Recitation #7

Irina Espejo (iem244@nyu.edu)

Center for Data Science

DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

Announcement

Next recitation session on Wed 28th October will be a midterm review.

The office hours on Thr 29th October is moved to Wed 28th 9 AM EST after the recitation

Start to study early, the midterm will have time pressure

recall: how to find an eigenvalue

Exercise 0

Let $A \in \mathbb{R}^{n\times n}$ have an eigenvalues . How would you find its eigenvector $v \in \mathbb{R}^n$ that is $Av = \lambda v$

$$\begin{pmatrix} \cdot & - & - \\ \cdot & - & - \\ - & - & - \end{pmatrix}$$

$$Av = \lambda v$$

$$Av - \lambda v = 0$$

$$\underbrace{(A - \lambda Id)}_{=0} v = 0$$

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 - \lambda \cdot 1 & 0 \\ 3 & 1 - \lambda \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

recall: spectral theorem

Spectral theorem

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, Then there is a orthonormal basis of $\mathbb{R}^n$ composed of eigenvectors of A.

Let's get things clear

- If A is symmetric then $A^T = A$ and it diagonalizes $A = PDP^{-1}$
- P is an orthonormal matrix therefore $P^T = P^{-1}$

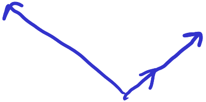
Exercise 1

A matrix $M \in \mathbb{R}^{n\times n}$ is diagonalizable ~~if M has a basis of eigenvectors that span $\mathbb{R}^n$.~~

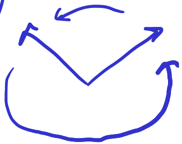
- If M is diagonalizable, then M is invertible ✓True ✗False
- If M is invertible, then M is diagonalizable ✓True ✗False

1) $M = P\underbrace{D}P^{-1}$ $\quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Counterexample $\quad M = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



2) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



Exercise 2

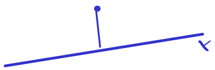
- The matrix corresponding to an orthogonal projection is symmetric ✓True ✗False
- The matrix corresponding to an orthogonal projection is orthogonal ✓True ✗False
- There can exists a set of n non-zro orthogonal vectors in $\mathbb{R}^m$ if $n > m$ ✓True ✗False

1) $P = VV^T$

2) $\mathbb{R}^2$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$

rank no full

$\begin{pmatrix} | & | & | & | \end{pmatrix}$ → orthogonal vectors



practice: spectral theorem

Exercise 2

- The matrix corresponding to an orthogonal projection is symmetric ✓True ✗False
- The matrix corresponding to an orthogonal projection is orthogonal ✓True ✗False
- There can exists a set of n non-zro orthogonal vectors in $\mathbb{R}^m$ if $n > m$ ✓True ✗False

lin ind

3)

$\mathbb{R}^m$ $v_1 \dots v_m$

$n > m$ ——— $v_{m-1} \dots v_n$

very useful

Exercise 3

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $\lambda_1, ..., \lambda_n \in \mathbb{R}$ be its eigenvalues. Show that
A is positive semi-definite if and only if $\lambda_i \geq 0$ for all $i = 1..n$

$\Rightarrow$ Assume A PSD $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$

A diagonalizes $A = PDP^T = Q^T D Q$ $(Q = P^T)$

$x^T A x = x^T Q^T D Q x = y^T D y = \sum \lambda_i \underbrace{y_i^2}_{+} \geq 0$ $\square$

So $\lambda_i \geq 0$ $i = 1 \dots n$

$\Leftarrow$ Assume $\lambda_i \geq 0$ for $i = 1, \dots, n$ show A PSD

$A = Q^T D Q$

$x^T A x = x^T Q^T D Q x = y^T D y = \sum_i \lambda_i y_i^2 \geq 0$ $\square$

$\lambda_i \geq 0$ (hypothesis)

Exercise 4

Let $A \in \mathbb{R}^{n\times n}$ be a positive semi-definite matrix. Show that then there exists a matrix $S \in \mathbb{R}^{n\times n}$ also positive-semidefinite such that $A = S^2$

A is PSD $\Rightarrow$ all $\lambda_i \geq 0$ $i = 1, \dots, n$

A diagonalizes $\boxed{A = P\underbrace{D}_{S?}P^T}$ S? Consider $D^{1/2} = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$

then if $S = PD^{1/2}$ we have $A = SS^T$ but A is symetric therefore $S = S^T$ and we have $\boxed{A = S^2}$ $\square$

