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grupa 144

Geometrie și algebră liniară

(1) a)

(2) b)

(3) b)

(4) a)

(5) $\Gamma: f(x) = x_1^2 - 8x_1x_2 + 7x_2^2 - 12x_1 - 6x_2 - 9 = 0$

a) $A = \begin{pmatrix} 1 & -4 \\ -4 & 7 \end{pmatrix}$ $B = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$ $\tilde{A} = \begin{pmatrix} 1 & -4 & -6 \\ -4 & 7 & -3 \\ -6 & -3 & -9 \end{pmatrix}$

$\delta = \det A = \begin{vmatrix} 1 & -4 \\ -4 & 7 \end{vmatrix} = 7 - 16 = -9 \neq 0 \Rightarrow$ conică cu centru unic

$\Delta = \det \tilde{A} = \begin{vmatrix} 1 & -4 & -6 \\ -4 & 7 & -3 \\ -6 & -3 & -9 \end{vmatrix} = -324 \neq 0 \Rightarrow$ conică nedegenerată

Fie P_0 centrul conicei.

$P_0 : \begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} 2x_1 - 8x_2 - 12 = 0 \\ -8x_1 + 14x_2 - 6 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - 4x_2 = 6 \mid \cdot 4 \\ -4x_1 + 7x_2 = 3 \end{cases}$ (+)

$-9x_2 = 27$

$\Rightarrow x_2 = -3, x_1 = -6 \Rightarrow \boxed{P_0(-6, -3)}$

$R = \{O; e_1, e_2\} \xrightarrow[\text{translație}]{\theta} R' = \{P_0; e_1, e_2\} \xrightarrow[\text{rotație}]{\tau} R'' = \{P_0; e_1', e_2'\}$

$\theta: X = X' + X_0,\ X_0 = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$

$\theta(\Gamma): X'^T A X' + \frac{\Delta}{\delta} = 0$

$Q: \mathbb{R}^2 \to \mathbb{R}, \; Q(x) = x'^T A x' \Rightarrow Q(x) = x_1'^2 - 8x_1'x_2' + 7x_2'$

Aducem Q la forma canonică, folosind metoda valorilor proprii.

$\lambda^2 - Tr(A)\lambda + \det A = 0$

$\lambda^2 - 8\lambda - 9 = 0$

$(\lambda - 9)(\lambda + 1) = 0 \Rightarrow \lambda_1 = 9, \; \lambda_2 = -1$

• $V_{\lambda_1} = \{x \in \mathbb{R}^2 \mid AX = 9X\}$

$(A - 9I_2)X = 0 \quad \begin{cases} -8x_1 - 4x_2 = 0 \\ -4x_1 - 2x_2 = 0 \end{cases} \Rightarrow -2x_1 - x_2 = 0 \Rightarrow$

$\Rightarrow x_2 = -2x_1$

$V_{\lambda_1} = \{(x_1, -2x_1) \mid x_1 \in \mathbb{R}\} = \langle\{(1,-2)\}\rangle \Rightarrow \boxed{e_1' = \frac{1}{\sqrt{5}}(1,-2)}$

• $V_{\lambda_2} = \{x \in \mathbb{R}^2 \mid AX = -X\}$

$(A + I_2)X = 0 \quad \begin{cases} 2x_1 - 4x_2 = 0 \\ -4x_1 + 8x_2 = 0 \end{cases} \Rightarrow x_1 - 2x_2 = 0$

$\Rightarrow x_1 = 2x_2$

$V_{\lambda_2} = \{(2x_2, x_2) \mid x_2 \in \mathbb{R}\} = \langle\{(2,1)\}\rangle \Rightarrow \boxed{e_2' = \frac{1}{\sqrt{5}}(2,1)}$

$\mathcal{T}: x' = RX'', \; R = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$\mathcal{T} \circ \theta(\Gamma): x = RX'' + X_0$

$\lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \frac{\Delta}{\delta} = 0 \Rightarrow$

$\Rightarrow \mathcal{T} \circ \theta(\Gamma): \; 9x_1''^2 - x_2''^2 + 36 = 0$

$\frac{x_1''^2}{4} - \frac{x_2''^2}{36} + 1 = 0$

$-\frac{x_1''^2}{4} + \frac{x_2''}{36} = 1 \Rightarrow \Gamma$ hiperbolă.

