{I, N, N\}$.

For each set, the number of words that we can form is $\dfrac{3!}{2!} = 3$.

Since there are 3 sets, there are $3 \cdot 3 = 9$ possible words:

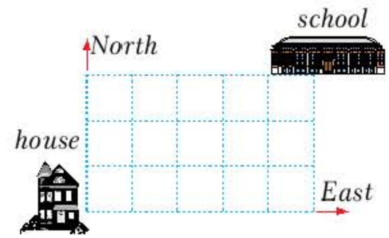
ANN, *NAN*, *NNA*, *RNN*, *NRN*, *NNE*, *INN*, *NIN*, *NNI*.

EXAMPLE 59 I toss a coin successively 7 times. In how many ways can I get 4 heads and 3 tails?

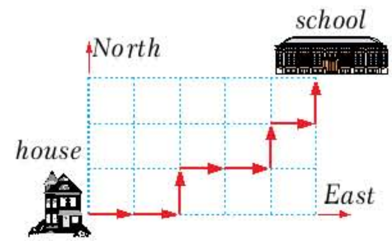
Solution Let H represent heads and T represent tails. Then we can write the result of 7 tosses as a sequence of 7 letters. So the problem is equivalent to finding the number of seven-letter words which contain 4 H letters and 3 T letters, such as *HHTHTTH*, *HTTHHTH* or *TTTHHHH*.

So the answer is $\dfrac{7!}{3! \cdot 4!} = \dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ different ways.

EXAMPLE 60 Every day, Lazar walks from his house to school. Lazar's neighborhood has streets laid out in a grid system, as shown by the grid lines opposite. If Lazar is only allowed to walk eastward and northward along the streets, in how many different ways can he walk to school?



Solution There are 3 northward paths and 5 eastward paths. Let N stand for a northward path and E stand for an eastward path. Then, since Lazar can walk only northward and eastward, any of Lazar's routes can be represented by any word formed of 3 N's and 5 E's. For example, the word *EENEENEN* represents the path opposite.



So the total number of ways will be $\dfrac{8!}{3!5!} = 56$.

Check Yourself 9

1. In a competition, 2 students will receive a gold medal, 3 students will receive a silver medal and 4 students will receive a bronze medal. In how many ways can the medals be awarded to 9 students?

2. A restaurant prepares a shish kebab with 5 identical pieces of meat, 3 slices of tomato and 2 identical pieces of pepper. In how many different ways can these pieces be put on the skewer

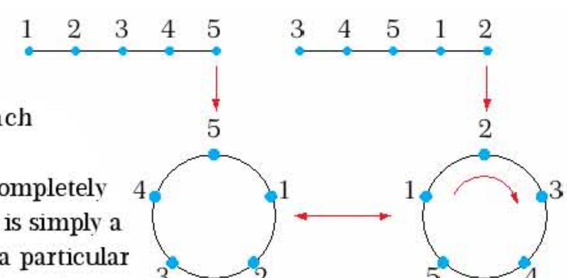
3. Esra is writing a test of multiple choice questions. Each question has 5 possible choices, and the test will include 20 questions. Esra will also prepare an answer sheet. How many different possible answer sheets can she prepare if the number of correct choices are equally distributed?

Answers

1. 1260 2. 2520 3. $\dfrac{20!}{4!4!4!4!4!}$

3. Circular Permutations

Consider the numbers 1, 2, 3, 4 and 5 and look at two different ways of arranging them in a line:



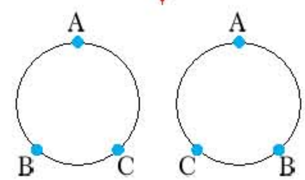
Now imagine that we connect the ends of each line to make the arrangements circular.
We can see that the linear arrangements are completely different, but the second circular arrangement is simply a rotation of the first. Provided we do not mark a particular position as the top of the circle, we can say that the two circular arrangements are identical.

Let us look at another example. In how many ways can we arrange the letters A, B and C in a line and in a circle?

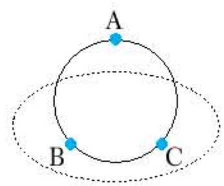
Linear arrangements

ABC	ACB
BAC	BCA
CAB	CBA

Circular arrangements



There are $3! = 6$ linear arrangements. To calculate the number of circular arrangements, we keep one letter in the same position and permute the rest. For example, let us keep A at the top and consider the linear permutations of the remaining letters B and C:



So there are two circular arrangements.

This is a simple example of a circular permutation of three things. What about the general case for n things?

Let us try to see a pattern. In the following circles, each arc represents a possible place for an object around the circle:

1^{st} object	2^{nd} object	3^{rd} object	4^{th} object	n^{th} object

1 possible arc (place)	1 possible place	2 possible places	3 possible places	$n - 1$ possible places
$(1 - 1)$	$(2 - 1)$	$(3 - 1)$	$(4 - 1)$	

Total number of permutations of three objects $= 1 \cdot 1 \cdot 2 = (3 - 1)!$

Total number of permutations of four objects $= 1 \cdot 1 \cdot 2 \cdot 3 = (4 - 1)!$

Total number of permutations of n objects $= 1 \cdot 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n - 1) = (n - 1)!$

Conclusion

The total number of circular permutations of n distinct objects is $(n - 1)!$.

EXAMPLE 61 In how many different ways can 5 girls sit around a circular table for dinner?

Solution Since there are 5 distinct people, the number of possible arrangements is $(5 - 1)! = 4! = 24$.

EXAMPLE 62 In how many different ways can a family of 2 parents and 4 children sit around a circular dining table if the parents must sit together?

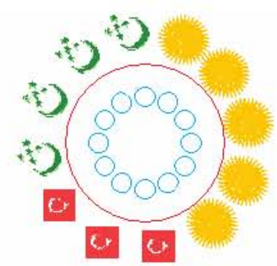
Solution Four children together with the parents make 6 people. In this question we can think of the parents as a single member of the group. So there are 5 members.

There are $(5 - 1)! = 4! = 24$ ways of sitting 5 people around a table. However, within the group of parents there will be $2! = 2$ different possible linear arrangements. So the family can sit in $(5 - 1)! \cdot 2! = 24 \cdot 2 = 48$ ways around the table.

EXAMPLE 63 In how many ways can 3 Turkish, 5 Kyrgyz and 4 Turkmen diplomats be seated around a circular table if diplomats from the same country must sit next to each other?

Solution There are 3 groups of diplomats so the groups can be seated in $(3 - 1)!$ different ways around the table. However, we must also consider the linear permutation of the diplomats in each group.

So there are $(3 - 1)! \cdot 3! \cdot 5! \cdot 4! = 2 \cdot 6 \cdot 120 \cdot 24 = 5760$ different possible ways of seating the diplomats.

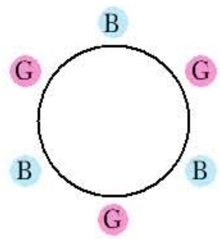


EXAMPLE 64 A room contains a circular table with 5 chairs and a bench for 3 people. In how many ways can 8 students be seated in the room?

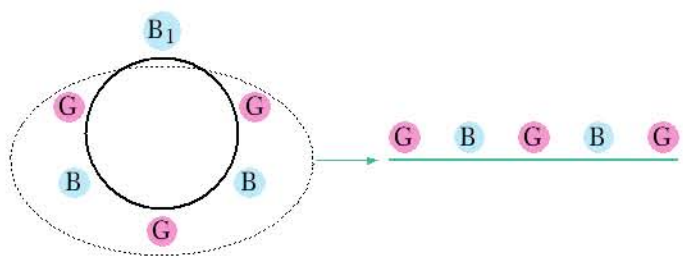
Solution Let us first sit 3 students on the bench. This is a linear permutation, so there are $P(8, 3)$ ways to do this. Then the remaining 5 students can be arranged around the circular table in $(5 - 1)!$ different ways. So the total number of arrangements is $P(8, 3) \cdot (5 - 1)! = 8064$.

EXAMPLE 65 In how many different ways can 3 girls and 3 boys be seated around a circular table with 6 chairs if no two girls must sit together?

Solution 1 If no two girls can sit together, the students must sit alternately:

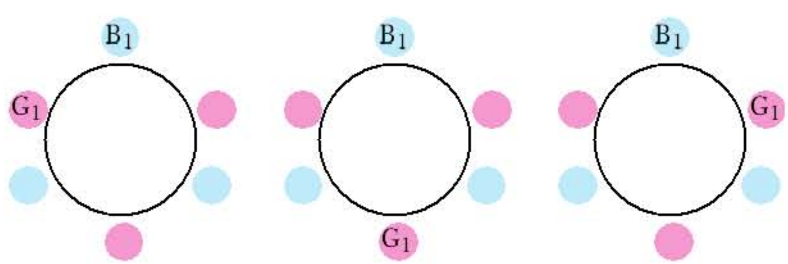


We can seat the boys in $(3 - 1)!$ different ways. However, if we fix the position of one boy (say B_1), we can think of the rest of the seats as a linear arrangement:



In this linear arrangement, the boys can be seated in 2! ways and the girls can be seated in 3! ways. So there are a total of $2! \cdot 3! = 12$ different ways to seat the students.

Solution 2 Alternatively, we can make use of circular permutation. The boys can sit in $(3 - 1)!$ different ways and the girls can also sit in $(3 - 1)!$ different ways. However, a certain girl (say G_1) can sit in three different positions with respect to a certain boy (say B_1):



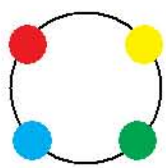
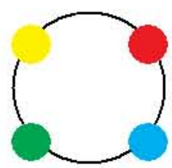
In conclusion, the children can be seated in $(3 - 1)! \cdot (3 - 1)! \cdot 3 = 12$ different ways.

Some permutation problems ask us to arrange objects on a chain or on a circular string, for example: arrange a set of beads on a necklace, or arrange a set of keys on a key ring. In these problems we must divide the total number of circular permutations by 2 since two permutations can simply be the same arrangement viewed from the front and back.

EXAMPLE 66 In how many ways can a red bead, a blue bead, a green bead and a yellow bead be arranged on a necklace?

Solution The answer is $\frac{(4-1)!}{2} = 3$ ways. We divide by 2 because there are two different points of view, from the front and from the back.

For instance,  and  are the same arrangement viewed from the front and the back.

Check Yourself 10

1. The construction department on Planet Zop is building a new flying saucer with 5 windows equally spaced around it. Each window will have a different color. In how many different ways can the windows be arranged?
2. In how many different ways can we put 6 keys on a key ring?
3. Three families will have a supper at a large round table. Each family has 2 parents and the families have 1, 2 and 3 children respectively. In how many different ways can all the people be seated around the table if members of the same family must sit together and the children must sit between their parents?

Answers

1. $(5-1)! = 24$ 2. $\frac{(6-1)!}{2} = 60$ 3. 192

EXERCISES 1.2

A. Factorial Notation

1. Write $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19$ using factorial notation.

2. Evaluate $\dfrac{5!+7!}{5!+6!}$.

3. Evaluate $\dfrac{10!-2\cdot 8!}{(5!+2\cdot 3!)\cdot 7!}$.

4. Simplify the expressions.

   a. $\dfrac{n!}{(n-2)!}$ b. $\dfrac{(n+2)!+(n+1)!}{2\cdot n!+(n+1)!}$

   c. $\dfrac{(2n+2)!}{(2n-1)!}\cdot\dfrac{(n-1)!}{(n+2)!}$

5. Solve the equations.

   a. $\dfrac{(n+1)!}{n!}=17$ b. $\dfrac{(x-2)!}{(x-4)!}=18-x$

   c. $\dfrac{(n+1)!}{(n-1)!}=56$

6. Simplify $(1\cdot 1!)+(2\cdot 2!)+(3\cdot 3!)+\ldots+(n\cdot n!)$.

B. Permutation Functions

7. $g=\begin{pmatrix}\Delta & \square & \bigcirc & \star\\ \square & \star & \Delta & \bigcirc\end{pmatrix}$ is a permutation function defined in $Q=\{\bigcirc, \star, \square, \Delta\}$. Find g^{-1}.

8. The functions $f=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$ and $g=\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$ are defined in $A=\{1, 2, 3\}$. Find $f\circ g^{-1}$

9. Two permutation functions f and g are defined in $D=\{1, 2, 3, 4\}$ such that $f(x)=5-x$ and $g(x)=x$. Write $(f\circ g)(x)$ as a permutation function.

10. The functions $f=\begin{pmatrix}a&b&c&d&e\\d&a&e&c&b\end{pmatrix}$ and g are defined in $S=\{a, b, c, d, e\}$ such that $(f\circ g)=\begin{pmatrix}a&b&c&d&e\\b&e&a&d&c\end{pmatrix}$. Find g.

C. Permutations of n Elements

11. How many different five-digit numbers can be formed by using the digits in the number 75491 once?

12. In how many ways can a group of 7 students be seated in a row of 7 chairs if a particular student insists on being in the first chair?

13. In how many different ways can we name a regular pentagon using letters P, Q, R, S, T?

14. Seven people will be in a group photograph. In how many different ways can the photograph be set up if 3 people must be in front and 4 must be at the back?

15. A group photograph will be taken of 5 boys and 5 girls. Five people must be in the front and 5 people must be at the back. If the girls must sit together, in how many ways can the photograph be taken?

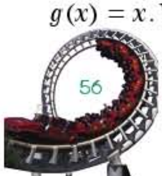
D. Permutations of r Elements Selected from n Elements

16. Evaluate the expressions.

    a. $P(11, 2)$ b. $P(8, 3)\cdot P(5, 4)$

    c. $\dfrac{P(n, 4)}{P(n, 3)}$ d. $\dfrac{P(4, 3)+P(8, 3)}{P(6, 3)}$

    e. $\dfrac{P(5, 5)}{P(7, 7)}$



17. Solve the equations.

a. $P(n+2, 2) = 12$ b. $P(n, 4) = 12 \cdot P(n, 2)$

c. $P(x, 2) = 72$ d. $\dfrac{P(n+1, 2)}{P(n, 3)} = 2$

18. In how many ways can the first, second and third places be decided in an 8-horse race if there is no tie?

19. In a computer shop, 7 out of 10 different laptop computers can be displayed in a row in the shop window. Find the number of possible window displays.

20. Three students will be selected from a group of 10 students such that one student will study physics, one will study math and the other will study chemistry. If any student can study any subject, in how many ways can the students be selected?

21. How many different three-digit numbers can be formed using the digits in the number 8479235 without repetition?

22. How many combinations of at most 5 letters can be formed using the letters in the word CHARITY without repetition?

23. Murat set a password on his e-mail account using the letters of the English alphabet. The password is between 1 and 3 letters long and does not contain repeated letters. How many possible passwords are there?

24. In how many different ways can 2 parents and 3 children be seated in a car if one of the parents must be in the driver's seat and the children sit in the back while the parents sit in front?

25. A four-letter password will be set up of the letters in the set $\{A, B, C, K, V, X\}$. How many of the possible passwords contain repeated letters?

26. Iona, Florica, Anton and their four friends are in a group. Four students are selected from the group to line up in a row. How many of the possible rows

a. do not include Iona and Florica?

b. include Anton?

27. In how many different ways can 5 students be seated in a row of 8 chairs if there must be no empty chairs between them?

28. A computer is generating palindromic sequences of letters using the 26 letters of the English alphabet. Each palindrome must be at least 7 letters long and at most 9 letters long with no repeated letters. How many different palindromes can be formed?

29. A meeting room has 5 seats in the front row and 4 seats in the back row. In how many different ways can 3 friends be seated in the two rows if they must all sit together?

30. A shopkeeper has 7 different pairs of slippers. She wants to display 4 pairs of slippers on a shelf so that the pairs are kept together. Each pair can be placed facing the wall or facing out. In how many ways can she display them?

E. Permutations with Restrictions

31. How many six-digit numbers can be formed from the digits {2, 3, 5, 6, 7, 9} if the digits 5 and 7 must be together and no digit is repeated?

32. Mulan has 4 different types of rose bulb, 6 different types of lily bulb and 7 different types of violet seed. In how many different ways can she plant them in a flower bed row if

a. she can plant them in any order?

b. flowers of the same type must be planted together in a row?

33. Jabari, Valery and their four friends need to be seated in a row. In how many ways can this be done if Jabari and Valery refuse to sit together?

34. In a class photograph of 9 students, 4 students must be in front and 5 must be at the back. In how many different ways can the students be seated if the 3 friends Mariam, Katyusha and Nataly want to sit next to each other?

35. A car dealer has 3 identical red cars, 4 identical blue cars and 2 identical white cars. In how many different ways can he display them in a row?

36. How many different permutations of the letters in the word GALATASARAY are possible if the vowels must be kept together?

37. How many ten-letter words can be formed from the letters in the word TAKLAMAKAN if each letter is used only once?

38. Mulan has 3 identical rose bulbs, 4 identical lily bulbs and 6 identical violet seeds. In how many different ways can she plant them in a row in a flower bed?

39. How many different seven-digit even numbers can be made by rearranging the digits in the number 2352547?

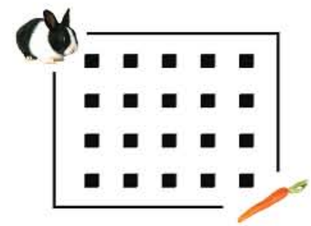
40. How many different six-digit numbers can be made by rearranging the digits in the number 335505?

41. *Humuhumunukunukuapua* is the native Hawaiian name for the triggerfish.

Find the number of different permutations of the letters in this word.

42. Find the number of eight-letter words that can be formed by using each of the letters of the word AMUDERIA once if each word must begin with DER.

43. A rabbit is trained to move only down and right through the maze opposite. In how many different ways can the rabbit reach the carrot?



44.

The figure above shows the road network in a part of Ankara. Nuran must go from point A to point B on foot via point C using the shortest possible route. Find the number of routes that Nuran can take.

45. A child has 4 different math books, 5 identical dictionaries and 3 different story books. In how many different ways can she put them on a bookshelf if

a. they can be put in any order?

b. books of the same kind must be kept together?

46. A restaurant offers a breakfast of 5 different items arrranged in a circle on a plate. How many different breakfast plates can be made?

47. Three Turkish and 5 German diplomats will have a meeting around a circular table. In how many ways can they be seated if diplomats from the same country must sit together?

48. A hot air balloon is made up of 24 pieces of material and each piece is a different color. How many different balloons can be made?

49. In how many ways can 5 boys and 5 girls be seated around a circular table if children of the same gender must not sit next to each other?

50. A group contains 5 boys and 5 girls. In how many different ways can the boys and girls be seated around 2 different circular tables if each table has 5 seats and the boys and girls must sit at separate tables?

51. A company is making a merry-go-round with 3 different wooden horses, 2 different miniature cars and 3 different miniature planes. In how many ways can these things be arranged on the merry-go-round if the planes must be kept together?

52. A Formula 1 racetrack is a simple closed curve. Seven racing cars are racing round the track. If there is no tie and without considering which car is first, second, third, etc., in how many ways can the cars be arranged on the racetrack?

53. 3 identical wooden horses, 4 identical cars and 2 identical swans are arranged on merry-go-round. In how many ways can this be done?

54. A lion, a parrot, a wolf, a dragonfly and a rabbit are attending a meeting. In how many ways can they be seated around a circular table if the wolf and the rabbit must not sit together?

55. In how many different ways can Aygerim put 7 different keys on his key ring?

Mixed Problems

56. ✪ How many zeros are there at the end of $(4!)! + (5!)! + 240!$?

57. In how many different ways can the letters in the word MATHEMATICS be arranged if the vowels must be kept together?

58. A ferris wheel has 22 cars and each car has 2 seats. 22 couples will ride on the wheel so that each couple rides in different car. In how many different ways can they do this?

59. Faruk has 5 different neckties. On condition that he doesn't wear the same tie on two consecutive days, in how many ways can Faruk wear his ties an his five days at the office?

60. ✪ A group of 20 people includes 2 brothers. In how many ways can this group be seated in a circle such that there is exactly one person between the brothers?

3 COMBINATIONS

When the order of the elements chosen from a set is important, we use permutation. However, order is not always important when we are choosing elements. For example, we may want to choose a certain number of people from a group to form a committee. The order of the chosen members is not important since the result is a group of people, not an ordered set. An unordered selection of elements like this is called a combination.

An r-element subset is a subset with r elements.

When we talk about a combination of n objects taken r at a time, we mean the r-element subsets of a set with n elements. We write total the number of such combinations as

$$C(n, r) \text{ or } \binom{n}{r} \ (n, r \in \mathbb{Z} \text{ and } 0 \leq r \leq n).$$

For example, if we are asked to choose two digits from the set $\{2, 3, 5\}$, we might choose $\{3, 5\}$ or $\{5, 3\}$. These are the same combination. This is very different to the problem of forming a two-digit number using the digits 3 and 5 because 35 and 53 are two different outcomes.

A. COMBINATIONS OF r ELEMENTS SELECTED FROM n ELEMENTS

Consider the set $K = \{1, 2, 3\}$. Let us compare the two-element combinations with the two-element permutations of the set K in a table:

$K = \{1, 2, 3\}$		
Combinations with 2 elements	Permutations with 2 elements	
$\{1, 2\}$	12	21
$\{2, 3\}$	23	32
$\{1, 3\}$	13	31

We can see that the number of permutations with two elements is twice the number of the combinations with two elements: $2 \cdot C(3, 2) = P(3, 2)$.

If we now consider the three-element combinations and permutations of the set $A = \{a, b, c, d\}$, we get the following table:

$A = \{a, b, c, d\}$						
Combinations with 3 elements	Permutations with 3 elements					
$\{a, b, c\}$	abc	acb	bac	bca	cab	cba
$\{a, b, d\}$	abd	adb	bad	bda	dab	dba
$\{a, c, d\}$	acd	adc	cad	cda	dac	dca
$\{b, c, d\}$	bcd	bdc	cbd	cdb	dbc	dcb

There are four combinations and 24 permutations. We can see that the number of permutations with three elements is 3! times the number of combinations with three elements: $3! \cdot C(4, 3) = P(4, 3)$.

If we repeated this exercise for two-element permutations and combinations we would find $2! \cdot C(4, 2) = P(4, 2)$.

We can generalize this pattern as

$$\underbrace{C(n, r)}_{\substack{\text{ways of choosing}\\ \text{a group}\\ \text{with } r \text{ elements}}} \cdot \underbrace{r!}_{\substack{\text{ways of arranging}\\ \text{arranging those}\\ \text{elements}}} = P(n, r),$$ which gives us the formula

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!/(n-r)!}{r!} = \frac{n!}{(n-r)! \cdot r!}.$$

Definition

combination

Let n and r be non-negative integers such that $0 \le r \le n$.

A subset of r elements chosen from a set of n elements is called an r-element combination of that set.

The number of r-element combinations of a set of n elements is

$$C(n, r) = \frac{n!}{r! \cdot (n-r)!} \quad (n, r \in \mathbb{Z} \text{ and } 0 \le r \le n).$$

$C(n, r)$ is sometimes written as C_r^n, $C\binom{n}{r}$, ${}_nC_r$ or ${}^nC_r$.

${}_nC_r$ is sometimes read as 'n, choose r'.

EXAMPLE 67 Calculate $C(8, 3)$.

Solution By the formula, $C(8,3) = \dfrac{8!}{3! \cdot (8-3)!} = \dfrac{8!}{3! \cdot 5!} = \dfrac{8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5!}}{\cancel{3!} \cdot \cancel{5!}} = 56.$

EXAMPLE 68 Evaluate $C(12, 5) \cdot C(7, 2)$.

Solution

$$C(12, 5) \cdot C(7, 2) = \frac{12!}{5! \cdot (12-5)!} \cdot \frac{7!}{2!(7-2)!} = \frac{12!}{5! \cdot \cancel{7!}} \cdot \frac{\cancel{7!}}{2! \cdot 5!} = \frac{12!}{5! \cdot 2! \cdot 5!}$$

$$= \frac{\cancel{12} \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 2} \cdot \cancel{5!}} = 11 \cdot 2 \cdot 9 \cdot 2 \cdot 7 \cdot 6 = 16632$$

EXAMPLE 69 Find the number of groups of 3 students which can be chosen from a class of 10 students.

Solution The number of such groups is $C(10,3)=\dfrac{10!}{3!\cdot(10-3)!}=\dfrac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}=120$.

EXAMPLE 70 There are 8 fruit pieces of different kinds including an apple on a tray. How many selections of 4 pieces of fruit can we make if we have to include the apple?

Solution If we have to include the apple, we need to select three pieces of fruit from the seven remaining: $C(7,\ 3)=35$.

EXAMPLE 71 There are 10 players in a list. A basketball coach will choose 6 players from the list for a school team and make one of them the captain. In how many ways can the coach form the team?

Solution The coach can choose 6 players in 210 ways $\left(C(10,6)=\dfrac{10!}{6!\cdot(10-6)!}=210\right)$.

