School Assignment by Match Quality*

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Abstract

We introduce match quality to two-sided matching and study the problem of maximizing aggregate match quality subject to stability constraints. We characterize subsets of stable assignments through cutoffs and admissible schools, and solve the optimization problem for each cutoff via a minimum-cost flow formulation. In comparison to the widely used Deferred Acceptance with random tie-breaking algorithm, match quality optimization in New York City public school assignment increases estimated Math and English standardized test scores for middle school applicants by about 0.11 standard deviation, and reduces average travelling distance for high school students by about 1 mile.

1 Introduction

Parental choice over public schools has become an integral education reform tool around the world. Market Design for school choice (Abdulkadiroğlu and Sönmez, 2003) has led to creation and implementation of efficient and transparent admissions processes in school choice programs.

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Stability, also more aptly referred to as justified-envy-freeness in the context of school assignment, has become critical in the design of school admissions processes (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Namely, an applicant can only be assigned to a school listed in her application form, and if an applicant prefers a school to her assignment, the preferred school must be fully assigned to applicants with better or equal admissions priorities. Stability alone does not pin down a unique assignment. Moreover, typically priorities are coarsely defined with several categories, such as sibling priority for applicants whose siblings are already enrolled at a school, neighborhood priority for applicants who reside within a certain distance from school. Such priorities are shared by multiple applicants. The set of stable assignments becomes large and complex when priorities are coarse. In practice, districts break ties among equal priority applicants by a lottery and find a stable assignment by employing the student-proposing Deferred Acceptance (DA) algorithm.

To some extent, priorities reflect district policies and societal preferences over admissions. However, the stability notion does not take into account finer considerations. For instance, some schools may be a better fit for certain types of students in terms of educational achievement. Likewise, assigning students closer to their homes reduces transportation cost for families and bussing cost for districts. We refer to applicant-school specific criteria, such as value added of a school for a specific student or travel distance between a student's home and a school, as **match quality** between the student and school. DA finds a stable assignment without any regard to match quality. In this paper, we introduce match quality to two-sided matching and study optimization with respect to the sum of match quality over assigned student-school pairs, i.e., **aggregate match quality**, among the set of stable assignments.

A **match quality optimal assignment** maximizes aggregate match quality. When priorities are coarse, finding a match quality optimal assignment is an NP-hard problem.[1] Hence, in general, this problem is not computationally tractable. We develop a solution that is polynomial time in the number of students, but potentially exponential in the number of schools. Since the number of schools in school districts is typically small, the algorithm is applicable for the majority of US

[1] When preferences and priorities are strict, the set of stable assignments can be formulated as a linear programming problem (Roth et al., 1993), so a match quality optimal assignment can be computed in polynomial time. However, school choice programs commonly feature weak priorities. Schools typically sort students into thick priority classes based on residential address, status of sibling enrollment etc.

school districts. Building on that solution, we also study a random search algorithm for large school districts.

The first building block is a novel characterization of a subset of stable assignment through 'admissible' schools, and an algorithm that maximizes aggregate match quality in this subset. The subset includes all stable assignments with a given cutoff profile, and therefore, the algorithm finds a stable assignment with maximum aggregate match quality among all stable assignments with the given cutoffs.[2] This step involves formulating the maximization problem as a minimum-cost flow problem. To the best of our knowledge, our paper is the first application of the minimum-cost flow formulation for optimization under stability constraints. Then, a match quality optimal assignment can be computed by exhaustively searching within subsets of stable assignments corresponding to all cutoff profiles.

The number of potential cutoff profiles, and therefore the computational burden of the solution rises exponentially in the number of schools. When the number of schools is large, the solution becomes computationally intractable. Therefore, we develop a random search version of our match quality maximization algorithm for large school districts. Instead of searching within bounds, the algorithm first runs the student proposing DA with some random tie-breakers to identify cutoffs of some stable assignment. Then, using the minimum-cost flow formulation, it finds a stable assignment with a weakly higher match quality than any stable assignment with the given cutoffs. The random search algorithm finds a match quality optimal assignment if it uses the cutoffs of a match quality optimal assignment. Remarkably, the simulations in Appendix E show that for most parameter values the algorithm yields locally optimal solutions that are very close to the global optimum even after a single search.

The minimum-cost flow method can be used to find a globally optimal outcome in moderately small school districts. To this end, we introduce two algorithms that establish lower and upper bounds for cutoff profiles corresponding to stable assignments. Match quality optimal assignment is then found by only searching within the cutoff profiles in these bounds.

We discuss two major applications of the solutions. The first application concerns optimizing the

[2] Admission cutoff, or simply cutoff, at a school is the lowest priority admitted by the school. For example, consider a school that grants priorities A, B and C such that A is better than B and B is better than C. Suppose that the worst priority students assigned to the school have priority B. Then, the cutoff at the school is B.

education production. It has been argued that parental choice boosts educational outcomes for students by creating competitive pressure on schools (Friedman, 1962; Tweedie et al., 1990; Hoxby, 2003), by allowing students to sort into schools with better match quality (Hoxby, 2000), and by allowing parents to act on local information, in turn providing better incentives for schools to invest in educational effectiveness than a centralized accountability system would do (Peterson and Campbell, 2001). These mechanisms would be at work only if parents choose schools partly based on school effectiveness and match quality. Contrary to these claims, empirical evidence suggests that parents do not seem to incorporate school effectiveness when forming preferences (Abdulkadiroğlu et al., 2020; Beuermann and Jackson, 2018; Ainsworth et al., 2020). In particular, they do not prefer schools that are especially effective for their own children and do not enroll their children in schools that are a better-than-average match for them (Abdulkadiroğlu et al., 2020). Match-quality optimization can address this issue as follows. First, one can estimate student-specific school effectiveness using historical data on students' assignments and educational outcomes. Then, one can choose a stable assignment that maximizes the estimated outcomes.

The second application is minimizing travel distance subject to stability constraints. The need to minimize busing costs has led to multiple public debates and admission reforms in Boston (Dur et al., 2018). Excessive busing costs still remains a major issue in the district. For example, during fiscal year 2017, more than 10 percent of Boston school district's budget - about \$116 million dollars - has been spent on busing children to schools. This is about \$2,000 per kid per year (WGBH, 2017). In 2018, the average cost of school transportation per student in the US exceeds \$1,000.[3] These costs constitute a major part of the school districts' budget, exceeding 4 percent of all expenditures. The rise in the oil prices and difficulties in finding bus drivers has further magnified the burden of transportation costs on school districts. To this end, school districts have been charging students/parents if they want to use school transportation,[4] which will likely leave disadvantaged students with no choice. These developments necessitate choice-based assignment solutions that reduce travel and busing costs.[5] If we define match quality as the negative of the

[3]This statistics include the students who do not benefit/use school transportation. See https://nces.ed.gov/programs/digest/d20/tables/dt20_236.90.asp?current=yes. Last accessed on January 2023.

[4]See https://www.publicschoolreview.com/blog/pay-to-ride-many-school-districts-now-charge-fees-to-ride-school-buses. Last accessed on January 2023.

[5]An easy way to minimize the transportation costs is assigning students to their neighborhood schools. Such an assignment can be done via simple linear optimization methods. However, under that assignment, student/parent preferences and other objectives of the school districts are totally neglected.

distance between a student's residential address and the school's location, a match quality optimal solution would give the least costly way to assign students to schools, subject to stability constraints.

We quantify the match quality solution using rich student-level data sets provided by the New York City Department of Education. Match quality optimal assignment can increase estimated average test scores for middle school students by about 0.11 standard deviation, and decrease the average travelling distance for high school students by about 1 mile compared to the student-proposing DA with random tie-breaking. The gains are substantial for both applications.

Match quality optimization can address other objectives of school districts, including maximizing student welfare (Erdil and Ergin, 2008, 2017; Abdulkadiroğlu et al., 2009) or improving diversity (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, 2005; Hafalir et al., 2013; Dur et al., 2018). These extensions are discussed in Section 8. It is important to note that in this work we do not account for parents' incentives to misreport their true preferences. As discussed in Section 8, match quality optimization under stability constraints is not compatible with strategyproofness. A potential compromise solution is to improve match quality under both stability and incentives constraints. We leave this question for future research.

The remainder of this work is organized as follows. Section 2 reviews the related literature. Section 3 describes the school choice problem and discusses the NP-hardness of finding match quality optimal assignment. Section 4 discusses the minimum-cost flow formulation, and Section 5 outlines our main solution. In Section 7 we evaluate the solution in the context of the New York City public schools. Section 6 studies global match quality optimization for smaller districts. Section 8 concludes. All proofs are in Appendix A.

2 Related Literature

Our work contributes to the literature that advocates match quality- or effectiveness-based solutions to allocation problems. Refugee resettlement is a major example (Bansak et al., 2018; Trapp et al., 2018). The Hebrew Immigrant Aid Society (HIAS) uses a machine-learning based algorithm called *Annie* MOORE (Matching and Outcome Optimization for Refugee Empowerment), named after Annie Moore, the first immigrant on record at Ellis Island, New York in 1892, which, according to developers Trapp et al. (2018), is *"... the first software designed for resettlement agency* *pre-arrival staff to recommend data-driven, optimized matches between refugees and local affiliates while respecting refugee needs and affiliate capacities."* Organ transplantation is another example of an match quality-based assignment. The algorithm of United Network for Organ Sharing (UNOS) prioritizes those patients who are in most urgent need of the transplant, and/or who are *"... most likely to have the best chance of survival if transplanted"*.[6] Slaugh et al. (2016) develop a match quality-based algorithm to assign children to families for adoption. The algorithm evaluates the probability of a successful adoption based on child's traits and family's preferences and incorporates this information to match them more effectively. Grigoryan (2020) provides a match quality maximization algorithm for pandemic rationing problems. In contrast to these works above, our problem involves optimization under stability constraints. The solution employs novel algorithms for bounding cutoffs of stable assignments and a minimum-cost flow formulation.

Our question is related to the problem of finding a stable assignment of largest size, i.e., one that assigns the most number of students, in a setting with weak priorities. This is a special case of our match quality maximization problem where match quality is binary, and equals zero if and only if a student is unassigned. In general, this problem is NP-hard (Manlove et al., 2002). Several heuristic and approximation algorithms have been discussed in the computer science literature. For example, Kwanashie and Manlove (2014) show that in practice the problem may be solved through integer programming. However, there are no theoretical guarantees that this approach is successful. Other papers provide α-approximation algorithms (McDermid, 2009; Király, 2011; Iwama et al., 2014), which guarantee that the value of the optimal solution is no more than α times larger than that of the approximating algorithm. These approximation bounds are typically not tight and the solutions may be far from the global optimum. Moreover, although some approximation bounds are possible for the problem above, no computationally feasible approximation algorithms exist for our general problem.

Another strand of papers provide solutions to NP-hard matching problems in a 'nearby' economy. For example, Nguyen and Vohra (2013) study a multi-unit allocation problem and maximize social welfare under fairness and incentive constraints. They assume that the number of objects each agent can receive is small relative to the market size. As a result, the number of fractional allocations in the solution of the relaxed linear program is guaranteed to be small. After applying a rounding algorithm, the resulting solution is 'approximately feasible', in a sense that constraints

[6] See https://unos.org/about/national-organ-transplant-system/. Last accessed on January 2023.

are only ‘slightly’ violated. Similar approximately feasible solutions are obtain stable allocations in environments with complementarities (Nguyen and Vohra, 2018) and complex distributional constraints (Nguyen et al., 2020; Nguyen and Vohra, 2019; Lin et al., 2022). We have a different framework and objectives compared all these papers, and unlike those works, we are interest in exact solutions.

Methodologically, our work is related to matching theory papers that use network flows, such as Katta and Sethuraman (2006); Yeon-Koo Che and Mierendorff (2013); Chandramouli and Sethuraman (2020) and Grigoryan (2020). The network flow approach allows to solve various optimization problem with integer solutions. This makes the method highly useful for matching problems with indivisible goods. Our setting, and therefore, the corresponding network flow formulation is different from all the works above.

