PMATH333: Introduction to Real Analysis

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Fall 2019

Contents

Chapter 1

Real Number

1.1 Lecture 2, Sept. 6th

1.1.1 Dedekind Cut

Definition 1.1 (Dedekind cut)**.** A **Dedekind cut** is a subset $A \subseteq \mathbb{Q}$, such that

1. $\varnothing \neq A \neq \mathbb{Q}$
2. If $x \in A$ and $q \in \mathbb{Q}$ with $q \leq x$, then $q \in A$
3. A has no largest element. That is, if $x \in A$, then there exists $y \in A$ with $x < y$

Example 1.1. If $q \in \mathbb{Q}$ is given, the $A_q := \{x \in \mathbb{Q} : x < q\}$ is a Dedekind cut

Proof. Let $q \in \mathbb{Q}$ and consider $A_q \subseteq \mathbb{Q}$

1. Since $q - 1$ is rational and $q - 1 < q$, we have $q - 1 \in A_q$ and $\varnothing \neq A_q$. Similarly, $q + 1$ is rational and $q + 1 \not< q$, So $q + 1 \notin A_q$, $A_q \neq \mathbb{Q}$. Hence $\varnothing \neq A_q \neq \mathbb{Q}$
2. Suppose $x \in A_q$ and $r \in \mathbb{Q}$ with $r \leq x$. By definition, $x < q$, so $r < q$ and $r \in A_q$.
3. Suppose $x \in A_q$, so by definition, $x < q$. Then $a < \frac{x+q}{2} < q$ and $\frac{x+}{2}$ is rational, so $\frac{x+q}{2} \in A_q$. That is, x is not a largest element of A_q

$\square$

Definition 1.2 (Rational Cut)**.** A Dedekind cut is called a **rational cut** if $X = A_q$ for some $q \in \mathbb{Q}$. Otherwise we sat the Dedekind cut X is an **irrational cut**

Rational Operations

For $q, r \in \mathbb{Q}$

$$q < r \iff r - q = \frac{a}{b} \text{ for some } a, b \in \mathbb{Z}^+$$
$$q \leq r \iff q < r \text{ or } q = r$$

For all $q, r, s \in \mathbb{Q}$

- Either $q < r$ or $r \leq q$. (any two of rational numbers are comparable)

- $q \leq r$ <u>and</u> $r \leq q \iff r = q$
- If $q \leq r$ and $r \leq s$, then $q \leq s$

For all $q, r, s, t \in \mathbb{Q}$

- $q \leq r$ and $s \leq t \implies q + s \leq r + t$
- If $0 \leq q$ and $0 \leq r$, then $\leq qr$
- If $q \leq r$, then $-r \leq -q$

Lemma. *Let q be a rational number with $0 < q$ and $q^2 < 2$. Then there exists some $r \in \mathbb{Q}$ with $q < r$ and $r^2 < 2$*

Proof. If $q \leq 1$, choose $r = 1.3$
Consider $r = q + \frac{1}{n}, n \in \mathbb{Z}^+$

$$\begin{aligned} (q + \frac{1}{n})^2 = &\quad q^2 + 2q\tfrac{1}{n} + \tfrac{1}{n^2}, &\quad \leftarrow \text{want it} < 2 \\ < &\quad q^2 + 4\tfrac{1}{n} + \tfrac{1}{n}, &\quad \text{since } q < q^2, \text{ q¡2} \\ < &\quad q^2 + \tfrac{5}{n} \end{aligned}$$

By definition, since $q^2 < 2$, we can write $2 - q^2 = \frac{a}{b}$ for $a, b \in \mathbb{Z}^+$. Choose $n = 5b$, then $r_n^2 < q^2 + \frac{1}{b} \leq q^2 + \frac{a}{b} = 2$
We have found a rational number r_n with $q < r$ and $r_n^2 < 2$ □

Goal

We want to show that the collection of Dedekind cuts may be reasonably thought of as a set of numbers. We must be able to define an <u>ordering</u> and <u>algebraic operations</u> on cuts.
We also need the <u>rational cuts</u> A_q to behave just like the rational numbers under these new operations.
The ordering we will use on Dedekind cuts is $\subseteq$

Theorem 1.1. If X and Y are Dedekind cuts, then either $X \subseteq Y$ or $Y \subseteq X$

Proof Skeleton. Suppose $X \nsubseteq Y$. Then there exists some $x \in X$ such that $x \notin Y$. Let $y \in Y$

I. If $x \leq y$, then because Y is a Dedekind cut, $x \in Y$, contradiction

II. $y < x$. Since $x \in X$ and X is a Dedekind cut, we conclude $y \in X$.
We have proved $y \in Y \implies y \in X$, hence $Y \subseteq X$

□

1.2 Lecture 3, Sept. 9th

1.2.1 Real Number

Definition 1.3 (Real Number)**.** A **real number** is a Dedekind cut.

