Optimization and Decision Support Methodologies

Practical Sheet #1

Post-optimization and sensitivity analysis

1. Consider the following linear programming problem:

Maximize z = 2x1 + x2

subject to

x1 + 2x2≤ 6 ***(1)***

5x1 + 3x2≤ 15 ***(2)***

-2x1 + x2≤ two ***(3)***

x1≥ 0, x2 ≥ 0

Considering x3, x4 and x5 the slack variables of the functional constraints (1), (2) and (3), respectively, the *simplex* optimal tableau is:

	cj	2	1	0	0	0	
xB	**cB \ xj**	**x1**	**x2**	**x3**	**x4**	**x5**	**B**
x3	0	0	7/5	1	-1/5	0	3
x1	2	1	3/5	0	1/5	0	3
x5	0	0	11/5	0	2/5	1	8
zj-cj		0	1/5	0	2/5	0	6

Determine the implications for the optimal solution arising from the variations referred to in the different paragraphs below.

a)

1) Suppose that the vector of the independent terms of the constraints has changed from $\begin{bmatrix} 6 \\ 15 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 9 \\ 15 \\ 5 \end{bmatrix}$.

2) Suppose that the vector of the independent terms of the constraints has changed from $\begin{bmatrix} 6 \\ 15 \\ 2 \end{bmatrix}$ para $\begin{bmatrix} 3 \\ 20 \\ 2 \end{bmatrix}$.

b)

1) Suppose that the coefficient of variable x2 in the objective function has changed from c2=1 to c2=1/2.

2) Suppose that the coefficient of variable x1 in the objective function has changed from c1=2 to c1=4.

c)

1) Suppose that the vector of the coefficients in the constraints of variable x2 has changed from $\begin{bmatrix}2\\3\\1\end{bmatrix}$ to $\begin{bmatrix}3\\4\\1\end{bmatrix}$.

2) Suppose that the vector of the coefficients in the constraints of variable x1 is changed from $\begin{bmatrix}1\\5\\-2\end{bmatrix}$ to $\begin{bmatrix}1\\6\\-3\end{bmatrix}$

d) Suppose now that a new functional constraint, x1 ≤ 2, has been added to the problem.

e) Determine for which interval of c1 (coefficient in the objective function of variable x1), the previously presented solution will remain optimal.

f) Determine for which interval of b3 (independent term of the 3rd constraint), the optimal basis previously presented will remain optimal.

2. Now consider the following linear programming problem:

Maximize z = x1 + 2x2

subject to

-x1 + x2≤ 2 ***(1)***

x1 + 3x2≤ 12 ***(2)***

-x1 + 2x2≥ 1 ***(3)***

x1≥ 0, x2 ≥ 0

a) Solve it by the graphical method;

Since x3 and x6 are the surplus and artificial variables of the functional constraint (3), and x4 and x5 are the slack variables of the functional constraints (1) and (2), respectively, the *simplex* optimal tableau for this problem is presented below.

	c_j	1	2	0	0	0	-M	
$\mathbf{x_B}$	$\mathbf{c_B}$ \ x_j	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	**b**
$\mathbf{x_4}$	0	0	0	4/5	1	1/5	-4/5	18/5
$\mathbf{x_1}$	1	1	0	3/5	0	2/5	-3/5	21/5
$\mathbf{x_2}$	2	0	1	-1/5	0	1/5	1/5	13/5
zj-cj		0	0	1/5	0	4/5	M-1/5	47/5

For each of the following changes in the initial problem, determine, using the post-optimization analysis, what happens to the optimal tableau, the optimal solution and the value of z*, previously obtained.

b) Changing the vector of independent terms of constraints, from $\begin{bmatrix} 2 \\ 12 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 13 \\ 1 \end{bmatrix}$.

c) Changing the coefficient of variable x2 in the objective function, from 2 to -2.

d) Introduction of a new variable x_{New} with coefficients in the constraints equal to $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and coefficient in the objective function equal to 3.

