Exam Solution
Course: AE4870A Rocket Motion
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0 Introduction

This contains an exam solution. If you wish to contribute to this exam solution:

1. Create a github account, (you can create an "anonymous" one).
2. git clone ...
3. edit your changes in the document.
4. open cmd, and browse to inside the folder you downloaded and edited
5. git pull (updates your local repository=copy of folder, to the latest version in github cloud)
6. git status shows which files you changed.
7. git add "/some folder with a space/someFileYouChanged.tex"
8. git commit -m "Included solution to question 1c."
9. git push

It can be a bit initimidating at first, so feel free to click on "issue" in the github browser of this repository and ask :) (You can also use that to say "Hi, I'm having a bit of help with this particular equation, can someone help me out?")

If you don't know how to edit a latex file on your own pc iso on overleaf, look at the "How to use" section of `https://github.com/a-t-0/AE4872-Satellite-Orbit-Determination`.

0.1 Consistency

To make everything nice and structured, please use very clear citations:

1. If you copy/use an equation of some slide or document, please add the following data:
   (a) Url (e.g. if simple wiki or some site)
   (b) Name of document
   (c) (Author)
   (d) PAGE/SLIDE number so people can easily find it again
   (e) equation number (so people can easily find it again)
2. If you use an equation from the slides/a book that already has an equation number, then hardcode that equation number in this solution manual so people directly see which equation in the lecture material it is, this facilitates remembering the equations.
3. Here is an example is given in eq. (10.32[1]) (See file references.bib [1]).

$$R_n^E = \sum_{t=1}^{n} l_m(p_t, z_t) - \min_i \sum_{t=1}^{n} {z_t}^i \tag{10.32[1]}$$

1 Fundamentals:MC understanding

A

False, the general military aircraft have an initial velocity gain of up to 800m/s. Including other parameters, the answer is correct. Ishim project is mentioned.

B

True, Propellant mass is not included in the mass fraction.

C

False, payload is not included

D

True, inertial range is larger than rotating range since impact point is rotating away.

E

True, It is always for the maximum shooting range. It uses the formula for $X_i g_0/V_{e_{id}}$. by taking the derivative wrt the launch angle, the optimal launch angle is found. Usually 45 degrees or larger.

F

False, Due to atmosphere and wind, the accuracy will be lower the longer thorught the lower parts of the atmosphere.

G

True, The drag of mass ratio decrease if the frontal area remains constant but the mass of the rocket increases. With this ratio, the drag loss is computed.

H

False, it is used to minimize aerodynamic losses.

I

True, by applying the expulsed mass flow times the exhaust velocity an extra force is derived.

J

False, it is shown in the slide for the Unconventional Space Launch. Reasoning is that launching at 0 velocity from a 10 km up does not contribute in the same way as already having an initial velocity of 500+ m/s.

2 Multi stage rockets:Multi stage. EOM vert flight.mass

2.1 a

We start from the general equations of motion, which are split in EoM in x and z direction.

$$M\frac{dV_x}{dt} = T\cos\theta = mc_{eff}\cos\theta \tag{2.1.1a[2]}$$

$$M\frac{dV_z}{dt} = T\sin\theta = mc_{eff}\sin\theta - Mg_0 \tag{2.1.1b[2]}$$

For vertical flight, the flight path angle is equal to the pitch angle $\theta_0 = \gamma_0 = \pi/2$ ($\alpha = 0$). Furthermore, the mass flow can be rewritten as (also on formula sheet)

$$m = -\frac{dM}{dt} \tag{1}$$

Thus, the only equation that remains is the vertical velocity component.

$$M\frac{dV_z}{dt} = T = mc_{eff} - Mg_0 = -\frac{dM}{dt}c_{eff} - Mg_0 \tag{2}$$

Burnout velocity

Now, they ask first for an expression of the velocity at burn out. This means that entire duration, the flight is powered (With thrust). Thus, only a single equation is needed. The equation for the velocity can be derived from eq. (2).

