How to Use this Reviewer

Hello! This is a compilation of solved exercises for module 3 of MATH 51.4. All of these exercises are taken straight from Aldrich and Cisco's course notes, so you can expect tests to be very similar to the items given. Some items are bound to be a little bit trickier than others, so I'll note when these items show up.

Normal items will look like this:

1 A very easy math problem. What's $1 + 1$?

whereas difficult problems will be soulless, like this:

2 A very difficult math problem. Prove that $\binom{2n}{n} < 2^{2n-2}, \ \forall n \geq 5$ using induction.

I might also include warnings in my **Nerd Interjections!**

Nerd Interjection![a] These sections are for me to remind you of some necessary information to solve the problems, elaborate on something that I think isn't all that clear with just pure math symbols, give a helpful theorem, be an annoying piece of shit, anything, really! Just think of it as a tips and tricks section.

[a] Image from @Ellem__ on Twitter.

I also have another section called **Can we Prove it?** (unfortunately lacking a cute picture to go along with it; Mikh was nice enough to edit one up for me, but I haven't been able to format it in a way I like), where I include some interesting, not really necessary, but nonetheless relevant proofs. So far, these two are my only two gimmicks, but I might add more in the future.

Can we Prove it? This is just a random proof I yoinked from our homeworks.

Proof. ($\Longrightarrow$) Let $x \in (A \cap B) \setminus C$. Then, $x \in (A \cap B)$ and $x \notin C$.
Since $x \in (A \cap B)$, $x \in A$ and $x \in B$.
Since $x \in A$ and $x \notin C$, $x \in (A \setminus C)$.
Since $x \in B$ and $x \notin C$, $x \in (B \setminus C)$.
Thus, $x \in (A \setminus C) \cap (B \setminus C)$.

($\Longleftarrow$) Let $x \in (A \setminus C) \cap (B \setminus C)$. Then, $x \in (A \setminus C)$ and $x \in (B \setminus C)$.
Since $x \in (A \setminus C)$, $x \in A$ and $x \notin C$.
Since $x \in (B \setminus C)$, $x \in B$ and $x \notin C$.
Since $x \in A$ and $x \in B$, $x \in (A \cap B)$.
Thus, $x \in (A \cap B) \setminus C$.

Since both sides of the conditional are true, it holds that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$. $\square$

Finally, there are blue boxes to indicate when instructions aren't obvious from the question itself, or if there are similar items that can be grouped together.

For items #7 to #12, we need to reevaluate our life decisions.

It's very important to note that this is a *work in progress!* I am human, and I will make mistakes, and I cannot finish doing all the exercises within the span of one day. If you spot anything wrong, please feel free to message me; I will correct it as soon as possible.

As a final note, these are not replacements for the modules/paying attention in class, these are supplements for them. I won't explain all the topics here, and I'll assume that you at least have read the basics, so don't treat these reviewers as your only source of information. Our teachers spends a lot of time on the handouts, they're really good! (except when they're wrong) With that, though, I think I've covered all pertinent points. Good luck, and happy studying!

3.1.1: Linear combinations

Some notes on notation: I'm not a big fan of the boldface lowercase letters representing matrices, particularly vectors, mostly because it's a little difficult to tell (and we love clarity here) so I'll be using the arrow on top instead to denote vectors.

For each set of vectors V and given vector $\vec{w}$, determine if $\vec{w}$ can be expressed as a linear combination of the vectors in V.

1 $V = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}; \qquad \vec{w} = \begin{pmatrix} 30 \\ 30 \end{pmatrix}$

Solution. We need to figure out if there exist two coefficients k_1 and k_2 such that $k_1\vec{v_1} + k_2\vec{v_2} = \vec{w}$. Expressing it in a linear equation like this, we can then solve for the coefficients using Gaussian elimination, like so

$$\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 30 \\ 30 \end{pmatrix} \iff \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 30 \\ 12 \end{pmatrix}.$$

This tells us that $k_2 = 12$, and substituting this into the first row gives us $k_1 = 6$. Thus, there exists a unique linear combination of vectors in V that we can express $\vec{w}$ as. ■

Nerd Interjection! The Gaussian elimination here is fairly straightforward since we're just dealing with two variables and two equations, so I won't bother mucking up the pages with all the steps. I might elaborate a little bit more later on, when we get to the sets with vectors in $\mathbb{R}^3$, but for these first three at least, it shouldn't be too hard to see how we got from one step to the next.

2 $V = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right\}; \qquad \vec{w} = \begin{pmatrix} -10 \\ 20 \end{pmatrix}$

Solution. We need to figure out if there exist two coefficients k_1 and k_2 such that $k_1\vec{v_1} + k_2\vec{v_2} = \vec{w}$. Expressing it in a linear equation like this, we can then solve for the coefficients using Gaussian elimination, like so:

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -10 \\ 20 \end{pmatrix} \iff \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -10 \\ 0 \end{pmatrix}.$$

This tells us that there are infinitely many solutions, and thus, infinitely many linear combinations of vectors in V that we can express $\vec{w}$ as. If we let $k_2 := r,\ r \in \mathbb{R}$, then $k_1 = 2r - 10$. ■

3 $V = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right\}; \qquad \vec{w} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Solution. We need to figure out if there exist two coefficients k_1 and k_2 such that $k_1\vec{v_1} + k_2\vec{v_2} = \vec{w}$. Expressing it in a linear equation like this, we can then solve for the coefficients using Gaussian elimination, like so:

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \iff \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This tells us that there is no solution, or that there is no possible linear combination of vectors in V that we can express $\vec{w}$ as. Wow! Three for three on the three cases of solutions! That's totally not intentional. ■

4 $V = \left\{ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} \right\}; \qquad \vec{w} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$

Solution. We need to figure out if there exist two coefficients k_1 and k_2 such that $k_1\vec{v_1} + k_2\vec{v_2} = \vec{w}$. Expressing it in a linear equation like this, we can then solve for the coefficients using Gaussian elimination, like so:

$$\begin{pmatrix} 2 & 5 \\ -1 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}.$$

By inspection, though, we can see that from the second row, $-k_1 = -2$, so $k_1 = 2$.[1] Substituting this into either of the other two rows gives us $k_2 = -1$. Thus, there exists a unique linear combination of vectors in V that we can express $\vec{w}$ as. ■

5 $V = \left\{ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} \right\}; \qquad \vec{w} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Solution. We need to figure out if there exist two coefficients k_1 and k_2 (I'm getting so bored of saying that) such that $k_1\vec{v_1} + k_2\vec{v_2} = \vec{w}$. Then,

$$\begin{pmatrix} 2 & 5 \\ -1 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

