Magnetic Resonance Imaging

Physical Principles and Sequence Design

Irina Grigorescu
UCL
irina.grigorescu.15@ucl.ac.uk

Contents

1 Magnetic Resonance Imaging: A Preview

1.1 Summary

1. The name **MRI**
   - **M**agnetic - the use of magnetic fields in the field
   - **R**esonance - the need to match the frequency of an oscillating magnetic field to the precessional frequency of the spins of some nucleus
   - **I**maging
2. **MRI** Concepts
   - Based on the *interaction of a nuclear spin with an external magnetic field* $\vec{B_0}$
   - Precession - *the circular motion of the axis of rotation of a spinning body about another fixed axis caused by the application of a torque in the direction of the precession*
   - Imaging - based on the *bulk precession of the hydrogen spins about the field direction* (Figure 1.1)
   - Precession angular frequency (Larmor Equation):

     $$\omega_0 = \gamma B_0 \tag{1.1}$$

     where:

     $\gamma =$ gyromagnetic ratio ($2.68 \times 10^8 rad/s/tesla$)

   - The Boltzmann Distribution shows that the spins can align both parallel and antiparallel with to the magnetic field. The spin excess is very small:

     $$\text{spin excess } \simeq N\frac{\hbar\omega_0}{2kT} \tag{1.2}$$

     where:

     N = total number of spins present in the sample
     k = Boltzmann constant
     kT = average thermal energy

   - M_0 - the average magnetic dipole density (longitudinal equilibrium relaxation)

     $$M_0 = \frac{\rho_0\gamma^2\hbar^2}{4kT}B_0 \tag{1.3}$$

   - $\vec{M}$ has been rotated by an rf pulse to a direction orthogonal to $\vec{B_0} = B_0\hat{z}$. The resulting transvserse magnetization has magnitude M_0 and precesses clockwise in the x-y plane. The complex magnetization is:

     $$M_+(t) \equiv M_x(t) + iM_y(t) = M_0e^{-i\omega t+i\phi_0} \tag{1.4}$$

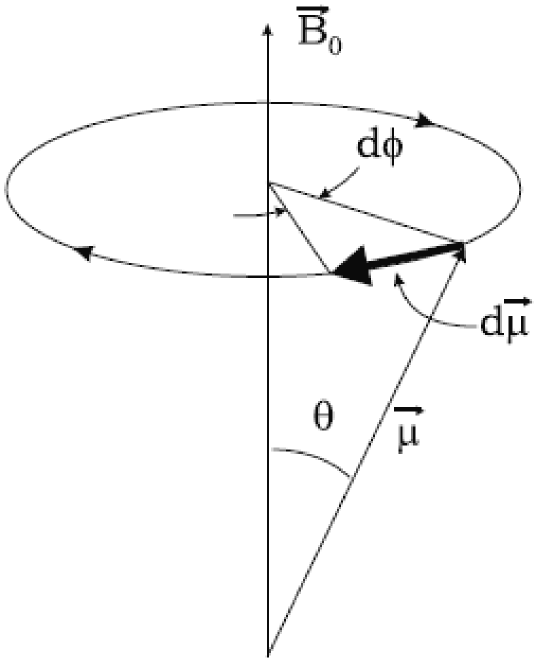


Figure 1.1: *By definition, precession is the circular motion of the axis of rotation of a spinning body about another fixed axis caused by the application of a torque in the direction of the precession. The interaction of the proton's spin with the magnetic field produces the torque, causing it to precess about $\vec{B}_0$ as the fixed axis. When looking down from above the vector $\vec{B}_0$, the precession of the magnetic moment vector $\vec{\mu}$, which is proportional to the spin vector, is clockwise. For the customary counterclockwise definition of polar angles, the differential $d\phi$ shown is negative.*

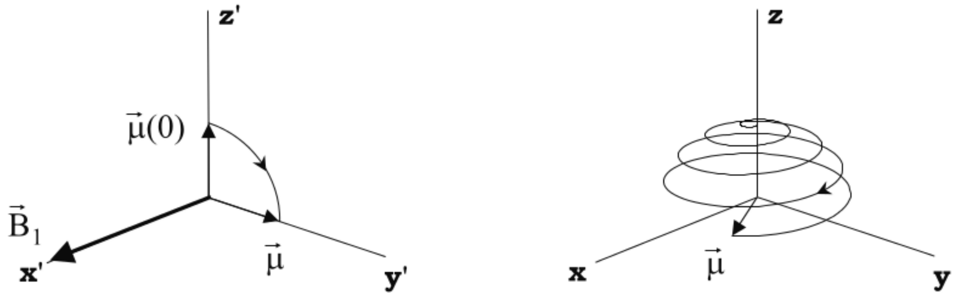


Figure 1.2: *Illustration of the effect of an rf pulse on an individual magnetic moment μ.*
(a) In a frame rotating about B_0 (which is along $\hat{z}$, say) at the Larmor frequency (with coordinates x', y' and z' = z), there is no observed precession about B_0. Upon application of an rf magnetic field pulse applied along $\hat{x}'$, the magnetic moment is rotated about $\hat{x}'$ at a rate corresponding to the frequency $\omega_1 = \gamma B_1$ determined by the amplitude of the rf field, B_1. A $\pi/2$ flip relative to its starting position along $\hat{z}'$ is achieved in a time τ_{rf} provided that $\omega_1 \tau_{rf} = \pi/2$.
(b) The behavior of the same magnetic moment rotation is observed to be more complicated in the fixed laboratory frame. This picture has been constructed for the case $\omega_1 = 0.06\omega_0$. In actual MR applications, the frequency ω_1 would be much smaller in relation to ω_0, but then the spiraling would be too dense to illustrate.

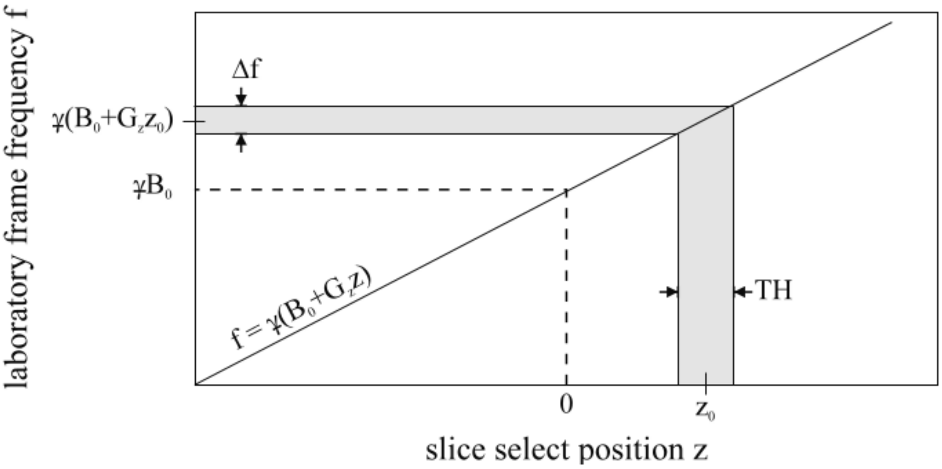


Figure 1.3: *The precession frequency ($f = \omega/(2\pi)$) in the laboratory frame is a function of position along the slice select axis. The original static field B_0 has been supplemented with a field with constant gradient G_z in the z-direction. The central frequency and spectral bandwidth of the rf pulse ($\Delta f \equiv BW_{rf}$, the shaded horizontal strip) are such that the slice of thickness $\Delta z \equiv TH$ (the shaded vertical strip) is uniformly 'excited' (i.e., all spins in the slice have the resonance condition satisfied). The fact that the slice is offset from the origin in the z-direction by z_0 implies that the center frequency of the rf pulse must be offset from the static Larmor frequency $f_0 = \gamma\!\!\!- B_0$ by $\gamma\!\!\!- G_z z_0$ as has been shown along the frequency axis.*

Category	Subcategory	Frequency (MHz)	Field strength (T)	Wavelength (m)
radio waves	LF (long wave)	0.03–0.3	7×10^{-4}–7×10^{-3}	10^4–10^3
	MF (medium wave)	0.3–3	7×10^{-3}–0.07	10^3–10^2
	AM radio (MF)	0.52–1.71	0.012–0.04	577–175
	HF (short wave)	3–30	0.07–0.7	10^2–10
	VHF	30–300	0.7–7	10–1
	FM radio (VHF)	87.5–108	2.05–2.54	3.43–2.78
	UHF	300–3×10^3	7–70	1–0.1
	SHF	3×10^3–3×10^4	70–700	0.1–0.01
microwaves		10^4–3×10^5	235–7×10^3	0.03–10^{-3}

Figure 1.4: *Range of radio and microwave frequencies from Wikipedia at en.wikipedia.org. Under the subcategory heading, the letter F refers to frequency. The letters L, M, H, V, U, and S in front of the letter F refer to low, medium, high, very, ultra, and super, respectively. Associated free-space wavelengths and NMR field strengths for protons are given here.*

1.2 Exercises

Problem 1.1

Using $\hbar = 1.05 \times 10^{-34}$ joule·s, $k = 1.38 \times 10^{-23}$ joule/K and $T = 300$ K, find the spin excess as a fraction of N for protons at 0.3 tesla.

Remember:

- This problem shows how to calculate the excess amount of spins occupying a lower energy level relative to the higher energy level, in a sample immersed in a static magnetic field.

```
function spexc = excessSpins(B0, T, gamma)
%
% Function that calculates the spin excess as a fraction of N
%  for protons at a given external magnetic field value (tesla)
%
% Input:
%     B0     = External magnetic field (Tesla)
%     T      = Temperature (Kelvin)
%     gamma = Gyromagnetic ratio(rad/s/T)
% Output:
%     spexc = Calculated spin excess (as a fraction)
%
% Author: Irina Grigorescu, irina.grigorescu.15@ucl.ac.uk
%                           irinagry@gmail.com

% Parameters:
reducedPlanck  = 1.05*(10^-34); % Reduced Planck constant (J*s)
k              = 1.38*(10^-23); % Boltzmann's constant (J/K)

if nargin < 2
    T               = 300;       % Temperature (K)
end

if nargin < 3
    gamma           = 2.68*1E08; % Gyromagnetic ratio
end

% Angular frequency
omega0 = gamma*B0;

% Spin excess
spexc = (reducedPlanck*omega0) / (2*k*T);
```

Problem 1.2

Find the frequency and free-space wavelength associated with the rf field required for proton magnetic resonance at each of the different B_0 values of a) 0.04 T, b) 0.2 T, c) 1.5 T, and d) 8 T.

Remember:

- This problem computes the resonance frequency of the oscillating magnetic field for a given static polarising magnetic field.

```
function freq = resonanceFrequency(B0, gamma)
%
% Function that calculates the frequency associated
%  with the RF field at a given static magnetic field value
%
% Input:
%    B0    = External magnetic field (Tesla)
%    gamma = Gyromagnetic ratio (rad/s/T)
% Output:
%    freq  = Calculated frequency (MHz)
%
% Author: Irina Grigorescu, irina.grigorescu.15@ucl.ac.uk
%                           irinagry@gmail.com

% Parameters:
if nargin < 2
    gamma = 2.68*1E08; % gyromagnetic ratio
end

% Frequency
freq = gamma .* B0 ./ (2*pi) .* 1E-06; % transforms values to MHz
```

Remember: (For the opposite problem)

- This problem computes the field strength associated with the frequency of an RF field for a given gyromagnetic ratio.

```
function B0 = fieldStrength(freq, gamma)
%
% Function that calculates the field strength associated
%  with the frequency of an RF field for a given spin species.
%
% Input:
%    freq  = RF frequency (MHz)
%    gamma = Gyromagnetic ratio (rad/s/T)
% Output:
%    B0    = External magnetic field (Tesla)
%
% Author: Irina Grigorescu, irina.grigorescu.15@ucl.ac.uk
%                           irinagry@gmail.com

% Parameters:
if nargin < 2
    gamma = 2.68*1E08; % gyromagnetic ratio
end

% Field Strength
B0 = freq * (2*pi) / gamma * 1E06; % in Tesla
```

2.1 Summary

1. This chapter investigates a single proton's response to an external magnetic field.
2. The magnetic moment of the spin in the presence of an external magnetic field will align with the external field as this is its equilibrium state.
   - A current loop of current I (and area A) placed in an external magnetic field B will experience a differential force on the loop determined by the cross product between the differential current and the magnetic field.

$$d\vec{F} = Id\vec{l} \times \vec{B} \tag{2.1}$$

   - The total force on this loop (or any closed loop) will be zero. This can be derived by taking the integral over the equation above. This leads to a zero change in the total momentum p as:

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{2.2}$$

   - In conclusion, a current loop found at rest in a constant external magnetic field will remain at rest unless external forces are applied, depending on the orientation (like in Figure 2.1a). Otherwise, they will experience a torque which will rotate the loop.

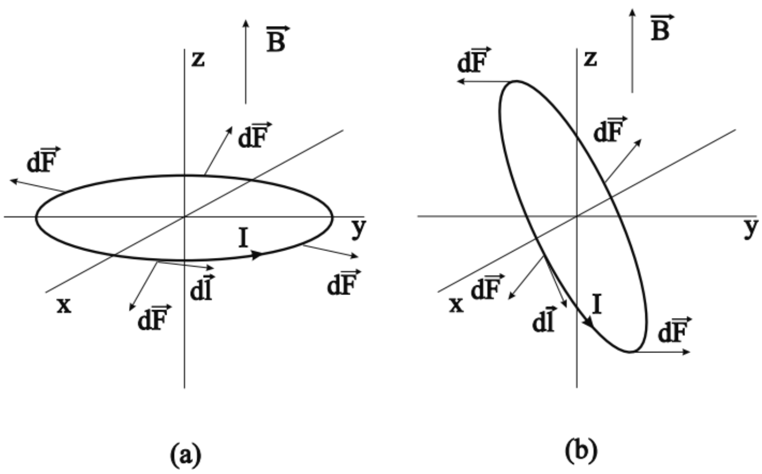


Figure 2.1: *Circular current loop depicted in two different orientations relative to a uniform magnetic field. The forces on representative differential current segments are shown (one d is explicitly shown in each case). The first (a) shows the current plane perpendicular to the field where there is no net twist (torque); the second (b) shows the current plane at an arbitrary angle to the field where there is a nonzero torque.*

   - In other words, if the current loop is not at rest, then it will try to rotate itself to be at rest. Rotations can also arise from forces applied off-centre even when the sum of all differential forces is zero. Rotations are described by a force called **torque**. Each differential segment experiences a differential torque:

$$d\vec{N} = \vec{r} \times d\vec{F} \tag{2.3}$$

   - The total sum of these differential torques is zero for non-rotating objects, and nonzero for rotating ones. This is exemplified in Figure 2.1 and in Exercises **??** and **??**.
   - For any arbitrary current distribution, the net torque has the following formula:

$$\vec{N} = \vec{\mu} \times \vec{B} \tag{2.4}$$

   where $\vec{\mu}$ is called the magnetic dipole moment or magnetic moment.

- The magnetic moment vector for planar loops is given in terms of the current going through the loop I, the area of the loop A and the unit vector $\vec{n}$ perpendicular to the current loop plane

$$\vec{\mu} = IA\hat{n} \tag{2.5}$$

This can be visualised in Figure 2.2.

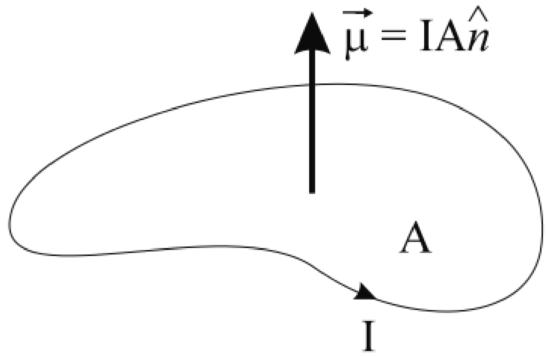


Figure 2.2: *A loop with current I lying in a plane*

- An example
For Figure 2.3, the total torque for planar loops will depend on the current I (anticlockwise), the magnetic field strength B, the angle θ between the plane in which the current loop lies and the magnetic field and the total area of the loop.

To get to this we begin with:

$$d\vec{N} = \vec{r} \times (Id\vec{l} \times \vec{B}) = Id\vec{l}(\vec{B} \cdot \vec{r}) - I\vec{B}(d\vec{l} \cdot \vec{r}) \tag{2.6}$$

where $\vec{B}$ and the cylindrical unit vectors looking like:

$$\vec{B} = B(cos\theta\ \hat{z} + sin\theta\ \hat{y}) \tag{2.7}$$

$$\hat{\rho} = cos\phi\ \hat{x} + sin\phi\ \hat{y} \tag{2.8}$$

$$\hat{\phi} = -sin\phi\ \hat{x} + cos\phi\ \hat{y} \tag{2.9}$$

Calculating and integrating over we arrive at:

$$\vec{N} = I\pi R2Bsin\theta\hat{x} \tag{2.10}$$

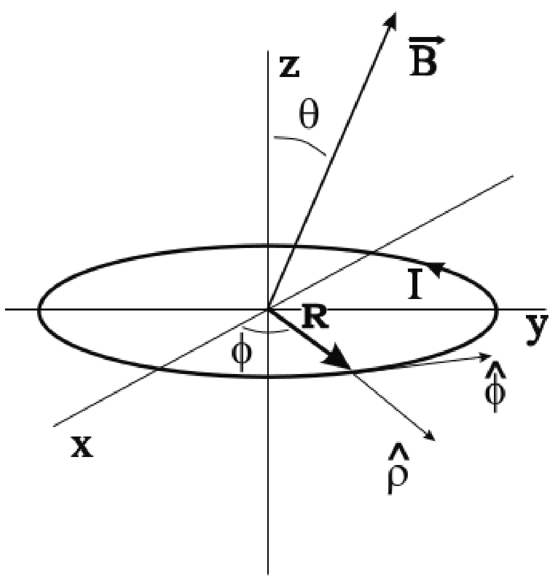


Figure 2.3: *A circular loop in x-y plane and the magnetic field lying in the y-z plane*

3. Spin has intrinsic angular momentum. Magnetic moment is proportional to it. In this case the motion is changed.
   - In the picture painted so far we introduce 'spinning', or **angular momentum**.
     - The change in the angular momentum of the spin will equal torque.

$$\frac{d\vec{J}}{dt} = \vec{N} \tag{2.11}$$

     - This comes from (a system of many point particles with respect to some origin):

$$\vec{J} = \sum_i \vec{r}_i \times \vec{p}_i \tag{2.12}$$

     where $d\vec{p} = \vec{F}dt$

     Sidenote:
     $\vec{p}$ is linear momentum
     $\vec{r} \times \vec{p}$ is angular momentum

   - There exists a connection between intrinsic angular momentum and its moment
     - The direct relationship between the magnetic moment and the spin angular momentum vector is found from experiment:

$$\vec{\mu} = \gamma \vec{J} \tag{2.13}$$

     where γ is called the *gyromagnetic ratio* and it depends on the particle or nucleus. For the proton nucleus it is:

$$\gamma = 2.675 \times 10^8 \text{rad/s/T} \tag{2.14}$$

   - The equation of motion of a precessing spin immersed in a static polarising magnetic field can be found using Eq 2.13 and Eq 2.4 in Eq 2.11 (also known as a simple version of the Bloch equation):

$$\frac{d\vec{\mu}}{dt} = \gamma \vec{\mu} \times \vec{B} \tag{2.15}$$

4. Geometrical Representation

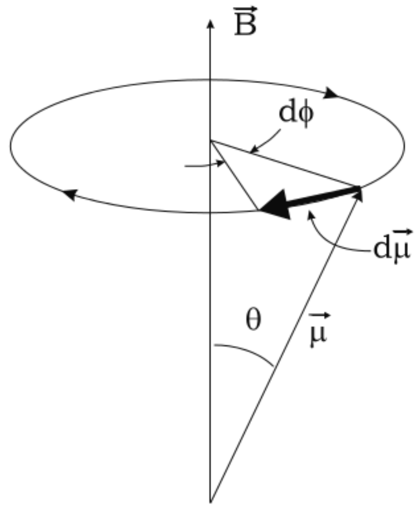


Figure 2.4: *Clockwise precession of a proton's spin about a magnetic field.*

$$R = |\mu| \cdot sin\theta$$
$$d\vec{\mu} = R \cdot |d\phi|$$
$$d\vec{\mu} = |\vec{\mu}| \cdot sin\theta \cdot |d\phi|$$

$$|d\vec{\mu}| = \mu\, sin\theta\, |d\phi| \tag{2.16}$$

From Eq 2.15:

$$|d\vec{\mu}| = \gamma \left| \vec{\mu} \times \vec{B} \right| dt = \gamma\, \mu\, B\, sin\theta\, dt \tag{2.17}$$