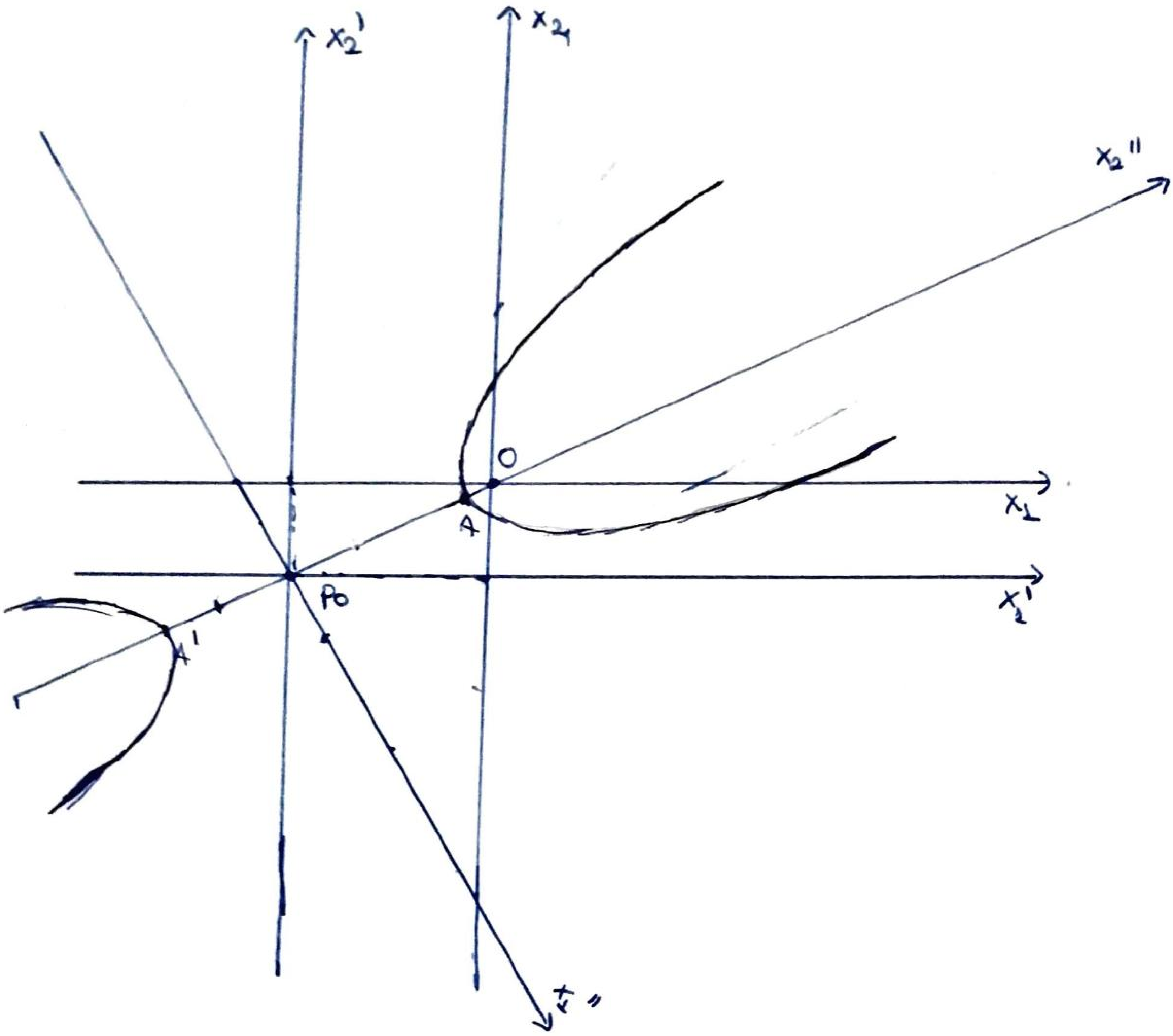
$\mathcal{H}: -\frac{x_1''^2}{4} + \frac{x_2''}{36} = 1$