Additionally, any one of these six chosen players can be the captain. By the multiplication property, the coach can form the team in $C(10,\ 6)\cdot 6=1260$ ways.

EXAMPLE 72 In a group of 9 children, 4 children will be given apples, another 3 children will be given oranges and the rest will be given peaches. In how many ways can these fruits be given?

Solution We can choose four children from nine in $\binom{9}{4}$ ways and from the remaining five children we can choose three in $\binom{5}{3}$ ways. There will only be one way to choose the other two children. So the total number of possible groupings is $\binom{9}{4}\cdot\binom{5}{3}\cdot 1=1260$.

Note that we can also solve this problem by treating it as a permutation with some identical elements.

EXAMPLE 73 A cafe offers chocolate, lemon, sour cherry and vanilla flavors of ice cream. A customer can choose one, two or three scoops but the flavours must all be different. How many different possible ice creams can a customer order?

Solution There are four types of ice cream.

The number of possible ice creams is $\binom{4}{1}+\binom{4}{2}+\binom{4}{3}=4+6+4=14.$

Notice that

$\binom{4}{1}$ and $\binom{4}{3}$ are equal: $\binom{4}{1}=\frac{4!}{1!\cdot 3!}=\binom{4}{3}.$

EXAMPLE 74 Classes 10A and 10B have 12 and 18 students respectively. A basketball team of 5 players will be formed by choosing 2 students from 10A and 3 students from 10B. How many different teams can be formed?

Solution The basketball team has five players.

We can choose 2 students from 12 students in $\binom{12}{2}$ ways.

We can choose 3 students from 18 students in $\binom{18}{3}$ ways.

So the team can be formed in $\binom{12}{2}\cdot\binom{18}{3}=66\cdot 816=53856$ ways.

EXAMPLE 75 How many three-digit numbers abc can we write which satisfy the condition $c < b < a$?

Solution Notice that the digits a, b and c must all be different. So any three-element set of digits $\{a, b, c\}$ will be enough to form a valid number, because we can just arrange the digits to satisfy the condition. For example, the digits set is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and from the chosen subset $\{3, 5, 8\}$ we can form the number 853. So we just need to find the total number of three-digit subsets of the set of digits: $C(10, 3) = 120$ different numbers can be formed.

EXAMPLE 76 A watchmaker has 7 different jewels. He wants to choose four of them to decorate the quarters (3, 6, 9, 12) on the face of a clock. How many different decorations are possible?

Solution The watchmaker can choose four jewels in $\binom{7}{4}$ different ways and set them around the quarters on the dial in 4! different ways. So the total number of possible decorations is $\binom{7}{4} \cdot 4! = 840$.

(Notice that we cannot use circular permutation in this problem. Can you see why?)

EXAMPLE 77 A room contains a circular table with 5 chairs and a bench for 3 people. In how many ways can 8 students be seated in the room?

Solution We studied this problem in Example 64. In that solution we began by sitting three of the students on the bench. Now let us begin by sitting five students at the circular table. We can choose these students in $\binom{8}{5}$ different ways. Then they can be arranged around the table in $(5-1)!$ ways. Finally, we have 3! ways for the remaining students to sit at the bench. So the total number of arrangements is $\binom{8}{5} \cdot 4! \cdot 3! = 56 \cdot 24 \cdot 6 = 8064$.

EXAMPLE 78 In a queue of 5 students at the canteen, 2 students are from class A and the rest are from class B. If we know that there are 7 students in class A and 9 students in class B, find the total number of possible ways to form the queue.

Solution In a queue, the order is significant. There are $\binom{7}{2}$ possible two-student groups from class A and $\binom{9}{3}$ possible three-student groups from class B. Those five students can form a queue in 5! different ways.

So the total number of possibilities is $\binom{7}{2} \cdot \binom{9}{3} \cdot 5! = 21 \cdot 84 \cdot 120 = 211680$.

Remark

The number of r-element subsets of a set of n elements is equal to the number of $(n-r)$-element subsets: $C(n, r) = C(n, n-r)$ $(n, r \in \mathbb{Z}$ and $0 \le r \le n)$.

Check: $C(n, n-r) = \dfrac{n!}{(n-r)! \cdot |n-(n-r)|!} = \dfrac{n!}{(n-r)! \cdot |n-n+r|!} = \dfrac{n!}{(n-r)! \cdot r!} = C(n, r)$.

EXAMPLE 79 The number of 3-element subsets of a set is equal to the number of 6-element subsets of the same set. Find the number of 7-element subsets of this set.

Solution We know that $C(n, r) = C(n, n - r)$ and we have $C(n, 3) = C(n, 6)$. So $n = 9$.

So the number of 7-element subsets of this set is $\binom{9}{7} = 36$.

EXAMPLE 80 There are 9 students in a class. Four of them will be chosen to go on a picnic and the rest of the students will form a basketball team. In how many ways can the picnic group be chosen?

Solution Let us calculate the number of ways of establishing the basketball team, since each student is only involved in one activity:

$$\binom{9}{5} = \frac{9!}{5!\,(9-5)!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 126.$$

Note that would have found the same answer if we had begun by considering the picnic group.

EXAMPLE 81 On Planet Zop there are 7 Zozos and 5 Yoyos. The army wants to form a crew of 3 members for a spacecraft. How many different crews can be formed if there must be at least one Yoyo in the crew?

Solution The number of groups with three members is $\binom{12}{3}$. This number is the sum of the number of groups with only Zozos ($\binom{7}{3}$) and the number of groups with at least one Yoyo.

Let us call this second group y. Then we can write

$\binom{12}{3} = \binom{7}{3} + y$, so $y = \binom{12}{3} - \binom{7}{3}$ which means $y = 220 - 35 = 185$.

So there are 185 crews which contain at least one Yoyo.

As an exercise, try to develop an alternative way of solving this problem.

EXAMPLE 82 Esma has 4 pigeons and 5 parrots. How many different pairs containing a parrot and a pigeon can Esma choose?

Solution 1 We can solve this question by using the multiplication principle:

First bird	Second bird
$\binom{4}{1}$	$\binom{5}{1}$

$\rightarrow 4 \cdot 5 = 20.$

Solution 2 Alternatively, there are $\binom{9}{2}$ ways of choosing a pair of birds. Of these, $\binom{4}{2}$ pairs contain only pigeons and $\binom{5}{2}$ pairs contain only parrots. Let d be the remaining number of mixed pairs.

Then $\binom{9}{2}=\binom{4}{2}+\binom{6}{2}+d$, so $d=\binom{9}{2}-\left[\binom{4}{2}+\binom{5}{2}\right]=36-(6+10)=20$.

This second solution may seem a bit longer than the first one. However, the strategy we have just used will be helpful in other questions.

EXAMPLE 83 A football team is made up of 10 players plus a goalkeeper. Five more players are reserves. The team coach wants to substitute 2 team players (not including the goalkeeper) with 2 reserves and then choose 3 forward players from the resulting team. If each player can play any position, in how many ways can the coach choose the 3 forward players?

Solution The coach can choose the players to substitute in $\binom{10}{2}$ different ways. He can choose the reserve players in $\binom{5}{2}$ different ways. Finally, he can arrange the forwards in $\binom{10}{3}$ different ways. So the forwards can be chosen in $\binom{10}{2}\binom{5}{2}\binom{10}{3}=54000$ different ways.

EXAMPLE 84 In how many different ways can 10 people be separated into 2 equivalent groups if

a. one group travels to Izmir and the other group travels to Kayseri?

b. the two groups play basketball together?

Solution

a. The group which goes to Izmir can be chosen in $\binom{10}{5}=252$ ways.

The rest of these ten people will go to Kayseri. So the answer is 252 ways.

b. There will be two teams of five players each. We can choose the first team in $\binom{10}{5}=252$ ways and the rest of the people will be in the second team.

However, half of these 252 possible teams will be the same as the other half.

So the ten people can be separated into two teams in $\dfrac{\binom{10}{5}}{2}=126$ ways.

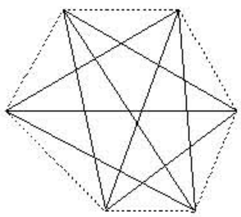
In the same way, 15 people can be separated into three teams of five members in

$\dfrac{\binom{15}{5}\binom{10}{5}\binom{5}{5}}{3!}$ ways.

In how many ways could 12 students be separated into four equal teams?

EXAMPLE 85 Derive a formula that gives the number of diagonals in a convex polygon with n sides.

Solution A polygon with n sides has n vertices A diagonal is a line segment which joins two non-adjacent vertices. The total number of line segments which join any of the vertices is determined by the two-element subsets of all the vertices: $\binom{n}{2} = \frac{n!}{2!\cdot(n-2)!} = \frac{n\cdot(n-1)}{2}$.



However, line segments which join two adjacent vertices are not counted as diagonals. They are the sides.

If we subtract n sides from the first formula, the formula for the number of diagonals is

$$\binom{n}{2} - n = \frac{n\cdot(n-1)}{2} - n = \frac{n\cdot(n-3)}{2}.$$

For example, in a hexagon there are $\frac{6\cdot(6-3)}{2} = 9$ diagonals.

EXAMPLE 86 I toss a coin successively 7 times. In how many ways can I get 4 heads and 3 tails?

Solution We have already solved this question as a permutation in Example 59. Now we can solve it as a combination. Let the numbers 1, 2, 3, 4, 5, 6, 7 represent each toss of the coin. Then any four-element subset chosen from this set will represent a group of outcomes in which the coin is heads. For example, 1, 2, 4, 6 means heads on the first, second, fourth and sixth toss. 2, 3, 4, 5 is another possibility.

There are $\binom{7}{4} = 35$ such groups, so can get the result in 35 ways.

EXAMPLE 87 A canteen has 2 circular tables with 4 seats each and a bench with 5 seats. In how many different ways can 13 students chosen from a class of 15 students sit in the canteen?

Solution We can choose 13 students in $\binom{15}{13}$ different ways.

From these students, $\binom{13}{4}$ students will sit at the first circular table and $\binom{9}{4}$ students will sit at the second circular table. These students can be seated in 3! and 3! ways respectively.

The remaining five students can sit on the bench in 5! different ways.

So the total number of possible arrangements is a product:

$$\binom{15}{13}\cdot\binom{13}{4}\cdot\binom{9}{4}\cdot 3!\cdot 3!\cdot 5! = 105\cdot 715\cdot 126\cdot 6\cdot 6\cdot 120 = 40\,864\,824\,000.$$

EXAMPLE 88 There are 13 bulbs in a box. Five of the bulbs are defective. We will select a set of 4 bulbs from the box. In how many ways can we do this if

a. none of the bulbs must be defective?

b. we want at least half of the set to be defective?

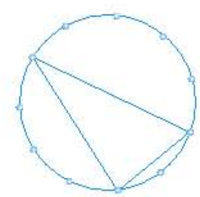
Solution a. There are eight working bulbs, so we must select four bulbs from eight bulbs: $\binom{8}{4}=70$ possible sets.

b. There are four bulbs in the set. If the set contains at least two defective bulbs, there must be 2 or 3 or 4 defective bulbs and 2 or 1 or zero working bulbs in each set:

$$\underbrace{\binom{5}{2}\binom{8}{2}}_{\substack{\text{groups with 2}\\\text{defective bulbs}}}+\underbrace{\binom{5}{3}\binom{8}{1}}_{\substack{\text{groups with 3}\\\text{defective bulbs}}}+\underbrace{\binom{5}{4}\binom{8}{0}}_{\substack{\text{groups with 4}\\\text{defective bulbs}}}=(10\cdot 28)+(10\cdot 8)+(5\cdot 1)=365.$$

EXAMPLE 89 There are 11 points on a circle. How many triangles can we form using any three of these points as the vertices?

Solution Any three points on the circle are not collinear. So any three points will make a triangle. So the total number of triangles is $\binom{11}{3}=165$.



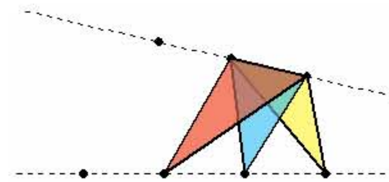
EXAMPLE 90 Seven points are given as shown in the adjacent figure.

a. How many lines can be draw which pass through at least two of the points?

b. How many triangles can be formed using the points as vertices?

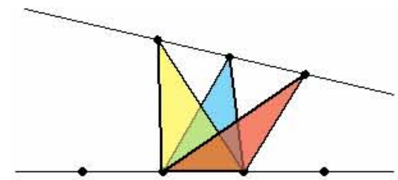
Solution a. Two lines are already given. There are three collinear points on the top line and four collinear points on the bottom line. Other lines can pass through one of the top and one of the bottom points. There are $3 \cdot 4 = 12$ such lines. Including the top and bottom line, there are $12 + 2 = 14$ possible lines.

b. For any triangle we want to draw, there are two cases:



Case **1**: A side is on the upper row and the vertex is a point on the lower row. Since two points determine a side, there can be $\binom{3}{2} \cdot 4 = 12$ such triangles.

Some of them are shown in the figure.



Case **2**: A side is on the lower row and the vertex is on the upper row. There are $\binom{4}{2} \cdot 3 = 18$ such triangles.

Some of them are shown in the figure.

In conclusion, we can form 12 + 18 = 30 triangles.

EXAMPLE 91 There are 4 permanent members and 9 elected members on a company's board of directors. For a decision to be passed, there must be at least 8 votes in favor of the decision and all of the permanent members must vote in favor. In how many ways can a decision be passed by the board, assuming that all members vote?

Solution For a decision to be passed, all the permanent members must vote in favor. This is possible in $\binom{4}{4} = 1$ way.

But this is not enough. At least four more votes are necessary since at least eight votes are needed in favor. These four or more votes can be provided by the nine elected members of the board in $\binom{9}{4}$, $\binom{9}{5}$, $\binom{9}{6}$, ... or $\binom{9}{9}$ different ways.

So there are $\binom{9}{4}+\binom{9}{5}+\binom{9}{6}+\binom{9}{7}+\binom{9}{8}+\binom{9}{9}$ ways in which at least four of the nine elected members can vote in favor of the decision. As a result, a decision can be passed in

$$\binom{4}{4}\cdot\binom{9}{4}+\binom{4}{4}\cdot\binom{9}{5}+\binom{4}{4}\cdot\binom{9}{6}+\binom{4}{4}\cdot\binom{9}{7}+\binom{4}{4}\cdot\binom{9}{8}+\binom{4}{4}\cdot\binom{9}{9}$$

$$=\binom{4}{4}\cdot\left[\binom{9}{4}+\binom{9}{5}+\binom{9}{6}+\binom{9}{7}+\binom{9}{8}+\binom{9}{9}\right]$$

$$=1\cdot(126+126+84+36+9+1)=382 \text{ ways.}$$

EXAMPLE 92 Prove that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, where $n, r \in \mathbb{N}$ with $r \leq n$.

Solution This solution is left as an exercise for you.

Theorem number of subsets of a set

The total number of subsets of a set with n elements is 2^n:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n, \; n \in \mathbb{Z} \;\; (0 \leq n).$$

Proof For a given subset there are two possibilities for each element of the main set: either it is in the subset or not in the subset. If the n elements are $a_1, a_2, a_3, \ldots, a_n$ we can construct the following table:

a_1	a_2	a_3	...	a_n
2	2	2	...	2

$\underbrace{\qquad\qquad\qquad\qquad}_{n \text{ elements}}$

So the total number of subsets is 2^n.

EXAMPLE 93 Find the total number of subsets of the set $K = \{a, b, c, d, e, f, g, h\}$.

Solution Since the set K has 8 elements, the total number of subsets is $2^8 = 256$.

EXAMPLE 94 How many subsets of the set $P = \{1, 2, 3, 4, 5, 6, 7\}$ contain 2 or 6 or both 2 and 6?

Solution The total number of subsets of P is 2^7. The subsets which do not contain either of the digits 2 and 6 are in fact the subsets of the set $\{1, 3, 4, 5, 7\}$, which has 2^5 subsets.

So the number of subsets that contain 2 or 6 or both is $2^7 - 2^5 = 128 - 32 = 96$.

Check Yourself 11

1. $P(n, 3) = (n + 1) \cdot C(n, 3)$ is given. Find n.
2. Find the number of subsets of the set $\{a, b, c, d, e\}$ which contain at least 3 elements.
3. There are 12 people in a room. Each person shakes hands with all the other people. How many handshakes are there?
4. In how many different ways can Hunfrid distribute 9 different toys among 3 children so that each child gets 3 toys?
5. A box holds 7 red cards and 5 green cards. How many different groups of 6 cards can be selected from the box if the selection must contain at least 3 red cards?

Answers

1. 5 2. $\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$ 3. 66 4. $\binom{9}{3}\binom{6}{3}\binom{3}{3}$ 5. 812

B. COMBINATIONS WITH IDENTICAL ELEMENTS *(OPTIONAL)*

In the previous section, the problems asked us to choose elements from a set of distinguishable elements. We could only usually choose an element once. However, in some cases we may be able to choose an element more than once, or the set we choose from may contain identical (non-distinguishable) elements. In this case, the number of selections we can make may change.

Let us look at an example. A set $\{A, B, C\}$ is given.

The permutations in A are AB, AC, BA, BC, CA, CB $\left(\dfrac{3!}{(3-2)!}=6\right)$.

The permutations with repetition are AA, AB, AC, BA, BB, BC, CA, CB, CC $(3 \cdot 3 = 9)$.

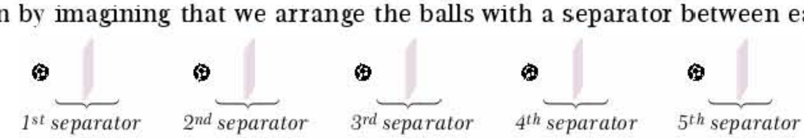
The combinations of the elements of A are AB, AC, BC $\left(\binom{3}{2}=3\right)$.

The combinations with identical elements are AA, AB, AC, BB, BC, CC.

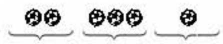
As we can see, if we are allowed to choose an element more than once in a combination, the number of combinations is different.

Now look at another example of combinations with repeated elements. We want to distribute six identical balls between three boxes. Each box must contain at least one ball. In many different ways can this be done?

Let us begin by imagining that we arrange the balls with a separator between each ball.



If we select two of the separators in the figure and remove the others, we have divided the balls into three groups. Suppose that we choose the second and fifth separators. Then the figure becomes

So the distribution is  .

So the question is equivalent to selecting two separators from five, and the answer is $\binom{5}{2} = 10$.

Theorem **de Moivre's theorem**

Let n be a positive integer. Then the number of positive integer solutions (each $x_i > 0$ where $i = 1, 2, 3, \ldots, r$) to $x_1 + x_2 + x_3 + \ldots + x_r = n$ is $\binom{n-1}{r-1}$. This theorem is called de Moivre's theorem.

Proof We can write n as the sum of n 1's as $n = 1 + 1 + 1 + \cdots + 1$. There are $(n - 1)$ plus signs $(+)$. In order to write n as the sum of r addends as indicated in the theorem we need to choose $(r - 1)$ plus signs from $n - 1$ plus signs.

This is the same as selecting $r - 1$ objects from $n - 1$ objects, i.e. $\binom{n-1}{r-1}$.

A positive solution is greater than or equal to 1.

EXAMPLE 95 Find the number of positive integer solutions to the equation $a + b + c + d + e = 15$.

Solution By the formula, there are $\binom{15-1}{5-1} = \binom{14}{4} = 1001$ solutions.

For instance, one of the solutions is $2 + 3 + 1 + 4 + 5 = 15$.

EXAMPLE 96 We want to divide 17 identical marbles among 4 children so that each child takes at least one marble. In how many different ways can we do this?

Solution Suppose the four children take a, b, c and d marbles respectively. Then $a + b + c + d = 17$ where $a, b, c, d \neq 0$ (can you see why?). So there are $\binom{17-1}{4-1} = \binom{16}{3}$ ways to distribute the marbles.

EXAMPLE 97 Find the number of integer solutions to the equation $x + y + z = 85$ such that $x \geq 13$, $y \geq 15$ and $z \geq 6$.

Solution Let us write $x = x' + 12$, $y = y' + 14$ and $z = z' + 5$ where $x', y', z' \geq 1$.

Then we have $x' + 12 + y' + 14 + z' + 5 = 85$, i.e. $x' + y' + z' = 64$.

So we are looking for the number of positive solutions to this equation. Therefore the answer is $\binom{64-1}{3-1} = \binom{63}{2}$.

Corollary corollary to de Moivre's theorem

Let n be a positive integer. The number of non-negative integer solutions (each $x_i \geq 0$ where $i = 1, 2, 3, \ldots, r$) to $x_1 + x_2 + x_3 + \ldots + x_r = n$ is $\binom{n+r-1}{r-1}$.

Proof Let us add 1 to each addend:

$\underbrace{x_1 + 1}_{y_1} + \underbrace{x_2 + 1}_{y_2} + \underbrace{x_3 + 1}_{y_3} + \ldots + \underbrace{x_r + 1}_{y_r} = n + r$ which we can write as

$y_1 + y_2 + y_3 + \ldots + y_r = n + r$.

> A non-negative solution is greater than equal to zero.

By de Moivre's theorem, the number of positive integer solutions to this equation is $\binom{n+r-1}{r-1}$.

This is the same as the number of non-negative integer solutions to the original equation.

Recall that at the beginning of this section we considered the number of pairs that could be formed from the elements of the set $\{A, B, C\}$ if repetition was allowed. We found that there were six possible pairs. Since this task is equivalent to selecting two elements from three elements if repetition is allowed, we can find the number of possibilities by using the corollary above. So the answer is $\binom{3+2-1}{2} = \binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6$.

EXAMPLE 98 Find the number of non-negative integer solutions to the equation $a + b + c + d + e + f = 12$.

Solution $\binom{12+6-1}{6-1} = \binom{17}{5}$. For instance, one of the solutions is $2 + 3 + 0 + 4 + 2 + 1 = 12$.

EXAMPLE 99 Thirteen people are traveling on a bus. There are 7 bus stops left on the route. The driver is making a list of the number of passengers which get off at each bus stop. How many different lists can be made?

Solution Let x_i be the number of passengers that get off at bus stop i.

Then we are looking for the non-negative solutions to the equation
$x_1 + x_2 + x_3 + ... + x_7 = 13$.

So the answer is $\binom{13+7-1}{7-1} = \binom{19}{6} = 27132$ possible lists.

EXAMPLE 100 Eight identical invitation letters have to be mailed into 5 mailboxes, although not all the mailboxes need to be used. In how many ways can we do this?

Solution Each mailbox either receives letters or does not receive letters. So we are looking for the number of groups of five such that the sum of the number of letters in each group is 8, which is the same as the number of non-negative solutions to $m_1 + m_2 + m_3 + m_4 + m_5 = 8$.

So the answer is $\binom{8+5-1}{5-1} = \binom{12}{4} = 495$ ways.

EXAMPLE 101 An ice cream stand sells 15 different flavors of ice cream. How many different four-scoop servings are possible at this stand?

Solution Since we can choose any flavor more than once the question is asking us to calculate the number of combinations with repetition. Any flavor can be selected or not in a four-scoop ice-cream.

So the number of scoops is non-negative and the same as the number of non-negative solutions to $f_1 + f_2 + ... + f_{15} = 4$.

So the answer is $\binom{15+4-1}{4-1} = \binom{18}{3} = \frac{18 \cdot 17 \cdot 16}{3 \cdot 2 \cdot 1} = 816$.

Check Yourself 12

1. Find the number of positive integer solutions to the equation $a + b + c + d + e + f = 80$.
2. Find the number of non-negative integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 32$.
3. There are 15 people in a lift in a building which has 9 floors. In how many ways can the people leave the lift if at least one person leaves the lift on each floor?

Answers

1. $\binom{79}{5}$ 2. $\binom{35}{3}$ 3. $\binom{14}{8}$

THE PIGEONHOLE PRINCIPLE

A pigeonhole is a small box or hole which is big enough for a pigeon to rest in. It has given its name to an important principle in combinatorics: the *pigeonhole principle*. We can describe the idea behind the pigeonhole principle simply as follows: If we have a number of pigeons and a smaller number of pigeonholes, then for all the pigeons to rest we must put at least two pigeons in one hole. We can also state this mathematically:

Let A and B be two finite sets, and let $f: A \to B$ be a function. If $n(A) > n(B)$ then there exist at least two distinct elements p and q in A such that $f(p) = f(q)$.

This apparently simple principle has some surprising consequences and applications. Here are just a few:

1. If there are three people in a room, at least two of them have the same gender.

In this case, the 'pigeons' are the three people and the 'pigeonholes' are the two possible genders: male or female. There are more pigeons than pigeonholes, so at least two pigeons (i.e. people) must be in the same pigeonhole (i.e. have the same gender).

