Recitation #3 (Section 03)

Irina Espejo (iem244@nyu.edu)

Center for Data Science

DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

recall: the Rank

Rank

Let $x_1, .., x_k \in \mathbb{R}^n$ we define the rank as

$$\boxed{\text{rank}(x_1, .., x_k) = \dim(Span(x_1, .., x_k))}$$

Informally: rank = "the number of linearly independent vectors among $x_1, .., x_k$"

practice: the Rank

Exercise 1

Calculate the rank of

$$A = \begin{pmatrix} 7 & 3 \\ 19 & 19 \\ -1 & 5 \end{pmatrix}$$

practice: the Rank

Exercise 2

Let $A, B \in \mathbb{R}^{n \times n}$ be two matrices then

$$rank(A + B) = rank(A) + rank(B)$$

True or false? ✓True ✗False

Rank nullity theorem

Rank nullity theorem

Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation then

$$rank(L) + dim(ker(L)) = m$$

practice: the Rank

Exercise 3

Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Prove that $rank(A) = rank(A^T)$.
(That is "the rank by columns = the rank by rows")

We will proceed by steps:

1. Prove that $x^T A^T A x \geq 0$ for all $x \in \mathbb{R}^n$
2. When is $x^T A^T A x = 0$?
3. Prove that $ker(A) = ker(A^T A)$
4. Use this to show $rank(A) = rank(A^T A)$
5. Show that $rank(A) = rank(A^T)$

Exercise 3.1: Prove that $x^T A^T A x \geq 0$ for all $x \in \mathbb{R}^n$

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

When is $x^T A^T A x = 0$?

Exercise 3.3: Prove that $ker(A) = ker(A^T A)$

Exercise 3.4: Use this to show $rank(A) = rank(A^T A)$

Exercise 3.5: Show that $rank(A) = rank(A^T)$

Solution

1. Prove $rank(A) \geq rank(A^T)$
   Recall that $\boxed{rank(T_1 T_2) \leq min(rank(T_1), rank(T_2))}$.
   Then, starting from $rank(A) = rank(A^T A)$ we can say that

   $$rank(A) = rank(A^T A) \leq min(rank(A^T), rank(A)) \leq rank(A^T)$$

2. Prove $rank(A^T) \geq rank(A)$
   Same as in 1) but interchanging the roles of A and A^T.
   Note that $(A^T)^T = A$.
   Strating from $rank(A) = rank(A^T A)$ and switching roles we have $rank(A^T) = rank(AA^T)$. Now, same as before

   $$rank(A^T) = rank(AA^T) \leq min(rank(A), rank(A^T)) \leq rank(A)$$

And we are done.

recall: invertible matrix

Invertible matrix

A matrix $M \in \mathbb{R}^{n \times n}$ is invertible if there exists another matrix $M^{-1} \in \mathbb{R}^{n \times n}$ such that

$$M\, M^{-1} = M^{-1}\, M = Id_n$$

Exercise 4

Let $M \in \mathbb{R}^{n \times n}$ be an invertible matrix, then its transpose M^T is also invertible.

Exercise 5

Which of the following are equivalent for $A \in \mathbb{R}^{m \times n}$?

1. The columns of A are linearly independent
2. The rows of A are linearly independent
3. $rank(A) = m$
4. $rank(A) = n$
5. The equation $Ax = 0$ has one solution
6. The equation $Ax = b$ has at least one solution for all b (assuming $b \neq 0$)
7. $Im(A^T) = \mathbb{R}^n$
8. $Im(A) = \mathbb{R}^m$
9. The linear transformation corresponding to A is injective (one-to-one)
10. The linear transformation corresponding to A is exhaustive (onto)
11. $ker(A) = \{0\}$
12. The span of the columns of A is $\mathbb{R}^m$

Exercise 5

Solution

1. Equivalent to $rank(A) = n$
2. Equivalent to $rank(A) = m$
3. $rank(A) = m$
4. $rank(A) = n$
5. Equivalent to $ker(A) = \{0\}$, by rank nullity theorem equivalent to $rank(A) = n$
6. This means $dim(Im(A)) = m$ so equivalent to $rank(A) = m$
7. Equivalent to $rank(A) = n$
8. Equivalent to $rank(A) = m$
9. This means $ker(A) = \{0\}$ so by rank nullity theorem is equivalent to $rank(A) = n$
10. This means $dim(Im(A) = m)$ so equivalent to $rank(A) = m$
11. By rank nullity theorem equivalent to $rank(A) = n$
12. Equivalent to $rank(A) = m$