How to Use this Reviewer

Hello! This is a compilation of solved exercises for chapter 2 of "An introduction to mathematical statistics and its applications" by Larsen and Marx, sixth ed. All of these exercises are taken straight from the book. There are certain items that are much more difficult than normal. I'll note when these items show up, so that you can get a heads-up and skip them if you're not too comfortable with the topic yet.

Normal items will look like this:

1 A very easy math problem. What's $1 + 1$?

whereas difficult problems will be soulless, like this:

2 A very difficult math problem. Prove that $\binom{2n}{n} < 2^{2n-2}, \ \forall n \geq 5$ using induction.

I might also include warnings in my **Nerd Interjections!**

Nerd Interjection![a] These sections are for me to remind you of some necessary information to solve the problems, elaborate on something that I think isn't all that clear with just pure math symbols, give a helpful theorem, be an annoying piece of shit, anything, really! Just think of it as a tips and tricks section.

[a] Image from @Ellem__ on Twitter.

I also have another section called **Can we Prove it?** (unfortunately lacking a cute picture to go along with it), where I include some interesting, not really necessary, but nonetheless relevant proofs. So far, these two are my only two gimmicks, but I might add more in the future.

Can we Prove it? This is just a random proof I yoinked from our homeworks.

Proof. ($\Longrightarrow$) Let $x \in (A \cap B) \setminus C$. Then, $x \in (A \cap B)$ and $x \notin C$.
Since $x \in (A \cap B)$, $x \in A$ and $x \in B$.
Since $x \in A$ and $x \notin C$, $x \in (A \setminus C)$.
Since $x \in B$ and $x \notin C$, $x \in (B \setminus C)$.
Thus, $x \in (A \setminus C) \cap (B \setminus C)$.

($\Longleftarrow$) Let $x \in (A \setminus C) \cap (B \setminus C)$. Then, $x \in (A \setminus C)$ and $x \in (B \setminus C)$.
Since $x \in (A \setminus C)$, $x \in A$ and $x \notin C$.
Since $x \in (B \setminus C)$, $x \in B$ and $x \notin C$.
Since $x \in A$ and $x \in B$, $x \in (A \cap B)$.
Thus, $x \in (A \cap B) \setminus C$.

Since both sides of the conditional are true, it holds that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$. $\square$

Finally, there are blue boxes to indicate when instructions aren't obvious from the question itself, or if there are similar items that can be grouped together.

For items #7 to #12, we need to reevaluate our life decisions.

It's very important to note that this is a *work in progress!* I am human, and I will make mistakes, and I cannot finish doing all the exercises within the span of one day. If you spot anything wrong, please feel free to message me; I will correct it as soon as possible.

2.3: The Probability Function

1 According to a family-oriented lobbying group, there is too much crude language and violence on television. Forty-two percent of the programs they screened had language they found offensive, 27% were too violent, and 10% were considered excessive in both language and violence. What percentage of programs did comply with the group's standards?

Solution. Consider selecting a random televsion show among the ones the lobbying group surveyed, and let A be the event in which it has offensive language, and B be the event in which it has excessive violence. Then, $P(A) = 0.42$, $P(B) = 0.27$, and $P(A \cap B) = 0.1$. We are asked to look for the percentage of programs that complied with the group's standards, i.e. ones that do not contain either offensive language or excessive violence. The percentage of shows that contain either is given by $P(A \cup B)$, so the percentage that we want is $1 - P(A \cup B)$. By the formula for $P(A \cup B)$, we have

$$1 - \big(P(A) + P(B) - P(A \cap B)\big) = 1 - (0.42 + 0.27 - 0.1) = 1 - 0.59 = 0.41.$$

Thus, the percentage of shows that complied with the group's standards is 41%. ■

2 Let A and B be any two events defined on S. Suppose that $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.1$. What is the probability that A or B but not both occur?

Solution. Symbolically, we can express the probability of either but not both occuring as $P(A \cup B) - P(A \cap B)$. Note that even though the probability for $P(A \cup B)$ already subtracts $P(A \cap B)$ in its formula, this is to avoid double-counting it. To fully eradicate this probability from the expression, we have to subtract it again, since by definition, any event that is in both A and B is also in their union. Thus, by simple substitution, we have

$$P(A \cup B) - P(A \cap B) = P(A) + P(B) - P(A \cap B) - P(A \cap B) = 0.4 + 0.5 - 0.1 - 0.1 = 0.7.$$

Therefore, the probability than either A or B occur but not both is 0.7 or 70%. ■

3 Express the following probabilities in terms of $P(A)$, $P(B)$, and $P(A \cap B)$.

a $P(A^C \cup B^C)$.