2. An ice cream shop sells six types of ice cream. At least how many students need to buy an ice cream to ensure that at least two students buy the same type of ice cream?

The answer is seven: we need more pigeons (students) than pigeonholes (types of ice cream).

3. We put only dimes, nickels and quarters in an empty box. Edip wants to find at least 5 dimes or at least 6 nickels or at least 8 quarters when he opens the box. How many coins must we put in the box so that we don't disappoint him?

If we put $(5 - 1)$ dimes, $(6 - 1)$ nickels and $(8 - 1)$ quarters in the box, any one extra coin will grant Edip's wish. So we must put at least $(5 - 1) + (6 - 1) + (8 - 1) + 1 = 18$ coins in the box.

Note: this does not really make sense since we can just put e.g. 5 dimes in the box and nothing else!

4. 12 different integers are chosen at random. Is it possible that two of them have the same remainder when they are divided by 11?

There are 11 possible remainders when a number is divided by 11: 0, 1, 2, 3, ... , 9 or 10. But we have twelve numbers. If we think of the remainders as the pigeonholes and the numbers as the pigeons then by the pigeonhole principle we have at least two pigeons for the same hole, i.e. two numbers with the same remainder. So the answer to the question is yes. Try it!

5. A normal person has an average of approximately 150 thousand hairs on his head. Can you explain why at least two people in Istanbul have the same number of hairs on their head?

6. Ten people are chosen at random. Can you explain why at least two of them must have their birthday on the same day of the week this year?

EXERCISES 1.3

A. Combinations of *r* Elements Selected from *n* Elements

1. Evaluate the expressions.

a. $C(4, 2)$

b. $C(6, 2) + C(8, 3)$

c. $\dfrac{P(7,4)}{C(7,4)}$

d. $\binom{9}{0}+\binom{9}{1}+\binom{9}{2}+\binom{9}{3}+\ldots+\binom{9}{9}$

e. $\binom{13}{0}+\binom{13}{2}+\binom{13}{4}+\ldots+\binom{13}{12}$

2. Simplify the expressions.

a. $2C(4, 2)$ b. $\dfrac{C(n,\ 3)}{P(n,\ 2)}$ c. $C(n, 2) + C(n, n-2)$

3. Solve the equations.

a. $C(n, 2) = 15$ b. $\binom{n}{2}+20=\binom{8}{3}$

c. $\binom{n+1}{n-1}=\binom{n-1}{2}+17$

4. List all the three-element subsets of the set $G = \{k, l, m, n, r\}$.

5. How many subsets of at least 4 elements does the set $K = \{$★, ●, ☆, □, ○, ▶, ⬟$\}$ have?

6. How many of the three-element subsets of the set $H = \{a, b, c, d, e, f\}$ include the letter e?

7. In how many different ways can one novel, one biography and one poetry book be chosen from 3 novels, 4 biographies and 5 poetry books?

8. Snow White wants to choose 3 of the 7 dwarfs to clean her house. How many different groups can she choose?

9. A computer programmer wants to set a key combination for an operation in a program. For this purpose, he will use two of the keys Shift, Ctrl or Alt together with one of 26 letters. How many different key combinations can he choose from?

10. In a group of 10 people, everybody shakes hands with everybody else. How many handshakes are there?

11. A, B, C, D and E are 5 distinct points on a circle. Assuming that an arc is represented by two letters and the order of the letters is important, how many different arcs can we define?

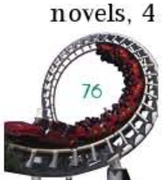
12. Three different digits a, b and c are chosen from the set $\{2, 3, 4, 5, 6,\}$. How many three-digit numbers abc can be formed in this way such that $a < b < c$?

13. In an aquarium there are 5 different white fish, 7 different goldfish and 4 different black fish. Four of the fish must be moved to a different aquarium. In how many ways can this be done?

14. Joo-Chan is making a 6×8 crossword puzzle in which he wants to put 12 black squares. How many different crossword patterns can he make?

15. In an attempt to travel around the world, you must visit 36 out of 40 predetermined cities which are all located near the Equator. In how many ways can you choose the group of cities you will not visit?

16. There are 11 children of different heights in a class. We want to choose any 4 children and line them up according to their height. In how many different ways can this be done?



17. Junko must choose two courses from the list M, N, Q, P and R for this semester. How many choices does he have if the lessons for M and Q are given at the same time?

18. A ship has 12 different barrels of oil. Eight of them will be unloaded at a port. Of these 8 barrels, 3 barrels will be painted. In how many ways can the 3 barrels be chosen?

19. In a class of 55 students, the number of possible pairs which can be formed by the girls is equal to the total number of boys in the class. Find the number of boys in this class.

20. Two basketball teams of 5 players each are playing a match. We want to choose 2 players from each team and seat them around a circular table after the match. In how many different ways can this be done?

21. In classes 9A, 9B and 9C there are 12 students, 10 students and 9 students respectively. A football team will be formed by choosing 5 students from 9A, 4 students from 9B and 2 students from 9C. How many different football teams can be formed?

22. Ali and his 7 friends are in group. Three people are chosen from the group to form a committee. How many of the possible committees include Ali?

23. In a group of 7 people, 3 people have a driver's license. How many different groups of 5 people can be chosen to travel in a car if the group must have at least one driver?

24. A research group of 4 people will be chosen from a board of 3 professors and 5 research managers. If the group must include at least one professor, how many different groups can be chosen?

25. An urn contains 5 green marbles and 7 red marbles. A student takes a sample of 7 marbles from the urn. How many different samples are possible if

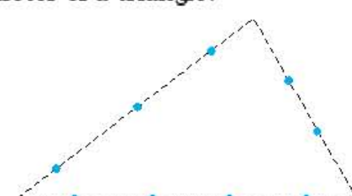
a. the sample must contain exactly three green marbles?

b. the sample must include at least five red marbles?

26. There are 5 candidates for the presidency of a school committee. A voter is allowed to vote for at most 3 candidates. In how many different ways can a student prepare her vote?

27. There are 7 points on a circle. How many pentagons can we form by joining 5 of the points?

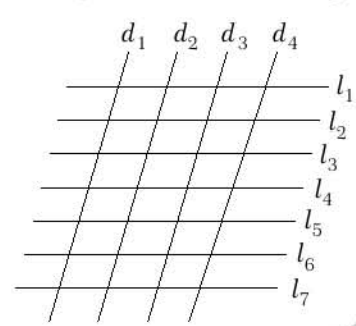
28. The figure below shows nine points on the perimeter of a triangle.



a. How many different lines can be drawn which pass through at least two of the given points?

b. Using the given points as vertices, how many different triangles can be constructed?

29. How many parallelograms are formed by the intersection of the parallel lines in the figure below?



30. a, b, c and d are 4 digits such that $a < b < c < d$. If we write all the possible four-digit numbers $abcd$ in ascending order, in which position will the number 3458 appear?

31. A student must select 4 out of 6 essay questions and 10 out of 15 multiple-choice questions in his mid-term examination. In how many different ways can the student select the 14 questions?

32. A company has 4 workers who can build a wall, 5 workers who can paint a wall and 2 workers who can do both. A job requires 3 builders and 3 painters. In how many ways can the company choose the workers for the job?

B. Combinations with Identical Elements

33. Find the number of different positive integer solutions to $a + b + c + d + e = 27$.

34. Find the number of different non-negative integer solutions to $a + b + c + d = 18$.

35. Find the number of different non-negative integer solutions to $a + b + c + d \leq 2006$.

36. Fifty identical marbles are to be distributed among 6 children. If each child must get at least one marble. How many possible distributions are possible?

37. Twenty-five identical red marbles and 33 identical yellow marbles are to be distributed among 5 children so that each child should take any kind, how many different distributions are there?

38. A chocolate shop sells 12 different types of chocolate. A customer wants to buy a selection of 20 chocolates. How many different selections are possible?

39. A teacher wants to distribute 20 identical gifts among his 16 students. In how many different ways can he do this if

a. he must give each student at least one gift?

b. there is no restriction on the number of gifts a student can receive?

Mixed Problems

40. If $P(n + 1, 4) = 40 \cdot C(n - 1, 2)$, find n.

41. In how many ways can 4 people be seated on a sofa for 6 people?

42. In how many ways can 4 out of 6 people be seated on a sofa for 4 people?

43. In how many ways can 3 out of 5 people be seated on 5 chairs in a row?

44. In how many ways can 5 out of 7 people be seated around a circular table?

45. In how many ways can we arrange 4 out of 5 different math books and 4 out of 6 physics books on a shelf if

a. the books can be put in any order?

b. books on the same subject must be together?

46. A software package consists of 7 programs including an antivirus program and a word processor. Ali's computer has 3 available hard drives. Ali must install 4 different programs from the package, two of which must be the antivirus program and the word processor. In how many different ways can Ali do this?

47. A restaurant offers a self-service buffet lunch. For lunch there are 3 choices of soup, 6 choices of main meal and 5 choices of dessert. Customers can also make a salad from 3 out of 18 things served at the salad bar. In how many different ways can a customer make a four-course meal of soup, a main meal, salad and dessert?

48. Five countries each send 4 diplomats to an international meeting. After the meeting, 2 diplomats from each country are chosen for another meeting around a circular table. How many different seating arrangements are possible around the table if diplomats from the same country must sit together?

49. Jandos has 6 rubber stamps and a circular holder that can hold 12 rubber stamps. In how many ways can Jandos put his stamps in the holder?

50. A scuba diving school has 7 teachers and 2 student teachers. Five of these people will dive in a coral reef and the rest will teach in a classroom. In how many ways can the groups be arranged if the two student teachers must be kept together?

51. Ali, Veli and their 6 friends are in a sports club. A teacher wants to choose a group of 3 people from the club so that Ali is chosen but Veli is not chosen. In how many ways can she do this?

52. Four women and 7 men are at a restaurant. One of the men is married to one of the women. In how many different ways can 5 people from this group be seated around a circular table if the group must contain the couple and the couple must sit together?

53. Selman has a box containing one coin each of the values 1¢, 10¢, 25¢, 50¢ and $1. He takes at least three coins from the box. How many different total values can he take?

54. Six friends are staying at a hotel on a rainy day. They want to go out but they only have 3 different umbrellas. So each pair takes an umbrella. In how many different ways can the friends be paired, if the person holding the umbrella in each pair is significant?

55. There are 6 men and 5 women in a room, including a couple. A group of 8 people will be chosen and seated around a circular table so that the group contains the couple and the couple will sit together. In how many ways can this be done?

56. In how many different ways can the letters in the word ASHGABAT be arranged such that all three A's will not be together?

57. A computer generates all the possible three-letter permutations of the set $A = \{a, b, c, d, e\}$. How many of the permutations contain b but do not contain e?

58. There are 7 captains, 5 sergeants and 6 lieutenants in a group. 3 captains, 3 sergeants and 3 lieutenants are chosen for a specific mission on an island. They are given 9 life jackets: 3 are green, 3 are black and 3 are blue. People of the same rank must all wear the same color jacket. Before they set out for the mission, they are asked to make a line for a photo so that the people of the same color stand together. While the photo is transferred to the headquarters, the commander calculates the number of possible different photos he might receive. What is the answer?

59. ✪ A building has 5 floors with 4 apartments on each floor. Each apartment is for one family. In how many ways can 20 families occupy the building

a. if any family can occupy any apartment?

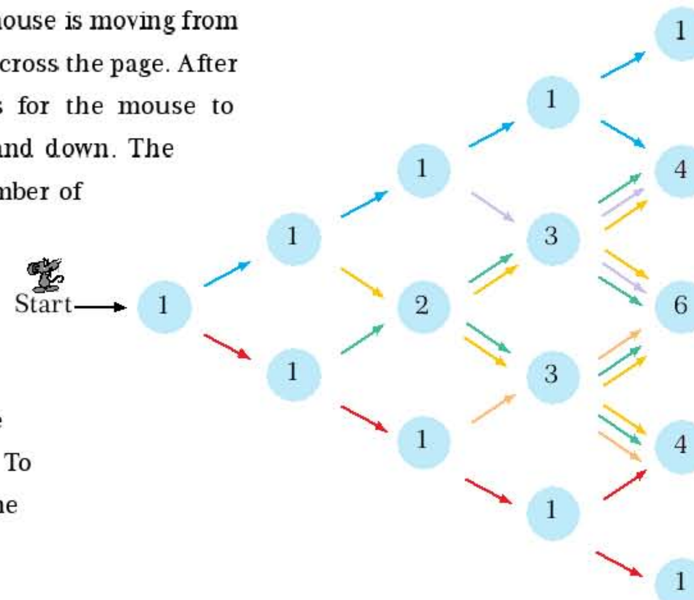
b. if three of the families are related and must live on the same floor?

4 BINOMIAL EXPANSION

A. PASCAL'S TRIANGLE AND BINOMIAL EXPANSION

Look at the picture opposite. A mouse is moving from circle to circle from left to right across the page. After each circle there are two ways for the mouse to proceed: right and up or right and down. The number in each circle is the number of ways in which the mouse can reach that circle.

The numbers in the circles show a pattern which is known as Pascal's triangle. This triangle has many interesting properties. To understand them, let us move the triangle to an upright position.



Row																				Sum
0										1										1
1									1		1									2
2								1		2		1								4
3							1		3		3		1							8
4						1		4		6		4		1						16
5					1		5		10		10		5		1					32
6				1		6		15		20		15		6		1				64
7			1		7		21		35		35		21		7		1			128
8		1		8		28		56		70		56		28		8		1		256
	↓																		↓	

First notice that each row begins and ends with 1.

Secondly, notice that the sum of any two consecutive terms in a row gives us the term between them on the next row. For instance, the number 15 marked in red in the sixth row is the sum of the 10 and 5 located above it. We can extend the triangle infinitely downwards by using this rule.

Notice also that the first row is row zero. For convenience, when we count the positions of the numbers in each row we also begin with zero (not 1). For example, number 21 marked in green is the second entry (not the third entry) in the seventh row. The entries in Pascal's triangle are related to the coefficients of the expansion of a binomial with a non-negative integer power. To understand the relationship, look at some binomial expansions with the first few powers and notice their coefficients.

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b = 1a+1b$$

$$(a+b)^2 = a^2+2ab+b^2 = 1a^2+2ab+1b^2$$

$$(a-b)^3 = a^3-3a^2b+3ab^2-b^3 = 1a^3-3a^2b+3ab^2-1b^3$$

$$(x+2y)^4 = x^4+8x^3y+24x^2y^2+32xy^3+16y^4 = 1x^4+4x^3(2y)+6x^2(2y)^2+4x(2y)^3+1(2y)^4$$

We can see that the coefficients in each expansion are the same as the entries in the corresponding row of Pascal's triangle.

Remark

The expansion of a binomial expression to the n^{th} power has the following properties:

- There are $n+1$ terms in the expansion.
- The coefficients of the terms in the expansion correspond to the entries in the n^{th} row of Pascal's triangle.
- The power of the first term in the binomial expression begins at n in the expansion and decreases by 1 in each term down to zero.
- The power of the second term in the binomial expression begins at zero in the expansion and increases by 1 in each term up to n.
- In the expansion of $(x+y)^n$, the sum of the exponents of x and y in each term is n.
- The sum of the coefficients of an expansion can be found by substituting 1 for each variable in the binomial expression.
- If the binomial expression is a polynomial then substituting zero for each variable in the binomial expression gives us the constant term of the expansion.

A constant term in an expression is a term that does not change with the variable.

EXAMPLE 102 How many terms are there in the expansion of $(x + y)^{12}$?

Solution Since the power is 12 ($n = 12$) there will be $n + 1 = 13$ terms.

EXAMPLE 103 Expand $(2x + y)^6$.

Solution The primary coefficients of the terms in the expansion will be the entries in the sixth row of Pascal's triangle: 1, 6, 15, 20, 15, 6, 1.

The first term of the binomial is $2x$. To avoid any mistakes, let us keep $2x$ in parantheses to begin with:

$$\begin{aligned}(2x + y)^6 &= (2x)^6 + 6(2x)^5y + 15(2x)^4y^2 + 20(2x)^3y^3 + 15(2x)^2y^4 + 6\cdot 2xy^5 + y^6\\ &= 64x^6 + 6(32x^5)y + 15(16x^4)y^2 + 20(8x^3)y^3 + 15\cdot 4x^2y^4 + 6\cdot 2xy^5 + y^6\\ &= 64x^6 + 192x^5y + 240x^4y^2 + 160x^3y^3 + 60x^2y^4 + 12xy^5 + y^6.\end{aligned}$$

EXAMPLE 104 Expand $(x^2 - 3y)^4$.

Solution We will use the entries in the fourth row as the coefficients: 1, 4, 6, 4, 1. If we write consider $(x^2 - 3y)^4$ as $(x^2 + (-3y))^4$ then

$$\begin{aligned}(x^2 - 3y)^4 &= (x^2)^4 - 4(x^2)^3(3y) + 6(x^2)^2(3y)^2 - 4(x^2)(3y)^3 + (3y)^4\\ &= x^8 - 12x^6y + 6x^4 9y^2 - 4x^2 27y^3 + 81y^4\\ &= x^8 - 12x^6y + 54x^4y^2 - 108x^2y^3 + 81y^4.\end{aligned}$$

EXAMPLE 105 Expand $\left(n + \frac{1}{n}\right)^7$.

Solution Use the seventh row:

$$\begin{aligned}\left(n + \frac{1}{n}\right)^7 &= n^7 + 7n^6\frac{1}{n} + 21n^5\left(\frac{1}{n}\right)^2 + 35n^4\left(\frac{1}{n}\right)^3 + 35n^3\left(\frac{1}{n}\right)^4 + 21n^2\left(\frac{1}{n}\right)^5 + 7n\left(\frac{1}{n}\right)^6 + \left(\frac{1}{n}\right)^7\\ &= n^7 + 7n^6\frac{1}{n} + 21n^5\frac{1}{n^2} + 35n^4\frac{1}{n^3} + 35n^3\frac{1}{n^4} + 21n^2\frac{1}{n^5} + 7n\frac{1}{n^6} + \frac{1}{n^7}\\ &= n^7 + 7n^5 + 21n^3 + 35n + \frac{35}{n} + \frac{21}{n^3} + \frac{7}{n^5} + \frac{1}{n^7}.\end{aligned}$$

EXAMPLE 106 Find the sum of the coefficients in the expansion of $(3x + y^2)^6$.

Solution If we substitute 1 for each variable in $(3x + y^2)^6$ we get $(3 \cdot 1 + 1^2)^6 = 4^6 = 4096$.

EXAMPLE 107 Find the constant term in the expansion of $(7x + 3)^5$.

Solution Substitute zero for each variable in $(7x + 3)^5$: $(7 \cdot 0 + 3)^5 = 3^5 = 243$.

Check Yourself 13

Expand the binomials.

1. $(3x + y)^4$ 2. $\left(x - \dfrac{1}{x}\right)^3$ 3. $(2 - \sqrt{2})^5$

Answers

1. $81x^4 + 108x^3y + 54x^2y^2 + 12xy^3 + y^4$ 2. $x^3 - 3x + \dfrac{3}{x} - \dfrac{1}{x^3}$ 3. $232 - 164\sqrt{2}$

B. FINDING BINOMIAL TERMS USING COMBINATION

A laboratory mouse has been exposed to five types of virus. A scientist wishes to find out how many viruses are now present in the mouse. In how many ways could the mouse have been infected?

Infected with no viruses: $C(5,0) = 1$ (mouse is clean)

Infected with 1 type of virus: $C(5,1) = 5$

Infected with 2 types of virus: $C(5,2) = 10$

Infected with 3 types of virus: $C(5,3) = 10$

Infected with 4 types of virus: $C(5,4) = 5$

Infected with 5 types of virus: $C(5,5) = 1$.

Can you see the similarity between the number of combinations and the entries in the fifth row of Pascal's triangle?

Perhaps one of the most interesting characteristics of Pascal's triangle is its relationship with combination. We can describe this relationship simply: entry number r in row n is the number of subsets of r elements which can be taken from a set with n elements.

This gives us another interesting characteristic of Pascal's triangle: the sum of the terms in the n^{th} row of the triangle is 2^n (can you see why?).

The symmetrical property of Pascal's triangle can also be related to the combination rule

$$\binom{n}{r} = \binom{n}{n-r} \quad (n,\ r \in \mathbb{Z} \text{ and } 0 \le r \le n).$$

For example, the third entry in the eighth row of the triangle is the same as the fifth entry in the same row since $C(8, 3) = C(8, 5) = 56$.

Let us now redraw Pascal's triangle using combination:

Row															
0								$\binom{0}{0}$							
1							$\binom{1}{0}$		$\binom{1}{1}$						
2						$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$					
3					$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$				
4				$\binom{4}{0}$		$\binom{4}{1}$		$\binom{4}{2}$		$\binom{4}{3}$		$\binom{4}{4}$			
5			$\binom{5}{0}$		$\binom{5}{1}$		$\binom{5}{2}$		$\binom{5}{3}$		$\binom{5}{4}$		$\binom{5}{5}$		
6		$\binom{6}{0}$		$\binom{6}{1}$		$\binom{6}{2}$		$\binom{6}{3}$		$\binom{6}{4}$		$\binom{6}{5}$		$\binom{6}{6}$	
	$\swarrow$														$\searrow$

Using the above triangle, we can generalize the expansion of a binomial to any power n as

$$\boxed{(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n}\ .$$

Notice that the coefficient 1 in the first and last terms of the expansion is obtained from $\binom{n}{0}$ and $\binom{n}{n}$ respectively.

EXAMPLE 108 Expand $(x + y)^6$ using combination.

Solution

$$(x+y)^6 = x^6 + \binom{6}{1}x^{6-1}y + \binom{6}{2}x^{6-2}y^2 + \binom{6}{3}x^{6-3}y^3 + \binom{6}{4}x^{6-4}y^4 + \binom{6}{5}x^{6-5}y^5 + y^6$$

$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

EXAMPLE 109 Expand $(2a-b)^5$.

Solution

$$(2a-b)^5=(2a)^5+\binom{5}{1}(2a)^{5-1}(-b)+\binom{5}{2}(2a)^{5-2}(-b)^2+\binom{5}{3}(2a)^{5-3}(-b)^3+\binom{5}{4}(2a)^{5-4}(-b)^4+(-b)^5$$

$$=32a^5-5\cdot16a^4b+10\cdot8a^3b^2-10\cdot4a^2b^3+5\cdot2ab^4-b^5$$

$$=32a^5-80a^4b+80a^3b^2-40a^2b^3+10ab^4-b^5$$

EXAMPLE 110 Expand $\left(\sqrt{x}+\frac{1}{x}\right)^7$.

Solution

$$\left(\sqrt{x}+\frac{1}{x}\right)^7=(\sqrt{x})^7+\binom{7}{1}(\sqrt{x})^6\left(\frac{1}{x}\right)+\binom{7}{2}(\sqrt{x})^5\left(\frac{1}{x}\right)^2+\binom{7}{3}(\sqrt{x})^4\left(\frac{1}{x}\right)^3+\binom{7}{4}(\sqrt{x})^3\left(\frac{1}{x}\right)^4$$

$$+\binom{7}{5}(\sqrt{x})^2\left(\frac{1}{x}\right)^5+\binom{7}{6}(\sqrt{x})\left(\frac{1}{x}\right)^6+\left(\frac{1}{x}\right)^7$$

$$=\sqrt{x^7}+7\sqrt{x^6}\,\frac{1}{x}+21\sqrt{x^5}\,\frac{1}{x^2}+35\sqrt{x^4}\,\frac{1}{x^3}+35\sqrt{x^3}\,\frac{1}{x^4}+21\sqrt{x^2}\,\frac{1}{x^5}+7\sqrt{x}\,\frac{1}{x^6}+\frac{1}{x^7}$$

$$=x^3\sqrt{x}+7x^2+21\sqrt{x}+\frac{35}{x}+\frac{35\sqrt{x}}{x^3}+\frac{21}{x^4}+\frac{7\sqrt{x}}{x^6}+\frac{1}{x^7}$$

The relation between combination and Pascal's triangle helps us to calculate any particular term in a binomial expansion without writing out the entire expansion.

For example, suppose that we are asked to find the third term in the expansion of $(x-2y)^3$. Using our knowledge of the properties of binomial expansion, we can say that $2y$ will have exponent 2 in this term and x will have exponent 1 since the sum of the exponents must be 3. Now we only need to find the coefficient, which we can calculate as $\binom{3}{2}=3$.

So the third term is $\binom{3}{2}x(2y)^2=3x\cdot4y^2=12xy^2$. We can easily check this against the full expansion: $(x-2y)^3=x^3-6x^2y+\underline{12xy^2}-8y^3$.

We can formulize our findings as follows:

Remark

The r^{th} entry in the expansion of $(x + y)^n$ is $\binom{n}{r-1}x^{n-(r-1)}y^{r-1} = \binom{n}{r-1}x^{n-r+1}y^{r-1}$.

