Department of Computer and Systems Engineering

Optimization and Decision Support Methodologies

Chapter I

Post-Optimization and Sensitivity Analysis

Introduction

In linear programming models the parameters involved (coefficients c, b and A) are rarely known exactly. It may also become necessary to add a new variable or constraint to the model.

It is fundamental to know if these modifications alter the optimal solution initially obtained.

It can also become interesting for the decision maker, the knowledge of solutions that, not being the optimum of the formalized problem, represent possible solutions to the real problem, providing a broader view of the consequences of the decision.

Therefore, it is important to have a type of analysis that incorporates in the model the "uncertainty" of real problems, while allowing a broader view of the spectrum of solutions.

Traditionally, 2 types of study are usually carried out:

- *Post-optimization analysis*, in which the impact of discrete changes in the model parameters on the optimal solution is studied;
- *Sensitivity analysis*, in which the main objective is to determine the variation intervals for the parameters, in such a way that there is no change in the optimal solution found.

Introductory concepts

Consider the following example (Example 1 of Chapter 2 of Operations Research).

"*A farmer wants to optimize his farm's rice and maize plantations, that is, he wants to know which areas to plant rice and maize in order to maximize the profit obtained from the plantations.*

The profit per unit of planted area of rice and maize is, respectively, 5 and 2 currency units (CU).

The areas to be planted for rice and maize should not be larger than 3 and 4 area units, respectively.

The total labor consumption (measured in men/hour) in the two plantations must not be greater than 9. Each unit of rice planted area requires 1 man/hour and each unit of corn planted area requires 2 men/hour."

The mathematical model of LP that describes it, is the following:

To determine

$x1$ = area to be planted with rice

$x2$ = area to be planted with corn

in order to

$$\text{maximize } z = 5\,x1 + 2\,x2$$

subject to

$$x1 \le 3$$

$$x2 \le 4$$

$$x1 + 2\,x2 \le 9$$

$$x1 \ge 0,\ x2 \ge 0$$

The problem in augmented form consists of:

To determine

$\mathbf{x} = (x1, x2, x3, x4, x5)$

in order to

$\max z = 5\,x1 + 2\,x2$

subject to

$$
\begin{array}{lllllll}
x1 & & + x3 & & & = 3 & (1) \\
 & x2 & & + x4 & & = 4 & (2) \\
x1 & + 2x2 & & & + x5 & = 9 & (3)
\end{array}
$$

$x1 \geq 0;\ x2 \geq 0;\ x3 \geq 0;\ x4 \geq 0;\ x5 \geq 0$

Consider the following tableaus, initial and optimal, resulting from the resolution of the previous model by the *simplex* method.

<u>Initial tableau</u>:

A

		cj	5	2	0	0	0	
xB	cB	xj	x1	x2	x3	x4	x5	b
x3	0		1	0	1	0	0	3
x4	0		0	1	0	1	0	4
x5	0		1	2	0	0	1	9
zj - cj			-5	-2	0	0	0	0

b

Initial feasible basic solution:

$$
\mathbf{x}_B \begin{cases} x3 = 3 \\ x4 = 4 \\ x5 = 9 \end{cases} \qquad \text{with} \qquad z = 0
$$

$$
\mathbf{x}_N \begin{cases} x1 = 0 \\ x2 = 0 \end{cases}
$$

Optimal tableau:

		cj	5	2	0	0	0	
xB	cB	xj	x1	x2	x3	x4	x5	B
x1	5		1	0	1	0	0	3
x4	0		0	0	1/2	1	-1/2	1
x2	2		0	1	-1/2	0	1/2	3
zj - cj			0	0	4	0	1	21

B-1 $\mathbf{x}_B$*

Optimal feasible basic solution:

$$\mathbf{x}_B^* \begin{cases} x1^* = 3 \\ x4^* = 1 \\ x2^* = 3 \end{cases} \quad \text{with} \quad z^* = 21$$

$$\mathbf{x}_N^* \begin{cases} x3^* = 0 \\ x5^* = 0 \end{cases}$$

It is verified that $\mathbf{x}$ = [$\mathbf{x_B}$, $\mathbf{x_N}$], that is, the vector of the problem variables, can be subdivided into the vectors $\mathbf{x_B}$ and $\mathbf{x_N}$, the first with the basic variables and the second with the non-basic variables.