Solution. The book actually gives a simplified expression for this already, but let's reason it out here.

The problem is asking for events that are either not in A or not in B. If an event is in either one of these, as long as they are not in the other, they are still a member of $A^C \cup B^C$. The only way for something to be excluded from this is if it is in *both* A and B. Thus, we can express $P(A^C \cup B^C)$ as $1 - P(A \cap B)$.

This is essentially an intuitive explanation of DeMorgan's Law: $x \in (A \cap B)^C \iff x \in A^C \cup B^C$. ■

b $P\big(A^C \cap (A \cup B)\big)$.

Solution. This solution is different from the one above in that it involves pure algebra.

$$\begin{aligned} P\big(A^C \cap (A \cup B)\big) &= P(A^C) + P(A \cup B) - P\big(A^C \cup (A \cup B)\big) \\ &= 1 - P(A) + P(A) + P(B) - P(A \cap B) - P(A^C \cup A \cup B) \\ &= 1 + P(B) - P(A \cap B) - \big(P(A^C) + P(A) + P(B) - P(A^C \cap A) - P(A^C \cap B) \\ &\quad - P(A \cap B) + P(A^C \cap A \cap B)\big) \end{aligned}$$

Here, the terms with $A^C \cap A$ cancel out because elements cannot be both in and out of A.

$$\begin{aligned} &= 1 + P(B) - P(A \cap B) - 1 - P(B) + P(A^C \cap B) + P(A \cap B) \\ &= P(A^C \cap B). \end{aligned}$$

From here though, we have to use reasoning to try and express this with our allowed terms. How can something be in B and not A? Well, we can take all the elements of B and remove those which are also in A. We know that elements in both sets are given by $A \cap B$. Thus, $A^C \cap B = B - A \cap B$, which gives our final answer of $P(B) - P(A \cap B)$. ■

Nerd Interjection! If you're wondering where I pulled the expansion out of in the previous problem, recall that, by **Theorem 2.3.7** in the book,

$$P(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} P(A_i) - \sum_{i<j} P(A_i \cup A_j) + (-1)^{3+1} P(A_1 \cap A_2 \cap A_3).$$

4 Let A and B be two events defined on S. If the probability that at least one of them occurs is 0.3 and the probability that A occurs but B does not occur is 0.1, what is $P(B)$?

Solution. In this problem, we are given $P(A \cup B) = 0.3$, which is the probability that at least one occurs, and $P(A) - P(A \cap B) = 0.1$, which is the probability that A but not B occurs. We know that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(B) + P(A) - P(A \cap B) \end{aligned}$$

So now we can substitute the values that we've already been given and solve for $P(B)$.

$$\begin{aligned} 0.3 &= P(B) + 0.1 \\ P(B) &= 0.2. \end{aligned}$$

Therefore, the probability of B occurring by itself is 0.2 or 20%. ■

5 Suppose that three fair dice are tossed. Let A_i be the event that a 6 shows on the i^{th} die, $i = 1, 2, 3$. Does $P(A_1 \cup A_2 \cup A_3) = \frac{1}{2}$? Explain.

Explanation. At first glance, this might seem true because A_1, A_2, and A_3 all have the same probability of rolling a 6, which is $\frac{1}{6}$, so multiplying this probability by 3 (for the three dice) gives us $\frac{1}{2}$. However, this wouldn't represent $P(A_1 \cup A_2 \cup A_3)$, because these events are not mutually exclusive. You can roll a 6 on any one of these dice and still get a 6 on another. We can only add probabilities like that for unions when they are exclusive.

Rather, this probability should be interpreted as the probability that *at least* one of the dice shows up as 6. The only way this wont happen is if all of them aren't 6, which we can interpret as $P(A_1^C \cap A_2^C \cap A_3^C)$. Subtracting this by 1 should give us the probability of $A_1 \cup A_2 \cup A_3$. Thus,

$$P(A_1 \cup A_2 \cup A_3) = 1 - P(A_1^C \cap A_2^C \cap A_3^C) = 1 - \left(\frac{5}{6}\right)^3 \neq \frac{1}{2},$$

which proves the statement is false. □

Note that you could also solve this by using the expansion given by **Theorem 2.3.7**, but this way is a little bit more intuitive (just a little bit more). It *should* give the same answer as the one above.

