Departamento de Engenharia Informática e de Sistemas

Optimization and Decision Support Methodologies

Operations Research

Review

Formulation of the LP model

Consider the following problem:

A small wooden toy factory plans to produce three new toys: trains, horses and huts. The production of these toys requires skilled labor in carpentry and finishing. The production of a train requires 1 hour of carpentry and 1 hour of finishing. The production of a horse requires 3 hours of carpentry and 2 hours of finishing. Producing a hut requires 2 hours of carpentry and 1 hour of finishing. The factory has 10 employees in the carpentry section, and 7 in the finishing section, with the weekly working hours of any of the employees being 40 hours.

With the sale of trains, horses and huts, the factory has a unitary profit of €20, €50 and €25, respectively.

The factory wants to know what quantities of each type of toy it should produce in order to maximize its weekly profit. (Assume the factory sells everything it produces.)

To help the factory get the desired answer, formulate the problem in terms of a linear programming model.

The graphical method

Solve each of the following problems using the graphical method:

Minimize $z = 3x_1 + 2x_2$ subject to $2x_1 + 2x_2 \leq 8$ $x_1 + 5x_2 \geq 10$ $-x_1 + 3x_2 = 6$ $x_1 \geq 0\ ,\ x_2 \geq 0$	*Maximize* $z = 3x_1 - x_2$ subject to $2x_1 + x_2 \geq 2$ $x_1 + 3x_2 \geq 3$ $x_2 \leq 4$ $x_1 \geq 0\ ,\ x_2 \geq 0$

The Simplex method

Consider the following linear programming problem:

Maximize $z = -x_1 + 2x_2$

subject to

$$x_1 + 3x_2 \geq 6$$

$$x_1 - x_2 \leq 1$$

$$x_1 \leq 5$$

$$x_1 \geq 0,\ x_2 \geq 0$$

- Solve it by the Simplex method using the “Big M” technique

Now consider the following linear programming problem:

Minimize $z = 3x_1 + 2x_2 + 4x_3$

subject to

$$2x_1 + x_2 + 3x_3 = 60$$

$$3x_1 + 3x_2 + 5x_3 \geq 120$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

- Solve it by the Simplex method using the “Two Phases” technique

Now consider the following linear programming problem:

Maximize $z = x_1 + x_2$

subject to

$x_1 + 2x_2 \leq 4$

$x_1 + \ \ x_2 = 3$

$x_1 \geq 0,\ x_2 \geq 0$

- Solve it by the Simplex method using the “Two Phases” technique

Now consider the following linear programming problem:

Minimize $z = 3x_1 + 2x_2 + 4x_3$

subject to

$2x_1 + x_2 + 3x_3 = 60$

$3x_1 + x_2 + 5x_3 \geq 120$

$x_1 , x_2 , x_3 \geq 0$

- Solve it by the Simplex method using the "Big M" technique

  (***Hint***: *Solve this exercise and conclude that this is an impossible, unsolvable problem, as you will reach the optimal tableau with an artificial variable in the basis*)

Duality – Dual Simplex method

Solve the following linear programming problem by the dual Simplex method:

Minimize $z = 3x_1 + 2x_2$

subject to

$$2x_1 + x_2 \geq 10$$

$$-3x_1 + 2x_2 \leq 6$$

$$x_1 + x_2 \geq 6$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Duality - Formulation of the dual problem

1.

Maximize $z = 3x_1 - 2x_2$

subject to

$x_1 \leq 4$

$x_1 + 3x_2 \leq 15$

$2x_1 + x_2 \leq 10$

$x_1 \geq 0$, $x_2 \geq 0$

2.

Minimize $z = x_1 + 9x_2 + x_3$

subject to

$x_1 + 2x_2 + 3x_3 \geq 9$

$3x_1 + 2x_2 + 2x_3 \leq 15$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$

Relationship Primal-Dual

MAXIMIZATION PROBLEM		OBTAINING DUAL ⟷	MINIMIZATION PROBLEM
i^{th} contraint	$\leq$ $\geq$ $=$	≥ 0 ≤ 0 free	i^{th} variable
j^{th} variable	≥ 0 ≤ 0 free	$\geq$ $\leq$ $=$	j^{th} constraint
Matrix A		Matrix A^T	
Coefficients of the OF		Independent terms	
Independent terms		Coefficients of the OF	

Duality - Obtaining the dual solution

Maximize $z = 2x_1 - x_2$

subject to

$2x_1 + 4x_2 \geq 8$ **(1)**
$x_1 + 2x_2 \geq 4$ **(2)**
$2x_1 + 2x_2 \leq 6$ **(3)**
$x_1 \geq 0$, $x_2 \geq 0$

Considering $\mathbf{x_3}$ and $\mathbf{x_5}$ the surplus and artificial variables of the functional constraint **(1)**, $\mathbf{x_4}$ and $\mathbf{x_6}$ the surplus and artificial variables of the functional constraint **(2)**, and $\mathbf{x_7}$ the slack variable of the functional constraint **(3)**, the Simplex optimal tableau is:

	c_i	2	-1	0	0	-M	-M	0	
$\mathbf{x_B}$	c_B / x_i	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_7}$	**b**
$\mathbf{x_2}$	-1	0	1	0	-1	0	1	-1/2	1
$\mathbf{x_3}$	0	0	0	1	-2	-1	2	0	0
$\mathbf{x_1}$	2	1	0	0	1	0	-1	1	2
	zj-cj	0	0	0	3	M	M-3	5/2	3

Duality - Complete exercise

Minimize $z = x_1 + 2x_2 + 4x_3$

subject to

$$\begin{aligned} x_1 + 3x_2 \quad\quad &\leq 5 \\ x_1 \quad\quad + 3x_3 &\geq 4 \\ x_2 + x_3 &\leq 9 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

- Solve it by the Dual Simplex method
- Formulate the dual problem associated with it
- From the results obtained in the 1st paragraph, indicate which is the optimal solution of the dual problem