How to Use this Reviewer

Hello! This is a compilation of solved exercises for chapter 2 of "An introduction to mathematical statistics and its applications" by Larsen and Marx, sixth ed. All of these exercises are taken straight from the book. There are certain items that are much more difficult than normal, and are mostly for nerds like me to geek out about on documents like these. I'll note when these items show up, so that you don't spend energy that you don't really need or want to trying to understand them.

Normal items will look like this:

1 A very easy math problem. What's $1 + 1$?

whereas difficult problems will be soulless, like this:

2 A very difficult math problem. Prove that $\binom{2n}{n} < 2^{2n-2}$, $\forall n \geq 5$ using induction.

I might also include warnings in my **Nerd Interjections!**

Nerd Interjection![a] These sections are for me to remind you of some necessary information to solve the problems, elaborate on something that I think isn't all that clear with just pure math symbols, give a helpful theorem, be an annoying piece of shit, anything, really! Just think of it as a tips and tricks section.

[a] Image from @Ellem__ on Twitter.

I also have another section called **Can we Prove it?** (unfortunately lacking a cute picture to go along with it), where I include some interesting, not really necessary, but nonetheless relevant proofs. So far, these two are my only two gimmicks, but I might add more in the future.

Can we Prove it? This is just a random proof I yoinked from our homeworks.

Proof. ($\implies$) Let $x \in (A \cap B) \setminus C$. Then, $x \in (A \cap B)$ and $x \notin C$.
Since $x \in (A \cap B)$, $x \in A$ and $x \in B$.
Since $x \in A$ and $x \notin C$, $x \in (A \setminus C)$.
Since $x \in B$ and $x \notin C$, $x \in (B \setminus C)$.
Thus, $x \in (A \setminus C) \cap (B \setminus C)$.

($\impliedby$) Let $x \in (A \setminus C) \cap (B \setminus C)$. Then, $x \in (A \setminus C)$ and $x \in (B \setminus C)$.
Since $x \in (A \setminus C)$, $x \in A$ and $x \notin C$.
Since $x \in (B \setminus C)$, $x \in B$ and $x \notin C$.
Since $x \in A$ and $x \in B$, $x \in (A \cap B)$.
Thus, $x \in (A \cap B) \setminus C$.

Since both sides of the conditional are true, it holds that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$. □

Finally, there are blue boxes to indicate when instructions aren't obvious from the question itself, or if there are similar items that can be grouped together.

For items #7 to #12, we need to reevaluate our life decisions.

It's very important to note that this is a *work in progress!* I am human, and I will make mistakes, and I cannot finish doing all the exercises within the span of one day. If you spot anything wrong, please feel free to message me; I will correct it as soon as possible.

2.3: The Probability Function

1 According to a family-oriented lobbying group, there is too much crude language and violence on television. Forty-two percent of the programs they screened had language they found offensive, 27% were too violent, and 10% were considered excessive in both language and violence. What percentage of programs did comply with the group's standards?

Solution. Consider selecting a random televsion show among the ones the lobbying group surveyed, and let A be the event in which it has offensive language, and B be the event in which it has excessive violence. Then, $P(A) = 0.42$, $P(B) = 0.27$, and $P(A \cap B) = 0.1$. We are asked to look for the percentage of programs that complied with the group's standards, i.e. ones that do not contain either offensive language or excessive violence. The percentage of shows that contain either is given by $P(A \cup B)$, so the percentage that we want is $1 - P(A \cup B)$. By the formula for $P(A \cup B)$, we have

$$1 - \big(P(A) + P(B) - P(A \cap B)\big) = 1 - (0.42 + 0.27 - 0.1) = 1 - 0.59 = 0.41.$$

Thus, the percentage of shows that complied with the group's standards is 41%. ■

2 Let A and B be any two events defined on S. Suppose that $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.1$. What is the probability that A or B but not both occur?

Solution. Symbolically, we can express the probability of either but not both occuring as $P(A \cup B) - P(A \cap B)$. Note that even though the probability for $P(A \cup B)$ already subtracts $P(A \cap B)$ in its formula, this is to avoid double-counting it. To fully eradicate this probability from the expression, we have to subtract it again, since by definition, any event that is in both A and B is also in their union. Thus, by simple substitution, we have

$$P(A \cup B) - P(A \cap B) = P(A) + P(B) - P(A \cap B) - P(A \cap B) = 0.4 + 0.5 - 0.1 - 0.1 = 0.7.$$

Therefore, the probability than either A or B occur but not both is 0.7 or 70%. ■

3 Express the following probabilities in terms of $P(A)$, $P(B)$, and $P(A \cap B)$.

a $P(A^C \cup B^C)$.

