Recitation #3 (Section 03)

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DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

recall: the Rank

Rank

Let $x_1, .., x_k \in \mathbb{R}^n$ we define the rank as

$$\boxed{\text{rank}(x_1, .., x_k) = \dim(Span(x_1, .., x_k))}$$

Informally: rank = "the number of linearly independent vectors among $x_1, .., x_k$"

Exercise 1

Calculate the rank of

$$A = \begin{pmatrix} 7 & 3 \\ 19 & 19 \\ -1 & 5 \end{pmatrix}$$

① Gaussian elimitation

② rank(A) = # col/rows that are non zero

$$A \to \begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix} \qquad \text{rank}(A) = 2$$

Exercise 2

Let $A, B \in \mathbb{R}^{n\times n}$ be two matrices then

$$rank(A+B) = rank(A) + rank(B)$$

True or false? ✓True ✗False

$n \times 1$ $\quad$ 1×1

$A = (1) \quad B = (-1)$

$A + B = (0)$

$rank(A+B) = 0$

$rank(A) = rank(B) = 1$

$0 \neq 1$

Rank nullity theorem

Rank nullity theorem

Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation then

$$rank(L) + dim(ker(L)) = m$$

$\tilde{L} = n \times m$

m in $\mathbb{R}^m$

n in $\mathbb{R}^n$

$$\begin{pmatrix} | & | & | & | \end{pmatrix}$$

practice: the Rank

Exercise 3

Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Prove that $rank(A) = rank(A^T)$.
(That is "the rank by columns = the rank by rows")

We will proceed by steps:

1. Prove that $x^T A^T A x \geq 0$ for all $x \in \mathbb{R}^n$
2. When is $x^T A^T A x = 0$?
3. Prove that $ker(A) = ker(A^T A)$
4. Use this to show $rank(A) = rank(A^T A)$
5. Show that $rank(A) = rank(A^T)$

$x = 0$

$Ax = 0$

$x \in ker(A)$

Exercise 3.1: Prove that $x^T A^T A x \geq 0$ for all $x \in \mathbb{R}^n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$A \quad m \times n$

$x \quad n \times 1$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$\boxed{\text{When is } x^T A^T A x = 0?}$

$x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ x is generic

$$x^T A^T A x = (Ax)^T (Ax) = (y_1 \dots y_m) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = y_1^2 + \dots + y_m^2 \geq 0$$

$$Ax = \begin{pmatrix} \sum_j a_{1j} x_j \\ \dots \\ \sum_j a_{mj} x_j \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

Exercise 3.3: Prove that $ker(A) = ker(A^T A)$

① $Ker(A) \subseteq Ker(A^T A)$

$v \in Ker(A)$: $Av = 0 \rightarrow A^T(Av) = A^T 0 = 0$

$(A^T A)v = 0 \rightarrow v \in Ker(A^T A)$ □

② $Ker(A^T A) \subseteq Ker(A)$

$v \in Ker(A^T A) \rightarrow (A^T A)v = 0 \rightarrow \underset{\neq 0}{A^T} \underbrace{(Av)}_{= 0} = 0$

$A \neq 0 \rightarrow Av = 0 \rightarrow v \in Ker(A)$

□

Exercise 3.4: Use this to show $rank(A) = rank(A^T A)$

$Ker(A) = Ker(A^T A)$

A $m \times n$

- Rank nullity theorem:

$rank(A) + \dim(Ker(A)) = n$

$A^T A \to n \times n$

$rank(A^T A) + \dim(Ker(A^T A)) = n$ $\to n \times m \quad m \times n$

$\hookrightarrow rank(A) = rank(A^T A)$ $\square$

Exercise 3.5: Show that $rank(A) = rank(A^T)$

$$\boxed{rank(A) = rank(A^TA)}$$

We know

A $m \times n$

A^T $n \times m$

① $\boxed{rank(A) \leq rank(A^T)}$

Recall: $rank(T_1 T_2) \leq \min(rank(T_1), rank(T_2))$

$rank(A) = rank(A^TA) \leq \min(rank(A), rank(A^T)) \leq rank(A^T)$

② $\boxed{rank(A^T) \leq rank(A)}$

A^TA $A A \leftrightarrow A^T$

$rank(A^T) \geq rank(A^TA)$ replace $rank(A^T)$ by the $rank(A) \rightarrow rank(A) \geq rank(A^T)$

$\square$

Invertible matrix

A matrix $M \in \mathbb{R}^{n\times n}$ is invertible if there exists another matrix $M^{-1} \in \mathbb{R}^{n\times n}$ such that

$$M\,M^{-1} = M^{-1}\,M = Id_n$$

Exercise 4

Let $M \in \mathbb{R}^{n\times n}$ be an invertible matrix, then its transpose M^T is also invertible.

$$\boxed{MM^{-1} = Id}$$

$$(MM^{-1})^T = Id^T$$

$$M^T (M^{-1})^T = Id$$

$$M^T (M^T)^{-1} = Id$$

↳ M^T has an inverse

The same goes for $M^{-1}M = Id$ $\square$

Exercise 5

Which of the following are equivalent for $A \in \mathbb{R}^{m \times n}$?

1. The columns of A are linearly independent
2. The rows of A are linearly independent
3. $rank(A) = m$
4. $rank(A) = n$
5. The equation $Ax = 0$ has one solution
6. The equation $Ax = b$ has at least one solution
7. $Im(A^T) = \mathbb{R}^n$
8. $Im(A) = \mathbb{R}^m$
9. The linear transformation corresponding to A is injective (one-to-one)
10. The linear transformation corresponding to A is exhaustive (onto)
11. $ker(A) = \{0\}$
12. The span of the columns of A is $\mathbb{R}^m$

Exercise 5

Goal

$\text{rank}(A^T) \leq \text{rank}(A)$

Know $\text{rank}(A) = \text{rank}(A^T A)$ ✓

$A \leftrightarrow A^T$

$\text{rank}(A^T) \neq \text{rank}(\;\cdot\;)$ ✓

$m \times n$

AA^T

$\text{rank}(A^T) = \text{rank}(AA^T) \leq \min\left(\text{rank}(A), \text{rank}(A^T)\right)$

$\leq \text{rank}(A)$

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