

Recitation #3 (Section 03)

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Rank

Let $x_1, \dots, x_k \in \mathbb{R}^n$ we define the rank as

$$\text{rank}(x_1, \dots, x_k) = \dim(\text{Span}(x_1, \dots, x_k))$$

Informally: rank = "the number of linearly independent vectors among x_1, \dots, x_k "

practice: the Rank

Exercise 1

Calculate the rank of

$$A = \begin{pmatrix} 7 & 3 \\ 19 & 19 \\ -1 & 5 \end{pmatrix}$$

① Gaussian elimination

$$A = \begin{pmatrix} 7 & 3 \\ 19 & 19 \\ -1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 7 & 3 \\ 19 & 19 \\ 0 & 26 \end{pmatrix} \rightarrow \begin{pmatrix} 7 & 3 \\ 0 & 76 \\ 0 & 26 \end{pmatrix} \rightarrow \begin{pmatrix} 7 & 3 \\ 0 & 76 \\ 0 & 0 \end{pmatrix}$$

② $\text{rank}(A) = \text{"# cols/row nonzero"}$

$$\boxed{\text{rank}(A) = 2}$$

Exercise 2

Let $A, B \in \mathbb{R}^{n \times n}$ be two matrices then

$$\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$$

True or false? ✓True ✗False

$$\begin{matrix} 1 \times 1 & & 1 \times 1 \\ A = (1) & B = (-1) \end{matrix}$$

$$A + B = (0)$$

$$\text{rank}(A + B) = 0$$

$$\text{rank}(A) = \text{rank}(B) = 1 \quad \left\{ \begin{array}{l} 0 \neq 1 \end{array} \right.$$

Rank nullity theorem

Rank nullity theorem

Let $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation then

$$\text{rank}(L) + \dim(\ker(L)) = m$$



$$\tilde{L} = n \times m \quad m \text{ in } \mathbb{R}^m$$

$$\begin{matrix} n \\ \text{in } \mathbb{R}^n \end{matrix} \uparrow \left(\begin{array}{c|c|c|c|c} | & | & | & | & | \\ \hline \end{array} \right)$$

practice: the Rank

Exercise 3

Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Prove that $\text{rank}(A) = \text{rank}(A^T)$.
(That is "the rank by columns = the rank by rows")

We will proceed by steps:

- 1 Prove that $x^T A^T A x \geq 0$ for all $x \in \mathbb{R}^n$
 - 2 When is $x^T A^T A x = 0$?
 - 3 Prove that $\ker(A) = \ker(A^T A)$
 - 4 Use this to show $\text{rank}(A) = \text{rank}(A^T A)$
 - 5 Show that $\text{rank}(A) = \text{rank}(A^T)$
- Handwritten notes:*
Step 1: $x=0$
Step 2: $Ax=0$
Step 2: $x \in \ker(A)$

Exercise 3.1: Prove that $x^T A^T A x \geq 0$ for all $x \in \mathbb{R}^n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \begin{array}{l} A \text{ } m \times n \\ x \text{ } n \times 1 \\ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \end{array}$$

When is $x^T A^T A x = 0$? $x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ x is generic

$$x^T A^T A x = \underbrace{(Ax)^T}_{x} (Ax) = (y_1 \dots y_m) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = y_1^2 + \dots + y_m^2 \geq 0$$

$$Ax = \begin{pmatrix} \sum_j a_{1j} x_j \\ \vdots \\ \sum_j a_{mj} x_j \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

Exercise 3.3: Prove that $\ker(A) = \ker(A^T A)$

$$\textcircled{1} \ker(A) \subseteq \ker(A^T A)$$

$$v \in \ker(A) \rightarrow Av = 0 \rightarrow A^T(Av) = A^T 0 = 0$$

$$(A^T A)v = 0 \rightarrow v \in \ker(A^T A) \quad \square$$

$$\textcircled{2} \ker(A^T A) \subseteq \ker(A)$$

$$v \in \ker(A^T A) \rightarrow (A^T A)v = 0 \rightarrow \underbrace{A^T}_{\neq 0} \underbrace{(Av)}_{=0} = 0$$

$$A \neq 0 \rightarrow Av = 0 \rightarrow v \in \ker(A)$$

\square

Exercise 3.4: Use this to show $\text{rank}(A) = \text{rank}(A^T A)$

$$\text{Ker}(A) = \text{Ker}(A^T A)$$

$$A \text{ } m \times n$$

• Rank nullity theorem:

$$\text{rank}(A) + \dim(\text{Ker}(A)) = n$$

$$\text{rank}(A^T A) + \dim(\text{Ker}(A^T A)) = \textcircled{n}$$

$A^T A \rightarrow m \times m$
 $\begin{matrix} \parallel & \parallel \\ m \times m & m \times n \end{matrix}$

$$\rightarrow \text{rank}(A) = \text{rank}(A^T A) \quad \square$$

Exercise 3.5: Show that $\text{rank}(A) = \text{rank}(A^T)$

Solution

- ① Prove $\text{rank}(A) \geq \text{rank}(A^T)$

Recall that $\text{rank}(T_1 T_2) \leq \min(\text{rank}(T_1), \text{rank}(T_2))$.

Then, starting from $\text{rank}(A) = \text{rank}(A^T A)$ we can say that

$$\text{rank}(A) = \text{rank}(A^T A) \leq \min(\text{rank}(A^T), \text{rank}(A)) \leq \text{rank}(A^T)$$

- ② Prove $\text{rank}(A^T) \geq \text{rank}(A)$

Same as in 1) but interchanging the roles of A and A^T .

Note that $(A^T)^T = A$.

Starting from $\text{rank}(A) = \text{rank}(A^T A)$ and switching roles we have $\text{rank}(A^T) = \text{rank}(A A^T)$. Now, same as before

$$\text{rank}(A^T) = \text{rank}(A A^T) \leq \min(\text{rank}(A), \text{rank}(A^T)) \leq \text{rank}(A)$$

And we are done.

recall: invertible matrix

Invertible matrix

A matrix $M \in \mathbb{R}^{n \times n}$ is invertible if there exists another matrix $M^{-1} \in \mathbb{R}^{n \times n}$ such that

$$M M^{-1} = M^{-1} M = Id_n$$

Exercise 4

Let $M \in \mathbb{R}^{n \times n}$ be an invertible matrix, then its transpose M^T is also invertible.

$$M M^{-1} = Id$$

$$(M M^{-1})^T = Id^T$$

$$M^T (M^{-1})^T = Id$$

$$M^T (M^T)^{-1} = Id$$

$\hookrightarrow M^T$ has an inverse

The same goes for $M^{-1} M = Id$



Exercise 5

Which of the following are equivalent for $A \in \mathbb{R}^{m \times n}$?

- ❶ The columns of A are linearly independent
- ❷ The rows of A are linearly independent
- ❸ $\text{rank}(A) = m$
- ❹ $\text{rank}(A) = n$
- ❺ The equation $Ax = 0$ has one solution
- ❻ The equation $Ax = b$ has at least one solution
- ❼ $\text{Im}(A^T) = \mathbb{R}^n$
- ❽ $\text{Im}(A) = \mathbb{R}^m$
- ❾ The linear transformation corresponding to A is injective (one-to-one)
- ❿ The linear transformation corresponding to A is exhaustive (onto)
- ⓫ $\ker(A) = \{0\}$
- ⓬ The span of the columns of A is \mathbb{R}^m

Exercise 5

Main result to solve the exercise :

$$B \in \mathbb{R}^{p \times 9}$$

$$\text{Ker}(B) = \{0\} \Leftrightarrow \text{rank}(B) = 9$$



$$\dim(\text{Im}(B)) = 9$$

onto function

Exercise 5

Solution

- ① Equivalent to $\text{rank}(A) = n$
- ② Equivalent to $\text{rank}(A) = m$
- ③ $\text{rank}(A) = m$
- ④ $\text{rank}(A) = n$
- ⑤ Equivalent to $\ker(A) = \{0\}$, by rank nullity theorem equivalent to $\text{rank}(A) = n$
- ⑥ This means $\dim(\text{Im}(A)) = m$ so equivalent to $\text{rank}(A) = m$
- ⑦ Equivalent to $\text{rank}(A) = n$
- ⑧ Equivalent to $\text{rank}(A) = m$
- ⑨ This means $\ker(A) = \{0\}$ so by rank nullity theorem is equivalent to $\text{rank}(A) = n$
- ⑩ This means $\dim(\text{Im}(A)) = m$ so equivalent to $\text{rank}(A) = m$
- ⑪ By rank nullity theorem equivalent to $\text{rank}(A) = n$
- ⑫ Equivalent to $\text{rank}(A) = m$