PMATH 333: Intro to Real Analysis Final Summary

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Some Thoughts

The course starts from generating the real line, using the idea of Dedekind cut. Then it introduces some basic properties about $\mathbb{R}$, e.g. boundedness, supremum/infimum, and then it introduces sequence and series, (limit points, convergence, completeness). And then move to functions on $\mathbb{R}$, continuity, differentiability, etc. Note that up until this point, we have two ways to talk about continuity: epsilon-delta definition, and sequence.
The last half is a more general application on $\mathbb{R}^n$. Starting from topology, we now can also discuss continuity using the idea of topology, which becomes more abstract. Then it introduces the properties of functions on $\mathbb{R}^n$, pointwise/uniform continuity. And the idea of pointwise/uniform convergence of functions.

1 Named Theorems PreMid

Theorem 1.1. Direct Comparison Test
Suppose $a_k, b_k \in \mathbb{R}$ with $0 \leq a_k \leq b_k$ for all $k \in \mathbb{Z}$. Then

$$\sum_{k=1}^{\infty} b_k \text{ converge} \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges}$$

Theorem 1.2. Squeeze Theorem
Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, and $(x_n)_{n=1}^{\infty}$ be $\mathbb{R}$-sequence. Suppose $a_n \leq x_n \leq b_n$ for all n. If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge to the same limit L, then $x_n \to L$ as well

Theorem 1.3. Bolzano-Weierstrass Theorem
Suppose $(\vec{x}_n)_{n=1}^{\infty}$ is a <u>bounded</u> $\mathbb{R}$-sequence. Then there exists a convergent <u>subsequence</u> $(\vec{x}_{n_k})_{k=1}^{\infty}$

Theorem 1.4. Alternating Series Test
Suppose $(a_k)_{k=1}^{\infty}$ is an $\mathbb{R}$-sequence with

- $a_k \geq 0$ for all k
- $a_1 \geq a_2 \geq a_3 \geq \cdots$
- $\lim\limits_{k\to\infty} a_k = 0$

Then the <u>alternating series</u> $\sum\limits_{k=1}^{\infty}(-1)^k a_k$ converges

Theorem 1.5. Rolle's Theorem
Suppose $a, b \in \mathbb{R}$ with $a < b$. Suppose $g : [a, b] \to \mathbb{R}$ s.t.

- g is differentiable at every $x \in (a, b)$
- g is continuous at a and b
- $g(a) = g(b)$

Then there is some $c \in (a, b)$ with $g'(c) = 0$

Theorem 1.6. Mean Value Theorem
Let $a, b \in \mathbb{R}$ with $a < b$, suppose $f : [a, b] \to \mathbb{R}$ is s.t.

- f is differentiable at every $x \in (a, b)$ and
- f is continuous at a and b

Then there exists some $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$

Theorem 1.7. Extreme Value Theorem
Let $a, b \in \mathbb{R}$ with $a \leq b$. Suppose $f : [a, b] \to \mathbb{R}$ is continuous at every $x \in [a, b]$. Then the range of f, $\text{Ran}(f) = \{f(x) : x \in [a, b]\}$ is bounded and there is a maximum/minimum value.

Theorem 1.8. Intermediate Value Theorem
Let $a, b \in \mathbb{R}$ and $a \leq b$. Suppose $f : [a, b] \to \mathbb{R}$ is continuous everywhere. If y is between $f(a)$ and $f(b)$, then there exists some $c \in [a, b]$ with $f(c) = y$

2 Continuity

Definition 2.1. Limit point and Convergence in $\mathbb{R}^n$
Let $X \subseteq \mathbb{R}^n$, $f : X \to \mathbb{R}^m$, $\vec{\alpha} \in \mathbb{R}^n$, $\vec{L} \in \mathbb{R}^m$. Then $\lim\limits_{\vec{x} \to \vec{\alpha}} f(\vec{x}) = \vec{L}$ means:

1. $\vec{\alpha}$ is a limit point of X
2. For any $\epsilon > 0$, there exists some $\delta > 0$, such that for all $x \in X$ with $0 < ||\vec{x} - \vec{\alpha}|| < \delta$, then $||f(\vec{x}) - \vec{L}|| < \epsilon$

