Back to the Future

Two-Way Timed Automata

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2023

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Timed Automata

We already know what timed automata are:

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Goal for today: define and investigate two way timed automata and a related real time logic.

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- Clocks don't really 'elapse time', but rather store timestamps of letters
  - Machine reads through an 'event log'
- Notion of determinism: one action possible from a given configuration

Definition

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- Constraints are boolean combinations of $x \leq T + c$, for $x \in C$
- T is a special variable which takes on the value of the current timestamp
- c is an integer

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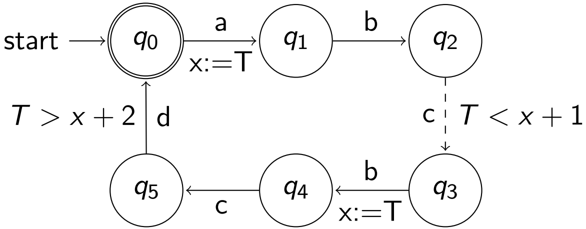
  - Move the tape head forward or backward as per d
- Computation stops when the automaton has no possible transitions to take
- Accept if in a final state, reject otherwise

Example 1



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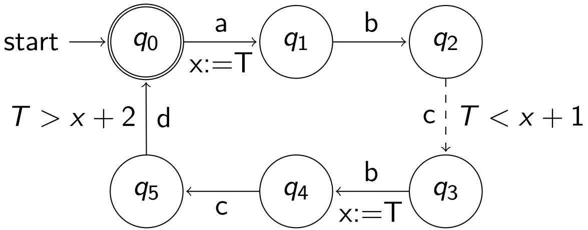

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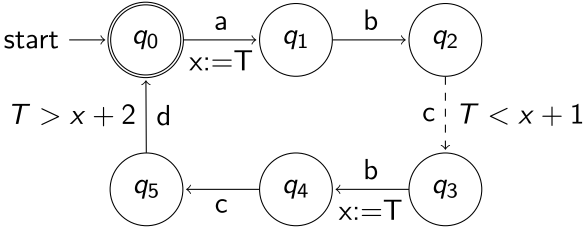

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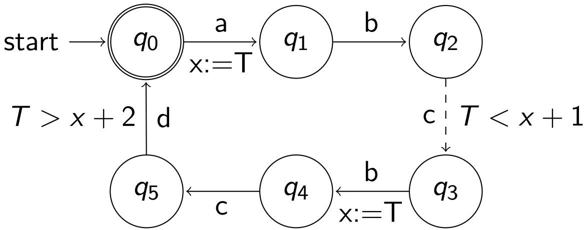

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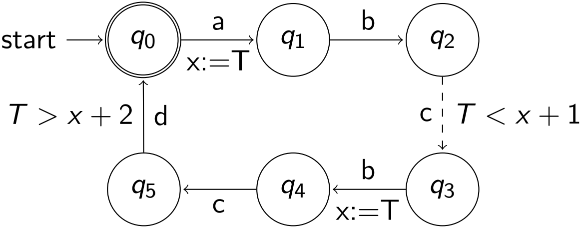

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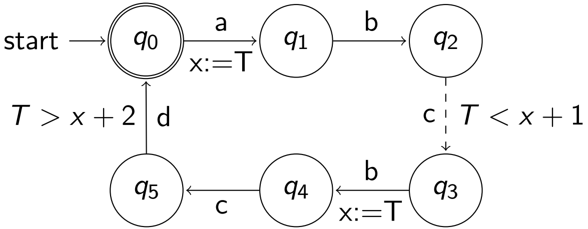