$$M\frac{dV_z}{dt} = -\frac{dM}{dt}c_{eff} - Mg_0 \tag{3}$$

$$\frac{dV_z}{dt} = -\frac{1}{M}\frac{dM}{dt}c_{eff} - g_0 \tag{4}$$

$$dV_z = -\frac{1}{M}dMc_{eff} - g_0 dt \tag{5}$$

$$V_z - V_{z_0} = -(\ln M - \ln M_0)c_{eff} - g_0(t - t_0) \tag{6}$$

$$V_z - V_{z_0} = (\ln M_0 - \ln M)c_{eff} - g_0(t - t_0) \tag{7}$$

$$V_z - V_{z_0} = \ln\frac{M_0}{M}c_{eff} - g_0(t - t_0) \tag{8}$$

where the initial values for velocity and time are zero.

$$V_z = \ln\frac{M_0}{M}c_{eff} - g_0 t \tag{9}$$

Furthermore, they ask to put the equation in terms of c_{eff}, Λ, ψ_0 the first two are used. The mass fraction is in terms of the initial mass and mass at burnout M_0/M_e. Thus, the equation can be expressed as the velocity at burnout.

$$V_e = \ln\Lambda c_{eff} - g_0 t_b \tag{10}$$

Burnout height

The standard equation for the distance is given by

$$Z = \int_0^{t_b} = V dt \text{ where } V = \ln\frac{M_0}{M}c_{eff} - g_0 t \tag{11}$$

The integral can be rewritten for the first part of the velocity by knowing $dt = -dM/m$.

$$Z = -\int_{M_0}^{M_e} \ln\frac{M_0}{M}c_{eff}\frac{dM}{m} - \int_0^{t_b} g_0 t dt \tag{12}$$

First consider the first integral.

$$-\int_{M_0}^{M_e} \ln\frac{M_0}{M}c_{eff}\frac{dM}{m} = -\frac{c_{eff}}{m}\int_{M_0}^{M_e}(\ln M_0 - \ln M)dM \tag{13}$$

With the identity given in the question, the integral is solved to:

$$-\frac{c_{eff}}{m}\int_{M_0}^{M_e}(\ln M_0 - \ln M)dM = -\frac{c_{eff}}{m}[(M\ln M_0) - (M\ln M - M)]_{M_0}^{M_e} \tag{14}$$

Writing this out becomes

$$-\frac{c_{eff}}{m}[(M_e\ln M_0 - M_0\ln M_0) - ((M_e\ln M_e - M_e) - (M_0\ln M_0 - M_0))] \tag{15}$$

By eliminating the obvious ones.

$$-\frac{c_{eff}}{m}[(M_e\ln M_0) - (M_e\ln M_e - M_e + M_0)] = -\frac{c_{eff}}{m}[M_e\ln\frac{M_0}{M_e} + M_e - M_0] \tag{16}$$

Now the tricky part starts. We multiply the equation by M_0/M_0

$$-\frac{c_{eff}M_0}{m}[\frac{M_e}{M_0}\ln\frac{M_0}{M_e} + \frac{M_e}{M_0} - \frac{M_0}{M_0}] = -\frac{c_{eff}M_0}{m}[\frac{1}{\Lambda}\ln\Lambda + \frac{1}{\Lambda} - 1] = \frac{c_{eff}M_0}{m}[1 - \frac{1}{\Lambda}(\ln\Lambda + 1)] \tag{17}$$

The second integral is easily found to be

$$\int_0^{t_b} g_0 t dt = 0.5 g_0 t_b^2 \tag{18}$$

Thus, the total height becomes

$$Z = \frac{c_{eff}M_0}{m}[1 - \frac{1}{\Lambda}(\ln\Lambda + 1)] - 0.5g_0t_b^2 \tag{19}$$

where

$$\frac{c_{eff}M_0}{m} = \frac{c_{eff}^2 M_0 g_0}{mc_{eff}g_0} = \frac{c_{eff}^2}{\psi_0 g_0} \text{ where } \frac{M_0 g_0}{mc_{eff}} = \frac{1}{\psi_0} \tag{20}$$

$$Z = \frac{c_{eff}^2}{\psi_0 g_0}[1 - \frac{1}{\Lambda}(\ln\Lambda + 1)] - 0.5g_0t_b^2 \tag{21}$$