EXAMPLE 111 Find the ninth term in the expansion of $(a + b)^{12}$.

Solution Substitute $n = 12$ and $r = 9$ in the formula: $\binom{12}{9-1}x^{12-9+1}y^{9-1} = 495a^4b^8$.

EXAMPLE 112 Find the sixth term in the expansion of $(2x + y)^9$.

Solution We will use $n = 9$ and $r = 6$. So $r - 1 = 5$ and the sixth term is

$$\binom{9}{5}(2x)^{9-5}y^5 = 126 \cdot 16x^4y^5 = 2016x^4y^5.$$

EXAMPLE 113 What is the coefficient of the fourth term in the expansion of $(2x - 4y)^7$?

Solution $r = 4$ means $r - 1 = 3$. So the fourth term is $\binom{7}{3}(2x)^{7-3}(-4y)^3 = 35 \cdot 16x^4(-64)y^3$. From this we can calculate the coefficient to be -35840.

EXAMPLE 114 Find the middle term in the expansion of $(2x^3 + 3y)^8$.

Solution Since there are nine terms in the expansion of a binomial to the eighth power, the middle term will be the fifth term. So the middle term is $\binom{8}{4}(2x^3)^{8-4}(-3y)^4 = 90720x^{12}y^4$.

EXAMPLE 115 2^7xy^3 is a term in the expansion of $(ax + 2y)^4$. Find a.

Solution The exponent of y is 3 and this is one less than the order of the term. Using $r - 1 = 3$ in the formula gives the term as $\binom{4}{3}(ax)^{4-3}(2y)^3$.

If we equate this with the given term we have

$$\binom{4}{3}(ax)^{4-3}(2y)^3 = 2^7xy^3$$

$$4(ax)^{4-3}(2y)^3 = 2^7xy^3$$

$$4(ax)2^3y^3 = 2^7xy^3$$

$$2^5(ax)y^3 = 2^7xy^3.$$ So a is $2^2 = 4$.

EXAMPLE 116 What is the constant term in the expansion of $\left(\frac{2}{x^2} + \frac{x^3}{2}\right)^{10}$?

Solution Let the constant term be $\binom{10}{r-1}\left(\frac{2}{x^2}\right)^{10-r+1}\left(\frac{x^3}{2}\right)^{r-1}$.

This becomes $\dfrac{\binom{10}{r-1}2^{11-r}x^{5r-25}}{2^{r-1}}$ after simplification.

If this is the constant term then the power of x must be zero. This gives us $5r - 25 = 0$ and $r = 5$. In other words, the constant term is the fifth term.

Substituting $r = 5$ in the expression gives us $\dfrac{\binom{10}{5-1}2^{11-5}x^{5\cdot5-25}}{2^{5-1}} = \dfrac{2^6 \cdot 210}{2^4} = 840$, which is the constant term.

EXAMPLE 117 Evaluate $\binom{5}{0}+2\binom{5}{1}+2^2\binom{5}{2}+2^3\binom{5}{3}+2^4\binom{5}{4}+2^5\binom{5}{5}$.

Solution Notice that the expression is the same as $(x+y)^n=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+...+\binom{n}{n}y^n$ for $n = 5$.

Since there is nothing but the coefficient $\binom{5}{0}$ in the first term, the first term in the binomial is 1. The term with the coefficient $\binom{5}{5}$ is 2^5.

We can rewrite the expression as follows:

$$\binom{5}{0}\cdot 1^5+\binom{5}{1}\cdot 1^4\cdot 2^1+\binom{5}{2}\cdot 1^3\cdot 2^2+\binom{5}{3}\cdot 1^2\cdot 2^3+\binom{5}{4}\cdot 1\cdot 2^4+\binom{5}{5}\cdot 2^5.$$

By the formula we have $(1 + 2)^5 = 3^5 = 243$.

EXAMPLE 118 What is the rational term in the expansion of $(\sqrt[3]{x}+2\sqrt{x})^7$?

Solution Let us write the rational term as $\binom{7}{r-1}(\sqrt[3]{x})^{7-r+1}(2\sqrt{x})^{r-1}$ and try to solve for r.

A rational term is a term that contains no irrational expressions.

Simplifying gives us $\binom{7}{r-1}2^{r-1}x^{\frac{8-r}{3}}x^{\frac{r-1}{2}}=\binom{7}{r-1}2^{r-1}x^{\frac{13+r}{6}}$. We need to find a value of r to make this rational.

In $\binom{7}{r-1}2^{r-1}x^{\frac{13+r}{6}}$, the factors $\binom{7}{r-1}$ and 2^{r-1} have rational values for any integer r.

However, $x^{\frac{13+r}{6}}$ is only rational for the value $r = 5$, so $\binom{7}{5-1}2^{5-1}x^{\frac{13+5}{6}}=560x^3$ is the rational term.

Check Yourself 14

1. What is the coefficient of the fourth term in the expansion of $(x+b)^{15}$?

2. What is the seventh term in the expansion of $(3x+y)^{11}$?

3. What is the constant term in the expansion of $\left(x^4+\dfrac{2}{x^2}\right)^9$?

Answers

1. $\binom{15}{3}=455$ 2. $112266x^5y^5$ 3. 5376

EXERCISES 1.4

A. Pascal's Triangle and Binomial Expansion

1. Expand each expression.

a. $(3x + 5)^5$ b. $(2x^2 - 5)^4$ c. $(4x^3 - 3y^2)^5$

2. What is the constant term in the expansion of $(2x - 3)^5$?

3. Find the middle term in the expansion of $(2x - 5y)^4$.

4. Find the sum of the coefficients in each expansion.

a. $(3x + 2)^4$ b. $(3x^2 + 2y)^{15}$

5. Find the twelfth term in the expansion of $(3x + 1)^{15}$ when the terms are written in order of decreasing powers of x.

6. Find the sixth term in the expansion of $(3x^4 + y^3)^8$ when the terms are written in order of decreasing powers of x.

B. Finding Binomial Terms Using Combination

7. $\left(\dfrac{3}{x^3} + 2x\right)^5 = \ldots + (a \cdot x) + \ldots$ is given. Find a.

8. Evaluate

$$\binom{9}{0} + 4\binom{9}{1} + 4^2\binom{9}{2} + 4^3\binom{9}{3} + \ldots + 4^9\binom{9}{9}.$$

9. What is the constant term in the expansion of $\left(2x^3 + \dfrac{1}{x^2}\right)^{10}$?

10. The sum of the coefficients in the expansion of $(x + y)^n$ is 1024. What is the greatest coefficient in this expansion?

CHAPTER SUMMARY

- Counting principles help us find the number of possible results of a task or choice. The most important counting principles are the addition principle and the multiplication principle.
- Addition Principle

  Let A and B be two actions that cannot both be performed at the same time. If action A can be performed in m ways and action B can be performed in n ways, then the action A or B can be performed in $m + n$ ways.
- A possible result of a task or a set of choices is called an outcome.
- Systematic listing is a way of finding all the possible outcomes of a task.
- A product table is a way of listing the outcomes of a task which has two parts or choices.
- Multiplication Principle

  Let a multiple-part task which satisfies the uniformity criterion consist of k parts. If the first part of the task can be performed in n_1 ways, the second part can be performed in n_2 ways and so on, then the number of ways to perform the entire task is $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$.
- It can be useful to use a set of boxes to count the number of ways n_1, n_2, n_3, etc.
- If a task does not satisfy the uniformity criterion directly, we can break it up into separate cases which satisfy the uniformity criterion and then add the number of outcomes of each case.
- A permutation is an ordered arrangement of some or all of the elements in a given set. The order of a set of books on a shelf is an example of a permutation.
- For any counting number n, the product of all positive integers less than or equal to n is called n factorial and denoted by $n!$: $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$.
- $0! = 1$
- For all positive integers, $n! = n \cdot (n-1)!$
- Let A be a non-empty set. A one-to-one and onto function from A to A is called a permutation function in A.
- There are $n!$ different permutation functions in a set with n elements.
- Let I be a permutation function defined in the set A. If $I(x) = x$ for every $x \in A$ then I is called the identity permutation function in A.
- The number of permutations of all of the n distinct elements in a set is as $P(n, n) = n!$
- 1. The composition the permutation functions is not commutative: $f \circ g \neq g \circ f$.
  2. The composition of permutation functions is associative: $(f \circ g) \circ h = f \circ (g \circ h)$ because $(f \circ g) \circ h = f(g(h(x))) = f \circ (g \circ h)$.
  3. For any permutation f and identity permutation I in a set A, $f \circ I = I \circ f = f$ since $\forall x \in A$, $f \circ I(x) = f(I(x)) = f(x)$ and $I \circ f(x) = I(f(x)) = f(x)$.
- The number of permutations of r elements selected from a set of n elements is
  $P(n, r) = \dfrac{n!}{(n-r)!}$ ($n, r \in \mathbb{Z}$ and $0 \le r \le n$).
- When a group of objects must be kept together in a permutation, this group can be considered as a single element. However, we must remember to count the possible permutations of the elements in the combined group.
- A circular permutation is an arrangement of a set of elements in a circle. Two circular permutations are said to be identical if one permutation is simply a rotation of the other.
- The total number of circular permutations of n distinct objects is $(n-1)!$.

- A combination is an unordered selection of some or all of the elements in a given set. The set of people chosen to play on a basketball team is an example of a combination.
- Let n and r be non-negative integers such that $0 \le r \le n$. A subset of r elements chosen from a set of n elements is called an r-element combination of the set.
- The number of r-element combinations of a set of n elements is

$$C(n, r) = \frac{n!}{r!\cdot(n-r)!} \quad (n, r \in \mathbb{Z} \text{ and } 0 \le r \le n).$$

- $C(n, r) = C(n, n-r) \quad (n, r \in \mathbb{Z} \text{ and } 0 \le r \le n)$
- The total number of subsets of a set with n elements is 2^n:

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n,\ n \in \mathbb{Z}\ (0 \le n).$$

- De Moivre's theorem: Let n be a positive integer. Then the number of positive integer solutions (each $x_i > 0$ where $i = 1, 2, 3, \dots, r$) to $x_1 + x_2 + x_3 + \dots + x_r = n$ is $\binom{n-1}{r-1}$.
- Let n be a positive integer. The number of non-negative integer solutions (each $x_i \ge 0$ where $i = 1, 2, 3, \dots, r$) to $x_1 + x_2 + x_3 + \dots + x_r = n$ is $\binom{n+r-1}{r-1}$.
- Pascal's triangle is useful for finding the coefficients of the terms in a binomial expansion.
- We can construct Pascal's triangle by using combinations.
- The expansion of a binomial $(x + y)^n$ for any power n is

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

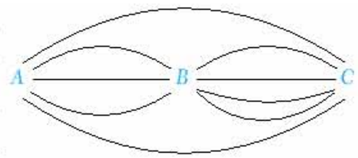
- The r^{th} entry in the expansion of $(x + y)^n$ is $\binom{n}{r-1}x^{n-r+1}y^{r-1}$.
- The sum of the coefficients of an expansion can be found by substituting 1 for each variable in the expansion.
- If the binomial expression is a polynomial then substituting zero for each variable gives us the constant term of the expansion.

Concept Check

- If we count the leaves on a tree, which counting principle do we apply?
- Why is systematic listing important?
- I have a black and a white bag, a book and a file. I will put my book or my file in one of my bags and go to school or to my office. Can I demonstrate all the outcomes of this in a single table?
- What is the uniformity criterion?
- Can there be a constant permutation function?
- How many elements does a set have if it has two subsets?
- What is the difference between a permutation and combination?
- Three students are chosen from a group of 5 students. Give an example of a permutation of these students. Give another example of a combination.
- Can you suggest a better name for a combination lock?
- Which is greater: the number of linear permutations of n objects or the number of circular permutations of the same objects. Explain your answer?
- To find the sum of the coefficients of the terms in a binomial expansion, why do we substitute 1 for the variables?
- Why is the number of combinations of r elements chosen from a set of n elements equal to the number of combinations of $(n-r)$ elements chosen from the same set?
- What does Pascal's triangle help us to find for a binomial expansion?
- Which combination formula illustrates the symmetric property of Pascal's triangle?

CHAPTER REVIEW TEST 1A

1. There are 3 routes from city A to city B, 4 routes from city B to city C and 2 direct routes from A to C. In how many different ways can Serkan travel from A to C?



A) 7 B) 9 C) 10 D) 12 E) 14

2. In each sequence below, L represents one of the letters in the 26-letter English alphabet and D represents a digit. A factory wants to register 10 million different product items. Which sequence shows a possible registry format?

A) *LLDDD* B) *LDDL* C) *DDDDL*
D) *DDDLLL* E) *DDLLL*

3. How many two-digit odd numbers can be formed from the digits {1, 2, 3, 4, 5, 6, 7} if repeated digits are allowed?

A) 14 B) 42 C) 28 D) 21 E) 49

4. Evaluate $\dfrac{(n+2)!(n-2)!}{(n+1)!(n-1)!}$.

A) $(n-3)$ B) $(n-1)$ C) $\dfrac{(n+1)}{(n+2)}$
D) $\dfrac{(n+2)}{(n-1)}$ E) 49

5. A box contains 5 different green balls and 9 different blue balls. In how many different ways can Salim pick out a green and a blue ball in any order?

A) 54 B) 45 C) 40 D) 35 E) $9! - 5!$

6. In how many ways can we name the vertices of a pentagon using any five of the letters O, P, Q, R, S, T, U in any order?

A) 2520 B) 9040 C) 5140
D) 4880 E) 3600

7. The permutation functions $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$ are given. Find the permutation function h such that $f \circ g = h$.

A) $h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ B) $h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

C) $h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ D) $h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$

E) $h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

8. How many six-digit numbers can be formed from the digits {2, 3, 4, 6, 7, 8} without repetition if the digits 3 and 7 must be together?

A) 120 B) 180 C) 144 D) 96 E) 240

9. How many eight-letter sequences can be formed from the letters in the word ALTAYLAR?

A) 2120 B) 2480 C) 3200 D) 3360 E) 3640

10. In how many different ways can 5 couples be seated around a circular table if the couples must not be separated?

A) 768 B) 724 C) 844 D) 696 E) 576

11. Which one of the following is a mathematical combination?

A) a social security number
B) the key for a combination lock
C) a committee chosen from a group of 10 people
D) the PIN code for a cellular phone
E) your name and surname

12. There are 5 different green balls and 9 different blue balls in a box. In how many different ways can Salih pick out two balls?

A) 132 B) 124 C) 111 D) 104 E) 91

13. There are 11 students of different heights in a class. We choose any 4 students and line them up from tallest to shortest in order. How many different orders are possible?

A) 280 B) 330 C) 480 D) 660 E) 7920

14. What is the middle term in the expansion of $(2x + 5y)^4$?

A) $600x^2y^2$ B) $120xy^2$ C) $5000xy^3$ D) $6x^2y^2$ E) $160x^2y^2$

15. There are 8 different math books, 4 different biology books and 6 different geometry books on a table. Nuran wants to select 4 math books, 3 geometry books and 2 biology books and then arrange them on a bookshelf so that books on the same subject are together. How many different arrangements are possible?

A) $\binom{8}{4}\binom{4}{2}\binom{6}{3}9!\cdot 3!\cdot 4!\cdot 2!$

B) $\binom{8}{4}\binom{4}{2}\binom{6}{3}3!\cdot 3!\cdot 4!\cdot 2!$

C) $\binom{8}{4}\binom{4}{2}\binom{6}{3}3!$

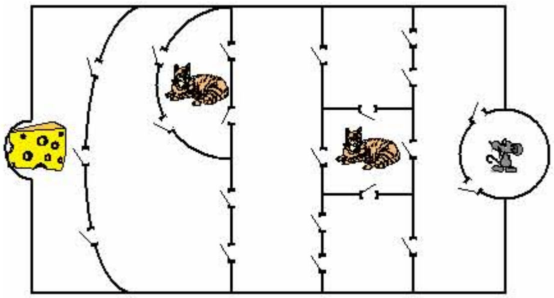
D) $\binom{8}{4}\binom{4}{2}\binom{6}{3}9!$

E) $3!\cdot 4!\cdot 2!$

16. A company in Ankara will send a group of 10 managers to participate in a seminar in Istanbul. However, only 6 plane tickets are available. The rest of the managers will go by bus. Two people are afraid of flying so they do not want to go by plane. In how many ways can the group be divided for the journey to Istanbul?

A) 18 B) 24 C) 28 D) 42 E) 56

1.



In how many different ways can the mouse in the picture get to the cheese without passing by a cat if all the gates allow only one direction pass?

A) 48 B) 36 C) 24 D) 44 E) 60

2. What is the sum of all the four-digit numbers which can be made from the elements of the set {1, 2, 3, 4} with no repeated digits?

A) 66660 B) 62600 C) 57420 D) 48800 E) 36000

3. How many three-digit even numbers smaller than 550 can be formed from the digits in the set {0, 3, 4, 5, 6, 7, 8} if no digit can be used more than once in a number?

A) 24 B) 32 C) 35 D) 42 E) 45

4. Solve $\frac{(2n)!}{(2n-3)!} \div \frac{n!}{(n-2)!} = 28$ for n.

A) 2 B) 3 C) 4 D) 5 E) 6

5. How many four-letter sequences can be formed from the letters in the sequence THISWORD if no letter can be used more than once?

A) 1680 B) 1540 C) 1420 D) 136 E) 1260

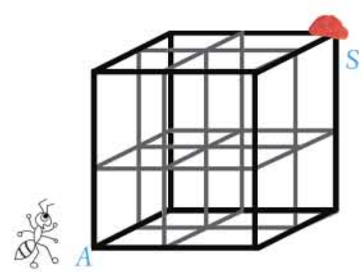
6. Fatma has 5 different history books and 4 different math books. In how many ways can she arrange them all on a shelf so that the history books are together and all the books are between two math books?

A) 2640 B) 2880 C) 5160 D) 8640 E) 12520

7. In how many ways can we exhibit 5 of 8 different new cars in a row if a certain car must be the first on the right?

A) 720 B) 760 C) 840 D) 900 E) 960

8.



The cube in the figure is made up of 27 sticks. An ant is trained to walk up and right along the sticks, relative to the cube. If it starts at point A and walks only along the sticks, in how many different shortest ways can it get to piece of the sugar at point S?

A) 90 B) 120 C) 144 D) 150 E) 180

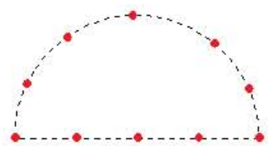
9. How many seven-digit odd numbers can be formed by rearranging the digits in the number 5321233?

A) 144 B) 168 C) 196 D) 225 E) 240

10. How many different necklaces can we make by threading 3 different red beads, 3 different green beads and 3 different yellow beads onto a chain if beads of the same color must be kept together?

A) 240 B) 216 C) 196 D) 164 E) 144

11. The figure shows ten points. Three of the points are chosen at random to form a triangle. How many different triangles can be constructed?



A) 80 B) 90 C) 100 D) 110 E) 120

12. Six women and 8 men are in a table tennis club. If each game needs 2 players for each team, how many different games between men and women can be arranged?

A) 240 B) 276 C) 360 D) 420 E) 480

13. A committee of 4 people will be selected from 8 girls and 12 boys in a class. How many different selections are possible if at least one boy must be selected?

A) 2865 B) 3755 C) 4225 D) 4455 E) 4775

14. What is the coefficient of the term containing $x^{12}y^6$ in the expansion of $(x^3 - 2y^2)^7$?

A) 84 B) –280 C) 560 D) 448 E) 35

15. A group of 5 people will be selected from 4 doctors and 7 nurses such that the group contains at least one doctor. After the selection, the group will hold a meeting around a circular table. How many different seating arrangements are possible around the table?

A) $\binom{4}{1}\binom{7}{4}4!$ B) $\binom{11}{5}4!$

C) $\binom{7}{5}4!$ D) $\left[\binom{11}{5}-\binom{7}{5}\right]4!$

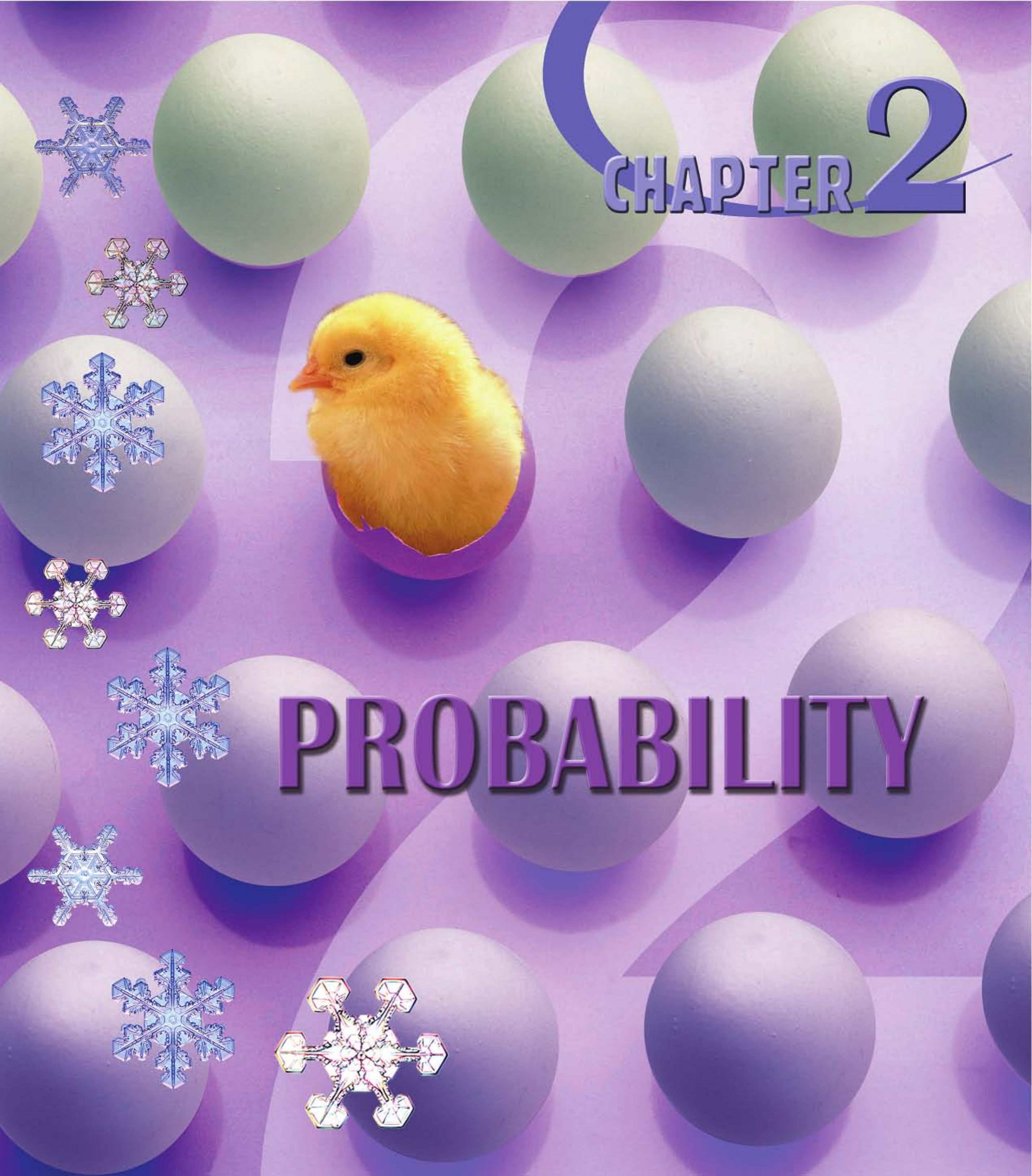
E) $\binom{4}{2}\binom{7}{3}4!$

16. A group of 12 people booked tickets to the theater. When they arrive they find that 5 places in the front row and 7 places in the back row have been reserved for them. 3 people do not want to sit at the front and 2 of them do not want to sit at the back. In how many different ways can the 12 people be seated?

A) $\binom{10}{3}\binom{7}{4}$ B) $5! \cdot 1!$

C) $\binom{7}{3}12!$ D) $\binom{12}{3}\binom{9}{4}$

E) $\binom{7}{3}5! \cdot 7!$


CHAPTER 2
PROBABILITY

BASIC CONCEPTS AND DEFINITIONS

Definition experiment, outcome, sample space, event, simple event

An experiment is an activity or a process which has observable results. For example, rolling a die is an experiment.

The possible results of an experiment are called outcomes. The outcomes of rolling a die once are 1, 2, 3, 4, 5, or 6.

The set of all possible outcomes of an experiment is called the sample space for the experiment. The sample space for rolling a die once is {1, 2, 3, 4, 5, 6}.

An event is a subset of (or a part of) a sample space. For example, the event of an odd number being rolled on a die is {1, 3, 5}.

