# Recitation #4

Irina Espejo (iem244@nyu.edu)

Center for Data Science

DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

# recall: norm and inner product

## Inner product

Let $V$ be a vector space. An inner product on $V$ is a function $\langle , \rangle$ from pairs of vectors $V \times V$ to $\mathbb{R}$ that holds the following points

1. Symmetry: $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$
2. Linearity: $\langle u + w, v\rangle = \langle u, v\rangle + \langle w, v\rangle$ and same for scalar multiplication.
3. Positive-defined: $\langle v, v\rangle \geq 0$ with equality if and only if $v = 0$

**Exercise 1**

Explain why the following functions $\langle \cdot, \cdot \rangle$ are not an inner product

1. $\langle x, y \rangle = x_1 y_2 + x_2 y_3 + x_3 y_1$
2. $\langle x, y \rangle = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$
3. $\langle x, y \rangle = x_1 y_1 + x_2 y_2$

1) Symmetry ✗ because

$\langle y, x \rangle = y_1 x_2 + y_2 x_3 + y_3 x_1$

and

$y_1 x_2 + y_2 x_3 + y_3 x_1 \neq x_1 y_2 + x_2 y_3 + x_3 y_1$

# practice: inner product

**Exercise 1**

Explain why the following functions $\langle \cdot, \cdot \rangle$ are not an inner product

1. $\langle x, y \rangle = x_1 y_2 + x_2 y_3 + x_3 y_1$
2. $\langle x, y \rangle = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$
3. $\langle x, y \rangle = x_1 y_1 + x_2 y_2$

2) linearity X

$\langle x+w, y \rangle \neq \langle x, y \rangle + \langle w, y \rangle$

because $(a+b)^2 \neq a^2 + b^2$

**Exercise 1**

Explain why the following functions $\langle \cdot, \cdot \rangle$ are not an inner product

1. $\langle x, y \rangle = x_1 y_2 + x_2 y_3 + x_3 y_1$
2. $\langle x, y \rangle = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$
3. $\langle x, y \rangle = x_1 y_1 + x_2 y_2$ for $x, y \in \mathbb{R}^3$ !!

3) Typo: $x, y \in \mathbb{R}^3$ if $x, y \in \mathbb{R}^2$ then it would be an inner product !!

Not positive definite because take $x \in \mathbb{R}^3$

$x = (0\ 0\ 1)$

$\langle x, x \rangle = 0$ but $x \neq 0$

**Norm induced by inner product**

(Proposition) If $\langle \cdot, \cdot \rangle$ is an inner product on $V$ then $\|v\| = \sqrt{\langle v, v \rangle}$ is its induced norm.

$v$

**Exercise 2**

Compute $\|ax\|$ for $a \in \mathbb{R}$ scalar and $x \in \mathbb{R}^n$ vector.

$$\|ax\| = \sqrt{\langle ax, ax \rangle} \underset{\substack{\uparrow \\ \text{linearity}}}{=} \sqrt{a^2 \langle x, x \rangle} = |a|\,\|x\|$$

# practice: norm

$\|v\| = \sqrt{\langle v, v\rangle}$

**Exercise 3**

When does $\|x + y\| = \|x\| + \|y\|$ for $x, y \in \mathbb{R}^n$?

$\|x+y\| = \sqrt{\langle x+y, x+y\rangle} = \sqrt{\langle x,x\rangle + \langle y,y\rangle + 2\langle x,y\rangle}$

- $\|x+y\|^2 = \langle x,x\rangle + \langle y,y\rangle + 2\langle x,y\rangle$
- $(\|x\| + \|y\|)^2 = \langle x,x\rangle + \langle y,y\rangle + 2\underbrace{\sqrt{\langle x,x\rangle \cdot \langle y,y\rangle}}_{\|x\| \cdot \|y\|}$

$\cancel{\langle x,x\rangle} + \cancel{\langle y,y\rangle} + \cancel{2}\langle x,y\rangle = \cancel{\langle x,x\rangle} + \cancel{\langle y,y\rangle} + \cancel{2}\sqrt{\langle x,x\rangle\langle y,y\rangle}$