We denote the set of real numbers by $\mathbb{R}$, and we use the following notation:

- If $q \in \mathbb{Q}$, we use "q" to mean A_q
- $\alpha \leq \beta$ means $\alpha \subseteq \beta$

The collection of rational cuts gives us a copy of $\mathbb{Q}$ in $\mathbb{R}$, we write $\mathbb{Q} \subseteq \mathbb{R}$

Theorem 1.2 (($\mathbb{R}, \leq$) is a **totally ordered** space)**.** If $\alpha, \beta, \gamma \in \mathbb{R}$, then

1. Either $\alpha \leq \beta$ or $\beta \leq \alpha$
2. If $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$
3. If $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$

Proof. Translation into Dedekind Cuts, If W, X, Y are Dedekind cuts, then

1. Either $W \subseteq X$ or $X \subseteq W$
2. If $W \subseteq X$ and $X \subseteq W$, then $X = W$
3. If $W \subseteq X$ and $X \subseteq Y$, then $W \subseteq Y$

$\square$

We also set

$$\alpha < \beta \iff \alpha \leq \beta \text{ and } \alpha \neq \beta$$

In the language of Dedekind cuts, this is $\subsetneq$ ("proper subset").

Goal

The goal of $\mathbb{R}$ is to fill the gaps in $\mathbb{Q}$

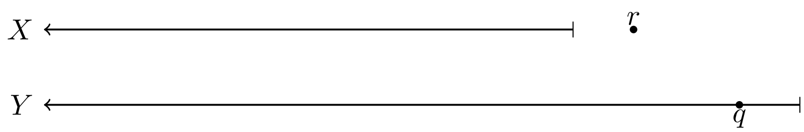
1. $\mathbb{R}$ has no gap
2. Real number is not too large

We will prove that between any two real numbers, there are rational numbers

1.2.2 Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 1.3 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** If $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, then there exists $q \in \mathbb{Q}$ with $\alpha < q < \beta$

Proof. We need to show that if X and Y are Dedekind cuts with $X \subsetneq Y$, then there exists a rational cut A_q with $X \subsetneq A_q \subsetneq Y$. By assumption, $X \subsetneq Y$ implies that there is some $r \in Y \subseteq \mathbb{Q}$ with $r \notin X$. Since $r \in Y$ and Dedekind cuts have no largest element, there exists some $q \in Y$ with $r < q$. We will prove $X \subsetneq A_q \subsetneq Y$



Let x be an arbitrary element of X. Suppose $q \leq x$, so $r < x$. Since X is a Dedekind cut, $r < x$ implies $r \in X$, a contradiction. We conclude $q \nleq x$ and hence $x < q$. By definition, we have $x \in A_q$. Since $x \in X \implies x \in A_q$. We have $X \subsetneq A_q$. Further, $r \notin X$, but $r < q$, so $r \in A_q$, we therefore have $X \subsetneq A_q$. Suppose $s \in A_q$, so by definition $s \in \mathbb{Q}$ with $s < q$. Since $q \in Y$, and Y is a Dedekind cut, this implies $s \in Y$. Since $s \in A_q \implies s \in Y$, we have $A_q \in Y$. Since $q \in Y$ but $q \notin A_q$, we have $A_q \subsetneq Y$ ☐

We just showed: for any <u>open interval</u> $(\alpha, \beta) = \{x \in \mathbb{R} : \alpha < x < \beta\}$ with $\alpha < \beta$ has $\mathbb{Q} \bigcap (\alpha, \beta) \neq \varnothing$. (This will later tell us that every real number is a limit of rational numbers)

1.2.3 Bound

Definition 1.4 (Bound)**.** If $\mathcal{S} \subseteq \mathbb{R}$, we say $\beta \in \mathbb{R}$ is an **upper bound** of $\mathcal{S}$ if every $x \in \mathcal{S}$ has $x \leq \beta$. If an upper bound for $\mathcal{S}$ exists, we say $\mathcal{S}$ is **bounded above**

Definition 1.5 (Supremum)**.** Let $\mathcal{S} \subseteq \mathbb{R}$ and $\beta \in \mathbb{R}$. Then β is a **supremum**, or **least upper bound** of $\mathcal{S}$ if

1. β is an upper bound of $\mathcal{S}$
2. If γ is an upper bound for $\mathcal{S}$, then $\beta \leq \gamma$

Theorem 1.4. Let $\mathcal{S} \subseteq \mathbb{R}$. If a supremum of $\mathcal{S}$ exists, it is unique.