Furthermore, we need to find an expression for the burn time. This can be obtained from the fact that the mass flow is constant over time.

$$t = \frac{M_0 - M}{m} = c_{eff}\frac{M_0 - M}{T} \tag{22}$$

By creating the collective term M_0 and multiplying the equation by g_0/g_0

$$t = c_{eff}\frac{M_0 g_0(1 - \frac{M}{M_0})}{Tg_0} \tag{23}$$

and realizing that

$$\frac{M_0 g_0}{Tg_0} = \frac{1}{g_0\psi_0} \tag{24}$$

we get that the burn time is

$$t_b = c_{eff}\frac{(1 - \frac{1}{\Lambda})}{\psi_0 g_0} \tag{25}$$

By integrating the expression for the burn time the burnout height can be expressed as

$$Z = \frac{c_{eff}^2}{\psi_0 g_0}[1 - \frac{1}{\Lambda}(\ln\Lambda + 1)] - \frac{1}{2}g_0[c_{eff}\frac{(1 - \frac{1}{\Lambda})}{\psi_0 g_0}]^2 \tag{26}$$

$$Z = \frac{c_{eff}^2}{\psi_0 g_0}[1 - \frac{1}{\Lambda}(\ln\Lambda + 1)] - \frac{c_{eff}^2}{2\psi_0^2 g_0}(1 - \frac{1}{\Lambda})^2 \tag{27}$$

Which can be simplified even further

$$Z = \frac{c_{eff}^2}{\psi_0 g_0}\left([1 - \frac{1}{\Lambda}(\ln\Lambda + 1)] - \frac{1}{2\psi_0}(1 - \frac{1}{\Lambda})^2\right) \tag{28}$$

2.2 b

For this question, the mass of the boosters is know and the mass after the burn is known for the core stage. So we need to find the amount of propellant that is used by the core stage. This can be done with the specific impulse equation, given in the formula sheet.

$$I_{sp} = \frac{T}{mg_0} \tag{29}$$

We know the burn time, the thrust, the specific impulse and the gravitational parameter. This means that we can find the mass flow.

$$m = \frac{T}{I_{sp}g_0} = \frac{1000000}{300 \cdot 9.81} = 339.789\text{kg/s} \tag{30}$$

Multiplying the mass flow with the burn time yields the total propellant mass that is used by the core stage.

$$M = mt_b = 339.789 \cdot 260 = 88345.14\text{kg} \tag{31}$$

Adding all the boosters and the mass after the burnout, yields the total mass of the rocket.

$$M_{tot} = 9M_b + M_e + M_p = 9 \cdot 10000 + 10000 + 88345.14 = 188345.14\text{kg} \tag{32}$$

2.3 c

For the acceleration at launch from the standard EoM, where the thrust is due to the core and SIX boosters

$$M\frac{dV}{dt} = T - Mg_0 \;\rightarrow\; \frac{dV}{dt} = \frac{T}{M} - g_0 \tag{33}$$

All parameters are known so just fill in

$$\frac{dV}{dt} = \frac{T}{M} - g_0 = \frac{2800000}{188345.14} - g_0 = 5.056\text{m/s}^2 \tag{34}$$

For the acceleration after ignition of the second set of boosters. The total mass is reduced. OBviously six boosters are gone plus for the core stage, some propellant is gone. First to find the mass of the rocket after the ignition of the second set of boosters.

$$M_core = M_e + M_p - m \cdot t = 10000 + 88345.14 - 339.789 \cdot 60 = 77957.8\text{kg} \tag{35}$$

Now the thurst is determined by the core plus three boosters

$$\frac{dV}{dt} = \frac{T}{M} - g_0 = \frac{1900000}{77957.8 + 30000} - g_0 = 7.7985\text{m/s}^2 \tag{36}$$

For the burnout at the core stage, no thurst is applied anymore. So only the gravitational acceleration is acting.

$$\frac{dV}{dt} = \frac{T}{M} - g_0 = \frac{0}{10000} - g_0 = -9.81\text{m/s}^2 \tag{37}$$

2.4 d

The equations found for the first sub question is used. Make sure to take into account that for the second part of this question, the initial velocity and height is not equal to zero!