By inspection, we can see that from the second row, $-k_1 = 1$, so $k_1 = -1$. However, substituting this into the other two rows gives us different answers, namely $k_2 = 0.6$ and $k_2 = 0.5$. k_2 can't have two values, so this is a contradiction, implying that there is no solution, and that $\vec{w}$ cannot be expressed as a linear combination of vectors in V. ■

6 $V = \left\{ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}; \qquad \vec{w} = \begin{pmatrix} 1 \\ -8 \\ 12 \end{pmatrix}$

Solution. We need to figure out of there exist *three* (YIPPIE!) coefficients k_1, k_2, and k_3 such that $k_1\vec{v_1} + k_2\vec{v_2} + k_3\vec{v_3} = \vec{w}$. Expressing it in a linear equation like this, we can then solve for the coefficients using Gaussian elimination, like so (let's actually go through the Gaussian elimination for this one):

$$\begin{pmatrix} 2 & 5 & 3 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \\ 12 \end{pmatrix} \qquad \begin{matrix} 2R_2 + R_1 \implies R_1 \\ 3R_2 + R_3 \implies R_3 \end{matrix} \qquad \begin{pmatrix} 0 & 5 & 5 \\ -1 & 0 & 1 \\ 0 & 4 & 4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -15 \\ -8 \\ -12 \end{pmatrix}$$

$$\begin{matrix} \frac{1}{5}R_1 \implies R_1 \\ \frac{1}{4}R_3 \implies R_3 \end{matrix} \qquad \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ -3 \end{pmatrix}.$$

We see here that there are two identical rows, which should already clue us in to the fact that there are infinitely many solutions. Thus, if we choose $k_3 := r,\ r \in \mathbb{R}$, then $k_2 = -r - 3$ and $k_1 = r + 8$. ■

[1] LOL, I said I'd be more detailed with the Gaussian elimination but we don't even have to do it for these first two items. Love it!

Each of the following sets of vectors $\{\vec{v_1}, \vec{v_2}\}$ below is special in that any vector $(x, y) \in \mathbb{R}^2$ can be uniquely expressed as a linear combination of $\vec{v_1}$ and $\vec{v_2}$. Find the *coefficients* k_1 and k_2 in terms of x and y such that

$$k_1\vec{v_1} + k_2\vec{v_2} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

7 $\vec{v_1} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \qquad \vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution. Let (x, y) be any arbitrary vector in $\mathbb{R}^2$. Setting up the linear equation gives us

$$\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad R_1 - R_2 \implies R_2 \qquad \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} x \\ x-y \end{pmatrix},$$

so $k_1 = x - y$. Substituting this into the first row gives us $k_2 = 4y - 3x$. ■

8 $\vec{v_1} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}, \qquad \vec{v_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Solution. Let (x, y) be any arbitrary vector in $\mathbb{R}^2$. Setting up the linear equation gives us

$$\begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad R_1 - 3R_2 \implies R_2 \qquad \begin{pmatrix} 7 & 3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} x \\ x-3y \end{pmatrix},$$

so $k_1 = x - 3y$. Substituting this into the first row gives us $k_2 = 7y - 2x$. ■

9 $\vec{v_1} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \qquad \vec{v_2} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

Solution. Let (x, y) be any arbitrary vector in $\mathbb{R}^2$. Setting up the linear equation gives us

$$\begin{pmatrix} 3 & 5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad R_1 + 3R_2 \implies R_1 \qquad \begin{pmatrix} 0 & 11 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} x+3y \\ y \end{pmatrix},$$

so $k_2 = \frac{1}{11}(x + 3y)$. Substituting this into the first row gives us $k_1 = \frac{1}{11}(2x - 5y)$. ■

10 $\vec{v_1} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}, \qquad \vec{v_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Solution. Let (x, y) be any arbitrary vector in $\mathbb{R}^2$. Setting up the linear equation gives us

$$\begin{pmatrix} 5 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad 5R_2 + R_1 \implies R_2 \qquad \begin{pmatrix} 5 & -2 \\ 0 & 13 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} x \\ x+5y \end{pmatrix},$$

so $k_2 = \frac{1}{13}(x + 5y)$. Substituting this into the first row gives us $k_1 = \frac{1}{13}(3x + 2y)$.[2] ■

[2] Yes, these answers with fractions of *prime numbers larger than 10* are correct, I double-checked.

3.1.2: Norm and dot product

We learned about this stuff in calculus so it shouldn't be that new, but to be honest I've forgotten everything since vectors were an afterthought in 30.24, so here's a refresher, I guess.

For each of the following vectors $\vec{v}$ find a unit vector $\hat{v}$ pointing in the same direction as $\vec{v}$.

Nerd Interjection! To find a unit vector pointing in the same direction, we just divide the vector by its norm. Recall that the norm of a vector in $\mathbb{R}^n$ is given by

$$\|\vec{v}\| = \sqrt{v_1^2, + v_2^2 + \cdots + v_n^2}.$$

11 $\vec{v} = (4, 3)$ [3]

Solution. The norm of $\vec{v}$ is $\sqrt{4^2 + 3^2} = 5$. Thus, $\hat{v} = \left(\frac{4}{5}, \frac{3}{5}\right)$. ■

12 $\vec{v} = (0, 1)$

Solution. The norm of $\vec{v}$ is $\sqrt{0^2 + 1^2} = 1$. Thus, $\vec{v}$ is already a unit vector, and $\hat{v} = \vec{v} = (0, 1)$. ■

13 $\vec{v} = (5, \sqrt{5}, 0)$

Solution. The norm of $\vec{v}$ is $\sqrt{5^2 + \sqrt{5}^2 + 0^2} = \sqrt{30}$. Thus, $\hat{v} = \left(\frac{5}{\sqrt{30}}, \sqrt{\frac{1}{6}}, 0\right)$. ■

14 $\vec{v} = (-1, 4, 1)$

Solution. The norm of $\vec{v}$ is $\sqrt{(-1)^2 + 4^2 + 1^2} = 3\sqrt{2}$. Thus, $\hat{v} = \left(-\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)$. ■

15 $\vec{v} = (0, 2, 1, -1)$

Solution. The norm of $\vec{v}$ is $\sqrt{0^2 + 2^2 + 1^2 + (-1)^2} = \sqrt{6}$. Thus, $\hat{v} = \left(0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$. ■

16 $\vec{v} = (1, 1, 2, 2)$

Solution. The norm of $\vec{v}$ is $\sqrt{1^2 + 1^2 + 2^2 + 2^2} = \sqrt{10}$. Thus, $\hat{v} = \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$. ■

Nerd Interjection! Some formulas for vectors in $\mathbb{R}^n$ (needed for the next page):

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$
$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$$

[3] *Norm*ally the lack of internal consistency with how vectors are represented would be killing me but I actually like the diversification of math notation. She says, after proclaiming a bold dislike (pun not intended) for boldface letters representing vectors. Maybe I need to do some self-reflection.