Or equivalently,

1. There exists some $(X \backslash \{\vec{\alpha}\})$-sequence $(\vec{x}_k)_{k=1}^{\infty}$ with $\lim\limits_{k \to \infty} \vec{x}_k = \vec{\alpha}$
2. For every $(X \backslash \{\vec{\alpha}\})$-sequence $(\vec{x}_k)_{k=1}^{\infty}$ with $\lim\limits_{k \to \infty} \vec{x}_k = \vec{\alpha}$, it follows $\lim\limits_{k \to \infty} f(\vec{x}) = \vec{L}$

Definition 2.2. Continuity on $\mathbb{R}^n$
Let $X \subseteq \mathbb{R}^n$, $f : X \to \mathbb{R}^n$, $\vec{x} \in X$

- If $\vec{x} \in X \backslash \mathrm{Lim}(X)$, we say f is continuous at $\vec{x}$ automatically
- If $\vec{x} \in (X \cap \mathrm{Lim}(X))$, we say : f is continuous at $\vec{x} \iff \lim\limits_{\vec{y} \to \vec{x}} f(\vec{y}) = f(\vec{x})$

Theorem 2.1. Let $X \subseteq \mathbb{R}^n$, $f : X \to \mathbb{R}^n$, $\vec{x} \in X$. Then

$$\begin{aligned}
f \text{ is } \underline{\text{continous}} \text{ at } \vec{x} \quad &\iff \quad \text{For any } \epsilon > 0 \text{, there exists some } \delta > 0 \text{, such that} \\
&\qquad\; \vec{y} \in X, ||\vec{x} - \vec{y}|| < \delta \Rightarrow ||f(\vec{x}) - f(\vec{y})|| < \epsilon \\
&\iff \quad \text{For } \underline{\text{any}}\ X\text{-sequnce } (\vec{x}_k)_{k=1}^{\infty} \text{ which converges to } \vec{x}, \\
&\qquad\; (f(\vec{x}_k))_{k=1}^{\infty} \text{ converges to } f(\vec{x})
\end{aligned}$$

Definition 2.3. Uniformly Continuous
Let $X \subseteq \mathbb{R}^n$, $f : X \to \mathbb{R}^m$. Then f is uniformly continuous on X is for any $\epsilon > 0$, there exists a $\delta > 0$, such that for ALL $\vec{x}, \vec{y} \in X$ with $||\vec{x} - \vec{y}|| < \delta$, it follows $||f(\vec{x}) - f(\vec{y})|| < \epsilon$

Theorem 2.2. Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Then

$$\begin{aligned}
f \text{ is } \underline{\text{uniformly continuous}} \text{ on } X \quad &\iff \quad \text{For any } \epsilon > 0 \text{, there exists } \delta > 0 \text{, such that for all} \\
&\qquad\; \vec{x}, \vec{y} \in X \text{ with } ||\vec{x} - \vec{y}|| < \delta \text{, we have } ||f(\vec{x}) - f(\vec{y})|| < \epsilon \\
&\iff \quad \text{For } \underline{\text{all}}\ X\text{-sequence } (\vec{x}_k)_{k=1}^{\infty} \text{ and } (\vec{y}_k)_{k=1}^{\infty} \text{ with} \\
&\qquad\; \lim_{k \to \infty} ||\vec{x}_k - \vec{y}_k|| = 0 \text{, it holds that} \\
&\qquad\; \lim_{k \to \infty} ||f(\vec{x}_k) - f(\vec{y}_k)|| = 0
\end{aligned}$$

Theorem 2.3. Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$, assume f is uniformly continuous on X. For any limit point $\vec{\alpha}$ of X, $\lim\limits_{\vec{x} \to \vec{\alpha}} f(\vec{x})$ exists in $\mathbb{R}^m$

3 Topology

Definition 3.1. Open Ball
Let $X \in \mathbb{R}^n$ and $r > 0$, $B_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^n : ||\vec{x} - \vec{y}|| < r\}$ is the open ball of radius r around $\vec{x}$

Definition 3.2. Openness
Let $X \subseteq \mathbb{R}^n$, then X is open if for every $\vec{x} \in X$, there **exists** some $\epsilon > 0$ such that $B_\epsilon(\vec{x}) \subseteq X$

