

Chapter 1. Review of Probability Theory and Statistics

1 Probability Space and Rules of Probability

To any experiment we assign its **sample space**, denoted by S , consisting of all its possible outcomes (called **elementary events**, denoted by e_i , $i \in \mathbb{N}$).

An **event** is a subset of S (events are denoted by capital letters, A_i , $i \in \mathbb{N}$).

Since events are defined as sets, we use set theory in describing them.

- two special events associated with every experiment:
 - the **impossible** event, denoted by \emptyset (“never happens”);
 - the **sure (certain)** event, denoted by S (“surely happens”).
- for events, we have the usual operations of sets:
 - **complementary** event, \overline{A} ,
 - **union** of A and B , $A \cup B = \{e \in S \mid e \in A \text{ or } e \in B\}$, the event that occurs if either A or B or both occur;
 - **intersection** of A and B , $A \cap B = \{e \in S \mid e \in A \text{ and } e \in B\}$, the event that occurs if both A and B occur;
 - **difference** of A and B , $A \setminus B = \{e \in S \mid e \in A \text{ and } e \notin B\} = A \cap \overline{B}$, the event that occurs if A occurs and B does not;
 - A **implies (induces)** B , $A \subseteq B$, if every element of A is also an element of B , or in other words, if the occurrence of A induces (implies) the occurrence of B ; A and B are equal, $A = B$, if A implies B and B implies A ;
- two events A and B are **mutually exclusive (disjoint, incompatible)** if A and B cannot occur at the same time, i.e. $A \cap B = \emptyset$;
- three or more events are mutually exclusive if any two of them are;
- events $\{A_i\}_{i \in I}$ are **collectively exhaustive** if $\bigcup_{i \in I} A_i = S$;

- events $\{A_i\}_{i \in I}$ form a **partition** of S if the events are collectively exhaustive and mutually exclusive, i.e.

$$\bigcup_{i \in I} A_i = S, \text{ and } A_i \cap A_j = \emptyset, \forall i, j \in I, i \neq j.$$

- we consider all events relating to an experiment to belong to a **σ -field**, \mathcal{K} , a collection of events from from S , an algebraic structure that allows all the usual set operations (mentioned above) within itself (e.g. the power set $\mathcal{P}(S) = \{S' | S' \subseteq S\}$).

Definition 1.1. Let \mathcal{K} be a σ -field over S . A mapping $P : \mathcal{K} \rightarrow \mathbb{R}$ is called **probability** if it satisfies the following conditions:

- (i) $P(S) = 1$;
- (ii) $P(A) \geq 0$, for all $A \in \mathcal{K}$;
- (iii) for any sequence $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{K}$ of mutually exclusive events,

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n). \quad (1.1)$$

The triplet (S, \mathcal{K}, P) is called a **probability space**.

Theorem 1.2. (Rules of Probability)

Let (S, \mathcal{K}, P) be a probability space, and let $A, B \in \mathcal{K}$. Then the following properties hold:

- a) $P(\overline{A}) = 1 - P(A)$.
- b) $0 \leq P(A) \leq 1$.
- c) $P(\emptyset) = 0$.
- d) $P(A \setminus B) = P(A) - P(A \cap B)$.
- e) If $A \subseteq B$, then $P(A) \leq P(B)$, i.e. P is monotonically increasing.
- f) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

g) more generally,

$$\begin{aligned} P\left(\bigcup_{i=1}^n A_i\right) &= \sum_{i=1}^n P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) \\ &+ \dots + (-1)^{n-1} P\left(\bigcap_{i=1}^n A_i\right), \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Definition 1.3. Let (S, \mathcal{K}, P) be a probability space and let $B \in \mathcal{K}$ be an event with $P(B) > 0$. Then for every $A \in \mathcal{K}$, the **conditional probability of A given B** (or the **probability of A conditioned by B**) is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \quad (1.2)$$

Theorem 1.4. (Rules of Probability – Continued)

h) $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B).$

i) *Multiplication Rule*

$$P(A_1 \cap \dots \cap A_n) = P(A_1)P(A_2|A_1) \dots P(A_n|A_1 \cap \dots \cap A_{n-1}).$$

j) *Total Probability Rule*

$$- P(A) = P(B)P(A|B) + P(\overline{B})P(A|\overline{B}).$$

- in general, if $\{A_i\}_{i \in I}$ is a partition of S ,

$$P(A) = \sum_{i \in I} P(A_i)P(A|A_i). \quad (1.3)$$

Definition 1.5. Two events $A, B \in \mathcal{K}$ are **independent** if

$$P(A \cap B) = P(A)P(B). \quad (1.4)$$

- A, B independent $\Leftrightarrow P(A|B) = P(A) \Leftrightarrow P(B|A) = P(B).$
- $A = \emptyset$ or $A = S$ and $B \in \mathcal{K}$, then A, B independent.
- A, B independent $\Leftrightarrow \overline{A}, B$ independent $\Leftrightarrow \overline{A}, \overline{B}$ independent.