For the velocity at the instant of burnout (assumed that mass of the boosters is gone). The velocity only depends on the burn time,c_{eff} and the mass ratio.

The effective exhaust velocity can be found with the specific impulse and gravitational paramter. The burn time is given $t_b = 60$ s.

$$V_e = \ln \Lambda c_{eff} - g_0 t_b \tag{38}$$

For effective velocity, a combination can be computed since the thurst is constant.

$$\bar{c}_{eff} = \frac{T_1 + T_2 + ...}{m_1 + m_2 + ...} \tag{39}$$

The mass flow is found since the thrust and the specific impulse is known.

$$m = \frac{T}{I_{sp} g_0} \tag{40}$$

The mass flow for the booster is

$$m = \frac{T}{I_{sp} g_0} = \frac{300000}{200 \cdot 9.81} = 152.905\text{kg/s} \tag{41}$$

The combined effective velocity is

$$\bar{c}_{eff} = \frac{300000 \cdot 6 + 1000000}{152.905 \cdot 6 + 339.789} = 2227.138\text{m/s} \tag{42}$$

The mass at burnout is

$$M_e = M_0 - (6 \cdot m_{Booster} - m_{core}) t_{burn} = 188345.14 - 75433.14 = 112912\text{kg} \tag{43}$$

Filling in the velocity equation, becomes.

$$V_e = \ln \Lambda c_{eff} - g_0 t_b = \ln \frac{188345.14}{112912} \cdot 2227.138 - 9.81 \cdot 60 = 550.954\text{m/s} \tag{44}$$

For the height, the initial thrust load needs to be known. This can be found with

$$\psi_0 = \frac{T}{M_0 g_0} = \frac{2800000}{188345.14 \cdot 9.81} = 1.5154 \tag{45}$$

With the initial thrust load, the effective exhaust velocity and the mass ratio $\Lambda = \frac{188345.14}{77957.8+30000}$, the height is found.

$$Z = \frac{c_{eff}^2}{\psi_0 g_0}\left([1 - \frac{1}{\Lambda}(\ln\Lambda + 1)] - \frac{1}{2\psi_0}(1 - \frac{1}{\Lambda})^2\right) \tag{46}$$

$$Z = \frac{2227.138^2}{1.5154 \cdot 9.81}\left([1 - \frac{1}{1.7446}(\ln 1.7446 + 1)] - \frac{1}{2 \cdot 1.5154}(1 - \frac{1}{1.7446})^2\right) = \tag{47}$$

$$333655.25(0.1078 - 0.0601) = 15915.356\text{m} = 15.915\text{km} \tag{48}$$

For the second burnout, the velocity increment is computed in the same way. First find the effective exhaust velocity.

$$c_{eff} = \frac{T_{core} + 3T_{booster}}{m_{core} + 3m_{booster}} = \frac{1000000 + 3 \cdot 300000}{339.789 + 3 \cdot 152.905} = 2379.449571\text{m/s} \tag{49}$$

The end mass, is the initial mass after the seperation of the first 6 boosters, minus the propellant used by the core and the 3 boosters.

$$M_e = M_0 - m_{core}2t_{burn} - 6M_{booster} - 3m_{booster}t_{burn} = 60047.56\text{kg} \tag{50}$$

The initial mass for the second burnout stage is

$$M_{0_2} = M_0 - m_{core}t_{burn} - 6M_{booster} = 107957.8\text{kg} \tag{51}$$

Based on the mass and effective velocity the velocity increment is

$$\Delta V_2 = \ln\Lambda c_{eff} - g_0 t_b = \ln\frac{107957.8}{60047.56} \cdot 2379.449571 - 9.81 \cdot 60 = 807.19\text{m/s} \tag{52}$$

This means that the velocity after the second burnout is

$$V_{e_2} = 550.954 + 807.19 = 1358.147 \tag{53}$$

For the height we need to know the initial thrust load again.