Solution. The book actually gives a simplified expression for this already, but let's reason it out here.

The problem is asking for events that are either not in A or not in B. If an event is in either one of these, as long as they are not in the other, they are still a member of $A^C \cup B^C$. The only way for something to be excluded from this is if it is in *both* A and B. Thus, we can express $P(A^C \cup B^C)$ as $1 - P(A \cap B)$.

This is essentially an intuitive explanation of DeMorgan's Law: $x \in (A \cap B)^C \iff x \in A^C \cup B^C$. ■

b $P\big(A^C \cap (A \cup B)\big)$.

Solution. This solution is different from the one above in that it involves pure algebra.

$$\begin{aligned} P\big(A^C \cap (A \cup B)\big) &= P(A^C) + P(A \cup B) - P\big(A^C \cup (A \cup B)\big) \\ &= 1 - P(A) + P(A) + P(B) - P(A \cap B) - P(A^C \cup A \cup B) \\ &= 1 + P(B) - P(A \cap B) - \big(P(A^C) + P(A) + P(B) - P(A^C \cap A) - P(A^C \cap B) \\ &\quad - P(A \cap B) + P(A^C \cap A \cap B)\big) \end{aligned}$$

Here, the terms with $A^C \cap A$ cancel out because elements cannot be both in and out of A.

$$\begin{aligned} &= 1 + P(B) - P(A \cap B) - 1 - P(B) + P(A^C \cap B) + P(A \cap B) \\ &= P(A^C \cap B). \end{aligned}$$

From here though, we have to use reasoning to try and express this with our allowed terms. How can something be in B and not A? Well, we can take all the elements of B and remove those which are also in A. We know that elements in both sets are given by $A \cap B$. Thus, $A^C \cap B = B - A \cap B$, which gives our final answer of $P(B) - P(A \cap B)$. ■

Nerd Interjection! If you're wondering where I pulled the expansion out of in the previous problem, recall that, by **Theorem 2.3.7** in the book,

$$P(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} P(A_i) - \sum_{i<j} P(A_i \cup A_j) + (-1)^{3+1} P(A_1 \cap A_2 \cap A_3).$$

4 Let A and B be two events defined on S. If the probability that at least one of them occurs is 0.3 and the probability that A occurs but B does not occur is 0.1, what is $P(B)$?

Solution. In this problem, we are given $P(A \cup B) = 0.3$, which is the probability that at least one occurs, and $P(A) - P(A \cap B) = 0.1$, which is the probability that A but not B occurs. We know that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(B) + P(A) - P(A \cap B) \end{aligned}$$

So now we can substitute the values that we've already been given and solve for $P(B)$.

$$\begin{aligned} 0.3 &= P(B) + 0.1 \\ P(B) &= 0.2. \end{aligned}$$

Therefore, the probability of B occurring by itself is 0.2 or 20%. ■

5 Suppose that three fair dice are tossed. Let A_i be the event that a 6 shows on the i^{th} die, $i = 1, 2, 3$. Does $P(A_1 \cup A_2 \cup A_3) = \frac{1}{2}$? Explain.

Explanation. At first glance, this might seem true because A_1, A_2, and A_3 all have the same probability of rolling a 6, which is $\frac{1}{6}$, so multiplying this probability by 3 (for the three dice) gives us $\frac{1}{2}$. However, this wouldn't represent $P(A_1 \cup A_2 \cup A_3)$, because these events are not mutually exclusive. You can roll a 6 on any one of these dice and still get a 6 on another. We can only add probabilities like that for unions when they are exclusive.