If the sample space of an experiment with n outcomes is $S = \{e_1, e_2, e_3, e_4, \ldots, e_n\}$ then the events $\{e_1\}$, $\{e_2\}$, $\{e_3\}$, ..., $\{e_n\}$ which consist of exactly one outcome are called simple events.

EXAMPLE 1 What is the sample space for the experiment of tossing a coin?

Solution There are two possible outcomes: tossing heads and tossing tails. So the sample space is {heads, tails}, or simply $\{H, T\}$.

EXAMPLE 2 Write the sample space for tossing a coin three times.

Solution The sample space is $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

EXAMPLE 3 The sample space for an experiment is {1, 2, 3, 4, 5, 6, 7, 8, 9}. Write the event that the result is a prime number.

Solution The event is {2, 3, 5, 7}.

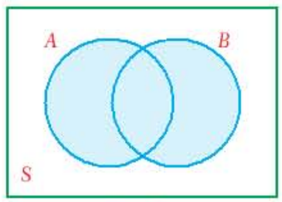
Definition

union and intersection of events, complement of an event

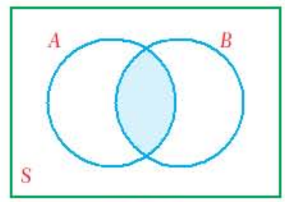
The union of two events A and B is the set of all outcomes which are in A and/or B. It is denoted by $A \cup B$.

The intersection of two events A and B is the set of all outcomes in both A and B. It is denoted by $A \cap B$.

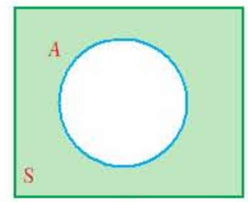
The complement of an event A is the set of all outcomes in the sample space that are not in the event A. It is denoted by A' (or A^c).



$A \cup B$
(union of A and B)



$A \cap B$
(intersection of A and B)



A'
(complement of A)

EXAMPLE 4 Consider the events $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$ in the experiment of rolling a die. Write the events $A \cup B$, $A \cap B$ and A'.

Solution The sample space for this experiment is $\{1, 2, 3, 4, 5, 6\}$. Therefore,

$A \cup B = \{1, 2, 3, 4, 5, 6\}$ (the set of all outcomes in events A and/or B);

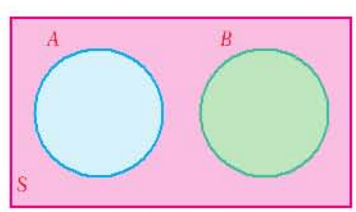
$A \cap B = \{4\}$ (the set of all common outcomes in A and B);

$A' = \{5, 6\}$ (the set of all outcomes in the sample space that are not in event A).

Definition

mutually exclusive events

Two events which cannot occur at the same time are called mutually exclusive events. In other words, if two events have no outcome in common then they are mutually exclusive events.



A and B are mutually exclusive events.

For example, consider the sample space for rolling a die. The event that the number rolled is even and the event that the number rolled is odd are two mutually exclusive events, since $E = \{2, 4, 6\}$ and $O = \{1, 3, 5\}$ have no outcome in common.

Now we are ready to define the concept of probability of an event.

Definition

probability of an event

Let E be an event in a sample space S in which all the outcomes are equally likely to occur. Then the probability of event E is $P(E) = \frac{n(E)}{n(S)}$, where $n(E)$ is the number of outcomes in event E and $n(S)$ is the number of outcomes in the sample space S.

EXAMPLE 5 A coin is tossed. What is the probability of obtaining a tail?

Solution The sample space for this experiment is $\{H, T\}$ and the event is $\{T\}$, so $n(S) = 2$ and $n(E) = 1$. So the desired probability is $P(E) = \frac{n(E)}{n(S)} = \frac{1}{2}$.

EXAMPLE 6 I roll a die. What is the probability that the number rolled is odd?

Solution The sample space is $S = \{1, 2, 3, 4, 5, 6\}$ and the event that the number is odd is $E = \{1, 3, 5\}$. So the probability is $P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$.

EXAMPLE 7 A coin is tossed three times. What is the probability of getting only one head?

Solution The sample space is

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ and the desired event is $E = \{HTT, THT, TTH\}$. So the probability is $P(E) = \frac{3}{8}$.

EXAMPLE 8 The integers 1 through 15 are written on separate cards. You are asked to pick a card at random. What is the probability that you pick a prime number?

Solution There are fifteen numbers in the sample space. The primes in the set are 2, 3, 5, 7, 11 and 13. So the desired probability is $\frac{6}{15}=\frac{2}{5}$.

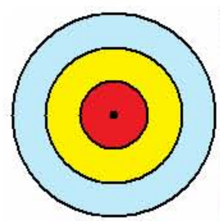
Remark

Since the number of outcomes in an event is always less than or equal to the number of outcomes in the sample space, $\frac{n(E)}{n(S)}$ is always less than or equal to 1.

Also, the smallest possible number of outcomes in an event is zero. So the smallest possible probability ratio is $\frac{n(E)}{n(S)}=\frac{0}{n(S)}=0$.

In conclusion, the probability of an event always lies between 0 and 1, i.e. $0 \leq P(E) \leq 1$.

EXAMPLE 9 A child is throwing darts at the board shown in the figure. The radii of the circles on the board are 3 cm, 6 cm and 9 cm respectively. What is the probability that the child's dart lands in the red circle, given that it hits the board?



Solution We know from geometry that the area of a circle with radius r is πr^2. Hence the area of the red circle is $\pi 3^2 = 9\pi$ cm^2 and the area of the pentire board is $\pi 9^2 = 81\pi$ cm^2.

We can consider the area of each region as the number of outcomes in the related event.

So the probability that the dart lands in the red circle is $P(\text{red}) = \frac{n(\text{red})}{n(\text{board})}=\frac{9\pi}{81\pi}=\frac{1}{9}$.

As the probability of an event gets closer to 1, the event is more likely to occur. As it gets closer to zero, the event is less likely to occur. In the previous example, the probability is close to zero so the event is not very likely. However, note that $\frac{1}{9}$ does not tell us anything about what will actually happen as the child is throwing the darts. The child will not necessarily hit the red circle once every nine darts. He might hit it three times with nine darts, or not at all. But if the child played for a long time and we looked at the ratio of the red hits, to the other hits we would find that it is close to $\frac{1}{9}$.

Definition **certain event, impossible event**

An event whose probability is 1 is called a certain event. An event whose probability is zero is called an impossible event.

EXAMPLE 10 A student rolls a die. What is the probability of each event?

a. the number rolled is less than 8

b. the number rolled is 9

Solution The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

a. We can see that every number in the sample space is less than 8.

So the event is $E = \{1, 2, 3, 4, 5, 6\}$.

Therefore the probability that the number is less than 8 is

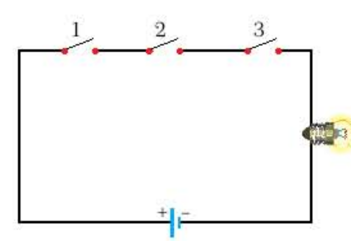
$P(E) = \frac{6}{6} = 1,$ which means the event is a certain event.

b. Since it is not possible to roll a 9 with a single die, the event is an empty set ($E = \varnothing$). So the probability is $P(E) = \frac{0}{6} = 0,$ which means the event is an impossible event.

EXAMPLE 11 A card is drawn from a deck of 52 cards. What is the probability that the card is a spade?

Solution Since there are 13 spades in a deck of 52 cards, the number of outcomes is 13. So the probability is $\frac{13}{52} = \frac{1}{4}$.

EXAMPLE 12 A small child randomly presses all the switches in the circuit shown opposite. What is the probability that the bulb lights?



Solution Each switch can be either open or closed. Let us write O to mean an open switch and C to mean a closed switch. Then the sample space contains $2 \cdot 2 \cdot 2 = 8$ outcomes, namely

$\{O_1O_2O_3,\ O_1O_2C_3,\ O_1C_2C_3,\ O_1C_2O_3,\ C_1O_2O_3,\ C_1O_2C_3,\ C_1C_2O_3,\ C_1C_2C_3\}$.

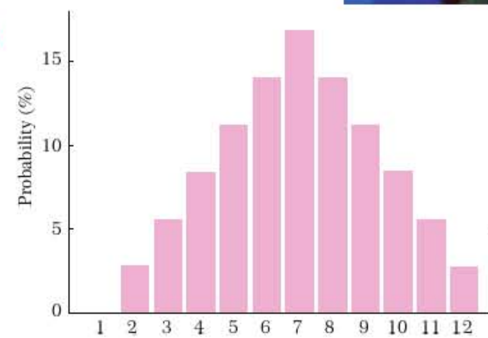
The bulb only lights when all the switches are closed. So the desired probability is $\frac{1}{8}$.

EXAMPLE 13 In a game, a player bets on a number from 2 to 12 and rolls two dice. If the sum of the spots on the dice is the number he guessed, he wins the game. Which number would you advise the player to bet on? Why?

Solution There is no difference between rolling a die twice and rolling two dice together. Let us make a table of the possible outcomes of rolling the dice:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

→ 7



We can see that there are six ways of rolling 7 with two dice. This is the most frequent outcome of the game, so the player should bet on 7. As there are $6 \cdot 6 = 36$ outcomes in the sample space, the probability of rolling 7 is $\frac{6}{36} = \frac{1}{6}$, which is the highest probability in the game.

Check Yourself 1

1. A family with three children is selected from a population and the genders (male or female) of the children are written in order, from oldest to youngest. If M represents a male child and F represents a female child, write the sample space for this experiment.
2. A student rolls a die which has one white face, two red faces and three blue faces. What is the probability that the top face is blue?
3. Two dice are rolled together. What is the probability of obtaining a sum less than 6?
4. A box contains 15 light bulbs, 4 of which are defective. A bulb is selected at random. What is the probability that it is not defective?
5. Three dice are rolled together. What is the probability of rolling a sum of 15?

Answers

1. $\{MMM, MMF, MFM, FMM, MFF, FMF, FFM, FFF\}$ 2. $\frac{1}{2}$ 3. $\frac{5}{18}$ 4. $\frac{11}{15}$ 5. $\frac{5}{108}$

EXERCISES 2.1

1. A coin is flipped three times. Specify the outcomes in each event.
   a. the same face occurs three times
   b. at least two tails occur

2. A pair of dice is rolled. Specify the outcomes in each event.
   a. the dice show the same number
   b. the sum of the numbers is greater than 7
   c. the dice show two odd numbers

3. There are 9 girls and 12 boys in a class. A student is called at random. Find the probability that the student is a boy.

4. A bag contains 3 red marbles, 4 blue marbles and 2 green marbles. Fırat takes a marble from the bag. Find the probability that he takes a red marble.

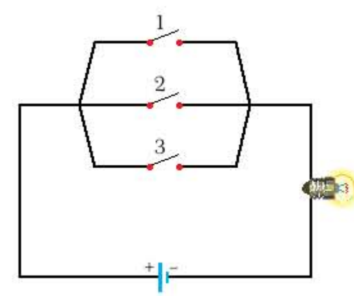
5. A continent name is chosen at random. What is the probability that the name begins with *A*?
   (The American continent is considered in two different parts.)

6. A number is drawn at random from the set $\{1, 2, 3, ..., 100\}$. What is the probability that the number is divisible by 3?

7. A pair of dice are rolled. What is the probability that their sum is greater than 6?

8. Two dice are rolled. What is the probability that their sum is a prime number?

9. Two dice are rolled. What is the probability that their sum is divisible by 4?

10. 

   A monkey is trained to press the switches in the circuit shown above. It presses all the switches many times. Find the probability that the bulb lights.

11. One-quarter of the Earth's surface is land and the rest is sea. A meteor hits the Earth. Find the probability that it lands in the sea.

12. ✪ A point is selected at random from the interior region of a circle with radius 4 cm. What is the probability that the distance between the selected point and the center of the circle is less than or equal to 2 cm?

2 WORKING WITH PROBABILITY

In this section we will learn some rules of probability which are frequently used for solving problems.

Rule

rules of probability

1. For every event E, $0 \le P(E) \le 1$.
2. For a sample space S, $P(S) = 1$.
3. For two mutually exclusive events A and B, $P(A \cup B) = P(A) + P(B)$.
4. For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
5. For any event A, $P(A) + P(A') = 1$. In other words, $P(A') = 1 - P(A)$.

Note

Many problems in probability are written in natural language. The key word for recognizing the union operation ($\cup$) in a written problem is 'or'. When we use the word 'or' (A or B) in mathematics, we mean A or B or both.

The key word for recognizing the intersection operation ($\cap$) in a written problem is 'and'. When we use the word 'and' (A and B) in mathematics, we mean both A and B.

EXAMPLE 14 A die is rolled. Find the probability that it shows 3 or 5.

Solution Let T mean the die shows 3 and F mean the die shows S. Then '3 or 5' means $T \cup F$.

Since T and F are mutually exclusive events, by the rules of probability we can write

$$P(T \cup F) = P(T) + P(F)$$
$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$ So the probability is $\frac{1}{3}$.

EXAMPLE 15 A die is rolled. Find the probability that it shows an even number or a prime number.

Solution The possible prime numbers are 2, 3 and 5 and the even numbers are 2, 4 and 6. Showing an even number (E) or a prime number (P) are not mutually exclusive events, since the outcome is in both events.

Since $P(2) = \frac{1}{6}$, $P(E \cap P) = \frac{1}{6}$.

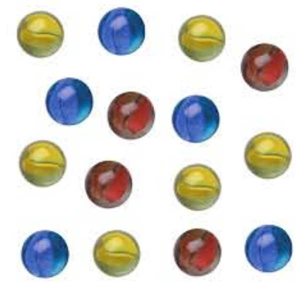
So the probability of E or P is $P(E \cup P) = P(E) + P(P) - P(E \cap P)$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{6}$$

$$= \frac{5}{6}.$$

EXAMPLE 16 An urn contains five blue marbles, four red marbles and six yellow marbles. We want to take one marble from the urn. What is the probability of taking a red or a yellow marble?

Solution 1 Since a marble cannot be both red and yellow, drawing a red marble and drawing a yellow marble are mutually exclusive events.



So the probability is

$$P(R \cup Y) = P(R) + P(Y) = \frac{4}{15} + \frac{6}{15} = \frac{10}{15} = \frac{2}{3}.$$

Solution 2 We can also solve the problem in another way. Let E be the event that a red or yellow marble is drawn. Then the complement of E (written E') is the event that neither a red nor a yellow marble is drawn. In other words, E' is the event that a blue marble is drawn.

We also know that $P(E) + \underbrace{P(E')}_{\substack{\text{drawing} \\ \text{a blue marble}}} = 1.$

So the probability of drawing a red or yellow marble is $P(E) = 1 - P(E') = 1 - \frac{5}{15} = \frac{10}{15} = \frac{2}{3}.$

EXAMPLE 17 We have twenty cards numbered from 1 to 20. A card is drawn at random. What is the probability of drawing an even number or a number divisible by 3?

Solution Let the event that an even number is drawn be $E = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ and the event that a number divisible by 3 is drawn be $T = \{3, 6, 9, 12, 15, 18\}$. We can see that $E \cap T = \{6, 12, 18\}$.

So we can write $P(E \cup T) = P(E) + P(T) - P(E \cap T) = \frac{10}{20} + \frac{6}{20} - \frac{3}{20} = \frac{13}{20}.$

EXAMPLE 18 A coin is tossed four times. What is the probability that the coin shows tails at least once?

Solution The sample space contains $2^4 = 16$ outcomes. If E is the event that we get tails at least once then E' is the event that we get no tails. In other words, E' is the event that we get heads three times (can you see why?). Since there is only one way to do this, $P(E') = \frac{1}{16}.$

So $P(E) = 1 - P(E') = 1 - \frac{1}{16} = \frac{15}{16}.$

We can check this result with the sample space:

$S = \{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT, TTHH, HTTH, THTH, HTTT, THTT, TTHT, TTTH, TTTT\}.$

EXAMPLE 19 A group of 6 people is selected at random. What is the probability that at least two of them have the same birthday?

Solution First let us assume that there are 365 days in a year. Then the sample space for one person's birthday has 365 outcomes because there are 365 possible dates for a contains. Let the desired event be A. Then A' is the event that none of these six people have a common birthday.

So A' contains $365 \cdot 364 \cdot 363 \cdot 362 \cdot 361 \cdot 360$ outcomes.

Let E be the sample space for the experiment. Then E contains 365^6 possible outcomes, because there are six people.

So the probability of A is $1 - \dfrac{n(A')}{n(E)} = 1 - \dfrac{365 \cdot 364 \cdot 363 \cdot 362 \cdot 361 \cdot 360}{365^6} \cong 0.05.$

Check Yourself 2

1. A die is rolled. What is the probability that the die shows a number greater than 3 or an even number?
2. A number is drawn at random from the set $A = \{1, 2, 3, \ldots, 100\}$.
   a. What is the probability that the number is divisible by both 2 and 3?
   b. What is the probability that the number is divisible by 2 or 3?

Answers

1. $\dfrac{2}{3}$ 2. a. $\dfrac{4}{25}$ b. $\dfrac{67}{100}$

A patient is talking to his doctor about a necessary operation.

— 'I'm worried about this operation, Doctor. They say it's 99 per cent risky.'

— 'That's true, but you needn't worry.'

— 'Why?'

— 'Because you are the hundredth patient. The other 99 have already died!'

PROTEIN FORMATION

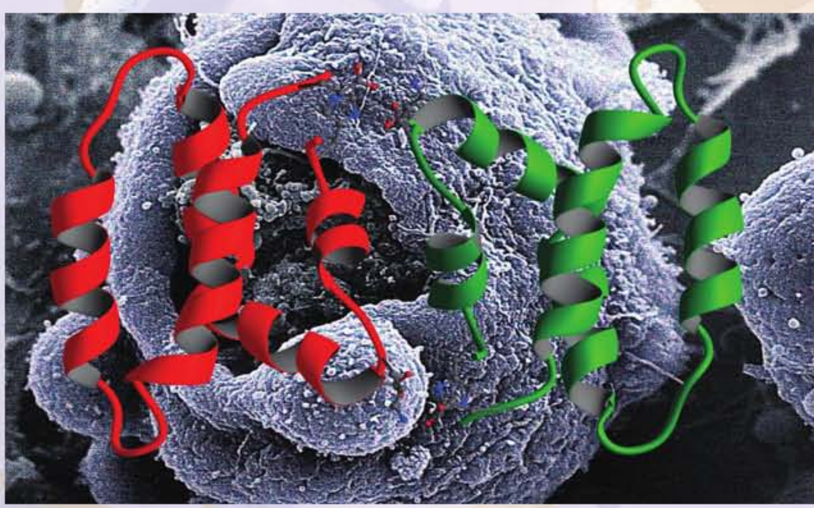
Proteins are the basic biological structures in your body. They help to create and regulate the cells inside you which make up your bones, skin and tissue. Enzymes and hormones are made up of proteins.

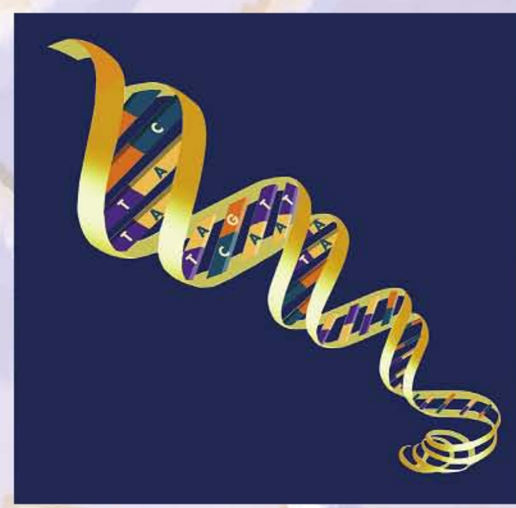
Proteins are made up of amino acids, using the information which is in your genes (your DNA). The simplest protein contains 20 different amino acids, although some complex proteins can contain thousands of amino acids.

Scientists have calculated that a theoretical complex protein comprising 288 amino acids taken from 12 children can be formed in 10^{300} different ways. This incredible number is the same as 1 followed by 300 zeros. However, only one of these possible amino acid arrangements is a real protein. The rest are biological waste material, and some may even be harmful for life. Therefore, the probability that a random arrangement of these amino acids forms a real protein is

$$\frac{1}{10^{300}} = \frac{1}{\underbrace{100...00}_{300\ zeros}}.$$

If we consider that even the smallest bacteria are made up of 600 different proteins, we can say that the chances of such bacteria forming randomly are zero: an impossible event. What do you think this says about humans, or other forms of life on Earth?





EXERCISES 2.2

1. An urn contains 5 red, 3 yellow and 6 white marbles. We draw a marble from the urn at random. What is the probability of drawing a red or a yellow marble?

2. An urn contains 3 black, 4 red and 2 blue marbles. What is the probability that a marble drawn at random is not blue?

3. A fair die is rolled. What is the probability of rolling an even number or a prime number?

4. A fair die is rolled. What is the probability of rolling a number which is less than 5 or greater than 2?

5. You draw a card from a well-shuffled deck of 52 cards. What is the probability of drawing a king or a queen?

6. Two dice are rolled. What is the probability of obtaining a sum of 6 or 10?

7. A bag contains 4 red balls, 3 yellow balls, 5 green balls and 2 black balls. Find the probability that a ball drawn at random is neither black nor red.

8. The spinner in the figure shows twelve numbers. What is the probability that

a. the spinner will stop on a red number?

b. the spinner will stop on a green number or an even number?

9. A number is drawn at random from the set {1, 2, 3, ..., 150}. Find the probability of drawing a number which is divisible by 3 or 5.

10. A number is drawn at random from the set {2, 4, 6, 8, ..., 100}. Find the probability of drawing a number which is not divisible by 3.

11. A traffic light is red for 25 seconds, amber for 5 seconds and green for 40 seconds. What is the probability of arriving at the traffic light when it is not red?

12. A survey of a group of 100 students shows that 70 students own a computer, 40 students own an MP3 player and 20 students own both. Find the probability that a student chosen at random from the group owns a computer or an MP3 player.

13. The *face cards* in a pack of playing cards are the jacks, queens and kings. A card is chosen at random from a well-shuffled deck of 52 playing cards. Find the probability that it is a black card or a face card.

14. ✪ A number is selected at random from the set $A = \{1, 2, 3, ..., 180\}$. What is the probability that the selected number is neither divisible by 3 nor divisible by 5?

15. ✪ A group of 7 people is selected at random. What is the probability that at least two of them were born in the same month?

3 COUNTING PRINCIPLES AND PROBABILITY

In our probability studies up to now, we have considered sample spaces with only a small number of outcomes. These outcomes can be listed easily. But sometimes the sample space of an experiment has a large number of outcomes. Determining this number might not always be easy or practical. In such cases we can use the counting methods we learned in Chapter 1 to determine the number of outcomes in a sample space and an event.

EXAMPLE 20 A number is selected at random from the three-digit numbers formed using the digits $\{1, 2, 3, 4, 5, 6, 7\}$ without repetition. Find the probability that the selected number is an even number.

Solution The number of outcomes in the sample space is $7 \cdot 6 \cdot 5 = 210$. There are also three even numbers in the set $\{1, 2, 3, 4, 5, 6, 7\}$. So there are three possible choices for the units digit. There are then six possible choices for the hundreds digit and five possible choices for the tens digit.

So the number of outcomes in the event is $6 \cdot 5 \cdot 3 = 90$ and the desired probability is $\dfrac{90}{210} = \dfrac{3}{7}$.

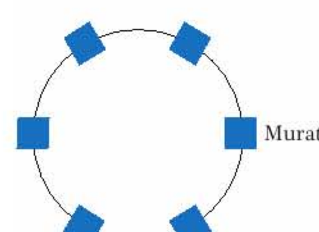
EXAMPLE 21 A machine generates all the three-letter sequences of the letters in the word *KAHVE*, with repetition allowed. Each sequence is written on a card and the cards are put in a box. Ömer draws a card. Find the probability that he draws a sequence beginning with the letter H.

Solution There are $5 \cdot 5 \cdot 5 = 125$ possible sequences, so $n(S) = 125$. If H is the event that the sequence begins with H, $n(H) = 1 \cdot 5 \cdot 5$. So the probability is $P(H) = \dfrac{n(H)}{n(S)} = \dfrac{25}{125} = \dfrac{1}{5}$.

EXAMPLE 22 Six people including Murat and Saim are to be seated around a circular table. Find the probability that Murat and Saim are seated next to each other.

Solution 1 We can use the formulas for circular permutation. The number of outcomes in the desired event is $(5 - 1)! \cdot 2! = 4! \cdot 2!$ and the number of outcomes in the sample space is $(6 - 1)! = 5!$.

So the probability is $\dfrac{4! \cdot 2!}{5!} = \dfrac{2}{5}$.