On the other hand, A = [B, N], that is, the matrix A (matrix of the coefficients of the variables in the constraints) can be subdivided in matrices B and N, the first with the columns of basic variables and the second with the columns of non-basic variables.

Being the matrix B^{-1} the inverse matrix of B, **b** the vector of terms independent of the constraints and $\mathbf{c_B}$ the vector of the coefficients of the basic variables in the objective function, in any basic solution, it appears that:

$$\boxed{\begin{cases} \mathbf{x}_B = B^{-1}b \\ \mathbf{x}_N = 0 \text{ with } z = c_B' \, x_B = c_B' \, B^{-1}b \end{cases}}$$

In the previous example, the B matrix of the optimal base is:

$$B = \begin{array}{ccc} (x_1) & (x_4) & (x_2) \\ \end{array}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

The inverse B^{-1}matrix of the optimal base is:

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & -1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

Matrix B^{-1} can be easily obtained from the simplex optimal tableau by selecting the columns corresponding to the basic variables of the initial tableau (i.e., the slacks and the artificial ones).

It is possible to confirm the above formulas when calculating the optimal solution and the value of z*:

$$\mathbf{x}_B{}^* = B^{-1}b$$

$$\mathbf{x}_B{}^* = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & -1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix} \text{x} \begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

with $z^* = c_B' \, x_B{}^* = \begin{bmatrix} 5 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = 21$

Consider again the initial and final tableaus of solving the previous problem by the *simplex* method.

Initial tableau:

xB	cB	cj / xj	5 x1 (P_1)	2 x2 (P_2)	0 x3 (P_3)	0 x4 (P_4)	0 x5 (P_5)	B
x3	0		1	0	1	0	0	3
x4	0		0	1	0	1	0	4
x5	0		1	2	0	0	1	9
zj - cj			-5	-2	0	0	0	0

It is designated by Pf, the vector of the coefficients of the variable xf in the functional constraints.

Optimal tableau:

xB	cB	cj / xj	5 x1 (X_1)	2 x2 (X_2)	0 x3 (X_3)	0 x4 (X_4)	0 x5 (X_5)	B
x1	5		1	0	1	0	0	3
x4	0		0	0	1/2	1	-1/2	1
x2	2		0	1	-1/2	0	1/2	3
zj - cj			0	0	4	0	1	21

The $\mathbf{X_f}$ vectors are the representation of the $\mathbf{P_f}$ vectors in the base B, verifying the following relationship:

$$\boxed{\mathbf{X}_f = B_{-1}\mathbf{P_f}}$$

Post-Optimization

Consider the following problem (taken from Chapter 2 of the Operations Research course unit), which will be used as the basis for the exercises in this chapter.

"*An office furniture company wants to launch a model of desks and shelves.*

It is believed that the market can absorb the entire production of shelves, but it is advised that the monthly production of desks does not exceed 160 units.

Both products are processed in two different units: stamping unit (SU) and assembly and finishing unit (AFU). Monthly availability on each of these units is of 720 machine-hours in the SU, and 880 machine-hours in the AFU. Each desk needs 2 machine-hours in the SU and 4 machine-hours in the AFU; each shelf requires 4 machine-hours in the SU and 4 machine-hours in the AFU.

The profit obtained for each desk produced is of 6 currency units (CU) and for each shelf produced is of 3 currency units.

The aim is to know which monthly production plan for desks and shelves maximizes the profit."

Formulating it in terms of a linear programming model, this problem consists of:

To determine

x_1 = number of desks to produce per month

x_2 = number of shelves to produce per month

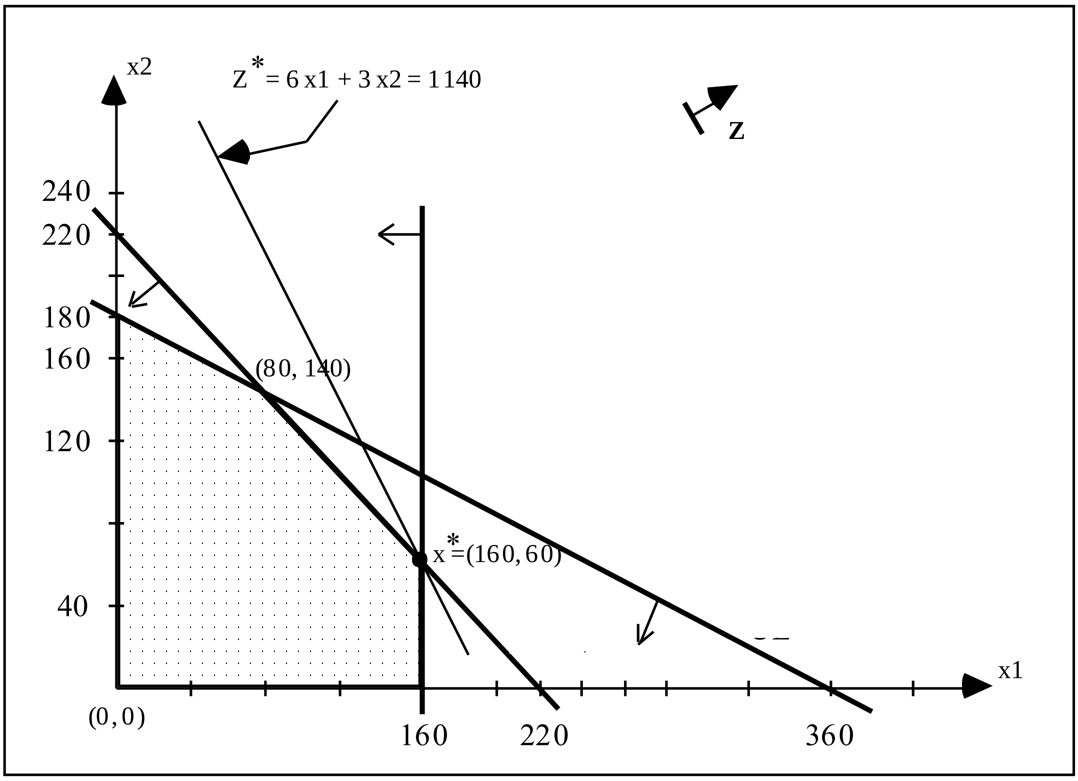
in order to maximize the monthly profit, that is,

$$\text{maximize } z = 6\, x_1 + 3\, x_2$$

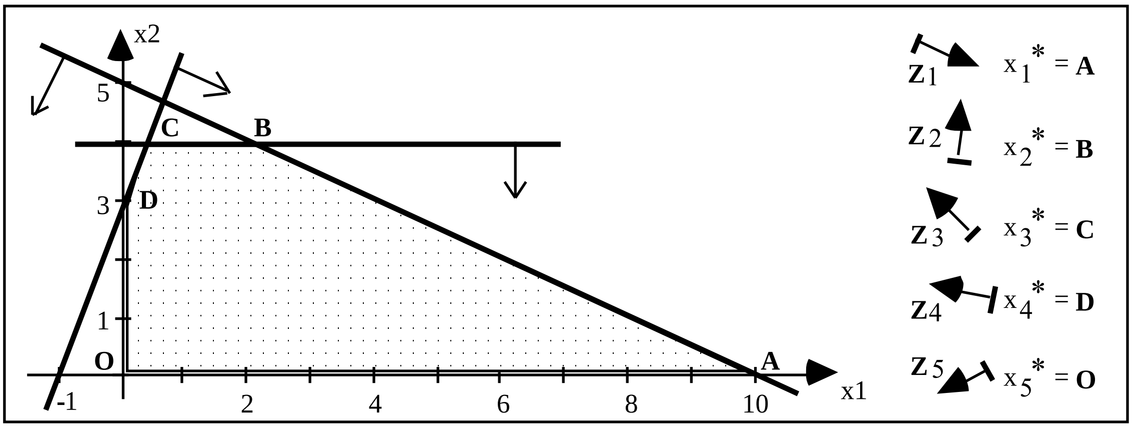
subject to

$$\begin{aligned}
2x_1 + 4x_2 &\leq 720 \\
4x_1 + 4x_2 &\leq 880 \\
x_1 \qquad &\leq 160 \\
x_1 \geq 0,\ x_2 &\geq 0
\end{aligned}$$

Solving by the graphical method we obtain:



a) ***Changes in objective function coefficients*** - c_j



In graphical terms, it means a change in the slope of the level lines of the objective function

Example

Considering the example previously presented (*page I-8*), suppose that the vector corresponding to the coefficients of the objective function was changed from [6 3] to [4 5].