For each of the following vectors in items #17 to #20 $\overrightarrow{u}$ and $\overrightarrow{v}$, find the distance between the two of them, the dot product of the two of them, and the angle between the two of them.

17 $\overrightarrow{u} = (1, -1)$, $\overrightarrow{v} = (-1, 1)$

Solution. The distance between the vectors is $\sqrt{\left(1-(-1)\right)^2 + (-1-1)^2} = 2\sqrt{2}$.

The dot product of the vectors is $1(-1) + (-1)1 = -2$.

The angle between the vectors is $\cos^{-1}\left(\frac{-2}{\sqrt{2}\cdot\sqrt{2}}\right) = \cos^{-1}(-1) = \pi$ rad or $180°$. ■

18 $\overrightarrow{u} = (0, 1, -1, 2)$, $\overrightarrow{v} = (1, 1, 2, 2)$

Solution. The distance between the vectors is $\sqrt{(0-1)^2 + (1-1)^2 + (-1-2)^2 + (2-2)^2} = \sqrt{10}$.

The dot product of the vectors is $0(1) + 1(1) + (-1)2 + 2(2) = 3$.

The angle between the vectors is $\cos^{-1}\left(\frac{3}{\sqrt{6}\cdot\sqrt{10}}\right) = \cos^{-1}\left(\frac{3}{\sqrt{60}}\right) \approx 1.173$ rad or $67.213°$.[4] ■

19 $\overrightarrow{u} = (0, 1, 0, 1)$, $\overrightarrow{v} = (3, 3, 3, 3)$

Solution. The distance between the vectors is $\sqrt{(0-3)^2 + (1-3)^2 + (0-3)^2 + (1-3)^2} = \sqrt{26}$.

The dot product of the vectors is $0(3) + 1(3) + 0(3) + 1(3) = 6$.

The angle between the vectors is $\cos^{-1}\left(\frac{6}{\sqrt{2}\cdot 6}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ rad or $45°$. ■

20 $\overrightarrow{u} = (1, \sqrt{2}, -1, \sqrt{2})$, $\overrightarrow{v} = (1, -\frac{1}{\sqrt{2}}, 1, -\frac{1}{\sqrt{2}})$

Solution. The distance between the vectors is

$$\sqrt{(1-1)^2 + \left(\sqrt{2} - \left(-\frac{1}{\sqrt{2}}\right)\right)^2 + (-1-1)^2 + \left(\sqrt{2} - \left(-\frac{1}{\sqrt{2}}\right)\right)^2} = \sqrt{13}.$$

The dot product of the vectors is $1(1) + \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + (-1)1 + \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 2$.

The angle between the vectors is $\cos^{-1}\left(\frac{2}{\sqrt{6}\cdot\sqrt{3}}\right) = \cos^{-1}\left(\frac{2}{3\sqrt{2}}\right) \approx 1.080$ rad or $61.874°$. ■

21 Consider the vector $\overrightarrow{v} = (-1, 3, 0, 4)$. Find a vector $\overrightarrow{u}$ which is twice as long as $\overrightarrow{v}$ and is pointing in the opposite direction.

Solution. We can start by finding the vector that is pointing in the opposite direction as $\overrightarrow{v}$ but with the same length, which we can get by negating $\overrightarrow{v}$. Then, we just multiply this vector by 2 to make its magnitude twice as much. Thus, the vector we want is $\overrightarrow{u} = (2, -6, 0, -8)$.

We can verify this answer by checking the norm of both vectors. The norm of $\overrightarrow{v}$ is $\sqrt{(-1)^2 + 3^2 + 0^2 + 4^2} = \sqrt{26}$, while the norm of $\overrightarrow{u}$ is $\sqrt{2^2 + (-6)^2 + 0^2 + (-8)^2} = 2\sqrt{26}$, which is indeed twice the norm of $\overrightarrow{v}$. ■

[4] Cheeky little Aldrich and Cisco making the vectors 4D so we can't just graph them in 2D or 3D space and visually check our answers. "Just use 3B1B's method for visualizing quaternions!" Be quiet, Thanie.

3.1.3: Standard matrices of linear transformations

Oh no! Drawings! Fuck! Even if it's kind of a pain, I'd much rather do all the graphs in LaTeX than use some external graphing software and take screenshots.[5] *I'm pretty familiar with tikz anyway (waw). Also, maybe the boldface font for vector notation isn't that bad. Let's try it for a little bit.*

For each of the following linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$, sketch and label the following six vectors. Sketch all the premiages in one axis and then all the images in another axis.
Preimages: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2$; *Images:* $T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_1 + \mathbf{e}_2)$

22 $T(x, y) = (x, -y)$

Solution. Solving for the vectors, we have $T(\mathbf{e}_1) = (1, 0)$, $T(\mathbf{e}_2) = (0, -1)$, and $T(\mathbf{e}_1 + \mathbf{e}_2) = (1, -1)$. ■

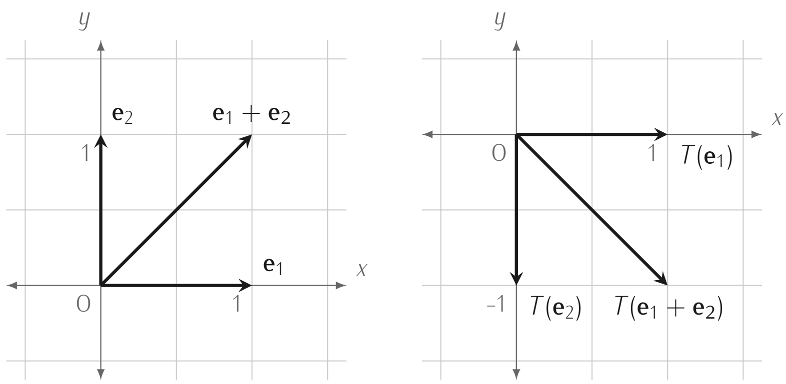


Figure 1: The function reflects vectors across the x-axis.

23 $T(x, y) = (-x, y)$

Solution. Solving for the vectors, we have $T(\mathbf{e}_1) = (-1, 0)$, $T(\mathbf{e}_2) = (0, 1)$, and $T(\mathbf{e}_1 + \mathbf{e}_2) = (-1, 1)$ ■

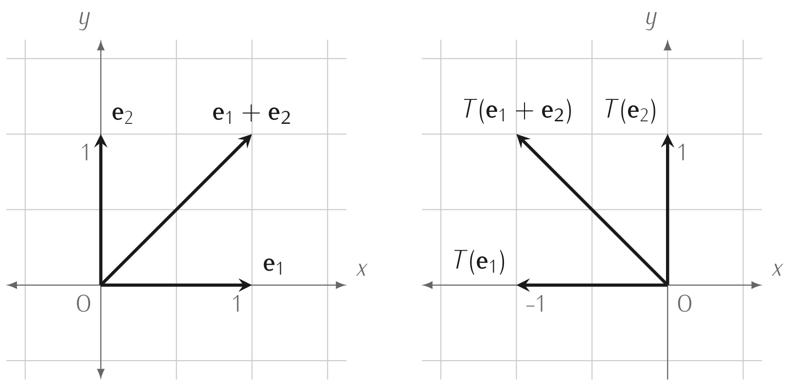


Figure 2: The function reflects vectors across the y-axis.

