Recitation #12

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DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues $\lambda_1, .., \lambda_n$.
Prove that $\|Ax\| \leq \max_i |\lambda_i| \|x\|$ for any $x \in \mathbb{R}^n$.

$A v_1 = \lambda_1 v_1$ by spectral th. $v_1, \dots, v_n$ orthonormal basis

$\Rightarrow x = \alpha_1 v_1 + \dots + \alpha_n v_n$

$$Ax = A(\alpha_1 v_1 + \dots + \alpha_n v_n)$$
$$= \alpha_1 \lambda_1 v_1 + \dots + \alpha_n \lambda_n v_n$$

$$\|Ax\| = \|\alpha_1 \lambda_1 v_1 + \dots + \alpha_n \lambda_n v_n\|$$

Exercise 4, 2018 review

$$\|x+y\| \leq \|x\| + \|y\|$$ triangular inequality

$$\|Ax\| = \|\alpha_1 \lambda_1 v_1 + \ldots + \alpha_n \lambda_n v_n\| \leq$$

$$|\lambda_1| \, \|\alpha_1 v_1\| + \ldots + |\lambda_n| \, \|\alpha_n v_n\| \leq$$

$$\max_i |\lambda_i| \left(\|\alpha_1 v_1\| + \ldots + \|\alpha_n v_n\| \right) =$$

$$\max_i |\lambda_i| \left(|\alpha_1| + \ldots + |\alpha_n| \right)$$

$$= \max_i |\lambda_i| \, \|x\|$$

Exercise 8, 2018 review

Suppose $A \in \mathbb{R}^{m \times n}$ has rank m. Prove AA^T is invertible

$\to$ dimensions?

$m \times m$

Show $\operatorname{rank}(AA^T) = m$

$A = U \Sigma V^T$ $\quad \operatorname{rank}(A) = \operatorname{rank}(\Sigma) = m$

Σ $m \times n$

$$\begin{pmatrix} \# & & & & 0 & - & 0 \\ & \# & & & | & & | \\ & & \# & & | & & | \\ & & & \# & 0 & & 0 \end{pmatrix}$$

$A^T = (U \Sigma V^T)^T = V \Sigma^T U^T$ $m \times n$

$$AA^T = U\Sigma \cancel{V^T}\cancel{V}\Sigma^T U^T \qquad U, V \text{ orthogonal}$$

$$= U \underbrace{\Sigma \Sigma^T}_{\Sigma_{AA^T}} U^T$$

Show $\Sigma_{AA^T} = \Sigma \Sigma^T$ has rank $= m$

$$\begin{pmatrix} * \neq 0 & & \\ & \ddots & \\ & & * \neq 0 \end{pmatrix} \begin{pmatrix} * \neq 0 & & \\ & \ddots & \\ & & * \end{pmatrix} = \begin{pmatrix} \ddots \end{pmatrix}$$

$m \times m$

m nonzero entrances

$\operatorname{rank}(\Sigma\Sigma^T) = m$

Therefore $\operatorname{rank}(AA^T) = m$

Exercise 9, 2018 review

Consider the optimization problem

$$\begin{aligned} &\text{minimize}_x \; \|x\|^2 \\ &\text{subject to } Ax = b \end{aligned}$$

$f(x) = \|x\|^2$

$h(x) = Ax - b$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are fixed and $b \in Im(A^T)$.

a) Prove that any minimizer x^* must belong to $Im(A)$

b) Give a formula for the minimizer x^* and show it is unique

a) Goal $x^* \in Im(A^T)$

Constraind opt $\to$ Lagrange multipliers

$\exists \lambda \in \mathbb{R}$ $\quad \nabla f(x) + \lambda \nabla h(x) = 0$ $\to$ Solve x^*, λ

Exercise 9, 2018 review

$$\nabla f(x) = 2x \qquad \nabla h(x) = A^T$$

$$2x_i^* + \lambda_i A_{i*} = 0$$

$$\boxed{2x^* + \vec{\lambda} A^T = 0}$$

$$x^* = -\frac{1}{2}\vec{\lambda} A^T \rightarrow x^* \in \mathrm{Im}(A^T)$$

b) We have done that.

Exercise 9, 2018 review

Exercise 10, 2018 review

Let $A \in \mathbb{R}^{\cancel{m \times n}\ n \times m}$ and $B \in \mathbb{R}^{n \times k}$, we define the block matrix $C \in \mathbb{R}^{n \times (m+k)}$ by

$$C = [A \ \ B]$$

Either prove the following statement or give a counterexample

$$rank(C) = rank(A) + rank(B)$$

False, counter example: $A = (1)$ $B = (1)$

$rank(A) = 1$

$rank(B) = 1$

$C = (1 \ 1)$ $rank(C) = 1 \neq 1 + 1$

Exercise 10, 2018 review

Let $A \in \mathbb{R}^{n \times n}$ have the unusual property that the image space (column space) $Im(A)$ is equal to its kernel.

a) What can we say about A^2?

b) Give an example of such an A

a) $\mathrm{Ker}(A) = \mathrm{Im}(A)$

$$Av = 0 \iff v = Aw$$

exists $w \in \mathbb{R}^n$

$$Av = 0$$

$$AAw = 0 \to A^2 w = 0$$

$$\mathrm{Ker}(A^2) = \mathbb{R}^n$$

Exercise 20, 2018 review

b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\dim(\mathrm{Im}(A)) + \dim(\mathrm{Ker}(A)) = n$$

$$\dim(\mathrm{Im}(A)) = \dim(\mathrm{Ker}(A)) = n/2$$

$n = 2$

Exercise 25, 2018 review

Let $A \in \mathbb{R}^{n \times n}$ and consider its SVD decomposition $A = U \Sigma V^T$. Let $A' = U \Sigma' V^T$ where Σ' is obtained from Σ by replacing every entry by zero except for the entry corresponding to the largets singular value.

a) Show that A‘ is the best rank 1 approximation of A in the Forbenius norm, meaning that A' is the solution to $\min_{B:rank(B)=1} \|B - A\|_F$
b) Show that A' is the best rank 1 approximation of A in the spectral norm, meaning that A' is the solution to $\min_{B:rank(B)=1} \|B - A\|_{\cancel{F}}$

Exercise 25, 2018 review

Exercise 25, 2018 review

Exercise 0.9, 2019 review

For each of the following statement, say if they are true or false and justify your answer

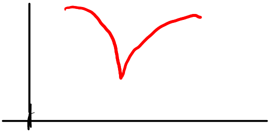
a) If a continuous function $f : \mathbb{R} \to \mathbb{R}$ has a unique minimizer then f is convex

b) If a continuous function $f : \mathbb{R} \to \mathbb{R}$ is such that f is decreasing on $(-\infty, x_0]$ and increasing on $(x_0, +\infty]$ is convex

c) A twixe differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative f' is non-decreasing is convex

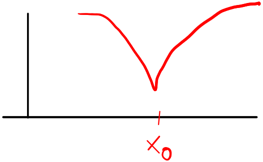
Exercise 0.9, 2019 review

a)



False

b)



False

c) f' non decreasing

this means $f'' \geq 0$

f convex $\iff f'' \geq 0$

True