6 Events A and B are defined on a sample space S such that $P\big((A \cup B)^C\big) = 0.5$ and $P(A \cap B) = 0.2$. What is the probability that either A or B but not both will occur?

Solution. We know that the probability that either one of the events but not both will occur is given by $P(A \cup B) - P(A \cap B)$, from item #2 above. Thus, we can solve for this by substituting values. We can deduce that $P(A \cup B) = 0.5$ since the probability of its complement is 0.5 as well, thus $1 - 0.5 = 0.5$. This gives us

$$P(\text{either occuring but not both}) = P(A \cup B) - P(A \cap B) = 0.5 - 0.2 = 0.3.$$

Therefore, the probability that either A or B occur but not both is 0.3 or 30%. ■

7 Let $A_1, A_2, \ldots, A_n$ be a series of events for which $A_i \cap A_j = \emptyset$ if $i \neq j$ and $A_1 \cup A_2 \cup \cdots \cup A_n = S$. Let B be any event defined on S. Express B as a union of intersections.

Solution. If B is any event defined on S, then it could have "components" all across $A_1, A_2, \ldots, A_n$, since the union of all these events is what makes up S. The problem also tells us that all these events are mutually exclusive, so the probability of any component of B that is a subset of a given A_i will likewise be exclusive from any component of B that is a subset of a different A_j. We can therefore fully express B as the union of all these components, giving us

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$$

as our expression for B as a union of intersections. ∎

8 Draw the Venn diagrams that would correspond to the following equations:

a $P(A \cap B) = P(B)$.

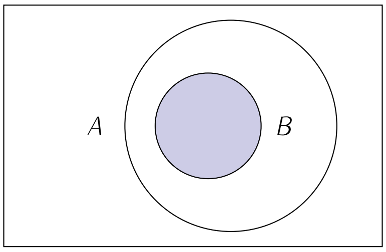

Figure 1: Venn diagram showing $P(A \cap B) = P(B)$, with the indicated probability highlighted.

b $P(A \cup B) = P(B)$.

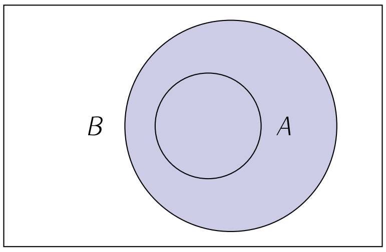

Figure 2: Venn diagram showing $P(A \cup B) = P(B)$, with the indicated probability highlighted.

9 In the game of "odd man out" each player tosses a fair coin. If all the coins turn up the same except for one, the player tossing the different coin is declared the odd man out and is eliminated from the contest. Suppose that three people are playing. What is the probability that someone will be eliminated on the first toss?

Solution. Since there are only three people playing, the only way for someone to not be eliminated is if all their coins are the same. If we let A be the event that someone is eliminated on the first toss, then A^C is the event that no one is, and $P(A)$ will be given by $1 - P(A^C)$. Out of all the possible outcomes of the coin flips (of which there are 8: {HHH, TTT, HTT, HHT, HTH, THH, TTH, THT}), two of them have all three coins showing up as the same face. Thus, $P(A^C) = \frac{1}{4}$, so the probability that someone is eliminated on the first round is $\frac{3}{4}$, or 0.75. ∎

Nerd Interjection! You'll notice that the previous problem essentially boiled down to a counting problem. If you're familiar with combinatorics, you might remember there's a handy way of counting ways to do something that involve successes and failures: the **binomial coefficients**. We'll see later that there's a way of formalizing a theorem for similar situations, which should give us solutions like this next one.