From Eq 2.16 and 2.17 we have: $|d\phi| = \gamma\, B\, dt$.
Because $\omega = \left|\frac{d\phi}{dt}\right|$, we get the famous Larmor precession formula:

$$\omega = \gamma\, B \tag{2.18}$$

Also, because we have a clockwise rotation in the figure:

$$\frac{d\phi}{dt} = -\omega \tag{2.19}$$

If the field is along the z-axis and constant in time, $\vec{B} = B_0\hat{z}$, the solution for Eq 2.19 is:

$$\phi = -\omega_0 t + \phi_0 \tag{2.20}$$

5. Cartesian Representation

$$\begin{aligned}
\vec{\mu_x}(t) &= \vec{\mu}(t) \cdot cos(\phi_0 - \xi) \\
\vec{\mu_x}(t) &= \vec{\mu}(t) \cdot (cos\phi_0 \cdot cos\xi + sin\phi_0 \cdot sin\xi) \\
\vec{\mu_x}(t) &= \vec{\mu}(t) \cdot cos\phi_0 \cdot cos\xi \\
&+ \vec{\mu}(t) \cdot sin\phi_0 \cdot sin\xi \\
\vec{\mu_x}(t) &= \vec{\mu}(t) \cdot \frac{\vec{\mu_x}(0)}{\vec{\mu}(0)} \cdot cos\xi \\
&+ \vec{\mu}(t) \cdot \frac{\vec{\mu_y}(0)}{\vec{\mu}(0)} \cdot sin\xi \\
\vec{\mu_x}(t) &= \vec{\mu_x}(0) \cdot cos\xi + \vec{\mu_y}(0) \cdot sin\xi
\end{aligned}$$

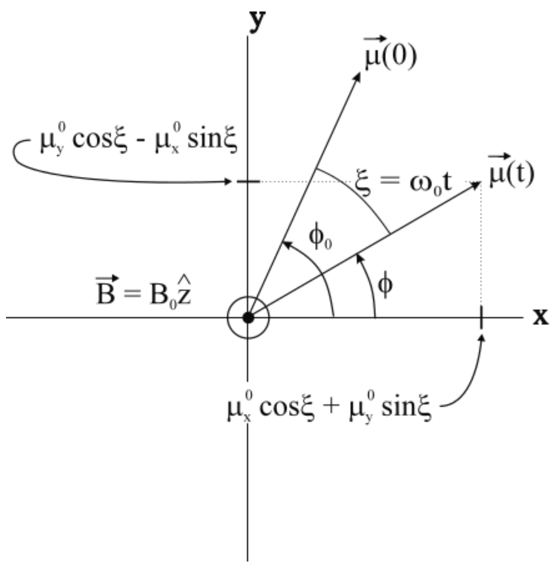


Figure 2.5

For $\vec{B} = B_0\hat{z}$ we have:

$$\vec{\mu}(t) = \vec{\mu_x}(t)\hat{x} + \vec{\mu_y}(t)\hat{y} + \vec{\mu_z}(t)\hat{z} \tag{2.21}$$

with:

$$\mu_x(t) = \mu_x(0) \cdot cos\omega_0 t + \mu_y(0) \cdot sin\omega_0 t \tag{2.22}$$
$$\mu_y(t) = \mu_y(0) \cdot cos\omega_0 t - \mu_x(0) \cdot sin\omega_0 t \tag{2.23}$$
$$\mu_z(t) = \mu_z(0) \tag{2.24}$$

6. Matrix Representation

$$\vec{\mu}(t) = R_z(\omega_0 t)\vec{\mu}(0) \tag{2.25}$$

with:

$$R_z(\theta) = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.26}$$

7. Complex Representation and Phase The two degrees of freedom, μ_x and μ_y, can be given in terms of the real and imaginary parts of:

$$\mu_+(t) = \mu_x(t) + i\mu_y(t) \tag{2.27}$$

with:

$$\mu_+(t) = \mu_+(0)e^{-i\omega_0 t} \tag{2.28}$$

In general, the complex number $\mu_+(t)$ can be written in terms of its magnitude and phase:

$$\mu_+(t) = \left|\mu_+(t)\right| e^{i\phi(t)} \tag{2.29}$$

So, the complex representation can be rewritten as:

$$\mu_+(t) = \left|\mu_+(0)\right| e^{i\phi_0(t)} \tag{2.30}$$

where the phase is:

$$\phi_0(t) = -\omega_0 t + \phi_0(0) \tag{2.31}$$

2.2 Exercises

Problem 2.1

If the total force on a current loop (or any system, for that matter) is zero ($\oint d\vec{F} = 0$), show that the total torque $\vec{N} = \oint \vec{r} \times d\vec{F}$ is independent of the choice of origin. Hint: Change to a primed coordinate system where $\vec{r} = \vec{r'} + \vec{r_0}$, with an arbitrary shift $\vec{r_0}$.

Remember:

- Zero total force means zero change in the total momentum (conservation of momentum).
- Rotations can arise (about the centre of mass) from forces applied off centre, even when the vector sum of all forces cancels out.
- The total torque is independent of the choice of origin.

$$\oint d\vec{F} = 0$$
$$\vec{N} = \oint \vec{r} \times d\vec{F}$$
$$\vec{r} = \vec{r'} + \vec{r_0}$$

Proof.

$$\vec{N'} = \oint (\vec{r} - \vec{r_0}) \times d\vec{F}$$
$$\vec{N'} = \oint \vec{r} \times d\vec{F} - \oint \vec{r_0} \times d\vec{F}$$

$$\left.\begin{array}{l} \vec{N'} = \oint \vec{r} \times d\vec{F} - \vec{r_0} \times \oint d\vec{F} \\ \text{We know that: } \oint d\vec{F} = 0 \end{array}\right\} \vec{N'} = \oint \vec{r} \times d\vec{F} = \vec{N}$$

Problem 2.2

Consider the constant magnetic field in the z-y plane at an angle θ with the z-axis as shown in Fig. 2.3b. A current I flows in a rectangular loop with sides a and b lying in the x-y plane.

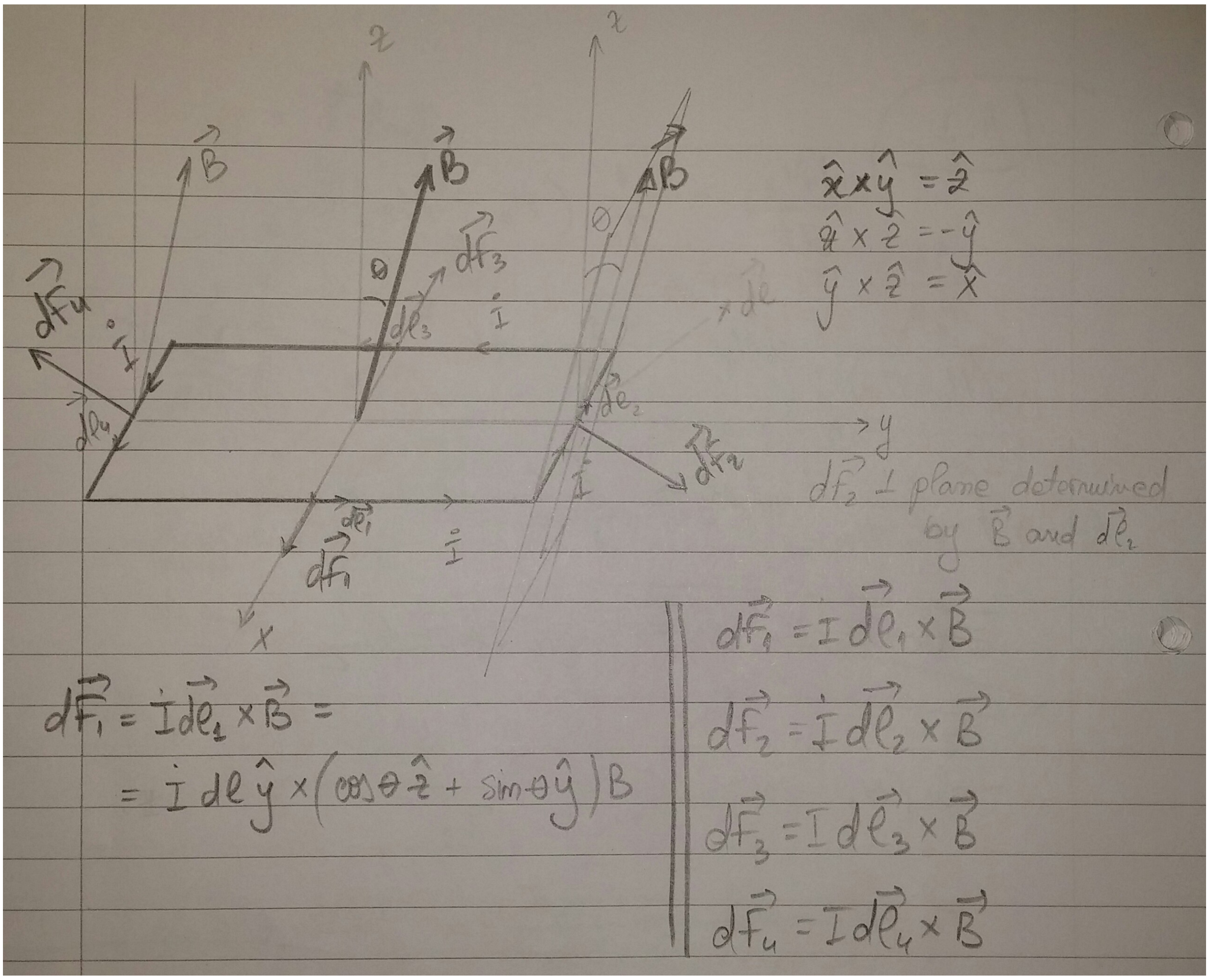
a) Show that the differential forces on the four respective current legs have the forms $IB(-\cos\theta\hat{y} + \sin\theta\hat{z})\,|dx|$, $-IB(\sin\theta\hat{z} - \cos\theta\hat{y})\,|dx|$, $IB\cos\theta\hat{x}\,|dy|$, and $-IB\cos\theta\hat{x}\,|dy|$.

b) Show that, after integration, the total force on the loop is zero.

c) Show that the differential torques on the four respective current legs relative to the center of the loop have the forms $[x\hat{x} - \frac{a}{2}\hat{y}] \times [IB(-\cos\theta\hat{y} + \sin\theta\hat{z})\,|dx|]$, $[x\hat{x} + \frac{a}{2}\hat{y}] \times [-IB(\sin\theta\hat{z} - \cos\theta\hat{y})\,|dx|]$, $[\frac{b}{2}\hat{x} + y\hat{y}] \times [IB\cos\theta\hat{x}\,|dy|]$, and $[-\frac{b}{2}\hat{x} + y\hat{y}] \times [-IB\cos\theta\hat{x}\,|dy|]$.

d) Show that the integration of the results in (c) is simplified by the vanishing of certain integrands odd in x or y, leading to a net torque on the loop given by $-IBab\sin\theta\hat{x}$. Again, the double cross product formula could have been used to bypass the force calculation, in achieving this result.

e) Find the magnetic moment vector for this loop from (2.5) and show that the total torque found in (d) agrees with the formula (2.4).



Remember:

- A rectangular loop behaves in a similar fashion as a circular one.
- The total force on the loop is zero.
- The total torque depends on the dimensions of the loop.

We know:

$$\hat{x} \times \hat{y} = \hat{z}$$
$$\hat{x} \times \hat{z} = -\hat{y}$$
$$\hat{y} \times \hat{z} = -\hat{x}$$

$$\vec{B} = \vec{B_y} + \vec{B_z}$$
$$\vec{B} = (cos\theta\hat{z} + sin\theta\hat{y})B$$

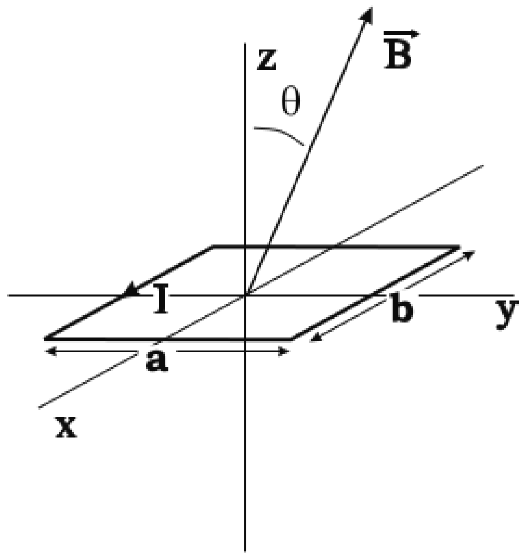


Figure 2.6: *current loops lying in the x-y plane and a magnetic charge pair along the z-axis*

a) Proof.

A current I flowing through a loop placed in an external magnetic field B will experience a differential force on the loop determined by:

$$d\vec{F} = Id\vec{l} \times \vec{B} \tag{2.32}$$

As this loop has a rectangular shape, each side of the loop will experience a differential force determined by its respective differential segments. These differential segments $d\vec{l}$ will

We get:

$$\begin{aligned}
d\vec{F_1} &= Id\vec{l_1} \times \vec{B} \\
&= I|dl|\,\hat{y} \times (cos\theta\hat{z} + sin\theta\hat{y})B \\
&= IBcos\theta\hat{x}|dl| \\
d\vec{F_2} &= Id\vec{l_2} \times \vec{B} \\
&= -I|dl|\,\hat{x} \times (cos\theta\hat{z} + sin\theta\hat{y})B \\
&= IBcos\theta\hat{y}|dl| - IBsin\theta\hat{z}|dl| \\
d\vec{F_3} &= Id\vec{l_3} \times \vec{B} \\
&= -I|dl|\,\hat{y} \times (cos\theta\hat{z} + sin\theta\hat{y})B \\
&= -IBcos\theta\hat{x}|dl| \\
d\vec{F_4} &= Id\vec{l_4} \times \vec{B} \\
&= I|dl|\,\hat{x} \times (cos\theta\hat{z} + sin\theta\hat{y})B \\
&= -IBcos\theta\hat{y}|dl| + IBsin\theta\hat{z}|dl|
\end{aligned}$$

b) Proof.

$$\begin{aligned}
\vec{F} &= d\vec{F_1} + d\vec{F_2} + d\vec{F_3} + d\vec{F_4} \\
\vec{F} &= IBcos\theta\hat{x}|dl| + IBcos\theta\hat{y}|dl| - IBsin\theta\hat{z}|dl| \\
&\quad - IBcos\theta\hat{x}|dl| - IBcos\theta\hat{y}|dl| + IBsin\theta\hat{z}|dl| \\
&= 0
\end{aligned}$$

c) Proof.

$$d\vec{N_i} = \vec{r_i} \times d\vec{F_i}$$

$$\begin{aligned}
\vec{r_1} &= \frac{b}{2}\hat{x} + y\hat{y} \\
\vec{r_2} &= x\hat{x} + \frac{a}{2}\hat{y} \\
\vec{r_3} &= -\frac{b}{2}\hat{x} + y\hat{y} \\
\vec{r_2} &= x\hat{x} - \frac{a}{2}\hat{y}
\end{aligned}$$

$$\begin{aligned}
d\vec{N_1} &= (\frac{b}{2}\hat{x} + y\hat{y}) \times (IBcos\theta\hat{x}|dl|) \\
&= -IBcos\theta y|dl|\,\hat{z} \\
d\vec{N_2} &= (x\hat{x} + \frac{a}{2}\hat{y}) \times (IBcos\theta\hat{y}|dl| - IBsin\theta\hat{z}|dl|) \\
&= IBcos\theta x|dl|\,\hat{z} - IBsin\theta|dl|\,(-x\hat{y} + \frac{a}{2}\hat{x}) \\
d\vec{N_3} &= (-\frac{b}{2}\hat{x} + y\hat{y}) \times (-IBcos\theta\hat{x}|dl|) \\
&= IBcos\theta y|dl|\,\hat{z} \\
d\vec{N_4} &= (x\hat{x} - \frac{a}{2}\hat{y}) \times (-IBcos\theta\hat{y}|dl| + IBsin\theta\hat{z}|dl|) \\
&= -IBcos\theta x|dl|\,\hat{z} + IBsin\theta|dl|\,(-x\hat{y} - \frac{a}{2}\hat{x})
\end{aligned}$$

d) Proof.

$$\begin{aligned}
\vec{N} &= \int_{-a/2}^{a/2} d\vec{N_1} + \int_{-b/2}^{b/2} d\vec{N_2} + \int_{-a/2}^{a/2} d\vec{N_3} + \int_{-b/2}^{b/2} d\vec{N_4} \\
&= \int_{-a/2}^{a/2} d\vec{N_1} + d\vec{N_3} + \int_{-b/2}^{b/2} d\vec{N_2} + d\vec{N_4} \\
&= 0 + \int_{-b/2}^{b/2} IBsin\theta|dl|\,(\cancel{-x\hat{y}} - \frac{a}{2}\hat{x}\cancel{+x\hat{y}} - \frac{a}{2}\hat{x})dx \\
&= -IBab\ sin\theta\hat{x}
\end{aligned}$$

e) Proof.

$$\begin{aligned}
\vec{N} &= \vec{\mu} \times \vec{B} \\
&= IA\ \hat{n} \times \vec{B} \\
&= Iab\ \hat{n} \times \vec{B} \\
&= -IBab\ sin\theta\hat{x}
\end{aligned}$$

Problem 2.3

> Consider a point mass m moving at velocity $\vec{v}(t)$ with position $\vec{r}(t)$ defined by some origin. Its angular momentum relative to that origin is therefore $\vec{J} = \vec{r} \times \vec{p}$ with $\vec{p} = m\vec{v}$. Derive (2.14) by taking the time derivative of this angular momentum. Note (2.2) and (2.3).

Remember:

- A magnetic moment will try to line up along the direction of an external magnetic field.
- If the system has nonzero total torque $\vec{N}$, then the system's total angular momentum $\vec{J}$ changes according to $\frac{d\vec{J}}{dt} = \vec{N}$.

(2.14) $\frac{d\vec{J}}{dt} = \vec{N}$

(2.2) $\vec{F} = \frac{d\vec{p}}{dt}$

(2.3) $d\vec{N} = \vec{r} \times d\vec{F}$.

Proof.

$$\begin{aligned}
\vec{J} &= \vec{r} \times \vec{p} \\
\frac{d\vec{J}}{dt} &= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \\
&= \vec{v} \times \vec{p} + \vec{r} \times \vec{F} \\
&= \vec{v} \times (m \cdot \vec{v}) + \vec{r} \times \vec{F} \\
&= \mathbf{0} + \vec{N} = \vec{N}
\end{aligned}$$

Problem 2.4

---

a) Show that the angular momentum $\vec{r} \times \vec{p}$ of the circulating particle with respect to the center is $mrv\hat{n}$ where $\hat{n}$ is a unit vector perpendicular to the plane of the circle. Here, $\hat{n}$ points in a direction given by the right-hand rule applied to the particle's motion.

b) Show that the magnetic moment associated with the motion of the point charge is $qvr/2$ and thus that the gyromagnetic ratio is given by (2.19).

c) Evaluate numerically the gyromagnetic ratio γ (2.19), choosing the same mass (1.67×10^{-27} kg) and charge (1.60×10^{-19} C) as for a proton. The difference between your answer and (2.17) is due to the more complicated motion of the proton constituents, the 'quarks.' For related reasons, a neutron has a nonvanishing magnetic moment despite its zero overall charge.