$a = 2$

$b = 6$

$e_1' = \frac{1}{\sqrt{5}}(1,-2)$

$e_2' = \frac{1}{\sqrt{5}}(2,1)$



⑥ a) $A = [f]_{R_0, R_0} = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{pmatrix}$

- $v_1 = e_1 + e_2$

  $f(v_1) = f(e_1 + e_2) = f(e_1) + f(e_2) = (3, 2, 2) + (-1, 0, -2) =$

  $= (2, 2, 0) = 2(1, 1, 0) = 2v_1 \Rightarrow v_1$ e vector propriu

  $\boxed{\lambda_1 = 2}$ val proprie coresp. lui v_1

- $v_2 = e_2 + e_3$

  $f(v_2) = f(e_2 + e_3) = f(e_2) + f(e_3) = (-1, 0, -2) + (1, 1, 3) = (0, 1, 1) = v_2$

  $\Rightarrow v_2$ e vector propriu

  $\boxed{\lambda_2 = 1}$ val. proprie coresp. lui v_2

• $v_3 = e_1 + e_2 + e_3$

$f(v_3) = f(e_1+e_2+e_3) = f(e_1)+f(e_2)+f(e_3) = (3,2,2)+(-1,0,-2)+(1,1,3)$

$= (3+1-1, 2+1, 2-2+3) = (3,3,3) = 3v_3 \Rightarrow v_3$ e vector propriu

$\boxed{\lambda_3 = 3}$ val. proprie coresp. lui v_3.

b) $\lambda_1 \neq \lambda_2 \neq \lambda_3 \Rightarrow v_1, v_2, v_3$ vectori proprii coresp. la val proprii distincte $\Rightarrow R = \{v_1, v_2, v_3\}$ formează un SLI

$|R| = 3 = \dim_{\mathbb{R}} \mathbb{R}^3$ $\Big| \Rightarrow R$ reper al lui $\mathbb{R}^3$.

• Fie $a, b, c \in \mathbb{R}$ a.î. $v = av_1 + bv_2 + cv_3$

$v = e_1 + 2e_2 + e_3 = a(e_1+e_2) + b(e_2+e_3) + c(e_1+e_2+e_3)$

$= ae_1 + ae_2 + be_2 + be_3 + ce_1 + ce_2 + ce_3$

$= e_1(a+c) + e_2(a+b+c) + e_3(b+c)$

$\Rightarrow \begin{cases} a+c = 1 \\ a+b+c = 2 \\ b+c = 1 \end{cases} \Rightarrow \begin{array}{l} a=1 \\ b=1 \\ c=0 \end{array} \Rightarrow (1,1,0)$ coord. lui v în raport cu R.

$f(v) = f(v_1+v_2) = f(v_1)+f(v_2) \xlongequal{\text{cf. a)}} 2v_1 + v_2 \Rightarrow (2,1,0)$ coord lui $f(u)$ în raport cu R!