Proof. If β_1 and β_2 are supremum of $\mathcal{S}$, then $\beta_1 \leq \beta_2$ because is a supremum and β_2 is an upper bound. Similarly, $\beta_2 \leq \beta_1$. Hence $\beta_1 = \beta_2$ ☐

1.3 Lecture 4, Sept. 11th

1.3.1 The Completeness of $\mathbb{R}$

Theorem 1.5 (The Completeness of $\mathbb{R}$)**.** Let $\mathcal{S} \subseteq \mathbb{R}$. If $\mathcal{S}$ is nonempty and bounded above ($\exists$ at least one upper bound), then the supremum of $\mathcal{S}$ exists.

Proof. We seek a number $\beta \in \mathbb{R}$ which is an upper bound of s and has $\beta \leq \gamma$ for every upper bound γ.
Translation in to Dedekind cuts: If $\mathcal{S}$ is a collection of Dedekind cuts which has

1). There exists at least one Dedekind cut $X \in \mathcal{S}$

2). There is some Dedekind cut $\mathcal{D}$ with $X \in \mathcal{S} \implies X \subseteq \mathcal{D}$

We are looking for a Dedekind cut $\mathcal{B}$ with

a). $X \in \mathcal{S} \implies X \subseteq \mathcal{B}$

b). Every $\mathcal{D}$ as in property 2 has $\mathcal{B} \subseteq \mathcal{D}$

We define $\mathcal{B} = \bigcup \mathcal{S} = \bigcup\limits_{X \in \mathcal{S}} X = x \in \mathbb{Q} : \exists X \in \mathcal{S}, x \in X$.
Suppose $X \in \mathcal{S}$. Consider an arbitrary element $x \in X \subseteq \mathbb{Q}$. By definition, $x \in \mathcal{B}$, hence $X \subseteq \mathcal{B}$
Suppose $\mathcal{D}$ has the property that $X \in \mathcal{S} \implies X \subseteq \mathcal{D}$, we want $\mathcal{B} \subseteq \mathcal{D}$. Consider ant $x \in \mathcal{B} \subseteq \mathbb{Q}$. By definition of $\mathcal{B}$, there is some $X \in \mathcal{S}$ such that $x \in X$. Since $X \subseteq \mathcal{D}$, we conclude $x \in \mathcal{D}$, hence $\mathcal{B} \subseteq \mathcal{D}$
The only thing left to prove is that $\mathcal{B} \in \mathbb{R}$, i.e. $\mathcal{B}$ is a Dedekind cut.

1). We want to show $\varnothing \neq \mathcal{B} \neq \mathbb{Q}$

- Since $\mathcal{S} \neq \varnothing$, there is at least one Dedekind cut $X \in \mathcal{S}$. Since Dedekind cuts are nonempty, there exists $x \in X$. By definition, $x \in \mathcal{B}$. We proved $\mathcal{B} \neq \varnothing$
- By assumption, there is a Dedekind cut $\mathcal{D}$ such that if $X \in \mathcal{S}$ then $X \subseteq \mathcal{D}$. Since $\mathcal{D}$ is a Dedekind cut, there is some rational number q with $q \notin \mathcal{D}$. Since $X \subseteq \mathcal{D}$, we must have $q \notin X$. That is $X \in \mathcal{S} \implies q \notin X$. Since $\mathcal{B}$ contains precisely all of the elements of any $X \in \mathcal{S}$, we must have $q \notin \mathcal{B}$, hence $\mathcal{B} \neq \mathbb{Q}$

2). We want to show if $x \in \mathcal{B}, q \in \mathbb{Q}, q \leq x$, then $q \in \mathcal{B}$
If $x \in \mathcal{B}, q \in \mathbb{Q}, q \leq x$, by definition of $\mathcal{B}$, there exists some $X \in \mathcal{S}$ such that $x \in X$, $\mathcal{S} \subseteq \mathbb{R}$, so $X \in \mathbb{R}$ is a Dedekind cut. Since $q \in \mathbb{Q}$ and $q \leq x$, we have $q \in X$, but $X \in \mathcal{B}$, so $q \in \mathcal{B}$

