Recitation #3 (Section 03)

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DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

recall: the Rank

Rank

Let $x_1, .., x_k \in \mathbb{R}^n$ we define the rank as

$$\boxed{\text{rank}(x_1, .., x_k) = \dim(Span(x_1, .., x_k))}$$

Informally: rank = "the number of linearly independent vectors among $x_1, .., x_k$"

practice: the Rank

Exercise 1

Calculate the rank of

$$A = \begin{pmatrix} 7 & 3 \\ 19 & 19 \\ -1 & 5 \end{pmatrix}$$

① Gaussian elimination

$$A = \begin{pmatrix} 7 & 3 \\ 19 & 19 \\ -1 & 5 \end{pmatrix} \to \begin{pmatrix} 7 & 3 \\ 19 & 19 \\ 0 & 26 \end{pmatrix} \to \begin{pmatrix} 7 & 3 \\ 0 & 76 \\ 0 & 26 \end{pmatrix} \to \begin{pmatrix} 7 & 3 \\ 0 & 76 \\ 0 & 0 \end{pmatrix}$$

② rank(A) = "# cols/row nonzero"

$$\boxed{\text{rank}(A) = 2}$$

practice: the Rank

Exercise 2

Let $A, B \in \mathbb{R}^{n\times n}$ be two matrices then

$$rank(A+B) = rank(A) + rank(B)$$

True or false? ✓True [✓ ✗False]

$$\overset{n\times 1}{A=(1)} \qquad \overset{1\times 1}{B=(-1)}$$

$$A+B=(0)$$

$$\left.\begin{array}{l} rank(A+B)=0 \\ rank(A)=rank(B)=1 \end{array}\right\} 0 \neq 1$$

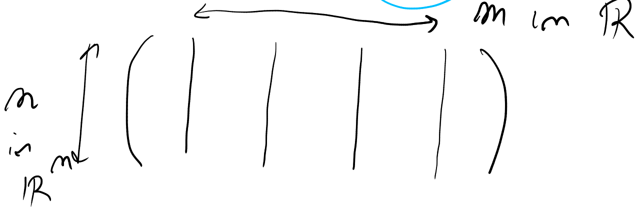
Rank nullity theorem

Rank nullity theorem

Let $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation then

$$rank(L) + dim(ker(L)) = m$$

$$\tilde{L} = n \times \boxed{m}$$



practice: the Rank

Exercise 3

Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Prove that $rank(A) = rank(A^T)$.
(That is "the rank by columns = the rank by rows")

We will proceed by steps:

1. Prove that $x^T A^T A x \geq 0$ for all $x \in \mathbb{R}^n$
2. When is $x^T A^T A x = 0$?
3. Prove that $ker(A) = ker(A^T A)$
4. Use this to show $rank(A) = rank(A^T A)$
5. Show that $rank(A) = rank(A^T)$

$x = 0$

$Ax = 0$

$x \in ker(A)$

Exercise 3.1: Prove that $x^T A^T A x \geq 0$ for all $x \in \mathbb{R}^n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$A \quad m \times n$

$x \quad n \times 1$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$\boxed{\text{When is } x^T A^T A x = 0?}$

$x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ x is generic

$$x^T A^T A x = (Ax)^T (Ax) = (y_1 \cdots y_m) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \underset{\overset{\vee\!/}{0}}{y_1^2 + \dots + y_m^2}$$

$$Ax = \begin{pmatrix} \sum_j a_{1j} x_j \\ \dots \\ \sum_j a_{mj} x_j \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

Exercise 3.3: Prove that $ker(A) = ker(A^T A)$

① $Ker(A) \subseteq Ker(A^T A)$

$v \in Ker(A) : Av = 0 \rightarrow A^T(Av) = A^T 0 = 0$

$(A^T A)v = 0 \rightarrow v \in Ker(A^T A)$ □

② $Ker(A^T A) \subseteq Ker(A)$

$v \in Ker(A^T A) \rightarrow (A^T A)v = 0 \rightarrow \underset{\neq 0}{A^T} \underbrace{(Av)}_{= 0} = 0$