Rather, this probability should be interpreted as the probability that *at least* one of the dice shows up as 6. The only way this wont happen is if all of them aren't 6, which we can interpret as $P(A_1^C \cap A_2^C \cap A_3^C)$. Subtracting this by 1 should give us the probability of $A_1 \cup A_2 \cup A_3$. Thus,

$$P(A_1 \cup A_2 \cup A_3) = 1 - P(A_1^C \cap A_2^C \cap A_3^C) = 1 - \left(\frac{5}{6}\right)^3 \neq \frac{1}{2},$$

which proves the statement is false. □

Note that you could also solve this by using the expansion given by **Theorem 2.3.7**, but this way is a little bit more intuitive (just a little bit more). It *should* give the same answer as the one above.

6 Events A and B are defined on a sample space S such that $P\big((A \cup B)^C\big) = 0.5$ and $P(A \cap B) = 0.2$. What is the probability that either A or B but not both will occur?

Solution. We know that the probability that either one of the events but not both will occur is given by $P(A \cup B) - P(A \cap B)$, from item #2 above. Thus, we can solve for this by substituting values. We can deduce that $P(A \cup B) = 0.5$ since the probability of its complement is 0.5 as well, thus $1 - 0.5 = 0.5$. This gives us

$$P(\text{either occuring but not both}) = P(A \cup B) - P(A \cap B) = 0.5 - 0.2 = 0.3.$$

Therefore, the probability that either A or B occur but not both is 0.3 or 30%. ■

7 Let $A_1, A_2, \ldots, A_n$ be a series of events for which $A_i \cap A_j = \emptyset$ if $i \neq j$ and $A_1 \cup A_2 \cup \cdots \cup A_n = S$. Let B be any event defined on S. Express B as a union of intersections.

Solution. If B is any event defined on S, then it could have "components" all across $A_1, A_2, \ldots, A_n$, since the union of all these events is what makes up S. The problem also tells us that all these events are mutually exclusive, so the probability of any component of B that is a subset of a given A_i will likewise be exclusive from any component of B that is a subset of a different A_j. We can therefore fully express B as the union of all these components, giving us

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$$

as our expression for B as a union of intersections. ■

8 Draw the Venn diagrams that would correspond to the following equations:

a $P(A \cap B) = P(B)$.

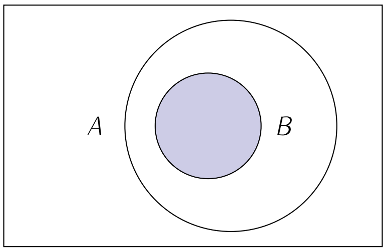

Figure 1: Venn diagram showing $P(A \cap B) = P(B)$, with the indicated probability highlighted.

b $P(A \cup B) = P(B)$.

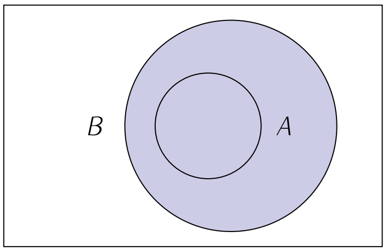

Figure 2: Venn diagram showing $P(A \cup B) = P(B)$, with the indicated probability highlighted.

9 In the game of "odd man out" each player tosses a fair coin. If all the coins turn up the same except for one, the player tossing the different coin is declared the odd man out and is eliminated from the contest. Suppose that three people are playing. What is the probability that someone will be eliminated on the first toss?

Solution. Since there are only three people playing, the only way for someone to not be eliminated is if all their coins are the same. If we let A be the event that someone is eliminated on the first toss, then A^C is the event that no one is, and $P(A)$ will be given by $1 - P(A^C)$. Out of all the possible outcomes of the coin flips (of which there are 8: {HHH, TTT, HTT, HHT, HTH, THH, TTH, THT}), two of them have all three coins showing up as the same face. Thus, $P(A^C) = \frac{1}{4}$, so the probability that someone is eliminated on the first round is $\frac{3}{4}$, or 0.75. ■

Nerd Interjection! You'll notice that the previous problem essentially boiled down to a counting problem. If you're familiar with combinatorics, you might remember there's a handy way of counting ways to do something that involve successes and failures: the **binomial coefficients**. We'll see later that there's a way of formalizing a theorem for similar situations, which should give us solutions like this next one.