Our work is potentially closest to papers that solve constrained optimization problems in school choice. Bodoh-Creed (2020) (hereafter, BC) offers a solution for small school districts. BC is different from our paper in several important ways. First and foremost, our theoretical question is different from that of BC, and we use different methodologies. BC studies a continuum model, and his method finds fractional solutions. In finite markets, his fractional solutions correspond to ex-ante stable stochastic assignments (Kesten and Ünver, 2015), but his solution does not necessarily give a stable assignment in the standard sense. We directly study a finite market setting and perform the optimization under (exact) stability constraints via the minimum-cost flow. Second, the computational tractability of BC’s algorithms relies on the assumption that if a school has empty seats under student proposing DA with some tie-breaker, it also has empty seats in all stable assignments. This assumption allows him to reduce the problem’s size. We do not require such an assumption for match quality optimization. Instead, we apply algorithms that reduce the computational burden of the solution to handle realistic problems without additional assumptions. Last but not least, we study a random-search algorithm for (local) match quality optimization for handling large school districts. The solution is applicable for any school choice problem, whereas BC’s solution can only be used for small and moderate-sized ones. Two other papers that study optimization problems in the school choice setting are Ashlagi and Shi (2016) and Shi (2019). These papers do not assume exogenously given priorities, and therefore, there are no stability constraints.

3 The Problem

A typical school choice problem with weak priorities consists of a finite set of students S, a finite set of schools C, a profile of strict preferences of students $P = (P)_{s\in S}$, a vector of school capacities $\kappa = (\kappa_c)_{c\in C}$ and a profile of priorities granted to students at schools $\rho = (\rho_{sc})_{s\in S,c\in C}$. Let $c\ P_s\ c'$ denote that s prefers c to c', and let R_s denote the weak preference relation, i.e., $c\ R_s\ c'$ if and only if $c\ P_s\ c'$ or $c = c'$. Capacity $\kappa_c \in \mathbb{N}$ denotes the maximum number of students that can be assigned to c. We assume that $\sum_{c\in C} \kappa_c \geq |S|$, i.e., the total number of school seats exceed the total number of students.[7] We model priorities via integer numbers: $\rho_{sc} \in \{1, 2, ..., K\}$ denotes student s's priority at school c. Without loss of generality, we assume that a smaller number indicates better priority. That is, if $\rho_{sc} < \rho_{s'c}$, then s has better priority than s' at c, and if $\rho_{sc} = \rho_{s'c}$, then s and s' have equal priority at c. Note that we allow for weak priorities by allowing students to have the same priority number at schools. For any $k \in \mathbb{N}$, we say student s has the k-th best priority at school c among students in $\tilde{S} \subseteq S$, if the number of students in $\tilde{S}$ with strictly better priorities at c is equal to $k - 1$, i.e., $\left|\left\{s' \in \tilde{S} : \rho_{s'c} < \rho_{sc}\right\}\right| = k - 1$. We add **match quality** to this standard model: $q(s, c) \in \mathbb{R}$ denotes the quality of match between s and c. A higher value of $q(s, c)$ indicates a better match quality. For the travel distance application, we may think of $q(s, c)$ as the negative of the distance between s and c. Let $q = \big(q(s, c)\big)_{s\in S,c\in C}$ be the match quality profile. We represent a school choice problem, or simply a problem, with tuple $(S, C, \kappa, P, \rho, q)$. We will fix the problem and omit references to it in the rest of the paper.

An **assignment** is a mapping $\mu : S \cup C \to C \cup 2^S$, such that for all $s \in S$ and $c \in C$,

- $\mu(s) \in C$,
- $\mu(c) \subseteq 2^S$, $|\mu(c)| \leq \kappa_c$,
- $c = \mu(s)$ if and only if $s \in \mu(c)$.

An assignment μ is **stable** if there is no **blocking pair** $(s, c) \in S \times C$ such that $c\ P_s\ \mu(s)$ and either $|\mu(c)| < \kappa_c$ or $\rho_{sc} < \rho_{s'c}$ for some $s' \in \mu(c)$. Let $\mathscr{A}$ denote the set of all stable assignments. It is well-known that $\mathscr{A}$ is not empty for any problem (Gale and Shapley, 1962; Irving, 1994).

[7] Assuming that students rank all school as acceptable, and that there are enough seats for all students is without loss of generality as we can add a school that represents the unassigned option and that has enough capacity for every student.

Our objective is to find a stable assignment that maximizes the aggregate match quality among all stable assignments. Namely, a **match quality optimal** assignment μ^* is a stable assignment such that

$$\sum_{s \in S} q(s, \mu^*(s)) \geq \sum_{s \in S} q(s, \mu(s)),$$

for any stable $\mu \in \mathscr{A}$.

Since the set of stable assignments is non-empty and finite, there always exists at least one match quality optimal assignment. When preferences and priorities are strict, which is a special case of our setting, a match quality optimal assignment can be found in polynomial time by formulating the set of stable assignments as linear programming constraints via Roth et al. (1993). When priorities are weak, finding a match quality optimal assignment is an NP-hard problem, and therefore unlikely to be polynomial time solvable. The problem is NP-hard even for simple special cases, such as when students' preferences are consistent with match quality and/or schools have common priority rankings. Moreover, (unless P = NP) there are no polynomial time stable algorithms that approximate the optimal solution for any level of approximation. These computational complexity results are stated and proved in Appendix C. In the next sections we introduce practical solutions which is polynomial time in the number of students.

4 Local Optimization with Minimum-Cost Flow

Stable assignments can be characterized by admissions cutoffs at schools (Abdulkadiroğlu et al., 2015; Azevedo and Leshno, 2016). Given a stable cutoff profile, we formulate a linear program to maximize aggregate match quality in a subset of stable assignments that includes all stable assignments with the given cutoff profile. This has a polynomial time solution for a locally optimal assignment. Equipped with this result, we develop a Random-Cutoff (Locally) Match Quality Optimal Algorithm (or R-MQO, in short), which operates as follows: (1) first, it runs the DA with random tie-breakers to find stable cutoff profiles, and (2) it applies the local optimization within each of these cutoffs. The solution finds a globally match quality optimal assignment if it uses a cutoff profile of a match quality optimal assignment.

In addition to R-MQO, we develop a solution for finding a globally match quality optimal assignment which is practical for smaller school districts. To this end, we introduce two further algorithms, based on student- and school-proposing DAs, that bound the set of cutoff profiles supporting a stable assignment. Then, the globally match quality optimal assignment is found by tracing all cutoff profiles within certain bounds. This solution is polynomial time in the number of students, but not schools, and can be applied for school districts with 30 or fewer schools, which makes it highly applicable.[8] Remarkably, our simulations in Appendix E show that match quality under R-MQO is very close to the global optimum.

First we define (priority) cutoffs at schools. Given a stable assignment $\mu \in \mathscr{A}$, let cutoff at school c, $\rho_c(\mu) \in \{1, 2, ..., K+1\}$, be

$$\rho_c(\mu) := \begin{cases} \max_{s \in \mu(c)} \rho_{sc} & \text{if } |\mu(c)| = \kappa_c, \\ K+1 & \text{otherwise.} \end{cases}$$

In words, under assignment μ, if school c fills its capacity, then its cutoff is equal to the priority of the assignee with the worst priority at the school, otherwise the cutoff is set to $K+1$. Let $\rho(\mu) = (\rho_c(\mu))_{c \in C}$ denote the cutoff profile. Then, the following result is immediate.

Observation 1. *An assignment μ is stable if and only if there is no pair $(s, c) \in S \times C$ such that $c \; P_s \; \mu(s)$ and $\rho_{sc} < \rho_c(\mu)$.*

For a vector $r \in \{1, 2, ..., K+1\}^{|C|}$, let $\mathscr{A}_r \subseteq \mathscr{A}$ be the set of stable assignments with cutoff profile equal to r, i.e., $\mu \in \mathscr{A}_r$ if and only if μ is stable and $\rho(\mu) = r$. We say that $r \in \{1, 2, ..., K+1\}^{|C|}$ supports a stable assignment if $\mathscr{A}_r \neq \emptyset$.

For a given $r \in \{1, 2, ..., K+1\}^{|C|}$, we first define two disjoint subsets of schools: $C^+(r) := \big\{c \in C : r_c < K+1\big\}$ and $C^-(r) := C \setminus C^+(r)$. Next, for every student $s \in S$, we define $C_s(r) \subseteq C$ as

$$C_s(r) = \{c \in C : \rho_{sc} \leq r_c, \text{ and } \rho_{sc'} \geq r_{c'} \text{ for all } c' \in C \text{ such that } c' \; P_s \; c\}.$$

[8] Only 323 of around 13,000 school districts in the US had more than 30 schools in the school year 2000-2001 (National Center for Education Statistics 2002). Even when the school district is relatively large, the number of schools serving the same grade may be below 30, in which case the district can be handled by our algorithm. For example, Boston Public Schools has around 130 schools, but only around 30 of those are high schools. Hence, match quality optimization will potentially be tractable for assigning high school students in Boston. Moreover, not all schools in a school district admits students via school choice. For instance, in Wake County public school system, which is the 15th largest school system in US, there are more than 170 schools (only magnet schools are admitting students via choice-based assignment). There are 36 magnet schools, 23 elementary schools, 9 middle schools and 4 high schools.

We refer to $C_s(r)$ as the set of 'admissible' schools for student s. An important next step is a characterization of a stable assignments $\bar{\mathscr{A}_r}$, which includes $\mathscr{A}_r$, via admissible schools. Let the set of assignments $\bar{\mathscr{A}_r}$ be such that $\mu \in \bar{\mathscr{A}_r}$ if and only if

1. $\mu(s) \in C_s(r)$ for all $s \in S$,
2. $|\mu(c)| = \kappa_c$ for all $c \in C^+(r)$.

That is, under any assignment in $\bar{\mathscr{A}_r}$ each student s is assigned a school in $C_s(r)$ and each school $c \in C^+(r)$ fills its capacity. Then,

Proposition 1. *Each assignment in $\mathscr{A}_r$ is an element of $\bar{\mathscr{A}_r}$ and each assignment in $\bar{\mathscr{A}_r}$ is stable. That is,*

$$\mathscr{A}_r \subseteq \bar{\mathscr{A}_r} \subseteq \mathscr{A}.$$

Proposition 1 justifies search for match quality maximizing assignments within $\bar{\mathscr{A}_r}$. Maximizing aggregate match quality in $\bar{\mathscr{A}_r}$ yields a stable assignment that has a weakly higher match quality than any assignment in $\mathscr{A}_r$. We refer to this solution as a **locally match quality optimal assignment** for the vector r.

Our final step formulates this optimization problem as a **minimum-cost flow** problem, which finds a locally match quality optimal assignment, whenever one exists, i.e., whenever $\bar{\mathscr{A}_r} \neq \emptyset$,[9] in polynomial time. The minimum-cost flow problem has numerous practical applications, including the design of optimal electrical network, transportation system or house allocation (Ahuja et al., 1993). To the best of our knowledge, ours is the first application of minimum-cost flow method for optimization under stability constraints.

To formulate the minimum-cost flow problem, consider the following components:

- a set of vertices $V = S \cup C \cup \{t\}$, for some $t \notin S \cup C$,
- a set of edges

$$E = \big\{(c,t) : c \in C^-(r)\big\} \cup \big\{(s,c) : c \in C_s(r)\big\} \subseteq V \times V,$$

[9] Otherwise, our minimum-cost flow algorithm/solver outputs that $\bar{\mathscr{A}_r} = \emptyset$.

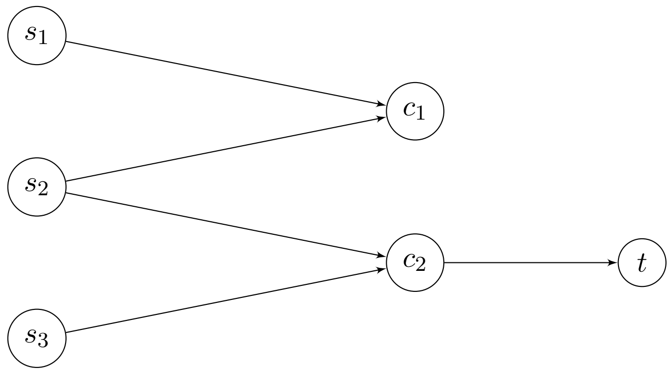

Figure 1: Minimum-cost flow graph

– a capacity $u(e)$ for each edge $e \in E$, given by

$$u(e) = \begin{cases} 1 & \text{if } e \in S \times C \\ \kappa_c & \text{if } e = (c, t) \in C^{-}(r) \times \{t\} \end{cases},$$

– a cost $l(e)$ for each edge $e \in E$, given by

$$l(e) = \begin{cases} -q(e) & \text{if } e \in S \times C \\ 0 & \text{if } e \in C^{-}(r) \times \{t\} \end{cases},$$

– a value $b(v)$ for each vertex $v \in V$, given by

$$b(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \in C^{-}(r) \\ -\kappa_v & \text{if } v \in C^{+}(r) \\ -|S| + \sum_{c \in C^{+}(r)} \kappa_c & \text{if } v = t \end{cases}.$$

Figure 1 illustrates the constructed minimum-cost flow graph using for an example with three students $S = \{s_1, s_2, s_3\}$ and two schools $C^{+} = \{c_1\}$ and $C^{-} = \{c_2\}$.

