Midterm Review

Irina Espejo (iem244@nyu.edu)

Center for Data Science

DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

Problem 0.1

Let $A, B \in^{n \times n}$. For each the following subset of ^{n} below, say whether it is a subspace of ^{n} and justify your answer:

1. $E_1 = \{x \in^n \mid Ax = 0\}$.
2. $E_2 = \{x \in^n \mid Ax = Bx\}$.
3. $E_3 = \{x \in^n \mid Ax = e_1\}$.
4. $E_4 = \{x \in^n \mid Ax \in (e_1)\}$.

Problem 0.1

Problem 0.2

True or False: There exists matrices $M \in^{2\times 3}$ such that $\dim((M)) = 1$ and $(M) = 2$.

Problem 0.2

Problem 0.3

Let $n > m$ and $A \in^{n \times m}$. Assume that A has "full rank", meaning that $(A) = \min(n, m) = m$.

1. Does $Ax = b$ has a solution for all $b \in^n$? (Prove or give a counter example)
2. Can there exists two vectors $x \neq x'$ such that $Ax = Ax'$? (Prove or give a counter example).

Problem 0.3

Problem 0.4

Let $n < m$ and $A \in^{n\times m}$. Assume that A has "full rank", meaning that $(A) = \min(n, m) = n$.

1. Does $Ax = b$ has a solution for all $b \in^{n}$? (Prove or give a counter example)
2. Can there exists two vectors $x \neq x'$ such that $Ax = Ax'$? (Prove or give a counter example).

Problem 0.4

Problem 0.5

True or False: There exists a family of k non-zero orthogonal vectors of ^{n}, for some $k > n$.

Problem 0.5

Problem 0.6

Let $A \in^{n \times m}$.

1. Prove that (A) and $\Im(A)$ are orthogonal to each other, i.e. that for all $x \in (A)$ and $y \in \Im(A)$ we have $x \perp y$.
2. Show that $(A) = \Im(A)^{\perp}$.

Problem 0.6

Problem 0.7

True or False: The matrix of an orthogonal projection is symmetric.

Problem 0.7

Problem 0.8

True or False: The matrix of an orthogonal projection is orthogonal.

Problem 0.8

Problem 0.9

Let S be a subspace of ^{n} and let P_S be the orthogonal projection onto S. Show that $\dim(S) = (P_S)$.

Problem 0.9

Problem 0.10

True or False: Let $A, B \in^{n \times n}$. Assume that $v \in^n$ is an eigenvector of A and B.

1. Is v an eigenvector of $A + B$?
2. Is v an eigenvector of AB ?

Problem 0.10

Problem 0.11

Let $A \in^{n\times n}$ and let $v_1, v_2 \in^n$ be two eigenvectors of A, associated with the same eigenvalue λ.
Show that any non-zero eigenvector in (v_1, v_2) is an eigenvector of A, associated with λ.

Problem 0.11

Problem 0.12

Let $A \in^{n \times n}$ be a symmetric matrix. Let $(v_1, v_2, \ldots, v_n)$ be an orthonormal family of eigenvectors of A, associated to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Give an orthonormal basis of (A) and $\Im(A)$ in terms of the v_i's.

Problem 0.12

Problem 0.13

Let $A \in^{n\times n}$ be a symmetric matrix, that satisfies $A^2 =$. Show that the matrix

$$M = \frac{1}{2}(A+)$$

is the matrix of an orthogonal projection.

Problem 0.13

Problem 0.14

Let $\rho \in (0,1)$. Let $v_1, \ldots, v_k \in^n$ such that

$$\|v_i\| = 1 \qquad \text{and} \qquad \langle v_i, v_j \rangle = \rho \qquad \text{for all} \quad i \neq j.$$

Show that $k \leq n$.

Problem 0.14