Recitation #4

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Center for Data Science

DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

recall: norm and inner product

Inner product

Let V be a vector space. An inner product on V is a function $\langle , \rangle$ from pairs of vectors $V \times V$ to $\mathbb{R}$ that holds the following points

1. Symmetry: $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$
2. Linearity: $\langle u + w, v \rangle = \langle u, v \rangle + \langle w, v \rangle$ and same for scalar multiplication.
3. Positive-defined: $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$

practice: inner product

Exercise 1

Explain why the following functions $\langle \cdot, \cdot \rangle$ are not an inner product

1. $\langle x, y \rangle = x_1 y_2 + x_2 y_3 + x_3 y_1$
2. $\langle x, y \rangle = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$
3. $\langle x, y \rangle = x_1 y_1 + x_2 y_2$

practice: inner product

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recall: Norm

Norm induced by inner product

(Proposition) If $\langle \cdot, \cdot \rangle$ is an inner product on V then $\|v\| = \sqrt{\langle v, v \rangle}$ is its induced norm.

Exercise 2

Compute $\|ax\|$ for $a \in \mathbb{R}$ scalar and $x \in \mathbb{R}^n$ vector.

practice: norm

Exercise 3

When does $\|x + y\| = \|x\| + \|y\|$ for $x, y \in \mathbb{R}^n$?

Exercise 4

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $x \in \mathbb{R}^n$ a vector. Show that

$$\|Ax\| \leq \|x\| \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} {A_{ij}}^2}$$

Hint: Cauchy-Schwarz inequality $\|\langle u, v \rangle\|^2 \leq \langle u, u \rangle \cdot \langle v, v \rangle$

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recall: orthogonal projection

Orthogonality

Two vectors $u, v \in \mathbb{R}^n$ are orthogonal if and only if $\langle u, v \rangle = 0$

Projection

Let S be a subspace of $\mathbb{R}^n$. The orthogonal projection of a vector x onto S is defined as the vector $P_S(x) \in S$ such that minimizes the distance to x:

$$\boxed{P_S(x) = \text{argmin}_{y \in S} \|x - y\|}$$

practice: orthogonality

Exercise 5

Prove that if $v_1, ..., v_k \in \mathbb{R}^n$ are orthogonal vectors then they also are linearly independent.

practice: orthogonal projection

Exercise 6

Show that if $P_S(x)$ denotes the orthogonal projection onto subspace S then

1. $\|P_S(x)\| \leq \|x\|$
2. $x - P_S(x)$ is orthogonal to S

Recall: if $v_1, .., v_k$ is an orthonormal basis of S then the projection onto S can be written as $P_S(x) = \langle x, v_1 \rangle v_1 + ... + \langle x, v_k \rangle v_k$ (Exercise: prove it).

practice: orthogonal projection

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practice: orthogonal spaces

Exercise 7

Let S, U be subspaces of a vector space V. Prove the following statement: $S \subset U \longrightarrow U^{\perp} \subset S^{\perp}$

Exercise 8

Let $A \in \mathbb{R}^{n \times m}$ be a matrix. Assume the Euclidean inner product. Prove that

$$Im(A^T) = ker(A)^\perp$$

Hint: This is an equality between sets so you need to prove that one is inside the other and viceversa. Start with $\subset$ and use Ex. 6 for the other.

practice: orthogonal spaces

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practice: orthogonal spaces

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Questions

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