$\boxed{\langle x,y\rangle = \|x\| \cdot \|y\|}$


$y = ax$

$\langle x,y\rangle = \|x\|\,\|y\| \cos\theta \qquad \cos\theta = 1 \rightarrow \theta = 0^\circ$

## Exercise 4

Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $x \in \mathbb{R}^n$ a vector. Show that

$$\|Ax\| \leq \|x\| \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} {A_{ij}}^2}$$

Hint: Cauchy-Schwarz inequality $\|\langle u, v\rangle\|^2 \leq \langle u, u\rangle \cdot \langle v, v\rangle$

$A = \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}$ as row vectors

$$\|Ax\|^2 = \underbrace{\langle r_1, x\rangle^2}_{} + \dots + \underbrace{\langle r_m, x\rangle^2}_{}$$

$(\equiv)(|) = (|)$

$$\leq \|r_1\|^2 \|x\|^2 + \dots + \|r_m\| \|x\|^2$$

$$= \|x\|^2 \left( \|r_1\|^2 + \dots + \|r_m\| \right)$$

**Exercise 4**

Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $x \in \mathbb{R}^n$ a vector. Show that

$$\|Ax\| \leq \|x\| \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} {A_{ij}}^2}$$

$$\|Ax\|^2 \leq \|x\|^2 \sum \sum {A_{ij}}^2$$

$$= \|x\|^2 \left( \|r_1\|^2 + \dots + \|r_m\|^2 \right)$$

$$A$$

$$= \|x\|^2 \left( \sum_{i=1}^{m} \underbrace{\|r_i\|^2} \right) \qquad \left( \cancel{r_i} \right)$$

$$= \|x\|^2 \left( \sum_{i=1}^{m} \sum_{j=1}^{n} {A_{ij}}^2 \right) \quad \square$$

# recall: orthogonal projection

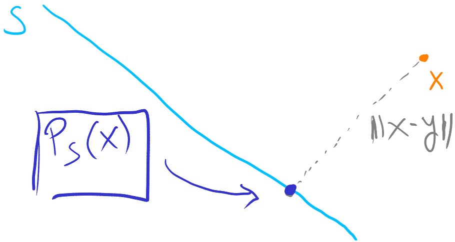
## Orthogonality

Two vectors $u, v \in \mathbb{R}^n$ are orthogonal if and only if $\langle u, v \rangle = 0$

## Projection

Let $S$ be a subspace of $\mathbb{R}^n$. The orthogonal projection of a vector $x$ onto S is defined as the vector $P_S(x) \in S$ such that minimizes the distance to $x$:

$$\boxed{P_S(x) = \operatorname{argmin}_{y \in S} \|x - y\|}$$



**Exercise 5**

Prove that if $v_1, \ldots, v_k \in \mathbb{R}^n$ are orthogonal vectors then they also are linearly independent.

We know $\langle v_i, v_j \rangle = 0$ for $i,j = 1 \ldots k$ $i \neq j$

GOAL: $\alpha_1 v_1 + \ldots + \alpha_k v_k = 0 \Rightarrow \alpha_1 = \ldots = \alpha_k = 0$

$$\alpha_1 v_1 + \ldots + \alpha_k v_k = 0$$

$$\times v_1 : \alpha_1 \langle v_1, v_1 \rangle + \ldots + \alpha_k \underbrace{\langle v_1, v_k \rangle}_{=0} = 0$$

$$\boxed{\alpha_1 \langle v_1, v_1 \rangle = 0} \rightarrow \boxed{\alpha_1 = 0}$$

+ do the same $v_2, \ldots, v_k$ $\square$

## Exercise 6

Show that if $P_S(x)$ denotes the orthogonal projection onto subspace $S$ then

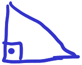
1. $\|P_S(x)\| \leq \|x\|$
2. $x - P_S(x)$ is orthogonal to $S$

$\langle\ \rangle = 0$ $\|v\| = 1$

Recall: if $v_1, .., v_k$ is an orthonormal basis of $S$ then the projection onto $S$ can be written as $P_S(x) = \langle x, v_1\rangle v_1 + ... + \langle x, v_k\rangle v_k$ (Exercise: prove it).

