Intro to Graph Theory

November 14, 2025

0.1 Seven Bridges of Königsberg

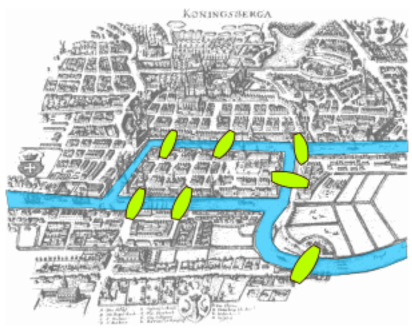

Figure 1: The seven bridges of Königsberg

Königsberg, Prussia (now known as Kaliningrad, Russia) is the setting for one of the most important problems in the development of modern mathematics. The city was laid out as pictured in Figure 1, with seven bridges connecting several landmasses around the Pregel River. A tale is told about how the citizens of this city used to walk around the bridges and wonder how one might plan a walk around the city crossing each bridge **exactly once**.

In 1736, Leonhard Euler proved that this would be impossible! In his proof, he recognized some simplifying assumptions that led to the modern development of graph theory and topology. In particular, he recognized that the relative shapes of each region is not really important, but what is only important are the connections between each region (the number of bridges connecting different regions). Mathematicians now recognize this as a graph-theoretic problem: given a set of vertices and some edges between those points, is there a path one can take that crosses each edge exactly once?

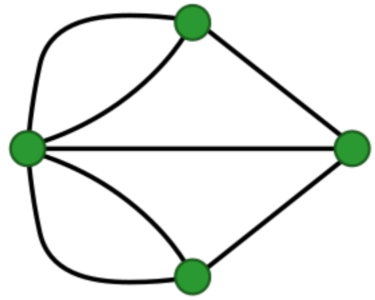

Figure 2: Graph representation of the Königsberg bridges problem

0.2 Definitions

Mathematically, we formalize the notion of graphs (vertices and edges between them) using set theory.

Definition 1. A **graph** G is an ordered pair (V, E), where V is a set of **vertices**, and E is a set of **edges**.

- In a **directed graph**, the edges are ordered pairs (u, v).
- In a **simple, undirected graph**, the edges are unordered pairs (sets of size 2), which we denote as $\{u, v\}$.
- If we use ordered pairs to represent an undirected graph, we must insist that the edge relation is symmetric: if $(a, b) \in E$, then $(b, a) \in E$.

For each $e \in E$, we associate one or two vertices, known as its **endpoints**. Two endpoints of an edge are adjacent: that is, two vertices u and v are called **adjacent** if there is an edge connecting u and v.

A vertex is most often depicted as a point in space ($\mathbb{R}^2$ or $\mathbb{R}^3$), and an edge is depicted as a curve joining the endpoints.

A graph is called **simple** if it has no self-loops and there is at most one edge between any two pair of vertices. (Note: the "Königsberg" graph in Figure 2 is not simple).

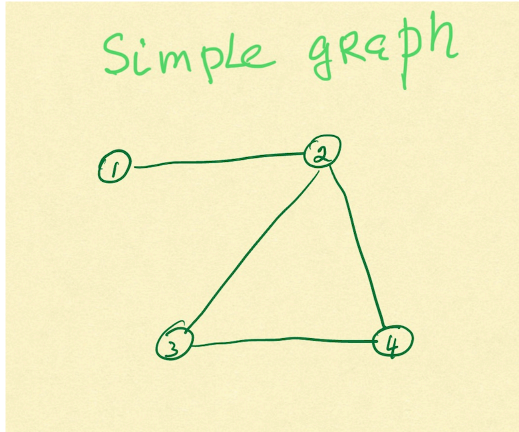

Figure 3: Examples of simple and non-simple graphs

Graphs can be either **directed** or **undirected**.

- In a directed graph, each edge has a direction. This is usually formalized as each edge being an ordered pair (a, b), where a is the start vertex and b is the terminal vertex.
- In an undirected graph, edges are bidirectional. That is, two vertices are either connected by an edge or they are not.