A positive value of $b(v)$ indicates that v is a supply vertex and a negative value of $b(v)$ indicates that v is a demand vertex. A vertex with $b(v) = 0$ is a transshipment vertex. We represent a flow function with a mapping $f : E \to \mathbb{R}$. Our goal is to find a least costly way of transferring values from supply vertices to demand vertices without exceeding the capacity of edges. Formally, we solve the following linear program:

$$\min_{f:E\to\mathbb{R}} \sum_{e\in E} l(e)f(e)$$

subject to

$$\sum_{v'\in V:(v,v')\in E} f(v,v') - \sum_{v'\in V:(v',v)\in E} f(v',v) = b(v), \forall v \in V, \tag{1}$$

$$0 \leq f(e) \leq u(e), \forall e \in E. \tag{2}$$

A flow function f is feasible if it satisfies constraints 1 and 2. A flow function is integral if $f(e) \in \mathbb{Z}$ for all $e \in E$. By condition 2, for any feasible and integral flow function f, $f(e)$ takes values 0 or 1 for each edge $e \in S \times C$. We establish a connection between the set of feasible and integral solutions of the minimum-cost flow problem and that set of locally match quality optimal assignments for vector r. Formally, with each feasible and integral flow function f, associate an assignment μ_f such that for each edge $(s, c) \in S \cup C$,

$$\mu_f(s) = c \text{ if and only if } f(s,c) = 1.$$[10]

Conversely, with each assignment μ, associate a flow function f_μ, where

$$f_\mu(e) = \begin{cases} 1 & \text{if } e = (s,c) \in S \times C, \mu(s) = c \\ 0 & \text{if } e = (s,c) \in S \times C, \mu(s) \neq c \cdot \\ |\mu(c)| & \text{if } e = (c,t) \in C^-(r) \times \{t\} \end{cases}$$

Then,

Lemma 1. *If f is feasible and integral, then $\mu_f \in \bar{\mathscr{A}_r}$. Conversely, if $\mu \in \bar{\mathscr{A}_r}$, then f_μ is feasible and integral.*

We briefly discuss the intuition behind Lemma 1. A non-zero flow from a student to a school is interpreted as the student being assigned to the school. Constraint 1 requires that each supply vertex has an outgoing flow that equals its value, and each demand vertex has an incoming flow that equals its value. Each student $s \in S$ has a value $b(s) = 1$, therefore, she is assigned to exactly one school in $C_r(s)$ in any integral solution of the minimum-cost flow problem. By definition, this is required for every assignment in $\bar{\mathscr{A}_r}$. Each school $c \in C^+(r)$ has a value $b(c) = -\kappa_c$, meaning that

[10]Note that for μ_f to be an assignment we need that for each $s \in S$, $f(s, c) = 1$ for exactly one $c \in C$. As we verify in Lemma 1, this is indeed the case.

κ_c students are assigned to c in any integral solution of the minimum-cost flow problem. Again, by definition, this is required for every assignment in $\bar{\mathscr{A}_r}$. There are no such requirements for schools in $C^-(r)$. However, these schools are connected to vertex t, and $b(t) = -|S| + \sum_{c \in C^+(r)} \kappa_c$. This means that schools in $C^-(r)$ cumulatively accommodate all students who are not assigned to a school in $C^+(r)$ in any integral solution of the minimum-cost flow problem. Finally, constraint 2 guarantees that no school $c \in C^-(r)$ is assigned more than κ_c students.

Given the equivalence established in Lemma 1, we conclude that there is a feasible and integral flow function if and only if $\bar{\mathscr{A}_r} \neq \emptyset$. Moreover, if f^* is an optimal integral solution to the minimum-cost flow problem, then $\mu_{f^*} \in \bar{\mathscr{A}_r}$ maximizes match quality in $\bar{\mathscr{A}_r}$. There are known polynomial time algorithms that find an optimal integral solution to the minimum-cost flow problem, whenever one exists. For example, the problem can be solved by a cycle-canceling algorithm (Ahuja et al., 1993; Sokkalingam et al., 2000). Hence, the following result is an immediate consequence of Lemma 1 and polynomial time solvability of the minimum-cost flow problem.

Proposition 2. *There are polynomial time algorithms that find a locally match quality optimal assignment for a vector r, whenever one exists.*

In Example 1 in Appendix B we illustrate how a locally match quality optimal assignment is found using the minimum-cost flow formulation.

5 Random-Cutoff (Locally) Match Quality Optimal Algorithm (R-MQO)

In this section, we introduce a polynomial time algorithm that maximizes aggregate match quality in a subclass of stable assignments. We start with a cutoff profile resulting from student proposing DA with a random tie-breaker. Then, we find a locally match quality optimal assignment for cutoff profile using the minimum-cost flow algorithm of Section 4. The cutoff profile of the resulted stable assignment may be different from the original one. In that case, we repeat the local optimization, and check the new cutoff profile. When the cutoff profile does not change at some step, we terminate the procedure. Here is the formal description of the algorithm.

Random Cutoff Locally Match Quality Optimal Algorithm (R-MQO)

Step* 0*: *Let μ^* be the outcome of student proposing DA with a random tie-breaker. Let r_0 be the cutoff profile of μ^*, i.e., $r_0 = \rho(\mu^*)$.*

Step* $t \geq 1$*: *Let $\mu_{r_{t-1}}$ be the locally match quality optimal assignment for r_{t-1}, found by the minimum-cost flow algorithm described in Section 4.*[11] *and let $r_t = \rho(\mu_{r_{t-1}})$ be its cutoff profile. If $\sum_{s\in S} q(s, \mu_{r_{t-1}}(s)) > \sum_{s\in S} q(s, \mu^*(s))$, we set $\mu^* = \mu_{r_{t-1}}$. If $r_t = r_{t-1}$, then the algorithm terminates and its outcome is assignment μ^*. Otherwise, continue with Step $t + 1$.*

By construction of $\bar{\mathscr{A}}_r$, $r_t = \rho(\mu_r^t) \leq r_{t-1}$ for each $t > 1$, i.e., the cutoff profile increases throughout the algorithm. Therefore, the number of steps it takes the algorithm to terminate is bounded by $|K| \times |C|$, which means that R-MQO terminates in polynomial time.

Let $\bar{R}$ denote the set of vectors that have been considered during the implementation of R-MQO, i.e., $\bar{R} = \{r_t\}_{t=0}^T$, where T is the last step of R-MQO. The outcome μ^* of the R-MQO is not necessarily (globally) match quality optimal. However, it is immediate from its description that the algorithm creates weakly higher aggregate match quality than any assignment in $\cup_{r\in\bar{R}} \bar{\mathscr{A}}_r$. Therefore, we can state the following result.

Proposition 3. *Suppose there is a match quality optimal assignment μ such that $\rho(\mu) \in \bar{R}$. Then, R-MQO gives a match quality optimal assignment.*

In simulations of Section E we report the results for a simpler version of the procedure that only includes Steps 0 and 1, i.e., locally match quality optimal assignment is only found in the cutoffs identified by the DA. Even for this simple version, for most parameter values the aggregate match quality is remarkably close to the (globally) match quality optimal assignment.

6 Solving for the Global Optimum

We provide a procedure for finding the globally match quality optimal assignment. To this end, we introduce novel algorithms for identifying upper and lower bounds for cutoff profiles that support a stable assignment. Then, we solve the minimum-cost flow problem for all cutoff profiles within these bounds, and pick the one with the largest aggregate match quality. We call this Match Quality

[11] A locally match quality optimal assignment exists since $\bar{\mathscr{A}}_{r_{t-1}} \neq \emptyset$ for any $t \geq 1$. In fact, $\mu^* \in \bar{\mathscr{A}}_{r_0}$ and $\mu_{r_{t-2}} \in \bar{\mathscr{A}}_{r_{t-1}}$ for $t \geq 2$.

Optimal (MQO) solution.

The bounding algorithms are based on student proposing and school proposing versions of DA. Applying either version of DA with an arbitrary tie-breaker results in a cutoff profile that supports a stable assignment. Identifying all cutoff profiles that support a stable assignment by applying different tie-breakers is computationally intractable. In contrast, our algorithms utilize no tie-breaker and operate in polynomial time. Although they do not eliminate all cutoff profiles not supporting a stable assignment, simulations in Appendix E show that the amount of eliminations is substantial.

We first describe the conventional DA algorithms. At every step of the student proposing DA, each student applies to her most preferred school that has not rejected her yet. Each school c provisionally accepts from all of its applicants up to κ_c in the order of priorities and tie-breakers, and rejects the rest. The algorithm terminates when there is no rejection, the provisional acceptances at that point are finalized. At every step of the school proposing DA, each school proposes to up to κ_c students who have not rejected the school in the order of priorities and tie-breakers. Students provisionally accept the proposal of their most preferred school and reject the rest. The algorithm terminates when there is no rejection, the provisional acceptances at that point are finalized.

Unlike the student proposing DA, our DR algorithm only rejects students who cannot be assigned to the school at any stable assignment. Consequently, the number of students provisionally held by each school may exceed the school's capacity. The resulting assignment provides upper bounds for cutoff profiles that support a stable assignment.

DR runs through multiple rounds until it cannot reject any more students. In each round, it goes through two stages. The first stage identifies 'immediate unstable demand' at every school by students who cannot be assigned to the school at any stable assignment.[12] The second stage identifies 'future unstable demand' at each school by students who have not been considered at the school yet but will apply to the school under DA with any tie-breaker.

Deferred Rejection (DR) Algorithm:

[12]Kwanashie and Manlove (2014) introduce algorithms analogous to the first stages of our algorithms. The authors use the algorithms to reduce the problem's size by shortening students' preference lists. Our goal is to bound the stable assignment cutoffs.

The following two stages are run in each round until the algorithm terminates.

Stage 1: Immediate Unstable Demand

Step $t \geq 1$***:*** *Each student s applies to her most preferred school which has not rejected her yet. Let A_c be the set of students applying to school c in this step. For every student $s \in A_c$, her demand for c is stable among A_c if $|\{s' \in A_c : \rho_{s'c} < \rho_{sc}\}| < \kappa_c$; otherwise her demand is unstable. Each school c rejects students in A_c with unstable demand and provisionally holds the remaining students in A_c.*

Stage 1 proceeds to the next step and terminates when no student is rejected.

By the end of Stage 1, the number of students a school holds may exceed its capacity. Some of these students with the worst priority need to be rejected, who then will try to apply to their next best choice. Stage 2 identifies such students and their unstable demand at their next best choice.

Stage 2: Future Unstable Demand

Let $\mu : S \cup C \rightarrow C \cup 2^S$ denote the outcome of Stage 1.[13]

Step 0***:*** *Let D_c denote the students who prefer c to their tentative assignment at μ, i.e., $D_c = \{s \in S : c \; P_s \; \mu(s)\}$. For each school c, we define the school's threshold priority $\bar{r}_c(\mu)$ at μ as follows:*

1. *if $|\mu(c)| \geq \kappa_c$, then $\bar{r}_c(\mu) = \max_{s \in \mu(c)} \rho_{sc}$,*
2. *if $|\mu(c)| < \kappa_c$ and $D_c = \emptyset$, then $\bar{r}_c(\mu) = K + 1$,*
3. *if $|\mu(c)| < \kappa_c$ and $D_c \neq \emptyset$, then $\bar{r}_c(\mu) = \min_{s \in D_c} \rho_{sc}$.*[14]

Find students who will be rejected by c' and whose next best alternative is c. To this end, let $M_{c'}$ be the set of students in $\mu(c')$ who have priorities equal to $\bar{r}_c(\mu)$. Refer to them as marginal students at c'. Let $D^c_{c'} \subseteq M_{c'}$ be the subset of marginal students whose next best alternative is c, i.e., each $s \in D^c_{c'}$ weakly prefers c to any school $c'' \in C \setminus \{c'\}$ which has not rejected her yet. Also define $g^1_{c'}$ as the number of students in $\mu(c')$ who have priorities strictly better than $\bar{r}_{c'}(\mu)$.

[13] Note that μ is not an assignment as the number of students at schools may exceed its capacity.

[14] This last case is not relevant in the initial round.