Remember:

- The relationship between angular momentum and magnetic moment is given by $\vec{\mu} = \gamma\vec{J}$
- In classic terms, the angular momentum depends on the mass and velocity of the circulating particle.
- The current $I = \frac{q}{t}$ is the flux of charge in time.
- There are differences due to mass between different types of of circulating particles.

a) Proof.

$$\begin{aligned} \vec{r} \times \vec{p} &= \vec{r} \times (m \cdot \vec{v}) \\ &= mrv\ sin\theta\hat{n} \\ \text{with } \theta &= 90^0 \Rightarrow \\ \vec{r} \times \vec{p} &= mrv\hat{n} \end{aligned}$$

b) Proof.

$$\begin{aligned} \vec{\mu} &= \gamma\vec{J} \\ &= \gamma\vec{r} \times \vec{p} \\ &= \gamma mrv\hat{n}\ (1) \\ \vec{\mu} &= IA\hat{n} \\ &= \frac{q}{T}\ \pi r^2\hat{n} \\ &= \frac{q\,v}{2\cancel{\pi}\cancel{r}}\ \cancel{\pi} r^{\cancel{2}}\hat{n} \\ &= \frac{qvr}{2}\ \hat{n}\ (2) \end{aligned}$$

$$\begin{aligned} \text{From (1) and (2)} \Rightarrow \frac{q\cancel{vr}}{2}\ \hat{n} &= \gamma m\cancel{vr}\hat{n} \\ \hat{n}\,(\frac{q}{2} - \gamma m) &= 0 \\ \frac{q}{2} - \gamma m &= 0 \\ \gamma m &= \frac{q}{2} \\ \gamma &= \frac{q}{2m} \end{aligned}$$

Problem 2.5

Demonstrate that (2.24) implies $d\mu/dt = 0$. Hint: Form a scalar (dot) product of both sides of (2.24) with $\vec{\mu}$.

Remember:

- The magnitude of $\vec{\mu}$ does not change in time.

Proof.
Dot product on the right hand side:

$$\frac{d\vec{\mu}}{dt} = \gamma\vec{\mu} \times \vec{B}$$
$$\frac{d\vec{\mu}}{dt} \cdot \vec{\mu} = \gamma(\vec{\mu} \times \vec{B}) \cdot \vec{\mu}$$
$$\frac{d\vec{\mu}}{dt} \cdot \vec{\mu} = (\gamma\, \mu\, B\, sin\theta\, \hat{n}) \cdot \vec{\mu}$$
$$\text{Obs: } \hat{n} \text{ is orthogonal to } \vec{\mu} \Rightarrow cos(\angle(\hat{n}, \vec{\mu})) = 0 \Rightarrow$$
$$\frac{d\vec{\mu}}{dt} \cdot \vec{\mu} = 0\, (1)$$

Dot product on the left hand side:

$$\vec{\mu} \cdot \frac{d\vec{\mu}}{dt} = \gamma(\vec{\mu} \times \vec{B}) \cdot \vec{\mu}$$
$$\vec{\mu} \cdot \frac{d\vec{\mu}}{dt} = \vec{\mu} \cdot (\gamma\, \mu\, B\, sin\theta\, \hat{n})$$
$$\text{Obs: } \hat{n} \text{ is orthogonal to } \vec{\mu} \Rightarrow cos(\angle(\hat{n}, \vec{\mu})) = 0 \Rightarrow$$
$$\vec{\mu} \cdot \frac{d\vec{\mu}}{dt} = 0\, (2)$$

From (1) + (2) we get:

$$\frac{d\vec{\mu}}{dt} \cdot \vec{\mu} + \vec{\mu} \cdot \frac{d\vec{\mu}}{dt} = 0$$
$$\frac{d(\vec{\mu} \cdot \vec{\mu})}{dt} = 0$$
$$\frac{d\mu^2}{dt} = 0$$
$$\frac{d\mu}{dt}\,\mu + \mu\,\frac{d\mu}{dt} = 0$$
$$2\,\mu\,\frac{d\mu}{dt} = 0$$
$$\frac{d\mu}{dt} = 0$$

Problem 2.6

It will be useful in later discussions to have the answer (2.33) rederived as a solution to the differential equation (2.24).

a) For $\vec{B} = B_0\hat{z}$, show that the vector differential equation (2.24) decomposes into the three Cartesian equations

$$\begin{aligned} \frac{d\mu_x}{dt} &= \gamma\mu_y B_0 = \omega_0\mu_y \\ \frac{d\mu_y}{dt} &= -\gamma\mu_x B_0 = -\omega_0\mu_x \\ \frac{d\mu_z}{dt} &= 0 \end{aligned} \tag{2.34}$$

b) By taking additional derivatives, show that the first two equations in (2.34) can be decoupled to give

$$\begin{aligned} \frac{d^2\mu_x}{dt^2} &= -\omega_0^2\mu_x \\ \frac{d^2\mu_y}{dt^2} &= -\omega_0^2\mu_y \end{aligned} \tag{2.35}$$

These decoupled second-order differential equations have familiar solutions of the general form $C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$.

c) By putting the general solutions back into the first-order differential equations (2.34), and by assuming the initial conditions used previously, show that you recover (2.33).

Remember:

- The magnitude of the z-component of the magnetic moment does not change in time, while the xy-components rotate about the z axis.

(2.33)

$$\begin{aligned} \vec{\mu_x}(t) &= \vec{\mu_x}(0) \cdot cos\omega_0 t + \vec{\mu_y}(0) \cdot sin\omega_0 t \\ \vec{\mu_y}(t) &= \vec{\mu_y}(0) \cdot cos\omega_0 t - \vec{\mu_x}(0) \cdot sin\omega_0 t \\ \vec{\mu_z}(t) &= \vec{\mu_z}(0) \end{aligned}$$

(2.24)

$$\frac{d\vec{\mu}}{dt} = \gamma\vec{\mu} \times \vec{B}$$

a) Proof.

$$\begin{aligned} \frac{d}{dt}(\vec{\mu_x} + \vec{\mu_y} + \vec{\mu_z}) &= \gamma(\vec{\mu_x} + \vec{\mu_y} + \vec{\mu_z}) \times (\vec{B_x} + \vec{B_y} + \vec{B_z}) \\ &= \gamma\, B_0\, (\mu_x\hat{x} + \mu_y\hat{y} + \mu_z\hat{z}) \times \hat{z} \\ &= \gamma\, B_0\, (-\mu_x\hat{y} + \mu_y\hat{x}) \end{aligned}$$

$$\begin{aligned} \frac{d\mu_x}{dt} &= \gamma\mu_y B_0 = \omega_0\, \mu_y \\ \frac{d\mu_y}{dt} &= -\gamma\mu_x B_0 = -\omega_0\, \mu_x \\ \frac{d\mu_z}{dt} &= 0 \end{aligned}$$

b) Proof.

$$\begin{aligned}\frac{d^2\mu_x}{dt^2} &= \frac{d}{dt}(\frac{d\mu_x}{dt}) = \omega_0\frac{d\mu_y}{dt} = -\omega_0^2\mu_x\\ \frac{d^2\mu_y}{dt^2} &= \frac{d}{dt}(\frac{d\mu_y}{dt}) = -\omega_0\frac{d\mu_x}{dt} = -\omega_0^2\mu_y\end{aligned}$$

c) Proof.

$f(t) = C_1\, cos\omega_0 t \;+\; C_2\, sin\omega_0 t$

$f(0) = C_1 \qquad \Rightarrow C_1 = f(0)$

$f'(t) = -C_1\omega_0\, sin\omega_0 t + C_2\omega_0\, cos\omega_0 t$

$f'(0) = C_2\omega_0 \qquad \Rightarrow C_2 = \dfrac{f'(0)}{\omega_0}$

$$\begin{aligned}\mu_x(t) &= C_{1x}\, cos\omega_0 t + C_{2x} sin\omega_0 t\\ C_{1x} &= \mu_x(0)\\ C_{2x} &= \frac{d\mu_x(0)}{dt}\frac{1}{\omega_0} = \mu_y(0)\\ &\Rightarrow\\ \mu_x(t) &= \mu_x(0)\, cos\omega_0 t + \mu_y(0) sin\omega_0 t\end{aligned}$$

$$\begin{aligned}\mu_y(t) &= C_{1y}\, cos\omega_0 t + C_{2y} sin\omega_0 t\\ C_{1y} &= \mu_y(0)\\ C_{2y} &= \frac{d\mu_y(0)}{dt}\frac{1}{\omega_0} = -\mu_x(0)\\ &\Rightarrow\\ \mu_y(t) &= \mu_y(0)\, cos\omega_0 t - \mu_x(0) sin\omega_0 t\\ \\ \mu_z(t) &= \mu_z(0)\end{aligned}$$

Problem 2.7

Show that, if (2.38) is substituted into (2.36), then (2.32) and (2.33) are recovered, assuming $\theta = \omega_0 t$.

Remember:

- The precession motion of a particle placed in a static magnetic field can be described using a matrix representation.

Proof.

$$\vec{\mu}(t) = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \vec{\mu}(0)$$

$$\begin{pmatrix} \mu_x(t) \\ \mu_y(t) \\ \mu_z(t) \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu_x(0) \\ \mu_y(0) \\ \mu_z(0) \end{pmatrix} \Rightarrow$$

$$\mu_x(t) = \mu_x(0) \cdot cos\omega_0 t + \mu_y(0) \cdot sin\omega_0 t$$

$$\mu_y(t) = \mu_y(0) \cdot cos\omega_0 t - \mu_x(0) \cdot sin\omega_0 t$$

$$\mu_z(t) = \mu_z(0)$$

Problem 2.8

Show that the real and imaginary parts of (2.41) agree with the first two solutions in (2.33).

Remember:

- The 2 degrees of freedom, μ_x and μ_y can be given in terms of real and imaginary part of μ_+.
- Phase directly relates to position and is of utmost importance in the description of spin motion.

Proof.

$$\begin{aligned}
\mu_+(t) &= \mu_+(0)e^{-i\omega_0 t} \\
&= (\mu_x(0) + i\,\mu_y(0))\,e^{-i\omega_0 t} \\
&= (\mu_x(0) + i\,\mu_y(0))\,(cos\omega_0 t\; - \; i\,sin\omega_0 t) \\
&= \mu_x(0)\,cos\omega_0 t - i\,\mu_x(0)\,sin\omega_0 t + i\,\mu_y(0)\,cos\omega_0 t + \mu_y(0) sin\omega_0 t \\
&= \mu_x(t) + i\,\mu_y(t)
\end{aligned}$$

3 Rotating Reference Frames and Resonance

3.1 Summary

1. This chapter investigates the response of a spin immersed in a static magnetic field and an additional osciallating magnetic field (going beyond static magnetic field).

3.2 Exercises

Problem 3.1

Reversing the argument given in the discussion of Sec. 2.3 and defining the unit vector $\hat{n}$ parallel to $\vec{\Omega} \times \vec{C}$, show that

$$d\vec{C}_\perp = \Omega dt |\vec{C}_\perp| \hat{n} = \vec{\Omega} \times \vec{C} dt \tag{3.4}$$

Remember:

- The perpendicular component (perpendicular to $\vec{\Omega}$, the rotational angular velocity vector of the rotating frame) of some vector $\vec{C}$ at rest in the rotating frame has a differential change whose magnitude can be calculated.
- The parallel component does not change through time.
- $\vec{\Omega} \times \vec{C}$ = anti-clockwise rotation

Proof.

$$\begin{aligned}
\frac{d\vec{C}}{dt} &= \vec{\Omega} \times \vec{C} \\
\frac{d(\vec{C}_\perp + \vec{C}_{||})}{dt} &= \vec{\Omega} \times (\vec{C}_\perp + \vec{C}_{||}) \\
\frac{d\vec{C}_\perp}{dt} + \frac{d\vec{C}_{||}}{dt} &= \vec{\Omega} \times \vec{C}_\perp + \vec{\Omega} \times \vec{C}_{||} \\
\frac{d\vec{C}_{||}}{dt} &= \vec{\Omega} \times \vec{C}_{||} \\
&= \Omega \, |\vec{C}_{||}| \, sin \, 0^o \, \hat{n} = \mathbf{0} \\
&\Rightarrow \\
\frac{d\vec{C}_\perp}{dt} &= \vec{\Omega} \times \vec{C}_\perp \\
d\vec{C}_\perp &= \Omega \, dt |\vec{C}_\perp| \, sin\theta \, \hat{n} \\
\text{But } \theta = 90^o &\Rightarrow sin \, \theta = 1 \Rightarrow \\
d\vec{C}_\perp &= \Omega \, dt |\vec{C}_\perp| \, \hat{n}
\end{aligned}$$

Problem 3.2

Show that the average (3.22) over time interval T for $f(t) = \sin(n\omega t)$ or $f(t) = \cos(n\omega t)$ for any positive integer n is zero when

a) $T = 2m\pi/\omega$ for any positive integer m (or $T = m\pi/\omega$ for even n)

or when the following limit is taken

b) $T \gg 2\pi/\omega$

Remember:

- In the rotating reference frame, only half of the original linearly polarized field ($\vec{B_1}^{lin} = b_1^{lin}\, cos\,(\omega t)\,\hat{x}$) is available to tip the spin ($\vec{B_1}^{lin} = \frac{1}{2}\, b_1^{lin}[\hat{x}'(1 + cos\, 2\omega t) + \hat{y}' sin\, 2\omega t])$.
- The idea is that the effects of the oscillating terms average to zero over times that are half multiples of the rf period, or for all times large compared to the rf period.

Also remember:

$$\int_0^{2\pi} sin(x)dx = 0$$
$$\int_0^{2\pi} cos(x)dx = 0$$
$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cos(x)dx = 2$$
$$\int_0^{\pi} sin(x)dx = 2$$

a) Proof.

$$\begin{aligned}
< f(t) > \equiv \frac{1}{T}\int_0^T f(t)dt &= \frac{1}{T}\int_0^T sin(n\omega t)dt \\
&= \frac{\omega}{2m\pi}\int_0^{\frac{2m\pi}{\omega}} sin(n\omega t)dt \\
&= \frac{\omega}{2m\pi}[-cos(n\omega t)\frac{1}{n\omega}]_0^{\frac{2m\pi}{\omega}} \\
&= \frac{1}{2nm\pi}(-cos(2nm\pi) + cos(0)) \\
&= \frac{1}{2nm\pi} - \frac{1}{2nm\pi}cos(2nm\pi)
\end{aligned}$$

We know that: $cos(2N\pi) = 1$ for $N \in \mathbb{Z} \Rightarrow$

$$\frac{1}{T}\int_0^T f(t)dt = 0$$

b) Proof.

$$
\begin{aligned}
< f(t) > \equiv \frac{1}{T}\int_0^T f(t)dt &= \frac{1}{T}\int_0^T sin(n\omega t)dt \\
&= \frac{1}{T}[-cos(n\omega t)\frac{1}{n\omega}]_0^T \\
&= \frac{1}{n\omega T}(-cos(n\omega T) + cos(0)) \\
&= \frac{1}{n\omega T}(1 - cos(n\omega T))
\end{aligned}
$$

Problem 3.3

In many MR experiments it is necessary to flip a spin, which is initially along the z-axis, into the x'-y' plane by using an appropriate rf pulse. This is referred to as a '90°' or '$\pi/2$' pulse. If the desired rf pulse time interval is 1.0 ms, what B_1 magnitude in μT (1 T = 10,000 G) is required for

a) a proton spin? (answer given above)

b) an electron spin? (See (2.22).)

c) How many Larmor precession cycles take place in the laboratory frame, for $B_0 = 1.0\,\text{T}$, during the $\pi/2$ flip of the proton spin in (a)? See Fig. 3.3.

Remember:

- For the same angle of tipping, the rf magnetic field will depend on the type of particle (electron/proton).

a) Proof.

$$
\begin{aligned}
\Delta\theta &= \gamma_p B_1 \tau \\
\cancel{\pi}/2\,\cancel{rad} &= 42.6 \times 10^6 \times 2\cancel{\pi}\,\cancel{rad}\,T^{-1}\cancel{s^{-1}}\,B_1\,10^{-3}\cancel{s} \\
B_1 &= \frac{1}{4 \times 42.6} \times 10^{-3}\,T \\
B_1 &= 5.87\mu T
\end{aligned}
$$

b) Proof.

$$
\begin{aligned}
\Delta\theta &= \gamma_e B_1 \tau \\
\cancel{\pi}/2\,\cancel{rad} &= 658 \times 42.6 \times 10^6 \times 2\cancel{\pi}\,\cancel{rad}\,T^{-1}\cancel{s^{-1}}\,B_1\,10^{-3}\cancel{s} \\
B_1 &= \frac{1}{4 \times 658 \times 42.6} \times 10^{-3}\,T \\
B_1 &= 0.009\mu T
\end{aligned}
$$

c) Proof.

$$
\begin{aligned}
\omega_0 &= \gamma B_0 \\
\cancel{2\pi\,rad}\,\nu_0 &= 42.6 \times 10^6 \times \cancel{2\pi\,rad}\,\cancel{T^{-1}}\,s^{-1} \times 1\cancel{T} \\
\nu_0 &= 42.6\,MHz
\end{aligned}
$$

Problem 3.4

A static field points uniformly along the positive z-axis. A classical spinning particle, with positive gyromagnetic ratio γ and fixed magnetic moment magnitude μ, has its spin initially in the direction of the static field. A circularly polarized rf field points along the $\hat{y}'$ axis with time-dependent amplitude $B_{1y'}(t)$ (e.g., the rf field may be turned off at a later time) applied on-resonance starting at $t = 0$.

a) Give expressions analogous to (3.33) for all three magnetic-moment vector components in the rotating (prime) reference frame for time $t > 0$. Your answer will be in terms of a definite integral.

b) Show that the equation of motion (2.24) is satisfied by your answers in (a) for $\vec{B} \to B_{1y'}\hat{y}'$.

c) Find the generalization of (2.35) needed for this time-dependent case.

Remember:

- The magnetic moment motion behaves similarly for an on-resonance effective field.

a) You are asked to: give solutions for the motion through time of the magnetic moment vector components when the B_1 field is a circularly polarised rf field pointing along the $\vec{y}'$ axis.
Proof.

$$\begin{aligned}
\mu_x(t) &= \mu_x(0)\cos\omega_0 t + \mu_y(0)\sin\omega_0 t \\
\mu_y(t) &= \mu_y(0)\cos\omega_0 t - \mu_x(0)\sin\omega_0 t \\
\mu_z(t) &= \mu_z(0)
\end{aligned} \qquad (2.33)$$

The total effective field on-resonance will be given by $\vec{B} = B_{1y'}\hat{y}'$. The magnetic moment vector motion is found by transcribing the solution (2.33) according to the substitutions: $z \to y'$, $x \to z'$, $y \to x'$

$$\begin{aligned}
&\mu_{x'}(t) = \mu_{x'}(t_0)\,cos\,\phi_1(t) - \mu_{z'}(t_0)\,sin\,\phi_1(t) \\
&\mu_{y'}(t) = \mu_{y'}(t_0) \\
&\mu_{z'}(t) = \mu_{z'}(t_0)\,cos\,\phi_1(t) + \mu_{x'}(t_0)\,sin\,\phi_1(t) \\
&\text{with :} \\
&\phi_1(t) = \int_{t_0}^{t} dt'\omega_1(t') \\
&\text{in which :} \\
&\omega_1(t) = \gamma\, B_1(t)
\end{aligned}$$

b) Proof.

$$\frac{d\vec{\mu}}{dt} = \gamma\vec{\mu} \times \vec{B} \qquad (2.24)$$

$$
\begin{aligned}
\vec{B}_{1y'}(t) &= B_{1y'}(t)\,\hat{y}' \\
&= B_{1y'}(t)\,(\hat{x}\,sin\omega t + \hat{y}\,cos\omega t)
\end{aligned}
$$

We know that:

$$
\begin{aligned}
(\frac{d\vec{\mu}}{dt})' &= \gamma\vec{\mu}' \times \vec{B}_{1y'}(t) \\
\frac{d\mu_{x'}(t)}{dt}\hat{x}' + \frac{d\mu_{y'}(t)}{dt}\hat{y}' + \frac{d\mu_{y'}(t)}{dt}\hat{z}' &= \gamma\,(\mu_{x'}(t)\hat{x}' + \mu_{y'}(t)\hat{y}' + \mu_{z'}(t)\hat{z}') \times B_{1y'}(t)\hat{y}' \\
&= \gamma B_{1y'}(t)\,(\mu_{x'}(t)\hat{x}' \times \hat{y}' + \mu_{y'}(t)\hat{y}' \times \hat{y}' + \mu_{z'}(t)\hat{z}' \times \hat{y}') \\
&= \gamma B_{1y'}(t)\,(\mu_{x'}(t)\hat{z}' - \mu_{z'}(t)\hat{x}') \\
\Rightarrow & \\
\frac{d\mu_{x'}(t)}{dt} &= -\gamma\mu_{z'}(t)B_{1y'}(t) = -\omega_1(t)\mu_{z'}(t) \\
\frac{d\mu_{y'}(t)}{dt} &= 0 \\
\frac{d\mu_{z'}(t)}{dt} &= \gamma\mu_{x'}(t)B_{1y'}(t) = \omega_1(t)\mu_{x'}(t)
\end{aligned}
$$

Taking the first derivative of the solutions from a) we get :

$$
\begin{aligned}
\frac{d\mu_{x'}(t)}{dt} &= -\,\mu_{x'}(t_0)\,\omega_1(t)\,sin\,\phi_1(t)\; - \mu_{z'}(t_0)\,\omega_1(t)\,cos\,\phi_1(t) \\
&= -\,\omega_1(t)\,(\mu_{x'}(t_0)\,sin\,\phi_1(t) + \mu_{z'}(t_0)\,cos\,\phi_1(t)) \\
&= -\,\omega_1(t)\,\mu_{z'}(t) \\
\frac{d\mu_{y'}(t)}{dt} &= 0 \\
\frac{d\mu_{z'}(t)}{dt} &= -\,\mu_{z'}(t_0)\,\omega_1(t)\,sin\,\phi_1(t)\; + \mu_{x'}(t_0)\,\omega_1(t)\,cos\,\phi_1(t) \\
&= \omega_1(t)\,(\mu_{x'}(t_0)\,cos\,\phi_1(t) - \mu_{z'}(t_0)\,sin\,\phi_1(t)) \\
&= \omega_1(t)\,\mu_{x'}(t)
\end{aligned}
$$

c) Proof.

$$
\begin{aligned}
\frac{d^2\mu_x}{dt^2} &= -\omega_0^2\mu_x \\
\frac{d^2\mu_y}{dt^2} &= -\omega_0^2\mu_y \qquad (2.35)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2\mu_{x'}(t)}{dt^2} &= \frac{d}{dt}\,(\frac{d\mu_{x'}(t)}{dt}) \\
&= -\frac{d}{dt}\,(\omega_1(t)\mu_{z'}(t)) \\
&= -\frac{d\omega_1(t)}{dt}\,\mu_{z'}(t) - \omega_1(t)\,\frac{d\mu_{z'}(t)}{dt} \\
&= -\frac{d\omega_1(t)}{dt}\,\mu_{z'}(t) - \omega_1^2(t)\mu_{x'}(t) \\
\frac{d^2\mu_{z'}(t)}{dt^2} &= \frac{d}{dt}\,(\frac{d\mu_{z'}(t)}{dt}) \\
&= \frac{d}{dt}\,(\omega_1(t)\mu_{x'}(t)) \\
&= \frac{d\omega_1(t)}{dt}\,\mu_{x'}(t) + \omega_1(t)\,\frac{d\mu_{x'}(t)}{dt} \\
&= \frac{d\omega_1(t)}{dt}\,\mu_{x'}(t) - \omega_1^2(t)\mu_{z'}(t)
\end{aligned}
$$

Problem 3.5

Show that

$$\hat{x}^{right} = \hat{x}' \cos 2\omega t + \hat{y}' \sin 2\omega t \tag{3.43}$$

using steps like those used in deriving (3.21). Its time average is clearly zero.

