Recitation #5

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DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

recall: Orthogonal matrices

Orthogonal matrix

Let $A \in \mathbb{R}^{n\times n}$ be a matrix. A is an orthogonal matrix if its columns form an orthogonal family of $\mathbb{R}^n$ (therefore linearly independent and basis of $\mathbb{R}^n$)

Alert!: orthogonal vectors given an inner product

$$\langle x, y\rangle = x^T y$$

$$\begin{pmatrix} | & | & | & | \end{pmatrix}$$

Properties of Orthogonal matrices

The following are equivalent:

1. A is orthogonal
2. $AA^T = Id$
3. $A^TA = Id$

$$A^T = A^{-1}$$

Exercise for home: re-prove it.

Exercise 1

Prove that the product of two orthogonal matrices is also an orthogonal matrix.

Goal : $A \cdot B$ is orthogonal

Start : $A, B \in \mathbb{R}^{n \times n}$ are orthogonal

Show: $\underbrace{(A \cdot B)(A \cdot B)^T}$ $= Id$

$$A \cdot \underbrace{B\, B^T}_{Id} A^T \underset{\uparrow}{=} A\, A^T \underset{\uparrow}{=} Id \qquad \square$$

B orthogonal

A orthogonal

Exercise 2

Let $Q \in \mathbb{R}^{n \times n}$ be an orthogonal matrix and let $x, y \in \mathbb{R}^n$.
Show that $\langle Qx, Qy \rangle = \langle x, y \rangle$

$$\langle x, y \rangle = x^T y$$

$$\langle Qx, Qy \rangle = (Qx)^T Qy = x^T \underbrace{Q^T Q}_{Id} y = x^T y \quad \square$$

Q is orthogonal

QR decomposition

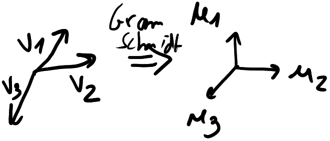
Exercise 3

Let $A \in \mathbb{R}^{m\times n}$ be a matrix with linearly independent columns. Show that A can be written as $A = Q\,R$ where $Q \in \mathbb{R}^{m\times m}$ is an orthogonal matrix and $R \in \mathbb{R}^{m\times n}$ an upper triangular matrix.
Hint: apply Gram-Schmidt to the columns of A.

$$R = \begin{pmatrix} \diagdown & * \\ 0 & \diagdown \end{pmatrix}$$

$$A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

$v_1, \dots, v_n$ lin. ind



Gram-Schmidt

$$\begin{cases} u_1 = v_1 \\ u_2 = v_2 - \overbrace{\mathrm{proj}_{u_1}(v_2)}^{*}\, u_1 \\ \vdots \\ u_m = v_n - \mathrm{proj}_{u_{n-1}}(v_n) - \dots - \mathrm{proj}_{u_1}(v_2)\, u_1 \end{cases}$$

practice: orthogonal matrices

Exercise 3

Let $A \in \mathbb{R}^{m \times n}$ be a matrix with linearly independent columns. Show that A can be written as $A = Q\,R$ where $Q \in \mathbb{R}^{m \times m}$ is an orthogonal matrix and $R \in \mathbb{R}^{m \times n}$ an upper triangular matrix.
Hint: apply Gram-Schmidt to the columns of A.

QR $\widetilde{=}$ Gram-Schmidt

$$Q = \begin{pmatrix} | & & | \\ u_1 & \dots & u_m \\ | & & | \end{pmatrix}$$

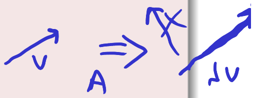
$$R = \begin{pmatrix} 1 & \neq & \neq \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$r_{ij} = \mathrm{proj}_{u_i}(v_j)$

Eigenvalue, eigenvector

Let $A \in \mathbb{R}^{n\times n}$. A non-zero vector $v \in \mathbb{R}^n$ ($v \neq 0$) is an eigenvector of A if there exists $\lambda \in \mathbb{R}$ such that

$$\boxed{A\underbrace{v}_{\text{eigenvec}} = \underbrace{\lambda}_{\text{eigenval}} v}$$



The scalar λ is the eigenvalue of A associated to v.

important directions

Eigenspace

The following set is called eigenspace of A associated to λ.

$$\boxed{E_\lambda(A) = \{x \in \mathbb{R}^n | Ax = \lambda x\} = ker(A - \lambda Id)}$$

The dimension of the eigenspace is called multiplicity of λ.