Solution 2 Alternatively, let us seat Murat first. Then we need to seat five more people in the remaining chairs, as shown opposite. Saim can sit in any one of these chairs, so there are five possible places for him. But only two of the chairs are next to Murat, so the probability that he sits in one of these chairs is $\dfrac{2}{5}$.

EXAMPLE 23 A box contains 4 yellow marbles and 6 red marbles. Two marbles are drawn at random from the box. What is the probability that both marbles are yellow?

Solution We can draw any two marbles from ten marbles without restriction in $\binom{10}{2}$ ways.

Similarly, we can draw two yellow marbles from four yellow marbles in $\binom{4}{2}$ ways.

So the desired probability is $\dfrac{\binom{4}{2}}{\binom{10}{2}} = \dfrac{2}{15}$.

EXAMPLE 24 A box contains 18 light bulbs. 5 of these bulbs are defective. We choose 3 bulbs at random. What is the probability that

a. two of the chosen light bulbs are defective?

b. at least one of the chosen light bulbs is defective?

Solution a. We need to choose 2 defective bulbs from 5 bulbs and one working bulb from 13 bulbs.

So the number of outcomes in the event is $\binom{5}{2} \cdot \binom{13}{1} = 130$.

The number of outcomes in the sample space for selecting 3 bulbs from 18 is $\binom{18}{3} = 186$.

So the probability is $\dfrac{\binom{5}{2} \cdot \binom{13}{1}}{\binom{18}{3}} = \dfrac{65}{408}$.

b. Let the event be E. Then E' is the event that no defective bulbs are selected. In other words E' is the event that three working bulbs are selected.

So the answer is $P(E) = 1 - P(E') = 1 - \dfrac{\binom{13}{3}}{\binom{18}{3}} = \dfrac{265}{408}$.

EXAMPLE 25 A coin is tossed eight times. What is the probability of getting 5 heads and 3 tails?

Solution 1 The number of outcomes in the sample space is $n(S) = 2^8 = 256$. The number of outcomes in the desired event is $n(E) = C(8, 5) \cdot C(3, 3)$.

Therefore the probability is $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{\binom{8}{5}\binom{3}{3}}{256} = \dfrac{7}{32}$.

Solution 2 Alternatively, we can think of the desired event as an arrangement of the letters in $HHHHHTTT$. By permutation with repetition, $n(E) = \dfrac{8!}{5! \cdot 3!} = \dfrac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$.

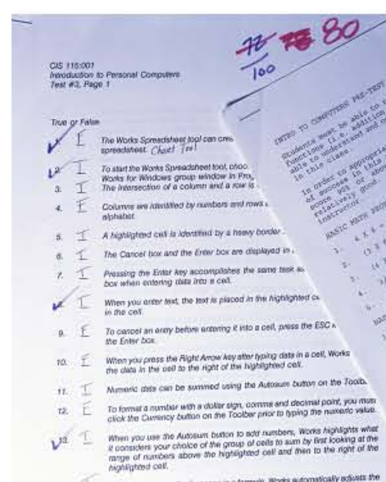
So $P(E) = \dfrac{56}{256} = \dfrac{7}{32}$.

EXAMPLE 26 In a group of 13 people, 4 people speak English, 6 people speak Turkish and 3 people speak German. A committee of 5 people is chosen at random from the group. What is the probability that the committee contains 2 English speakers, 2 Turkish speakers and one German speaker?

Solution We can choose 5 people from 13 in $C(13, 5)$ different ways without any restriction. So the number of outcomes in the sample space is $C(13, 5)$. However, the desired selection can be achieved in $C(4, 2) \cdot C(6, 2) \cdot C(3, 1)$ different ways.

So the probability is $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{\binom{4}{2}\binom{6}{2}\binom{3}{1}}{\binom{13}{5}} = \dfrac{\dfrac{4!}{2! \cdot 2!} \cdot \dfrac{6!}{2! \cdot 4!} \cdot \dfrac{3!}{2!}}{\dfrac{13!}{5! \cdot 8!}} = \dfrac{30}{143}$.

Example 27 A student is taking a test which has 15 true-or-false questions. If the student guesses every answer, what is the probability that he or she will answer exactly eleven questions correctly?



Solution There are two possible answers (true or false) for each question. Therefore the sample space has 2^{15} outcomes and the desired event has $\binom{15}{11}$ outcomes because the event is the same as selecting 11 questions among 15.

So the probability is $P(E) = \dfrac{\binom{15}{11}}{2^{15}} = \dfrac{1365}{32768}$, or approximately 0.04.

EXAMPLE 28 A machine generates a random four-letter sequence of letters from the letters in the word *KARTAL*. What is the probability that the word begins and ends with A?

Solution Let us find the sample space. We have three cases:

Case 1: $4!$ possible sequences do not contain an A, since there are four other letters to choose from $\{K, R, T, L\}$.

Case 2: $\binom{4}{3} \cdot 4!$ possible sequences contain only one A.

Case 3: $\binom{4}{2} \cdot \frac{4!}{2! \cdot 1! \cdot 1!}$ possible sequences contain two A's.

So the sample space is $4 + \binom{4}{3} \cdot 4! + \binom{4}{2} \cdot \frac{4!}{2! \cdot 1! \cdot 1!}$.

The number of outcomes of the desired event is $\binom{4}{2} \cdot 2!$ because there are four letters left to choose from if the two A's have to be chosen.

So the required probability is $$\frac{\binom{4}{2} \cdot 2!}{\underbrace{4!}_{\substack{\text{sequences}\\ \text{with no } A}} + \underbrace{\binom{4}{3} \cdot 4!}_{\substack{\text{sequences}\\ \text{with one } A}} + \underbrace{\binom{4}{2} \cdot \frac{4!}{2! \cdot 1! \cdot 1!}}_{\substack{\text{sequences}\\ \text{with two } A\text{'s}}}} = \frac{1}{16}.$$

Check Yourself 3

1. A student will choose 4 courses at random to study next term. There are 14 courses in the list. Six of them are science courses. What is the probability that the student chooses all science courses?
2. A machine generates a three-letter sequence of letters from the elements in the set $\{a, b, c, d, e\}$, with repetition. What is the probability that the letters in the word are all different?
3. Ahmet, Kemal and their seven friends are called randomly to sit in 9 chairs placed side by side. What is the probability that Ahmet and Kemal are seated next to each other?
4. Set A has 6 elements. Each subset of A is written on a card and all the cards are put in a box. A student chooses a card at random. What is the probability that he selects a card which shows four elements?

Answers

1. $\frac{15}{1001}$ 2. $\frac{12}{25}$ 3. $\frac{2}{9}$ 4. $\frac{15}{64}$

A STICKY PROBLEM!

A box contains seven sticks which are respectively 2 cm, 3 cm, 4 cm, 5 cm, 7 cm, 8 cm and 11 cm long.

Find the probability that any three sticks chosen at random from the box will form a triangle.

Two mathematicians were talking about how important their jobs were.

'My dear friend, our country is not taking math seriously. I think the government should tax people who can't do math,' complained one mathematician.

'That is what the lotto is for!' said the other.

WINNING THE LOTTERY

Many countries hold a 6/49 lottery. '6/49' means that you must correctly guess six numbers from the first 49 positive integers to win the first prize. Does it sound easy? Only six numbers!

In fact, it is not very easy to win a lottery like this. There are so many possible combinations of six numbers that the chances of choosing the right combination is very small indeed.

Here are two sets of six lottery numbers: $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{5, 12, 18, 23, 33, 41\}$. Which set of numbers do you think is more likely to win? Some people think that set B is more likely than set A. In fact, both sets of numbers have an equal chance, since six numbers are chosen from 49 at random.

So what is the real probability that you will win the lottery with one ticket? The answer is $\frac{1}{13983816}$. In other words, about one in 14 million! Can you believe it? Think about the math:

The number of different ways to choose six numbers from 49 is $\binom{49}{6}$.

This is equal to $\frac{49!}{(49-6)!\cdot 6!} = \frac{49\cdot 48\cdot 47\cdot 46\cdot 45\cdot 44}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 13983816$. Your lottery ticket is only one of these sets of six numbers, so the probability that you will win the lottery with one ticket is $\frac{1}{13983816} \cong 0.00000007$.

To be sure of winning the lottery, you would therefore need to fill in 13983816 tickets, using each possible combination of numbers just once. If it takes you 15 seconds to fill in one ticket, you would need approximately 58265 hours to complete them all. This is the same as 2427 days, or 6.65 years with no break. And of course, you would have to pay for all the tickets.

Do you still want to play the lottery?

EXERCISES 2.3

1. Ahmet, Berk, Can, Deniz and Engin are seated at random around a circular table. What is the probability that Ahmet and Can are seated next to each other?

2. Two integers are randomly selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ without repetition. What is the probability that the sum of the numbers is 10?

3. Five math books and 4 physics books are arranged randomly on a bookshelf. What is the probability that books about the same subject are all together?

4. An integer is randomly selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and then replaced. Then another integer is selected. What is the probability that the sum of the two integers is 10?

5. Nine people want to be seated around a circular table. What is the probability that two particular people will not be seated next to each other?

6. Two numbers are drawn at random from 5 positive and 3 negative numbers. What is the probability that the product of the selected numbers is positive?

7. Two numbers are selected randomly from 6 odd numbers and 5 even numbers. What is the probability that the product of the selected numbers is even?

8. In a lottery game, a player must pick 4 winning numbers from 42 numbers to win the prize. Find the probability of winning the prize.

9. 5 boys and 4 girls are seated at random in a row of 9 seats. What is the probability that the boys and girls are seated alternately?

10. Two dice are rolled together 900 times. How many times would you expect to get a sum of 7?

11. There are 4 different letters and 6 different mailboxes. Each letter will be put in a random mailbox. What is the probability that all four letters will be put in the same mailbox?

12. 200 patients were treated with a new medicine. 32 of them were not cured. Four patients from the 200 are selected at random. What is the probability that three of them were cured?

13. The first 5 positive integers are written randomly in a row. Find the probability that the numbers are written in either ascending or descending order.

14. A four-digit number is formed using the digits $\{1, 2, 3, 4, 5, 6, 7\}$ without repeating the digits. Find the probability that the number is an even number.

15. A box contains 4 red marbles and 8 other marbles of different colors. What is the probability that 4 marbles selected at random from the box are not red?

16. Seven points A, ..., G are arranged in a circle. A triangle is drawn by connecting three points chosen at random. What is the probability that point E is a vertex of the triangle?

17. Three hundred people apply for 4 jobs. Ninety of the applicants are women. Four people are selected at random for the jobs. Find the probability of each event.

a. all the selected people are men

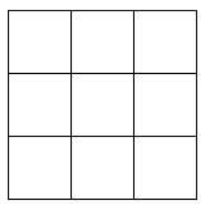
b. exactly two people are men

c. exactly one person is a man

d. no men are selected

18. A standard piano keyboard has 88 different keys. A cat jumps on 6 keys of the keyboard at random (possibly with repetition). Find the probability that the cat will strike the first six notes of Beethoven's Fifth Symphony.

19. The square in the figure is divided into 9 smaller squares. A quadrilateral is chosen at random from all the quadrilaterals in the figure. What is the probability that the quadrilateral is a square?



20. An urn contains 3 red, 3 blue and 4 yellow marbles. Three marbles are drawn at random. Find the probability no red marbles are drawn.

21. A box contains 4 red, 5 yellow and 4 white marbles. We draw 3 marbles at the same time. What is the probability that only one of them is a red marble?

22. Three numbers are selected from the set $\{1, 2, 3, ..., 15\}$. What is the probability that the product of the numbers is divisible by 3?

23. A subset is drawn from all of the four-element subsets of the set $A = \{a, b, c, d, e, f, g, h\}$. What is the probability that the selected subset does not contain the element d?

24. A committee of 4 men and 3 women is chosen at random from a group of 8 men and 5 women that includes Sami and Dilara. What is the probability that both Sami and Dilara are chosen?

25. A teacher distributes 20 questions before an exam and tells her students that 10 of them will be in exam. Mehmet can solve 15 of the questions. In order to pass the exam a student must answer at least 6 questions correctly. What is the probability that Mehmet will pass the exam?

26. Abraham and Bill are in a group of 12 people. Seven people are chosen randomly from the group and seated in a row. Find the probability that Abraham and Bill are seated next to each other.

27. Three numbers are selected from the set $\{1, 2, 3, ..., 15\}$. What is the probability that the sum of the selected numbers is divisible by 3?

28. We draw two numbers at random from the set $\{1, 2, 3, ..., 100\}$. What is the probability of drawing a pair of consecutive numbers?

29. A committee of 4 people is chosen from 6 men and 5 women. What is the probability that the committee contains at least one man?

30. A group of 4 people is chosen at random from 6 couples. What is the probability that there is no couple in the group?

31. A chessboard has 64 squares and the length of the side of each square is 1 unit. A rectangle is drawn at random on the chessboard. What is the probability that the perimeter of the rectangle is greater than 4 units?

32. Two numbers are drawn at random from the set $\{1, 2, 3, ..., 100\}$. Find the probability that one of the drawn numbers is half of the other.

4 CONDITIONAL PROBABILITY

Sometimes the outcome of an experiment is affected by a related event or by some additional conditions. These things can modify the sample space and therefore change the probability of an event. This leads us to the concept of conditional probability.

Definition

conditional probability

Let A and B be two events in an experiment such that $P(B) \neq 0$ and B has already occurred. Then the probability that the event A will occur is called the conditional probability of event A, written as $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$.

EXAMPLE 29 A fair die is rolled twice. What is the probability that the sum of the numbers rolled is 9 if it is known that one of the numbers is 4?

Solution Let A be the event that the sum of the numbers is 9 and let B be the event that one of the numbers is 4.

Then $A = \{(3, 6), (6, 3), (4, 5), (5, 4)\}$ and
$B = \{(1, 4), (4, 1), (2, 4), (4, 2), (3, 4), (4, 3), (4, 4), (4, 5), (5, 4), (4, 6), (6, 4)\}$.

We can see that $A \cap B = \{(4, 5), (5, 4)\}$. So by the definition of conditional probability, $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{2}{36}}{\frac{11}{36}} = \dfrac{2}{11}$.

EXAMPLE 30 Two dice are rolled. What is the probability that the sum of the numbers rolled will be greater than 9, given that the first die shows a 6?

Solution Let the event that the sum is greater than 9 be A and the event that the first number is 6 be B. Then $A = \{(6, 4), (4, 6), (6, 5), (5, 6), (6, 6), (5, 5)\}$ and $B = \{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$.

Hence the probability is $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{3}{36}}{\frac{6}{36}} = \dfrac{3}{6} = \dfrac{1}{2}$.

EXAMPLE 31 In a group of 80 students, 20 study physics, 45 study math and 14 study both. A student is chosen at random from the group.

a. What is the probability that the student studies only math?

b. What is the probability that the student studies physics if it is known that he studies math?

c. What is the probability that the student studies math if it is known that he studies physics?

d. What is the probability that the student is not studying math if it is known that he studies physics?

Solution

a. $P(\text{math}) = \dfrac{45}{80} = \dfrac{9}{16}$

b. $P(\text{physics}\,|\,\text{math}) = \dfrac{P(\text{physics and math})}{P(\text{math})} = \dfrac{\frac{14}{80}}{\frac{45}{80}} = \dfrac{14}{45}$

c. $P(\text{math}\,|\,\text{physics}) = \dfrac{P(\text{math and physics})}{P(\text{physics})} = \dfrac{\frac{14}{80}}{\frac{20}{80}} = \dfrac{14}{20} = \dfrac{7}{10}$

d. $P(\text{not math}\,|\,\text{physics}) = \dfrac{P(\text{not math and physics})}{P(\text{physics})} = \dfrac{\frac{6}{80}}{\frac{20}{80}} = \dfrac{6}{20} = \dfrac{3}{10}$

Rule

multiplication rule

If we multiply both sides of the equality in the formula for conditional probability by $P(B)$, we get $P(A \text{ and } B) = P(A \cap B) = P(B) \cdot P(A\,|\,B)$.

EXAMPLE 32 A box contains 6 white, 3 red and 5 blue balls. A person chooses a ball at random from the box and then replaces it. This is done three times. What is the probability that the person chooses first a white ball, then a blue ball, then a red ball?

Solution Since the person replaces the selected ball each time, the number of outcomes in the sample space in each case is 14. So the desired probability is

$P(\text{first white then blue then red}) = \dfrac{6}{14} \cdot \dfrac{5}{14} \cdot \dfrac{3}{14} = \dfrac{45}{1372}$.

EXAMPLE 33 Two cards are drawn from a shuffled deck of 52 cards, without replacement. What is the probability that the first card is a queen and the second is an ace?

Solution We want to find

$$P(\text{queen first and ace second}) = P(\text{queen first}) \cdot P(\text{ace second} \mid \text{queen first})$$
$$= \dfrac{4}{52} \cdot \dfrac{4}{51} = \dfrac{4}{663}.$$

EXAMPLE 34 Twenty cards numbered 1 through 20 are placed in an urn and mixed, then one card is drawn at random. If the card shows an odd number, what is the probability that this number is divisible by 3?

Solution Let A be the event 'the number is divisible by 3' and B be the event 'the number is odd.' We want to find $P(A|B)$.

Obviously $P(B) = \frac{10}{20} = \frac{1}{2}$ and $P(A \cap B)$ is the probability of drawing an odd number which is divisible by 3. These numbers are 3, 9 and 15. So $P(A \cap B) = \frac{3}{20}$.

So the desired probability is $P(A|B) = P(A \cap B) \div P(B) = \frac{3}{20} \div \frac{1}{2} = \frac{6}{20} = \frac{3}{10}$.

EXAMPLE 35 Fatih is asked to draw a card from a well-shuffled deck of 52 cards. If it is a diamond, he puts it back and draws another card. However, if it is not a diamond, he draws another card without replacing the first card. What is the probability that the second card is the king of diamonds?

Solution Let A be the event that the first card is a diamond and let B be the event that the second card is the king of diamonds. Then we have two mutually exclusive conditions: either the first card is a diamond or the first card is not a diamond. So we want the probability

$$P(B) = P(A|B) + P(A'|B) = P(A) \cdot P(B|A) + P(A') \cdot P(B|A') = \frac{1}{4} \cdot \frac{1}{52} + \frac{3}{4} \cdot \frac{1}{51}$$

$$= \frac{1}{108} + \frac{1}{68} = \frac{11}{459}.$$

Check Yourself 4

1. In a class of 32 students, 22 students passed a math exam and 18 students passed a physics exam. 2 students failed both exams. What is the probability that a student selected at random passed the math exam if it is known that he or she passed the physics exam?
2. A die is rolled. The number it shows is greater than 2. What is the probability that the number is prime?
3. Two dice are rolled together. The sum of the numbers they show is 8. What is the probability that one of the numbers is 5?
4. A box contains 5 red, 3 blue and 3 white marbles. A marble is chosen at random. If it is known that the chosen marble is not blue, what is the probability that it is white?

Answers

1. $\frac{5}{9}$ 2. $\frac{1}{2}$ 3. $\frac{2}{5}$ 4. $\frac{3}{8}$

EXERCISES 2.4

1. A person's birthday is known to be in September. What is the probability that it is
   a. September 12?
   b. in the first eleven days of September?
   c. in the first half of September?

2. A die is rolled and shows an odd number. Find the probability that this number is prime.

3. 60% of a store's customers are men and 80% of the men have a credit card. What is the probability that a customer selected at random is a man with a credit card?

4. Two dice are rolled and show different numbers. Find the probability that the sum of the numbers is 8.

5. Two fair dice are rolled together. What is the probability that the sum of the numbers showing is greater than 8, if it is known that one of the numbers is a 5?

6. A company has 150 employees. 95 of these employees are men, 12 of the employees are administrators, and 8 of the administrators are men. A person is selected at random from the employees. Find the probability that the selected person is an administrator, given that he is a man.

7. The probability that a person owns a notebook computer is 0.15 and the probability that the owner of a notebook computer also owns a wireless mouse is 0.6. What is the probability that a person selected at random owns both a notebook computer and a wireless mouse?

8. 65% of the students who take a university entrance exam pass the exam. 40% of the students who pass the exam are accepted by a university. What is the probability that a student selected at random is accepted by a university, given that he/she passed the exam?

9. Daniel wants to travel from city A to city B. For this trip he has a choice of 2 different ferry lines, 3 airlines, 2 train lines, 4 bus routes or 3 car routes. He selects a means of transport at random. Find the probability that Daniel travels by bus, given that he travels by land.

10. 60% of the students in a class are boys and 20% of the boys wear glasses. 70% of the girls do not wear glasses. What is the probability that a student selected at random is a boy, given that the student wears glasses?

11. A number is selected at random from the set $\{1, 2, 3, ..., 60\}$. What is the probability that the selected number is divisible by 5, given that it is greater than 40?

12. Two dice are rolled together. One of the dice shows a 4. What is the probability that the other die shows an odd number?

13. A manufacturer of CD players buys 65% of its parts from China and the rest from Japan. 1.4% of the Japanese parts are defective, and 4.3% of the Chinese parts are defective. Find the probability of each event.
    a. a part is defective and made in Japan
    b. a part is defective and made in China

14. A four-element subset is chosen from the subsets of the set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. What is the probability that the smallest element in the set is 4 if we known that 4 is included in the set?

15. Two urns contain marbles. The first urn contains 3 white and 5 red marbles. The second urn contains 4 white and 2 red marbles. First we roll a die. If a number less than 3 comes up we draw a marble from the first urn. Otherwise we draw a marble from the second urn. Elif drew a red marble. What is the probability that she rolled a number greater than 2?

DEPENDENT AND INDEPENDENT EVENTS

Imagine that we draw two cards one after the other from a well-shuffled deck of cards. Then the outcome of the second draw will be influenced by the outcome of the first, since the deck will be different on the second draw. We can say that these two events are dependent.

Now consider the example of rolling a die twice. The number we roll first has no effect on the number we roll after it. We say that these events are independent.

Definition

independent events

Two events A and B are independent events if the outcomes in A do not affect the outcomes in B. Since we know from the multiplication rule that $P(A \cap B) = P(A) \cdot P(A|B)$ we can write $P(A \cap B) = P(A) \cdot P(B)$.

EXAMPLE 36 A die is rolled and a coin is tossed. Find the probability that the die shows a 5 and the coin shows a head.

Solution Rolling a die and tossing a coin are two independent events, so the probability of rolling a five (F) and tossing a head (H) is $P(F \cap H) = P(F) \cdot P(H) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$.

EXAMPLE 37 A pair of dice is rolled twice. What is the probability that the numbers showing add up to 9 each time?

Solution Two die rolls in succession are independent events because the first one has no influence on the second. The probability that the numbers showing on two dice add up to 9 is $\frac{1}{9}$ (can you see why?).

So the desired probability is $\frac{1}{9} \cdot \frac{1}{9} = \frac{1}{81}$.

EXAMPLE 38 Six men and 7 women apply for the positions of manager and assistant manager in a company. Each name is written on a card and the cards are all put in a box. Then two names are selected in succession from the box at random, without replacement. What is the probability that both of the people selected are men?

Solution Since the name is not replaced after selecting the first person, the two events are dependent. Let the first event be A and the second event be B.

Then $P(A \cap B) = P(A) \cdot P(B) = \frac{6}{13} \cdot \frac{5}{12} = \frac{5}{26}$.

EXAMPLE 39 You buy a watch made by Brand A and then a watch made by Brand B. 3% of Brand A watches are defective, while 2% of Brand B watches are defective. What is the probability that both of the watches you buy are defective?

Solution Since the events are independent, the answer is $\frac{3}{100} \cdot \frac{2}{100} = \frac{3}{5000}$.

EXAMPLE 40 Three balls are drawn successively without replacement from a box which contains 5 red, 6 yellow and 4 green balls. What is the probability that all three balls are red?

Solution $\frac{5}{15} \cdot \frac{4}{14} \cdot \frac{3}{13} = \frac{2}{91}$.

EXAMPLE 41 A coin is tossed and a die is rolled. Find the probability that the coin shows heads or the die shows an even number.

Solution The events are independent. The probability of getting a head is $P(H) = \frac{1}{2}$ and the probability that the die shows an even number is $P(E) = \frac{1}{2}$.

So the probability that the coin comes up heads or the dice shows an even number is

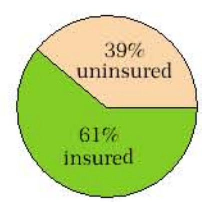
$$P(H \cup E) = P(H) + P(E) - P(H \cap E) = P(H) + P(E) - P(H) \cdot P(E)$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

EXAMPLE 42 An American company conducted a survey about health insurance. Two of the questions they asked are shown below, with the results.

A) In the past year, have you had problems paying medical bills?

B) If you have had problems paying your medical bills, what is your health insurance status?



