

Recitation #3 (Section 03)

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DS-GA 1014: Optimization and Computational Linear Algebra
for Data Science



recall: the Rank

Rank

Let $x_1, \dots, x_k \in \mathbb{R}^n$ we define the rank as

$$\text{rank}(x_1, \dots, x_k) = \dim(\text{Span}(x_1, \dots, x_k))$$

Informally: rank = "the number of linearly independent vectors among x_1, \dots, x_k "

practice: the Rank

Exercise 1

Calculate the rank of

$$A = \begin{pmatrix} 7 & 3 \\ 19 & 19 \\ -1 & 5 \end{pmatrix}$$

Exercise 2

Let $A, B \in \mathbb{R}^{n \times n}$ be two matrices then

$$\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$$

True or false? ✓True ✗False

Rank nullity theorem

Rank nullity theorem

Let $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation then

$$\text{rank}(L) + \dim(\ker(L)) = m$$

Exercise 3

Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Prove that $\text{rank}(A) = \text{rank}(A^T)$.
(That is "the rank by columns = the rank by rows")

We will proceed by steps:

- 1 Prove that $x^T A^T A x \geq 0$ for all $x \in \mathbb{R}^n$
- 2 When is $x^T A^T A x = 0$?
- 3 Prove that $\ker(A) = \ker(A^T A)$
- 4 Use this to show $\text{rank}(A) = \text{rank}(A^T A)$
- 5 Show that $\text{rank}(A) = \text{rank}(A^T)$

Exercise 3.1: Prove that $x^T A^T A x \geq 0$ for all $x \in \mathbb{R}^n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

When is $x^T A^T A x = 0$?

Exercise 3.3: Prove that $\ker(A) = \ker(A^T A)$

Exercise 3.4: Use this to show $\text{rank}(A) = \text{rank}(A^T)$

Exercise 3.5: Show that $\text{rank}(A) = \text{rank}(A^T)$

recall: invertible matrix

Invertible matrix

A matrix $M \in \mathbb{R}^{n \times n}$ is invertible if there exists another matrix $M^{-1} \in \mathbb{R}^{n \times n}$ such that

$$M M^{-1} = M^{-1} M = Id_n$$

Exercise 4

Let $M \in \mathbb{R}^{n \times n}$ be an invertible matrix, then its transpose M^T is also invertible.

Exercise 5

Which of the following are equivalent for $A \in \mathbb{R}^{m \times n}$?

- ❶ The columns of A are linearly independent
- ❷ The rows of A are linearly independent
- ❸ $\text{rank}(A) = m$
- ❹ $\text{rank}(A) = n$
- ❺ The equation $Ax = 0$ has one solution
- ❻ The equation $Ax = b$ has at least one solution
- ❼ $\text{Im}(A^T) = \mathbb{R}^n$
- ❽ $\text{Im}(A) = \mathbb{R}^m$
- ❾ The linear transformation corresponding to A is injective (one-to-one)
- ❿ The linear transformation corresponding to A is exhaustive (onto)
- ⓫ $\ker(A) = \{0\}$
- ⓬ The span of the columns of A is \mathbb{R}^m

Exercise 5

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