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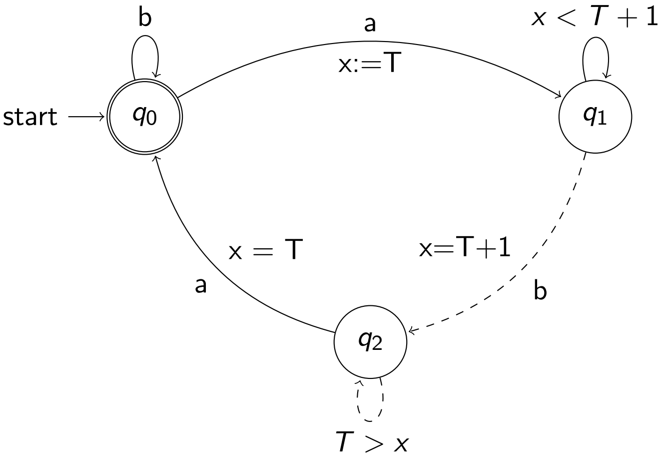
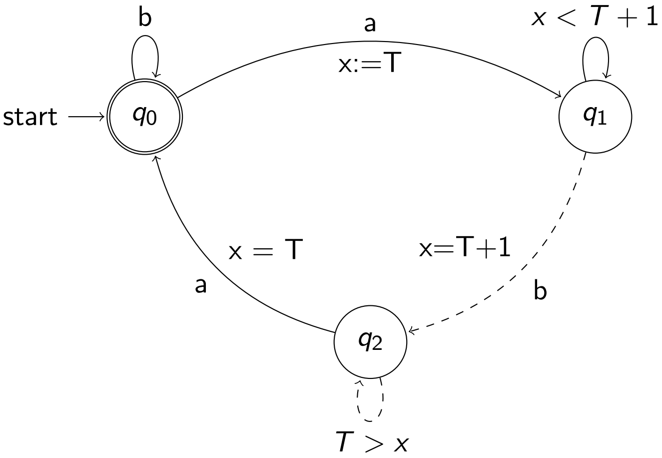
- Does it with only one clock!
- First checks the condition on c and a, then reuses the same clock to check the condition on b and d.
- Overall visits each symbol at most two times

Example 2



Example 2



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- Can use more clocks to ensure that no two events happen at the same timestamp
- This language is not accepted by NTAs (its untime is not regular)!
- Letters of the word are visited *arbitrarily many* times

Determinism

- Resets have to be deterministic; only allow resetting to the current timestamp

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Boundedness

Definition (Reversals, Bounded 2TAs)

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- The automaton of the first example was of type DTA_1
- The automaton of the second example is unbounded

Equivalence with the Standard Model

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- The point: NTAs store elapsed time information in clocks, $2NTA$s store timestamps
- Can convert one to the other by comparing with the current timestamp

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The major results:

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- $\cup_k NTA_k = NTA_0$: any bounded number of passes can be simulated by a single nondeterministic pass
- $\cup_k DTA_k$ is closed under boolean operations
- $NTA_0 \subsetneq 2NTA$: unbounded 2NTAs are stronger than bounded ones

Complexity

- Unbounded two-wayness: very bad
- The emptiness problem for $2DTA$s is undecidable
- Boundedness: desirable and harmless
- Emptiness (and therefore universality and inclusion) for DTA_ks are PSPACE-complete

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Motivation

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- For example: in the first two seconds of execution, every a is followed by a b 1 to 2 seconds later.

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- $\phi := p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \, \mathcal{U}_I \, \phi_2$
- I is a nonsingular interval with integer endpoints

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- Each letter represents the *world state* at the current time
- Each letter is a valuation from P to $\{\perp, \top\}$
- If a formula ϕ is true at the i^{th} symbol of a word w, we write $(w, i) \models \phi$

Semantics (contd.)

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  - $(w, k) \models \phi_1$ for all $i \leq k < j$

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    - $\tau_j \in \tau_i + I$
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A word w over models ϕ if $(w, 1) \models \phi$.

Examples!

- $p \, \mathcal{U}_{[1,2]} \, q$
- $\top \, \mathcal{U}_{[2,3)} \, (p \wedge q)$
- $p \, \mathcal{U}_{[1,2]} \, \left(q \, \mathcal{U}_{[1,2]} \, r\right)$

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  - $\tau_j \in \tau_i - I$
  - $(w, k) \models \phi_1$ for all $j < k \leq i$
- Note: this is simply a reversed 'until'!

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- This intuition is in fact what happens: alternations increase expressive power
- Let $MITL_k^P$ be the fragment of this extended logic which allows for k alternations between past and future operators
  - Note: This is not the exact definition, but close enough to convey the point

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Theorem (Alur, Henzinger)

- For every MITL formula ϕ, there exists a DTA_1 A_ϕ that accepts precisely the models of ϕ.
- $MITL_k^P \subset DTA_k$.

Thank You!

Vielen Dank!
Haben Sie Fragen?

Reference

These slides present the work in *Back to the Future: Towards a Theory of Timed Regular Languages* by R. Alur and T. Henzinger, published in 1992.