***Step** $t \geq 1$: The best priority $g^t_{c'}$ students in $\mu(c')$ are guaranteed to be held by c' at this step. This leaves $\kappa_{c'} - g^t_{c'}$ seats available for marginal students. Recall that $D^c_{c'}$ is the set of marginal students at $\mu(c')$ and whose next best alternative is c. If $|D^c_{c'}| > \kappa_{c'} - g^t_{c'}$, then some of these students must be rejected by c'. Let $\tilde{D}^c_{c'} \subseteq D^c_{c'}$ be such that $|\tilde{D}^c_{c'}| = \max\{0, |D^c_{c'}| - (\kappa_{c'} - g^t_{c'})\}$ and no student in $\tilde{D}^c_{c'}$ has strictly better priority than the ones in $D^c_{c'} \setminus \tilde{D}^c_{c'}$ for school c. Let $\tilde{D}^c = \cup_{c' \in C} \tilde{D}^c_{c'}$. For each school c, let $\hat{\rho}_c$ be the priority of κ_c-th best priority student in $\mu(c) \cup \tilde{D}^c$ if $|\mu(c) \cup \tilde{D}^c| \geq \kappa_c$. If $|\mu(c) \cup \tilde{D}^c| < \kappa_c$, let $\hat{\rho}_c = \bar{r}_c(\mu)$.*

We consider each school $c \in C$ one by one. If $|\mu(c) \cup \tilde{D}^c| \geq \kappa_c$, we set g^{t+1}_c to the number of students in $\mu(c) \cup \tilde{D}^c$ who have a strictly better priority than the κ_c-th priority student in $\mu(c) \cup \tilde{D}^c$. Otherwise, we set g^{t+1}_c to $|\mu(c) \cup \tilde{D}^c|$. If $\hat{\rho}_c < \bar{r}_c(\mu)$, then school c rejects all students with priorities strictly greater than $\hat{\rho}_c$ and continue with Stage 1 of the next round. If $\hat{\rho}_c \geq \bar{r}_c(\mu)$ for all $c \in C$ and $g^{t+1}_{c'} > g^t_{c'}$ for some $c' \in C$, then we continue with Step $t+1$ of Stage 2. Otherwise, the procedure terminates and the final outcome is μ.

Given the final outcome μ of DR, we calculate $\bar{r}(\mu)$ as described in Stage 2 (see Step 0) of DR. As stated in our next result, $\bar{r}(\mu)$ gives an upper bound on cutoff profiles that support a stable assignment.

Proposition 4. *If $\bar{\mu}$ is a stable assignment, then $\rho(\bar{\mu}) \leq \bar{r}(\mu)$.*

Here is a brief intuition behind the algorithm and the result. In Stage 1 of the DR algorithm, we run student proposing DA algorithm by only rejecting students who cannot be assigned to the schools they are applying to for any tie-breaker. Among the set of students who are provisionally held by school c in the outcome of Stage 1, we can determine the number of students who cannot be assigned to c for any tie-breaker and whose next achievable choice is c'. By using this information, we further update the set of schools that cannot be achieved by each students in any stable assignment in Stage 2 of DR algorithm. This information allows us to rerun Stage 1 and repeat the procedure.

Next, we define the Deferred Proposal Algorithm (DP), which is the schools proposing analog of the of DR. Unlike the school proposing DA, our DP algorithm only proposes to students who clear the school's (admissions) cutoff at any stable assignment. Consequently, the number students provisionally held by each school may be less than the school's capacity. The resulting assignment provides lower bounds for cutoff profiles that support a stable assignment.

DP runs through multiple rounds until no more school is rejected by students. In each round, it goes through two stages. Again, first stage identifies 'immediate unstable demand' of every school for students who cannot be assigned to the school at any stable assignment. The second stage identifies 'future unstable demand' at each school by students who have not been proposed by the school yet but will have a guaranteed proposal under DA with any tie-breaker.

Deferred Proposal (DP) Algorithm

The following two stages are run in each round until the algorithm terminates.

Stage 1: Immediate Unstable Demand

In the first stage, we run a modified version of school proposing DA. For each school $c \in C$, let A_c denote the set of students who have not rejected school c yet.

***Step** $t \geq 1$**:** Each school c considers students in A_c one by one. Each school c proposes to student $s \in A_c$ if the number of students in A_c with weakly better priority than s is less than or equal to κ_c, i.e., $|\{s' \in A_c : \rho_{s'c} \leq \rho_{sc}\}| \leq \kappa_c$. A school's demand for a student is unstable, if the student receives a proposal from a more preferred school. Each student s provisionally accepts her most preferred proposing school. All schools with unstable demand are rejected by corresponding students. If there are no more rejection, Stage 1 terminates.*

Stage 2: Future Unstable Demand

Let $\mu : S \cup C \to C \cup 2^S$ denote the outcome of Stage 1. In this stage, given the outcome of Stage 1, we identify schools that each student is guaranteed under any tie-breaker. For each $s \in S$ define D_s as the set of schools preferred to $\mu(s)$, i.e., $D_s = \{c \in C : c \ P_s \ \mu(s)\}$. Each student s rejects any school $c \notin D_s \cup \mu(s)$ and s is removed from A_c. We proceed with the following steps:

(i) *Select a student $s \in S$ such that $|D_s| \geq 2$ and who has not been considered before.* Define $E_s = (D_s \times D_s) \setminus \cup_{c \in D_s}\{(c, c)\}$.

(ii) *We consider each pair in E_s one by one. Let (c_1, c_2) be the pair under consideration. Let $\bar{S}$ be the set of students such that for all $s' \in \bar{S}$ we have $s' \in A_c$ and $\rho_{s'c} < \rho_{sc}$ for some $c \in \{c_1, c_2\}$.*

(iii) *If $|\bar{S}| \leq \kappa_{c_1} + \kappa_{c_2}$, then move s from A_c for any school c less preferred than c_1 and c_2. If*

$|\bar{S}| > \kappa_1 + \kappa_2$ and there is a school pair in D_s not considered, then go back to bullet (ii). Otherwise, go back to bullet (i).[15]

If A_c is updated during the Stage 2, we continue with Stage 1 of the next round. Otherwise, the algorithm terminates. Let μ denote the outcome of DP.

Due to finiteness of the sets of S and C, the algorithm terminates in finite number of rounds. For each $c \in C$, we define threshold priority $\bar{r}_c(\mu)$ as follows:

1. if $|\mu(c)| = \kappa_c$, then $\bar{r}_c(\mu) = \max_{s \in \mu(c)} \rho_{sc}$,
2. if $\mu(c) = \emptyset$, then $\bar{r}_c(\mu) = 0$,
3. if $\mu(c) \neq \emptyset$ and $|\mu(c)| < \kappa_c$, then $\bar{r}_c(\mu) = \max_{s \in \mu(c)} \rho_{sc} + 1$.

As stated in the next result, $\bar{r}(\mu)$ gives a lower bound on cutoff profiles that support a stable assignment.

Proposition 5. *If $\bar{\mu}$ is a stable assignment, then $\rho(\bar{\mu}) \geq \bar{r}(\mu)$.*

Here is a brief intuition behind the algorithm and the result. In Stage 1 of the DP algorithm, each school c proposes to the set of highest ranked students in its priority order up to its capacity (possibly fewer than its capacity) without using a tie-breaker. As a result, when Stage 1 terminates, some school c's proposals might be accepted by strictly less than κ_c students. Some students who have not been proposed during Stage 1 might prefer c to their match. Such students can be considered as the students in wait lists of the schools. In Stage 2, we determine the worst school that a student can guarantee by considering the overlaps in schools' wait lists. Hence, we can eliminate some schools from students consideration and repeat the procedure by using this updated information. We illustrate DR and DP through an example (Example 2) in Appendix B.

The DR and DP bounds help us eliminate a significant number of cutoff profiles that do not support a stable assignment. Then, we solve for a match quality optimal assignment by computing the minimum-cost flow solution only for cutoff profiles within these bounds.

[15] We can do this stage for any subset of schools, instead of pairs only, to determine the guaranteed schools for the students. Considering larger subsets of schools will give tighter bounds, however this will also increase the computational burden.

7 Match Quality Optimization in New York City Public Schools

We use the New York City (NYC) public schools data to evaluate the match quality gains from the R-MQO algorithm. We compare R-MQO with a benchmark DA and a heuristic solution.[16] The benchmark corresponds to the current application of DA in NYC public school assignment, in which ties are broken by lottery numbers. The heuristic solution breaks ties according to match quality, favoring applicants with better match quality, then applies DA. As motivated in the introduction, we focus on maximizing an education production function at middle schools and minimizing travel distance to high schools.[17]

7.1 NYC School District and Data

NYC is the largest school district in the US with about 1,000,000 students. Entry grade students at each level participate in a centralized school choice admission in which the student proposing DA algorithm is used. In this section, we focus on the middle and high school admissions.

In NYC public schools, middle schools include grades six to eight. There are about 700 programs served by about 500 middle schools. Every year, about 70,000 current fifth grade students participate in the centralized admission process. Students submit their rank order lists over programs, and programs rank applicants based on their priorities.[18] Priorities are set based on predetermined rules. Since many students may have the same priority, in order to apply the DA algorithm, schools break ties via random tie-breakers.

Admissions to high schools (grades nine to twelve) is decided analogously. There are about 700 programs served by about 400 high schools. Every year about about 80,000 current eight grade students participate in the centralized admission process.[19]

[16]Since the numbers of middle and high schools at NYC exceed 400, we do not search for a globally optimal assignment via the MQO.

[17]Due to the data availability, we cannot focus on student achievement at high schools. Moreover, since in the middle school assignment students are more limited to the schools in their district, we do not focus on travel distance as middle school assignment.

[18]Some programs might be only available to students living in the same district or borough. Hence, students' choice sets do not include all programs.

[19]Different from the middle school admission, some high schools rank students actively through screening procedures. Those high schools break ties by using non-random methods. Moreover, all high schools are available to all

The data used in this work is provided by the New York City Department of Education (NYC DOE) and covers all middle and high school students enrolled in a New York City (NYC) public school between 2011-2012 and 2017-2018 school years. The data includes demographic information, census tracts of residence, (middle school) students' standardized Math and ELA (English Language Arts) test scores, submitted rank order lists, and admission priorities. We supplement these datasets with publicly available information on schools' and census tracts' locations.

7.2 Maximizing Student Achievement

In this section, we restrict attention to middle school admissions and school years between 2015 and 2018. In our dataset, for this sample of students we observe rich set of characteristics, including gender, race, English language learner and disabilities status, whether the applicant qualifies for free or reduced-priced lunch, census tracts of residence, and Math and ELA standardized test scores.

We define match quality between a student-school pair as the estimated student-specific school effectiveness, measured by the student's sixth grade standardized test scores conditional on being assigned to the school during their sixth grade.

The sixth grade Math and ELA test scores are observed only for applicants who enroll to a school in the centralized system. This constitutes about 87.5% of total applicants. The remaining 12.5% either opt-out to a school outside the system (e.g., home schooling, private school, or a public school outside of NYC). In Table 1 we report the characteristics for two groups of applicants: (1) all middle school applicants in year 2017-2018, and (2) those applicants that enrol at a school from the system, and for whom we observe the next year's test scores. As the table illustrates, the characteristics of these two groups are very similar (including fifth grade test scores), and hence the sample of applicants for whom we observe the outcome of interest is potentially representative for the whole population. We however see some differences in the proportions of applicants who qualify for free or reduced lunch. This is intuitive given that higher income families are potentially more likely to opt-out for private schools.

In this section, our first goal is to estimate school effectiveness as a function of student character-

students independent of their addresses.

	All applicants	Applicants with sixth grade data
Female	0.491	0.491
Black	0.222	0.218
Hispanic	0.416	0.414
Asian	0.179	0.184
ELL	0.139	0.123
SWD	0.214	0.218
Free/reduced priced lunch	0.706	0.771
Fifth grade Math	0.014	0.020
Fifth grade ELA	0.014	0.016
Total	74,719	65,327

Table 1: Middle school applicant characteristics 2018-2019. For demographic variables Female, Black, Hispanic, Asian, ELL, and SWD we report the proportion of applicants from the respective categories. ELL indicates for applicants who have an English Language Learner Status, SWD indicates applicants with disabilities, and Free/Reduced Priced Lunch indicates applicants who receive free or reduced-price meals. The Fifth Grade Test Scores are in normalized values.

istics. Consider the following equation for each school c:

$$Y_{sc} = \alpha_c + \beta_c X_s + \epsilon_{sc}, \tag{3}$$

where

- Y_{sc} is the normalized sixth grade ELA or Math standardized test score of student s if enrolled at school c at the sixth grade,
- X_s is the vector of students' observed characteristics, which includes dummies for race, gender, and fifth grade (pre-assignment) standardized test scores,[20]
- α_c and β_c are coefficients to be estimated,
- ϵ_{sc} is the school-student specific error term.