$$\psi_0 = \frac{T}{M_0 g_0} = \frac{1900000}{107957.8 \cdot 9.81} = 1.794 \tag{54}$$

The height becomes:

$$Z = \frac{c_{eff}^2}{\psi_0 g_0}\left([1 - \frac{1}{\Lambda}(\ln\Lambda + 1)] - \frac{1}{2\psi_0}(1 - \frac{1}{\Lambda})^2\right) = \frac{2379.4495^2}{1.794 \cdot 9.81}(0.11753 - 0.0552079) = 20049.5\text{m} = 20.0495\text{km} \tag{55}$$

So adding the two heights, the total height after the second burnout becomes

$$h_{e_2} = 15.915 + 20.0495 = 35.965\text{km} \tag{56}$$

2.5 e

This question follows the same methods to find the velocity and altitude. Taking into account that the burn is 10 seconds shorter than designed.

First the effective velocity of the core stage is

$$c_{eff} = \frac{T}{m} = \frac{1000000}{339.789} = 2943\text{m/s} \tag{57}$$

Secondly, the mass at the start of the last burn phase is given by

$$M_{0_3} = M_0 - 9M_{booster} - m_{core}2t_{burn} = 57570.46\text{kg} \tag{58}$$

The end mass is

$$M_{e_3} = M_{0_3} - m_{core} \cdot 130 = 13397.89\text{kg} \tag{59}$$

The velocity follows from

$$\Delta V_3 = \ln\Lambda c_{eff} - g_0 t_b = \ln\frac{57570.46}{13397.89} \cdot 2943 - 9.81 \cdot 130 = 3015.336\text{m/s} \tag{60}$$

The total velocity is

$$V = 1358.147 + 3015.336 = 4373.48\text{m/s} \tag{61}$$

The initial thurst load is

$$\psi_0 = \frac{T}{M_0 g_0} = 1.7706 \tag{62}$$

The height follows from

$$Z = \frac{c_{eff}^2}{\psi_0 g_0}\left([1 - \frac{1}{\Lambda}(\ln\Lambda + 1)] - \frac{1}{2\psi_0}(1 - \frac{1}{\Lambda})^2\right) = \frac{2943^2}{1.7706 \cdot 9.81}(0.428 - 0.1662) = 130545.1372\text{m} = 130.545\text{km} \tag{63}$$

The total height after the core burnout is

$$h = 130.545 + 35.965 = 166.54\text{km} \tag{64}$$

2.6 f

Velocity at the maximum height is zero. With the energy equation, the height achieved after burnout is found

$$\Delta h = \frac{V^2}{2g_0} \tag{65}$$

Given the velocity and height after the core burnout the extra heigth until the velocity is zero becomes

$$\Delta h = \frac{4373.48^2}{2 \cdot 9.81} = 947.889\text{km} \tag{66}$$

So the total heigth achieved is

$$h = 947.889 + 166.54 = 1141.4\text{km} \tag{67}$$

The designed maximum heigth is given in the question. So the difference is

$$\Delta h_{error} = 1765 - 1141.4 = 623.6\text{km} \tag{68}$$

3 Multi stage rockets:V(height).alts.Multistage

3.1 a

The velocity can be expressed in terms of the different ΔV components that have been given in the question.

For the initial velcity,

$$V_{\text{ignition}} = 0 \text{ m/s} \tag{69}$$

For the first stage velocity is just the first ΔV. This includes the gravity losses.

$$V_{\text{burnout1}} = \Delta V_1 \text{ m/s} \tag{70}$$

For the second stage velocity is just the first and second ΔV. This includes the gravity losses.

$$V_{\text{burnout2}} = (\Delta V_1 + \Delta V_2) \text{ m/s} \tag{71}$$

Culmination point is per definition:

$$V_{\text{culimination}} = 0 \text{ m/s} \tag{72}$$

3.2 b

The height can also be expressed in terms of altitude differences.

$$h_{\text{ignition}} = 0 \text{ m} \tag{73}$$

For the first stage altitude is just the first Δh.

$$h_{\text{burnout1}} = \Delta h_1 \text{ m} \tag{74}$$

For the second stage it is just the first and second ΔV but now also includes the velocity due to the first stage times the burn time of the second stage.