Solution. The probability that the result of one player's coin toss is different from the other two's is described by the binomial distribution. We have 0.5 for the probability of either heads or tails, three trials (because there are three players), and we wish to find one "success" (i.e. that player's result is different). Thus,

$$\binom{3}{1}(1-0.5)^{3-1}0.5^1 = 3 \cdot 0.25 \cdot 0.5 = 0.375 \text{ or } \frac{3}{8}.$$

But wait! This isn't our answer from earlier. Well, that's because we need to multiply it by 2. "Success" here can mean either heads or tails, depending on what the other two flipped, but we've only calculated for one of the cases (let's say heads). Thus, without loss of generality[1], we say that the case in which the player eliminated flips tails is the same as the case where they flip heads, and we multiply this result by 2, giving us $\frac{3}{4}$ or 0.75. ■

We can actually also solve item #5 with the binomial distribution, but we'll save that for when we actually learn the binomial distribution because it's a bit more involved than this one is. But if you're curious, here are the funny numbers:

$$\binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{3-1} + \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{3-2} + \binom{3}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{3-3} = \frac{91}{216},$$

which is exactly the answer we got earlier (that I was too lazy to simplify).

10 An urn contains twenty-four chips, numbered 1 through 24. One is drawn at random. Let A be the event that the number is divisible by 2 and let B be the event that the number is divisible by 3. Find $P(A \cup B)$.

Solution. There are 12 numbers divisible by 2 within the range of $[1, 24]$, which means the probability of drawing one is $\frac{12}{24} = \frac{1}{2}$. Likewise, there are 8 numbers divisible by 3 within the range, which gives the probability of $\frac{8}{24} = \frac{1}{3}$. We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, which means we just need to solve for the intersection of these two events. We know that if a number is divisible by both 2 and 3, then it is divisible by 6. Thus, $P(A \cap B)$ is equivalent to P(the number drawn is divisible by 6). There are 4 such numbers in the given range, which gives $\frac{4}{24} = \frac{1}{6}$ for our probability. Finally, substituting these values gives us $\frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$ for the probability that a number is divisible by 2 or 3. ■

11 If State's football team has a 10% chance of winning Saturday's game, a 30% chance of winning two weeks from now, and a 65% chance of losing both games, what are their chances of winning exactly once?

Solution. Let A be the event that they win Saturday's game and B be the event that they win two weeks from now. Then, $P(A) = 0.1$, $P(B) = 0.3$, and $P(A^C \cap B^C) = 0.65$. We need to find the probability that they win one but not the other, which, as we've shown repeatedly thus far, is $P(A \cup B) - P(A \cap B)$. We know by DeMorgan's laws that $(A \cup B)^C = A^C \cap B^C$, so $1 - P(A^C \cap B^C) = 1 - 0.65 = 0.35 = P(A \cup B)$. Now, we just need to find $P(A \cap B)$, which thankfully we have a formula for, given $P(A)$, $P(B)$, and $P(A \cup B)$. Therefore,

$$P(A \cup B) - P(A \cap B) = 0.35 - (0.1 + 0.3 - 0.35) = 0.35 - 0.5 = 0.3$$

is the probability that the team will win exactly once. ■

12 Events A_1 and A_2 are such that $A_1 \cup A_2 = S$ and $A_1 \cap A_2 = \emptyset$. Find p_2 if $P(A_1) = p_1$, $P(A_2) = p_2$, and $3p_1 - p_2 = \frac{1}{2}$.

Solution. $A_1 \cup A_2 = S$ and $A_1 \cap A_2 = \emptyset$ tell us that p_1 and p_2 are mutually exclusive, and that $P(A_1 \cup A_2) = P(S) = 1$. Thus, $p_1 + p_2 = 1$. We can rewrite this equation as $p_1 = 1 - p_2$ and substitute it into the given equation. Then,

$$3(1-p_2) - p_2 = \frac{1}{2} \iff 3 - 4p_2 = \frac{1}{2} \iff -4p_2 = -\frac{5}{2} \iff p_2 = \frac{5}{8},$$

which gives us our value for p_2. ■

[1] These four words are *very* important in combinatorics. Look it up if you're interested! I'm not great at explaining it.

13 Consolidated Industries has come under considerable pressure to eliminate its seemingly discriminatory hiring practices. Company officials have agreed that during the next five years, 60% of their new employees will be women and 30% will be members of racial minorities. One out of four new employees, though, will be a white man. What percentage of their new hires will be minority women?[2]

Solution. Consider randomly selecting an employee out of Consolidated Industries' new hires. Let A be the event that the employee is a woman, B be the event that the employee is a member of a minority group, and C be the event that the employee is a white man. The probabilites for all these events are already given, with C obviously being exclusive from the other two. We are tasked to find $P(A \cap B)$, which we know will be equal to $P(A) + P(B) - P(A \cup B)$. We know all of these except the last term, so we will have to figure it out.