The optimal tableau of the *simplex* is:

x_B	c_B \ c_j / x_j	6 x_1	3 x_2	0 x_3	0 x_4	0 x_5	b
x_3	0	0	0	1	-1	2	160
x_2	3	0	1	0	1/4	-1	60
x_1	6	1	0	0	0	1	160
$z_j - c_j$		0	0	0	3/4	3	1140

$x_1 = 160$
$x_2 = 60$
$x_3 = 160$
$x_4 = 0$
$x_5 = 0$
$Z = 1140$

Since $\mathbf{x_1}$ and $\mathbf{x_2}$ <u>are at the basis</u> the entire row "zj - cj" has to be updated.

x_B	c_B \ c_j / x_j	4 x_1	5 x_2	0 x_3	0 x_4	0 x_5	b
← x_3	0	0	0	1	-1	$\underline{2}$ *	160
x_2	5	0	1	0	1/4	-1	60
x_1	4	1	0	0	0	1	160
$z_j - c_j$		0	0	0	5/4	-1 ↑	940

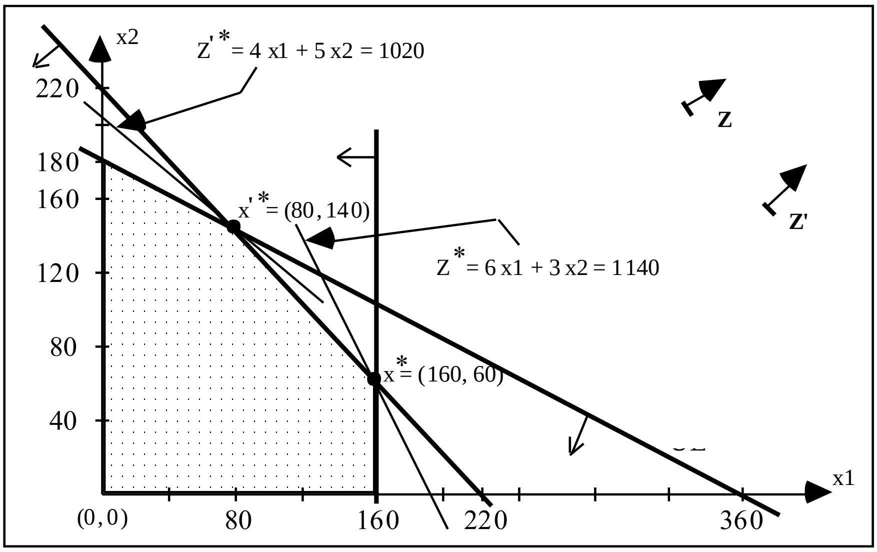
The tableau is no longer optimal => The optimal solution (x*), the value of z* and the optimal basis ($\{x_3, x_2, x_1\}$), do not remain the same!

The *simplex* algorithm is applied until a new optimal tableau is found.

x_5	0	0	0	1/2	-1/2	1	80
x_2	5	0	1	1/2	-1/4	0	140
x_1	4	1	0	-1/2	1/2	0	80
$z_j - c_j$		0	0	1/2	3/4	0	1020

$x_1 = 80$
$x_2 = 140$
$x_3 = 0$
$x_4 = 0$
$x_5 = 80$
$Z = 1020$

The change of the slope of the objective function, led to the optimum being reached at another extreme point: x*→ x'*.



That is, in graphical terms, a change in the coefficients c_j means a change in the slope of the level lines of the objective function

$$\Downarrow$$

- The optimal solution found remains feasible

  $\mathbf{x}_B^* = B^{-1}\mathbf{b}$ it's always non-negative

- The optimal solution found may be no longer optimal because there are changes in "$z_j - c_j$"

  (associated dual solution may become unfeasible)

Let c_f undergo an increase (or decrease) Δc_f

$$\tilde{c}_f = c_f + \Delta c_f$$

- If x_f does not belong to the optimal basis:

  ⇨ update the value of c_f and calculate the value "$z_j - c_j$" corresponding to column of x_f

  $$z_f - \tilde{c}_f = z_f - (c_f + \Delta c_f)$$

  - If ≥ 0, solution remains optimal*a)*;
  - Otherwise, apply *simplex* algorithm until obtaining a new optimal solution.

