Recitation #9

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DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

Hessian and Gradient

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. It's gradient is the column vector

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f(x)}{\partial x_n}(x) \end{pmatrix}$$

The gradient indicates the direction of maximum growth of $f(x)$.
The Hessian $H_f(x)$ is

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial^2 x_1} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial^2 x_2} & \dots & \frac{\partial^2 f(x)}{\partial x_n \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 f(x)}{\partial^2 x_n} \end{pmatrix}$$

$H_f(x)$ is symmetric.

practice: Hessian

Exercise 1

Recall that if the Hessian $H_f(x)$ is positive definite, then f is an strictly convex function. Let's counter example the converse statement.
Find a convex function f such that its Hessian $H_f(x)$ is not positive definite.

Exercise 1

recall: Convexity

Convex sets and functions

A set $C \subset \mathbb{R}^n$ is convex if for all $x, y \in C$ and all $\alpha \in [0, 1]$,

$$\alpha x + (1 - \alpha)y \in C$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for all $x, y \in \mathbb{R}^n$ and all $\alpha \in [0, 1]$ it holds that

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

practice: convexity

Exercise 2

If C is convex then all convex combination of elements in C remains in C.

Exercise 3

practice: convexity

Exercise 3

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $\alpha \in \mathbb{R}$. Show that the "α- subleveled set" :

$$C_\alpha = \{x \in \mathbb{R}^n \,|\, f(x) \leq \alpha\}$$

is convex.

Exercise 3

practice: convexity

Exercise 4

Show that the sum of two convex functions is also convex.

Exercise 4

recall: Taylor n-variables

Taylor's approximation

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, Taylor's formula for all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$ "small"

- Order 1: $f(x+h) \approx f(x) + \langle \nabla f(x), h \rangle$
- Order 2: $f(x+h) \approx f(x) + \langle \nabla f(x), h \rangle + \frac{1}{2} h^T H_f(x) h$

practice: Taylor's approximation

Exercise 5

Apply Taylor's formula to second order to the following functions with $h \in \mathbb{R}^n$ small

1. $f(x, y) = e^x \cos(y)$ at $(x, y) = (0, 0)$
2. $f(x) = \|x\|^2$ with $x \in \mathbb{R}^n$ at $x = 0$

Exercise 5

practice: useful derivatives

Exercise 6

Calculate the gradient and Hessian of the following functions.
Assume $X \in \mathbb{R}^n$

- $f(x) = \|x\|^2$
- $f(x) = \|Ax\|^2$ with $A \in \mathbb{R}^{n \times n}$
- $f(x) = x^T A x$

Exercise 6