If one out of every four employees will be a white man, then that means the remaining three will either be women or members of minority groups. Thus, $P(A \cup B) = 0.75$. Now that we have all our values, we can just substitute.

$$P(A \cap B) = 0.6 + 0.3 - 0.75 = 0.15$$

Thus, 15% is the percentage of new hires that will be women from minority groups. ■

Nerd Interjection! I thought it was very quaint that they included a situation like this in the sample problems, even if they did use the word "females" (shudders). People really need to recognize the biases, both personal and systemic, against members of minority groups. Especially those who try to gaslight everyone into thinking there isn't a problem!
What, you thought I could only talk about math in these?

14 Three events—A, B, and C—are defined on a sample space, S. Given that $P(A) = 0.2$, $P(B) = 0.1$, and $P(C) = 0.3$, what is the smallest possible value for $P\big((A \cup B \cup C)^C\big)$?

Solution. Asking what the smallest possible value for $P\big((A \cup B \cup C)^C\big)$ is is equivalent to asking what the *largest* possible value for $P(A \cup B \cup C)$ is, as any increase to the value of this probability will simultaneously decrease $P\big((A \cup B \cup C)^C\big)$. Then, the largest probability for the union of these three events is when all three of them are mutually exclusive from each other, and when we can thus add them all together to find $P(A \cup B \cup C)$. This value is 0.6, and subtracting it from 1 gives us 0.4 as the smallest possible value for $P\big((A \cup B \cup C)^C\big)$. ■

15 A coin is to be tossed four times. Define events X and Y such that

$$X: \text{ first and last coins have opposite faces}, \quad Y: \text{ exactly two heads appear}$$

Assume that each of the sixteen head/tail sequences has the same probability. Evaluate

a $P(X^C \cap Y)$

Solution. This probability is asking for the cases in which the first and last coins have the same faces, and exactly two heads appear. From inspection, we notice that this only happens when the sequence of tosses is $THHT$ or $HTTH$. Thus, dividing this by the total number of possible outcomes (16, I won't bother listing them but it's just 2^4 because 2 outcomes, 4 coins) gives us $\frac{2}{16} = \frac{1}{8} = 0.125$. ■

b $P(X \cap Y^C)$

Solution. This probability is asking for the cases in which the first and last coins are opposite, and there aren't exactly two heads. From inspection, we notice that there will never be four or zero heads, because the first and last faces have to be different. Then, the valid sequences are $HTTT$, $HHHT$, $THHH$, and $TTTH$. Thus, the probability of one of these occuring is simply $\frac{4}{16} = \frac{1}{4} = 0.25$. ■

[2] I changed the phrasing of this question because it goes against my morals to use the word "females".

16 Two dice are tossed. Assume that each possible outcome has a $\frac{1}{36}$ probability. Let A be the event that the sum of the faces showing is 6, and let B be the event that the face showing on one die is twice the face showing on the other. Calculate $P(A \cap B^C)$.

Solution. To be honest, I think this question is rather vague, particularly the "Assume that each possible outcome has a $\frac{1}{36}$ probability". I'm taking it to mean that each outcome of both dice rolled together has a probability of $\frac{1}{36}$, and not that the dice individually have 36 possible sides, because if that were the case then maybe they would've just said that. Also it works out nicely that there are 36 possible outcomes for two six-sided dice. Just very confusing why they chose to word it this way.

If we analyze $A \cap B$, we identify two cases: $(2,4),(4,2)$. If we exclude these but still consider the outcomes with a sum of 6, we have $\{(1,5),(5,1),(3,3)\}$, for a total of three possible outcomes wherein the sum is 6, but the face on one die is not twice the face on the other. Therefore, the probability of $A \cap B^C$ is $\frac{3}{36} = \frac{1}{12}$. ■

17 Let A, B, and C be three events defined on a sample space, S. Arrange the probabilities of the following events from smallest to largest: (a) $A \cup B$; (b) $A \cap B$; (c) A; (d) S; (e) $(A \cap B) \cup (A \cap C)$.