Exercise 5

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix then $\lambda_{min} = \min_{\|v\|=1} \; v^T A v$

$A = PDP^T = Q^T D Q$

$v^T A v = \underbrace{v^T Q^T}_{y} D \underbrace{Q v}_{y} = y^T D y =$

$\|y\| = 1$ because Q is orthonormal

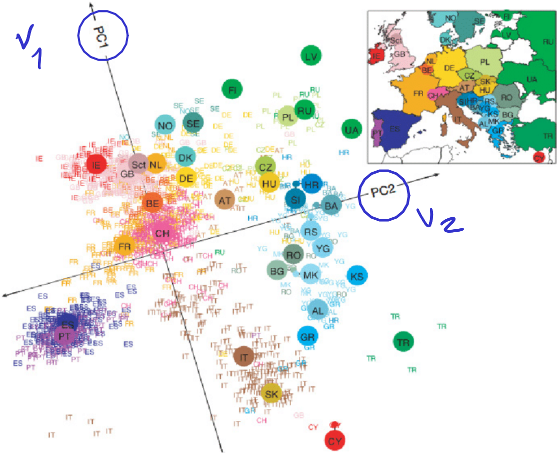
$= \sum_i \lambda_i \underbrace{y_i^2}_{+} = \underline{\underline{\lambda_{min}}}$ $\square$

$(1, 0 \cdots 0)$
λ_{min}

recall: Principal Component Analysis (PCA)

PCA

Problem: we have data $a_1, .., a_n \in \mathbb{R}^d$ with d very large. We wish to represent the dataset in a lower dimension.



recall: PCA

Method

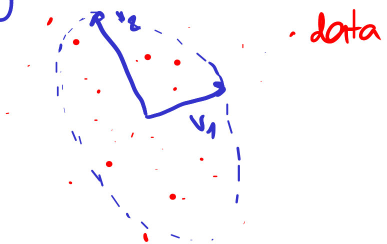
To perform PCA on a dataset we have to:

1. Calculate th sample covariance matrix S
2. Find the eigenvector v_1 of S with largest eigenvalue. v_1 is the first singular vector
3. Continue finding as many eigenvectors v_k of S as necessary by decreasing order with respect to the respective eigenvalues λ_k.
4. The dimensionally reduced data set will be composed by the projection of the data into the $v_1, ..v_k$ eigenbasis.

Exercise 6

Suppose there are two eigenvectors of the covariance matrix that correspond to large eigenvalues and the rest of eigenvalues are small. How do we interpret this geometrically?

This means the data can be "well" expressed in 2D the two large eigenvalue directions



recall: Singular Value decomposition

SVD

Let $A \in \mathbb{R}^{n\times m}$. Then there exists two orthogonal matrices $U \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^{m\times m}$ and a matrix $\Sigma \in \mathbb{R}^{n\times m}$ such that

$$A = U\Sigma V^T$$

with $\Sigma_{11}, \Sigma_{22}, ..., \Sigma_{11} \geq 0$ and $\Sigma_{ij} \neq 0$ for $i \neq j$

practice: SVD

Exercise 7

Let $A \in \mathbb{R}^{n\times m}$ give a method to compute numerically $rank(A)$ using SVD.

1) Decompose $A = \underbrace{U}_{\substack{\text{invertible} \\ \text{full rank}}} \Sigma \underbrace{V^T}_{\substack{\text{invertible} \\ \text{full rank}}}$

2) Count the non-zero entrances of Σ, this is the rank of A

Exercise 9

Let $A \in \mathbb{R}^{n \times m}$ explain why the set $\{Ax \text{ s.t. } \|x\| = 1\}$ is an ellipsoid.

SVD : $A = U \Sigma \underbrace{V}_{\text{orthogonal}}^T$

orthogonal preserves length

$\|x\| = 1$ is a circle

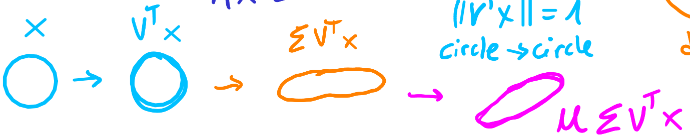
Let $x \in \mathbb{R}^m$ $\|x\| = 1$

$Ax = U \Sigma \underbrace{V^T x}_{\|V^T x\| = 1} = U \Sigma y = U y'$

circle → circle

dilation

rotation



Announcement

Next recitation session on Wed 28th October will be a midterm review.

The office hours on Thr 29th October is moved to Wed 28th 9 AM EST after the recitation

Start to study early, the midterm will have time pressure