One of the people who completed the survey is chosen at random. Find the probability that

a. he/she has had no problems paying medical bills.

b. he/she is an uninsured person who has had problems paying medical bills.

Solution a. $\frac{77}{100}$ b. P(having problems and uninsured) $= \frac{23}{100} \cdot \frac{39}{100} = \frac{897}{1000}$.

Check Yourself 5

1. A box contains 4 blue marbles and 6 white marbles. Two marbles are drawn one after the other, without replacement. What is the probability of drawing two blue marbles?
2. A box contains 5 Brand A floppy disks and 4 Brand B floppy disks. Two disks are selected from the box. What is the probability of selecting two disks of the same brand?
3. A die is rolled and a coin is tossed. What is the probability that the die shows a number greater than 4 and the coin shows a tail?
4. There are six couples in a room. A man and a woman are chosen at random. What is the probability that they are a couple?
5. A computer keyboard has 85 keys and a monkey is pressing the keys at random. Assuming that the monkey is equally likely to hit any key, what is the probability that the monkey types the word MATH with four consecutive hits?
6. Two urns contain red and white beads. The first urn contains 5 white and 3 red beads. The second urn contains 4 white and 2 red beads. A man draws a bead from the first urn and then puts it into the second urn without knowing its color. Then he draws a bead from the second urn. Find the probability that he draws a white bead from the second urn.

Answers

1. $\frac{2}{15}$ 2. $\frac{4}{9}$ 3. $\frac{1}{6}$ 4. $\frac{1}{11}$ 5. $\left(\frac{1}{85}\right)^4$ 6. $\frac{37}{56}$

FEAR and RISK

Once upon a time a friend of mine was afraid of traveling by rowboat. One evening we walked together through Istanbul to the Golden Horn. We had to get to Eyüp Sultan and there was no carriage to take us, so we had to take a boat.

'I'm frightened. What if the boat sinks?' my friend said.

I asked him, 'How many boats do you think there are here on the Golden Horn?'

'Perhaps a thousand,' he replied.

I continued, 'How many boats sink in a year?'

'One or two. Perhaps none at all,' he admitted.

I asked him, 'How many days are there in a year?'

'Three hundred and sixty,' he answered.

Then I said to him, 'So the chance that our boat will sink, which makes you so afraid, is one in three hundred and sixty thousand. Surely such a risk should not frighten a sensible human being.'

Then I asked my friend how long he thought he would live.

He replied, 'I am old. Perhaps I shall live another ten years.'

So I said to him, 'That is three thousand six hundred days. You could die on any of those days, since the time of your death cannot be known. You see: the chance that you will die today is one in three thousand six hundred, which is much greater than chance of this rowboat sinking. So write your will!'

He understood my argument and managed to get into the boat. As we were rowing across, I said to him, 'The sense of fear was given to preserve life, not to make it difficult and painful!'

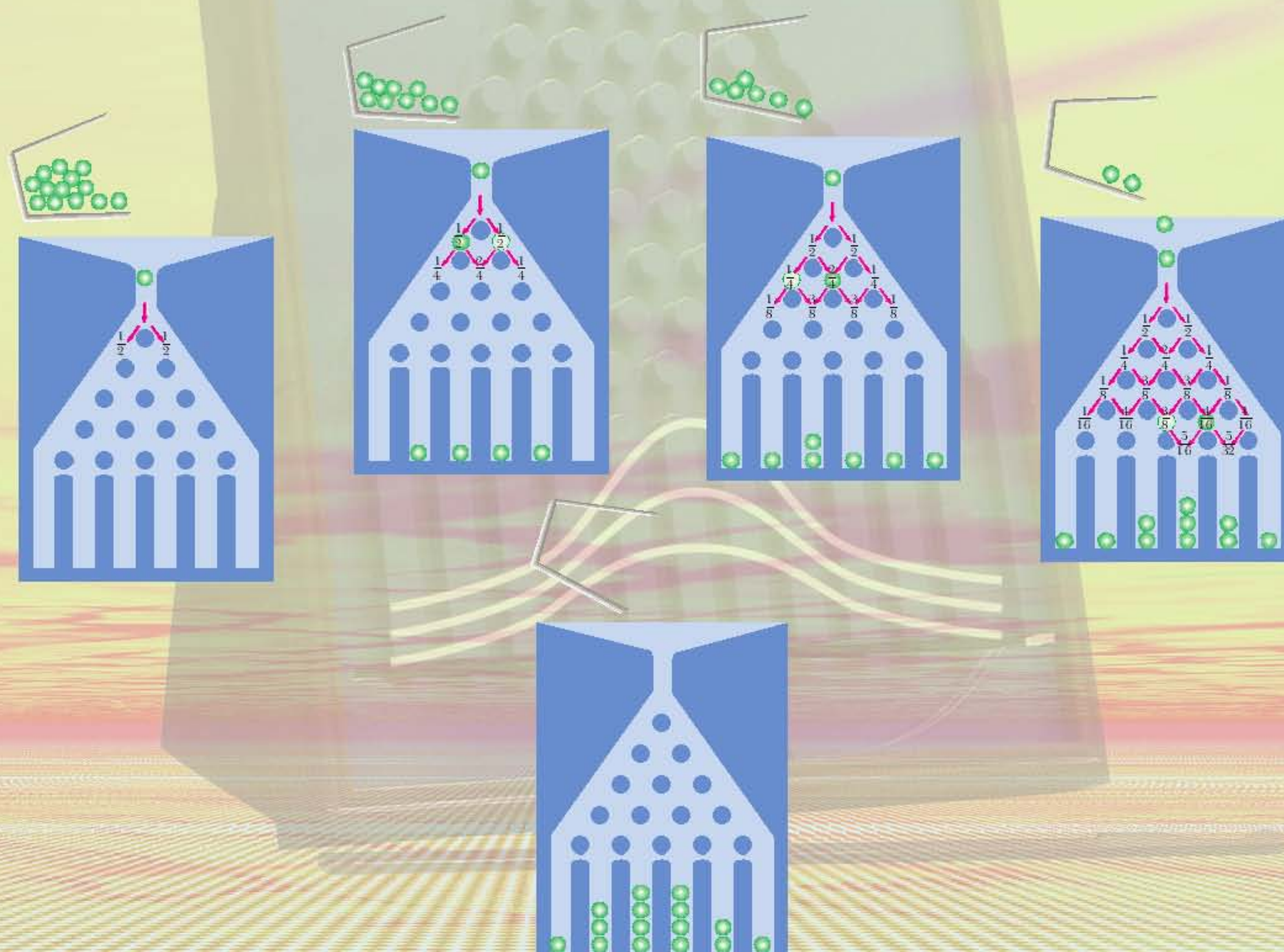
DEMONSTRATING PROBABILITY

Imagine you have a vertical maze made from five rows of pegs, as shown in the pictures below. A small marble is dropped into the maze and falls to the bottom. When the marble hits a peg, the probability that it moves left or right are equal. In other words, it is $\frac{1}{2}$. The marble falls through the maze until it lands in one of the slots at the bottom of the maze. If you continue to drop marbles into the maze, the marbles in each slot will show you the general probability that a marble lands in that slot.

For five rows of pegs, you will find that the probabilities for each slot are $\frac{1}{16}$, $\frac{5}{32}$, $\frac{5}{16}$, $\frac{5}{16}$, $\frac{5}{32}$ and $\frac{1}{16}$.

As the probabilities show, it is more likely for a marble to land in the two middle slots. As we move left or right, the probabilities decrease.

Can you see the relationship between this maze and Pascal's triangle?



EXERCISES 2.5

1. A box contains 2 blue, 3 white and 5 red marbles. A marble is drawn from the box and a die is rolled. What is the probability that the marble is red and the die shows a prime number?

2. An experiment consists of rolling a die and tossing a coin. What is the probability of getting a tail and rolling an odd number?

3. Michael rolls a red die and a yellow die. What is the probability that the red die shows a prime number and the yellow die shows a number greater than 2?

4. There are 7 cannons on the wall of a fortress. Two of the cannons are empty and 5 of them are ready to fire. There are 15 soldiers and only 7 of them know how to fire a cannon. A cannon and a soldier are selected at random. Find the probability that the soldier can fire the cannon.

5. A box contains 3 green marbles and 5 blue marbles. Another box contains 4 green marbles and 3 blue marbles. A student takes a marble from the first box and puts it into the second box without looking at its color. What is the probability that she draws a green marble from the second box?

6. An urn contains 7 red and 4 yellow beads. Two beads are taken out one after the other. What is the probability that the first bead is red and the second bead is yellow?

7. In a shipment of 100 laptop computers, 4 are defective. A person buys 2 laptops from the shipment. What is the probability that they are both defective?

8. There are 8 couples in a room. Two people are called at random. What is the probability that a couple is called?

9. A box contains 5 blue marbles and 3 black marbles and another box contains 3 blue marbles and 4 black marbles. A coin is tossed. If it shows heads we draw a marble from the first box, otherwise we draw one from the second box. What is the probability of drawing a blue marble?

10. Four people meet in a room. What is the probability that all of them were born on the same day of the week?

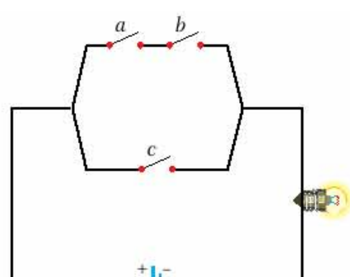
11. A private plane has 4 engines. The probabilities that the engines fail during a flight are 0.01, 0.02, 0.04 and 0.03 respectively. Find the probability that at least one engine engine fails during a flight.

12. In the NBA finals, team *A* has won 3 games and team *B* has won 2 games. A team must win 4 games to win the final. Assuming that the teams have an equal chance of winning each game, find the probability that team *A* wins the final.

13. Adam, Bill, Carla and David are asked to work together on a problem. The probabilities that each of them will be able to solve the problem alone are $\frac{1}{2}$, $\frac{2}{5}$, $\frac{3}{7}$ and $\frac{1}{3}$ respectively. Find the probability that they can solve the problem by working together.

14. Mustafa has a key ring with 6 keys. One of the keys unlocks a door. Mustafa tries each key one by one in the door. What is the probability that Mustafa unlocks the door on his fourth attempt?

15.



A machine presses the switches in the circuit above at random. The probability that each of the switches in the circuit is pressed is $\frac{1}{2}$. What is the probability that the bulb lights?

16. David and Joseph play a game with a die such that the person who rolls a 6 first wins the game. David rolls the die first. What is the probability that Joseph wins the game on his third turn?

17. The probability that an archer A hits a target is $\frac{1}{3}$ and the probability that an archer B hits the same target is $\frac{3}{4}$. Each archer shoots one arrow. What is the probability that the target is hit only once?

18. There are two boxes containing marbles. The first box contains 3 red and 4 yellow marbles. The second box contains 2 red and 5 yellow marbles. A marble is taken from each box and put into the other one, simultaneously. What is the probability that the marbles in each box remain the same color?

19. A box contains 5 red marbles and 6 blue marbles. Anna draws a marble at random from the box. If Anna draws a red marble, she replaces it with a blue one. If she draws a blue marble, he replaces it with a red one. Then she draws a second marble. What is the probability that the second marble is blue one?

20. Aslan, Bekir and Cihan take an exam which they never altogether fail. The probability that Ahmet passes the exam is three times that of Bekir and half of the probability that Cihan passes the exam. Find the probability that all of the students pass the exam.

21. Three boxes contain marbles. The first box contains 5 white and 3 red marbles, the second box contains 2 white and 4 red marbles and the third box contains 4 white and 3 red marbles. A box is selected at random and then a marble is drawn from the selected box. What is the probability of drawing a red marble?

BINOMIAL PROBABILITY

Suppose that we are rolling a die several times and we are interested in the number of times that we roll a 4. We can consider the outcome 4 as a success and the other five outcomes as failures. In other words, we can divide the outcomes of the experiment into two events. The probabilities of these outcomes are called binomial probabilities.

Definition

binomial probability

When the outcomes of an experiment are divided into two events, the probabilities of the events are called binomial probabilities.

Remember that the prefix 'bi' means two.

Let us look at some examples of binomial probabilities.

EXAMPLE 43 What is the probability of obtaining exactly two tails on six tosses of a fair coin?

Solution Suppose that tossing tails counts as a success and tossing heads counts as a failure. In order to calculate the required probability, we need to determine which two tosses are successful.

We can choose the two tosses in $\binom{6}{2}$ ways. We then calculate the probability of each success as $\frac{1}{2}$ and each failure as $\frac{1}{2}$.

So the probability of exactly two tails in six tosses is $\binom{6}{2}\underbrace{\left(\frac{1}{2}\right)^2}_{successes}\underbrace{\left(\frac{1}{2}\right)^4}_{failures}$.

This is one example of calculating a binomial probability. We can generalize the rule for binomial probability as follows:

Rule

Let n be the number of trials in an experiment and let s be the probability of success in a trial and f be the probability of failure in a trial such that $f = 1 - s$.

Then the probability of x successes in n trials is $P(x) = \binom{n}{x} \cdot s^x \cdot f^{n-x}$.

EXAMPLE 44 A die is rolled five times. What is the probability that a number greater than 4 comes up twice?

Solution We can group the outcomes into two events: rolling a number greater than 4 and rolling a number less than 4. So we are working with binomial probability. The probability of obtaining a number greater than 4 is $\frac{2}{6} = \frac{1}{3}$. So by the rule above, the answer is $\binom{5}{2} \cdot \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^3$.

EXAMPLE 45 The probability that a marksman hits a target is $\frac{2}{3}$. What is the probability that he hits the target at least 8 times in 10 trials?

Solution The desired probability involves three events: hitting the target 8 times, 9 times and 10 times. So we want to find the probability that the marksman hits the target 8 times or 9 times or 10 times.

Thus the answer is

$$\binom{10}{8}\left(\frac{2}{3}\right)^8\left(\frac{1}{3}\right)^2+\binom{10}{9}\left(\frac{2}{3}\right)^9\left(\frac{1}{3}\right)+\binom{10}{10}\left(\frac{2}{3}\right)^{10}.$$

EXAMPLE 46 Orlando Magic and the NY Knicks are two popular basketball teams. Orlando Magic player Hidayet Türkoğlu scores with 90.625% of his free throws. In a game at the end of the season the score is Orlando Magic 96 - NY Knicks 97. Hidayet has the opportunity to take 3 free throws. What is the probability that Orlando Magic wins the match?

Solution Orlando Magic will win the match if Hidayet scores at least twice. Therefore the answer is

$$P(E)=\underbrace{\binom{3}{2}\cdot\left(\frac{90}{100}\right)^2\cdot\left(\frac{10}{100}\right)}_{\substack{2\text{ successes}\\1\text{ failure}}}+\underbrace{\binom{3}{3}\cdot\left(\frac{90}{100}\right)^3}_{3\text{ successes}}$$

$$=\frac{972}{1000}$$

$$=0.972.$$

Check Yourself 6

1. A coin is flipped 5 times. What is the probability of obtaining exactly 2 tails?
2. A die is rolled 4 times. What is the probability of obtaining a number divisible by 3 exactly twice?
3. A and B play a game with a die. In the game, a player gets a point if he or she rolls a six. A and B roll the die 6 times each. A has 4 points at the end of the game. Find the probability that B wins the game.

Answers

1. $\frac{5}{16}$ 2. $\frac{8}{27}$ 3. $\frac{31}{6^6}$

EXERCISES 2.6

1. A die is rolled four times. What is the probability of rolling exactly one 4?

2. An exam is made up of 15 true-or-false questions. A student who hasn't studied for the exam answers all 15 questions by just guessing. What is the probability that the student correctly answers 6 questions?

3. The probability that a marksman hits a target is $\frac{1}{4}$. What is the probability that the marksman hits the target only once, on his six shot?

4. A coin is flipped 5 times. What is the probability of obtaining at least one head?

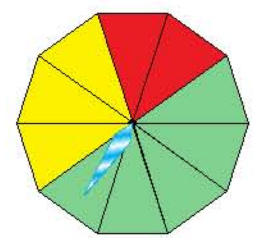
5. A die is rolled 4 times. What is the probability of rolling at least one 5?

6. A student takes a test which is made up of 20 multiple-choice questions. There are 5 choices for each question. If the student guesses all of the answers, what is the probability that she gets exactly 8 correct answers?

7. Adam and Brian are playing a game in which Adam holds five matchsticks and Brian picks one. If Brian picks the shortest matchstick he wins the game. Otherwise Adam wins it. They play the game 6 times. Find the probability that Brian wins the game exactly twice.

8. When a brick is released from a height of one meter, the probability that the brick will break into pieces is $\frac{5}{7}$. Seven bricks are dropped. Find the probability that exactly two bricks are not broken.

9. The figure shows a spinner which is divided into equal parts. Find the probability that the spinner will stop on yellow 3 times if it is spun 10 times.



10. A coin is flipped 7 times. What is the probability of obtaining at least 2 heads?

11. A bag contains 5 red and 6 yellow balls. A ball is drawn and then replaced. This is repeated 6 times. Find the probability of each event.
    a. the first five balls are red and the last one is yellow
    b. exactly five of the balls are red
    c. at least one ball is red
    d. at least five of the balls are red

CHAPTER SUMMARY

- An experiment is an activity or a process which has observable results.
- The observable results of an experiment are called outcomes.
- The set of all possible outcomes of an experiment is called the sample space for the experiment.
- An event is a subset of (or a part of) a sample space.
- The union of two events A and B is the set of all outcomes in A and/or B. It is denoted by $A \cup B$.
- The intersection of two sets A and B is the set of all outcomes in both A and B. It is denoted by $A \cap B$.
- The complement of an event A is the set of all outcomes in the sample space that are not in event A. It is denoted by A' or A^c.
- If two events have no outcome in common, they are called mutually exclusive events.
- The probability of an event E is $P(E) = \dfrac{n(E)}{n(S)}$, where $n(E)$ is the number of outcomes in event E and $n(S)$ is the number of outcomes in the sample space S.
- An event whose probability is 1 is called a certain event.
- An event whose probability is zero is called an impossible event.
- For two mutually exclusive events A and B, $P(A \cup B) = P(A) + P(B)$.
- For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- For any event A, $P(A) + P(A') = 1$. In other words, $P(A') = 1 - P(A)$.
- The conditional probability that an event A will occur given that an event B has occurred $(P(B) \neq 0)$ is $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.
- Two events A and B are said to be independent if the outcomes in A do not affect the outcomes in B. If A and B are independent, $P(A \cap B) = P(A) \cdot P(B)$
- When the outcomes of an experiment can be divided into two events, the probabilities of the events are called binomial probabilities.

Concept Check

- Why does the probability of an event take a value between zero and 1?
- Which set of six numbers is more likely to win the lottery: {1, 2, 3, 4, 5, 6} or {3, 9, 16, 25, 28, 41}? Explain your answer.
- Explain why the following statement is false: 'The probability that a bus will arrive late at Istanbul is 0.32, and the probability that it will be on time or early is 0.66.'
- Explain what it means to say that two events are independent.
- Explain the difference between mutually exclusive events and independent events.

CHAPTER REVIEW TEST 2A

1. Two dice are rolled and the numbers are added together. What is the probability that the sum is greater than 9?

A) $\frac{1}{4}$ B) $\frac{3}{11}$ C) $\frac{1}{3}$ D) $\frac{5}{12}$ E) $\frac{1}{6}$

2. A number is drawn at random from the set $\{1, 2, 3, ..., 150\}$. What is the probability that the number is divisible by 4?

A) $\frac{31}{150}$ B) $\frac{16}{75}$ C) $\frac{33}{150}$ D) $\frac{6}{25}$ E) $\frac{37}{150}$

3. A bag contains 4 green balls, 3 red balls and 3 black balls. A ball is picked at random. What is the probability that it is not black?

A) $\frac{4}{5}$ B) $\frac{7}{10}$ C) $\frac{3}{5}$ D) $\frac{1}{2}$ E) $\frac{2}{5}$

4. Two dice are rolled and the numbers are added together. What is the probability that the sum is prime or odd?

A) $\frac{5}{12}$ B) $\frac{1}{2}$ C) $\frac{11}{12}$ D) $\frac{17}{36}$ E) $\frac{19}{36}$

5. A number is drawn at random from the set $\{2, 3, ..., 50\}$. What is the probability that the selected number leaves remainder 1 when it is divided by 5?

A) $\frac{1}{7}$ B) $\frac{8}{49}$ C) $\frac{9}{49}$ D) $\frac{10}{49}$ E) $\frac{11}{49}$

6. A card is drawn from a well-shuffled deck of 52 cards and replaced. Then another card is drawn. What is the probability of drawing a red card?

A) $\frac{1}{2}$ B) $\frac{1}{52}$ C) $\frac{1}{13}$ D) $\frac{1}{4}$ E) 1

7. A card is drawn from a standard deck of 52 cards and replaced. Then another card is drawn. What is the probability that the first card is a diamond and the second card is a king?

A) $\frac{1}{4}$ B) $\frac{1}{13}$ C) $\frac{17}{52}$ D) $\frac{4}{13}$ E) $\frac{1}{52}$

8. A box contains 4 green balls, 6 pink balls and 5 red balls. Two balls are drawn successively from the box, without replacement. What is the probability of drawing a red ball and a green ball, in any order?

A) $\frac{1}{21}$ B) $\frac{2}{21}$ C) $\frac{4}{21}$ D) $\frac{8}{11}$ E) $\frac{16}{11}$

9. A pair of dice is rolled. What is the probability that the sum of the numbers showing is 8 if it is known that the numbers are different?

A) $\frac{1}{10}$ B) $\frac{2}{15}$ C) $\frac{1}{6}$ D) $\frac{1}{5}$ E) $\frac{7}{30}$

10. Nine people are seated around a circular table. What is the probability that any two people selected at random are sitting next to each other?

A) $\frac{1}{2}$ B) $\frac{1}{4}$ C) $\frac{1}{8}$ D) $\frac{1}{16}$ E) $\frac{2}{9}$

11. Three numbers are drawn at random from the set $\{1, 2, 3, \ldots, 40\}$. What is the probability that the product of the drawn numbers is an odd number?

A) $\frac{3}{26}$ B) $\frac{6}{13}$ C) $\frac{9}{13}$ D) $\frac{3}{78}$ E) $\frac{1}{13}$

12. Two cards are drawn at random from a standard deck of 52 playing cards. What is the probability of drawing two kings?

A) $\frac{1}{13}$ B) $\frac{1}{17}$ C) $\frac{1}{52}$ D) $\frac{1}{221}$ E) $\frac{1}{51}$

13. A box contains 8 red beads and 6 green beads. Four beads are drawn one by one without replacement. What is the probability of drawing 2 red beads followed by 2 green beads?

A) $\frac{50}{143}$ B) $\frac{40}{143}$ C) $\frac{30}{143}$ D) $\frac{20}{143}$ E) $\frac{10}{143}$

14. A pair of dice are rolled 4 times. What is the probability of obtaining a double (i.e. the same number on both dice) on all four rolls?

A) $\frac{1}{6}$ B) $\frac{1}{36}$ C) $\frac{1}{6^4}$ D) $\frac{1}{3^4}$ E) $\frac{1}{2^4}$

15. Five dice are rolled together. What is the probability that all of them show a different number?

A) $\frac{1}{36}$ B) $\frac{5}{54}$ C) $\frac{5}{324}$ D) $\frac{25}{216}$ E) $\frac{25}{36}$

16. A die is rolled 8 times. What is the probability of rolling a 5 exactly three times?

A) $\binom{8}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^5$ B) $\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^5$

C) $\binom{8}{3}\left(\frac{3}{6}\right)^3\left(\frac{3}{6}\right)^5$ D) $\left(\frac{3}{6}\right)^3\left(\frac{3}{6}\right)^5$

E) $\binom{8}{3}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^4$

CHAPTER REVIEW TEST 2B

1. A card is drawn from a well-shuffled deck of 52 cards. What is the probability of drawing a red queen?

A) $\frac{1}{4}$ B) $\frac{1}{13}$ C) $\frac{1}{16}$ D) $\frac{1}{26}$ E) $\frac{1}{52}$

2. A pencil is drawn at random from a bag which contains 4 red pencils, 5 white pencils and 7 green pencils. What is the probability of drawing a red pencil or a white pencil?

A) $\frac{1}{16}$ B) $\frac{5}{16}$ C) $\frac{1}{4}$ D) $\frac{7}{16}$ E) $\frac{9}{16}$

3. A pair of dice is rolled. What is the probability that at least one of the dice shows a six?

A) $\frac{5}{36}$ B) $\frac{1}{6}$ C) $\frac{7}{36}$ D) $\frac{2}{9}$ E) $\frac{11}{36}$

4. Two cards are drawn one after the other from a well-shuffled deck of 52 cards. What is the probability that at least one card is black?