Our identifying assumption is 'selection on observables', which says that school assignment is as good as random conditional on the covariates and assignment status. Formally, we assume that

$$\mathbb{E}\big[Y_{sc} \mid X_s,\ s \text{ is enrolled at } c\ \big] = \alpha_c + \beta_c X_s. \tag{4}$$

Equation 4 implies that the ordinary least squares (OLS) regression of the outcome Y_{sc} on covariates X_s interacted with school-enrollment indicator recovers unbiased estimators of parameters α_c and β_c. This corresponds to multiple-treatment analog of the Oaxara-Blinder treatment effects estimator (Oaxaca, 1973; Kline, 2011; Abdulkadiroğlu et al., 2020).[21]

The OLS regression produces estimates for α_c and β_c for each school c, which we denote by $\hat{\alpha}_c$ and $\hat{\beta}_c$, respectively. This is about 500 estimated parameter vectors for each school year. In Table 2 we provide a summary statistics (the mean and the standard deviation) for these parameters for the school year 2017-2018.

[20] We restrict attention to covariates with largest predictive powers. In Appendix D we consider richer set of covariates, which also include reduced lunch status, English Language learner (ELL) status, student with disabilities (SWD) status, and distance to school.

[21] The credibility of the selection on observable assumption in the school assignment context is a matter of ongoing debate (Deming, 2014; Guarino et al., 2015). Some recent findings suggest that the assumption is more plausible for short-term outcomes such as test scores, than longer-term ones (Chetty et al., 2014; Abdulkadiroğlu et al., 2020). Abdulkadiroğlu et al. (2020) show that OLS results do not substantially differ from those of more robust methods that control for selection bias. These findings justify our current specification.

Variable	Math		ELA	
	Mean	SD	Mean	SD
Female	0.029	0.132	0.160	0.153
Black	-0.080	0.403	-0.084	0.382
Hispanic	-0.058	0.383	-0.039	0.358
Asian	0.105	0.0.362	0.053	0.407
Fifth grade score	0.678	0.147	0.665	0.122

Table 2: Mean and standard deviation (SD) for estimated coefficients at 2017-2018 NYC middle schools admissions

We can see that gender, race and previous test scores are strong predictors for the performance in sixth grade tests. Moreover, the large standard deviations indicate that the effects of these covariates on school effectiveness substantially differ across schools.

Once we obtain parameter estimates $\hat{\alpha}_c$ and $\hat{\beta}_c$, we compute the match quality between every student-school pair (s, c) by

$$q(s, c) = \hat{Y}_{sc} = \hat{\alpha}_c + \hat{\beta}_c X_s.$$

Here, match quality corresponds to the estimated sixth grade test score of student s if enrolled at school c during the sixth grade.

In Table 3 we compare average school effectiveness under three mechanisms: (1) DA with random tie-breaking (DA-RTB), (2) DA with effectiveness-based tie-breaking (DA-EBTB), and (3) R-MQO. We report the results for school years 2015-2018. For each school year, we run 100 simulations with different randomly drawn tie-breakers.[22] Standard errors and in parentheses.

NYC uses the DA with random tie-breaking for student assignment. Due to the normalization, standardized test scores are close to zero under that mechanism. In the last column of Table 3,

[22] All three mechanisms involve a random component. Randomness manifests under the DA with match-quality-based tie-breaking by that ties need to be broken among students who have the same estimated match-quality in a given school (e.g., students with same characteristics). Under R-MQO, the random tie-breakers are used to obtain the initial cutoff profile.

Year	DA-RTB	DA-EBTB	R-MQO	DA-RTB	DA-EBTB	R-MQO
		Math			ELA	
2015-2016	0.017	0.095	0.131	0.012	0.085	0.120
	(0.0008)	(0.0005)	(0.0005)	(0.0008)	(0.0005)	(0.0004)
2016-2017	0.011	0.086	0.112	0.010	0.081	0.117
	(0.0008)	(0.0005)	(0.0005)	(0.0008)	(0.0005)	(0.0004)
2017-2018	0.005	0.080	0.116	0.005	0.074	0.108
	(0.0008)	(0.0004)	(0.0003)	(0.0007)	(0.0005)	(0.0003)

Table 3: Estimated average sixth grade test scores at NYC middle schools admissions

we see that R-MQO results in about 0.11 standard deviation higher average test scores compared to the DA with random tie-breaking. Our results suggests that match quality optimization can increase standardized test scores by amounts that are comparable in magnitudes to test scores declines following those major events.[23] Finally, we show that by applying match-quality-based tie-breaking, the DA outcome can be improved by about 0.08 standard deviation. This gain is substantial, but it is significantly smaller than the match quality optimal one.

7.3 Minimizing Travel Distance

In this section, we define match quality between a student-school pair as the corresponding travelling distance. We restrict attention to high school admissions where travel distance is typically larger.

We supplement our datasets with the publicly available data on NYC high school and census tract locations. First, we obtain the geographic coordinates of NYC census tract population centers from the census data[24] and the geographic coordinates of NYC schools from the public data of the NYC

[23]For example, the decrease in grade 3-8 test scores from COVID-pandemic in the US has been estimated at 0.20-0.27 standard deviation for Math, and 0.08-0.17 standard deviation for reading (Kuhfeld et al., 2022). After Hurricane Katrina, the immediate drop in the Math test scores was 0.10 standard deviation in New Orleans, and 0.09 standard deviation in Rita (Oaxaca, 2012).

[24]See https://www.census.gov/geographies/reference-files/time-series/geo/centers-population.html. Last accessed

Year	2011-2012	2012-2013	2013-2014	2014-2015	2015-2016	2016-2017	2017-2018
Mean	4.53	4.52	4.47	4.40	4.39	4.36	4.31
SD	3.32	3.37	3.33	3.29	3.29	3.28	3.23

Table 4: Mean and standard deviation (SD) of distance (in miles) to ranked choices in NYC high schools admissions

DOE.[25] Then, we use the database of OpenStreetMap to compute the minimum driving distance from each census tract's population center to each NYC high school. Then, we compute the match quality between each student-school pair as the negative of the driving distance between student's residential location (proxied by the population center of the census tract) and the school's (exact) location.

In Table 4, we report a summary statistics from students' distances to the schools in their choice list. The average distance to a ranked choice is about 4.43 miles. The standard deviations are relatively large, and hence, there is large possibility for distance minimization.

We compare average distance to the assigned schools under three mechanisms: (1) DA with a random tie-breaking, (2) DA with a distance-based tie-breaking (DA-DBTB), and (3) R-MQO. For each school year between 2011 and 2018, we run 100 simulations with different randomly drawn tie-breakers. The results are in Table 5.

The results reveal substantial reductions in average traveling distance from our optimization algorithm: the average travel distances under the R-MQO and the DA with a random tie-breaker are 2.80 miles and 3.78 miles, respectively. This is about 25% reduction of travel distance. The reductions are about twice larger under R-MQO compared to the DA with match-quality-based tie-breaking, which could be thought of as a 'greedy' solution to the problem.

on January 2023.

[25]See https://data.cityofnewyork.us/Education/School-Point-Locations/jfju-ynrr. Last accessed on January 2023.

Year	DA-RTB	DA-DBTB	R-MQO
2011-2012	3.80	3.11	2.75
	(0.005)	(0.001)	(0.004)
2012-2013	3.79	3.16	2.82
	(0.004)	(0.001)	(0.004)
2013-2014	3.80	3.14	2.78
	(0.004)	(0.001)	(0.003)
2014-2015	3.78	3.15	2.80
	(0.004)	(0.001)	(0.004)
2015-2016	3.78	3.16	2.82
	(0.004)	(0.001)	(0.004)
2016-2017	3.72	3.11	2.76
	(0.005)	(0.001)	(0.004)
2017-2018	3.74	3.11	2.75
	(0.004)	(0.001)	(0.003)

Table 5: Average travel distance (in miles) at NYC high schools admissions

8 Discussion

School choice programs enable students to attend schools outside of their neighborhoods. Ideally, if parents ranked schools by effectiveness, school choice would have improved student achievement. In contrast, empirical evidence suggests that parents' preferences do not reflect school effectiveness (Abdulkadiroğlu et al., 2020; Beuermann and Jackson, 2018; Ainsworth et al., 2020). Hence, DA with random tie-breaking is unlikely to create effective matches, whereas the match quality optimal assignment can maximize student achievement without compromising on choice.

Additionally, proponents argue that school choice will result in desegregation and district consolidation. These objectives cannot be achieved without school transportation, which creates huge costs for the district. Match quality optimization can substantially reduce travel costs, while simultaneously preserving stability. Hence, the solution will allow district to fulfil the essence of school choice, with minimal costs.

We are not aware of a better and practicable solution that incorporates match quality. Using data from the largest US public school system, we show that match quality optimization under stability constraints can have real impact on various policy dimensions.

Our work partially addresses another potential challenge in school choice. There is an ongoing academic debate on whether parents form their preferences based on peer composition and achievement, or whether they align with effectiveness (Hanushek, 1981; Jacob and Lefgren, 2007; Abdulkadiroğlu et al., 2020). Moreover, even when parents value effectiveness, informational and cognitive barriers may preclude separation of a school's effectiveness from achievement of its student body (Kane and Staiger, 2002). Then, higher demand for schools that recruit higher-achieving students may create incentives for school principals to devote resources to screening and selection rather than better instruction (Ladd, 2002; MacLeod and Urquiola, 2015). The match quality optimal assignment does not fully solve this issue, as preferences still play a role in determining the final assignment. However, match quality-based assignment can potentially incentivize schools to become more effective.

Although our primary motivation is finding a match quality optimal assignment, the methodology can be easily applied to optimize other policy objectives. For instance, student welfare has been an important consideration in school choice. Typically, there are multiple stable assignments to choose from, and there is no obvious selection rule for a policy maker who cares for both stability and student welfare. Erdil and Ergin (2008) provide an algorithm that finds a stable assignment that is not Pareto dominated by other stable assignment. Our method allows to find stronger results. For example, with an appropriate choice of match quality matrix, the match quality optimal assignment can maximize the number of students that are assigned to their first choices. Another common policy objective is achieving desirable distributional outcomes, such as a more diverse student body or more students assigned to neighborhood schools. Such goals, too, can be partially achieved using match quality optimization with the appropriate choice of the match quality matching. For example, one may assign higher match quality for low-income families to schools' in affluent neighborhoods. The applicability of the solutions is broader than the school choice problem. Match quality optimization under stability constraints may be applied to any two-sided or priority-based matching problem.

An important question in public school assignment is whether parents can improve their childrens' assignments by preference manipulation. The commonly applied DA algorithm with a random-tie breaker is strategyproof, meaning that parents have no incentives to submit false preferences. However, the algorithm completely ignores information on match quality. In contrast, maximizing aggregate match quality or minimizing travel distance under a stable solution conflicts with strategyproofness, i.e., the MQO and R-MQO algorithms are not strategyproof. This is true even in the special case of our problem where schools' priorities are strict: in that setting, if the match quality reflects schools' priorities, match quality optimal assignment coincides with school proposing DA, which is not strategyproof for students (Roth, 1982). In our work we abstract away from incentives issues, and we leave it for future research.

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A Proofs

A.1 Proof of Proposition 1

First we prove $\mathscr{A}_r \subseteq \bar{\mathscr{A}_r}$. Suppose $\mu \in \mathscr{A}_r$. We show that μ satisfies conditions in the definition of $\bar{\mathscr{A}_r}$ one by one.

We start with the first condition. On the contrary, suppose the first condition does not hold. That is, there exists a student s such that $c = \mu(s) \notin C_s(r)$. By definition of $\rho(\mu)$, $\rho_{sc} \leq \rho_c(\mu) = r_c$. Therefore, $c \notin C_s(r)$ implies that there is a school $c' \in C$ such that $c'\ P_s\ c$ and $\rho_{sc'} < r_{c'} = \rho_{c'}(\mu)$. By Observation 1, this contradicts the stability of μ.

We continue with the second condition. If $C^+(r) = \emptyset$, then second condition holds trivially. Suppose $C^+(r) \neq \emptyset$ and $c \in C^+(r)$. By definition of $C^+(r)$, $\rho_c(\mu) = r_c < K + 1$. Hence, by definition of cutoffs, $|\mu(c)| = \kappa_c$. This completes the proof of the first part, i.e., $\mathscr{A}_r \subseteq \bar{\mathscr{A}_r}$.