Remember:

- A left-circularly polarized field is maximally effective in tipping the spin around the x'-axis, while a right-circularly polarized field is completely ineffective.

Proof.
We know that:

$$\begin{aligned}\hat{x}^{right} &= \hat{x}\, cos\omega t + \hat{y}\, sin\omega t\\ \hat{x}' &= \hat{x}\, cos\omega t - \hat{y}\, sin\omega t\\ \hat{y}' &= \hat{x}\, sin\omega t + \hat{y}\, cos\omega t\end{aligned}$$

Therefore, we can write:

$$\begin{aligned}\hat{x}'\, cos2\omega t &= \hat{x}\, cos\omega t\,(1 - 2sin^2\omega t)\\ &- \hat{y}\, sin\omega t\,(2\,cos^2\omega t - 1)\\ &= \hat{x}\, cos\omega t - 2\hat{x}\, cos\omega t\, sin^2\omega t\\ &- 2\hat{y}\, sin\omega t\, cos^2\omega t + \hat{y}\, sin\omega t\\ \hat{y}'\, sin2\omega t &= 2\hat{x}\, sin^2\omega t\, cos\omega t\\ &+ 2\hat{y}\, sin\omega t\, cos^2\omega t\end{aligned}$$

By adding them together we get:

$$\begin{aligned}&\hat{x}'\, cos2\omega t + \hat{y}'\, sin2\omega t = \hat{x}\, cos\omega t + \hat{y}\, sin\omega t \Rightarrow\\ &\hat{x}'\, cos2\omega t + \hat{y}'\, sin2\omega t = \hat{x}^{right}\end{aligned}$$

Calculating the time average:

$$< f(t) > \equiv \frac{1}{T}\int_0^T f(t)dt \Rightarrow$$

$$\begin{aligned}< \hat{x}^{right} > &\equiv \frac{1}{T}\int_0^T \hat{x}'\, cos2\omega t + \hat{y}'\, sin2\omega t\ dt =\\ &= \frac{1}{T}\int_0^T \hat{x}\, cos\omega t + \hat{y}\, sin\omega t\ dt\end{aligned}$$

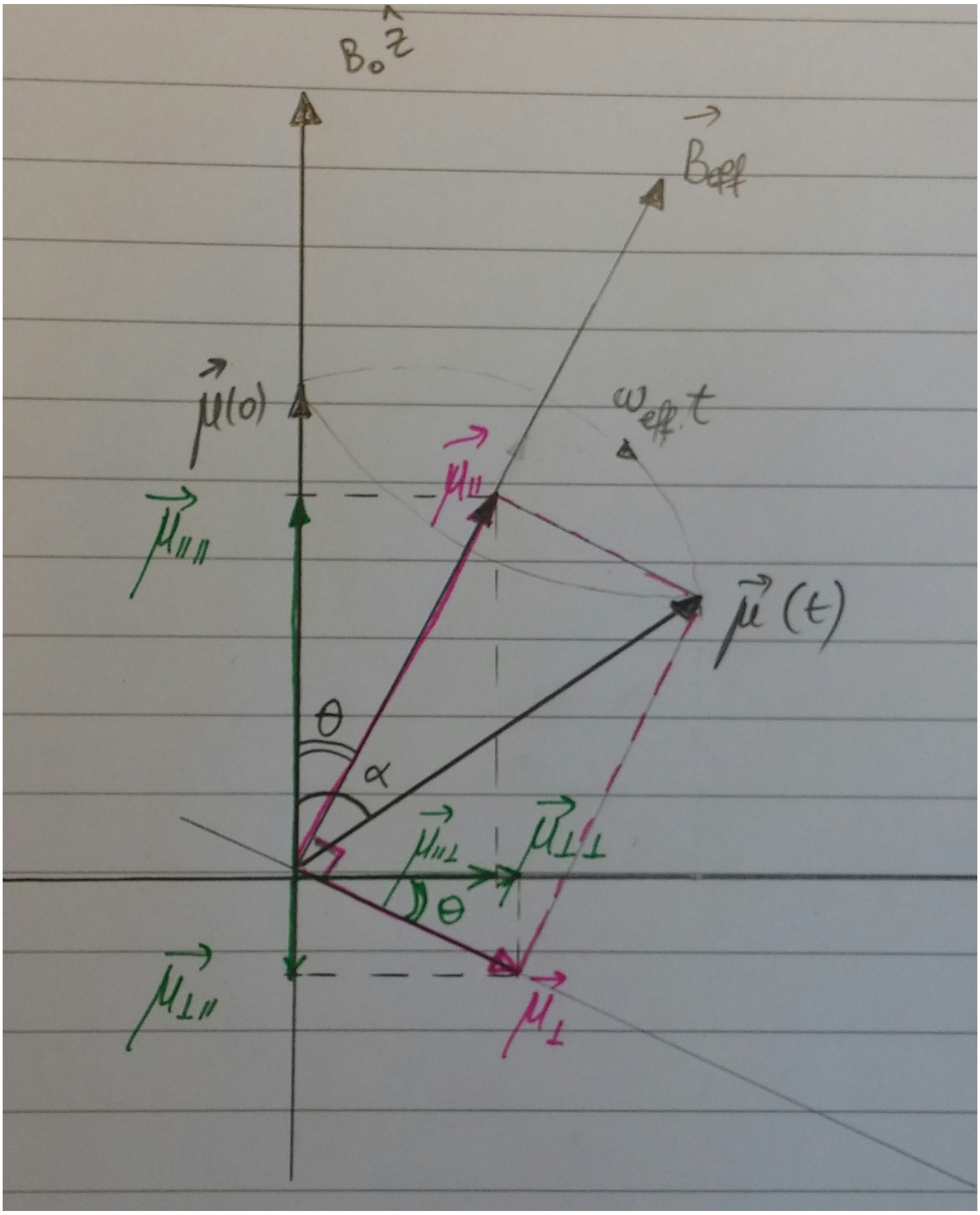
$$\begin{aligned}&= \hat{x}\,\frac{1}{T}\int_0^T cos\,\omega t\, dt + \hat{y}\,\frac{1}{T}\int_0^T sin\,\omega t\, dt \Rightarrow \text{(as shown in } \textit{Problem 3.2}\text{)}\\ &= \mathbf{0}\end{aligned}$$

Problem 3.6

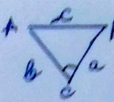
Let $\alpha(t)$ be the angle between $\vec{\mu}(0)$ and $\vec{\mu}(t)$ as shown in Fig. 3.4. Assume $\vec{\mu}(0)$ is parallel to the $\hat{z}$-direction. Show that $\cos\alpha(t)$, measuring the amount of magnetic moment, or spin, left along $\hat{z}$, is given in terms of θ and ω_{eff} by (3.54). Hint: Decompose $\vec{\mu}(t)$ into components parallel and perpendicular to $\vec{B}_{eff}$. Then project these components onto the z-axis.

Remember:

- The behaviour of the z-component of the magnetic moment is shown here.

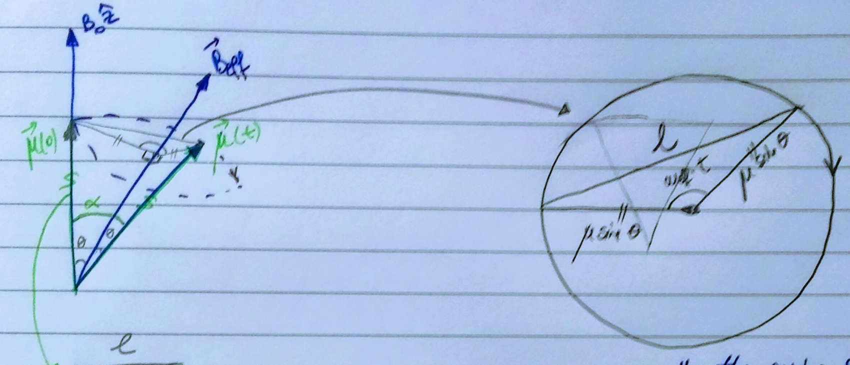


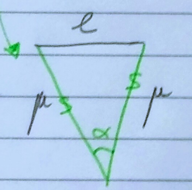
Proof.

1) The cosine rule:  $c^2 = a^2 + b^2 - 2ab\cos C$

2) Double angle formula: $\cos(a+a) = \cos a \cos a - \sin a \sin a$

Problem 3.6





⇓ the cosine rule

$$\ell^2 = \mu^2\sin^2\theta + \mu^2\sin^2\theta - 2\cdot\mu^2\sin^2\theta\cos(\omega_{eff}\cdot t) \quad (1)$$

⇓ the cosine rule

$$\ell^2 = 2\mu^2 - 2\mu^2\cos\alpha \quad (2)$$

From (1) and (2) ⇒ $\cancel{2\mu^2}\sin^2\theta\left(1-\cos(\omega_{eff}\,t)\right) = \cancel{2\mu^2}(1-\cos\alpha)$

$$\Rightarrow \sin^2\theta - \sin^2\theta\cos(\omega_{eff}\cdot t) = 1-\cos\alpha \Rightarrow$$

$$\Rightarrow \cos\alpha = 1 - \sin^2\theta\left(1-\cos(\omega_{eff}\cdot t)\right) \xRightarrow{\text{double angle formula}}$$

$$\Rightarrow \cos\alpha = 1 - \sin^2\theta\left(\underbrace{1-\cos^2(\omega_{eff}t/2)}_{\sin^2(\omega_{eff}t/2)} + \sin^2(\omega_{eff}\,t/2)\right)$$

$$\Rightarrow \boxed{\cos\alpha = 1 - 2\sin^2\theta\,\sin^2\frac{\omega_{eff}\cdot t}{2}}$$

4 Magnetization, Relaxation, and the Bloch Equation

4.1 Summary

1. This chapter investigates the

4.2 Exercises

Problem 4.1

Derive (4.12) by solving the first-order differential equation (4.11). Hint: One method is to use an integrating factor. Another is simply to put the magnetization and time variables on the opposite sides of the equation and integrate.

$$\frac{dM_z}{dt} = \frac{1}{T_1}(M_0 - M_z) \qquad (\vec{B}_{ext} \parallel \hat{z}) \qquad (4.11)$$

$$M_z(t) = M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1}) \qquad (\vec{B}_{ext} \parallel \hat{z}) \qquad (4.12)$$

Remember:

- After applying an RF pulse, the longitudinal magnetization relaxes back to its equilibrium value (M_0), following an exponential evolution from the initial value ($M_z(0)$).

Proof.

$$\begin{aligned}
\frac{dM_z}{dt} &= \frac{1}{T_1}(M_0 - M_z) \\
\frac{1}{M_0 - M_z}dM_z &= \frac{1}{T_1}dt \\
\int_{M_z(0)}^{M_z(t)} \frac{1}{M_0 - M_z}dM_z &= \int_t^0 \frac{1}{T_1}dt \\
[ln(M_0 - M_z)]_{M_z(0)}^{M_z(t)} &= [\frac{t}{T_1}]_t^0 \\
ln\frac{M_0 - M_z(t)}{M_0 - M_z(0)} &= -\frac{t}{T_1} \\
exp(ln\frac{M_0 - M_z(t)}{M_0 - M_z(0)}) &= exp(-\frac{t}{T_1}) \\
\frac{M_0 - M_z(t)}{M_0 - M_z(0)} &= exp(-\frac{t}{T_1}) \\
M_0 - M_z(t) &= M_0\, e^{-t/T_1} - M_z(0)\, e^{-t/T_1} \\
M_z(t) &= M_0(1 - e^{-t/T_1}) + M_z(0)e^{-t/T_1}
\end{aligned}$$

Problem 4.2

The key equation (4.12) can be used to investigate general questions. If unmagnetized material is placed in a region with a finite static field at $t = 0$ ($M_z(0) = 0$):

a) Find the time it takes, in units of T_1, for the longitudinal magnetization to reach 90% of M_0.

b) Find an approximate formula for $M_z(t)$ of this material in the limit that $t \ll T_1$. Use this formula to find the initial ($t = 0$) slope of $M_z(t)$, and compare the answer to the general formula indicated in Fig. 4.1, when $M_z(0) = 0$.

Remember:

- Investigating general questions using $M_z(t) = M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1})$
- *Taylor Series*:
  $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$
- *Maclaurin Series* (is Taylor series for $a = 0$):
  $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$
- If $f(x) = e^{-t/T_1}$, then: $e^{-t/T_1} \approx 1 - \frac{t}{T_1}$

a) Proof.
We know that:

$$M_z(t) = 0.9M_0$$
$$M_z(0) = 0$$

Therefore:

$$\begin{aligned} M_z(t) &= M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1}) \\ 0.9M_0 &= M_0(1 - e^{-t/T_1}) \\ e^{-t/T_1} &= 0.1 \\ -t/T_1 &= ln(0.1) \\ t &= 2.3\,T_1 \end{aligned}$$

b) Proof.
We know that:

$$t << T_1$$

Therefore:

$$\begin{aligned} M_z(t) &= M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1}) \\ [\frac{dM_z(t)}{dt}]_{t<<T_1} &= -[\frac{1}{T_1}M_z(0)e^{-t/T_1}]_{t<<T_1} + [\frac{1}{T_1}M_0e^{-t/T_1}]_{t<<T_1} \\ [\frac{dM_z(t)}{dt}]_{t<<T_1} &= -\frac{1}{T_1}M_z(0) + \frac{1}{T_1}M_0 \\ [\frac{dM_z(t)}{dt}]_{t<<T_1} &= \frac{1}{T_1}(M_0 - M_z(0)) \end{aligned}$$

Problem 4.3

A direct derivation of the steady-state solution, when it exists, of a system of differential equations can often be found by the following procedure. Assuming that the system evolves to constant values for large times, all time derivatives can be set to zero. The problem reduces to a system that can often be solved analytically. Use this procedure to find the steady-state solution directly from (4.22), (4.23), and (4.24), verifying (4.28).

Remember:

- Finding the steady-state solution.

$$\frac{dM_z}{dt} = \frac{M_0 - M_z}{T_1} \tag{4.22}$$
$$\frac{dM_x}{dt} = \omega_0 M_y - \frac{M_x}{T_2} \tag{4.23}$$
$$\frac{dM_y}{dt} = -\omega_0 M_x - \frac{M_y}{T_2} \tag{4.24}$$

$$M_x(\infty) = M_y(\infty) = 0, \quad M_z(\infty) = M_0 \tag{4.28}$$

Proof.

$$\begin{aligned}
\frac{M_0 - M_z}{T_1} &= 0 \\
\omega_0 M_y - \frac{M_x}{T_2} &= 0 \\
-\omega_0 M_x - \frac{M_y}{T_2} &= 0 \\
&\Rightarrow \\
M_z &= M_0 \\
M_x &= \omega_0 T_2 M_y \\
M_y &= \omega_0 T_2 M_x \\
&\Rightarrow \\
M_z &= M_0 \\
M_x &= \omega_0 T_2 M_y \\
M_y &= \omega_0 T_2 M_x \\
(M_x + M_y) &= \omega_0 T_2 (M_x + M_y) \rightarrow \omega_0 T_2 = 1 \\
&\Rightarrow
\end{aligned}$$

$$
\begin{aligned}
M_z &= M_0 \\
M_x &= M_y \\
\omega_0 M_x - \frac{M_x}{T_2} &= 0 \rightarrow M_x(\omega_0 - \frac{1}{T_2}) = 0 \rightarrow M_x = 0 \\
&\Rightarrow \\
M_x(\infty) &= M_y(\infty) = 0 \\
M_z(\infty) &= M_0
\end{aligned}
$$

Or:

$$
\begin{aligned}
\lim_{t\to\infty} M_x(t) &= \lim_{t\to\infty} e^{-t/T_2}(M_x(0)cos\omega_0 t + M_y(0)sin\omega_0 t) = 0 \\
\lim_{t\to\infty} M_y(t) &= \lim_{t\to\infty} e^{-t/T_2}(M_y(0)cos\omega_0 t - M_x(0)sin\omega_0 t) = 0 \\
\lim_{t\to\infty} M_z(t) &= \lim_{t\to\infty} M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1}) = M_0 \\
&\Rightarrow \\
M_x(\infty) &= M_y(\infty) = 0 \\
M_z(\infty) &= M_0
\end{aligned}
$$

Problem 4.4

a) Find the differential equation for $M_+(t)$ analogous to (2.40) and show that its solution is (4.30).

b) Show that (4.30) is equivalent to (4.25) and (4.26).

c) Repeat (a) for

$$M_- \equiv M_x - iM_y = -i(M_y + iM_x) \tag{4.34}$$

Remember:

- The complex representation solutions.

a) Proof.

$$\begin{aligned}
\frac{dM_+}{dt} &= (-i\omega_0 - \frac{1}{T_2})M_+ \\
\frac{1}{M_+}dM_+ &= (-i\omega_0 - \frac{1}{T_2})dt \\
\int_{M_+(0)}^{M_+(t)} \frac{1}{M_+}dM_+ &= \int_0^t (-i\omega_0 - \frac{1}{T_2})dt \\
ln\frac{M_+(t)}{M_+(0)} &= t(-i\omega_0 - \frac{1}{T_2}) \\
\frac{M_+(t)}{M_+(0)} &= e^{-i\omega_0 t - \frac{t}{T_2}} \\
M_+(t) &= M_+(0)e^{-i\omega_0 t - \frac{t}{T_2}}
\end{aligned}$$

Problem 4.5

Taking note of (3.26), demonstrate that in the primed basis (4.21) reduces to (4.37)–(4.39).