$A \neq 0 \rightarrow Av = 0 \rightarrow v \in Ker(A)$

□

Exercise 3.4: Use this to show $rank(A) = rank(A^T A)$

$Ker(A) = Ker(A^T A)$

$A \quad m \times n$

- Rank nullity theorem:

$$\begin{array}{ccccc} rank(A) & + & \dim(Ker(A)) & = & n \\ & & \| & & \| \\ rank(A^T A) & + & \dim(Ker(A^T A)) & = & \boxed{n} \xrightarrow{} n \times m \quad m \times n \end{array}$$

$A^T A \rightarrow n \times n$

$\hookrightarrow rank(A) = rank(A^T A) \qquad \square$

Exercise 3.5: Show that $rank(A) = rank(A^T)$

Solution

1. Prove $rank(A) \leq rank(A^T)$
   Recall that $\boxed{rank(T_1 T_2) \leq min(rank(T_1), rank(T_2))}$.
   Then, starting from $rank(A) = rank(A^T A)$ we can say that

   $$rank(A) = rank(A^T A) \leq min(rank(A^T), rank(A)) \leq rank(A^T)$$

2. Prove $rank(A^T) \leq rank(A)$
   Same as in 1) but interchanging the roles of A and A^T.
   Note that $(A^T)^T = A$.
   Strating from $rank(A) = rank(A^T A)$ and switching roles we have $rank(A^T) = rank(AA^T)$. Now, same as before

   $$rank(A^T) = rank(AA^T) \leq min(rank(A), rank(A^T)) \leq rank(A)$$

And we are done.

recall: invertible matrix

Invertible matrix

A matrix $M \in \mathbb{R}^{n \times n}$ is invertible if there exists another matrix $M^{-1} \in \mathbb{R}^{n \times n}$ such that

$$\boxed{M\,M^{-1}} = \boxed{M^{-1}\,M} = Id_n$$

Exercise 4

Let $M \in \mathbb{R}^{n \times n}$ be an invertible matrix, then its transpose M^T is also invertible.

$$\boxed{MM^{-1} = Id}$$

$$(MM^{-1})^T = Id^T$$

$$M^T(M^{-1})^T = Id$$

$$\underline{M^T(M^T)^{-1} = Id}$$

↳ M^T has an inverse

The same goes for $M^{-1}M = Id$

☐

Exercise 5

Which of the following are equivalent for $A \in \mathbb{R}^{m \times n}$?

1. The columns of A are linearly independent
2. The rows of A are linearly independent
3. $rank(A) = m$
4. $rank(A) = n$
5. The equation $Ax = 0$ has one solution
6. The equation $Ax = b$ has at least one solution
7. $Im(A^T) = \mathbb{R}^n$
8. $Im(A) = \mathbb{R}^m$
9. The linear transformation corresponding to A is injective (one-to-one)
10. The linear transformation corresponding to A is exhaustive (onto)
11. $ker(A) = \{0\}$
12. The span of the columns of A is $\mathbb{R}^m$

Exercise 5

Main result to solve the exercise :

$B \in \mathbb{R}^{p \times q}$

$$\mathrm{Ker}(B) = \{0\} \iff \mathrm{rank}(B) = q$$

$$\Updownarrow$$

$$\dim(\mathrm{Im}(B)) = q$$

onto function

Exercise 5

Solution

1. Equivalent to $rank(A) = n$
2. Equivalent to $rank(A) = m$
3. $rank(A) = m$
4. $rank(A) = n$
5. Equivalent to $ker(A) = \{0\}$, by rank nullity theorem equivalent to $rank(A) = n$
6. This means $dim(Im(A)) = m$ so equivalent to $rank(A) = m$
7. Equivalent to $rank(A) = n$
8. Equivalent to $rank(A) = m$
9. This means $ker(A) = \{0\}$ so by rank nullity theorem is equivalent to $rank(A) = n$
10. This means $dim(Im(A) = m)$ so equivalent to $rank(A) = m$
11. By rank nullity theorem equivalent to $rank(A) = n$
12. Equivalent to $rank(A) = m$