Nerd Interjection! Those two were really easy! Though, to be honest, most of these are fairly straightforward. To be even more honest, I'm just padding for space because I hate having massive white empty spaces. But also, these two are unique in that they are one of the fundamental transforms you can do: reflections! In fact, they're so fundamental that they're described in the handouts so why even have these problems as exercises SMH.

[5] Since we're mostly dealing with values of 1, graphs will be on a scale of one gridline is to $\frac{1}{2}$ units, just so the vectors will be more visible.

24 $T(x, y) = (x, x + y)$

Solution. Solving for the vectors, we have $T(\mathbf{e}_1) = (1, 1)$, $T(\mathbf{e}_2) = (0, 1)$, and $T(\mathbf{e}_1 + \mathbf{e}_2) = (1, 2)$. ∎

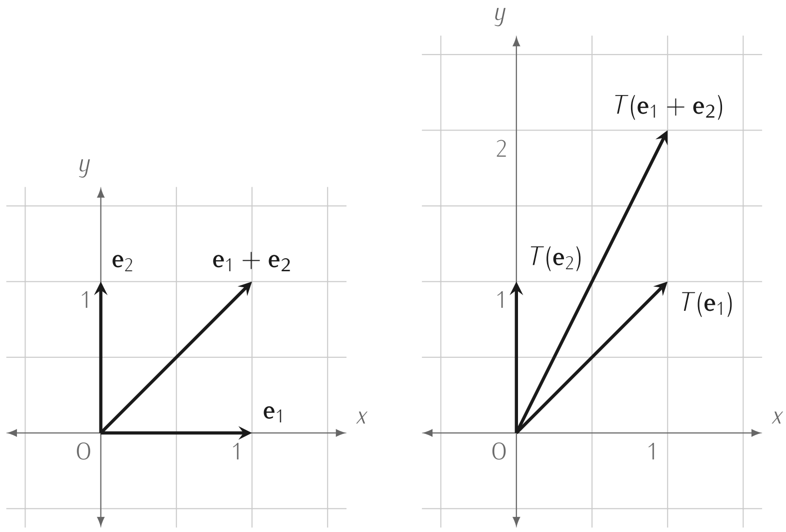


Figure 3: The function pulls or *shears* vectors in the y direction by a factor of 1 (or in other words, upwards).

25 $T(x, y) = (x + y, y)$

Solution. Solving for the vectors, we have $T(\mathbf{e}_1) = (1, 0)$, $T(\mathbf{e}_2) = (1, 1)$, and $T(\mathbf{e}_1 + \mathbf{e}_2) = (2, 1)$. ∎

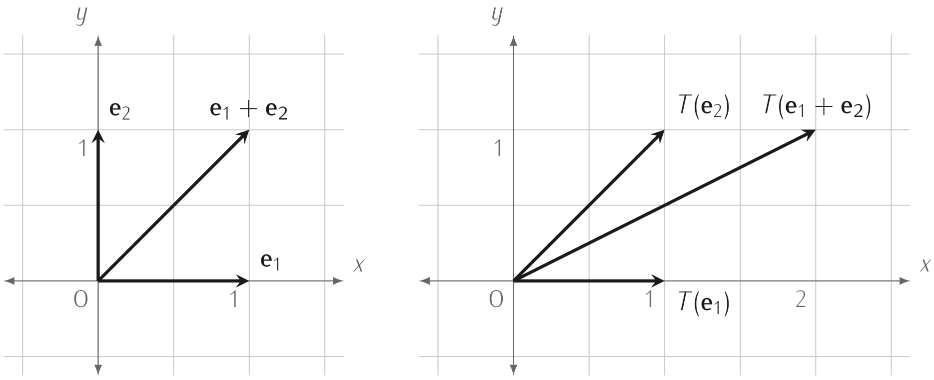


Figure 4: The function shears vectors in the x direction by a factor of 1 (or in other words, to the right).

Nerd Interjection! In preparation for transformations on the unit square later on, think about what it would look like if the three vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_1 + \mathbf{e}_2$ formed a square. You can imagine two lines connecting the arrowheads of the two basis vectors to the arrowhead of their sum. What would the square look like after these transformations? In the first two, it would still be a square, but in these previous two, clearly that isn't the case.

26 $T(x, y) = (-2x, 3y)$

Solution. Solving for the vectors, we have $T(\mathbf{e}_1) = (-2, 0)$, $T(\mathbf{e}_2) = (0, 3)$, and $T(\mathbf{e}_1 + \mathbf{e}_2) = (-2, 3)$. ∎

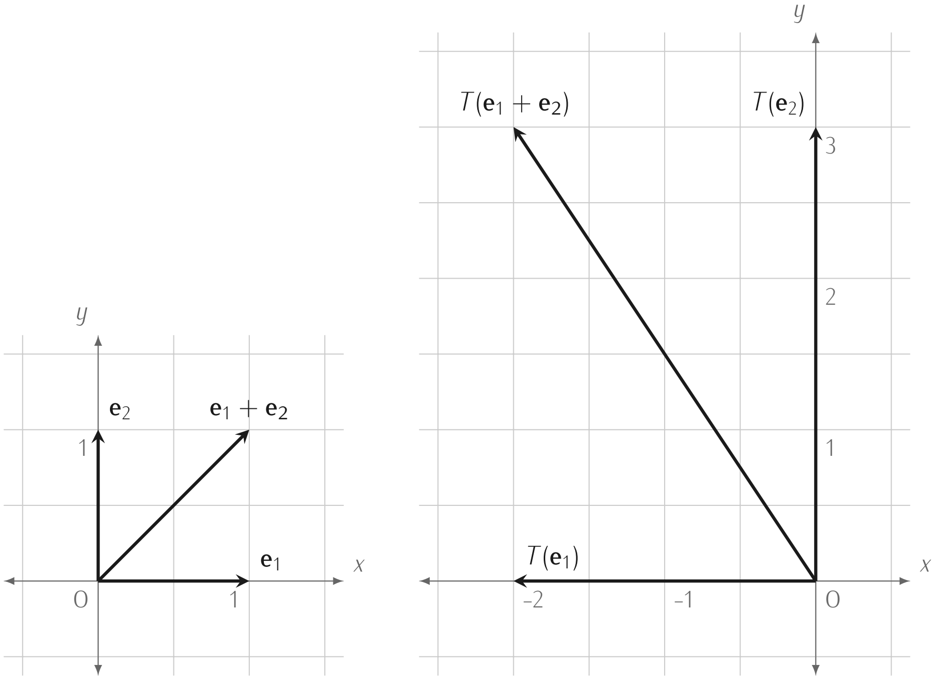


Figure 5: The function reflects vectors across the y-axis and scales the y-coordinate to be larger than the x-coordinate.