$$V_{\text{burnout2}} = (\Delta h_1 + \Delta h_2 + \Delta V_1 t_{b_2}) \text{ m} \tag{75}$$

Culmination point is the altitude up to burnout of stage 2. After that the altitude increase is computed with the energy equation:

$$\frac{1}{2}mV^2 = mg_0\Delta h \;\rightarrow\; \Delta h = \frac{(\Delta V_1 + \Delta V_2)^2}{2g_0} \tag{76}$$

So the total altitude in the culmination point becomes

$$\Delta h_c = (\Delta h_1 + \Delta h_2 + \Delta V_1 t_{b_2}) + \frac{(\Delta V_1 + \Delta V_2)^2}{2g_0} \tag{77}$$

3.3 c

For the velocity, the gravity losses in the coast period has to be included. For the sake of brevity, the culmination and ignition velocity are omitted. The velocity after the first stage is

$$V_{\text{burnout1}} = \Delta V_1 \text{ m/s} \tag{78}$$

The velocity after the second stage, is the increase in velocity due to the two stage and includes a term for the gravity losses during the coasting period.

$$V_{\text{burnout2}} = (\Delta V_1 - g_0 t_{co} + \Delta V_2) \text{ m/s} \tag{79}$$

For the altitude, the ignition altitude is once again 0. The altitude after the first stage is not different.

$$h_{\text{burnout1}} = \Delta h_1 \text{ m} \tag{80}$$

The altitude after the second stage is different. The velocity at the end of the second stage differs. There is an altitude increase due to the coast.

$$h_{\text{burnout2}} = \Delta h_1 + \Delta h_2 + \Delta V_1 t_{co} - \frac{1}{2} g_0 t_{co}^2 + (\Delta V_1 - g_0 t_{co}) t_{b_2} \tag{81}$$

For the culmination point, the theory remains the same. The altitude already reached plus some term that is computed with the energy equation. The latter is:

$$\Delta h_c = \frac{(\Delta V_1 - g_0 t_{co} + \Delta V_2)^2}{2 g_0} \tag{82}$$

$$h_c = \Delta h_1 + \Delta h_2 + \Delta V_1 t_{co} + (\Delta V_1 - g_0 t_{co}) t_{b_2} + \frac{(\Delta V_1 - g_0 t_{co} + \Delta V_2)^2}{2 g_0} \tag{83}$$

which can be simplified to

$$h_c = \Delta h_1 + \Delta h_2 + \Delta V t_{b_2} + \frac{(\Delta V_1 + \Delta V_2)^2}{2 g_0} - [\Delta V_2 + g_0 t_{b_2}] t_{co} \tag{84}$$

$$h_c' = h_c - [\Delta V_2 + g_0 t_{b_2}] t_{co} \tag{85}$$

3.4 d

The mass fraction Λ becomes smaller for an increase in burnout mass of the stages. This means that the amount ΔV decreases. This means that both coasting altitude and culmination altitude will be lower. The gravity losses during the coast and to the culmination point are not affected.

3.5 e

By subtracting the culmnation altitude h_c' found in (c) with the altitude h_c found in (b), it can be seen what the effect is of the coasting.

$$\Delta h = h_c' - h_c = -[\Delta V_2 + g_0 t_{b_2}] t_{co} \tag{86}$$

What can be seen is that the ΔV which includes the gravity losses loses the gravity losses. So the ΔV is just given by the mass ratio and effective exhaust velocity of the second stage

$$\Delta h_c = -c_{eff_2} \ln \Lambda_2 t_{co} \tag{87}$$

4 Balistic flight over earth:Spherical.Flight range phi-0

4.1 a

Sketch is just figure 4.2 in the lecute notes on page 77 [2].

4.2 b

The derivation is done in a couple of steps. First, determine a/R_e, p/R_e. From there, the eccentricity is found with the equation $e = \sqrt{1 - a/p}$. From there, the trajectory equation of conic sections can be filled in and by realizing that the angle $psi_0 = \pi/2 - \theta_0$, the equation is found that is asked.