Next we prove $\bar{\mathscr{A}_r} \subseteq \mathscr{A}$. Suppose $\mu \in \bar{\mathscr{A}_r}$. We first show that $\rho(\mu) \leq r$. By definition of $C^-(r)$, $r_c = K + 1$ for any $c \in C^-(r)$. Hence, $\rho_c(\mu) \leq K + 1 = r_c$ for any $c \in C^-(r)$.

Now consider a school $c \in C^+(r)$. Recall that, $|\mu(c)| = \kappa_c$. Therefore, by definition of $\rho_c(\mu)$,

$$\rho_c(\mu) = \max_{s \in \mu(c)} \rho_{sc}. \tag{5}$$

By definition of $\bar{\mathscr{A}_r}$,

$$\rho_{sc} \leq r_c, \forall s \in \mu(c). \tag{6}$$

Equations 5 and 6 imply that $\rho_c(\mu) \leq r_c$.

We now show that μ is stable. Consider an arbitrary $s \in S$ and $c = \mu(s)$. By definition of $\bar{\mathscr{A}_r}$, there is no $c' \in C$ such that $c'\ P_s\ c$ and $\rho_{sc'} < r_{c'}$. Since $\rho_{c'}(\mu) \leq r_{c'}$, there is no $c' \in C$ such that $c'\ P_s\ c$ and $\rho_{sc'} < \rho_{c'}(\mu)$. Hence, by Observation 1, μ is stable, i.e., $\mu \in \mathscr{A}$.

A.2 Proof of Lemma 1

First, suppose f is feasible and integral. By constraint 1,

$$\sum_{c \in C_r(s)} f(s, c) = b(s) = 1 \text{ for any student } s \in S.$$

Thus, μ_f is indeed an assignment, and $\mu_f(s) \in C_s(r)$ for any $s \in S$. This establishes the first condition in the definition of $\bar{\mathscr{A}_r}$. Also, for any $c \in C^+(r)$,

$$|\mu_f^{-1}(c)| = -\sum_{s\in S} f(s,c) = b(c) = -\kappa_c,$$

where the second equality follows from constraint 1. This establishes the second condition in the definition of $\bar{\mathscr{A}_r}$.

Now suppose $\mu \in \bar{\mathscr{A}_r}$. Integrality of f_μ, as well as equation 2 are immediate from the construction of f_μ. We now verify feasibility constraint 1. We check feasibility for each vertex type one by one.

- Let $v \in S$. By definition, each student is assigned to exactly one school. Therefore,

$$\sum_{c\in C} f_\mu(v,c) = 1 = b(v).$$

- Let $v \in C^-(r)$. Then,

$$\sum_{s\in S} f_\mu(s,v) - f_\mu(v,t) = |\mu(v)| - |\mu(v)| = 0 = b(v).$$

- Let $v \in C^+(r)$. Then,

$$-\sum_{s\in S} f_\mu(s,v) = -|\mu(v)| = -\kappa_c = b(v),$$

  where second equality follows from $v \in C^+(r)$.

- Finally, let $v = t$. Then,

$$-\sum_{c\in C^-(r)} f(c,v) = -\sum_{c\in C^-(r)} |\mu(c)| = -\sum_{c\in C} |\mu(c)| + \sum_{c\in C^+(r)} |\mu(c)|$$

$$= -|S| + \sum_{c\in C^+(r)} \kappa_c = b(v).$$

This completes the proof of Lemma 1.

A.3 Proof of Proposition 4

We first show that if a student s is rejected from school c during DR, then there is no stable assignment $\bar{\mu}$ such that $\bar{\mu}(s) = c$. By contradiction, suppose our claim does not hold. That is, there exists a stable assignment $\bar{\mu}$, a student s and a school c such that c rejects s during DR and $\bar{\mu}(s) = c$. Without loss of generality, we assume s is the first such student who is rejected from her assignment under $\bar{\mu}$ during DR, i.e., if s' is rejected by c' before s is rejected by c during DR, then $\bar{\mu}(s') \neq c'$. Suppose c rejects s in some Round m. We consider the following possible cases.

Case 1: c rejects s in Stage 1 of Round m. Suppose that s is rejected by c in Step t. Then, $s \in A_c$ and by our supposition $c \; R_{s'} \; \bar{\mu}(s')$ for all $s' \in A_c$. Since s is rejected by c in Step t of Stage 1, we have $|\{s' \in A_c : \rho_{s'c} < \rho_{sc}\}| \geq \kappa_c$. Stability of $\bar{\mu}$ implies that any student $s'' \in \{s' \in A_c : \rho_{s'c} < \rho_{sc}\}$ is assigned to c under $\bar{\mu}$. Then, $|\bar{\mu}^{-1}(c)| \geq \kappa_c + 1$, which contradicts the feasibility of $\bar{\mu}$.

Case 2: c rejects s in Step 0 of Stage 2 of Round m. Let μ be the outcome achieved at the end of Stage 1 in Round m. Since c rejects s in Step 0, then $\rho_{sc} > \bar{r}_c(\mu)$. Then, either $|\mu(c)| \geq \kappa_c$ and $\bar{r}_c(\mu) = \max_{s' \in \mu(c)} \rho_{s'c}$ or $D_c \neq \emptyset$ and $\bar{r}_c(\mu) = \min_{s' \in D_c} \rho_{s'c}$. If the former case holds, our supposition and stability of $\bar{\mu}$ imply that all students in $\mu(c)$ and s are assigned to c. This requires at least κ_c students to be assigned to c, violation of feasibility. Suppose the latter case holds. Our supposition implies that any student in D_c who has been rejected by c before s cannot be assigned to a school weakly better than c. Since $\bar{r}_c(\mu) = \min_{s' \in D_c} \rho_{s'c} < \rho_{sc}$, $\bar{\mu}$ cannot be stable, a contradiction.

Case 3: c rejects s in some Step $t \geq 1$ of Stage 2 of Round m. Then, there exists at least κ_c students in $\mu(c) \cup \tilde{D}^c$ who have strictly better priority than s. By our supposition at least κ_c students having better priority than ρ_{sc} cannot be assigned to schools weakly better than c under $\bar{\mu}$. Stability of $\bar{\mu}$ requires more than κ_c students to be assigned to c, a contraction.

Hence, if a school c has rejected student s during DR, student s cannot be assigned to school c in any stable assignment. Thus, no student s with $\rho_{s'c} > \bar{r}_c(\mu)$ can be assigned to c in any stable assignment. This concludes the proof.

A.4 Proof of Proposition 5

In order to prove this statement, we show that if a student s rejects a school c during DP, then there does not exist a stable assignment $\bar{\mu}$ such that $c \; R_s \; \bar{\mu}(s)$.

By contradiction, suppose a student s who rejects school c during this procedure is assigned to a school weakly worse than c under some stable assignment $\bar{\mu}$. Without loss of generality, let s be the first such a student who has rejected some school c and is assigned to a worse school under $\bar{\mu}$ when we apply DP algorithm. We consider the following possible cases.

Case 1: s rejects c in Stage 1 of Round m. Suppose s rejects c in Step t. Then, by definition $s \in A_c$ and she has received an offer from c' such that $c' \; P_s \; c$ in Step t and the number of students with better priority than s for c' and who has not rejected c' yet is strictly less than $\kappa_{c'}$. By our supposition, any student who has rejected c' earlier cannot be assigned to c' under $\bar{\mu}$. Therefore, stability implies that s cannot be assigned to a school worse than c' under $\bar{\mu}$. This is a contradiction.

Case 2: s rejects c in Stage 2 of Round m. By definition of the procedure, there exists at least two schools c_1 and c_2 such that the number of students with better priority than s either for c_1 or c_2 who has not rejected them yet is strictly less than the total capacity of c_1 and c_2. By our supposition, any student who has rejected either school before cannot be assigned to these schools. Therefore, by our supposition, s and either c_1 or c_2 would form a blocking pair at assignment $\bar{\mu}$ which contradicts stability of $\bar{\mu}$.

Then, our claim implies that if a student $s \in \mu(c)$, then at most $\kappa_c - 1$ students with weakly better priority than s can be assigned to c in stable assignment $\bar{\mu}$. Therefore, $\bar{\mu}(s) R_s \mu(s)$. This completes the proof.

B Examples

Example 1. *Let $C = \{c_1, c_2\}$, $S = \{s_1, s_2, s_3\}$ and $\kappa = (2, 2)$. Preferences of students are:*

s_1	s_2	s_3
c_1	c_1	c_2
c_2	c_2	c_1

There are 2 priority classes and school priorities are:

Priority Points	c_1	c_2
1	s_1	s_1, s_3
2	s_2, s_3	s_2

Match qualities are:

$$q(s_1, c_1) = 3 \qquad q(s_2, c_1) = 5 \qquad q(s_3, c_1) = 2$$

$$q(s_1, c_2) = 4 \qquad q(s_2, c_2) = 2 \qquad q(s_3, c_2) = 5$$

Consider the vector $r = (2, 3)$*. Then,*

$$\begin{aligned} C_{s_1}(r) &= \{c_1\} \\ C_{s_2}(r) &= \{c_1, c_2\} \\ C_{s_3}(r) &= \{c_2\} \end{aligned} \qquad \begin{aligned} C^+(r) &= \{c_1\} \\ C^-(r) &= \{c_2\} \end{aligned} .$$

The corresponding minimum-cost flow graph is depicted in Figure 1.

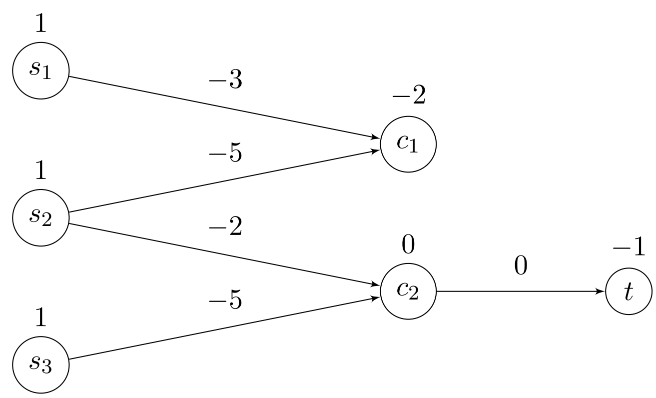

Figure 2: Minimum-cost flow graph

In Figure 2, numbers above the vertices denote their values (supply/demand), numbers above the edges denote their costs.
In this example, there is a unique feasible flow function f*, given by*

$$f(e) = \begin{cases} 0 & \text{if } e = (s_2, c_1) \\ 1 & \text{otherwise} \end{cases} .$$

Therefore, f *is the desired solution.*

Example 2. *Let* $C = \{c_1, c_2, c_3, c_4, c_5\}$*,* $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ *and* $\kappa = (1, 1, 1, 1, 2)$*. Preferences of students are:*

s_1	s_2	s_3	s_4	s_5	s_6
c_1	c_1	c_2	c_2	c_1	c_2
c_2	c_2	c_3	c_1	c_3	c_3
c_4	c_3	c_5	c_3	c_2	c_1
c_3	c_5	c_1	c_4	c_4	c_4
c_5	c_4	c_4	c_5	c_5	c_5

There are 4 priority classes and school priorities are:

Priority Points	c_1	c_2	c_3	c_4	c_5
1	s_4	s_1	s_5, s_6	s_1, s_5, s_6	s_2
2	s_1, s_2	s_2	s_4, s_1	s_2, s_3	s_3, s_4, s_5
3	s_3, s_5, s_6	s_3, s_4	s_3	s_4	s_1, s_6
4		s_5, s_6	s_2		

We first apply DR to this problem.

Round 1.

Stage 1.

We illustrate the steps of the first stage below. In each step, provisionally held students are given in bold.

	c_1	c_2	c_3	c_4	c_5
Step 1:	$\boldsymbol{s_1}, \boldsymbol{s_2}, s_5$	$\boldsymbol{s_3}, \boldsymbol{s_4}, s_6$			
Step 2:	$\boldsymbol{s_1}, \boldsymbol{s_2}$	$\boldsymbol{s_3}, \boldsymbol{s_4}$	$\boldsymbol{s_5}, \boldsymbol{s_6}$		

When Stage 1 terminates school c_1 holds s_1 and s_2, c_2 holds s_3 and s_4, and c_3 holds s_5 and s_6. We denote this outcome with μ.

Stage 2.