- If x_f belongs to the optimal basis:

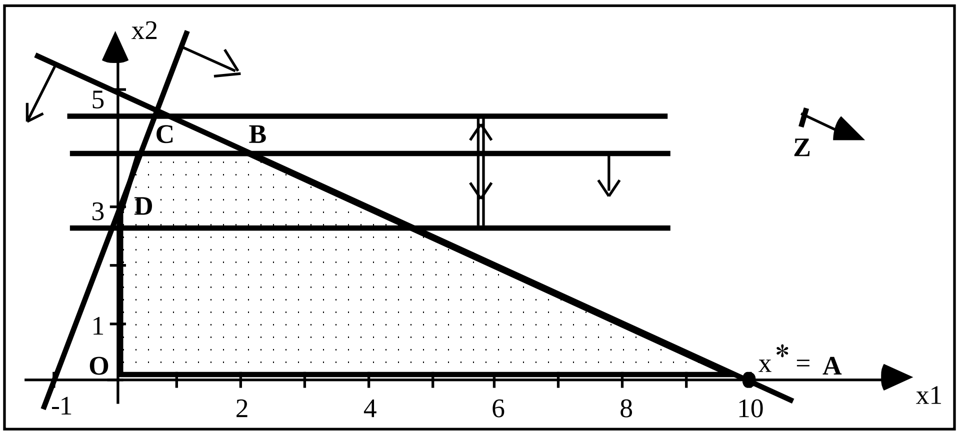
  ⇨ update the entire row "$z_j - c_j$"

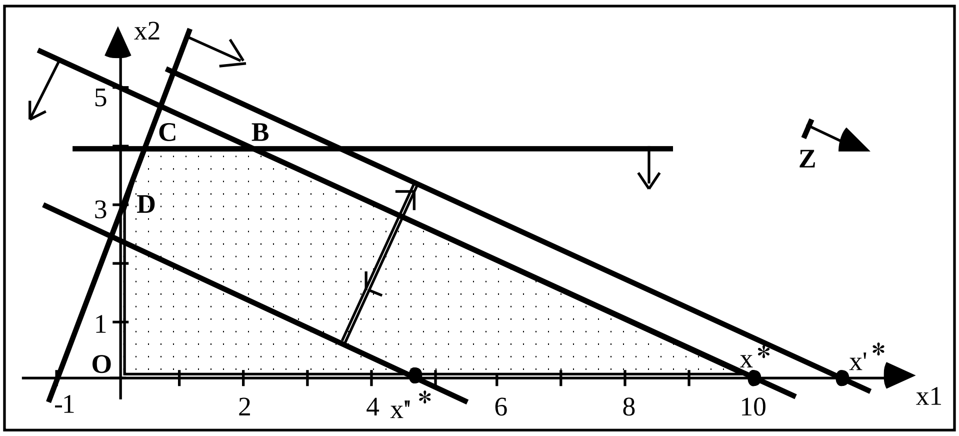
  - If solution remains optimal, calculate new $z\{c_f + \Delta c_f\}^* = z\{c_f\}^* + \Delta c_f \, x_f^*$;
  - If not, apply the *simplex* algorithm until obtaining a new optimal solution.

a) if = 0, the solution remains optimal, but there are new alternative optimal solutions that must be calculated!

b) ***Changes in the independent terms of constraints*** - bi

The dual problem can be studied or a direct approach may be used.





- The row "zj - cj" is not affected

  (the associated dual solution remains the same)

- The corresponding solution depends on the change under consideration.

  $$\mathbf{x}^{*B} = B\text{-}1b$$

  and may become unfeasible.