$v \in E_\lambda(A)$ $\quad \boxed{Av = \lambda v}$ $\quad Av - \lambda v = 0$

$\boxed{(A - \lambda Id)\, v = 0}$

Exercise 4

Show that the eigenspace $E_\lambda(A)$ is a subspace.

- $v, w \in E_\lambda(A)$ $\quad Av = \lambda v$

$\quad Aw = \lambda w$

Show : $A(v+w) = \lambda(v+w)$

$$A(v+w) = Av + Aw = \lambda v + \lambda w = \lambda(v+w) \square$$

- αv, $v \in E_\lambda(A)$ $\alpha \in \mathbb{R}$

$$A(\alpha v) = \alpha(Av) = \alpha(\lambda v) = \lambda(\alpha v) \square$$

Diagonalizable matrix

A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if and only if there exists a matrix $P \in \mathbb{R}^{n \times n}$ such that

$A \sim D$ P is a change of basis

$$\boxed{A = P^{-1}DP} \quad \text{where} \quad D = diag(\lambda_1, \ldots, \lambda_n)$$

eigenvalues

Exercise 5

Are all real $\mathbb{R}$ matrices diagonalizable? Why?

✓True ✗False $\mathbb{R}$

$A = (\quad)$ $\lambda = i$

$\times$ diagonalizable $\mathbb{R}$

✓ diagonalizable $\mathbb{C}$

Exercise 5

Are all real matrices diagonalizable? Why?

✓True ✗False

Counterexample: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\underset{\neq 0}{A} = P^{-1} \cancel{D}^{\,0} P$$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} v_2 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$\nearrow$ $\underline{\underline{\lambda = 0}}$ $\quad v_2 = \lambda v_1 \Rightarrow v_2 = 0$

$\searrow$ $\cancel{\lambda \neq 0 \quad 0 = \lambda v_2 \Rightarrow v_2 = 0 \quad v_2 = \lambda v_1 \Rightarrow v_1 = 0} \Big\} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Exercise 6

Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix.
Prove that the eigenvalues of Q can only be -1, $+1$.

$\Rightarrow$ Recall: $x \in \mathbb{R}^n \quad \|Qx\| = \|x\|$

Lets find $v \in \mathbb{R}^n \quad Qv = \lambda v$

$$\|Qv\| = \|\lambda v\|$$

recall $\rightarrow$ $\|$

$$\cancel{\|v\|} = |\lambda| \cancel{\|v\|} \Rightarrow |\lambda| = 1 \rightarrow \lambda = \pm 1 \quad \square$$

Exercise 7

Let $A \in \mathbb{R}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and respective eigenvectors $v_1, v_2, ..., v_n$. $\lambda_1 \neq \dots \neq \lambda_n$!!
Prove that $v_1, v_2, ..., v_n$ are linearly independent.

Show $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$ $\alpha_1 = \dots = \alpha_n = 0$

Note that we can rewrite equation in vector form $\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, $v = \begin{pmatrix} -v_1- \\ \vdots \\ -v_n- \end{pmatrix} \neq 0$ as $\alpha^T v = 0$

Now apply A at each side and use eigenvector hypothesis

Exercise 7

Let $A \in \mathbb{R}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and respective eigenvectors $v_1, v_2, \ldots, v_n$.
Prove that $v_1, v_2, \ldots, v_n$ are linearly independent.

$$A\alpha^T V = 0 \qquad \rightarrow \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$A(\alpha_1 v_1 + \ldots + \alpha_n v_n) = 0$$

$$\lambda_1 \alpha_1 v_1 + \ldots + \lambda_n \alpha_n v_n = 0$$

$$A\alpha^T V = \Lambda \alpha^T V = 0$$

But then A diagonalizes as $A = P \Lambda P^{-1}$

so $AP = P\Lambda$ $\quad$ $P = v$ (Think about this)

$A\alpha^T P = \underline{AP\alpha^T} = 0$ $\quad$ But $AP \neq 0$ so $\boxed{\vec{\alpha} = 0}$ $\square$

practice: diagonal matrices

Exercise 8 (*)

Suppose $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix.
Give a vector $v \in \mathbb{R}^n$ with $\|v\| = 1$ such that $\|Dv\|$ is maximized.

Note:

$$\|Dv\| = \sum_{i=1}^{n} (d_{ii} v_i)^2 \leq (\max_i d_{ii}^2) \overbrace{\sum_{i=1}^{n} v_i^2}^{\|v\|=1} =$$

$$= \max_i d_{ii}^2$$

Take $v = e_j$ where $|d_{jj}|$ is the max absolute value of D $\square$