27 $T(x, y) = (x, 0)$

Solution. Solving for the vectors, we have $T(\mathbf{e}_1) = (1, 0)$, $T(\mathbf{e}_2) = (0, 0)$, and $T(\mathbf{e}_1 + \mathbf{e}_2) = (1, 0)$. ∎

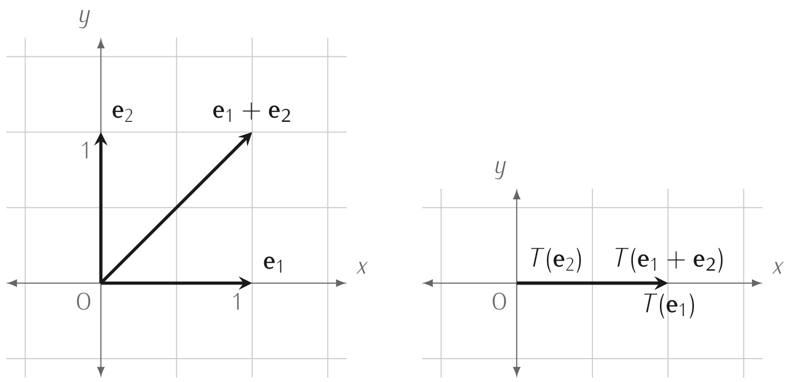


Figure 6: The function annihilates the y-coordinate of the vectors (that's the term for turning a vector into $\vec{0}$).

State the domain and codomain of each of the following linear transformations T, then find the standard matrix for T.

28 $T(x, y) = (x + 2y, x - 2y)$

Solution. By inspection, the transformation takes in a vector with two elements and outputs a vector with two elements. Thus, its domain and codomain are both $\mathbb{R}^2$.

To find the standard matrix of T, we need to apply T to the standard basis vectors of $\mathbb{R}^2$ and combine them into one 2×2 matrix. Thus,

$$\left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \middle| T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \right] = \begin{bmatrix}1 & 2\\1 & -2\end{bmatrix}$$

is the standard matrix for T. ∎

Nerd Interjection! "Applying" the function just means inputting the basis vectors and taking the output. So, in the previous example, just to be explicit, we have

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix}\text{ and }T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}2\\-2\end{bmatrix},$$

which we then combine simply by putting them next to each other to get our standard matrix.

29 $T(x, y) = (2x - 3y, x - y, y - 4x)$

Solution. By inspection, the transformation takes in a vector with two elements and outputs a vector with three elements. Thus, its domain is $\mathbb{R}^2$ and its codomain is $\mathbb{R}^3$.

To find the standard matrix of T, we need to apply T to the standard basis vectors of $\mathbb{R}^2$ and combine them into one 3×2 matrix[6]. Thus,

$$\left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\middle|T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right]=\begin{bmatrix}2 & -3\\1 & -1\\-4 & 1\end{bmatrix}$$

is the standard matrix for T. ■

30 $T(x, y, z) = (x + y, x - y, z - x)$

Solution. By inspection, the transformation takes in a vector with three elements and outputs a vector with three elements. Thus, its domain and codomain are both $\mathbb{R}^3$.

To find the standard matrix of T, we need to apply T to the standard basis vectors of $\mathbb{R}^3$ and combine them into one 3×3 matrix. Thus,

$$\left[T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)\middle|T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)\middle|T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right]=\begin{bmatrix}1 & 1 & 0\\1 & -1 & 0\\-1 & 0 & 1\end{bmatrix}$$

is the standard matrix for T. ■

31 $T(x, y) = (5x + y, 0, 4x - 5y)$

Solution. By inspection, the transformation takes in a vector with two elements and outputs a vector with three elements. Thus, its domain is $\mathbb{R}^2$ and its codomain is $\mathbb{R}^3$.

To find the standard matrix of T, we need to apply T to the standard basis vectors of $\mathbb{R}^2$ and combine them into one 3×2 matrix. Thus,

$$\left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\middle|T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right]=\begin{bmatrix}5 & 1\\0 & 0\\4 & -5\end{bmatrix}$$

is the standard matrix for T. ■

32 $T(x, y, z) = (3x - 2z, 2y - z)$

Solution. By inspection, the transformation takes in a vector with three elements and outputs a vector with two elements. Thus, its domain is $\mathbb{R}^3$ and its codomain is $\mathbb{R}^2$.

To find the standard matrix of T, we need to apply T to the standard basis vectors of $\mathbb{R}^3$ and combine them into one 2×3 matrix. Thus,

$$\left[T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)\middle|T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)\middle|T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right]=\begin{bmatrix}3 & 0 & -2\\0 & 2 & -1\end{bmatrix}$$

[6] Matrices are always notated in row by column. Why? Probably has something to do with how you multiply them.

is the standard matrix for T. ■

33 $T(x_1, x_2, x_3, x_4) = (x_4, x_3, x_1 + x_2, x_1)$

Solution. By inspection, the transformation takes in a vector with four elements and outputs a vector with four elements. Thus, its domain and codomain are $\mathbb{R}^4$.

To find the standard matrix of T, we need to apply T to the standard basis vectors of $\mathbb{R}^4$ and combine them into one 4×4 matrix. Thus,

$$\left[T\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) \middle| T\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) \middle| T\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) \middle| T\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) \right] = \begin{bmatrix}0&0&0&1\\0&0&1&0\\1&1&0&0\\1&0&0&0\end{bmatrix}$$

is the standard matrix for T. ■

34 $T(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ [This is **vector annihilation**.]

Solution. We *could* do the whole she-bang (who she bangin? HOHOHO) that we did earlier, but since we recognize this as turning all the elements to zero, it should be pretty obvious what matrix we'd have to multiply a vector by to do that, which is the 4×4 zero matrix. ■

35 $T(x, y) = (x + 2y, x - 2y, x, y)$

Solution. By inspection, the transformation takes in a vector with two elements and outputs a vector with three elements. Thus, its domain is $\mathbb{R}^2$ and its codomain is $\mathbb{R}^4$.

To find the standard matrix of T, we need to apply T to the standard basis vectors of $\mathbb{R}^2$ and combine them into one 4×2 matrix. Thus,

$$\left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \middle| T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \right] = \begin{bmatrix}1&2\\1&-2\\1&0\\0&1\end{bmatrix}$$

is the standard matrix of T. ■

Use a counterexample to show that the following functions are not linear transformations.