3). We want to show that if $x \in \mathcal{B}$, then $\exists y \in \mathcal{B}$ with $x < y$
If $x \in \mathcal{B}$, by definition, there exists some $X \in \mathcal{S}$ such that $x \in X$ and X is a Dedekind cut, then $\exists$ some $y \in X$ with $x < y$. But $X \subseteq \mathcal{B}$, so $y \in \mathcal{B}$

$\square$

Dictionary: $\sup(\mathcal{S})$ exists in $\mathbb{R} \implies \bigcup \mathcal{S}$ is a Dedekind cut

Note. *sup*$([0,1)) = 1$, *but* $1 \notin [0,1)$

1.4 Lecture 5, Sept. 13th

If we switch all inequalities ($\leq$ to *geq*)

Definition 1.6. A lower bound of a set $\mathcal{S} \subseteq \mathbb{R}$ is some $\gamma \in \mathbb{R}$ such that $x \in \mathcal{S} \implies \gamma \leq x$. If some lower bound exists, $\mathcal{S}$ is bounded below.
An infimum of $\mathcal{S}$ is a lower bound β with $\gamma \leq \beta$ for all lower bounds γ. If an infimum exists, it is unique and hence denoted by $\inf(\mathcal{S})$

Theorem 1.6. If $\mathcal{S} \subseteq \mathbb{R}$ is nonempty and bounded below, then $\inf(\mathcal{S})$ exists.

Note. *Proof goes roughly the same with $\mathcal{B} = \bigcap \mathcal{S}$. However, $\mathcal{B}$ might not be a Dedekind cut, from existence of a largest element q. In this case, $\mathcal{B} \backslash \{q\}$ is the Dedekind cut you are looking for.*

Operations

If $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, then

1). $\alpha + \beta = \beta + \alpha$

2). $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

3). If $\alpha \leq \beta$ and $\gamma \leq \delta$, then $\alpha + \gamma \leq \beta + \delta$

$\mathbb{Q}$ also has an **additive identity**: $q \in \mathbb{Q} \implies q + 0 = q$

Theorem 1.7. If $\alpha \in \mathbb{R}$, then $\alpha + 0 = \alpha$. Moreover, if $\beta \in \mathbb{R}$ has $\alpha + \beta = \alpha$ for all $\alpha \in \mathbb{R}$, then $\beta = 0$

Proof. The official version of the first claim is $\alpha + A_0 = \alpha$, where $A_0 = \{q \in \mathbb{Q} : q < 0\}$, and α is any Dedekind cut. If $a \in \alpha$ and $q \in A_0$, then $a + q$ is rational with $a + q < a$ and $a + q \in \alpha$. That is $\alpha + A_0 \subseteq \alpha$.
If $a \in \alpha$, ten since α is a Dedekind cut, there is some $b \in \alpha$ with $a < b$, then $a = b + (a - b) \in \alpha + A_0$, hence $\alpha \subseteq \alpha + A_0$ and $\alpha = \alpha + A_0$
For the second claim, suppose $\beta \in \mathbb{R}$ with $\alpha + \beta = \alpha$ for all $\alpha \in \mathbb{R}$. Then $\beta = \beta + 0 = 0 + \beta = 0$ $\square$

1.5 Lecture 6, Sept. 16th

Theorem 1.8. Fix $\alpha \in \mathbb{R}$, then there exists a unique additive inverse: some $\beta \in \mathbb{R}$ with $\alpha + \beta = 0$. We denote it by $-\alpha$

Proof. Uniqueness: Suppose $\alpha + \beta_1 = \alpha + \beta_2 = 0$ for $\beta_1, \beta_2 \in \mathbb{R}$,

$$\beta_1 = \beta_1 + 0 = \beta_1 + (\alpha + \beta_2) = (\beta_1 + \alpha) + \beta_2 = (\alpha + \beta_1) + \beta_2 = 0 + \beta_2 + \beta_2 + 0 = \beta_2$$

$\square$

Theorem 1.9. If $\alpha, \beta \in \mathbb{R}$ have $\alpha < \beta$, then $-\beta > -\alpha$

Proof. Suppose for contradiction, $\alpha < \beta$ but $-\alpha \leq -\beta$. We know it is valid to add inequalities, so $0 < 0$. $\square$