Remember:

- The Bloch equations magnetization components in the primed basis.

$$\frac{d\vec{M}}{dt} = \gamma \vec{M} \times \vec{B}_{ext} + \frac{1}{T_1}(M_0 - M_z)\hat{z} - \frac{1}{T_2}\vec{M}_\perp \tag{4.21}$$

$$\left(\frac{dM_z}{dt}\right)' = -\omega_1 M_{y'} + \frac{M_0 - M_z}{T_1} \tag{4.37}$$

$$\left(\frac{dM_{x'}}{dt}\right)' = \Delta\omega\ M_{y'} - \frac{M_{x'}}{T_2} \tag{4.38}$$

$$\left(\frac{dM_{y'}}{dt}\right)' = -\Delta\omega\ M_{x'} + \omega_1 M_z - \frac{M_{y'}}{T_2} \tag{4.39}$$

Proof.

$$\begin{aligned}
(\frac{d\vec{M}}{dt})' &= \gamma \vec{M}' \times \vec{B}_{eff} + \frac{1}{T_1}(M_0 - M_z)\hat{z} - \frac{1}{T_2}(M_{x'}\hat{x}' + M_{y'}\hat{y}') \\
&= \gamma \vec{M}' \times [(B_0 - \omega/\gamma)\hat{z} + B_1\hat{x}'] + \frac{1}{T_1}(M_0 - M_z)\hat{z} - \frac{1}{T_2}(M_{x'}\hat{x}' + M_{y'}\hat{y}') \\
&= \gamma(B_0 - \omega/\gamma)(-M_{x'}\hat{y}' + M_{y'}\hat{x}') + \gamma B_1(-M_{y'}\hat{z} + M_z\hat{y}') \\
&+ \frac{1}{T_1}(M_0 - M_z)\hat{z} - \frac{1}{T_2}(M_{x'}\hat{x}' + M_{y'}\hat{y}') \Rightarrow \\
(\frac{dM_z}{dt})' &= -\omega_1 M_{y'} + \frac{M_0 - M_z}{T_1} \\
(\frac{dM_x}{dt})' &= (\omega_0 - \omega)M_{y'} - \frac{M_{x'}}{T_2} \\
(\frac{dM_y}{dt})' &= -(\omega_0 - \omega)M_{x'} + \omega_1 M_z - \frac{M_{y'}}{T_2} \\
&\Rightarrow \\
(\frac{dM_z}{dt})' &= -\omega_1 M_{y'} + \frac{M_0 - M_z}{T_1} \\
(\frac{dM_x}{dt})' &= \Delta\omega M_{y'} - \frac{M_{x'}}{T_2} \\
(\frac{dM_y}{dt})' &= -\Delta\omega M_{x'} + \omega_1 M_z - \frac{M_{y'}}{T_2}
\end{aligned}$$

Problem 4.6

Return to the Bloch equations (4.37)–(4.39) and solve them in the steady state for arbitrary B_1, obtaining

$$M_{x'}^{ss} = M_0 \frac{\Delta\omega\, T_2}{D} \omega_1 T_2 \tag{4.50}$$

$$M_{y'}^{ss} = M_0 \frac{1}{D} \omega_1 T_2 \tag{4.51}$$

$$M_z^{ss} = M_0 \frac{1 + (\Delta\omega\, T_2)^2}{D} \tag{4.52}$$

with

$$D = 1 + (\Delta\omega\, T_2)^2 + \omega_1^2 T_1 T_2 \tag{4.53}$$

Notice that these reduce to (4.48), (4.49), and (4.45), respectively, for small ω_1. In particular, show that $M_{x'}^{ss}$ and $M_{y'}^{ss}$ are $\mathcal{O}(\omega_1)$ and that $M_0 - M_z^{ss} = \mathcal{O}(\omega_1^2)$, consistent with the previous discussion. Also, show that the steady-state magnetization develops a phase shift in the x'-y' plane

$$\Delta\phi = \cot^{-1}(\Delta\omega T_2) \qquad (\text{mod } \pi) \tag{4.54}$$

Remember:

-

Proof.

Summary

- This chapter is focused on the continuous and discrete Fourier Transforms relevant to MR imaging.
- In the absence of relaxation effects, $s(\vec{k})$ and $\rho(\vec{r})$ are **Fourier transform pairs**.
- The reconstructed image, $\hat{\rho}(\vec{r})$, represents an **estimate** of the **effective spin density** due to various **numerical approximations** that take place during the MR imaging process (from **discretised** and **truncated** signal sampling).

Figure 5.1: *Jean-Baptiste Joseph Fourier [Image courtesy of Wikipedia]*

5.1 The Continuous Fourier Transform

- The **Fourier transform is a functional**, mapping a function to another function.

  Let:
  h be a function of x-space (or time space) and
  H be a function of k-space (or frequency space) then:

$$
\begin{aligned}
\mathfrak{F} &: (h) \rightarrow (H) \\
\mathfrak{F}^{-1} &: (H) \rightarrow (h) \\
\mathfrak{F} \circ \mathfrak{F}^{-1} &= \text{identity}
\end{aligned}
$$

  In MRI, the **Fourier transform maps a function** from **position space (x-space)** to its **'conjugate' space (k-space)**, and back, through the inverse Fourier transform.

- Fourier Transform Pairs in MRI: $\rho(\vec{r}) \overset{\mathfrak{F}}{\Leftrightarrow} s(\vec{k})$, where
  $\rho(\vec{r})$ = **spin density in x-space (position space)** and
  $s(\vec{k})$ = **signal in k-space (spatial frequency space)**

- Knowing the **signal equation** :

$$
s(\vec{k}) = \int_{-\infty}^{+\infty} d\vec{r}\ \rho(\vec{r}) e^{-i2\pi\vec{k}\cdot\vec{r}}
$$

  the **Fourier transform pairs** are shown for the general case:

$$
\begin{aligned}
H(\vec{k}) \equiv \mathfrak{F}(h(\vec{r})) &= \int_{-\infty}^{+\infty} d\vec{r}\ h(\vec{r}) e^{-i2\pi\vec{k}\vec{r}} \\
h(\vec{r}) \equiv \mathfrak{F}^{-1}(H(\vec{k})) &= \int_{-\infty}^{+\infty} d\vec{k}\ H(\vec{k}) e^{+i2\pi\vec{k}\vec{r}}
\end{aligned}
$$

  and for **Dirac delta functions**, knowing from (Book 9.24) that $\delta(\vec{r}) = \int_{-\infty}^{+\infty} d\vec{k}\ e^{i2\pi\vec{k}\cdot\vec{r}}$, we have:

$$
\begin{aligned}
\delta(\vec{k} - \vec{k}_0) &= \int_{-\infty}^{+\infty} d\vec{r}\ e^{-i2\pi(\vec{k}-\vec{k}_0)\cdot\vec{r}} \\
\delta(\vec{r} - \vec{r}_0) &= \int_{-\infty}^{+\infty} d\vec{k}\ e^{+i2\pi\vec{k}\cdot(\vec{x}-\vec{x}_0)}
\end{aligned}
$$

  Sidenote: $\delta(\vec{r}) \equiv \mathfrak{F}^{-1}(G(\vec{k})) = \int_{-\infty}^{+\infty} d\vec{k}\ G(\vec{k})\ e^{i2\pi\vec{k}\cdot\vec{r}} \Rightarrow G(\vec{k}) = 1$. This is very interesting as we have seen in Book Chapter 9.3.1, that $\delta(x) = \lim_{K\to\infty} 2K\ sinc(2\pi Kx)$ (see Figure 5.2), which shows that, at the limit, the Dirac delta function encompasses all possible frequencies. Therefore, its Fourier transform will be a straight line.

- The identity property ($\mathfrak{F} \circ \mathfrak{F}^{-1}$ = identity) can be derived like this:

$$
\begin{aligned}
h(x) &= \int_{-\infty}^{+\infty} dk\ H(k) e^{+i2\pi kx} \\
&= \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dx'\ h(x') e^{-i2\pi kx'}\ \ e^{+i2\pi kx} \\
&= \int_{-\infty}^{+\infty} dx'\ h(x') \int_{-\infty}^{+\infty} dk\ e^{+i2\pi k(x-x')} \\
&= \int_{-\infty}^{+\infty} dx'\ h(x') \delta(x - x') \text{ (See Book 9.27)} \\
&= h(x)
\end{aligned}
$$

- A continuous signal $s(k)$, known for all k, can be used to retrieve the **spin density** (Book 11.7):

$$\rho(\vec{r}) = \int_{-\infty}^{+\infty} d\vec{k}\ s(\vec{k}) e^{+i2\pi\vec{k}\cdot\vec{r}}$$

In fact, the signal is not known for all k, but it is an approximation of the continuous form.
The actual signal is finitely sampled, it is called $s_m(k)$ (measured signal) and it yields the reconstructed image $\hat{\rho(x)}$ through a continuous inverse Fourier transform:

$$\hat{\rho}(\vec{r}) = \int_{-\infty}^{+\infty} d\vec{k}\ s_m(\vec{k}) e^{+i2\pi\vec{k}\cdot\vec{r}}$$

- The differences between $\rho(\vec{r})$ and $\hat{\rho}(\vec{r})$ are very important in understanding imaging artifacts.

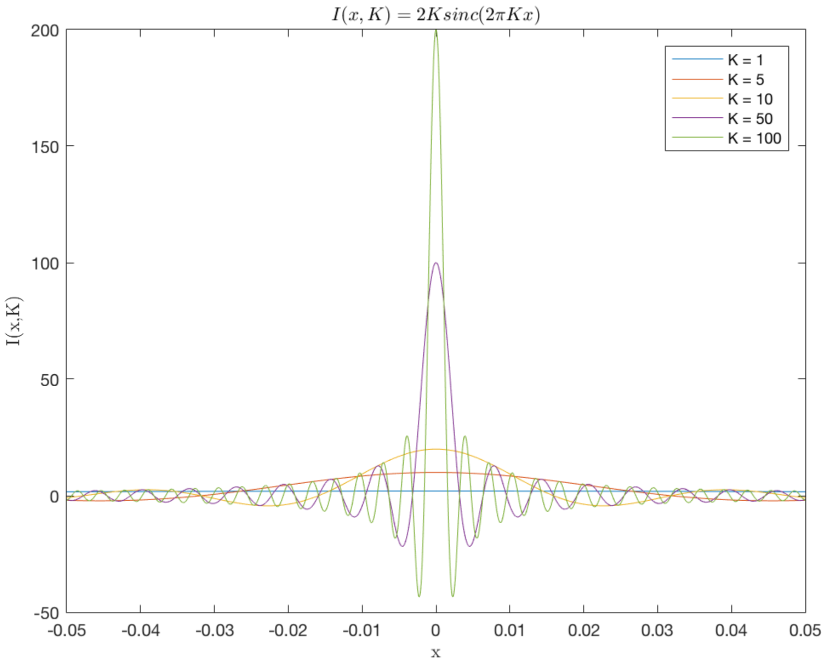


Figure 5.2: *When the sinc argument increases (K increases), the sinc function becomes narrower and narrower. The term outside of the sinc function increases its amplitude. Therefore, the larger the K, the closer the $I(x,K) = 2Ksinc(2\pi\ Kx)$ function gets to the Dirac delta.*

5.2 Continuous Fourier Transform Properties and Phase Imaging

Properties of the continuous Fourier transform are introduced here. It is again mentioned the fact that differences between $\hat{\rho}(x)$ and $\rho(x)$ are generally called artifacts.

5.2.1 Complexity of the Reconstructed Image

- Complexity of the reconstructed image refers to the fact that $\hat{\rho}(x)$ can be a complex-valued quantity unlike its ideal continuous version $\rho(x)$. This difference can be due to a switch between the real and imaginary RF receiver channels or an incorrect demodulation that can lead to the presence of a **global (constant) phase shift ϕ_0 in the signal**.
- This signal can be written as:

  $$\widetilde{s}(k) = e^{i\phi_0} s(k)$$

  where $s(k)$ is the real/true signal.

  The reconstructed image will then be:

  $$\begin{aligned} \hat{\rho}(x) &= \int_{-\infty}^{+\infty} dk\ \widetilde{s}(k) e^{+i2\pi kx} \\ &= e^{i\phi_0} \int_{-\infty}^{+\infty} dk\ s(k) e^{+i2\pi kx} \\ &= e^{i\phi_0} \rho(x) \end{aligned}$$

- This is the reason why, in practice, magnitude images are used in order to get $\rho(x)$ independent of ϕ_0. The magnitude image is calculated as:

  $$\begin{aligned} \hat{\rho}(x) &= e^{i\phi_0} \rho(x) \\ &= cos(\phi_0)\rho(x) + i\ sin(\phi_0)\rho(x) \\ |\hat{\rho}(x)| &= \sqrt{\rho(x)^2\ (cos^2(\phi_0) + sin^2(\phi_0))} = |\rho(x)| \end{aligned}$$

5.2.2 The Shift Theorem

- The Fourier Transform Shift Theorem shows that a shift in one space will translate into a global constant phase in its Fourier transform. In terms of MRI, the Fourier transform shift theorem is an important concept because it explains what happens to the reconstructed image when there is **a) a shift in the echo relative to its expected location** (see Figure 5.3) or due to **b) incorrect demodulation** (demodulation at the wrong frequency).
- Cases:
  1. **A linear shift of the k-space origin** will create an additional phase in the reconstructed image $\hat{\rho}(x)$. A magnitude image will faithfully reconstruct $\rho(x)$.

     The shift in k-space moves by k_0 from where it was expected. This means that $s_m(k) \rightarrow s_m(k-k_0)$. Taking the inverse Fourier transform of the shifted measured signal shows that the reconstruction will have an additional global phase in addition to the expected reconstruction for a non-shifted k-space.

     $$\begin{aligned} \hat{\rho}(x) &= \mathfrak{F}^{-1}(s_m(k-k_0)) \\ &= \int_{-\infty}^{+\infty} dk\ s_m(k-k_0) e^{i2\pi kx} \\ \text{Change of variable: } k' &= k - k_0 \\ &= e^{i2\pi k_0 x} \int_{-\infty}^{+\infty} dk'\ s_m(k') e^{i2\pi k' x} \\ &= e^{i2\pi k_0 x}\ \hat{\rho}_{expected}(x) \end{aligned}$$

2. **A linear phase shift in the k-space data** ($s_m(k) \rightarrow s_m(k)e^{-i2\pi kx_0}$) will lead to a spatial shift (of x_0) in the reconstructed image.

$$\rho(x') = \int_{-\infty}^{+\infty} dk\ s(k)e^{+i2\pi kx'}$$

Change of variable: $x' = x - x_0 \Rightarrow$

$$\begin{aligned}\rho(x - x_0) &= \int_{-\infty}^{+\infty} dk\ s(k)e^{+i2\pi k(x-x_0)} \\ &= \int_{-\infty}^{+\infty} dk\ s(k)e^{-i2\pi kx_0}\ e^{+i2\pi kx} \\ &= \mathfrak{F}^{-1}(s(k)\ e^{-i2\pi kx_0})\end{aligned}$$

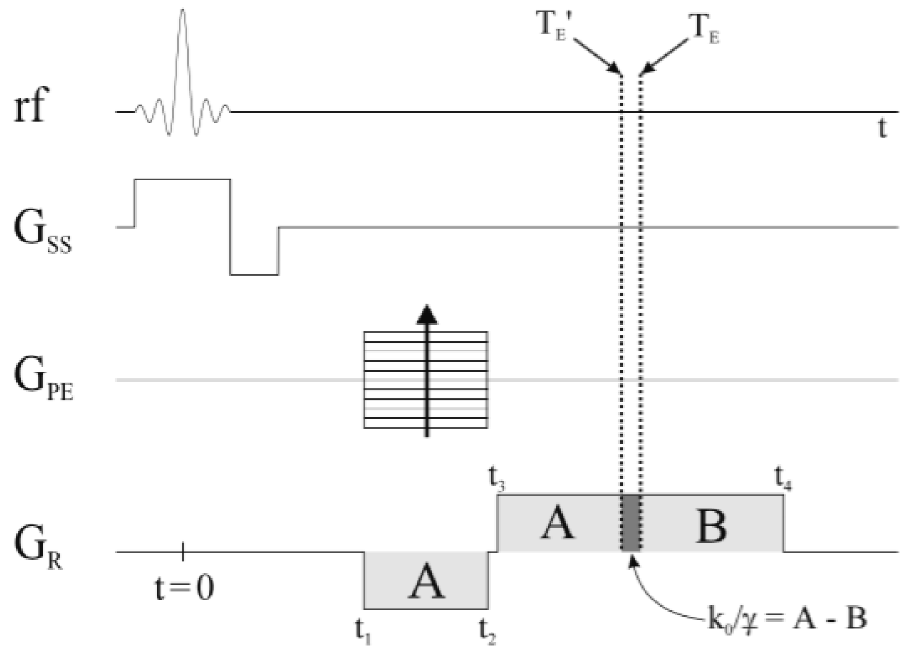


Figure 5.3: *Example of a read gradient structure leading to a shift of the echo in the sampling window. The area under the gradient between t_1 and t_2 is A and the area under the read gradient itself is 2B. The echo is shifted from TE to $TE' < TE$ if A is chosen to be less than B. This results in a shift of $k_0 < 0$ in the read direction k-space variable.*
[Image courtesy of Haacke et al. Magnetic Resonance Imaging: Physical Principles and Sequence Design, volume 1st. Wiley-Liss, 1999 [?]]

5.2.3 Phase Imaging and Phase Aliasing

- Phase imaging is a technique used in MRI where images of the accumulated phase are constructed instead of the classic anatomical or functional ones. Phase images are calculated with the inverse tangent as presented here:

$$\phi(x, y) \equiv Arg\ \hat{\rho}(x, y) = tan^{-1}\left(\frac{Im\ \hat{\rho}(x, y)}{Re\ \hat{\rho}(x, y)}\right)$$

  One particular usefulness is to see whether the expected gradient echo is uncentered.

- Phase Imaging artifact (see Figure 5.4):
  - *Zebra stripes* are a phase imaging artifact that can be caused by an **asymmetry in the echo position**. Because of the inverse tangents periodicity, spins with phase values differing by multiples of 2π will have the same intensity and this will cause the aforementioned artifact.
  - How? The centre of the sampling window is expected to be at $k = 0$, for which the readout gradient phase for all spins is 0. When this is no longer true, and a certain shift of k_0 has happened in the

way data was sampled, the measured signal $s_m(k)$ is no longer the true (expected) $s(k)$. Instead, $s_m(k) = s(k - k_0)$.

  – Applying the shift theorem here yields a reconstructed image in x-space that will also have an additional phase $\Delta\phi(x) = 2\pi k_0 x$.

- Besides the gradient echo asymmetry, local field inhomogeneities also cause changes in the phase images.

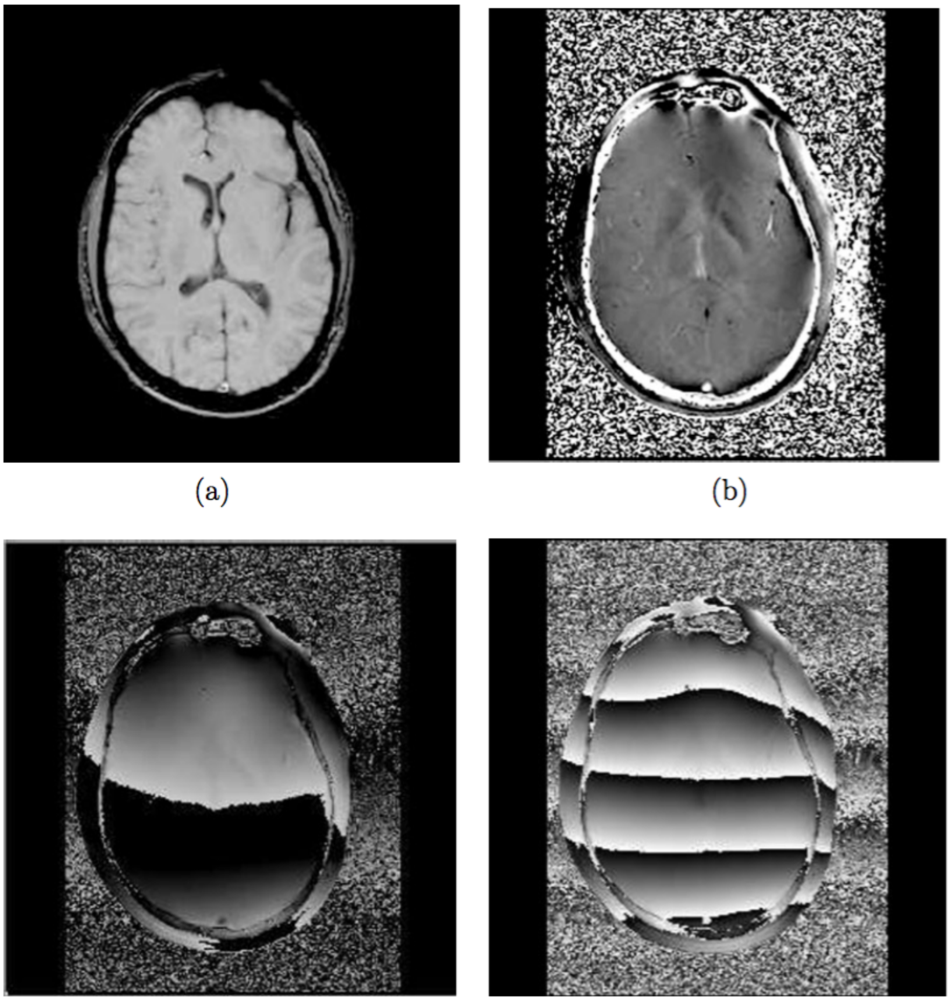


Figure 5.4: *(a) A 2D gradient echo magnitude image of the head. (b) The corresponding phase image of the head when the echo is centered in the read data acquisition window. (c) A phase image taken from the same data that produced (b) except that the echo is shifted 'one sample point' away from the center of the sampling window. (d) A phase image where the echo is shifted five points away from the center of the sampling window. The restricted variation of the phase from π to π creates the banding referred to as the 'zebra stripe' artifact. The phase shift occurs along the read (vertical) direction, and the bands are perpendicular to this direction. This is an example of how knowledge of the theory behind an artifact gives information about the sequence acquisition.*
[Image courtesy of Haacke et al. Magnetic Resonance Imaging: Physical Principles and Sequence Design, volume 1st. Wiley-Liss, 1999 [?]]