Let b_k undergo an increase (or decrease) Δb_k

$$\tilde{b}_k = b_k + \Delta b_k$$

$$\tilde{\mathbf{x}}_B = B^{-1}\tilde{\mathbf{b}}$$

$$\tilde{\mathbf{x}}_B = B^{-1}(\mathbf{b} + \Delta\mathbf{b})$$

$$\tilde{\mathbf{x}}_B = B^{-1}\mathbf{b} + B^{-1}\Delta\mathbf{b}$$

$$\tilde{\mathbf{x}}_B = x_B^* + B^{-1}\Delta\mathbf{b} \qquad \text{with } \Delta\mathbf{b} = \begin{bmatrix} 0 \\ : \\ \Delta b_k \\ : \\ 0 \end{bmatrix}$$

- if $\tilde{\mathbf{x}}_B \geq 0$, the solution is optimal and e $\tilde{z}^*$ is calculated.
- Otherwise, the solution is not feasible and the dual *simplex* algorithm is applied to calculate the new feasible solution (since the optimality condition is not violated).

Example

Go back to the previous example *(page I-8)*.

Suppose that the vector of the independent terms of the constraints has been changed from $\begin{bmatrix} 720 \\ 880 \\ 160 \end{bmatrix}$ to $\begin{bmatrix} 720 \\ 1280 \\ 160 \end{bmatrix}$.

Be the optimal tableau, the following:

x_B	c_B \ c_j / x_j	6 x_1	3 x_2	0 x_3	0 x_4	0 x_5	b
x_3	0	0	0	1	-1	2	160
x_2	3	0	1	0	1/4	-1	60
x_1	6	1	0	0	0	1	160
$z_j - c_j$		0	0	0	3/4	3	1140

$x_1 = 160$
$x_2 = 60$
$x_3 = 160$
$x_4 = 0$
$x_5 = 0$
$Z = 1140$

$$x_B{*} = \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 160 \\ 60 \\ 160 \end{bmatrix}$$

$$\Delta \mathbf{b} = \begin{bmatrix} 720 \\ 1280 \\ 160 \end{bmatrix} - \begin{bmatrix} 720 \\ 880 \\ 160 \end{bmatrix} = \begin{bmatrix} 0 \\ 400 \\ 0 \end{bmatrix}$$

The new solution, resulting from the changes, will be:

$$\tilde{\mathbf{x}}_B = \begin{bmatrix} 160 \\ 60 \\ 160 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1/4 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 400 \\ 0 \end{bmatrix} = \begin{bmatrix} -240 \\ 160 \\ 160 \end{bmatrix}$$

This solution is <u>not feasible</u> (concerning the problem with the change).

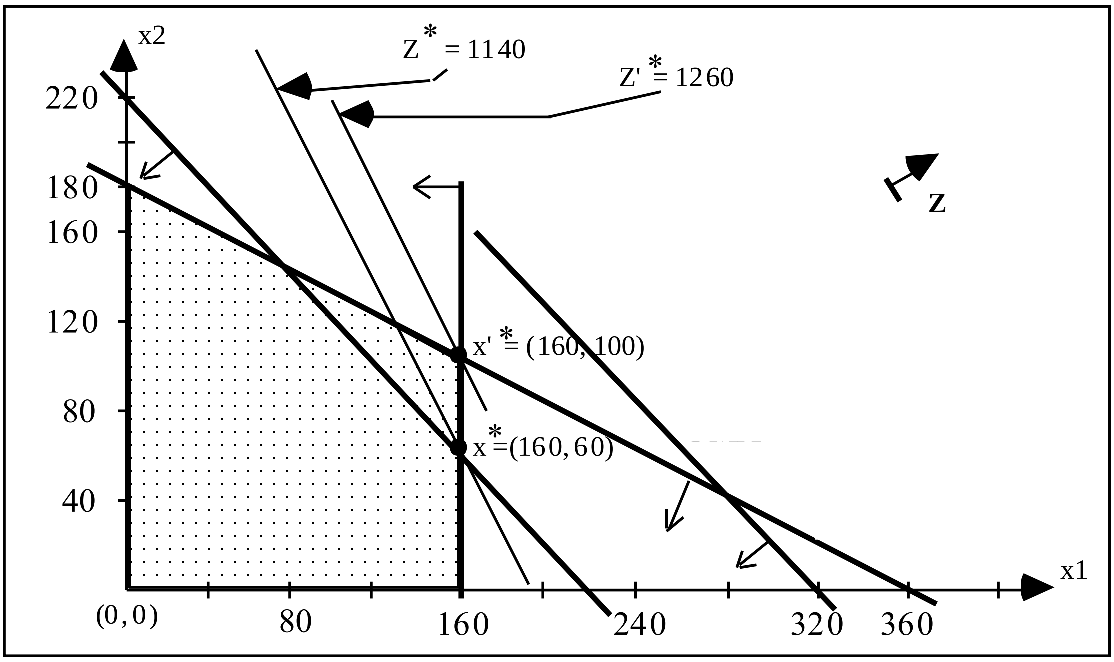
x_B	c_B	c_j / x_j	6 x_1	3 x_2	0 x_3	0 x_4	0 x_5	b
← x_3	0		0	0	1	$\underline{-1}$ *	2	-240
x_2	3		0	1	0	1/4	-1	160
x_1	6		1	0	0	0	1	160
$z_j - c_j$			0	0	0	3/4	3	1440
						↑		