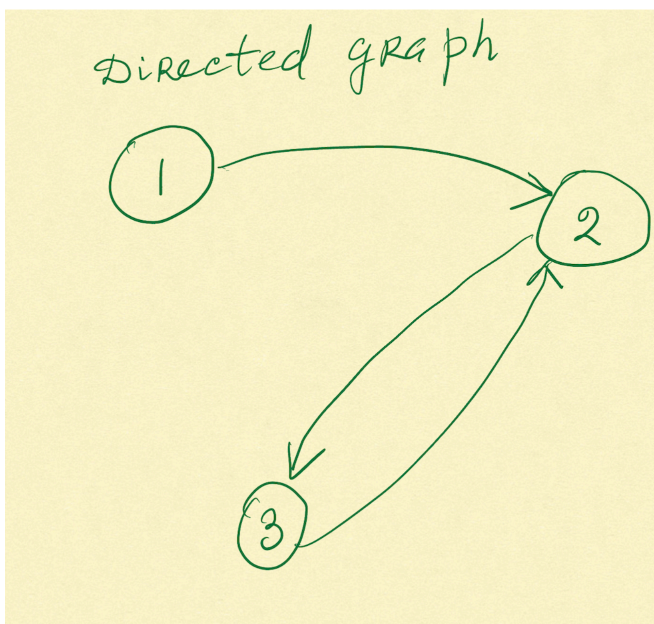

Figure 4: Directed graph example

Graphs can be **weighted** and/or **labeled**. We can label vertices and/or edges, and we can weight vertices and/or edges. Often this is done for path-finding algorithms, where the vertices are labeled by locations, and edges are weighted by the time (or distance) it would take to travel between those locations.

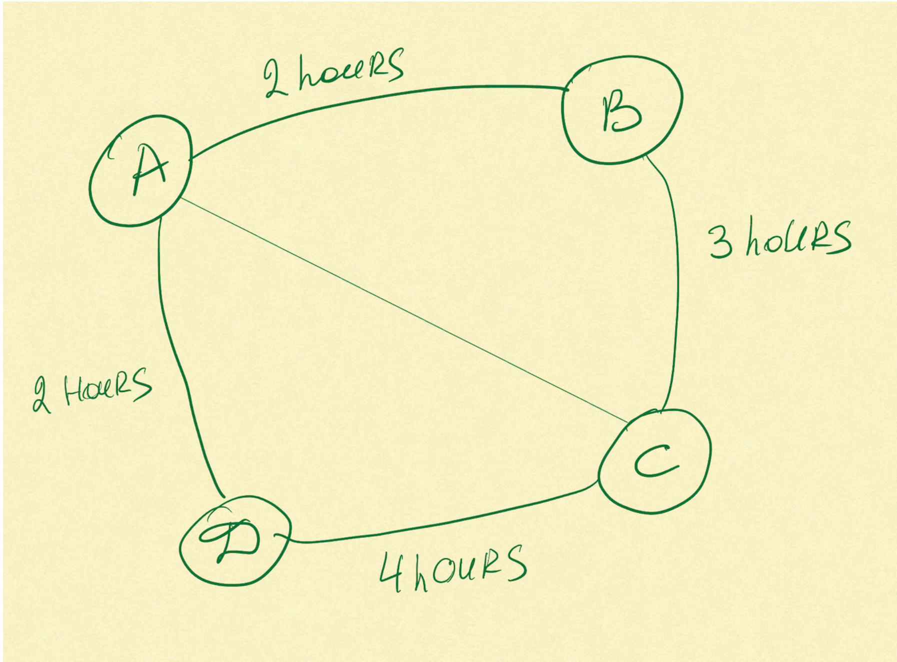

Figure 5: Labeled and weighted graph example

Given a graph $G = (V, E)$, the **degree** of a vertex v is the number of edges connected to v.

- If an undirected graph G contains **self-loops**, we count that as two edges connected to v.
- If G is a directed graph, we can define the **in-degree** and **out-degree** of v.
  - The **in-degree** of v is the number of edges that point **toward** v.
  - The **out-degree** of v is the number of edges that start at v.

The **degree-sequence** of a graph G is a sequence of all degrees in non-decreasing order. For example:

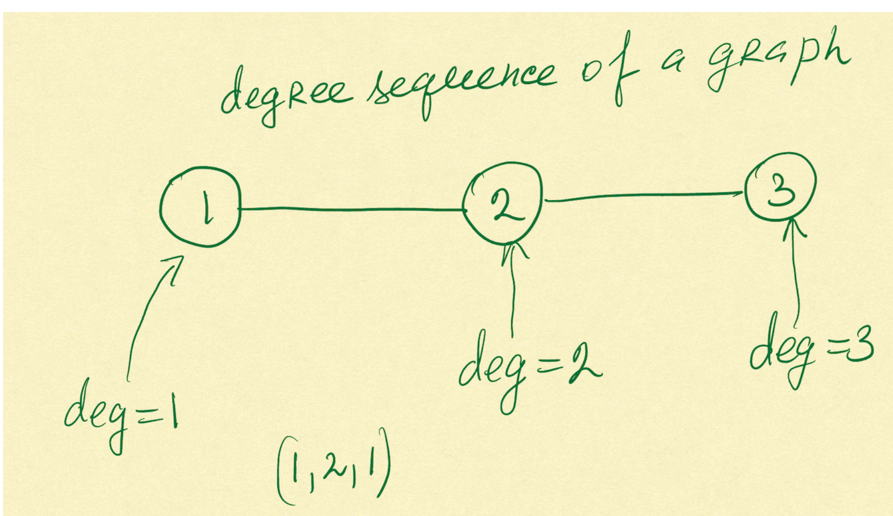

Figure 6: Graph with degree sequence (1,2,1)

The degree sequence of the graph in Figure 6 is $(1, 2, 1)$.

