Recitation #7

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Center for Data Science

DS-GA 1014: Optimization and Computational Linear Algebra
for Data Science

Next recitation session on Wed 28th October will be a midterm review.

The office hours on Thr 29th October is moved to Wed 28th 9 AM EST after the recitation

Start to study early, the midterm will have time pressure

recall: how to find an eigenvalue

Exercise 0

Let $A \in \mathbb{R}^{n \times n}$ have an eigenvalues . How would you find its eigenvector $v \in \mathbb{R}^n$ that is $Av = \lambda v$

recall: spectral theorem

Spectral theorem

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, Then there is a orthonormal basis of $\mathbb{R}^n$ composed of eigenvectors of A.

Let's get things clear

- If A is symmetric then $A^T = A$ and it diagonalizes $A = PDP^{-1}$
- P is an orthonormal matrix therefore $P^T = P^{-1}$

Exercise 1

A matrix $M \in \mathbb{R}^{n \times n}$ is diagonalizable if M has a basis of eigenvectors that span $\mathbb{R}^n$.

- If M is diagonalizable, then M is invertible ✓True   ✗False
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Exercise 2

- The matrix corresponding to an orthogonal projection is symmetric ✓True   ✗False
- The matrix corresponding to an orthogonal projection is orthogonal ✓True   ✗False
- There can exists a set of n non-zro orthogonal vectors in $\mathbb{R}^m$ if $n > m$ ✓True   ✗False

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Exercise 3

Let $A \in \mathbb{R}^{n \times n}$ be a positive semi-definite matrix. Show that then there exists a matrix $S \in \mathbb{R}^{n \times n}$ also positive-semidefinite such that $A = S^2$

Exercise 4

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and let $\lambda_1, ..., \lambda_n \in \mathbb{R}$ be its eigenvalues. Show that

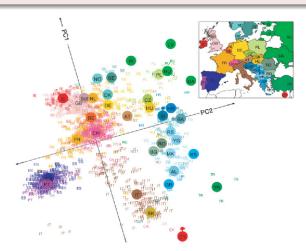A is positive semi-definite if and only if $\lambda_i \geq 0$ for all $i = 1..n$

Exercise 5

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix then $\lambda_{min} = \min_{\|v\|=1} v^T A v$

PCA

Problem: we have data $a_1, .., a_n \in \mathbb{R}^d$ with d very large. We wish to represent the dataset in a lower dimension.

Method

To perform PCA on a dataset we have to:

1. Calculate th sample covariance matrix S
2. Find the eigenvector v_1 of S with largest eigenvalue. v_1 is the first singular vector
3. Continue finding as many eigenvectors v_k of S as necessary by decreasing order with respect to the respective eigenvalues λ_k.
4. The dimensionally reduced data set will be composed by the projection of the data into the $v_1, .. v_k$ eigenbasis.

Exercise 6

Suppose there are two eigenvectors of the covariance matrix that correspond to large eigenvalues and the rest of eigenvalues are small. How do we interpret this geometrically?

recall: Singular Value decomposition

SVD

Let $A \in \mathbb{R}^{n \times m}$. Then there exists two orthogonal matrices $U \in \mathbb{R}^{n \times n}$, $V \in \mathbb{R}^{m \times m}$ and a matrix $\Sigma \in \mathbb{R}^{n \times m}$ such that

$$A = U \Sigma V^T$$

with $\Sigma_{11}, \Sigma_{22}, ..., \Sigma_{11} \geq 0$ and $\Sigma_{ij} \neq 0$ for $i \neq j$

Exercise 7

Let $A \in \mathbb{R}^{n \times m}$ give a method to compute numerically $rank(A)$ using SVD.

Exercise 8

Let $A \in \mathbb{R}^{n \times m}$. Show that AA^T and $A^T A$ have the same non-zero eigenvalues.

Exercise 9

Let $A \in \mathbb{R}^{n \times m}$ explain why the set $\{Ax \text{ s.t. } \|x\| = 1\}$ is an ellipsoid.

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