7 a) $A = [f]_{R_0, R_0} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

f proiecție $\Leftrightarrow f \circ f = f$

Fie $x \in \mathbb{R}^3$

$f \circ f(x) = f(f(x)) = f\left(\frac{1}{3}(x_1+x_2+x_3, x_1+x_2+x_3, x_1+x_2+x_3)\right) =$

$= f\left(\frac{x_1+x_2+x_3}{3}, \frac{x_1+x_2+x_3}{3}, \frac{x_1+x_2+x_3}{3}\right) =$

$= \frac{1}{3}\left(\cancel{3}\frac{x_1+x_2+x_3}{\cancel{3}}, \cancel{3}\frac{x_1+x_2+x_3}{\cancel{3}}, \cancel{3}\frac{x_1+x_2+x_3}{\cancel{3}}\right) =$

$= \frac{1}{3}(x_1+x_2+x_3, x_1+x_2+x_3, x_1+x_2+x_3) = f(x)$ -)

$\Rightarrow f\circ f = f \Rightarrow$ <u>f proiecție</u>

$f \in Sim(\mathbb{R}^3) \Leftrightarrow A^T = A$

$A^T = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = A \Rightarrow A^T = A \Rightarrow f \in Sim(\mathbb{R}^3)$

a) $P(\lambda) = \det(A - \lambda I_3) = 0$

$\lambda^3 - \sigma_1\lambda^2 + \sigma_2\lambda - \sigma_3 = 0$

$\sigma_1 = \frac{1}{3}(1+1+1) = 1$

$\sigma_2 = \frac{1}{9}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \frac{1}{9}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \frac{1}{9}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

$\sigma_3 = \det A = 0$

$\lambda^3 - \lambda^2 = 0 \Rightarrow \lambda^2(\lambda - 1) = 0 \begin{array}{l} \nearrow \lambda_1 = 0,\ m_1 = 2 \\ \searrow \lambda_2 = 1,\ m_2 = 1 \end{array}$

• $V_{\lambda_1} = \{X \in \mathbb{R}^2 \mid AX = 0\}$ $\begin{cases} \frac{1}{3}(x_1+x_2+x_3) = 0 \\ \frac{1}{3}(x_1+x_2+x_3) = 0 \\ \frac{1}{3}(x_1+x_2+x_3) = 0 \end{cases} \Rightarrow x_1+x_2+x_3 = 0 \Rightarrow x_3 = -x_1 - x_2.$

$V_{\lambda_1} = \{(x_1, x_2, -x_1-x_2) \mid x_1, x_2 \in \mathbb{R}\} = <\{(1,0,-1), (0,1,-1)\}>$

$R_1 = \{(1,0,-1), (0,1,-1)\}$ reper in V_{λ_1}

• $V_{\lambda_2} = \{X \in \mathbb{R}^2 \mid AX = X\}$

$(A - I_3)X = 0$ $\begin{cases} -\frac{2}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 = 0 \\ \frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3 = 0 \\ \frac{1}{3}x_1 + \frac{1}{3}x_2 - \frac{2}{3}x_3 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} -2x_1 + x_2 + x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \\ x_1 + x_2 - 2x_3 = 0 \end{cases} \Rightarrow x_2 = x_1, x_3 = x_1$

$V_{\lambda_2} = \{(x_1, x_1, x_1) \mid x_1 \in \mathbb{R}\} = <\{(1,1,1)\}>$

$R_2 = \{(1,1,1)\}$ reper in V_{λ_2}

$\exists$ un reper $R = R_1 \cup R_2$, R reper în $\mathbb{R}$ aî matricea asociată lui f în raport cu R este diagonală

$R = \{(1,0,-1),(0,1,-1)\} \cup \{(1,1,1)\} = \{(1,0,-1),(0,1,-1),(1,1,1)\}$

$$A' = [f]_{R,R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8) a) $d_1 : x_2 = x_3 = 0$

$$d_1 : \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0} \Rightarrow \begin{cases} x_1 = t \\ x_2 = 0 \\ x_3 = 0 \end{cases} \quad A_1(0,0,0) \quad u = (1,0,0)$$

$$d_2 : \begin{cases} x_2 - 1 = 0 \\ x_1 = x_3 \end{cases} \Rightarrow \begin{cases} x_1 = s \\ x_2 = 1 \\ x_3 = s \end{cases} \quad A_2(0,1,0) \quad v = (1,0,1)$$

$$\begin{vmatrix} u_1 & v_1 & b_1 - a_1 \\ u_2 & v_2 & b_2 - a_2 \\ u_3 & v_3 & b_3 - a_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow d_1, d_2 \text{ necoplan.}$$