5.2.4 The Duality Theorem

- The Duality Theorem states that if $h(x)$ and $H(k)$ are Fourier transform pairs, then $H(-x)$ and $h(k)$ are also Fourier transform pairs.
- Why this is important?
  – Knowing one pair of Fourier transforms will lead to knowing a different pair without any other calculations.
  – Explanation from here:

> Duality between the time and frequency domains is another important property of Fourier transforms. This property relates to the fact that the analysis equation and synthesis equation

look almost identical except for a factor of $1/2\pi$ and the difference of a minus sign in the exponential in the integral. As a consequence, if we know the Fourier transform of a specified time function, then we also know the Fourier transform of a signal whose functional form is the same as the form of this Fourier transform. Said another way, *the Fourier transform of the Fourier transform is proportional to the original signal reversed in time.* One consequence of this is that whenever we evaluate one transform pair we have another one for free. As another consequence, if we have an effective and efficient algorithm or procedure for implementing or calculating the Fourier transform of a signal, then exactly the same procedure with only minor modification can be used to implement the inverse Fourier transform. This is, in fact, very heavily exploited in discrete-time signal analysis and processing, where explicit computation of the Fourier transform and its inverse play an important role. [Quotation courtesy of MIT]

- Deriving the duality theorem (if $h(x) \stackrel{\mathfrak{F}}{\Leftrightarrow} H(k)$, then $H(-x) \stackrel{\mathfrak{F}}{\Leftrightarrow} h(k)$):

$$1)\text{ Starting from: } h(x) = \int_{-\infty}^{+\infty} dk\ H(k)e^{+i2\pi kx}$$

Change of variable $k \to -k$

$$h(x) = \int_{-\infty}^{+\infty} dk\ H(-k)e^{-i2\pi kx} = \mathfrak{F}(H(-k))$$

Change of variable $k \to x$

$$h(k) = \int_{-\infty}^{+\infty} dx\ H(-x)e^{-i2\pi kx} = \mathfrak{F}(H(-x))$$

$$2)\text{ Starting from: } H(k) = \int_{-\infty}^{+\infty} dx\ h(x)e^{-i2\pi kx}$$

Change of variable $k \to -k$

$$H(-k) = \int_{-\infty}^{+\infty} dx\ h(x)e^{+i2\pi kx} = \mathfrak{F}^{-1}(h(x))$$

Change of variable $k \to x$

$$H(-x) = \int_{-\infty}^{+\infty} dk\ h(k)e^{+i2\pi kx} = \mathfrak{F}^{-1}(h(k))$$

- Duality theorem for shifted Delta function ($\delta(x) \to \delta(xx_0)$):

$$h(x) = \delta(x - x_0) \stackrel{\mathfrak{F}}{\Leftrightarrow} H(k) = e^{i2\pi kx_0}$$

and

$$H(-x) = e^{+i2\pi kx_0} \stackrel{\mathfrak{F}}{\Leftrightarrow} h(k) = \delta(k - k_0)$$

The shifted Delta function gives rise to a periodic transform with period $1/x_0$ as seen in Figure 5.5.

- A classic Fourier transform pair for MRI can be seen in both Figure 5.7 and Figure 5.6, for the $rect(\frac{x}{W}) \stackrel{\mathfrak{F}}{\Leftrightarrow} W\ sinc(\pi Wk)$.

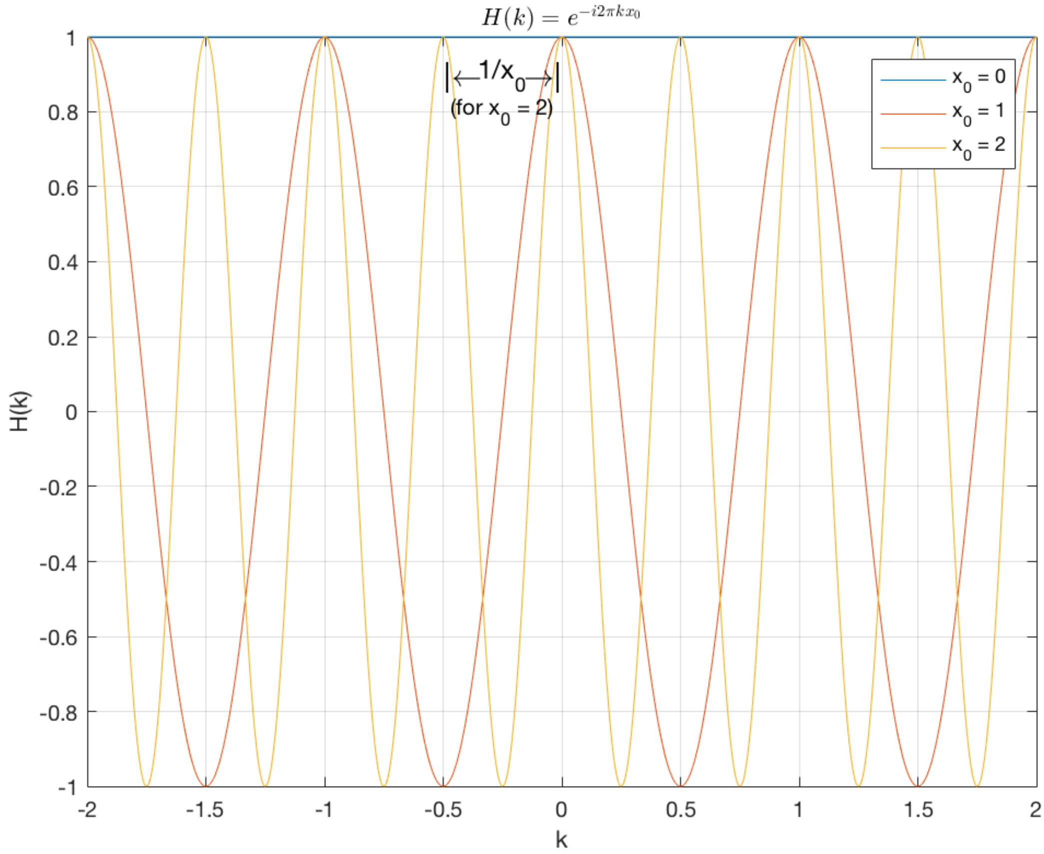


Figure 5.5: *The Fourier transform of a shifted Delta function ($H(k) \equiv \mathfrak{F}(\delta(x - x_0)) = e^{-i2\pi k x_0}$) is a periodic function (with period $1/x_0$)*

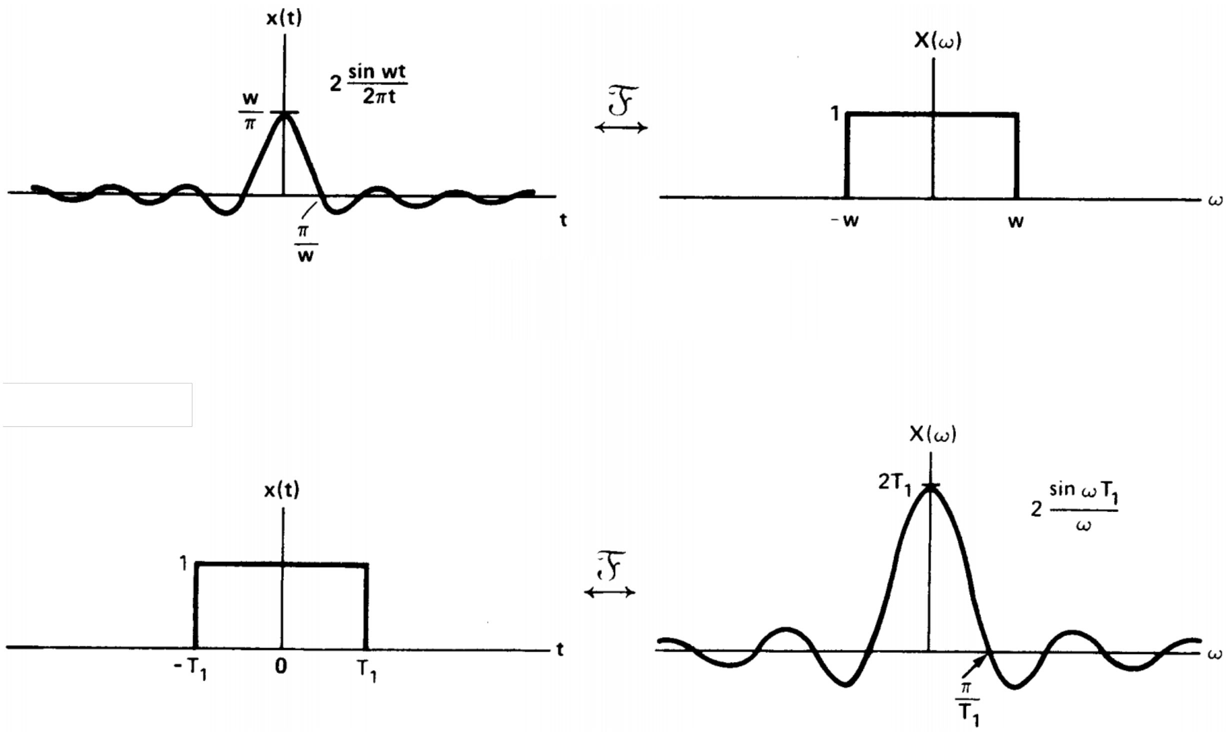


Figure 5.6: *The $rect(\frac{x}{W}) \overset{\mathfrak{F}}{\Leftrightarrow} W\ sinc(\pi W k)$ Fourier transform pair. Image courtesy of MIT*

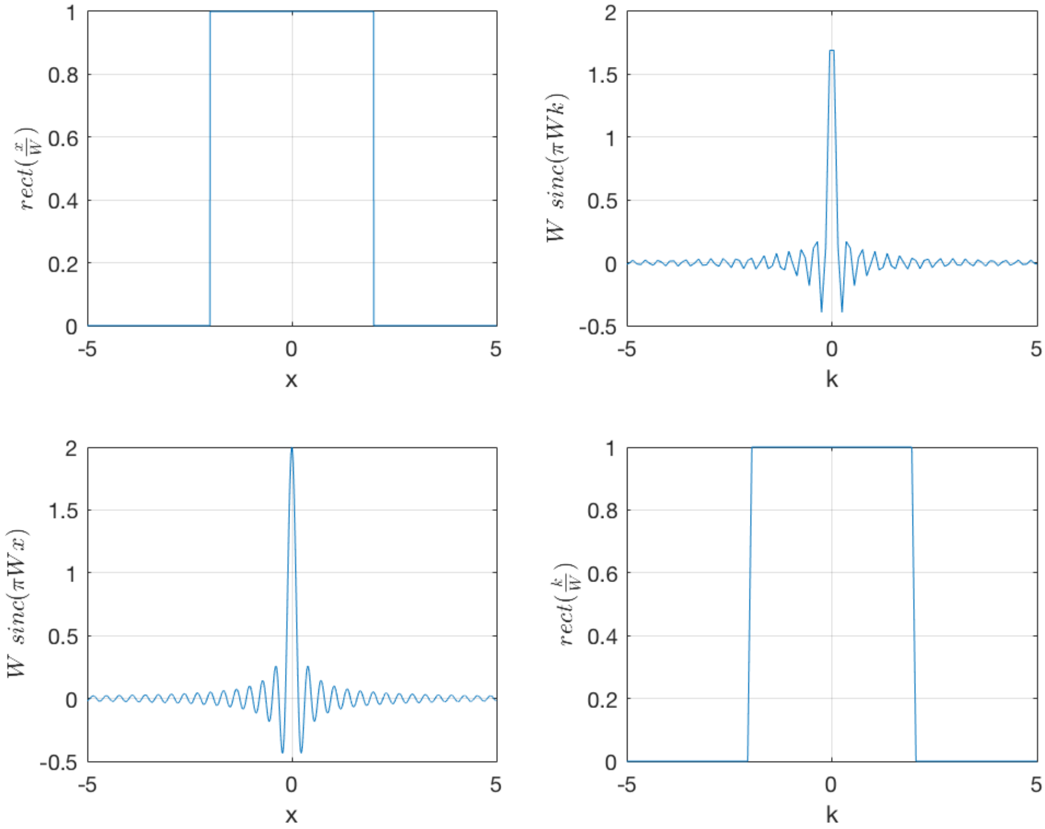


Figure 5.7: *The* $rect(\frac{x}{W}) \overset{\mathfrak{F}}{\Leftrightarrow} W\ sinc(\pi W k)$ *Fourier transform pair*

5.2.5 The Convolution Theorem

- The Convolution Theorem shows that the Fourier transform of the product of two functions is the convolution between the Fourier Transforms of each individual function.

$$\mathfrak{F}(g(x) \cdot h(x)) = G(k) * H(k)$$
$$\text{where: } G(k) * H(k) \equiv \int_{-\infty}^{+\infty} dk' G(k') H(kk')$$

- One example of why this is important to know is that, in MRI, the measured signal can be modelled as a multiplication between the ideal non-decaying signal $s(k)$ and a T_2^* decaying factor. The inverse Fourier transform of this product will yield a convolution between the spin density function and the Fourier transform of the decaying factor.
- The integral involved in the convolution operation shows that

  > Finding the convolution of two functions at the position x involves reflecting one function through the origin, displacing it by x, and finding the area of the product of the two functions. A graphical representation is useful for a qualitative, and sometimes quantitative, understanding of convolution. [Quotation courtesy of Haacke et al. *Magnetic Resonance Imaging: Physical Principles and Sequence Design, volume 1st. Wiley-Liss, 1999* [?]]

  The graphical representation is found in Figure 5.8.
- The convolution theorem and Parseval's theorem are found in the Problems section.

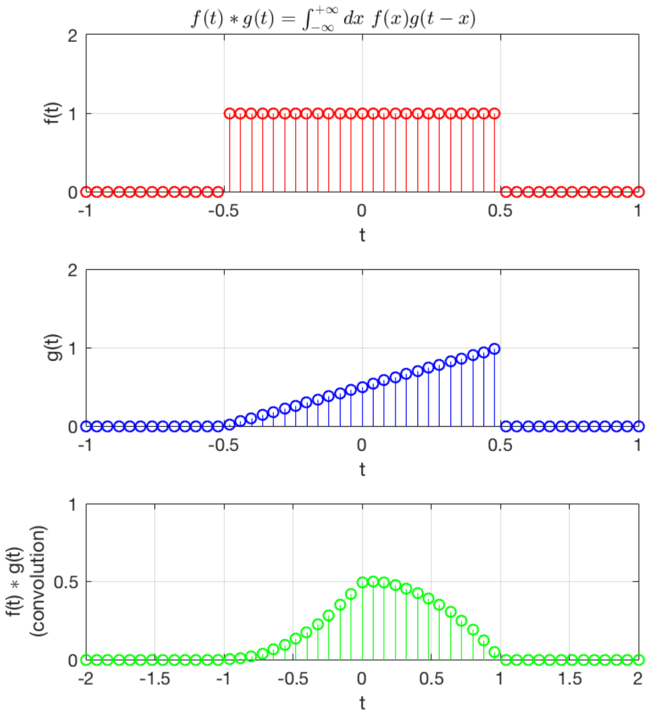


Figure 5.9: *The convolution between a ramp function and a rect function*

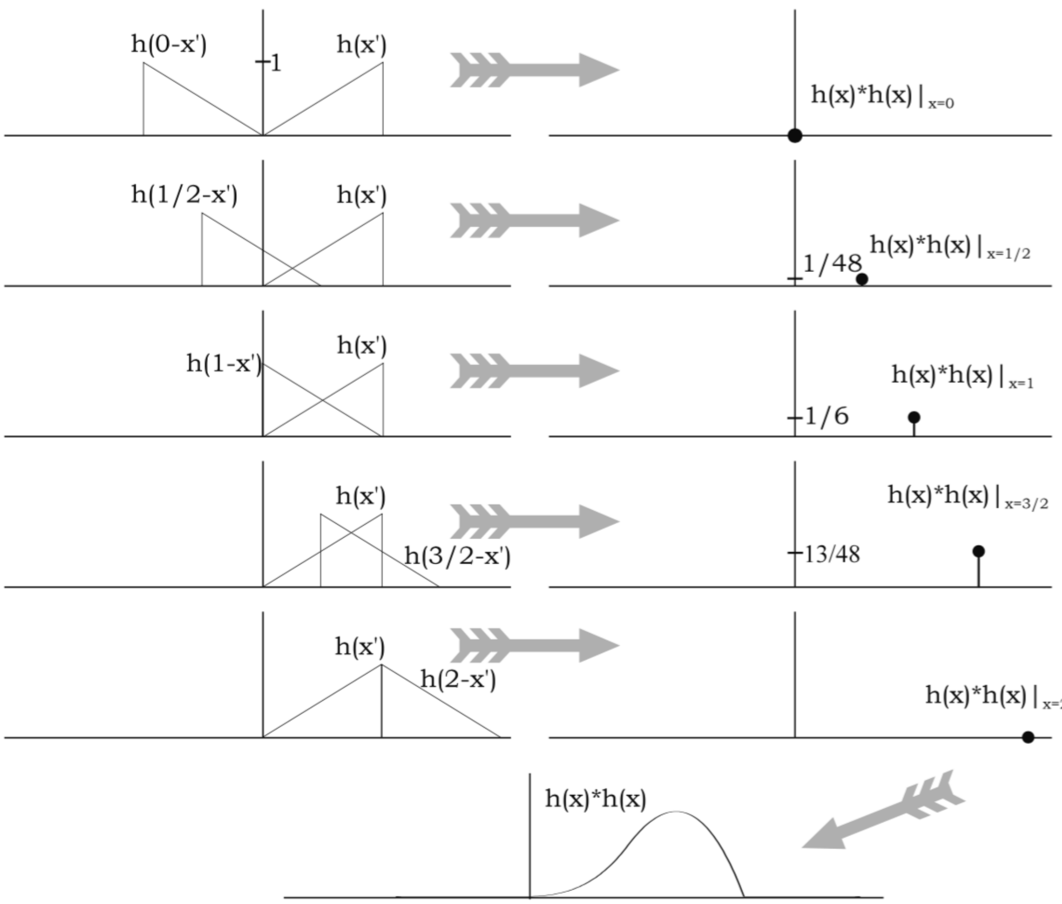


Figure 5.8: *A combined graphical and numerical example in convolution. The convolution of a ramp function with itself involves an integration over the product (which is not shown) of the mirror-reflected ramp functions $h(x')$ and $h(xx')$ both of which are plotted for five different x values. [Image courtesy of Haacke et al. Magnetic Resonance Imaging: Physical Principles and Sequence Design, volume 1st. Wiley-Liss, 1999 [**?**]]*

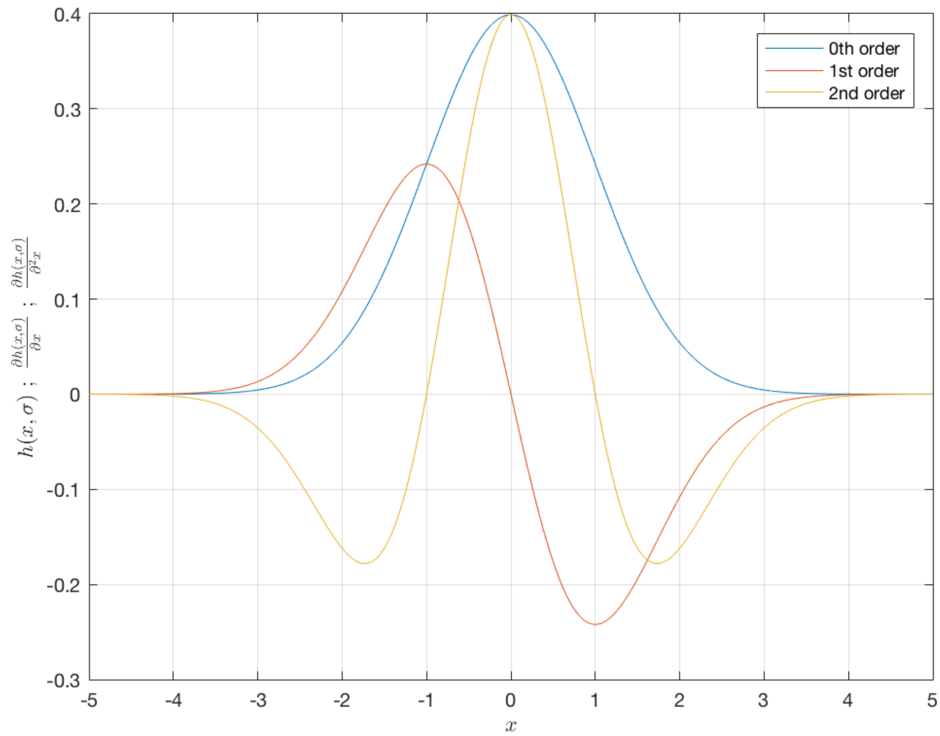


Figure 5.10: *The 0th, 1st and 2nd order 1D Gaussian*

5.2.6 Convolution Associativity

- Convolution is commutative and associative.

$$a(x) * b(x) = b(x) * a(x) \ (commutativity)$$
$$[a(x) * b(x)] * c(x) = a(x) * [b(x) * c(x)] \ (associativity)$$

- These properties are important because later on we shall see that finite sampling and truncation of the signal can be written as a convolution between the ideal signal $s(k)$, a sampling function and a truncation function. The order in which this is done does not matter.