3. Consider the following linear programming problem:

Maximize z = 5x1 + 2x2

subject to

x1 + 2x2≤ 2 ***(1)***

3x1 + x2≥ 3 ***(2)***

x1≥ 0, x2 ≥ 0

a) Solve it by the graphical method;

Consider that the same problem was solved by the simplex method, with x3 and x5 the surplus and artificial variables of the functional constraint (2), and x4 the slack variable of the functional constraint (1), having obtained the following optimal *simplex* tableau:

	c_j	5	2	0	0	-M	
$\mathbf{x_B}$	$c_B \backslash x_j$	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$	**b**
$\mathbf{x_3}$	0	0	5	1	3	-1	3
$\mathbf{x_1}$	5	1	2	0	1	0	2
zj-cj		0	8	0	5	M	10

For each of the following changes in the initial problem, determine, using the post-optimization analysis, what happens to the optimal picture, the optimal solution x* and the value of z*, previously obtained.

b) Changing the objective function to Minimize z=2x1 + 6x2.

c) Introduction of a new variable x_{New}, with coefficients in the constraints equal to $\begin{bmatrix} \mathbf{-1} \\ \mathbf{3} \end{bmatrix}$, and coefficient 1 in the objective function.

d) Changing the vector of the coefficients of the variable x1 in the constraints, from $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ for $\begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$.

e) Introduction of a new functional constraint in the problem: x1 + x2 ≥ **3**.

4. Consider the following linear programming problem:

Maximize z = 4x1 + 5x2 - 2x3 + x4
subject to

2x1 + 2x2 + 3x3≤ 10 ***(1)***
2x1 - x2 + 2x4≤ 5 ***(2)***
x1≥ 0, x2 ≥ 0, x3 ≥ 0, x4 ≥ 0

Considering x5 and x6 the slack variables of the functional constraints (1) and (2), respectively, the *simplex* optimal tableau is:

	c_j	4	5	-2	1	0	0	
$\mathbf{x_B}$	$\mathbf{c_B}$ \ x_j	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	**b**
$\mathbf{x_2}$	5	1	1	3/2	0	1/2	0	5
$\mathbf{x_4}$	1	3/2	0	3/4	1	1/4	1/2	5
z_j-c_j		5/2	0	41/4	0	11/4	1/2	30

a) Suppose that a new variable x_{New} was added to the problem, whose coefficient in the objective function is 5, and the coefficients in the functional constraints are $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Determine the implications for the optimal solution resulting from this alteration.

b) Determine for which interval of b1 (independent term of the 1st constraint), the optimal basis presented above will remain optimal.

c) Determine for which intervals of c1 and c4 (coefficients of x1 and x4 in the objective function, respectively), the solution above will remain optimal. Analyze the coefficients separately.

5. Consider the following linear programming problem:

Maximize z = 3x1 + x2 + 3x3
subject to

x_1 + 2x3 ≤ 3 ***(1)***
x_1 + 3x2 + 3x3 ≤ 6 ***(2)***
5x_1 + 5x2 + x3 ≥ 5 ***(3)***
x_1, x2, x3≥ 0

Considering x4 and x7 the surplus and artificial variables of the functional constraint (3), and x5 and x6 the slack variables of the functional constraints (1) and (2), respectively, the optimal *simplex* tableau is as follows in the next page.

x_B	c_B \ x_i (c_i)	x_1 (3)	x_2 (1)	x_3 (3)	x_4 (0)	x_5 (0)	x_6 (0)	x_7 (- M)	**b**
x_4	0	0	0	32/3	1	10/3	5/3	-1	15
x_2	1	0	1	1/3	0	-1/3	1/3	0	1
x_1	3	1	0	2	0	1	0	0	3
zj-cj		0	0	10/3	0	8/3	1/3	M	**10**

a) Determine, by carrying out a sensitivity analysis study, for which interval of c2, coefficient of x2 in the objective function, the optimal solution presented above remains optimal.

b) Determine, by carrying out a sensitivity analysis study, for which interval of b2 (independent term of the 2nd constraint), the optimal basis presented above remains optimal.

