Recitation #6

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DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

recall: Markov chain

Markov chain

A sequence of random variables $(X_0, X_1, ...)$ is a Markov chain with state space E and transition matrix P if for all $t \geq 0$,

$$\mathbb{P}(X_{t+1} = y | X = x_0, ..., X_t = x_t) = P(x_t, y)$$

for all $x_0, ..., x_t$ such that $\mathbb{P}(X_0 = x_0, .., X_t = x_t) > 0$
Intuitively, "if the future only depends on the present and not the past"

recall: Markov chain

Stochastic matrix

Let $P \in \mathbb{R}^{n \times n}$ be a matrix, we say P is stochastic if:

- $P_{i,j} \geq 0$ for all $1 \leq i, j \leq n$
- $\sum_{i=1}^{n} P_{i,j} = 1$ for all $1 \leq j \leq n$

practice: stochastic matrix

Exercise 1

Let $A, B \in \mathbb{R}^{n\times n}$ be stochastic matrices then

1. A is invertible ✓True ✗False
2. The eigenvector corresponding to the largest eigenvalue of A is unique ✓True ✗False
3. A does not have $\lambda = 0$ as eigenvalue ✓True ✗False

practice: stochastic matrix

Exercise 2

Let $A, B \in \mathbb{R}^{n \times n}$ then be stochastic matrices then AB is also a stochastic matrix.
Hint: express the condition "sum of each column $= 1$" as a matrix multiplication.

recall: Markov chain

Proposition

For a Markov chain with the notation above, for all $t \geq 0$

$$\boxed{x^{(t+1)} = Px^{(t)}} \quad \text{and consequently,} \quad \boxed{x^{(t)} = P^t x^{(0)}}$$

and recall that the limit $t \to \infty$ is $\boxed{x^{(t)} \to \mu}$ for some $\mu \in \Delta_n$ (probability vector).

Perron-Frobenius Theorem

Let P be a stochastic matrix such that exists $k \geq 1$ such that all the entries of P^k are strictly positive. Then,

- $\lambda = 1$ is an eigenvalues of P with μ its an eigenvector.
- The eigenvalue $\lambda = 1$ has multiplicity equal to $ker(P - Id) = Span(\mu)$
- For all probability vectors $x \in \Delta_n$ we have $P^t x \to \mu$ in the limit $t \to \infty$

practice: change of basis and diagonalization

Exercise 3

Let $A \in \mathbb{R}^{n \times n}$ be a matrix with eigenvectors $v_1, .., v_n$ and associated eigenvalues $\lambda_1, .., \lambda_n$. Let $x = \alpha_1 v_1 + ... + \alpha_n v_n$ be a vector in $\in \mathbb{R}^n$. Show

1. Let P be a linear transformation that maps the canonical basis $e_1, .., e_n$ of $\mathbb{R}^n$ to the eigenvector basis of A: $v_1, .., v_n$. Write P explicitly.
2. What is PDP^{-1}? ($D = diag(\lambda_1, .., \lambda_n)$)
3. Simplify $(PDP^{-1})^k$ for $k \in \mathbb{N}$
4. If $A = PDP^{-1}$, give an interpretation of the action of A

practice: change of basis and diagonalization

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practice: change of basis and diagonalization

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What is PDP^{-1}? ($D = diag(\lambda_1, .., \lambda_n)$)
Simplify $(PDP^{-1})^k$ for $k \in \mathbb{N}$

practice: change of basis and diagonalization

Exercise 3

If $A = PDP^{-1}$, give an interpretation of the action of A

recall: Spectral theorem

Spectral theorem !!

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then, there exists an orthonormal basis of $\mathbb{R}^n$ composed of eigenvectors of A

Exercise 4

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Give a vector $v \in \mathbb{R}^n$ with $\|v\| = 1$ such that $\|Av\|$ is maximized.

practice: Spectral theorem

Exercise 5

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with eigenvalues $\lambda_1, .., \lambda_n$ and orthonormal eigenvectors $v_1, .., v_n$.
Give an orthonormal basis of $Ker(A)$ and $Im(A)$ in terms of $v_1, .., v_n$ only.

practice: Spectral theorem

Exercise 6(*)

Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices. Show that $AB = BA$ if and only if A and B diagonalize in the same basis. (Does the same hold if we just assume that A, B are diagonalizable?)