Exercises:
At least one of the following problems is impossible! Can you figure out which one?

- Find an undirected graph whose degree sequence is $(1, 1, 2, 2)$
- Find an undirected graph whose degree sequence is $(1, 1, 1, 2)$
- Find an undirected graph whose degree sequence is $(2, 2, 2, 2)$
- Find a simple undirected graph whose degree sequence is $(0, 1, 2, 3)$

Answers:

- The second one is impossible! Take the vertex of degree 2. It is connected to two other vertices, each has degree one. Then there is no room for a third vertex of degree one to be placed anywhere: it must be connected to some other vertex, but all the other vertices have been exhausted!
- The fourth one is also impossible. If our vertices have no self-loops, and one vertex has degree 3, it must be connected to three others. That would mean none of the vertices can have degree 0.

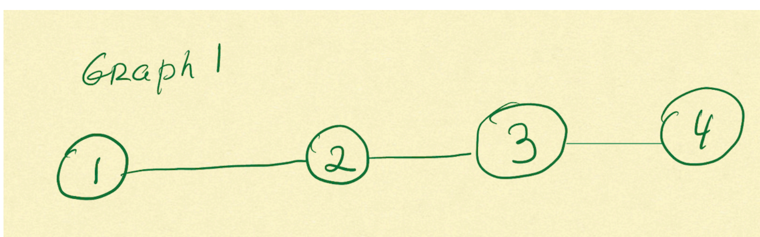

Figure 7: Possible graph for degree sequence (1,1,2,2)

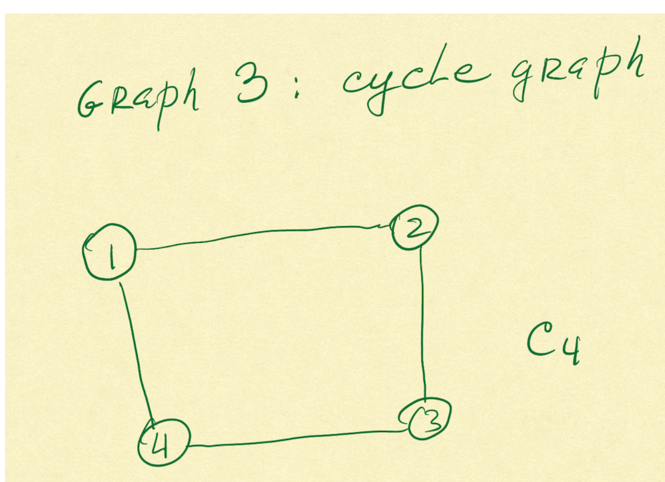

Figure 8: Possible graph for degree sequence (2,2,2,2)

0.3 Some Quick Theorems

One of the benefits of formalizing graphs as vertices and edges is that the combinatorics of graph theory is sometimes quite simple. For example, in the previous exercise, we saw that it was not possible for a graph to have exactly three vertices of degree one and one vertex of degree 2. We can actually see a more general property using a counting argument.

Take any finite graph $G = (V, E)$. What can we say about the degrees of all the vertices? If we count vertices, it's hard to say, but instead we can count edges. For each edge e, we see that it counts toward the degree of two vertices: each of its endpoints (or, if it's a self-loop, by definition it counts twice toward the degree of one vertex). That means each edge is counted twice when we add up all the degrees. Which gives us the following result:

Theorem 2 (Handshaking Lemma)**.** *For any finite graph $G = (V, E)$, the sum of the degrees of all vertices is equal to twice the number of edges:*

$$\sum_{v \in V} \deg(v) = 2|E|$$

This is a fairly innocent theorem, but it has some important consequences. First of all, that means that the sum of the degrees is even.

Theorem 3. *For any finite graph $G = (V, E)$, there is an even number of vertices of odd degree.*

Proof: Since the sum of the degree is even, we can really think of this as a question of adding numbers mod 2. If we add an odd number of odd numbers, we get an odd number. Adding any even number after that would keep it as an odd number. Therefore, there must be an even number of odd numbers in that sum.