For each of the following changes in the initial problem, determine, by carrying out a post-optimization study, what are the implications for the optimal solution presented (in the value of x*, in the value of z* and in the optimal basis), resulting from the variation:

c) Changing of independent terms of constraints from $\begin{bmatrix}3\\6\\5\end{bmatrix}$ to $\begin{bmatrix}3\\3\\5\end{bmatrix}$;

d) Changing the coefficients of variable x1 in the constraints from $\begin{bmatrix}1\\1\\5\end{bmatrix}$ to $\begin{bmatrix}2\\-1\\17\end{bmatrix}$;

e) Introduction of a new variable x_{New} with coefficients in the functional constraints, $P_{New} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$, and coefficient in the objective function, c_{New} =2;

f) Changing the coefficient of variable x1 in the objective function from 3 to 4.

6. Consider the following linear programming problem:

Maximize z = x1 + 3x2

subject to

x_1 + 2x2 ≤ 16 ***(1)***

x_1 + x2 ≤ 12 ***(2)***

x_1 - 2x2 ≤ 8 ***(3)***

x1 ≥ 0, x2 ≥ 0

Considering x3, x4 and x5 the slack variables of the functional constraints (1), (2), and (3) respectively, the *simplex* optimal tableau is the one presented in the next page.

	c_i	1	3	0	0	0	
x_B	c_B \ x_i	x_1	x_2	x_3	x_4	x_5	**b**
x_2	3	1/2	1	1/2	0	0	8
x_4	0	1/2	0	-1/2	1	0	4
x_5	0	2	0	1	0	1	24
zj-cj		1/2	0	3/2	0	0	24

a) For each of the following changes in the initial problem, determine, by carrying out a post-optimization study, what are the implications for the optimal solution presented (in the value of x*, in the value of z* and in the optimal basis), resulting from the variation:

1) Change in the vector of the independent terms of the constraints from $\begin{bmatrix}16\\12\\8\end{bmatrix}$ to $\begin{bmatrix}12\\10\\8\end{bmatrix}$;

2) The coefficient in the objective function of variable x2 has changed from c2=3 to c2=2;

3) Addition of a new functional constraint to the initial problem: 2x1 + x2 ≤ 10;

4) Introduction of a new variable x_{New} with coefficients in the functional constraints equal to $\begin{bmatrix}2\\1\\2\end{bmatrix}$, and coefficient in the objective function equal to 2.

b) Determine, by carrying out a sensitivity analysis study, for which interval of c1 (coefficient of x1 in the objective function), the optimal solution presented above will remain optimal.

7. Consider the following linear programming problem:

Maximize z = -1x1 + 5x2 + 2x3 – x4

subject to

x_1 + x2 - 2x3 + 3x4 ≤ 60 ***(1)***

5x_1 + 4x2 + x3 + 2x4 ≤ 100 ***(2)***

2x_1 + x2 + 3x3 + 5x4 ≤ 20 ***(3)***

x1 ≥ 0, x2 ≥ 0, x3 ≥ 0, x4 ≥ 0

Considering x5, x6 and x7 the slack variables of the functional constraints (1), (2) and (3) respectively, the *simplex* optimal tableau is the one presented in the next page.

x_B	c_B \ x_i (c_i)	x_1 (-1)	x_2 (5)	x_3 (2)	x_4 (-1)	x_5 (0)	x_6 (0)	x_7 (0)	b
x_5	0	-1	0	-5	-2	1	0	-1	40
x_6	0	-3	0	-11	-18	0	1	-4	20
x_2	5	2	1	3	5	0	0	1	20
zj-cj		11	0	13	26	0	0	5	100

a) For each of the following changes in the initial problem, determine, by carrying out a post-optimization study, what are the implications for the optimal solution presented (in the value of x*, in the value of z*, and in the optimal basis), resulting from the variation:

1) Change in the vector of the terms independent of the constraints from $\begin{bmatrix} 60 \\ 100 \\ 20 \end{bmatrix}$ to $\begin{bmatrix} 30 \\ 120 \\ 20 \end{bmatrix}$;

2) The vector of the coefficients of variable x1 in the functional constraint has changed from $\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$;

3) Introduction of a new variable x_{New} with coefficients in the functional constraints equal to $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and coefficient in the objective function equal to 3.

b) Determine, by carrying out a sensitivity analysis study, for which interval of c2 (coefficient of x2 in the objective function), the optimal solution presented above will remain optimal.

c) Determine, by carrying out a sensitivity analysis study, for which interval of c3 (coefficient of x3 in the objective function), the optimal solution presented above will remain optimal.

8. Consider the following linear programming problem:

Maximize z = x1 + 3x2

subject to

$2x1 + x2 \geq 5$ ***(1)***

$x_1 + 2x2 \leq 20$ ***(2)***

$x_1 \leq 10$ ***(3)***

$x1 \geq 0,\ x2 \geq 0$

Considering x3 and x4 the surplus and artificial variables of the functional constraint (1), and x5 and x6 the slack variables of the functional constraints (2) and (3), respectively, the *simplex* optimal tableau is the one presented in the next page.

x_B	c_B \ x_i \ c_i	1 x_1	3 x_2	0 x_3	-M x_4	0 x_5	0 x_6	b
x_2	3	1/2	1	0	0	½	0	10
x_3	0	-1/2	0	1	-1	½	0	5
x_6	0	1	0	0	0	0	1	10
	zj-cj	1/2	0	0	M	3/2	0	30

a) For each of the following changes in the initial problem, determine, by carrying out a post-optimization study, what are the implications for the optimal solution presented (in the value of x*, in the value of z* and in the optimal basis), resulting from the variation:

1) Change in the vector of terms independent of the constraints from $\begin{bmatrix}5\\20\\10\end{bmatrix}$ to $\begin{bmatrix}6\\10\\8\end{bmatrix}$;

2) Change in the coefficient of variable x2 in the objective function, from 3 to 2;

3) Addition of a functional new constraint to the problem: 2x1 + x2 ≤ **20**;

b) Determine, by carrying out a sensitivity analysis study, for which interval of b1 (independent term of the 1st constraint), the optimal basis presented above will remain optimal.

c) Determine, by carrying out a sensitivity analysis study, for which interval of b2 (independent term of the 2nd constraint), the optimal basis presented above will remain optimal.

9. Consider the following single-purpose linear programming problem:

Maximize z = 2x1 + x2
subject to

x1 + x2 ≤ 12	***(1)***
2x1 + x2 ≥ 6	***(2)***
x2 ≤ 9	***(3)***
x1≥ 0, x2 ≥ 0	

Considering x3 and x5 the surplus and artificial variables of the functional constraint (2), and x4 and x6 the slack variables of the functional constraints (1) and (3) respectively, the *simplex* optimal tableau is the one presented in the next page.

	c_i	2	1	0	0	-M	0	
x_B	$c_B \setminus x_i$	x_1	x_2	x_3	x_4	x_5	x_6	**b**
x_3	0	0	1	1	2	-1	0	18
x_1	2	1	1	0	1	0	0	12
x_6	0	0	1	0	0	0	1	9
	zj-cj	0	1	0	2	M	0	24

a) For each of the following changes in the initial problem, determine, by carrying out a post-optimization study, what are the implications for the optimal solution presented (in the value of x*, in the value of z* and in the optimal basis), resulting from the variation:

1) Change in the coefficient of variable x2 in the objective function, from 1 to 2;

2) Introduction of a new variable, x_{New}, with coefficients in the constraints equal to $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, and coefficient in the objective function equal to 5;

3) Addition of a new functional constraint to the problem: x1 - 2x2 ≤ 6**;**

b) Determine, by carrying out a sensitivity analysis study, for which interval of c1 (coefficient of x1 in the objective function), the optimal solution presented above will remain optimal.