Definition 1.7 (Multiplication)**.** If $\alpha, \beta \in \mathbb{R}$ have $\alpha \geq 0$ and $\beta \geq 0$, define

$$\alpha\beta := \{qr : q, r \in \mathbb{Q}, 0 \leq q \leq \alpha, 0 \leq r \leq \beta\}$$

We extend this multiplication to all of $\mathbb{R}$ by

$$(-\alpha)\beta = \alpha(-\beta) = -(\alpha\beta)$$

$$(-\alpha)(-\beta) = \alpha\beta$$

The definition is designed so that $|\alpha\beta| = |\alpha||\beta|$
This new multiplication agrees with the old one if $\alpha, \beta \in \mathbb{Q}$

Operations

Let $\alpha.\beta, \gamma \in \mathbb{R}$, then

1. $\alpha\beta = \beta\alpha$
2. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$
3. $\alpha(\beta\gamma) = (\alpha\beta)\gamma$
4. $0\alpha = 0$
5. $1\alpha = \alpha$ and this multiplicative identity is unique
6. If $\alpha < \beta$ and $\gamma > 0$, then $\alpha\gamma < \beta\gamma$

1.5.1 Archimedean Property

Theorem 1.10 (Archimedean Property)**.** If $\alpha, \beta \in \mathbb{R}$ and $\alpha > 0$, then there exists some $n \in \mathbb{Z}^+$ with $n\alpha > \beta$

Proof. If $\beta \leq 0$, choose $n = 1$
If $\beta > 0$, note $\beta < \beta + 1$, so by the density of $\mathbb{Q}$ of $\mathbb{R}$, there is some $q \in \mathbb{Q}$ with $0 < \beta < q < \beta + 1$. Since $0 < \alpha$, choose a rational $r \in \mathbb{Q}$ with $0 < r < \alpha$. Write $q = \frac{a}{b}$, $r = \frac{c}{d}$ for $a, b, c, d \in \mathbb{Z}^+$. Choose $n = ad \in \mathbb{Z}^+$. Then $n\alpha = (ad)\alpha > (ad)r = ac \geq a \geq q > \beta$ $\square$

Theorem 1.11. If $\alpha \in \mathbb{R}$ and $\alpha \neq 0$, there is some $\beta \in \mathbb{R}\backslash\{0\}$ with $\alpha\beta = 1$

Proof. We may assume $\alpha > 0$. Let $\mathcal{S} = \{r \in \mathbb{R} : \gamma \geq 0, \alpha\gamma < 1\}$. $\mathcal{S} \subseteq \mathbb{R}$ is nonempty because $0 \in \mathcal{S}$. By the Archimedean Property, there is some $n \in \mathbb{Z}^+$ with $\alpha n > 1$.
We claim that n is an upper bound of $\mathcal{S}$. Indeed, if $\gamma in \mathcal{S}$ but $\gamma > n$, then $\alpha\gamma > \alpha n > 1$, contradicting $\gamma \in \mathcal{S}$. Thus, $\mathcal{S}$ is nonempty and bounded above, so $\sup(\mathcal{S})$ exixts. Let $\beta = \sup(\mathcal{S})$. We claim that $\alpha\beta = 1$.
If $\alpha\beta < 1$, write $\alpha\beta = 1 - \epsilon$ for $\epsilon > 0$. By the Archimedean Property, choose $m \in \mathbb{Z}^+$ with $m\epsilon > \alpha$. Since m is nonzero rational, $\frac{1}{m}$ exists, $a < \frac{1}{m}$. $\alpha\frac{1}{m} < \epsilon$. Then $\alpha(\beta + \frac{1}{m}) = \alpha\beta + \alpha\frac{1}{m} < 1$.
We conclude $\beta + \frac{1}{m} \in \mathcal{S}$, contradicting that β is an upper bound of $\mathcal{S}$.
If $\alpha\beta > 1$, write $\alpha\beta = 1 + \epsilon$ for $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ with $m\epsilon > \alpha$, so $\epsilon > \alpha\frac{1}{m}$. $\alpha(\beta - \frac{1}{m}) = \alpha\beta - \alpha\frac{1}{m} = (1 + \epsilon) + (-\alpha\frac{1}{m}) > 1$. Since $\beta - \frac{1}{m} < \beta = \sup(\mathcal{S})$, $\beta - \frac{1}{m}$ cannot be an upper bound. Hence $\exists\gamma \in \mathcal{S}$ with $\gamma > \beta = \frac{1}{m}$ $\square$