Definition 1.6. Consider an experiment whose outcomes are finite and equally likely. Then the **probability of the event A** is given by

$$P(A) = \frac{\text{number of favorable outcomes for the occurrence of } A}{\text{total number of possible outcomes of the experiment}} \stackrel{\text{not}}{=} \frac{N_f}{N_t}. \quad (1.5)$$

Remark 1.7. This notion is closely related to that of *relative frequency* of an event A : repeat an experiment a number of times N and count the number of times event A occurs, N_A . Then the relative frequency of the event A is

$$f_A = \frac{N_A}{N}.$$

Such a number is often used as an approximation to the probability of A . This is justified by the fact that

$$f_A \xrightarrow{N \rightarrow \infty} P(A).$$

The relative frequency is used in computer simulations of random phenomena.

2 Probabilistic Models

Binomial Model

This model is used when the trials of an experiment satisfy three conditions, namely

- (i) they are independent,
- (ii) each trial has only two possible outcomes, which we refer to as “success” (A) and “failure” (\bar{A}) (i.e. the sample space for each trial is $S = A \cup \bar{A}$),
- (iii) the probability of success $p = P(A)$ is the same for each trial (we denote by $q = 1 - p = P(\bar{A})$ the probability of failure).

Trials of an experiment satisfying (i) – (iii) are known as **Bernoulli trials**.

Model: Given n Bernoulli trials with probability of success p , find the probability $P(n; k)$ of exactly k ($0 \leq k \leq n$) successes occurring.

We have

$$\begin{aligned} P(n; k) &= C_n^k p^k (1 - p)^{n-k} = C_n^k p^k q^{n-k}, \quad k = 0, 1, \dots, n \quad \text{and} \\ \sum_{k=0}^n P(n; k) &= 1. \end{aligned} \tag{2.1}$$

Pascal (Negative Binomial) Model

Model: Consider an infinite sequence of Bernoulli trials with probability of success p (and probability of failure $q = 1 - p$) in each trial. Find the probability $P(n, k)$ of the n th success occurring

after k failures ($n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$).

We have

$$\begin{aligned} P(n, k) &= C_{n+k-1}^k p^n q^k, \quad k = 0, 1, \dots \quad \text{and} \\ \sum_{k=0}^{\infty} P(n; k) &= 1. \end{aligned} \tag{2.2}$$

Geometric Model

Although a particular case for the Pascal Model (case $n = 1$), the Geometric model comes up in many applications and deserves a place of its own.

Model: Consider an infinite sequence of Bernoulli trials with probability of success p (and probability of failure $q = 1 - p$) in each trial. Find the probability p_k that the first success occurs after k failures ($k \in \mathbb{N} \cup \{0\}$).

Here, we have

$$\begin{aligned} p_k &= pq^k, \quad k = 0, 1, \dots \quad \text{and} \\ \sum_{k=0}^{\infty} p_k &= 1. \end{aligned} \tag{2.3}$$

3 Random Variables

3.1 Random Variables, PDF and CDF

Random variables, variables whose observed values are determined by chance, give a more comprehensive quantitative overlook of random phenomena. Random variables are the fundamentals of modern Statistics.

Definition 3.1. Let (S, \mathcal{K}, P) be a probability space. A **random variable** is a function $X : S \rightarrow \mathbb{R}$ satisfying the property that for every $x \in \mathbb{R}$, the event

$$(X \leq x) := \{e \in S \mid X(e) \leq x\} \in \mathcal{K}. \tag{3.1}$$

- if the set of values that it takes, $X(S)$, is at most countable in \mathbb{R} , then X is a **discrete random variable** (quantities that are counted);
- if $X(S)$ is a continuous subset of \mathbb{R} (an interval), then X is a **continuous random variable** (quantities that are measured).