Solution. We know that given two events A and B, $A \cap B \leq A, B \leq A \cup B$, and that S is always equal to 1, so we just need to figure out where to put in $(A \cap B) \cup (A \cap C)$. Well, from set theory (which you should know), we know that $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$, so we can see it's really just another intersection of A, but with a set that is larger than B, so we put it in between $A \cap B$ and A.

Therefore, our final ordering is $A \cap B \leq (A \cap B) \cup (A \cap C) \leq A \leq A \cup B \leq S$.[3] ■

18 Lucy is currently running two dot-com scams out of a bogus chatroom. She estimates that the chances of the first one leading to her arrest are one in ten; the "risk" associated with the second is more on the order of one in thirty. She considers the likelihood that she gets busted for both to be 0.0025. What are Lucy's chances of avoiding incarceration?

Solution. Let A be the event that she gets caught for the first scam, and B be the event that she gets caught for the second. We are looking for the probability that she doesn't get caught for either, or symbolically, $P\big((A \cup B)^C\big)$. Since we have the probabilites for both events and their intersection. We can just do simple substitution.

$$P\big((A \cup B)^C\big) = 1 - P(A \cup B) = 1 - \left(\frac{1}{10} + \frac{1}{30} - \frac{25}{10000}\right) = 1 - \frac{157}{1200} = \frac{1043}{1200} \approx 0.8691666\ldots$$

Therefore, Lucy has a $\frac{1043}{1200}$ or roughly 87% chance of evading capture. Go Lucy! ■

[3] You'll notice I used less than or equal to instead of strictly less than. Technically speaking, any of these events could be equal to the next one, because the question didn't go the tiny extra step of saying $A \neq B \neq C$, but that's just me being picky about semantics. We got it anyway.

2.4: Conditional Probability

This section has 54 exercises, so it's probably unreasonable for me to finish them all, but I'll see how far I can get before getting bored.

1 Suppose that two fair dice are tossed. What is the probability that the sum equals 10 given that it exceeds 8?

Solution. Let the probability that the result is 10 be A. $P(A) = \frac{3}{36}$, because there are three combinations that give 10, namely $\{(4,6),(5,5),(6,4)\}$. Meanwhile, let the probability that the sum exceeds 8 be B. $P(B) = \frac{10}{36}$, because there are ten combinations that result in a sum above 8 (which I won't list anymore). Since 10 is always greater than 8, $P(A \cap B) = P(A)$. Thus, substituting, we have

$$P(A|B) = P(A)/P(B) = \frac{3}{36} \cdot \frac{36}{10} = \frac{3}{10},$$

for the probability that the sum of the dice is 10 given that it is greater than 8. ■

2 Find $P(A \cap B)$ if $P(A) = 0.2$, $P(B) = 0.4$, and $P(A|B) + P(B|A) = 0.75$.

Solution. We can solve for this by simply substituting the first two equations into the last one.

$$\frac{P(A \cap B)}{0.4} + \frac{P(B \cap A)}{0.2} = 0.75 \iff P(A \cap B)\big(7.5\big) = 0.75 \iff P(A \cap B) = 0.1.$$

Simplification of fractions has been omitted for space. ■

3 If $P(A|B) < P(A)$, show that $P(B|A) < P(B)$.

Proof. Finally, some proving so I can feel alive again. Here, we just do straightforward algebra. By the given,

$$\frac{P(A \cap B)}{P(B)} < P(A) \iff P(A \cap B) < P(A)\,P(B)$$

Now, if we try to solve for $P(B|A)$, we get

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} < \frac{P(A)\,P(B)}{P(A)} = P(B),$$

which proves the statement. □

4 Let A and B be two events such that $P\big((A \cup B)^C\big) = 0.6$ and $P(A \cap B) = 0.1$. Let E be the event that either A or B but not both occur. Find $P(E|A \cup B)$.

