Recitation #3 (Section 03)

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DS-GA 1014: Optimization and Computational Linear Algebra for Data Science



recall: the Rank

Rank

Let $x_1, ..., x_k \in \mathbb{R}^n$ we define the rank as

$$\mathsf{rank}(x_1,..,x_k) = \mathsf{dim}(\mathit{Span}(x_1,..,x_k))$$

Informally: rank = "the number of linearly independent vectors among $x_1, ..., x_k$ "

practice: the Rank

Exercise 1

Calculate the rank of

$$A = \begin{pmatrix} 7 & 3 \\ 19 & 19 \\ -1 & 5 \end{pmatrix}$$

practice: the Rank

Exercise 2

Let $A, B \in \mathbb{R}^{n \times n}$ be two matrices then

$$rank(A + B) = rank(A) + rank(B)$$

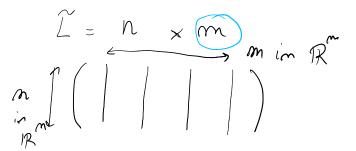
True or false?
$$\sqrt{\text{True}}$$
 $\sqrt{\text{False}}$
 $A = (A)$ $B = (-1)$
 $A + B = (0)$
 $(m \times (A + B) = 0$
 $(m \times (A) = (m \times (B) = 1)$

Rank nullity theorem

Rank nullity theorem

Let $L: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation then

$$rank(L) + dim(ker(L)) = m$$



practice: the Rank

Exercise 3

Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Prove that $rank(A) = rank(A^T)$. (That is "the rank by columns = the rank by rows")

We will proceed by steps:

- Prove that $x^T A^T A x \ge 0$ for all $x \in \mathbb{R}^n$ When is $x^T A^T A x = 0$?
- 3 Prove that $ker(A) = ker(A^T A)$
- **4** Use this to show $rank(A) = rank(A^T A)$
- **3** Show that $rank(A) = rank(A^T)$

Exercise 3.1: Prove that $x^T A^T A x \ge 0$ for all $x \in \mathbb{R}^n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad A \sim M$$

$$X \sim M$$

$$X$$

Exercise 3.3: Prove that $ker(A) = ker(A^T A)$

$$0 \text{ Ker}(A) \subseteq \text{Ker}(A^TA)$$

$$v \in \text{Ker}(A) \circ Av = 0 \rightarrow A^TAv) = A^T0 = 0$$

$$(A^TA) v = 0 \rightarrow v \in \text{Ker}(A^TA) D$$

$$2 kn(A^{T}A) \subseteq kn(A)$$

$$v \in kn(A^{T}A) \Rightarrow (A^{T}A) v = 0 \Rightarrow A^{T}(Av) = 0$$

$$A \neq 0 \Rightarrow Av = 0 \Rightarrow v \in kn(A)$$

Exercise 3.4: Use this to show $rank(A) = rank(A^{\prime})$

Amxm

· Rank mullity theorem:

romk (AA) +dim (Ker(AA)) = (m)

 $L \rightarrow ram K(\underline{A}) = ram K(\underline{A}'\underline{A})$

Exercise 3.5: Show that $rank(A) = rank(A^T)$

Solution

- Prove $rank(A) \ge rank(A^T)$ Recall that $rank(T_1T_2) \le min(rank(T_1), rank(T_2))$. Then, starting from $rank(A) = rank(A^TA)$ we can say that $rank(A) = rank(A^TA) \le min(rank(A^T), rank(A)) \le rank(A^T)$
- Prove $rank(A^T) \ge rank(A)$ Same as in 1) but interchanging the roles of A and A^T . Note that $(A^T)^T = A$. Strating from $rank(A) = rank(A^TA)$ and switching roles we have $rank(A^T) = rank(AA^T)$. Now, same as before

$$rank(A^T) = rank(AA^T) \le min(rank(A), rank(A^T)) \le rank(A)$$

And we are done.

recall: invertible matrix

Invertible matrix

A matrix $M \in \mathbb{R}^{n \times n}$ is invertible if there exists another matrix $M^{-1} \in \mathbb{R}^{n \times n}$ such that

$$MM^{-1} = M^{-1}M = Id_n$$

Exercise 4

Let $M \in \mathbb{R}^{n \times n}$ be an invertible matrix, then its transpose M^T is also invertible.

$$(MM) = Id$$
 $MM = Id$
 $M = Id$

Exercise 5

Which of the following are equivalent for $A \in \mathbb{R}^{m \times n}$?

- The columns of A are linearly independent
- 2 The rows of A are linearly independent
- rank(A) = m
- \bullet rank(A) = n
- **5** The equation Ax = 0 has one solution
- **1** The equation Ax = b has at least one solution

- The linear transformation corresponding to A is injective (one-to-one)
- \odot The linear transformation corresponding to A is exhaustive (onto)
- **1** $ker(A) = \{0\}$
- **4** The span of the columns of A is \mathbb{R}^m

Exercise 5

Exercise 5

Solution

- Equivalent to rank(A) = n
- 2 Equivalent to rank(A) = m
- rank(A) = m
- \bullet rank(A) = n
- Equivalent to $ker(A) = \{0\}$, by rank nullity theorem equivalent to rank(A) = n
- This means dim(Im(A)) = m so equivalent to rank(A) = m
- **1** Equivalent to rank(A) = n
- **3** Equivalent to rank(A) = m
- This means $ker(A) = \{0\}$ so by rank nullity theorem is equivalent to rank(A) = n
- This means dim(Im(A) = m) so equivalent to rank(A) = m
- **1** By rank nullity theorem equivalent to rank(A) = n
- ② Equivalent to rank(A) = m