Recitation #2 (Section 03)

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DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

Linear transformations: recall & practice

Linear transformation L

A function $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if

- for all $v \in \mathbb{R}^n$ and for all $\lambda \in \mathbb{R}$ it is true that $L(\lambda \cdot v) = \lambda \cdot L(v)$
- for all $v, w \in \mathbb{R}^n$ it is true that $L(w + v) = L(v) + L(w)$

Note that if L is a linear transformation *then* $L(\vec{0}_n) = \vec{0}_m$ (useful to quickly see if a function is NOT linear) if $L(0) \neq 0$ L NOT linear

Exercise 1

Is the function f linear? $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f((a, b)) = (2a, a + b)$

2) $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$

$f((a_1, b_1) + (a_2, b_2)) = f((a_1 + a_2, b_1 + b_2))$

$= (2(a_1 + a_2), (a_1 + a_2) + b_1 + b_2) = (2(a_1), a_1 + b_1)$

$+ (2(a_2), a_2, b_2)$

Exercise 2

Is the function f linear? $f : \mathbb{R}^2 \to \mathbb{R}^3$,
$f((a, b)) = (a + b, 2a + 2b, 1)$

$f((0,0)) = (0,0,\underline{\underline{1}}) \rightarrow$ NOT linear

What about $f((a,b)) = (a+b, 2a+2b, \underline{\underline{0}})$?

$\hookrightarrow$ Yes, Now f is linear $\longrightarrow$ exercise

Linear transformations: practice

Exercise 3

Is the function f linear? $f : \mathbb{R}^2 \to \mathbb{R}$, $f((a,b)) = \sqrt{a^2+b^2}$ NO

① $f(\lambda(a,b)) = f((\lambda a, \lambda b)) = \sqrt{\lambda^2 a^2 + \lambda^2 b^2} = \sqrt{\lambda^2} \cdot \sqrt{a^2+b^2}$

$= \lambda \sqrt{a^2+b^2} = \lambda f((a,b))$

② $f((a_1+a_2, b_1+b_2)) = \sqrt{(a_1+a_2)^2 + (b_1+b_2)^2}$

$(a+b)^2 \neq a^2+b^2$

$(a+b)^2 = a^2+b^2+2ab$ | $\sqrt{a_1^2+a_2^2 + b_1^2+b_2^2 + 2a_1a_2 + 2b_1b_2}$

Linear transformations: practice

Exercise 4.1

If $v, w \in \mathbb{R}^n$ are linearly independent vectors, are $v, v + w \in \mathbb{R}^n$ also *linearly* independent? yes

$$\alpha v + \beta w = 0 \iff \alpha, \beta = 0$$

$$\alpha v + \beta(v + w) = 0 \qquad \alpha v + \beta v + \beta w = 0$$

$$(\alpha + \beta) v + \beta w = 0 \implies \begin{cases} \beta = 0 \\ \alpha + \beta = 0 \end{cases} \implies \alpha = 0$$

Exercise 4.2

If $v, w \in \mathbb{R}^n$ are linearly independent vectors, are $v, \lambda w \in \mathbb{R}^n$ also *linearly* independent? ($\lambda \neq 0$) yes

Start: $\alpha v + \beta w = 0 \iff \alpha, \beta = 0$

Goal: $\underbrace{\alpha v}_{OK} + \underbrace{\beta(\lambda w)}_{\beta' = \lambda w} = 0 \iff \alpha, \beta = 0$

$\alpha v + \beta' w = 0$ But v, w are linearly ind !! $\to \boxed{\alpha = 0}, \beta' = 0$

$\beta' = \beta \lambda \implies \boxed{\beta = 0}$ $\square$

Linear transformations: matrices

Recall

All linear transformations (synonym map) $L : \mathbb{R}^n \to \mathbb{R}^m$ can be represented as a matrix $\tilde{L}$ with respect to the basis $\tilde{e}_1, \ldots, \tilde{e}_n$ of $\mathbb{R}^n$. If the selected basis of $\mathbb{R}^n$ is the canonical basis $e_1, .., e_n$ then the matrix $\tilde{L}$ is called *canonical matrix*.

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\tilde{L} = \begin{pmatrix} | & | & & | \\ L(e_1) & L(e_2) & \ldots & L(e_n) \\ | & | & & | \end{pmatrix}$$

Matrices: practice

Exercise 5

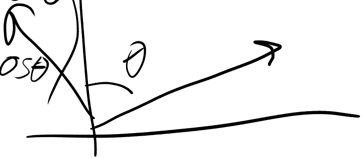
Given the linear transformtion $R : \mathbb{R}^2 \to \mathbb{R}^2$ with $R((x,y)) = (\frac{\sqrt{3}}{2}x - \frac{1}{2}y,\ \frac{1}{2}x + \frac{\sqrt{3}}{2}y)$ find the canonical matrix of R.