Step 0: We first determine $\rho_c(\mu)$ for all $c \in C$: $\rho_{c_1}(\mu) = 2$, $\rho_{c_2}(\mu) = 3$, $\rho_{c_3}(\mu) = 1$, $\rho_{c_4}(\mu) = \rho_{c_5}(\mu) = 5$. Then, in addition to the students rejected in Stage 1, c_1 rejects s_3 and s_6, c_2 rejects s_5 and c_3 rejects s_4, s_1, s_3 and s_2. For each school c we construct M_c, g_c^1 as follows:

	c_1	c_2	c_3	c_4	c_5
M	s_1, s_2	s_3, s_4	s_5, s_6	$\emptyset$	$\emptyset$
g^1	*0*	*0*	*0*	*0*	*0*

Step 1: Given M and g^1, we have $\tilde{D}^{c_2}_{c_1} = \{s_2\} = \tilde{D}^{c_2}$ and $\tilde{D}^{c_4}_{c_3} = \{s_5\} = \tilde{D}^{c_4}$. For all $c \notin \{c_2, c_4\}$ we have $\tilde{D}^c = \emptyset$. Then, we calculate $\hat{\rho}_{c_2} = 2$ and $\hat{\rho}_{c_4} = 1$. Since $\hat{\rho}_{c_2} < \rho_{c_2}(\mu)$, c_2 rejects s_3 and s_4. Since $\hat{\rho}_{c_4} < \rho_{c_4}(\mu)$, c_4 rejects s_2 , s_3 and s_4. We go Round 2.

Round 2.

Stage 1. *We illustrate the steps of the first stage below. In each step, provisionally held students are given in bold.*

	c_1	c_2	c_3	c_4	c_5
Step 1:	$s_1, s_2, \boldsymbol{s_4}$		$\boldsymbol{s_5}, \boldsymbol{s_6}$		$\boldsymbol{s_3}$
Step 2:	$\boldsymbol{s_4}$	$\boldsymbol{s_1}, s_2$	$\boldsymbol{s_5}, \boldsymbol{s_6}$		$\boldsymbol{s_3}$
Step 3:	$\boldsymbol{s_4}$	$\boldsymbol{s_1}$	$\boldsymbol{s_5}, \boldsymbol{s_6}$		$\boldsymbol{s_2}, \boldsymbol{s_3}$

When Stage 1 terminates school c_1 holds s_4, c_2 holds s_1, c_3 holds s_5 and s_6, and c_5 holds s_2 and s_3. We denote this outcome with μ.

Stage 2.

Step 0: We determine $\rho_c(\mu)$ for all $c \in C$: $\rho_{c_1}(\mu) = 1$, $\rho_{c_2}(\mu) = 1$, $\rho_{c_3}(\mu) = 1$, $\rho_{c_4}(\mu) = 1$, and $\rho_{c_5}(\mu) = 2$. Then, c_5 rejects s_1 and s_6. For each school c we construct M_c, g^1_c as follows:

	c_1	c_2	c_3	c_4	c_5
M	s_4	s_1	s_5, s_6	$\emptyset$	s_3
g^1	*0*	*0*	*0*	*0*	*1*

Step 1: Given M and g^1, we have $\tilde{D}^{c_4}_{c_3} = \{s_5\} = \tilde{D}^{c_4}$. For all $c \neq c_4$ we have $\tilde{D}^c = \emptyset$. Then, we calculate $\hat{\rho}_{c_4} = 1$. Since $\hat{\rho}_{c_4} = \rho_{c_4}(\mu)$ and $g^2_{c_4} = g^1_{c_4}$, the algorithm terminates here.

Final outcome of DR is

c_1	c_2	c_3	c_4	c_5
s_4	s_1	s_5, s_6	$\emptyset$	s_2, s_3

Next, we apply DP to the problem.

Round 1.

Stage 1. *We illustrate the steps of the first stage below. In each step, provisionally held colleges are given in bold.*

	s_1	s_2	s_3	s_4	s_5	s_6
Step 1:	$\boldsymbol{c_2}$	$\boldsymbol{c_5}$		$\boldsymbol{c_1}$		

Stage 1 terminates and the outcome is $\mu(s_1) = c_2$, $\mu(s_2) = c_5$, $\mu(s_4) = c_1$, *and* $\mu(s_3) = \mu(s_5) = \mu(s_6) = \emptyset$.

Stage 2.

We first construct D_s *for each* $s \in S$: $D_{s_1} = \{c_1\}$, $D_{s_2} = \{c_1, c_2, c_3\}$, $D_{s_4} = \{c_2\}$, *and* $D_{s_3} = D_{s_5} = D_{s_6} = C$. *We update* A_c *for each* $c \in C$: $A_{c_1} = A_{c_2} = S$, $A_{c_3} = \{s_2, s_3, s_5, s_6\}$, $A_{c_4} = \{s_3, s_5, s_6\}$ *and* $A_{c_5} = \{s_2, s_3, s_5, s_6\}$.

Once we follow the steps of Stage 2, we can see that both s_5 *and* s_6 *are guaranteed to be assigned to a school not worse than both* c_3 *and* c_4. *Then, we remove them from* A_{c_5} *and set it to be* $A_{c_5} = \{s_2, s_3\}$. *We continue with Round 2 with updated* A_c *for all* $c \in C$.

Round 2.

Stage 1. *We illustrate the steps of the first stage below. In each step, provisionally held colleges are given in bold.*

	s_1	s_2	s_3	s_4	s_5	s_6
Step 1:	$\boldsymbol{c_2}$	$\boldsymbol{c_5}$	$\boldsymbol{c_5}$	$\boldsymbol{c_1}$		

Stage 1 terminates and the outcome is $\mu(s_1) = c_2$, $\mu(s_2) = \mu(s_3) = c_5$, $\mu(s_4) = c_1$, *and* $\mu(s_5) = \mu(s_6) = \emptyset$.

Stage 2.

We first construct D_s *for each* $s \in S$: $D_{s_1} = \{c_1\}$, $D_{s_2} = \{c_1, c_2, c_3\}$, $D_{s_3} = \{c_2, c_3\}$, $D_{s_4} = \{c_2\}$, *and* $D_{s_5} = D_{s_6} = C$. *We update* A_c *for each* $c \in C$: $A_{c_1} = \{s_1, s_2, s_4, s_5, s_6\}$, $A_{c_2} = S$, $A_{c_3} = \{s_2, s_3, s_5, s_6\}$, $A_{c_4} = \{s_5, s_6\}$ *and* $A_{c_5} = \{s_2, s_3\}$.

Once we follow the steps of Stage 2, we can see that both s_5 and s_6 are guaranteed to be assigned to a school not worse than both c_3 and c_4. We continue with Round 3 with updated A_c for all $c \in C$.

Round 3.

Stage 1.

We illustrate the steps of the first stage below. In each step, provisionally held colleges are given in bold.

	s_1	s_2	s_3	s_4	s_5	s_6
Step 1:	$\mathbf{c_2}$	$\mathbf{c_5}$	$\mathbf{c_5}$	$\mathbf{c_1}$		

Stage 1 terminates and the outcome is $\mu(s_1) = c_2$, $\mu(s_2) = \mu(s_3) = c_5$, $\mu(s_4) = c_1$, and $\mu(s_5) = \mu(s_6) = \emptyset$.

Stage 2.

We first construct D_s for each $s \in S$: $D_{s_1} = \{c_1\}$, $D_{s_2} = \{c_1, c_2, c_3\}$, $D_{s_3} = \{c_2, c_3\}$, $D_{s_4} = \{c_2\}$, and $D_{s_5} = D_{s_6} = C$. We do not update A_c for any $c \in C$.

Once we follow the steps of Stage 2, we can see that both s_5 and s_6 are guaranteed to be assigned to a school not worse than both c_3 and c_4. Algorithm terminates since A_c stays the same for all $c \in C$.

C Computational Complexity Results

Finding a match quality optimal assignment is an NP-hard problem, even in some special cases, such as where priorities are common across schools and when preferences are aligned with match quality. Moreover, the problem of maximizing match quality under stability constraints is computationally hard to approximate for any level of approximation.

Proposition 6. *Finding a match quality optimal assignment is an NP-hard problem, even when*

- $\rho_{sc} = \rho_{sc'}$,
- $q(s, c) > q(s, c')$ *implies* $c \; P_s \; c'$,

for all $s \in S$ and $c, c' \in C$.

Proof. Consider a special case of our problem where there is a school $\bar{c}$ with $\kappa_{\bar{c}} \geq |S|$, and where for all $s \in S$ and $c \in C$,

$$q(s,c) = \begin{cases} 1 & \text{if } c \; P_s \; \bar{c}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the constructed match quality is aligned with preferences, i.e., $q(s,c) > q(s,c')$ implies $c \; P_s \; c'$.

If we interpret being assigned to $\bar{c}$ as being 'unassigned', then our problem of finding a match quality optimal assignment is equivalent to finding a stable matching that minimizes the number of students who are unassigned. The latter problem is known to be NP-hard, even when $\rho_{sc} = \rho_{sc'}$ for all $s \in S$ and $c, c' \in C$ (Irving et al., 2008). Therefore, so is the problem of finding a match quality optimal assignment. □

Our next result says that, unless P=NP, any polynomial-time algorithm can result in an assignment whose aggregate match quality is arbitrary times smaller than that of the match quality optimal one. In other words, the problem of maximizing match quality under stability constraints is not approximable for any level of approximation.

Proposition 7. *Unless P=NP, there is no polynomial-time stable algorithm that has an approximation ratio strictly larger than zero.*

Proof. We will use a result by Manlove et al. (2002), which says that for a given student-school pair (s,c), verifying whether there is a stable assignment that assigns s to c is an NP-hard problem. The problem is known as the stable pair problem.

Fix an arbitrary student-school pair (s,c), and let match quality be as follows: for all $s' \in S$ and $c' \in C$,

$$q(s',c') = \begin{cases} 1 & \text{if } (s',c') = (s,c), \\ 0 & \text{otherwise.} \end{cases}$$

Then, by NP-hardness of the stable pair problem, unless P=NP, there is no polynomial-time algorithm that assigns s to c, whenever that can be done in some stable assignment. Thus, any

Year	DA-RTB	DA-EBTB	R-MQO	DA-RTB	DA-EBTB	R-MQO
	Math			ELA		
2015-2016	0.003	0.081	0.120	-0.001	0.076	0.112
	(0.0009)	(0.0004)	(0.0004)	(0.0009)	(0.0004)	(0.0004)
2016-2017	0.000	0.077	0.114	0.001	0.074	0.111
	(0.0009)	(0.0005)	(0.0006)	(0.0008)	(0.0004)	(0.0004)
2017-2018	-0.006	0.072	0.108	-0.006	0.066	0.102
	(0.0008)	(0.0004)	(0.0003)	(0.0008)	(0.0005)	(0.0003)

Table 6: Estimated sixth grade test scores with all covariates at NYC middle schools admissions

polynomial-time stable algorithm may fail to match s to c at some problem where the match quality optimal assignment does so. Hence, for any $\epsilon > 0$, no polynomial-time stable algorithm has an approximation ratio of ϵ. $\square$

D Robustness Check

In this section we replicate the exercise in Section 7.2, with an additional set of covariates to control for potential selection bias. In addition to gender, race, and previous test scores, the covariates include reduced lunch status, English learner status, and Sensory Processing Disorder (SPD) status, and distance to school. As we see from Table 6, the match quality gains are similar to that in the main section.

E Additional Simulations

E.1 Setup

The algorithm described in Section 6 finds a match quality optimal assignment by searching in the set of all cutoff profiles. Without bounding the set of cutoff profiles to be searched within, we may need to consider $(K+1)^{|C|}$ cutoff profiles. Section 6 introduces two algorithms, DR and DP, that decrease the number of cutoff profiles that need to be considered. In this section, by using computer simulations we first measure the possible reduction in the number of cutoff profiles that need to be considered. Then, we find a match quality optimal assignment and compare it with the student proposing DA outcome with a random tie-breaker.

In our simulations we consider an environment that mimics a standard school choice setting with 750 students and 15 schools. Each school has 50 seats.

In the construction of school priorities we use two criteria: sibling status and distance between students and schools. We set the fraction of students with sibling status to 0.4 and determine the students with sibling priority and at which school they have the sibling priority randomly. Let $sib : S \times C \to \{0, 1\}$ be an indicator function such that $sib(s, c) = 1$ means student s has sibling priority at c and $\sum_{c \in C} sib(s, c) \leq 1$ for all $s \in S$.