So first the semi major axis. This is found with the 'vis-viva equation' The energy equation.

$$-\frac{\mu}{2a} = \frac{V^2}{2} - \frac{\mu}{r} \tag{88}$$

Using the initial condition $r = R_e$ and $V = V_0$, the semi major axis is found

$$-\frac{\mu}{2a} = \frac{V_0^2}{2} - \frac{\mu}{R_e} \tag{89}$$

$$\frac{1}{a} = -\frac{V_0^2}{\mu} + \frac{1}{2R_e} \tag{90}$$

$$\frac{a}{R_e} = -\frac{\mu}{V_0^2 R_e} + \frac{1}{2} = \frac{1}{2 - \frac{V_0^2}{\mu/R_e}} \tag{91}$$

using the fact that

$$S_0 = \frac{V_0}{\sqrt{\mu/R_e}} \tag{92}$$

$$\frac{a}{R_e} = \frac{1}{2 - S_0^2} \tag{93}$$

For the semi latus rectum, the angular momentum is used.

$$p = \frac{H^2}{r} \tag{94}$$

where

$$H = rV\cos\gamma \tag{95}$$

using the initial conditions this becomes:

$$H = R_e V_0 \cos\gamma_0 \tag{96}$$

Thus, the semi latus rectum becomes

$$p = \frac{R_e^2 V_0^2 \cos^2\gamma_0}{\mu} \tag{97}$$

$$\frac{p}{R_e} = \frac{R_e V_0^2 cos^2\gamma_0}{\mu} = S_0^2 \cos^2\gamma_0 \tag{98}$$

With these equations the eccentricity is found

$$\frac{p}{a} = \frac{S_0^2 \cos^2\gamma_0}{\frac{1}{2 - S_0^2}} = S_0^2 \cos^2\gamma_0 (2 - S_0^2) \tag{99}$$

The eccentricy beomces

$$e = \sqrt{1 - p/a} = \sqrt{1 - S_0^2 \cos^2\gamma_0 (2 - S_0^2)} \tag{100}$$

Now using the trajectory equation,

$$r = \frac{p}{1 + e\cos\theta} \tag{101}$$

$$\frac{r}{R_e} = \frac{\frac{p}{R_e}}{1 + \sqrt{1 - S_0^2 \cos^2\gamma_0 (2 - S_0^2)}\cos\theta} \tag{102}$$

Plugging in the initial conditions

$$1 = \frac{S_0^2 \cos^2\gamma_0}{1 + \sqrt{1 - S_0^2 \cos^2\gamma_0 (2 - S_0^2)}\cos\theta} \tag{103}$$

We can rewrite this to isolate the cosine part.

$$-\cos\theta = \frac{1 - S_0^2 \cos^2\gamma_0}{\sqrt{1 - S_0^2 \cos^2\gamma_0 (2 - S_0^2)}} \tag{104}$$

Using the fact that $\psi_0 = \pi/2 - \theta_0$:

$$\cos\psi_0 = \frac{1 - S_0^2 \cos^2\gamma_0}{\sqrt{1 - S_0^2 \cos^2\gamma_0 (2 - S_0^2)}} \tag{105}$$

$$\psi_0 = \arccos[\frac{1 - S_0^2 \cos^2\gamma_0}{\sqrt{1 - S_0^2 \cos^2\gamma_0 (2 - S_0^2)}}] \tag{106}$$

c

For this part, it is meant to rewrite the equation found in part b to be able to find an expression for the initial flight path angle.

First we need to remove the square root out of the equation.