Solution. The probability that the result of one player's coin toss is different from the other two's is described by the binomial distribution. We have 0.5 for the probability of either heads or tails, three trials (because there are three players), and we wish to find one "success" (i.e. that player's result is different). Thus,

$$\binom{3}{1}(1-0.5)^{3-1}0.5^1 = 3 \cdot 0.25 \cdot 0.5 = 0.375 \text{ or } \frac{3}{8}.$$

But wait! This isn't our answer from earlier. Well, that's because we need to multiply it by 2. "Success" here can mean either heads or tails, depending on what the other two flipped, but we've only calculated for one of the cases (let's say heads). Thus, without loss of generality[1], we say that the case in which the player eliminated flips tails is the same as the case where they flip heads, and we multiply this result by 2, giving us $\frac{3}{4}$ or 0.75. ■

We can actually also solve item #5 with the binomial distribution, but we'll save that for when we actually learn the binomial distribution because it's a bit more involved than this one is. But if you're curious, here are the funny numbers:

$$\binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{3-1} + \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{3-2} + \binom{3}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{3-3} = \frac{91}{216},$$

which is exactly the answer we got earlier (that I was too lazy to simplify).

10 An urn contains twenty-four chips, numbered 1 through 24. One is drawn at random. Let A be the event that the number is divisible by 2 and let B be the event that the number is divisible by 3. Find $P(A \cup B)$.

Solution. There are 12 numbers divisible by 2 within the range of $[1, 24]$, which means the probability of drawing one is $\frac{12}{24} = \frac{1}{2}$. Likewise, there are 8 numbers divisible by 3 within the range, which gives the probability of $\frac{8}{24} = \frac{1}{3}$. We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, which means we just need to solve for the intersection of these two events. We know that if a number is divisible by both 2 and 3, then it is divisible by 6. Thus, $P(A \cap B)$ is equivalent to P(the number drawn is divisible by 6). There are 4 such numbers in the given range, which gives $\frac{4}{24} = \frac{1}{6}$ for our probability. Finally, substituting these values gives us $\frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$ for the probability that a number is divisible by 2 or 3. ■

11 If State's football team has a 10% chance of winning Saturday's game, a 30% chance of winning two weeks from now, and a 65% chance of losing both games, what are their chances of winning exactly once?

Solution. Let A be the event that they win Saturday's game and B be the event that they win two weeks from now. Then, $P(A) = 0.1$, $P(B) = 0.3$, and $P(A^C \cap B^C) = 0.65$. We need to find the probability that they win one but not the other, which, as we've shown repeatedly thus far, is $P(A \cup B) - P(A \cap B)$. We know by DeMorgan's laws that $(A \cup B)^C = A^C \cap B^C$, so $1 - P(A^C \cap B^C) = 1 - 0.65 = 0.35 = P(A \cup B)$. Now, we just need to find $P(A \cap B)$, which thankfully we have a formula for, given $P(A)$, $P(B)$, and $P(A \cup B)$. Therefore,

$$P(A \cup B) - P(A \cap B) = 0.35 - (0.1 + 0.3 - 0.35) = 0.35 - 0.5 = 0.3$$

is the probability that the team will win exactly once. ■

12 Events A_1 and A_2 are such that $A_1 \cup A_2 = S$ and $A_1 \cap A_2 = \emptyset$. Find p_2 if $P(A_1) = p_1$, $P(A_2) = p_2$, and $3p_1 - p_2 = \frac{1}{2}$.

Solution. $A_1 \cup A_2 = S$ and $A_1 \cap A_2 = \emptyset$ tell us that p_1 and p_2 are mutually exclusive, and that $P(A_1 \cup A_2) = P(S) = 1$. Thus, $p_1 + p_2 = 1$. We can rewrite this equation as $p_1 = 1 - p_2$ and substitute it into the given equation. Then,

$$3(1 - p_2) - p_2 = \frac{1}{2} \iff 3 - 4p_2 = \frac{1}{2} \iff -4p_2 = -\frac{5}{2} \iff p_2 = \frac{5}{8},$$

which gives us our value for p_2. ■

[1] These four words are *very* important in combinatorics. Look it up if you're interested! I'm not great at explaining it.

13 Consolidated Industries has come under considerable pressure to eliminate its seemingly discriminatory hiring practices. Company officials have agreed that during the next five years, 60% of their new employees will be women and 30% will be minorities. One out of four new employees, though, will be a white man. What percentage of their new hires will be minority women?[2]

Solution. ■

[2] I changed the phrasing of this question because it goes against my morals to use the word "females".