Theorem 3.1. Suppose $\mathcal{O}_1, ..., \mathcal{O}_k \subseteq \mathbb{R}^n$ are **finitely** many open sets. Then $\bigcap\limits_{j=1}^{k} \mathcal{O}_j$ is open

Theorem 3.2. Let Λ be an index set of any size. If $\mathcal{O}_\lambda \subseteq \mathbb{R}^n$ is open for each $\lambda \subseteq \Lambda$, then $\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$ is open

Definition 3.3. Limit Points
Let $X \in \mathbb{R}^n$, $\vec{\alpha} \in \mathbb{R}^n$, then $\vec{\alpha}$ is called a limit point of X is for every $\epsilon > 0$, there exists some $\vec{x} \in X$ with $0 < ||\vec{\alpha} - \vec{x}|| < \epsilon$
Or equivalently, there exists a $(X \backslash \vec{\alpha})$-sequence $(\vec{x}_k)_{k=1}^{\infty}$ with $\lim\limits_{k \to \infty} \vec{x}_k = \vec{\alpha}$

Definition 3.4. Closeness
A set $X \subseteq \mathbb{R}^n$ is called closed if every limit point of X is a member of X. (i.e. $\text{Lim}(X) \subseteq X$)

Theorem 3.3. Let $X \subseteq \mathbb{R}^m$

$$\begin{array}{lcl} X \text{ is closed} & \iff & R^n \backslash X \text{ is open} \\ X \text{ is open} & \iff & R^n \backslash X \text{ is closed} \end{array}$$

Corollary.

- *If $\mathcal{C}_1, ..., \mathcal{C}_k \subseteq \mathbb{R}^n$ are finitely many closed sets, then $\bigcup\limits_{j=1}^{k} \mathcal{C}_j$ is closed*
- *If $\{\mathcal{C}_\lambda : \lambda \in \Lambda\}$ is any collection of closed sets $\mathcal{C}_\lambda \subseteq \mathbb{R}^n$, then $\bigcap\limits_{\lambda \in \Lambda} \mathcal{C}_\lambda$ is closed*
- *If $X \subseteq \mathbb{R}^n$, then there exists a **smallest** closed set $\mathcal{C}$ with $X \subseteq \mathcal{C}$. We denote $\mathcal{C} = \overline{X}$ and call it the **closure** of X*

Corollary. $\overline{X} = X \cup Lim(X) = \{\vec{\alpha} \in \mathbb{R}^n : \text{ there exists a converget } X\text{-sequence with limit } \vec{\alpha}\}$

Remark. *3 expressions of $\overline{X}$:*

- *The smallest closed set containing X*
- *$X \cup Lim(X)$*
- *For every convergent X-sequence, the limit is in $\overline{X}$*

Definition 3.5. Preimage
If $f : X \to Y$ and $B \subseteq Y$, then the preimage is $f^{-1}(B) = \{x \in X : f(x) \in B\}$

Corollary. *If $X \subseteq \mathbb{R}^n$, then there exists a largest open subset V of X, which we call the* ***interior*** *of X and denote as X°*
$X^\circ = \bigcup\{W : W \text{ open and } W \subseteq X\}$

Theorem 3.4. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and let $\vec{x} \in \mathbb{R}^n$, then

$$f \text{ is continous at } \vec{x} \iff \begin{array}{l} \text{For every open set } \mathcal{O} \subseteq \mathbb{R}^m \text{ which includes } f(\vec{x}), \\ \vec{x} \text{ is an } \underline{\text{interior}} \text{ point of } f^{-1}(\mathcal{O}) \end{array}$$

Corollary. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Then f is continuous on $\mathbb{R}^n$ if and only if for every open set $\mathcal{O} \subseteq \mathbb{R}^m$, $f^{-1}(\mathcal{O})$ is open.*

Definition 3.6. Relative Topology
Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq X$

1. Y is relatively open in X if $X = X \cap \mathcal{O}$ for some open set $\mathcal{O} \subseteq \mathbb{R}^n$
2. Y is relatively closed in X if $X = X \cap \mathcal{C}$ for some closed set $\mathcal{C} \subseteq \mathbb{R}^n$

