Recitation #5

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DS-GA 1014: Optimization and Computational Linear Algebra for Data Science

recall: Orthogonal matrices

Orthogonal matrix

Let $A \in \mathbb{R}^{n \times n}$ be a matrix. A is an orthogonal matrix if its columns form an orthogonal family of $\mathbb{R}^n$ (therefore linearly independent and basis of $\mathbb{R}^n$)
Alert!: orthogonal vectors given an inner product

Properties of Orthogonal matrices

The following are equivalent:

1. A is orthogonal
2. $AA^T = Id$
3. $A^T A = Id$

Exercise for home: re-prove it.

practice: orthogonal matrices

Exercise 1

Prove that the product of two orthogonal matrices is also an orthogonal matrix.

Exercise 2

Let $Q \in \mathbb{R}^{\times n}$ be an orthogonal matrix and let $x, y \in \mathbb{R}^n$.
Show that $\langle Qx, Qy \rangle = \langle x, y \rangle$

practice: orthogonal matrices

Exercise 3

Let $A \in \mathbb{R}^{m \times n}$ be a matrix with linearly independent columns. Show that A can be written as $A = Q\,R$ where $Q \in \mathbb{R}^{m \times m}$ is an orthogonal matrix and $R \in \mathbb{R}^{m \times n}$ an upper triangular matrix. Hint: apply Gram-Schmidt to the columns of A.

practice: orthogonal matrices

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recall: eigenvalues and eigenvectors

Eigenvalue, eigenvector

Let $A \in \mathbb{R}^{n\times n}$. A non-zero vector $v \in \mathbb{R}^n$ is an eigenvector of A if there exists $\lambda \in \mathbb{R}$ such that

$$\boxed{Av = \lambda v}$$

The scalar λ is the eigenvalue of A associated to v.

Eigenspace

The following set is called eigenspace of A associated to λ.

$$\boxed{E_\lambda(A) = \{x \in \mathbb{R}^n | Ax = \lambda x\} = ker(A - \lambda Id)}$$

The dimension of the eigenspace is called multiplicity of λ.

practice

Exercise 4

Show that the eigenspace $E_\lambda(A)$ is a subspace.

Diagonalizable matrix

A matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable if and only if there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\boxed{A = P^{-1}DP} \text{ where } D = diag(\lambda_1, ..., \lambda_n)$$

Exercise 5

Are all real matrices diagonalizable? Why?

✓True ✗False

practice: diagonalizable matrices

Exercise 5

Are all real matrices diagonalizable? Why?

✓True ✗False

practice: orthogonal matrices and eigenvalues

Exercise 6

Let $Q \in \mathbb{R}^{n \times n}$ be an orthogonal matrix.
Prove that the eigenvalues of Q can only be -1, $+1$.

Exercise 7

Let $A \in \mathbb{R}^{n \times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ with $\lambda_1 \neq ... \neq \lambda_n$ and respective eigenvectors $v_1, v_2, ..., v_n$.
Prove that $v_1, v_2, ..., v_n$ are linearly independent.

Exercise 7

Let $A \in \mathbb{R}^{n \times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ with $\lambda_1 \neq ... \neq \lambda_n$ and respective eigenvectors $v_1, v_2, ..., v_n$.
Prove that $v_1, v_2, ..., v_n$ are linearly independent.

practice: diagonal matrices

Exercise 8 (*)

Suppose $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix.
Give a vector $v \in \mathbb{R}^n$ with $\|v\| = 1$ such that $\|Dv\|$ is maximized.