5.2.7 Derivative Theorem

- The derivative theorem states that the Fourier transform of the derivative of a function is the Fourier transform of that function multiplied with $i2\pi k$

$$\mathfrak{F}\left[\frac{dh}{dx}\right] = i2\pi kH(k)$$

- This is an interesting result, as a boundary image can be extracted easily by taking the inverse Fourier transform of the product between the signal $s(k)$ and $i2\pi k$.
- Also,

> upon multiplication by k, the measured signal, $s_m(k)+\eta_m(k)$, is enhanced at large k values, where noise is relatively more important. [Quotation courtesy of Haacke et al. *Magnetic Resonance Imaging: Physical Principles and Sequence Design, volume 1st. Wiley-Liss, 1999* [?]]

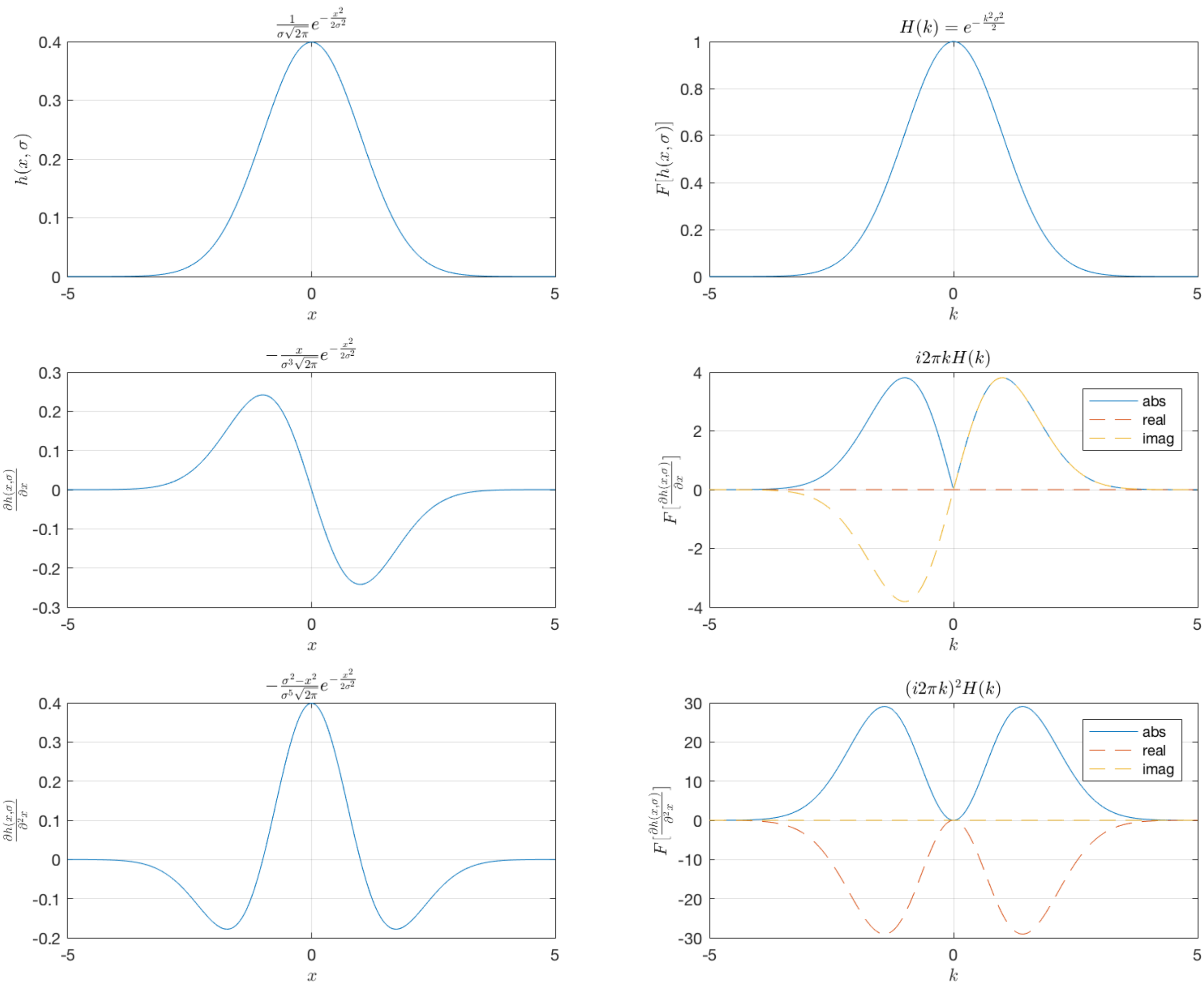


Figure 5.11: *The Derivative Theorem shown for a 1D Gaussian*

5.2.8 Fourier Transform Symmetries

- The Fourier transform symmetries refers to the symmetry property of a Fourier transform of a real function. More specifically, for $h : \mathfrak{R} \rightarrow \mathfrak{R}$, and $H(k) \equiv \mathfrak{F}(h(x))$, we have:
$$\begin{aligned} Re[H(k)] &= \quad Re[H(-k)] \text{ (even symmetry) and} \\ Im[H(k)] &= -Im[H(-k)] \text{ (odd symmetry)} \end{aligned}$$
- A useful consequence of this property is seen in *partial Fourier imaging.* This technique takes advantage of the complex conjugate symmetry ($H(k) = H^*(-k)$) of the k-space data for purely real spin density functions. This means that you can acquire half of k-space and generate the missing quantities by assuming conjugate symmetry.
- This property also tells us that the Fourier transform of an even/odd function will also be an even/odd function. Let's test this with the cosine and sine functions:

$$\begin{aligned}
1)\ C(k) &= \mathfrak{F}[cos(2\pi k_0 x)] = \\
&= \int_{-\infty}^{+\infty} dx\ cos(2\pi k_0 x) e^{-i2\pi kx} \\
&= \int_{-\infty}^{+\infty} dx\ \frac{e^{i2\pi k_0 x} + e^{-i2\pi k_0 x}}{2} e^{-i2\pi kx} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} dx\ e^{-i2\pi(k-k_0)x} + \frac{1}{2}\int_{-\infty}^{+\infty} dx\ e^{-i2\pi(k+k_0)x} \\
&= \frac{1}{2}\delta(k-k_0) + \frac{1}{2}\delta(k+k_0) \\
C(-k) &= \frac{1}{2}\delta(-(k+k_0)) + \frac{1}{2}\delta(-(k-k_0)) \\
&= \frac{1}{2}\delta(k+k_0) + \frac{1}{2}\delta(k-k_0) = C(k)
\end{aligned}$$

$$\begin{aligned}
2)\ S(k) &= \mathfrak{F}[sin(2\pi k_0 x)] = \\
&= \int_{-\infty}^{+\infty} dx\ sin(2\pi k_0 x) e^{-i2\pi kx} \\
&= \int_{-\infty}^{+\infty} dx\ \frac{e^{i2\pi k_0 x} - e^{-i2\pi k_0 x}}{2i} e^{-i2\pi kx} \\
&= \frac{1}{2i}\int_{-\infty}^{+\infty} dx\ e^{-i2\pi(k-k_0)x} - \frac{1}{2i}\int_{-\infty}^{+\infty} dx\ e^{-i2\pi(k+k_0)x} \\
&= \frac{i}{2}\delta(k+k_0) - \frac{i}{2}\delta(k-k_0) \\
S(-k) &= -\frac{i}{2}\delta(-(k+k_0)) + \frac{i}{2}\delta(-(k-k_0)) \\
&= -\frac{i}{2}\delta(k+k_0) + \frac{i}{2}\delta(k-k_0) = -S(k)
\end{aligned}$$

Sidenote:

$$\begin{aligned}
e^{ix} &= cos(x) + isin(x) \\
e^{-ix} &= cos(x) - isin(x) \\
e^{ix} + e^{-ix} &= 2cos(x) \Rightarrow cos(x) = \frac{e^{ix} + e^{-ix}}{2} \\
e^{ix} - e^{-ix} &= 2i\ sin(x) \Rightarrow sin(x) = \frac{e^{ix} - e^{-ix}}{2i}
\end{aligned}$$

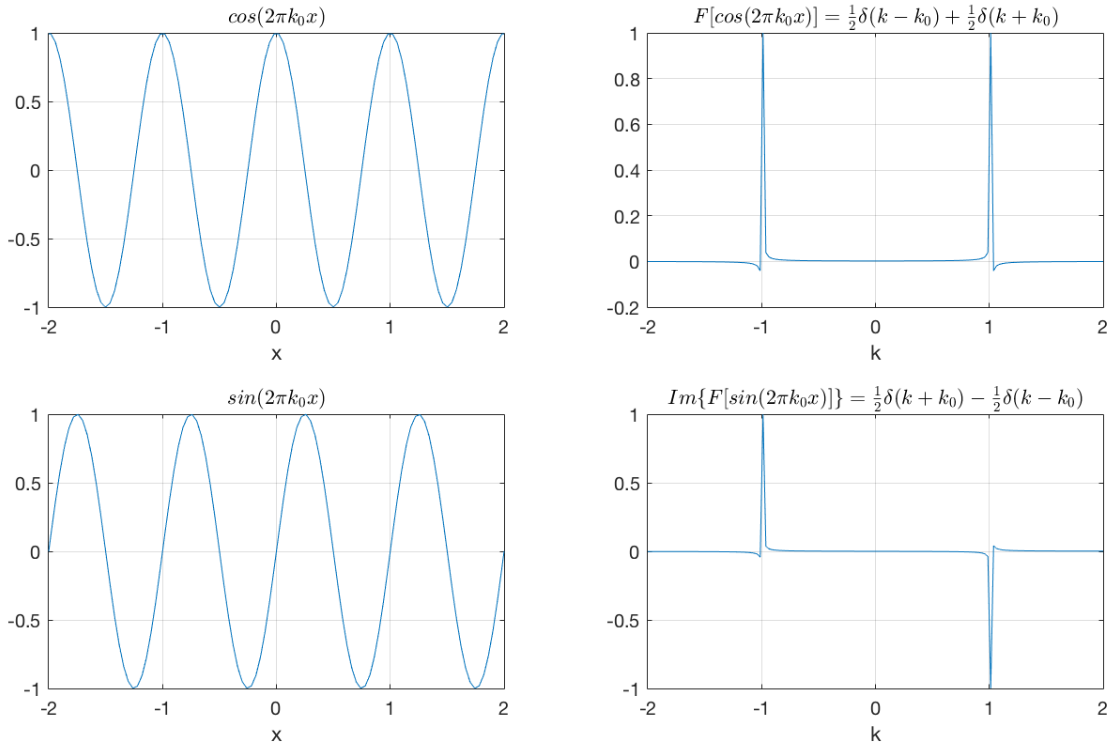


Figure 5.12: *The odd/even behaviour of sin/cos*

5.2.9 Summary of the mathematical properties of the continuous Fourier transform

Property	Mathematical Expression	Proof
linearity	$\mathfrak{F}[\alpha h(x)] = \alpha\mathfrak{F}[h(x)]$	$\mathfrak{F}[\alpha h(x)] =$ $= \int_{-\infty}^{+\infty} dx\ \alpha h(x) e^{-i2\pi kx}$ $= \alpha \int_{-\infty}^{+\infty} dx\ h(x) e^{-i2\pi kx}$ $= \alpha\mathfrak{F}(h(x))$
„	$\mathfrak{F}[h_1(x) + h_2(x)] =$ $\mathfrak{F}[h_1(x)] + \mathfrak{F}[h_2(x)]$	$\mathfrak{F}[h_1(x) + h_2(x)] =$ $= \int_{-\infty}^{+\infty} dx\ (h_1(x) + h_2(x)) e^{-i2\pi kx}$ $= \int_{-\infty}^{+\infty} dx\ h_1(x) e^{-i2\pi kx} + \int_{-\infty}^{+\infty} dx\ h_2(x) e^{-i2\pi kx}$ $= \mathfrak{F}(h_1(x)) + \mathfrak{F}(h_2(x))$
duality	if $h(x) \overset{\mathfrak{F}}{\Leftrightarrow} H(k)$ then $H(-x) \overset{\mathfrak{F}}{\Leftrightarrow} h(k)$	Proof in Section 5.2.4
space scaling	$\mathfrak{F}(h(\alpha x)) = \frac{1}{\lvert a \rvert} H(\frac{k}{\alpha})$	TBA
space shifting	$\mathfrak{F}(h(x \pm x_0)) = H(k) e^{\pm i2\pi k x_0}$	$\mathfrak{F}(h(x - x_0)) =$ $= \int_{-\infty}^{+\infty} dx\ h(x - x_0) e^{-i2\pi kx}$ Change of variable: $x' = x - x_0$ $= e^{-i2\pi k x_0}\ \int_{-\infty}^{+\infty} dx'\ h(x') e^{-i2\pi kx'}$ $= e^{-i2\pi k x_0}\ H(k)$
frequency shifting	$\mathfrak{F}^{-1}(H(k \pm k_0)) = h(x) e^{\mp i2\pi k_0 x}$	$\mathfrak{F}^{-1}(H(k - k_0)) =$ $= \int_{-\infty}^{+\infty} dk\ H(k - k_0) e^{+i2\pi kx}$ Change of variable: $k' = k - k_0$ $= e^{+i2\pi k_0 x}\ \int_{-\infty}^{+\infty} dk'\ H(k') e^{-i2\pi k' x}$ $= e^{+i2\pi k_0 x}\ h(x)$
alternate form	$h^*(x) = \mathfrak{F}^{-1}[H^*(-k)]$	$h(x) = \int_{-\infty}^{+\infty} dk H(k) e^{+i2\pi kx} \Rightarrow$ $[h(x)]^* = [\int_{-\infty}^{+\infty} dk\ H(k) e^{+i2\pi kx}]^* \Rightarrow$ $h^*(x) = \int_{-\infty}^{+\infty} dk\ H^*(k) e^{-i2\pi kx} \Rightarrow$ Change of variable $k = -k \Rightarrow$ $h^*(x) = \int_{-\infty}^{+\infty} dk\ H^*(-k) e^{+i2\pi kx} \Rightarrow$ $h^*(x) = \mathfrak{F}^{-1}[H^*(-k)]$
even $h(x)$	$H(k) = 2\int_0^{+\infty} dx\ h(x)\ cos(2\pi kx)$	$H(k) = \int_{-\infty}^{+\infty} dx\ h(x)\ e^{-i2\pi kx}$ $= \int_{-\infty}^{0} dx\ h(x)\ e^{-i2\pi kx} + \int_0^{+\infty} dx\ h(x)\ e^{-i2\pi kx}$ Because $h(x) = h(-x) \Rightarrow$ $= \int_0^{+\infty} dx\ h(x)\ e^{+i2\pi kx} + \int_0^{+\infty} dx\ h(x)\ e^{-i2\pi kx}$ $= \int_0^{+\infty} dx\ h(x)\ (e^{+i2\pi kx} + e^{-i2\pi kx})$ $= 2\ \int_0^{+\infty} dx\ h(x)\ cos(2\pi kx)$
odd $h(x)$	$H(k) = -2i\int_0^{+\infty} dx\ h(x)\ sin(2\pi kx)$	$H(k) = \int_{-\infty}^{+\infty} dx\ h(x)\ e^{-i2\pi kx}$ $= \int_{-\infty}^{0} dx\ h(x)\ e^{-i2\pi kx} + \int_0^{+\infty} dx\ h(x)\ e^{-i2\pi kx}$ Because $-h(x) = h(-x) \Rightarrow$ $= -\int_0^{+\infty} dx\ h(x)\ e^{+i2\pi kx} + \int_0^{+\infty} dx\ h(x)\ e^{-i2\pi kx}$ $= \int_0^{+\infty} dx\ h(x)\ (-e^{+i2\pi kx} + e^{-i2\pi kx})$ $= -2i\ \int_0^{+\infty} dx\ h(x)\ sin(2\pi kx)$
convolution	$\mathfrak{F}[g(x)h(x)]$ $= G(k) * H(k)$ $\equiv \int_{-\infty}^{+\infty} dk' G(k') H(k - k')$	Proof in Section 5.6

derivative	$\mathfrak{F}[\frac{dh}{dx}] = i2\pi k H(k)$	$h(x) = \int_{-\infty}^{+\infty} dk\ H(k) e^{+i2\pi kx} \Rightarrow$ $[h(x)]' = [\int_{-\infty}^{+\infty} dk\ H(k) e^{+i2\pi kx}]' \Rightarrow$ $\frac{dh}{dx} = \int_{-\infty}^{+\infty} dk\ H(k)\ \frac{d}{dx} e^{+i2\pi kx} \Rightarrow$ $\frac{dh}{dx} = i2\pi k\ \int_{-\infty}^{+\infty} dk\ H(k) e^{+i2\pi kx} \Rightarrow$ $\frac{dh}{dx} = i2\pi k\ h(x) \Rightarrow$ $\mathfrak{F}[\frac{dh}{dx}] = i2\pi k\ \mathfrak{F}[h(x)]$ $\mathfrak{F}[\frac{dh}{dx}] = i2\pi k\ H(k)$
Parseval	$\int_{-\infty}^{+\infty} dx\ \lvert g(x)\rvert^2 = \int_{-\infty}^{\infty} dk\ \lvert G(k)\rvert^2$	Proof in Section 5.6

Table 5.1: *Important mathematical properties of the continuous Fourier transform*

5.3 Fourier Transform Pairs

5.3.1 The rect and sinc function

The rectangular function $h(x) = rect\Big(\dfrac{x}{W}\Big)$ has the following Fourier transform pair: $H(k) = W sinc(\pi k W)$

Proof:

$$\begin{aligned}
\text{Let: } h(x) = rect\Big(\frac{x}{W}\Big) \text{ where: } rect\Big(\frac{x}{W}\Big) &\equiv \begin{cases} 1, & -\frac{W}{2} \leq x \leq \frac{W}{2} \\ 0, & \text{otherwise} \end{cases} \\
\text{Then: } H(k) \equiv \mathfrak{F}(h(x)) &= \int_{-\infty}^{+\infty} dx\ h(x)\ e^{-i2\pi kx} \\
&= \int_{-\infty}^{+\infty} dx\ rect\Big(\frac{x}{W}\Big)\ e^{-i2\pi kx} \\
&= \int_{-W/2}^{+W/2} dx\ e^{-i2\pi kx} \\
&= -\frac{1}{i2\pi k} e^{-i2\pi kx}\Big|_{-W/2}^{W/2} \\
&= \frac{1}{i2\pi k}\Big(- e^{-i\pi kW} + e^{i\pi kW}\Big) \\
\text{Knowing that: } sin(k) &= \frac{e^{ik} - e^{-ik}}{2i} \text{ we get:} \\
&= \frac{1}{i2\pi k} 2i\ sin(\pi kW) \\
&= \frac{W}{\pi kW}\ sin(\pi kW) \\
&= W sinc(\pi kW)
\end{aligned}$$

Sidenote:
In x-space, the bigger the W, the wider the rectangular function,
In k-space, the bigger the W, the narrower the sinc function.