Nerd Interjection! As a refresher, these are the properties of linear transformations:

$$T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v}) \quad \text{and} \quad T_A(k\mathbf{v}) = kT_A(\mathbf{v}).$$

Thus, if we can prove that the following linear transformations break one of the properties, that will be sufficient to show that they are nonlinear.

36 $T(x, y) = (x^2, y)$

Proof. Let $\mathbf{v} = \mathbf{u} := (1, 1)$. Then, substituing $\mathbf{u} + \mathbf{v}$ into T gives us $T(\mathbf{u} + \mathbf{v}) = T(2, 2) = (4, 2)$. However, if we instead transform them individually and then get their sum, we have $T(\mathbf{u}) + T(\mathbf{v}) = (1, 1) + (1, 1) = (2, 2)$. Thus, the transformation is nonlinear. □

37 $T(x, y) = (x, y, xy)$

Proof. Let $\mathbf{v} = \mathbf{u} := (1, 1)$. Then, substituting $\mathbf{u}+\mathbf{v}$ into T gives us $T(\mathbf{u}+\mathbf{v}) = T(2, 2) = (2, 2, 4)$. However, if we instead transform them individually and then get their sum, we have $T(\mathbf{u}) + T(\mathbf{v}) = (1, 1, 1) + (1, 1, 1) = (2, 2, 2)$. Thus, the transformation is nonlinear. □

3.2.1: Transformations on the unit square

More drawings?! It's like they want me to suffer. My copy-pasting and Ctrl-F'ing skills can only take me so far.

Given the following matrices A and transformations T, sketch the effect of the corresponding linear transformation $T_A(x, y)$ on the unit square.

38 $A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$

Solution. We can start by figuring out how A changes the standard basis vectors, then form the square from there. Multiplying A to $\mathbf{e}_1$ and $\mathbf{e}_2$,

$$A\mathbf{e}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad A\mathbf{e}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

which now lets us draw the unit square. ■

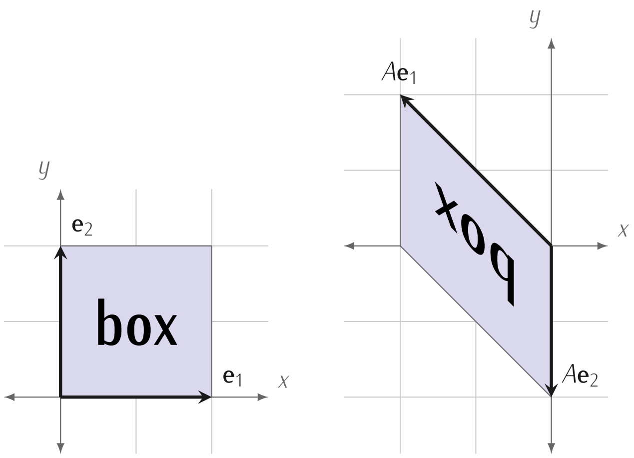


Figure 7: The function inverts the square to the negative quadrant then shears it upward in the y-direction.

39 $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

Solution. We can start by figuring out how A changes the standard basis vectors, then form the square from there. Multiplying A to $\mathbf{e}_1$ and $\mathbf{e}_2$,

$$A\mathbf{e}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \text{and} \quad A\mathbf{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which now lets us draw the unit square (see next page for figure). ■

40 $T(x, y) = (x + y, y)$

Solution. We can start by figuring out how T changes the standard basis vectors, then form the square from there. Inputting $\mathbf{e}_1$ and $\mathbf{e}_2$ into T,

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which now lets us draw the unit square (see next page for figure). ■

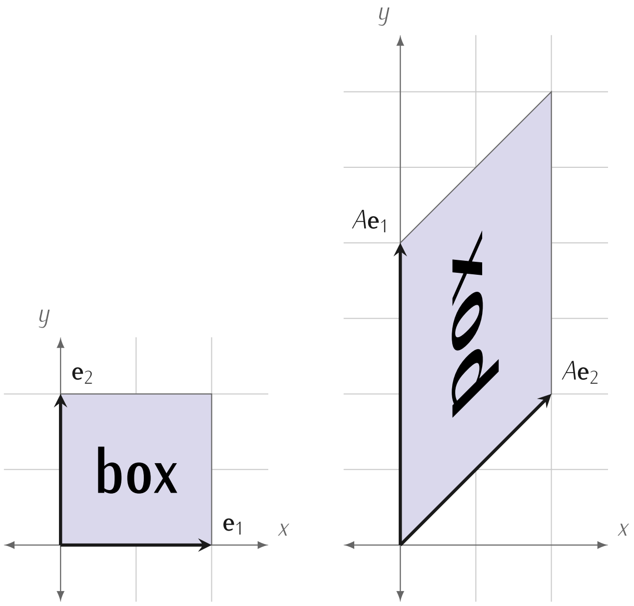


Figure 8: (#39) The function reflects the square then stretches and shears it upwards in the y-direction.

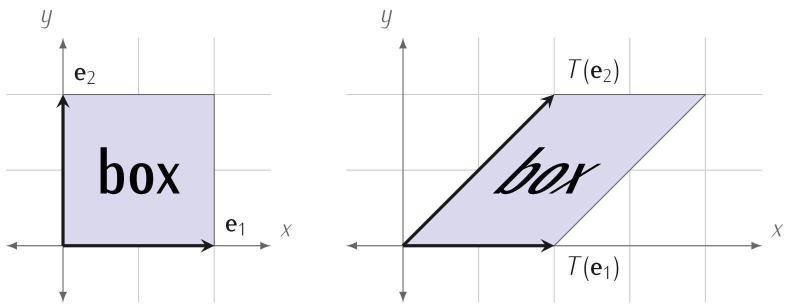


Figure 9: (#40) The function shears the unit square to the right in the x-direction.

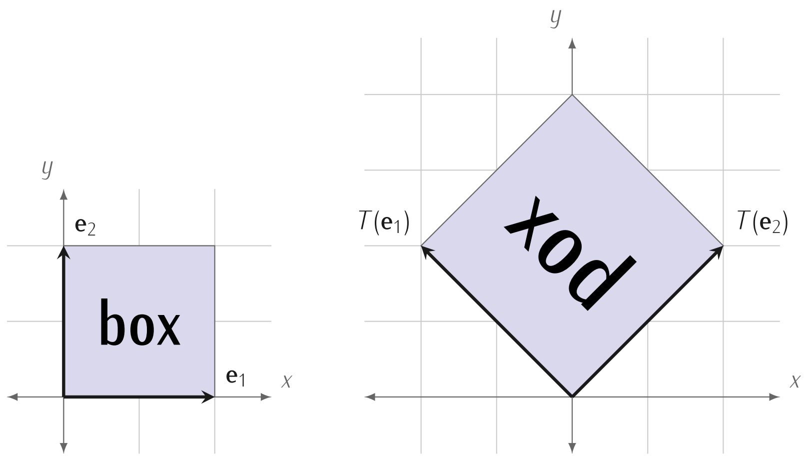


Figure 10: (#41) The function reflects the square across the y-axis then rotates the it by 45°, expanding it as it goes.