A) $\frac{77}{102}$ B) $\frac{73}{102}$ C) $\frac{51}{102}$ D) $\frac{25}{102}$ E) $\frac{1}{26}$

5. A computer program generates a list of all the possible four-digit numbers that can be formed from the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$. A number is chosen at random from the list. What is the probability that the chosen number is not divisible by 5?

A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{1}{2}$ D) $\frac{3}{4}$ E) $\frac{4}{5}$

6. A number is drawn at random from the set $\{1, 2, 3, ..., 90\}$. What is the probability that the drawn number is divisible by 3 if it is known that the number is an even number?

A) $\frac{1}{5}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{2}{3}$ E) $\frac{3}{4}$

7. Two balls are drawn from a bag which contains 4 white balls and 6 red balls. What is the probability that both balls are the same color?

A) $\frac{1}{3}$ B) $\frac{2}{15}$ C) $\frac{7}{30}$ D) $\frac{41}{90}$ E) $\frac{7}{15}$

8. Two numbers are drawn at random from a set of 5 different odd numbers and 7 different even numbers. What is the probability that the product of the drawn numbers is an even number?

A) $\frac{28}{33}$ B) $\frac{35}{66}$ C) $\frac{7}{22}$ D) $\frac{14}{33}$ E) $\frac{7}{33}$

9. An urn contains 5 red marbles, 3 brown marbles and 4 green marbles. Three marbles are drawn successively from the urn without replacement. What is the probability of drawing a red marble, a green marble and then another a red marble, in that order?

A) $\frac{1}{33}$ B) $\frac{2}{33}$ C) $\frac{1}{11}$ D) $\frac{4}{33}$ E) $\frac{5}{33}$

10. A box contains 3 yellow balls, 4 red balls and 2 blue balls. Three balls are drawn at random. What is the probability of drawing a ball of each color?

A) $\frac{5}{7}$ B) $\frac{4}{7}$ C) $\frac{3}{7}$ D) $\frac{2}{7}$ E) $\frac{1}{7}$

11. Five numbers are drawn at random from the set $\{1, 2, 3, \ldots, 30\}$. What is the probability that the greatest number is 23 and the smallest number is 15?

A) $\frac{1}{10}$ B) $\frac{7}{30}$ C) $\frac{\binom{9}{3}}{\binom{30}{5}}$ D) $\frac{\binom{7}{3}}{\binom{30}{5}}$ E) $\frac{\binom{9}{5}}{\binom{30}{5}}$

12. A *palindromic number* is a number that reads the same forward as backward. For example, 2, 4, 55, 33, 121, 343 and 707 are palindromic numbers. A number is drawn from the set $\{1, 2, 3, \ldots, 1000\}$. What is the probability that it is a palindromic number?

A) $\frac{3}{125}$ B) $\frac{9}{250}$ C) $\frac{27}{250}$ D) $\frac{27}{500}$ E) $\frac{27}{1000}$

13. A box contains 4 brown beads, 3 red beads and 6 green beads. A bead is drawn and replaced. Then another bead is drawn from the box. What is the probability that both of the drawn beads are the same color?

A) $\frac{2}{13}$ B) $\frac{3}{13}$ C) $\frac{61}{169}$ D) $\frac{71}{169}$ E) $\frac{4}{13}$

14. Seven people support one of 10 basketball teams in a league. What is the probability that each of the 7 people supports a different team?

A) $\frac{\binom{10}{7}\cdot 7!}{10^7}$ B) $\frac{\binom{10}{7}}{10^7}$ C) $\frac{1}{\binom{10}{7}\cdot 7!}$

D) $\frac{1}{\binom{10}{7}}$ E) $\frac{1}{10^7}$

15. Four different people write their names on a small piece of paper and put the papers in a box. Then each person takes a name from the box. What is the probability that they all get their own names?

A) $\frac{1}{4}$ B) $\frac{1}{12}$ C) $\frac{1}{24}$ D) $\frac{5}{48}$ E) $\frac{1}{48}$

16. A die is rolled ten times. What is the probability of rolling a 3 at most twice?

A) $\binom{10}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^9$

B) $\left(\frac{5}{6}\right)^{10}+\binom{10}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^9+\binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8$

C) $\binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8$

D) $\binom{10}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^9\cdot\binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8$

E) $\binom{10}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^9+\binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8+\binom{10}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^7$

ANSWERS TO EXERCISES

EXERCISES 1.1

1. 12 2. 10 ways

		Windows				
		1	2	3	4	5
Doors	A	1A	2A	3A	4A	5A
	B	1B	2B	3B	4B	5B

3. 4 5. 12 6. 6 7. $\{MMM, MMF, MFM, MFF, FMM, FMF, FFM, FFF\}$

8. 12 9. 13 10. 4 11. 624 12. 4^6 13. 24 14. 6 15. 14 16. 4^{12} 17. a. 4^{12} b. 5^{12} 18. 240 19. 28800

20. $26^4 = 456976$ 21. 5000 22. 4 000 000 23. a.10 b. $2^{10} = 1028$ 24. 5850 25. 74360 26. No

27. No 28. a. 25 b. 10 29. 1944 30. 48 31. 135 32. 159 33. 8500 34. 12 35. a. 1792

b. 320 36. a. 140 b. 36 37. 9990000 38. $36^7 - 26^7 - 10^7 = 70322353920$ 39. $5^2(26 + 26^2 + 26^3) = 456950$

40. Hint: Apply to the pigeonhole principle

EXERCISES 1.2

1. $\dfrac{24!}{18!}$ 2. $\dfrac{43}{7}$ 3. $\dfrac{16}{3}$ 4. a. $n^2 - n$ b. $n + 1$ c. $\dfrac{8n+4}{n+2}$ 5. a. 16 b. 6 c. 7 6. $(n + 1)! - 1$ 7. $g^{-1} = \begin{pmatrix} \triangle & \square & \bigcirc & \star \\ \bigcirc & \triangle & \star & \square \end{pmatrix}$

8. $\begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix}$ 9. $\begin{pmatrix} 1\,2\,3\,4 \\ 4\,3\,2\,1 \end{pmatrix}$ 10. $\begin{pmatrix} a\,b\,c\,d\,e \\ e\,c\,b\,a\,d \end{pmatrix}$ 11. $5! = 120$ 12. 720 13. 120 14. 5040 15. $5! \cdot 5! \cdot 2! = 28800$

16. a. 110 b. 81 c. $n - 3$ d. 3 e. $\dfrac{1}{42}$ 17. a. 2 b. 6 c. 9 d. 3 18. 336 19. $P(10, 7) = 604800$

20. $P(10, 3) = 720$ 21. $P(7, 3) = 210$ 22. $P(7, 1) + P(7, 2) + P(7, 3) + P(7, 4) + P(7, 5) = 3619$

23. $P(26, 1) + P(26, 2) + P(26, 3) = 16276$ 24. $2! \cdot 3! = 12$ 25. $6^4 - P(6, 4) = 576$ 26. a. 120

b. $4 \cdot P(6, 3) = 480$ 27. $4 \cdot 5! = 480$ 28. $2 \cdot P(26, 4) + P(26, 5)$ 29. $2 \cdot 3! + 3 \cdot 3! = 30$

30. $P(7, 4) \cdot 2^4$ 31. $5! \cdot 2! = 240$ 32. a. $17!$ b. $3! \cdot 4! \cdot 6! \cdot 7! = 522547200$ 33. $6! - 5! \cdot 2! = 480$

34. $(2! \cdot 3! \cdot 5!) + (3! \cdot 3! \cdot 4!) = 2304$ 35. $\dfrac{9!}{3! \cdot 4! \cdot 2!} = 1260$ 36. $7! = 5040$ 37. 75599

38. $\dfrac{13!}{3! \cdot 4! \cdot 6!} = 60060$ 39. 540 40. 50 41. $\dfrac{20!}{9! \cdot 2! \cdot 2! \cdot 2! \cdot 2! \cdot 2!}$ 42. 60 43. 126

44. $\left(\dfrac{8!}{6!\cdot 2!}\right)\left(\dfrac{4!}{2!\cdot 2!}\right)=168$ 45. a. $\dfrac{12!}{5!}$ b. $3!\cdot 4!\cdot 3!$ 46. $4!=24$ 47. $3!\cdot 5!=720$ 48. $23!$

49. $4!\cdot 5!=2880$ 50. $2!\cdot 4!\cdot 4!=1152$ 51. $5!\cdot 3!=720$ 52. $6!=720$ 53. $\dfrac{8!}{3!\cdot 4!\cdot 2!}=140$ 54. 12

55. 360 56. 4 57. $\dfrac{8!}{2!\cdot 2!}\cdot\dfrac{4!}{2!}=6\cdot 8!$ 58. $21!\cdot 2^{22}$ 59. $5\cdot 4^4=1280$ 60. $2\cdot 18!$

EXERCISES 1.3

1. a. 6 b. 71 c. 24 d. $2^9=512$ e. $2^{12}=4096$ 2. a. 12 b. $\dfrac{n-2}{6}$ c. n^2-n 3. a. 6 b. 9 c. 9

4. $\{(k, l, m), (k, l, n), (k, l, r), (k, m, n), (k, m, r), (k, n, r), (l, m, n), (l, n, r), (l, m, r), (m, n, r)\}$ 5. 64 6. 10

7. 60 8. 35 9. 78 10. 45 11. 20 12. 10 13. $\dbinom{16}{4}=1820$ 14. $\dbinom{48}{12}=69668534468$

15. $\dbinom{40}{4}=91390$ 16. $\dbinom{11}{4}=330$ 17. $\dbinom{3}{2}+\dbinom{3}{1}+\dbinom{3}{1}=9$ 18. $\dbinom{12}{8}\cdot\dbinom{8}{3}=27720$ 19. 45 20. $\dbinom{5}{2}\cdot\dbinom{5}{2}\cdot 3!=600$

21. $\dbinom{12}{5}\cdot\dbinom{10}{4}\cdot\dbinom{9}{2}$ 22. $\dbinom{7}{2}=21$ 23. $\dbinom{7}{5}$ or $\dbinom{3}{1}\dbinom{4}{4}+\dbinom{3}{2}\dbinom{4}{3}+\dbinom{3}{3}\dbinom{4}{2}=21$ 24. $\dbinom{8}{4}-\dbinom{5}{4}=65$

25. a. $\dbinom{5}{3}\dbinom{7}{4}=350$ b. $\dbinom{7}{5}\dbinom{5}{2}+\dbinom{7}{6}\dbinom{5}{1}+\dbinom{7}{7}=246$ 26. $\dbinom{5}{3}+\dbinom{5}{2}+\dbinom{5}{1}+\dbinom{5}{0}=26$ 27. 21

28. a. 29 b. $\dbinom{9}{3}-\dbinom{3}{3}-\dbinom{4}{3}=79$ 29. $\dbinom{7}{2}\cdot\dbinom{4}{2}=126$ 30. $\dbinom{8}{3}+\dbinom{7}{3}+3=94$ 31. 45045 32. 600

33. $\dbinom{26}{4}=14950$ 34. $\dbinom{21}{3}=1330$ 35. $\dbinom{2010}{4}=678072034710$ 36. $\dbinom{49}{5}=1906884$

37. $\dbinom{29}{5}\cdot\dbinom{37}{5}=7843173975$ 38. $\dbinom{19}{11}=75582$ 39. a. $\dbinom{19}{15}=3876$ b. $\dbinom{35}{15}=3247943160$ 40. 4

41. $\dbinom{6}{4}\cdot 4!=360$ 42. $P(6, 4)=360$ 43. $\dbinom{5}{3}\cdot P(5,3)=600$ 44. $\dbinom{7}{5}\cdot 4!=504$ 45. a. $\dbinom{5}{4}\cdot\dbinom{6}{4}\cdot 8!$

b. $P(5, 4)\cdot P(6, 4)\cdot 2!=86400$ 46. $\dbinom{5}{2}\cdot 3^4=810$ 47. $\dbinom{3}{1}\cdot\dbinom{6}{1}\cdot\dbinom{5}{1}\cdot\dbinom{18}{3}=146880$

48. $6^5\cdot 4!\cdot 2^5=5971968$ 49. $\dbinom{12}{6}\cdot 5!=110880$ 50. $\dbinom{7}{3}+\dbinom{7}{2}=56$ 51. 15 52. $\dbinom{9}{3}\cdot 3!\cdot 2!=1008$

53. $\binom{5}{3}+\binom{5}{4}+\binom{5}{5}=16$ 54. $\binom{6}{2}\binom{4}{2}\cdot 3!\cdot 2^4 = 40320$ 55. $\binom{9}{6}\cdot 6!\cdot 2!$ 56. $\frac{8!}{3!}-6=6714$ 57. $\binom{3}{2}\cdot 3!=18$

58. $\binom{7}{3}\cdot\binom{5}{3}\cdot\binom{6}{3}\cdot(3!)^5$ 59. a. $20!$ b. $\binom{5}{1}\binom{4}{3}\cdot 3!\cdot 17!$

EXERCISES 1.4

1. a. $243x^5 + 2025x^4 + 6750x^3 + 11250x^2 + 9375x + 3125$ b. $16x^8 - 160x^6 + 600x^4 - 1000x^2 + 625$

c. $1024x^{15} - 3840x^{12}y^2 + 5760x^9y^4 - 4320x^6y^6 + 1620x^3y^8 - 243y^{10}$ 2. -243 3. $600x^2y^2$

4. a. 625 b. 5^{15} 5. $\binom{15}{11}(3x)^4$ 6. $\binom{8}{5}(3x^4)^3\cdot(y^3)^5$ 7. 240 8. 5^9 9. 3360 10. 252

EXERCISES 2.1

1. a. $\{TTT, HHH\}$ b. $\{TTH, THT, HTT, TTT\}$ 2. a. $\{(1, 2), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

b. $\{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4), (3, 6), (6, 3), (4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)\}$

c. $\{(1, 3), (3, 1), (1, 5), (5, 1), (3, 5), (5, 3), (1, 1), (3, 3), (5, 5)\}$ 3. $\frac{4}{7}$ 4. $\frac{1}{3}$ 5. $\frac{4}{7}$ 6. $\frac{33}{100}$ 7. $\frac{7}{12}$

8. $\frac{5}{12}$ 9. $\frac{1}{4}$ 10. $\frac{7}{8}$ 11. $\frac{3}{4}$ 12. $\frac{1}{4}$

EXERCISES 2.2

1. $\frac{4}{7}$ 2. $\frac{7}{9}$ 3. $\frac{5}{6}$ 4. 1 5. $\frac{2}{13}$ 6. $\frac{2}{9}$ 7. $\frac{4}{7}$ 8. a. $\frac{1}{3}$ b. $\frac{2}{3}$ 9. $\frac{7}{15}$ 10. $\frac{17}{25}$ 11. $\frac{9}{14}$ 12. $\frac{9}{10}$

13. $\frac{8}{13}$ 14. $\frac{8}{15}$ 15. $1-\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{12^7}\cong 0.9$ A sticky problem (page 114): 40%

EXERCISES 2.3

1. $\frac{1}{2}$ 2. $\frac{2}{9}$ 3. $\frac{1}{63}$ 4. $\frac{1}{4}$ 5. $\frac{3}{4}$ 6. $\frac{13}{28}$ 7. $\frac{8}{11}$ 8. $\frac{1}{\binom{42}{4}} = \frac{1}{111930}$ 9. $\frac{5!\cdot 4!}{9!}$ 10. 150 11. $\frac{1}{6^3}$

12. $\frac{\binom{32}{1}\binom{168}{3}}{\binom{200}{4}} = \frac{12419456}{32342475}$ 13. $\frac{1}{60}$ 14. $\frac{3}{7}$ 15. $1-\frac{\binom{4}{4}}{\binom{12}{4}}$ 16. $\frac{3}{7}$ 17. a. $\frac{\binom{210}{4}}{\binom{300}{4}} = \frac{532}{2235}$

b. $\frac{\binom{210}{2}\binom{90}{2}}{\binom{300}{4}} = \frac{11837}{44551}$ c. $\frac{\binom{210}{1}\binom{90}{3}}{\binom{300}{4}} = \frac{9968}{133653}$ d. $\frac{\binom{90}{4}}{\binom{300}{4}} = \frac{5162}{668265}$ 18. $\frac{1}{88^6}$ 19. $\frac{7}{18}$ 20. $\frac{\binom{7}{3}}{\binom{10}{3}} = \frac{7}{24}$

21. $\frac{\binom{4}{1}\binom{5}{1}\binom{4}{1}+\binom{4}{1}\binom{5}{2}+\binom{4}{1}\binom{4}{2}}{\binom{13}{3}} = \frac{72}{143}$ 22. $1-\frac{\binom{10}{3}}{\binom{15}{3}} = \frac{67}{91}$ 23. $1-\frac{\binom{7}{3}}{\binom{8}{4}} = \frac{1}{2}$ 24. $\frac{\binom{7}{3}\binom{4}{2}}{\binom{8}{4}\binom{5}{3}} = \frac{3}{10}$

25. $\frac{\binom{15}{6}\binom{5}{4}+\binom{15}{7}\binom{5}{3}+\binom{15}{8}\binom{5}{2}+\binom{15}{9}\binom{5}{1}+\binom{15}{10}}{\binom{20}{10}}$ 26. $\frac{\binom{10}{5}\cdot 6!\cdot 2!}{\binom{12}{7}\cdot 7!} = \frac{1}{11}$ 27. $\frac{3\cdot\binom{5}{3}+\binom{5}{1}\binom{5}{1}\binom{5}{1}}{\binom{15}{3}} = \frac{31}{91}$

28. $\frac{99}{\binom{100}{2}} = \frac{1}{50}$ 29. $1-\frac{\binom{5}{4}}{\binom{11}{4}} = \frac{65}{66}$ 30. $\frac{\binom{6}{4}\cdot 2^4}{\binom{12}{4}} = \frac{16}{33}$ 31. $1-\frac{64}{\binom{9}{2}\binom{9}{2}} = \frac{77}{81}$ 32. $\frac{50}{\binom{100}{2}} = \frac{1}{99}$

EXERCISES 2.4

1. a. $\frac{1}{30}$ b. $\frac{11}{30}$ c. $\frac{1}{2}$ 2. $\frac{2}{3}$ 3. $\frac{12}{25}$ 4. $\frac{2}{15}$ 5. $\frac{5}{11}$ 6. $\frac{8}{95}$ 7. $\frac{9}{100}$ 8. $\frac{2}{5}$ 9. $\frac{4}{9}$ 10. $\frac{1}{2}$

11. $\frac{1}{5}$ 12. $\frac{1}{2}$ 13. a. $\frac{49}{1000}$ b. $\frac{559}{20000}$ 14. $\frac{5}{42}$ 15. $\frac{16}{31}$

EXERCISES 2.5

1. $\frac{1}{4}$ 2. $\frac{1}{4}$ 3. $\frac{1}{3}$ 4. $\frac{1}{3}$ 5. $\frac{35}{64}$ 6. $\frac{14}{55}$ 7. $\frac{1}{825}$ 8. $\frac{1}{15}$ 9. $\frac{59}{112}$ 10. $\frac{1}{7^3}$ 11. $1-\frac{99\cdot 98\cdot 97\cdot 96}{100^4}$ 12. $\frac{3}{4}$

13. $\frac{31}{35}$ 14. $\frac{1}{6}$ 15. $\frac{5}{8}$ 16. $\frac{5^5}{6^6}=\frac{3125}{46656}$ 17. $\frac{7}{12}$ 18. $\frac{26}{49}$ 19. $\frac{65}{121}$ 20. $\frac{1}{12}$ 21. $\frac{247}{504}$

EXERCISES 2.6

1. $\frac{4\cdot 5^3}{6^4}=\frac{125}{324}$ 2. $\binom{15}{6}\frac{1}{2^{15}}=\frac{5005}{32768}$ 3. $\frac{3^6}{2^{11}}$ 4. $\frac{31}{32}$ 5. $1-\left(\frac{5}{6}\right)^4=\frac{671}{1296}$ 6. $\binom{20}{8}\frac{4^{12}}{5^{20}}$ 7. $\binom{6}{2}\frac{4^4}{5^6}$ 8. $\binom{7}{2}\frac{2^2\cdot 5^5}{7^7}$

9. $\binom{10}{3}\left(\frac{3}{10}\right)^3\left(\frac{7}{10}\right)^7$ 10. $\frac{15}{16}$

11. a. $\frac{5^5\cdot 6}{11^6}=\frac{18750}{1771561}$ b. $\frac{6^2\cdot 5^5}{11^6}=\frac{112500}{1771561}$ c. $1-\left(\frac{6}{11}\right)^6=\frac{1724905}{17711561}$ d. $\frac{6^2\cdot 5^5+5^6}{11^6}=\frac{128125}{1771561}$

ANSWERS TO TESTS

TEST 1A

1.	E	9.	D
2.	D	10.	A
3.	C	11.	C
4.	D	12.	E
5.	B	13.	D
6.	A	14.	A
7.	B	15.	B
8.	E	16.	C

TEST 1B

1.	A	9.	E
2.	A	10.	B
3.	E	11.	D
4.	C	12.	D
5.	A	13.	E
6.	D	14.	B
7.	C	15.	D
8.	A	16.	E

TEST 2A

1.	E	9.	B
2.	E	10.	B
3.	B	11.	A
4.	E	12.	D
5.	C	13.	E
6.	A	14.	C
7.	E	15.	B
8.	C	16.	A

TEST 2B

1.	D	9.	B
2.	E	10.	D
3.	E	11.	D
4.	A	12.	C
5.	D	13.	C
6.	C	14.	A
7.	E	15.	C
8.	A	16.	B

GLOSSARY

A

addend: a number that is added to another number.

associative property: An operation is associative if you can group the numbers in the operation in any way without changing the answer. Addition and multiplication are both associative, since $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$.

B

binomial expansion: the set of terms in the expansion of a binomial expression which is raised to a particular power.

binomial expression: a mathematical expression which is the sum or difference of two terms.

bit: a basic unit of computer data that can take one of two values, such as 0 for false and 1 for true.

byte: a sequence of 8 bits which is processed as a single unit of information.

C

coefficient: a number which is multiplied by a variable in an algebraic expression.

collinear: If two or more points are collinear, they are on the same straight line.

combination: a selection of objects chosen from a group without considering the order.

combination lock: a lock which can be opened only by turning a set of dials in a special sequence.

commutative property: An operation is commutative if you can change the order of the numbers in the operation without changing the result. Subtraction is not commutative, since $3 - 5 \neq 5 - 3$.

complement of an event: the set of all outcomes in a sample space that are not in the event.

conditional probability: the probability of an event occurring if another related event has already occurred.

constant term: a term in an expression that does not change with the variable.

D

defective: if something is defective, it does not work properly.

dependent events: If the occurrence of an event A affects the probability of another event B then A and B are dependent events.

die (plural: dice): a small cube with spots on each face that show the numbers from 1 to 6.

dominant: In genetics, a dominant gene has priority over other genes.

E

equally likely: If the probabilities of two simple events are the same, then the events are said to be equally likely.

event: a set of possible outcomes resulting from a particular experiment.

experiment: an activity or process which has observable results. Rolling a die is an experiment.

F

factorial: the product of a given integer and all smaller positive integers, shown with an exclamation mark (!). For example, $3! = 3 \cdot 2 \cdot 1 = 6$.

H

head (heads): the side of a coin which often has a picture of a person's head on it. The opposite side is called tails.

I

identity function: a function whose output is identical to its input.

independent event: Two events are independent if the outcome of one event does not affect the outcome of the other.

intersection: the set of elements which are common to two or more sets.

L

lottery: a drawing of tickets or numbers at random in which prizes are given to the winning tickets or numbers.

M

mutually exclusive events: events that cannot happen together at the same time.

O

outcome: an observable result of an experiment.

P

palindrome: a word or phrase that reads the same backward as forward. 12421 is a palindromic number.

permutation: an ordered arrangement of some or all of the elements in a given set.

prime number: a number that can only be divided by itself and 1.

probability: a measure of how likely it is that an event will occur.

R

random: If something is done or chosen at random it is done or chosen without any plan.

recessive: A recessive gene is a gene which is present in an organism but not dominant

S

sample space: a list of all the possible outcomes of an experiment.

shuffle: if you shuffle a pack of cards, you mix it to make a random order or arrangement.

spinner: a small object in the shape of a polygon with a pin or stick through its center, which is used to make a random choice.

subset: a set which is contained in another set

successive: successive objects or actions follow in order one after another.

systematic: if an activity is systematic, it is done according to a careful plan in a methodical way.

T

tail (tails): the reverse side of a coin that usually shows how much the coin is worth and which does not show the head of a person.

tree diagram: a schematic diagram that starts from a set of roots and shows all the possible outcomes of an event.

trial: a systematic opportunity for an event to occur.

U

uniformity criterion: The uniformity criterion states that any part of a task can be done in the same number of ways.

union: the set of elements which are in any of a group of sets.

urn: a large, decorative container which is sometimes used to hold lottery tickets or other items which are chosen at random.