0.4 Eulerian Paths and Cycles

Before we tackle the Bridges of Königsberg problem, let's make a few more definitions:

Definition 4. Let $G = (V, E)$ be an undirected graph.

1. A **walk** in G is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that, for each $i < n$, the vertices v_i and v_{i+1} are adjacent (connected by an edge).
2. A **trail** is a walk in which no edge is repeated.
3. An **Eulerian path** is a walk in which every edge is used exactly once.
4. An **Eulerian circuit** or an **Eulerian tour** is an Eulerian path that starts and ends at the same vertex.

We now have enough information to explain Euler's proof that there is no solution to the Seven Bridges of Königsberg problem. Given a graph $G = (V, E)$, we ask: does this graph contain an Eulerian path? Does this graph contain an Eulerian circuit?

It turns out that these are questions that can be answered entirely by looking at the degree sequence of the graph. How would we use up all the edges at a particular vertex? Suppose that a vertex has degree 3. If we start at that vertex and take one of the edges, we have two more edges left to visit from that vertex. That means we have to return, which will will use one more edge, and leave it again, which will use the other edge. What if we didn't start at that vertex? If we take an edge in, we take one out, and then we still have one edge left to use. That means that we would need to end at that vertex!

We can generalize this. If a vertex has odd degree, we would need to either start or end at that vertex. This is because, besides the beginning or end of the walk, every time we enter a vertex through one edge, we would leave through another. That means visiting a vertex in the "middle" of the walk would use up an even number of its edges. If a vertex has odd degree, then, it has to use one edge to leave at the beginning of the walk, or it has to use one edge to return at the end of the walk.

Theorem 5 (Eulerian Path Theorem)**.** *A connected graph $G = (V, E)$ has an Eulerian path if and only if it has exactly two vertices of odd degree.*

In particular, any Eulerian path would start at one of the vertices that has odd degree, and end at the other.

Theorem 6 (Eulerian Circuit Theorem)**.** *A connected graph $G = (V, E)$ has an Eulerian circuit if and only if every vertex of G has even degree.*

Exercise: Determine if the graph in Figure 9 has an Eulerian path and/or an Eulerian circuit. If so, find one.

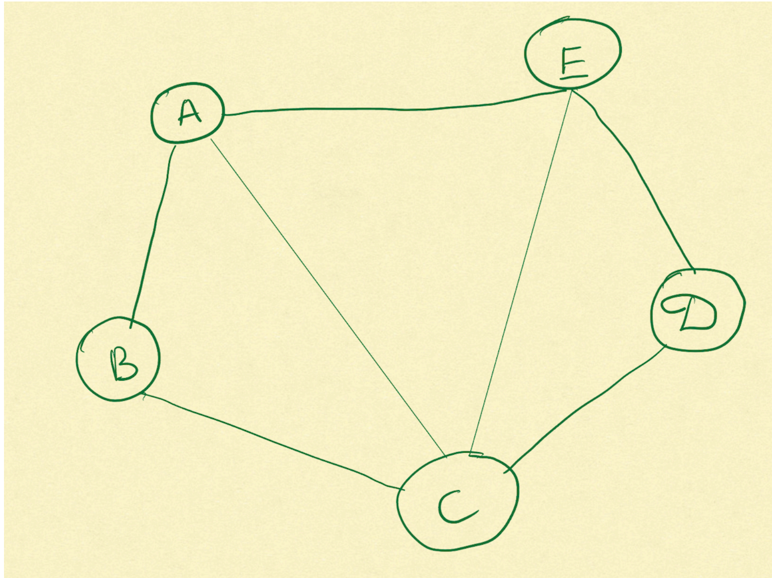

Figure 9: Graph for Eulerian path/circuit exercise

Answer: Notice that the degree sequence of this graph is $(2, 2, 3, 3, 4)$. So this should have an Eulerian path! One Eulerian path is: A - B - C - D - E - C - A - E. (This uses edges: AB, BC, CD, DE, EC, CA, AE).

0.5 Non-theorem: Hamiltonian Cycles

One of the amazing results in graph theory is that, while the Eulerian path problem has an easy solution involving just counting degrees, a related problem is not known to have any easy solution.

Definition 7. Let $G = (V, E)$. A **Hamiltonian path** is a walk in which every vertex in G is visited exactly once. A **Hamiltonian circuit** is a walk starting and ending in the same vertex, in which every vertex other than the starting vertex is visited exactly once.

From the definition it appears to be quite similar to the Eulerian path / circuit problem, but this problem is NP-complete. There is no known "efficient" algorithm which determines if a graph with n vertices has a Hamiltonian path or circuit whose running time is a polynomial function in n.