$\tilde{R}$ $\underline{2} \times \underline{2}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

rotation

$$\tilde{R} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



$\theta = 30^\circ$

Subspaces

Kernel

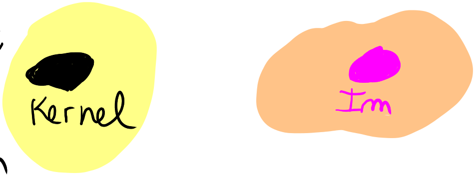
Given a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$, the kernel $Ker(L) = \{v \in \mathbb{R}^n | L(v) = \vec{0}_m\}$

Image

The image is $Im(L) = \{w \in \mathbb{R}^m |\ \text{exists}\ \ v \in \mathbb{R}^n\ \ \text{with}\ \ L(v) = w\}$

$L : \mathbb{R}^m \longrightarrow \mathbb{R}^m$

dim(Ker) can be from 0, 1, 2, . . . , n
Depends on the problem



dim(Im) can be from 0 to 1, 2, ... m
depends on the problem

Kernel and image: practice

Exercise 5

Find the kernel of the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}$,
$f((a, b)) = 2a - 3b$
Hint: The kernel is a subspace. $\dim(\operatorname{Ker}(f)) = 1, 2$

- $\operatorname{Ker}(f) = \{ (a,b) \in \mathbb{R}^2 \mid f(a,b) = 0 \}$

$= \{ \quad " \quad \mid 2a - 3b = 0 \}$

$= \{ \quad " \quad \mid a = \frac{3}{2} b \}$

$b = 1 \to a = 3/2 \quad (3/2, 1) \in \operatorname{Ker}(f)$

$\operatorname{Ker}(f) = \langle (3/2, 1) \rangle$

Exercise 6

Find the image of the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}$, $f((a, b)) = 2a - 3b$

$\dim(\mathrm{Im}(f)) = 0, 1$

$f((0,0)) = 0$

$0 \in \mathrm{Im}(f)$

$2a - 3b = \square \in \mathbb{R}$? I can output any number from $\mathbb{R}$ no constrains

$\square = \pi$

$a = 0 \quad b = -\frac{\pi}{3}$

$\dim(\mathrm{Im}(f)) = 1$

$\boxed{\mathrm{Im}(f) = \mathbb{R}}$

Kernel and image: practice

Exercise 7

Find the kernel of $\tilde{L} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

$$\mathrm{Kn}(\tilde{L}) = \{ v \in \mathbb{R}^2 \mid \tilde{L} v = 0 \}$$

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \underbrace{\boxed{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}}_{\text{Kernel}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \text{Gauss elimination}$$

Solve the system of equations in order to find the Kernel

Gauss elimination

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & -1 & 0 \end{array}\right)$$

get rid of this element by row#1 - 2row#2

$$\begin{array}{rrr|r} & 2 & -1 & 0 \\ + & -2 & 0 & 0 \\ \hline & 0 & -1 & 0 \end{array}$$

new row#2

Now we can solve the system

$$\left.\begin{array}{r} 2v_1 - v_2 = 0 \\ -v_2 = 0 \end{array}\right\}$$

$$\boxed{v_2 = 0,\ v_1 = 0}$$

Only solution!

$$\boxed{\mathrm{Ker}(\tilde{L}) = \{(0,0)\} \quad \dim(\tilde{L}) = 0}$$

Kernel and image: practice

Exercise 7: Another way of solving it

Note that $\tilde{L} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ is an invertible matrix!

Prove that for an invertible matrix A then always $ker(A) = \{\vec{0}\}$

In the last exercise 7, we saw $Ker(\tilde{L}) = \{(0,0)\}$ via Gaussian elimination

$\tilde{L}$ is invertible, all invertible matrices have $\{\vec{0}\}$ as a kernel

*Proof: A is invertible, has A^{-1} such that $A^{-1}A = Id_2$, $AA^{-1} = Id_2$

To find $Ker(A)$ we need to solve $\boxed{Av = 0}$ for v

But, $A^{-1}(Av) = A^{-1}0$

$Id_2 v = 0 \Rightarrow \boxed{v = 0}$ the Kernel(A) only has the zero

Questions