Theorem 3.5. Let $X \subseteq \mathbb{R}^n$

- The union of any relatively open subsets of X is relatively open in X
- The intersection of finitely many open subsets of X is relatively open in X
- If $Y \subseteq X$, then there is a largest subset of Y which is relatively open in X. This is called the relative interior of Y

Theorem 3.6. Let $\varnothing \neq X \subseteq \mathbb{R}^n$, $f : X \to \mathbb{R}^n$. Then

$$\begin{array}{lll} f \text{ is continous on } X & \iff & \text{For every open } \mathcal{O} \subseteq \mathbb{R}^m,\ f^{-1}(\mathcal{O}) \text{ is } \underline{\text{relatively open}} \text{ in } X \\ & \iff & \text{For every closed } \mathcal{C} \subseteq \mathbb{R}^m,\ f^{-1}(\mathcal{C}) \text{ is } \underline{\text{relatively closed}} \text{ in } X \end{array}$$

Definition 3.7. Open cover
Let $X \subseteq \mathbb{R}^n$ and let $C = \{\mathcal{O}_\lambda : \lambda \in \Lambda\}$ be a collection of subsets of $\mathbb{R}^n$. Then C is called an open cover of X is

1. Each $\mathcal{O}_\lambda$ is open
2. $X \subseteq \bigcup\limits_{\lambda \in \Lambda} \mathcal{O}_\lambda = \bigcup C$

Definition 3.8. Subcover
Let $X \subseteq \mathbb{R}^n$ and let C be an open cover of X. A subcover is a subset $S \subseteq C$ which is still an open cover of X
In particular, a finite subcover is a subcover S which contains only finitely many sets

Definition 3.9. Compact
A set $X \subseteq \mathbb{R}^n$ is called compact if every open cover of X has a finite subcover
If $\{\mathcal{O}_\lambda : \lambda \in \Lambda\}$ is a collection of open sets such that $X \subseteq \bigcup\limits_{\lambda \in \Lambda} \mathcal{O}_\lambda$, then there exists $m \in \mathbb{N}^+$ and $\lambda_1, ..., \lambda_m \in \Lambda$ with $X \subseteq (\mathcal{O}_{\lambda_1} \cup \cdots \mathcal{O}_{\lambda_m})$

Lemma. *Let $a, b \in \mathbb{R}$ with $a < b$. Then $[a, b]$ is compact*

Theorem 3.7. Extreme Value Theorem
If $X \subseteq \mathbb{R}^n$ is compact and $f : X \to \mathbb{R}^n$ is continuous, then $f(X)$ is compact

Lemma. *Let $X \subseteq \mathbb{R}^n$. If X is compact and Y is a closed subset of X, then Y is compact*

Lemma. *If $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are compact, then $A \times B \subseteq \mathbb{R}^{n+m}$ is compact*

Theorem 3.8. Heine-Borel Theorem

$$X \text{ is compact} \iff X \text{ is closed and bounded}$$

Theorem 3.9. If $X \subseteq \mathbb{R}^n$ is compact and $f : X \to \mathbb{R}^m$ is continuous, then f is uniformly continuous

Definition 3.10. Separation
Let $X \subseteq \mathbb{R}^n$. A separation of X is a choice of two subsets A and B such that

- $A \neq \varnothing \neq B$
- $A \cap B = \varnothing$
- $A \cup B = X$
- Both A and B are relatively open in X

Definition 3.11. Connected and Disconnected
Let $X \subseteq \mathbb{R}^n$. If there exists a separation of X, then X is called disconnected. Otherwise, we say X is connected.

Theorem 3.10. If $X \subseteq \mathbb{R}^n$ is path connected, then X is connected

Remark. *There exist connected sets which are NOT path connected*

Theorem 3.11. Let $X \subseteq \mathbb{R}^n$, then

$$\begin{aligned} X \text{ is disconnected} &\iff \text{There exists a continuous function } f : X \to \mathbb{R} \text{ with } f(X) = \{0, 1\} \\ &\iff \text{There exists } C \subseteq X \text{ such that } \varnothing \neq C \neq X \\ & \text{ and } C \text{ is both relatively open and relatively closed} \end{aligned}$$

Theorem 3.12. Intermediate Value Theorem
If $X \subseteq \mathbb{R}^n$ and X is connected, and $f : X \to \mathbb{R}^m$ is continuous on X, then $f(X)$ is connected.