5.3.2 Gaussian

The Gaussian function, $g(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}}$ has the following Fourier transform pair: $G(k) = e^{\frac{-4\pi^2\sigma^2k^2}{2}}$

Proof:

$$
\begin{aligned}
g(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}} \xRightarrow{\text{differentiate (1)}} \\
\frac{dg(x)}{dx} &= -\frac{x}{\sigma^3\sqrt{2\pi}}\ e^{\frac{-x^2}{2\sigma^2}} \\
\frac{dg(x)}{dx} &= -\frac{x}{\sigma^2}\ g(x) \xRightarrow{\text{Fourier transform}} \\
\mathfrak{F}[\frac{dg(x)}{dx}] &= -\frac{1}{\sigma^2}\ \mathfrak{F}[xg(x)] \xRightarrow{(2)} \\
i2\pi k\ G(k) &= -\frac{1}{\sigma^2}\ \frac{i}{2\pi}\ \frac{dG(k)}{dk} \\
\frac{\frac{dG(k)}{dk}}{G(k)} &= -4\pi^2\sigma^2 k \xRightarrow{integrate} \\
\int_0^k dk'\ \frac{\frac{dG(k')}{dk'}}{G(k')} &= -4\pi^2\sigma^2 \int_0^k dk'\ k' \\
ln\Big(G(k')\Big)\Big|_0^k &= -\frac{4\pi^2\sigma^2k^2}{2} \\
ln\Big(G(k)\Big) - ln\Big(G(0)\Big) &= -\frac{4\pi^2\sigma^2k^2}{2} \xRightarrow{G(0)=1} \\
ln\Big(G(k)\Big) &= -\frac{4\pi^2\sigma^2k^2}{2} \xRightarrow{\text{apply the exponent}} \\
e^{ln(G(k))} &= e^{-\frac{4\pi^2\sigma^2k^2}{2}} \\
G(k) &= e^{-\frac{4\pi^2\sigma^2k^2}{2}}
\end{aligned}
$$

Sidenote:
(1) $\frac{d}{dx}e^{f(x)} = \frac{df(x)}{dx}\ e^{f(x)}$
(2) $\mathfrak{F}[xg(x)] = \int_{-\infty}^{+\infty} dx\ xg(x)e^{-i2\pi kx} = \int_{-\infty}^{+\infty} dx\ \frac{i}{2\pi}\frac{d}{dk}\Big[g(x)e^{-i2\pi kx}\Big] = \frac{i}{2\pi}\ \frac{d}{dk}\Big[\int_{-\infty}^{+\infty} dx\ g(x)e^{-i2\pi kx}\Big] = \frac{i}{2\pi}\frac{dG(k)}{dk}$

5.3.3 Lorentzian form

The Gaussian function, $g(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}}$ has the following Fourier transform pair: $G(k) = e^{\frac{-4\pi^2\sigma^2k^2}{2}}$

Proof:

5.3.4 Sampling ('comb') function

5.3.5 Fourier transform pairs

rect function	$rect(\frac{x}{W})$	$\overset{\mathfrak{F}}{\Leftrightarrow}$	$W sinc(\pi W k)$
Gaussian	$rect(\frac{x}{W})$	$\overset{\mathfrak{F}}{\Leftrightarrow}$	$W sinc(\pi W k)$
Lorentzian	$rect(\frac{x}{W})$	$\overset{\mathfrak{F}}{\Leftrightarrow}$	$W sinc(\pi W k)$
Sampling ('comb') function	$rect(\frac{x}{W})$	$\overset{\mathfrak{F}}{\Leftrightarrow}$	$W sinc(\pi W k)$

Table 5.2: *Fourier transform pairs important to MRI*

5.4 The Discrete Fourier Transform

5.5 Discrete Transform Properties

Problem 11.1

---

a) Write the Cartesian 2D forms of (11.2) and (11.3).

b) If $s(k_x, k_y) = \rho_0 AB\ \mathrm{sinc}(\pi k_x A)\ \mathrm{sinc}(\pi k_y B)$, describe the object $\rho(x, y)$ that produced this signal. (See Ch. 9 or Table 11.3.) An illustration of this 2D Fourier pair is in Figs. 11.1a and 11.1b.

Remember:

- We used: $rect(\frac{x}{W}) \overset{\mathfrak{F}}{\Leftrightarrow} W sinc(\pi W k)$

Definitions:

$$H(k) \equiv \mathfrak{F}(h(x)) = \int_{-\infty}^{\infty} dx\ h(x) e^{-i2\pi kx}$$
$$h(x) \equiv \mathfrak{F}^{-1}(H(k)) = \int_{-\infty}^{\infty} dk\ H(k) e^{+i2\pi kx}$$

a) Proof.

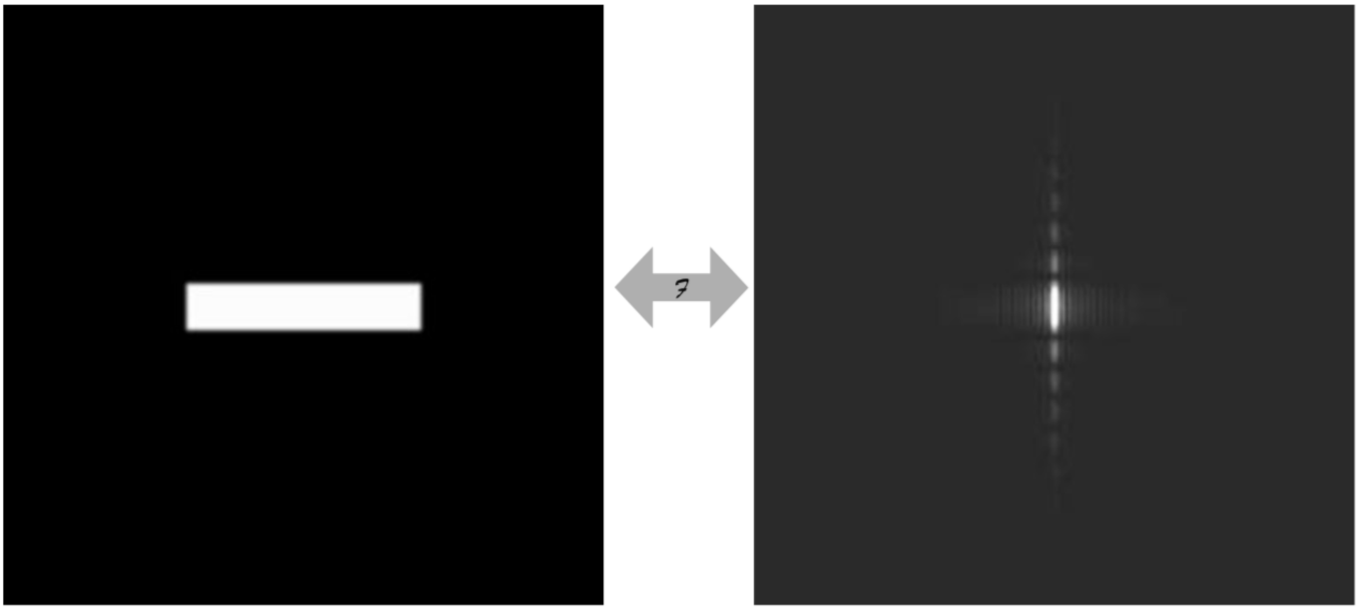
$$H(k_x, k_y) \equiv \mathfrak{F}(h(x, y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dxdy\ h(x, y) e^{-i2\pi(k_x x + k_y y)}$$
$$h(x, y) \equiv \mathfrak{F}^{-1}(H(k_x, k_y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y\ H(k_x, k_y) e^{+i2\pi(k_x x + k_y y)}$$

b) Proof.

$$\begin{aligned}
\rho(x, y) \equiv \mathfrak{F}^{-1}(s(k_x, k_y)) &= \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y\ s(k_x, k_y) e^{+i2\pi(k_x x + k_y y)} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y\ \rho_0 AB sinc(\pi k_x A)\ sinc(\pi k_y B) e^{+i2\pi(k_x x + k_y y)} \\
&= \rho_0 \int_{-\infty}^{\infty} dk_x\ A sinc(\pi k_x A)\ e^{+i2\pi k_x x} \int_{-\infty}^{\infty} dk_y\ B sinc(\pi k_y B)\ e^{+i2\pi k_y y} \\
&= \rho_0\ \mathfrak{F}(B sinc(\pi k_y B)) \int_{-\infty}^{\infty} dk_x\ A sinc(\pi k_x A) \\
&= \rho_0\ \mathfrak{F}(A sinc(\pi k_x A))\ \mathfrak{F}(B sinc(\pi k_y B)) \\
&= \rho_0\ rect(\frac{x}{A})\ rect(\frac{y}{B})
\end{aligned}$$

$$\rho(x,y) = \rho_0 \ rect(\frac{x}{A}) \ rect(\frac{y}{B})$$

$$s(k_x, k_y) = \rho_0 AB \ sinc(\pi k_x A) \ sinc(\pi k_y B)$$



Problem 11.2

Show that $h(x - x_0) = \mathcal{F}^{-1}(H(k)e^{-i2\pi k x_0})$. Explain how this result leads to a spatial shift in the reconstructed image when the signal is incorrectly demodulated, so that $s_m(t) \to s_m(t)e^{-i\Delta\omega t}$. To what demodulation frequency does this correspond?

Remember:

- The shift theorem:

$$\begin{aligned} \mathfrak{F}(h(x - x_0)) &= H(k)e^{-i2\pi k x_0} \\ h(x - x_0) &= e^{-i2\pi k x_0}\mathfrak{F}^{-1}(H(k)) \end{aligned}$$

Definitions:

$$\begin{aligned} H(k) \equiv \mathfrak{F}(h(x)) &= \int_{-\infty}^{\infty} dx\ h(x)e^{-i2\pi k x} \\ h(x) \equiv \mathfrak{F}^{-1}(H(k)) &= \int_{-\infty}^{\infty} dk\ H(k)e^{+i2\pi k x} \end{aligned}$$

Proof.

$$\begin{aligned} \text{Let } h(x') = \mathfrak{F}^{-1}(H(k)) \Rightarrow& \\ h(x') &= \int_{-\infty}^{+\infty} dk\ H(k)e^{+i2\pi k x'} \\ \text{Change of variable: } x' = x - x_0 \Rightarrow& \\ h(x - x_0) &= \int_{-\infty}^{+\infty} dk\ H(k)e^{+i2\pi k(x - x_0)} \\ &= \int_{-\infty}^{+\infty} dk\ H(k)e^{-i2\pi k x_0}\ e^{+i2\pi k x} \\ &= \mathfrak{F}^{-1}(H(k)\ e^{-i2\pi k x_0}) \end{aligned}$$

Problem 11.3

Explain how you could use the result (11.13) of the Fourier shift theorem analysis to find the time of the center of the echo given a zebra stripe (a region where phase changes from $-\pi$ to π) of width A and a readout gradient strength G.

Remember:

-

Definitions:

Proof.

a) Prove the convolution theorem given in (11.16) and Table 11.1,

$$\mathcal{F}(g(x)h(x)) = G(k) * H(k) \equiv \int_{-\infty}^{\infty} dk'\, G(k')H(k-k')$$

Hint: Replace $g(x)$ and $h(x)$ by their Fourier transform representations inside the transform of the product.

b) Show that Parseval's theorem of Table 11.1 follows directly from the convolution theorem upon the replacement of $h(x)$ by $g^*(x)$.

Remember:

-

Definitions:

$$H(k) \equiv \mathfrak{F}(h(x)) = \int_{-\infty}^{\infty} dx\ h(x)e^{-i2\pi kx}$$
$$h(x) \equiv \mathfrak{F}^{-1}(H(k)) = \int_{-\infty}^{\infty} dk\ H(k)e^{+i2\pi kx}$$

a) Proof.

$$\begin{aligned}
\mathfrak{F}(g(x)h(x)) &= \int_{-\infty}^{\infty} dx\ g(x)h(x)e^{-i2\pi kx} \\
&= \int_{-\infty}^{\infty} dx\ e^{-i2\pi kx} \int_{-\infty}^{\infty} dk'\ G(k')e^{+i2\pi k'x} \int_{-\infty}^{\infty} dk''\ H(k'')e^{+i2\pi k''x} \\
&= \int_{-\infty}^{\infty} dx\ e^{-i2\pi kx} \int_{-\infty}^{\infty} dk'\ G(k') \int_{-\infty}^{\infty} dk''\ H(k'')e^{+i2\pi(k'+k'')x} \\
&\text{Change of variable: } k'+k'' = k''' \rightarrow k'' = k'''-k' \\
&= \int_{-\infty}^{\infty} dx\ e^{-i2\pi kx} \int_{-\infty}^{\infty} dk'\ G(k') \int_{-\infty}^{\infty} dk'''\ H(k'''-k')e^{+i2\pi k'''x} \\
&= \int_{-\infty}^{\infty} dx\ e^{-i2\pi kx} \int_{-\infty}^{\infty} dk'''\ e^{+i2\pi k'''x} \int_{-\infty}^{\infty} dk'\ G(k')H(k'''-k') \\
&\text{Notation: } W(k''') = \int_{-\infty}^{\infty} dk'\ G(k')H(k'''-k') \\
&= \int_{-\infty}^{\infty} dx\ e^{-i2\pi kx} \int_{-\infty}^{\infty} dk'''\ W(k''')e^{+i2\pi k'''x} \\
&= \int_{-\infty}^{\infty} dx\ w(x)e^{-i2\pi kx} \\
&= W(k) \\
&= \int_{-\infty}^{\infty} dk'\ G(k')H(k-k') \equiv G(k) * H(k)
\end{aligned}$$

b) Proof.

Let $g(x) = \int_{-\infty}^{\infty} dk'\ G(k')e^{+i2\pi k'x}$ and $g^*(x) = \int_{-\infty}^{\infty} dk\ G^*(k'')e^{-i2\pi k''x}$ then:

$$\begin{aligned}
\int_{-\infty}^{+\infty} dx\ |g(x)|^2 &= \int_{-\infty}^{+\infty} dx\ g(x)g^*(x) \\
&= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{\infty} dk'\ G(k')e^{+i2\pi k'x} \int_{-\infty}^{\infty} dk''\ G^*(k'')e^{-i2\pi k''x} \\
&= \int_{-\infty}^{\infty} dk'\ G(k') \int_{-\infty}^{\infty} dk''\ G^*(k'') \int_{-\infty}^{+\infty} dx\ e^{-i2\pi(k''-k')x} \\
&\text{We know that: } \delta(k-k_0) = \int_{-\infty}^{+\infty} dx\ e^{-i2\pi(k-k_0)x} \Rightarrow \\
&= \int_{-\infty}^{\infty} dk'\ G(k') \int_{-\infty}^{\infty} dk''\ G^*(k'')\ \delta(k''-k') \\
&= \int_{-\infty}^{\infty} dk'\ G(k')\ G^*(k') \\
&\text{Substitute } k = k' \\
&= \int_{-\infty}^{\infty} dk\ G(k)\ G^*(k) = \int_{-\infty}^{\infty} dk\ |G(k)|^2
\end{aligned}$$

Problem 11.5

Derive (11.19) by direct integration.

Remember:

-

Definitions:

Proof.

a) Show that the (isosceles) triangle function $\Lambda(x)$ defined as

$$\Lambda(x) = \left\{ \begin{array}{cc} 1 - |x| & -1 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{array} \right\} \tag{11.20}$$

can be obtained as the convolution of two rect functions with both unit width and unit height

$$\Lambda(x) = \text{rect}(x) * \text{rect}(x) \tag{11.21}$$

as shown in Fig. 11.5. Do this either graphically or by direct integration.

b) Derive the Fourier transform of $\Lambda(x)$ by direct integration of (11.20).

c) Rederive the same Fourier transform using the convolution theorem.

Remember:

-

Definitions:

Proof.

Problem 11.7

Prove the associativity: $a(x) * (b(x) * c(x)) = (a(x) * b(x)) * c(x)$.

Remember:

-

Definitions:

Proof.

Problem 11.8

Derive the Fourier derivative theorem (see Table 11.1).

Remember:

-

Definitions:

Proof.

Problem 11.9

Find the even/odd symmetries for the real and imaginary parts of the transform of a purely imaginary function $h(x)$. Compare your answers with those given in Table 11.2.

Remember:

-

Definitions:

Proof.

Spectroscopists study 'metabolites' by consideration of the area under the frequency or spectral response of the MR signal. Assume that the effective signal decay in the time domain is of the form $h(t) = e^{-|t|/T_2}$. Note that f and t are the Fourier conjugate variables in this problem, instead of x and k.

a) Derive $H(f)$ and compare your answer to that in Table 11.3.

b) Using the form found in part (a), find the total area under the curve $H(f)$. How does it depend upon T_2?

c) Alternatively, can you find the area under $H(f)$ from $h(t)$?

d) In practice, the true $t = 0$ point is not sampled. Why does this imply that the area must be found in the frequency domain?

Remember:

-

Definitions:

Proof.

Problem 11.11

a) Prove the identity given in (11.30). Hint: Consider integrations over small intervals that either include or exclude the region where the argument of one of the δ-functions vanishes.

b) Show that the Fourier transform of $U(x)$ collapses back to (11.28).

Remember:

-

Definitions:

Proof.

a) Prove the identity (11.36).

b) Substitute (11.32) into (11.34) to show that the discrete Fourier transform in fact gives rise to a transform pair.

Remember:

-

Definitions:

Proof.

Assuming periodicity of $g(q)$

a) Prove the space shifting property of Table 11.4.

b) Prove Parseval's theorem as shown in Table 11.4.

Remember:

-

Definitions:

Proof.

Problem 11.14

Find the even/odd symmetries for the real and imaginary parts of the transform $G(p)$ of a purely real function $g(q)$. Compare your result with the entry in Table 11.2.

Remember:

-

Definitions:

Proof.

6 Fast Imaging in the Steady State

Summary

- This chapter is focused on the signal behaviour when $T_R << T_1$ and/or T_2.

6.1 Expression for the Steady-State Incoherent (SSI) Signal

6.2 Exercises

7 Appendix

7.1 Torque

Torque - the tendency of a force to rotate an object about an axis.

Loosely speaking, *torque* is a measure of the turning force on an object.

The magnitude of torque depends on 3 quantities: the force applied, the length of the lever arm connecting the axis to the point of force application, and the angle between the force vector and the lever arm (see Figure 7.1). In symbols:

$\tau = r \times F$

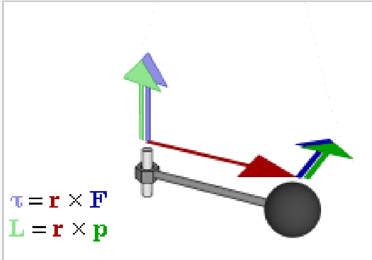
$\tau = \|r\| \cdot \|F\| \, sin\theta$

τ - torque vector / magnitude of torque vector

r - displacement vector

F - force vector

θ - angle between the force vector and the lever arm vector



Relationship between force **F**, torque **τ**, linear momentum **p**, and angular momentum **L** in a system which has rotation constrained in one plane only (forces and moments due to gravity and friction not considered).

Figure 7.1

7.2 Cross Product

Algebraic Properties

- If $a \times b = 0$ then $a = 0$ or $b = 0$ or the sine between them is 0 (they are parallel or antiparallel).
- The self cross product of a vector is the zero vector:
  $a \times a = 0$
- The cross product is **anticommutative**:
  $a \times b = -(b \times a)$
- It is **distributive** over addition:
  $a \times (b + c) = a \times b + a \times c$
- It is **compatible with scalar multiplication**:
  $(ra) \times b = a \times (rb) = r(a \times b)$
- It is **not associative** but satisfies the Jacobi identity:
  $a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$