41 $T(x, y) = (y - x, y + x)$

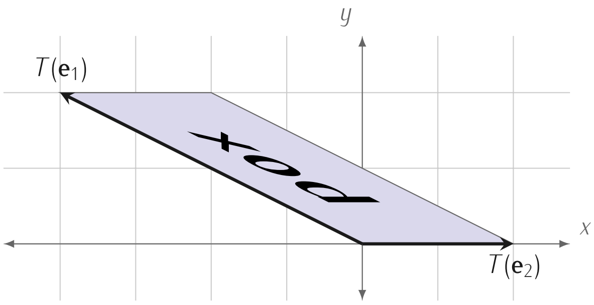
Solution. We can start by figuring out how T changes the standard basis vectors, then form the square from there. Inputting $\mathbf{e}_1$ and $\mathbf{e}_2$ into T,

$$T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which now lets us draw the unit square (see previous page for figure). ■

Given how the following linear transformations act on the unit square, identify the standard matrix A associated with this transformation, and also give an explicit formula for the output of $T_A(x, y)$ in terms of (x, y).

42



Solution. The linear transformation turns $\mathbf{e}_1$ into $(-2, 1)$ and $\mathbf{e}_2$ into $(1, 0)$. Putting these together gives us the standard matrix of the transformation, which is

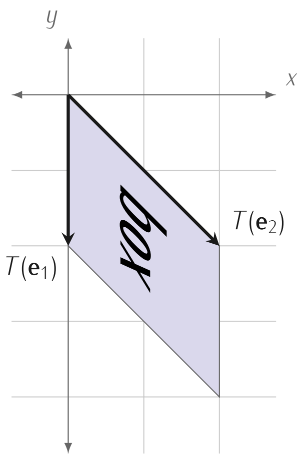
$$A = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}.$$

To find $T_A(x, y)$ in terms of x and y, we simply multiply A by an arbitrary given vector $\mathbf{v} = (x, y)$ and see what the result is. Doing so gives us

$$A\mathbf{v} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y - 2x \\ x \end{bmatrix},$$

and thus, $T_A(x, y) = (y - 2x, x)$ is the linear transformation that gives the result above. ■

43



Solution. The linear transformation turns $\mathbf{e}_1$ into $(0, -1)$ and $\mathbf{e}_2$ into $(1, -1)$. Putting these together gives us the standard matrix of the transformation, which is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

To find $T_A(x, y)$ in terms of x and y, we simply multiply A by an arbitrary given vector $\mathbf{v} = (x, y)$ and see what the result is. Doing so gives us

$$A\mathbf{v} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x - y \end{bmatrix},$$

and thus, $T_A(x, y) = (y, -x - y)$ is the linear transformation that gives the result above. ■

For items #44 to #48, all rotations mentioned are counterclockwise rotations with respect to the origin.

44 Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be perpendicular 2D vectors.[7] Use the rotation matrix to verify that they will still be perpendicular even if you rotate both by the same amount θ.

Proof. Recall that if two vectors are perpendicular, their dot product will be equal to 0. Then, let $\overrightarrow{u} := (u_1, u_2)$ and $\overrightarrow{v} := (v_1, v_2)$, such that $\overrightarrow{u} \cdot \overrightarrow{v} = 0$ (that is, $u_1v_1 + u_2v_2 = 0$). To rotate these by some angle θ, we multiply them by the rotation matrix, which is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Multiplying this to both $\overrightarrow{u}$ and $\overrightarrow{v}$ gives us

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1\cos\theta - u_2\sin\theta \\ u_1\sin\theta + u_2\cos\theta \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1\cos\theta - v_2\sin\theta \\ v_1\sin\theta + v_2\cos\theta \end{bmatrix}. \tag{2}$$

Finding the dot product of (1) and (2) (I'm doing this in two lines because they won't fit in one),

$$\begin{aligned}
(u_1\cos\theta - u_2\sin\theta)(v_1\cos\theta - v_2\sin\theta) &= u_1v_1\cos^2\theta + u_2v_2\sin^2\theta - u_1v_2\sin\theta\cos\theta - u_2v_1\sin\theta\cos\theta \\
&= u_1v_1\cos^2\theta + u_2v_2\sin^2\theta - \sin\theta\cos\theta\,(u_1v_2 + u_2v_1) \qquad (3) \\
(u_1\sin\theta + u_2\cos\theta)(v_1\sin\theta + v_2\cos\theta) &= u_1v_1\sin^2\theta + u_2v_2\cos^2\theta + u_1v_2\sin\theta\cos\theta + u_2v_1\sin\theta\cos\theta \\
&= u_1v_1\sin^2\theta + u_2v_2\cos^2\theta + \sin\theta\cos\theta\,(u_1v_2 + u_2v_1). \qquad (4)
\end{aligned}$$

Adding (3) and (4) together to complete the dot product,

$$\begin{aligned}
&= \sin^2\theta\,(u_1v_1 + u_2v_2) + \cos^2\theta\,(u_1v_1 + u_2v_2) - \sin\theta\cos\theta\,(u_1v_2 + u_2v_1) + \sin\theta\cos\theta\,(u_1v_2 + u_2v_1) \\
&= \sin^2\theta\,(0) + \cos^2\theta\,(0) = 0.
\end{aligned}$$

Thus, the dot product of the two rotated vectors is 0, which means they are perpendicular. □

45 Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be 2D vectors.[8] Use the rotation matrix to verify that their dot product is preserved even if you rotate both by the same amount θ.

Proof. We can extract a portion of the solution in item #44 after we find the dot product of the two rotated vectors and simply ignore the condition in that proof that $u_1v_1 + u_2v_2 = 0$. Instead, we simply consider a generalized dot product of $\overrightarrow{u} \cdot \overrightarrow{v} = u_1v_1 + u_2v_2$. Then

$$\begin{aligned}
\overrightarrow{u}_{\text{rot}} \cdot \overrightarrow{v}_{\text{rot}} &= \sin^2\theta\,(u_1v_1 + u_2v_2) + \cos^2\theta\,(u_1v_1 + u_2v_2) - \sin\theta\cos\theta\,(u_1v_2 + u_2v_1) + \sin\theta\cos\theta\,(u_1v_2 + u_2v_1) \\
&= (u_1v_1 + u_2v_2)(\sin^2\theta + \cos^2\theta).
\end{aligned}$$

Since $\sin^2\theta + \cos^2\theta = 1$, this simplifies to $u_1v_1 + u_2v_2$, which is the original dot product, $\overrightarrow{u} \cdot \overrightarrow{v}$. Thus, the dot product is preserved even when you rotate the vectors by some value θ. □

[7] Oh, so now they use the arrow notation.