In order to determine the neighborhood (walk-zone) priority, we randomly distribute schools and students on an 1×1 unit map. In particular, we represent the location of agent $i \in C \cup S$ with $\ell_i = (\ell_i^1, \ell_i^2)$ and both ℓ_i^1 and ℓ_i^1 are i.i.d. standard uniformly distributed random variables. Given the locations of each student s and school c, we calculate the euclidean distance and denote it with $d(s, c)$, i.e., $d(s, c) = \sqrt{(\ell_s^1 - \ell_c^1)^2 + (\ell_s^2 - \ell_c^2)^2}$. We set the neighborhood radius of each school to $rd \in \{0.2, 0.4\}$. If $d(s, c) \leq rd$, then student s has neighborhood priority at school s.

Given the sibling and neighborhood status, we group students into four priority classes. For each school $c \in C$, students having sibling priority at school c are ranked higher than students without sibling priority at c. Then, we subgroup students based on the neighborhood priority. That is, first priority group is composed of students with sibling and neighborhood priority. Second priority group is composed of students with sibling priority but not neighborhood priority. Third priority group is composed of students with neighborhood priority but not sibling priority. The remaining students constitute the fourth priority group.

Preferences of students are constructed by taking various criteria: students' common and individual taste over schools, siblings status, and distance from schools. To construct the preference of each student s we calculate her utility from being assigned to each school c, denoted by U_{sc}, as follows:

$$U_{sc} = \alpha X_c + (1-\alpha)Y_{sc} + \beta sib(s,c) - \gamma dist(s,c),$$

where $X_c \in (0,1)$ represents common taste for school c and $Y_{sc} \in (0,1)$ represents student s's individual taste over school c. Both X_c and Y_{sc} are i.i.d. standard uniformly distributed random variables. The level of correlation in the preferences of students is captured by $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$, i.e., as α increases preferences become more correlated. Variable $sib(s,c)$ takes value 0 or 1 and coefficient $\beta \in \{0, 0.25, 0.5, 0.75, 1\}$ is the additional utility received from attending the same school with sibling. Finally, travel is costly, we capture this by multiplying distance between students and schools with coefficient $-\gamma$. We report results for $\gamma \in \{0.2, 0.4\}$. The utility values of students are used to construct the ordinal preferences of students over the schools. Match quality of each student-school pair is drawn i.i.d. uniformly from the $(0,1)$ interval. Thus, in our simulations preferences are uncorrelated with the idiosyncratic match quality, which is consistent with the empirical evidence in Abdulkadiroğlu et al. (2020).

We run 100 simulations for each combination of parameter values.

E.2 Reductions in Cutoff Profiles by DR and DP

We first report the average number of cutoffs eliminated by DR and DP.

In our setting there are enough seats to accommodate all students, therefore all schools fill up their seats in any stable assignment. This means that there are at most 4^{15} cutoff profiles that our optimization algorithm need to consider to find a match quality optimal assignment. However, in our runs, the number of students having sibling priority at a school typically does not exceed the school's capacity. Therefore, for all our simulations DR and DP eliminate all cutoff profiles where a school has an admission cutoff equal to 1 or 2. This reduces the number of cutoff profiles to consider to 2^{15}. Additionally, DR and DP identify schools that have a single cutoff at any stable assignment (hereafter, a single stable cutoff). Identifying any such school reduces the computational burden exponentially.

Table 7 reports the percentage of schools with a single stable cutoff. For parameter value combinations $rd = 0.2, \alpha \geq 0.75$ and $rd = 0.4, \alpha \geq 0.25$, DR and DP identify a single stable cutoff for more than around 30% of the schools, or around 4 to 5 of the 15 schools. This decreases the computational burden around 2^4 to 2^5 times. Thus, the algorithm eliminates more than 94% of the remaining 2^{15} cutoff profiles.

We can observe from simulations that the amount of reductions increases with parameter α. This is intuitive as a larger α implies more homogeneous preferences, and therefore, potentially fewer cutoff profiles supporting a stable assignment. Additionally, the amount of reduction increases with γ. This is because a larger γ makes preferences more aligned with priorities, and again, potentially fewer cutoffs profiles support a stable assignment.

In our simulations, all students rank all schools acceptable and the total number of seats at schools are sufficient to accommodate all students. This fact allows us not to consider cutoffs equal to five, and potentially reduces the computational burden. Although, assuming more school seats than applicants is a realistic assumption in the public school choice setting, school districts commonly restrict the number of choices in the students' preference lists, therefore cutoffs at some schools can equal to five at some stable assignments. However, in the environment with restricted lists computational burden is not necessarily heavier than with unrestricted ones. The reason behind this is that when preference lists are restricted many schools do not fill their seats at any stable assignment, and DR and DP algorithms partially identify some of those schools. As a result, the number of cutoff profiles that can support stable assignments in a setting with restricted lists may decrease compared to our environment.

E.3 Comparing Match Quality

We report the percentage gains in match quality compared to student proposing DA algorithm with random tie-breaking for $rd = 0.2$ (Table 8) and $rd = 0.4$ (Table 9). For each combination of parameter values and each mechanism we run 100 simulations and report the averages. We consider three mechanisms, (1) MQO, (2) R-MQO, and DA with match-quality-based tie-breaking

		$rd = 0.2$		$rd = 0.4$	
α	β	$\gamma = 0.2$	$\gamma = 0.4$	$\gamma = 0.2$	$\gamma = 0.4$
	0.00	0.00	0.00	0.07	2.13
	0.25	0.13	0.13	0.33	3.13
0.00	0.50	0.27	0.20	0.80	5.60
	0.75	0.87	0.73	2.67	9.07
	1.00	1.93	1.67	3.73	10.33
	0.00	0.13	0.20	30.87	37.07
	0.25	0.27	0.67	32.60	38.47
0.25	0.50	0.80	1.33	33.27	38.07
	0.75	1.87	2.53	30.13	38.27
	1.00	1.73	2.47	28.13	36.40
	0.00	5.67	12.47	32.60	35.53
	0.25	12.47	20.73	37.13	38.73
0.50	0.50	11.13	22.26	37.27	38.93
	0.75	7.27	14.40	35.80	38.40
	1.00	6.47	11.73	35.40	37.93
	0.00	29.27	34.93	32.60	35.53
	0.25	35.80	39.07	34.67	36.27
0.75	0.50	35.80	39.87	33.67	36.27
	0.75	30.40	38.27	32.67	34.80
	1.00	29.20	37.53	32.67	34.80
	0.00	37.07	38.60	32.00	34.20
	0.25	40.00	41.47	33.73	35.13
1.00	0.50	40.93	43.47	33.80	35.80
	0.75	39.87	42.87	31.87	34.53
	1.00	39.20	40.80	31.13	34.40

Table 7: Percentage of schools with a single stable cutoff

(DA-MQ). For R-MQO, in each simulation we only consider a single cutoff for local maximization.

As Tables 8 and 9 show, we find substantial match quality gains across all parameter values. Across all parameter ranges, the MQO improves match quality by about 25-60% compared to the DA with random tie-breaking. Remarkably, the R-MQO algorithm with a single search performs almost equally well. For most parameter values considered, the performance of R-MQO is almost indistinguishable from that of MQO (within one percentage point). The performance gap between the MQO and R-MQO is substantial only for the case when the neighborhood radius is large and α is small. Intuitively speaking, in that environment, preferences and priorities are closer to uniform, which potentially makes the set of stable cutoffs more complex. We leave the formal analysis of this phenomenon for future research.

For the DA-MQ, we run the student proposing DA with all schools breaking ties in favor of higher match quality students. We can see that DA-MQ too considerable improves upon the DA with random tie-breaking, with match quality gains of 5-40%. However, match quality gains from this heuristic solution are less than two thirds of those of MQO and R-MQO.

When fixing other parameters, the match quality gains drop as β increases. For example, for $\alpha = 0, \gamma = 0.2$ and $rd = 0.2$, the gains from MQO drop from 46.64% at $\beta = 0$ to 29.15% at $\beta = 1$. This drop is dues to the change in the set of stable assignments. As students care more about schools their siblings attend, the set of stable assignments shrink, reducing the potential for gains. In fact, percentage of schools with a single stable cutoff increases as β increases for each value of α in Table 7. Likewise, match quality gains drop slightly as γ changes from 0.2 to 0.4 because the set of stable assignments shrinks as students care more about distance to school.

		MQO		R-MQO		DA-MQ	
α	β	$\gamma = 0.2$	$\gamma = 0.4$	$\gamma = 0.2$	$\gamma = 0.4$	$\gamma = 0.2$	$\gamma = 0.4$
	0.00	46.64	40.29	46.52	40.24	17.95	17.92
	0.25	41.35	35.84	41.27	35.74	16.79	16.43
0.00	0.50	36.38	31.63	36.32	31.46	15.53	15.38
	0.75	31.74	27.75	31.62	27.41	14.02	13.63
	1.00	29.15	25.43	28.98	25.08	13.02	12.31
	0.00	42.02	35.28	41.71	34.55	24.47	19.69
	0.25	35.86	30.47	35.50	29.85	20.50	16.55
0.25	0.50	30.40	26.11	29.98	25.52	17.33	14.22
	0.75	27.69	23.70	27.33	23.13	16.36	13.30
	1.00	27.43	22.03	27.06	21.46	16.54	13.27
	0.00	37.51	34.99	36.94	34.27	21.85	20.46
	0.25	31.49	30.40	31.08	29.96	18.44	18.24
0.50	0.50	26.13	25.14	25.66	24.64	15.11	14.81
	0.75	24.29	22.27	23.67	21.80	13.65	12.93
	1.00	24.03	21.61	23.41	21.04	13.50	12.36
	0.00	39.15	37.83	38.37	37.36	25.28	24.56
	0.25	33.63	33.28	33.04	32.82	21.70	21.88
0.75	0.50	28.72	28.29	28.14	27.87	18.28	18.35
	0.75	25.55	24.79	25.03	24.40	16.31	15.97
	1.00	24.93	23.47	24.38	23.06	15.76	15.11
	0.00	39.52	38.65	38.41	37.80	26.11	25.43
	0.25	35.06	34.61	34.21	33.88	23.52	23.06
1.00	0.50	31.19	30.76	30.36	30.19	20.59	20.31
	0.75	28.04	27.55	27.36	27.06	18.42	18.08
	1.00	26.11	25.24	25.32	24.66	17.16	16.48

Table 8: Results for $rd = 0.2$. The numbers denote the percentage gains in match quality compared to student proposing DA with a random tie-breaking.

		MQO		R-MQO		DA-MQ	
α	β	$\gamma = 0.2$	$\gamma = 0.4$	$\gamma = 0.2$	$\gamma = 0.4$	$\gamma = 0.2$	$\gamma = 0.4$
	0.00	58.53	59.41	17.76	12.97	10.28	7.21
	0.25	51.65	52.89	15.49	11.95	9.05	6.38
0.00	0.50	45.64	46.69	13.68	11.77	8.11	6.06
	0.75	40.85	41.70	12.18	11.35	7.03	5.72
	1.00	38.49	38.79	11.26	10.65	6.45	5.30
	0.00	58.62	59.77	38.21	40.34	22.06	23.42
	0.25	50.16	51.71	33.68	36.66	20.02	21.72
0.25	0.50	42.90	44.16	28.87	31.22	16.81	18.37
	0.75	38.99	39.56	24.38	26.42	13.80	15.28
	1.00	38.36	38.43	23.08	24.98	12.93	13.82
	0.00	59.14	60.05	58.45	59.13	42.17	42.97
	0.25	48.90	50.62	45.83	47.29	31.43	32.99
0.50	0.50	41.83	42.94	38.53	39.86	27.70	28.22
	0.75	38.96	39.35	35.57	36.13	24.81	25.56
	1.00	38.54	38.55	35.17	35.27	24.50	25.03
	0.00	59.14	60.05	58.45	59.13	42.17	42.97
	0.25	50.64	51.96	50.07	51.25	36.52	37.37
0.75	0.50	44.29	45.13	43.77	44.45	32.77	33.17
	0.75	40.41	40.93	39.91	40.20	30.82	30.85
	1.00	39.64	39.55	39.12	38.87	30.52	30.21
	0.00	59.64	60.24	59.30	59.86	43.85	44.26
	0.25	52.59	53.50	52.27	53.14	39.11	39.49
1.00	0.50	47.12	47.76	46.79	47.39	35.73	35.86
	0.75	42.97	43.50	42.69	43.20	33.51	33.57
	1.00	40.30	40.51	39.99	40.20	32.42	32.17

Table 9: Results for $rd = 0.4$. The numbers denote the percentage gains in match quality compared to student proposing DA with a random tie-breaking.