For each random variable, discrete or continuous, there are two important functions associated with it:

- **PDF (probability distribution/density function)**

- if X is discrete, then the pdf is an array

$$X \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I}, \quad (3.2)$$

where $x_i \in \mathbb{R}$, $i \in I$, are the values that X takes and $p_i = P(X = x_i)$

- if X is continuous, then the pdf is a function $f : \mathbb{R} \rightarrow \mathbb{R}$;

- **CDF (cumulative distribution function)** $F = F_X : \mathbb{R} \rightarrow \mathbb{R}$, defined by

$$F(x) = P(X \leq x). \quad (3.3)$$

- if X is discrete, then

$$F(x) = \sum_{x_i \leq x} p_i. \quad (3.4)$$

- if X is continuous, then

$$F(x) = \int_{-\infty}^x f(t) dt. \quad (3.5)$$

The pdf has the following properties:

- all values $x_i, i \in I$, are distinct and listed in increasing order;
- all probabilities $p_i > 0, i \in I$ and $f(x) \geq 0$, for all $x \in \mathbb{R}$;
- $\sum_{i \in I} p_i = 1$ and $\int_{\mathbb{R}} f(t) dt = 1$.

The cdf has the following properties:

- if $a < b$ are real numbers, then $P(a < X \leq b) = F(b) - F(a)$;
- $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$;

- if X is discrete, then $P(X < x) = F(x-0) = \lim_{y \nearrow x} F(y)$ and $P(X = x) = F(x) - F(x-0)$;
- if X is continuous, then $P(X = x) = 0$, $P(X < x) = P(X \leq x) = F(x)$ and

$$P(a < X \leq b) = P(a < X \leq b) = P(a < X < b) = P(a \leq X \leq b) = \int_a^b f(t) dt;$$
- if X is continuous, then $F'(x) = f(x)$, for all $x \in \mathbb{R}$.

3.2 Numerical Characteristics of Random Variables

The **expectation (expected value, mean value)** of a random variable X is a real number $E(X)$ defined by

- if X is a discrete random variable with pdf $\begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I}$,

$$E(X) = \sum_{i \in I} x_i P(X = x_i) = \sum_{i \in I} x_i p_i, \quad (3.6)$$

if it exists;

- if X is a continuous random variable with pdf $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$E(X) = \int_{\mathbb{R}} x f(x) dx, \quad (3.7)$$

if it exists.

The **variance (dispersion)** of a random variable X is the number

$$V(X) = E\left(X - E(X)\right)^2, \quad (3.8)$$

if it exists.

The **standard deviation** of a random variable X is the number

$$\sigma(X) = \text{Std}(X) = \sqrt{V(X)}. \quad (3.9)$$

Properties:

- $E(aX + b) = aE(X) + b$, for all $a, b \in \mathbb{R}$;

- $E(X + Y) = E(X) + E(Y)$;
- If X and Y are independent, then $E(X \cdot Y) = E(X)E(Y)$;
- If $X(e) \leq Y(e)$ for all $e \in S$, then $E(X) \leq E(Y)$;
- $V(X) = E(X^2) - E(X)^2$.
- If X and Y are independent, then $V(X + Y) = V(X) + V(Y)$.

Let X be a random variable with cdf $F : \mathbb{R} \rightarrow \mathbb{R}$ and $\alpha \in (0, 1)$. A **quantile of order α** is a number q_α satisfying the condition $P(X < q_\alpha) \leq \alpha \leq P(X \leq q_\alpha)$, or, equivalently,

$$F(q_\alpha - 0) \leq \alpha \leq F(q_\alpha). \quad (3.10)$$

If X is continuous, then for each $\alpha \in (0, 1)$, there is a unique quantile q_α , given by $F(q_\alpha) = \alpha$, or equivalently, $q_\alpha = F^{-1}(\alpha)$.

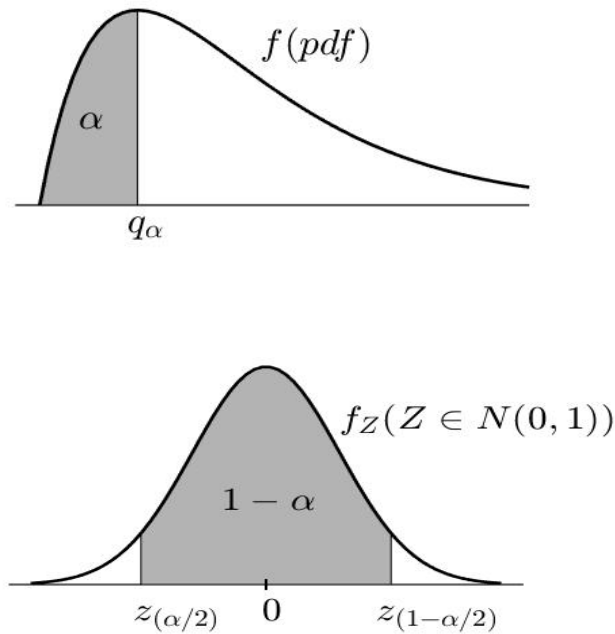


Fig. 1: Quantiles