[8] The original item said "perpendicular 2D vectors", but I removed it because that would just be the same as item #44. This way, it's a more relevant question of showing that the dot product is the same after rotation. And, it's a surprise tool that will help us later!

46 Let $\overrightarrow{u}$ be a 2D vector. Use the rotation matrix to verify that its length is preserved even if you rotate it by some angle θ.

Proof. Let $\overrightarrow{u} := (u_1, u_2)$, with a length of $\|\overrightarrow{u}\| = \sqrt{u_1^2 + u_2^2}$. Multiplying the rotation matrix by $\overrightarrow{u}$,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1\cos\theta - u_2\cos\theta \\ u_1\sin\theta + u_2\cos\theta \end{bmatrix}.$$

Finding the magnitude of this rotated vector gives us

$$\begin{aligned} \|\overrightarrow{u}_{\text{rot}}\| &= \sqrt{(u_1\cos\theta - u_2\sin\theta)^2 + (u_1\sin\theta + u_2\cos\theta)^2} \\ &= \sqrt{u_1^2\cos^2\theta + u_2^2\sin^2\theta - 2u_1u_2\sin\theta\cos\theta + u_1^2\sin^2\theta + u_2^2\cos^2\theta + 2u_1u_2\sin\theta\cos\theta} \\ &= \sqrt{u_1^2(\sin^2\theta + \cos^2\theta) + u_2^2(\sin^2\theta + \cos^2\theta)} = \sqrt{u_1^2 + u_2^2}, \end{aligned}$$

since $\sin^2\theta + \cos^2\theta = 1$. Thus, the magnitude of $\overrightarrow{u}$ is maintained even when it is rotated. □

47 Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be 2D vectors. Verify that the angle α between them is preserved even if you rotate both by the same amount θ.

Proof. Let $\overrightarrow{u} := (u_1, u_2)$ and $\overrightarrow{v} := (v_1, v_2)$. Then, the angle α between them is given by

$$\alpha = \cos^{-1}\left(\frac{\overrightarrow{u}\cdot\overrightarrow{v}}{\|\overrightarrow{u}\|\|\overrightarrow{v}\|}\right).$$

This proof actually becomes quite trivial because of all the previous items. We know that the dot product of two vectors stays the same even after rotating, and we know that the magnitude of a vector stays the same even after rotating. Thus, the fraction will not be changed at all, and the angle α between them will be preserved. □

48 Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be 2D vectors. Verify that projecting $\overrightarrow{u}$ onto $\overrightarrow{v}$ and then rotating the projection by θ gives the same result as rotating $\overrightarrow{u}$ and $\overrightarrow{v}$ by θ and then projecting the rotated $\overrightarrow{u}$ onto the rotated $\overrightarrow{v}$.

49 Show that:

a reflecting across the line $y = x$

b and then reflectiong across the line $x = 0$

produces the same result as a 90° counterclockwise rotation.

50 Show that:

a reflecting across the line $y = x$

b and then reflectiong across the line $y = -x$

produces the same result as a 180° rotation.

51 Show that:

a rotating 90° counterclockwise

b and then reflecting across the line $x = 0$

gives the same result as

a reflecting across the line $x = 0$

b and then rotating 90° **clockwise**.

Nerd Interjection! I've marked the next few items as difficult (first ones in the whole reviewer; did you notice?), but after the second LT, I don't know if I trust Aldrich enough to not give them as test items. They *are* considerably difficult, but technically speaking we do have all the tools we need to solve them. They just need some clever reasoning and intuition. I'll try my best to explain them, *just in case*. If you are going to study these, I'd advise looking at items #53 to #55 more than you do #52.

52 Use rotation matrices to prove the following trigonometric identities:

a $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Proof. Let A be the matrix for rotating a vector by an angle α and B be the matrix for rotating by an angle β. Finally, let X be the matrix for rotating by their sum, $\alpha+\beta$. Then,

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix};\ B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix};\ X = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}.$$

We know that rotation by α followed by rotation by β (or vice versa) is the same as rotating by their sum. This in turn means, that if we multiply A and B by any given vector $\mathbf{v}$ in $\mathbb{R}^2$, then the effect on $\mathbf{v}$ should be the same as if we had just multiplied X by it instead. Thus, $BA\mathbf{v} = X\mathbf{v}$, so $BA = X$. Performing the matrix multiplication, we have

$$\begin{array}{cc} & \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \\ \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} & \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} \end{array}.$$

Setting this to be equal to X and rearranging some terms,

$$\begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

gives us $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, which proves the identity. □

b $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Proof. Taking A from the previous item, let B^- be the matrix for rotating an angle by $-\beta$ (in other words, β in the other direction, clockwise). Finally, let X^- be the matrix for rotating by their sum, $\alpha-\beta$. Then, (Recall that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$.)

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix};\ B^- = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix};\ X^- = \begin{bmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{bmatrix}.$$

We know that rotation by α followed by rotation by $-\beta$ (or vice versa) is the same as rotating by their sum. Thus, by similar reasoning as **a** above, $B^-A = X^-$. Performing the matrix multiplication, we have

$$\begin{array}{cc} & \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \\ \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} & \begin{bmatrix} \cos\alpha\cos\beta + \sin\alpha\sin\beta & -\sin\alpha\cos\beta + \cos\alpha\sin\beta \\ -\cos\alpha\sin\beta + \sin\alpha\cos\beta & \sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} \end{array}.$$

Setting this to be equal to X and rearranging some terms,

$$\begin{bmatrix} \cos\alpha\cos\beta + \sin\alpha\sin\beta & -\sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin\alpha\cos\beta - \cos\alpha\sin\beta & \sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{bmatrix}$$

gives us $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$, which proves the identity. □

c $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Proof. From **a**, we have

$$\begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix},$$

so $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, which proves the identity. □

d $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Proof. From **b**, we have

$$\begin{bmatrix} \cos\alpha\cos\beta + \sin\alpha\sin\beta & -\sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin\alpha\cos\beta - \cos\alpha\sin\beta & \sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{bmatrix},$$

so $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, which proves the identity. □

53 Let $T(x, y)$ be a transformation that takes in a point (x, y) and reflects it across the line that is a θ counterclockwise rotation away from the positive x-axis. Find the standard matrix for T. (This is called the reflection matrix.)

54 Use the reflection matrix to verify that every reflection is its own inverse.

55 Show that, in general, the composition of any two reflections correspond to some rotation. If the reflections are represented using angles α and β, express the angle of rotation in terms of α and β.

3.2.2: Orthogonality and projection

blah blah blah

3.2.3: Spanning sets

blah blah blah