This is done with the following identity

$$\tan^2\psi_0 = \frac{\sin^2\psi_0}{\cos^2\psi_0} = \frac{1-\cos^2\psi_0}{\cos^2\psi_0} = \frac{1}{\cos^2\psi_0} - 1 \tag{107}$$

So first square the entire equation.

$$\cos^2\psi_0 = \frac{(1-S_0^2\cos^2\gamma_0)^2}{1-S_0^2\cos^2\gamma_0(2-S_0^2)} \tag{108}$$

$$\frac{1}{\cos^2\psi_0} = \frac{1-S_0^2\cos^2\gamma_0(2-S_0^2)}{(1-S_0^2\cos^2\gamma_0)^2} \tag{109}$$

$$\frac{1}{\cos^2\psi_0} - 1 = \frac{1-S_0^2\cos^2\gamma_0(2-S_0^2)-(1-S_0^2\cos^2\gamma_0)^2}{(1-S_0^2\cos^2\gamma_0)^2} \tag{110}$$

Simplifying this

$$\frac{1}{\cos^2\psi_0} - 1 = \frac{S_0^4\cos^2\gamma_0(1-\cos^2\gamma_0)}{(1-S_0^2\cos^2\gamma_0)^2} \tag{111}$$

Now de identity is used

$$\tan^2\psi_0 = \frac{S_0^4\cos^2\gamma_0(1-\cos^2\gamma_0)}{(1-S_0^2\cos^2\gamma_0)^2} \tag{112}$$

$$\tan^2\psi_0 = \frac{S_0^4\cos^2\gamma_0\sin^2\gamma_0}{(1-S_0^2\cos^2\gamma_0)^2} \tag{113}$$

$$\tan^2\psi_0 = \frac{S_0^4\sin^2 2\gamma_0}{4(1-S_0^2\cos^2\gamma_0)^2} \tag{114}$$

Now by square rooting both sides

$$\tan\psi_0 = \frac{S_0^2\sin 2\gamma_0}{2(1-S_0^2\cos^2\gamma_0)} \tag{115}$$

Now that the squareroot has been removed out of the fraction, it is needed that the two terms of the flight path angle is removed to a single term. First we do cross multiplication to remove the fractions

$$\tan\psi_0 = \frac{\sin\psi_0}{\cos\psi_0} = \frac{S_0^2\sin 2\gamma_0}{2(1-S_0^2\cos^2\gamma_0)} \tag{116}$$

We first reduce the $\cos^2\gamma_0$ term in the fraction. The identity used is

$$2\cos^2\gamma_0 - 1 = \cos 2\gamma_0 \;\rightarrow\; \cos^2\gamma_0 = \frac{\cos 2\gamma_0 + 1}{2} \tag{117}$$

$$\frac{\sin\psi_0}{\cos\psi_0} = \frac{S_0^2\sin 2\gamma_0}{2-2S_0^2\cos^2\gamma_0} = \frac{S_0^2\sin 2\gamma_0}{2-S_0^2(\cos 2\gamma_0+1)} \tag{118}$$

From there the cross multiplication is done. Make sure that the term with the flight path angle will be on one side of the equation. The other terms on the other side

$$S_0^2(\cos 2\gamma_0\sin\psi_0 + \sin 2\gamma_0\cos\psi_0) = (2-S_0^2)\sin\psi_0 \tag{119}$$

The terms with the flight path angle can be reduced with the identity

$$\cos 2\gamma_0\sin\psi_0 + \sin 2\gamma_0\cos\psi_0 = \sin(2\gamma_0 + psi_0) \tag{120}$$

Thus

$$S_0^2\sin(2\gamma_0+\psi_0) = (2-S_0^2)\sin\psi_0 \tag{121}$$

This is the reduced form of the equation with which the flight path angle can be computed.

$$2\gamma_0 + \psi_0 = \arcsin[\frac{(2-S_0^2)\sin\psi_0}{S_0^2}] \tag{122}$$

$$\gamma_0 = \frac{\arcsin[\frac{(2-S_0^2)\sin\psi_0}{S_0^2}] - \psi_0}{2} \tag{123}$$

Filling it in knowing that $\psi_0 = \frac{d/2}{R_e} = 0.7071182$, and $S_0 = \frac{V_0}{\sqrt{\mu/R_e}} = 1000$ yields:

$$\gamma_0 \approx -0.707\text{rad} \tag{124}$$

Conclusion

The exam consists of many derivations that have been done in the slides. Study those well!

References

[1] Some author. *Advanced tree dynamics*, volume lecture 5 of *AE2344 Some course*, page 15. Accessed: 2019-04-27.

[2] H Wittenberg. *Rocket Motion*, page 142. AE4870A.