$\in S$

1) $\|P_S(x)\|^2 = \left\| \sum_{i=1}^{k} \langle x, v_i \rangle v_i \right\|^2 \leq \sum_i \|\langle x, v_i\rangle v_i\|^2$



triangular/pythagorean inequality

$= \|x\|^2$ because orthonormal basis

# practice: orthogonal projection

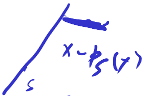
**Exercise 6**

Show that if $P_S(x)$ denotes the orthogonal projection onto subspace $S$ then

1. $\|P_S(x)\| \leq \|x\|$
2. $x - P_S(x)$ is orthogonal to $S$

Recall: if $v_1, .., v_k$ is an orthonormal basis of $S$ then the projection onto $S$ can be written as $P_S(x) = \langle x, v_1 \rangle v_1 + ... + \langle x, v_k \rangle v_k$

2)




$S$

Let $s \in S$ show $\langle x - P_S(x), s \rangle = 0$

Extend $\{v_1, .., v_k\} \in S$ to an orthogonal basis of $\mathbb{R}^n$ $\{v_{k+1}, ..., v_n\}$

$\boxed{x = \alpha_1 v_1 + ... + \alpha_n v_n}$ $\boxed{P_S(x) = \beta_1 v_1 + ... + \beta_k v_k}$ $\beta_1 = \alpha_1$, $\vdots$ $\beta_k = \alpha_k$

$x - P_S(x) = + \alpha_{k+1} v_{k+1} + ... + \alpha_n v_n$

# practice: orthogonal spaces

But $s \in S$ so $\langle x - P_S(x), s \rangle = 0$ $\square$
and orthogonal

**Exercise 7**

Let $S, U$ be subspaces of a vector space $V$. Prove the following statement: $S \subset U \longrightarrow U^\perp \subset S^\perp$


we know that $v \in S \to v \in U$

let $w \in U^\perp$ $\langle w, u \rangle = 0$ for all $u \in U$

We need to show $w \in S^\perp$

let $s \in S$
$\downarrow$
$s \in U$

$\langle w, s \rangle = 0$ $\Big\}$ Because of the two hypothesis. $\square$

## Exercise 8

Let $A \in \mathbb{R}^{n \times m}$ be a matrix. Assume the Euclidean inner product. Prove that

$$Im(A^T) = ker(A)^{\perp}$$

Hint: This is an equality between sets so you need to prove that one is inside the other and viceversa. Start with $\subset$ and use Ex. 6 for the other.

- Proof $Im(A^T) \subset Ker(A)^{\perp}$

Let $x \in Im(A^T)$ then $\exists y$ such that $A^T y = x$

Let $v \in Ker(A)$ We need to show that $\langle x, v \rangle = 0$

Calculate $\langle x, v \rangle = x^T v \underset{\uparrow \, x \in Im(A^T)}{=} (A^T y)^T v = y^T (Av) \underset{\uparrow \, v \in Ker(A)}{=} 0 \; \square$

## Exercise 8

Let $A \in \mathbb{R}^{n \times m}$ be a matrix. Assume the Euclidean inner product. Prove that

$$Im(A^T) = ker(A)^{\perp}$$

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- Proof $Ker(A)^{\perp} \subset Im(A^T)$

We will use Ex 7: $Im(A^T)^{\perp} \subset Ker(A)$ will imply automatically $Ker(A)^{\perp} \subset Im(A^T)$

Let $v \in Im(A^T)^{\perp}$ and $u \in Im(A^T)$ then $\langle v, u \rangle = 0$ and $\exists\, x$ such that $u = A^T x$

So, $\langle v, A^T x \rangle = \langle A^T x, v \rangle = x^T A v = 0$ implies $Av = 0$ $v \in Ker(A)$ $\square$