Fie d perpendiculara comună

Fie $P_1 = d \cap d_1$, $P_2 = d \cap d_2$

$P_1(t,0,0)$

$P_2(s,1,s)$

$\overrightarrow{P_1P_2} = (s-t, 1, s)$, $u = (1,0,0)$, $v = (1,0,1)$

$$\begin{cases} \langle \overrightarrow{P_1P_2}, u \rangle = 0 \\ \langle \overrightarrow{P_1P_2}, v \rangle = 0 \end{cases} \Rightarrow \begin{cases} s - t = 0 \\ s - t + s = 0 \end{cases} \Rightarrow \begin{matrix} t = 0 \\ s = 0 \end{matrix}$$

$P_1(0,0,0)$, $P_2(0,1,0)$, $\overrightarrow{P_1P_2} = (0,1,0)$

$$d : \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{0}$$

b) $dist(d_1, d_2) = dist(P_1, P_2) = \sqrt{1^2} = 1$

(9) a) $A = \frac{1}{7}\begin{pmatrix} -3 & -2 & 6 \\ 6 & -3 & 2 \\ 2 & 6 & 3 \end{pmatrix}$

$$A^T A = \frac{1}{49}\begin{pmatrix} -3 & 6 & 2 \\ -2 & -3 & 6 \\ 6 & 2 & 3 \end{pmatrix}\begin{pmatrix} -3 & -2 & 6 \\ 6 & -3 & 2 \\ 2 & 6. & 3 \end{pmatrix} = I_3 \quad (1)$$

$$\det A = \frac{1}{343}\begin{vmatrix} -3 & -2 & 6 \\ 6 & -3 & 2 \\ 2 & 6 & 3 \end{vmatrix} = \frac{1}{343}(27+216-8+36+36+36) = 1 \quad (2)$$

$(1),(2) \Rightarrow A \in SO(3) \Rightarrow f = R_\varphi$

e) $\operatorname{Tr} A = \frac{1}{7}(-3 \cancel{-3} + \cancel{3}) = -\frac{3}{7} = 1 + 2\cos\varphi \Rightarrow$

$\Rightarrow 2\cos\varphi = -\frac{3}{7} - 1 = -\frac{10}{7}$

$\cos\varphi = -\frac{5}{7} \Rightarrow \boxed{\varphi = \arccos\left(-\frac{5}{7}\right)}$

axa: $f(x) = x \Rightarrow \begin{cases} -3x_1 - 2x_2 + 6x_3 = 7x_1 \\ 6x_1 - 3x_2 + 2x_3 = 7x_2 \\ 2x_1 + 6x_2 + 3x_3 = 7x_3 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} -10x_1 - 2x_2 + 6x_3 = 0 \\ 6x_1 - 10x_2 + 2x_3 = 0 \\ 2x_1 + 6x_2 - 4x_3 = 0 \end{cases} \Rightarrow \begin{cases} -10x_1 - 2x_2 = -6x_3 \\ 6x_1 - 10x_2 = -2x_3 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} -5x_1 - x_2 = -3x_3 \quad | \cdot (-5) \\ 3x_1 - 5x_2 = -x_3 \end{cases} \Rightarrow \begin{cases} 25x_1 + 5x_2 = 15x_3 \\ 3x_1 - 5x_2 = -x_3 \end{cases}$

$\xrightarrow{(+)}$ $28x_1 = 14x_3 \Rightarrow x_1 = \frac{1}{2}x_3$

$x_2 = 3x_3 - 5x_1 = 3x_3 - \frac{5}{2}x_3 = \frac{1}{2}x_3$

$\langle\{(\frac{1}{2}, \frac{1}{2}, 1)\}\rangle = \langle\{(1, 1, 2)\}\rangle$ axa de rotație