Definition 3.12. Path Connected
A set $X \subseteq \mathbb{R}^n$ is path connected if for every $\vec{a}, \vec{b} \in X$, there exists a continuous function $\gamma : [0, 1] \to X$ such that $\gamma(0) = \vec{a}$ and $\gamma(1) = \vec{b}$

4 Convergence of Functions

Definition 4.1. Pointwise Convergence
Let X be a set and for each $n \in \mathbb{Z}^+$, suppose $f_n : X \to \mathbb{R}^m$. Let $f : X \to \mathbb{R}^m$, then $\underline{f_n \text{ converges to } f \text{ pointwise}}$ means:

$$\text{For each } x \in X, \lim_{n\to\infty} f_n(x) = f(x)$$

Or equivalently,

$$\forall x \in X, \forall \epsilon > 0, \exists N \in \mathbb{Z}^+, \forall n \in \mathbb{Z}^+, n > N \Rightarrow ||f_n(x) - f(x)|| < \epsilon$$

Remark. *N can depend on x!*

Definition 4.2. Uniform Convergence
Let X be a set and for each $n \in \mathbb{Z}^+$, suppose $f_n : X \to \mathbb{R}^m$. Let $f : X \to \mathbb{R}^m$, then $\underline{f_n \text{ converges to } f \text{ uniformly}}$ means:

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}^+, \forall x \in X, \forall n \in \mathbb{Z}^+, n > N \Rightarrow ||f_n(x) - f(x)|| < \epsilon$$

Or equivalently,

$$\lim_{n\to\infty} (\sup\{||f_n(x) - f(x)|| : x \in X\}) = 0$$

Remark. *N CANNOT depend on x! The supremum measures the worst discrepancy between f_n and f*

WARNING: The pointwise limit of continuous functions $f_n : X \to \mathbb{R}$ might not be continuous! e.g. $f_n(x) = x^n, f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

Theorem 4.1. Let $X \subseteq \mathbb{R}^n$, $g)n : X \to \mathbb{R}^m$. If each g_n is continuous on X and $g_n \xrightarrow{\text{uniformly}} g$, then $g : X \to \mathbb{R}^m$ is also continuous on X

Theorem 4.2. Let $X \subseteq \mathbb{R}^n$, $g)n : X \to \mathbb{R}^m$. If each g_n is <u>uniformly</u> continuous on X and $g_n \xrightarrow{\text{uniformly}} g$, then $g : X \to \mathbb{R}^m$ is also <u>uniformly</u> continuous on X

Definition 4.3. $C([a, b])$
Let $a, b \in R$, $a < b$. Then $\underline{C([a, b])}$ is the set of all continuous functions from $[a, b]$ to $\mathbb{R}$

Definition 4.4. Uniform Norm
Consider the <u>uniform norm</u> for $f \in C([a, b])$,

$$||f||_u = \sup(\{|f(x)| : x \in [a, b]\})$$

For $f_n, f \in C([a, b])$,

$$f_n \xrightarrow{\text{uniformly}} f \iff ||f_n - f||_u \to 0$$

Theorem 4.3. For $f, g \in C([a, b])$, $\alpha \in \mathbb{R}$

- $||\alpha f||_u = |\alpha|||f||_u$
- $||f+g||_u \leq ||f||_u + ||g||_u$
- $||fg||_u \leq ||f||_u||g||_u$

Definition 4.5. Uniformly Cauchy
Let $(f_n)_{n=1}^{\infty}$ be a $C([a,b])$-sequence. Then the sequence is uniformly Cauchy if

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}^+, \forall k, l > N \Rightarrow ||f_k - f_l||_u < \epsilon$$

Or equivalently

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}^+, \forall n > N \Rightarrow ||f_n - f_N||_u < \epsilon$$

Theorem 4.4. Completeness of $C[a,b]$
Let $(f_n)_{n=1}^{\infty}$ be a $C([a,b])$-sequence. Then it is uniformly convergent if and only if it is uniformly Cauchy

Theorem 4.5. Weierstrass M-Test
Suppose $f_k \in C([a,b])$, and suppose $M_k \in [0,+\infty)$ have $||f_k||_u \leq M_k$. If $\sum\limits_{k=1}^{\infty} M_k$ converges, then $f(x) = \sum\limits_{k=1}^{\infty} f_k(x)$ defines a continuous function on $[a,b]$.
In particular, the partial sums converge absolutely, and converge uniformly to f.