Solution. We know that the probability of either but not both occuring is $P(A \cup B) - P(A \cap B)$. With the given values, this equates to

$$P(E) = \big[1 - P\big((A \cup B)^C\big)\big] - P(A \cap B) = 1 - 0.6 - 0.1 = 0.3.$$

By the conditional probability, $P(E|A \cup B) = P\big(E \cap (A \cup B)\big)/P(A \cup B)$, so we need to find $P\big(E \cap (A \cup B)\big)$. However, we know that E is a subset of $A \cup B$, so the probability of their intersection is just equal to the probability of the smaller set, which is 0.3. Thus, dividing this by the probability of $A \cup B$, which is 0.4, gives us 0.75 or $\frac{3}{4}$ for the probability of only one occuring given that at least one has already occurred. ■

5 Suppose that in **Example 2.4.2** (the one with children, gender, and birth order), we ignored the ages of the children and distinguished only *three* family types: (boy, boy), (girl, boy), and (girl, girl). Would the conditional probability of both children being boys given that at least one is a boy be different from the answer found in the book? Explain.

Explanation. The probabilites would not change. Even if we remove the distinction between (boy, girl) and (girl, boy) families by placing them all under a single (girl, boy) category, families of this type would be twice as likely to appear as the other two. All we are really changing is the way we approach the problem. □

Nerd Interjection! The book example and this problem distinguish between combinations and permutations, which are more studied in combinatorics. For our purposes, it's important to note that if we are looking at probabilities of permutations but are only interested in how many of a certain item appear per permutation, we have to account for how many permutations will have the same amount of items that we desire. This idea will become more important when we look at the binomial distribution.

6 Two events, A and B, are defined on a sample space S such that $P(A|B) = 0.6$, $P(\text{At least one event occurs}) = 0.8$, and $P(\text{Exactly one event occurs}) = 0.6$. Find $P(A)$ and $P(B)$.

Solution. From the given, we have $P(A \cup B) = 0.8$ and $P(A \cup B) - P(A \cap B) = 0.6$, so $P(A \cap B) = 0.2$. Then, $P(A|B) = P(A \cap B)/P(B) = 0.2/P(B) = 0.6$, so $P(B) = \frac{1}{3}$. Substituting this into our formula for $P(A \cup B)$ gives us $0.8 = P(A) + \frac{1}{3} - 0.2$, so $P(A) = \frac{2}{3}$. ■

7 An urn contains one red chip and one white chip. One chip is drawn at random. If the chip selected is red, that chip together with two additional red chips are put back into the urn. If a white chip is drawn, the chip is returned to the urn. Then a second chip is drawn. What is the probability that both selections are red?

Solution. ■

8 Given that $P(A) = a$ and $P(B) = b$, show that

$$P(A|B) \geq \frac{a+b-1}{b}.$$

Proof. We know that $P(A|B) = P(A \cap B)/b$. We also know that $P(A \cap B) = a + b - P(A \cup B)$. Since $P(A \cup B)$ is a probability, it must be less than or equal to 1. Thus, $P(A \cap B) \geq a + b - 1$. Substituting this into the conditional probability formula gives us

$$P(A|B) = \frac{P(A \cap B)}{b} \geq \frac{a+b-1}{b},$$

which proves the statement. □

9 An urn contains one white chip and a second chip that is equally likely to be white or black. A chip is drawn at random and returned to the urn. Then a second chip is drawn. What is the probability that a white appeared on the first draw?

Solution. ■

10 Suppose events A and B are such that $P(A \cap B) = 0.1$ and $P\big((A \cup B)^C\big) = 0.3$. If $P(A) = 0.2$, what does $P\big((A \cap B)|(A \cup B)^C\big)$ equal?

Solution. ■

11 One hundred voters were asked their opinions of two candidates, A and B, running for mayor. Their response to three questions are summarized below:

	Number Saying "Yes"
Do you like A?	65
Do you like B?	55
Do you like both?	25

a What is the probability that someone likes neither?

b What is the probability that someone likes exactly one?

c What is the probability that someone likes at least one?

d What is the probability that someone likes at most one?

e What is the probability that someone likes exactly one given that they like at least one?

f Of those who like at least one, what proportion like both?

g Of those who do not like A, what proportion like B?

12 A fair coin is tossed three times. What is the probability that at least two heads will occur given that at most two heads have occurred?

13 Two fair dice are rolled. What is the probability that the number on the first die was at at least as large as 4 given that the sum of the two dice was 8?

14 Four cards are dealt from a standard fifty-two-card poker deck. What is the probability that all four are aces given that at least three are aces? (Note: There are 270,725 different sets of four cards that can be dealt. Assume that the probability associated with each of those hands is $1/270,725$).