5 Power Series

Definition 5.1. Power Series
Let $c \in \mathbb{R}$, then a power series centred at c is an expression $\sum_{n=1}^{\infty} a_n(x-c)^n$ for some $\mathbb{R}$-sequence $(a_n)_{n=0}^{\infty}$

Definition 5.2. Convergence of Power Series
Suppose $\sum_{n=0}^{\infty} a_n(x-c)^n$ is a power series, and let

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \in [0, +\infty]$$

Then,

- $\sum_{n=1}^{\infty} a_n(x-c)^n$ converges pointwise, absolutely, for $x \in (c-R, c+R)$
- If $0 \leq r < R$ is fixed, $\sum_{n=1}^{\infty} a_n(x-c)^n$ converges uniformly, on $[c-r, c+r]$
- No information if $x = c \pm R$
- $\sum_{n=1}^{\infty} a_n(x-c)^n$ divergences if $|x-c| > R$

Remark. *By ratio test,*

$$R = \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

Theorem 5.1. Let $f_k \in C([a,b])$ and assume each f_k is differentiable on (a,b). If

- The f_k' converge uniformly on (a,b)
- There exists one point $x_0 \in [a,b]$ such that $f_k(x_0)$ converges as $k \to \infty$

Then f_k converges uniformly on $[a,b]$ to some $f \in C([a,b])$, and f is differentiable on (a,b) with $f'(x) = \lim_{k\to\infty} f_k'(x)$

Corollary. *If $\sum_{n=0}^{\infty} a_n(x-c)^n$ is a power series with strictly positive radius of convergence R, then the function $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is differentiable on $(c-R, c+R)$, with $f'(x) = \sum_{n=0}^{\infty} na_n(x-c)^{n-1}$*

Corollary. *Power series with R.O.C $R > 0$ are infinitely differentiable at every $x \in (c-R, c+R)$, with derivative computed term-by-term*

Definition 5.3. Taylor Polynomial and Taylor Series
Suppose f is N-times differentiable on an open interval around c. Then the Nth Taylor polynomial (for f around c) is

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If f is infinitely differentiable, the Taylor series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Theorem 5.2. Upgrade Rolle's Theorem
Let $\mathcal{J}$ be an interval including distinct $a, b \in R$, let $N \in \mathbb{N}$. Suppose $g : \mathcal{J} \to \mathbb{R}$

- is $(N+1)$-times differentiable on $\mathcal{J}^\circ$
- is continuous at a and b
- $g(a) = g(b)$
- g has continuous one sided derivatives at a, up to order N, and for $1 \leq k \leq N$, $g^{(k)}(a) = 0$

Then, $\exists y \in \mathcal{J}^\circ$ with $g^{(N+1)}(y) = 0$

Definition 5.4. Taylor's Remainder
The remainder

$$R_N(x) = f(x) - T_N(x)$$

- $0 \leq k \leq N \Rightarrow R_N^{(k)}(c) = 0$
- $k > N \Rightarrow R_N^{(k)}(c) = f^{(k)}(x)$

Definition 5.5. Taylor's Theorem
Let $\mathcal{J}$ be an interval with distinct $b, c \in \mathcal{J}$. Suppose $f : \mathcal{J} \to \mathbb{R}$ is continuous on $\mathcal{J}$ with

- f is $(N+1)$-times differentiable on $\mathcal{J}^\circ$
- f has continuous (possibly one-sided) derivatives at c up to order N

Then $\exists y \in \mathcal{J}^\circ$ such that $R_N(b) = \frac{f^{(N+1)}(y)}{(N+1)!}(b-c)^{N+1}$