In this case, the dual *simplex* algorithm is applied starting from the previous tableau:

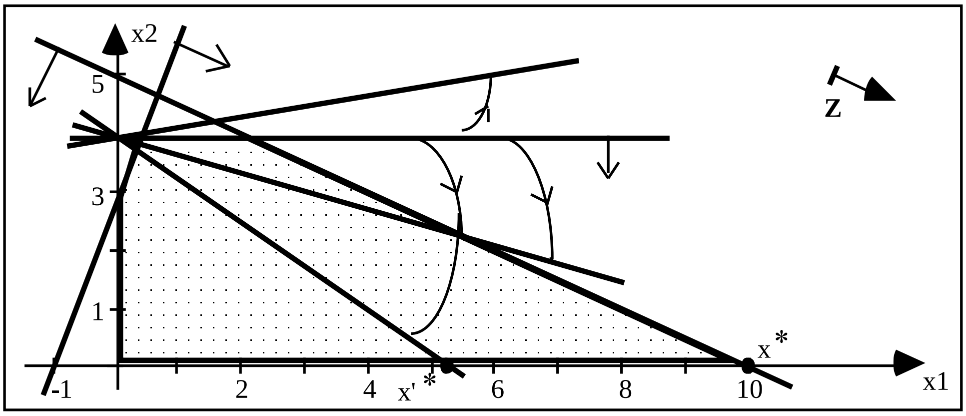
x_B	c_B	x_1	x_2	x_3	x_4	x_5	b
x_4	0	0	0	-1	1	-2	240
x_2	3	0	1	1/4	0	-1/2	100
x_1	6	1	0	0	0	1	160
$z_j - c_j$		0	0	3/4	0	9/2	1260

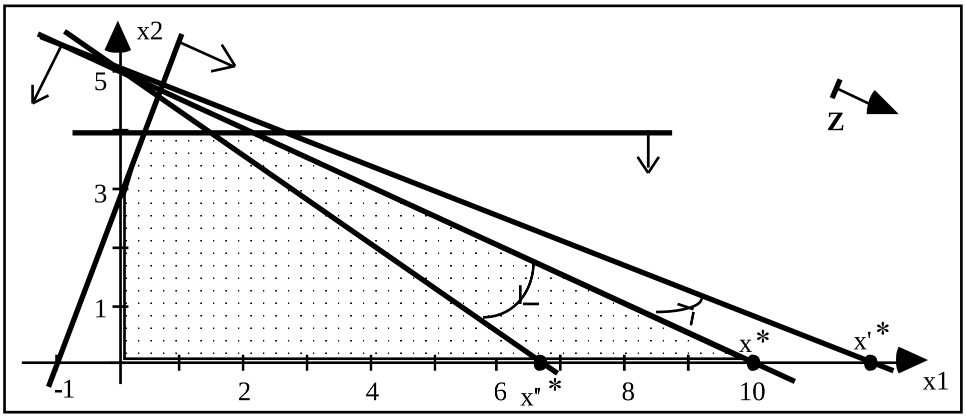
$x_1 = 160$
$x_2 = 100$
$x_3 = 0$
$x_4 = 240$
$x_5 = 0$
$Z = 1260$

The new optimal solution corresponds to the extreme point x'*:



c) ***Changes in the coefficients of matrix A*** **- aij**





Let akf undergo an increase (or decrease) Δakf

$$\tilde{a}_{kf} = akf + \mathbf{\Delta}akf$$

There are two situations to consider:

A) akf is a coefficient of xf not included in the optimal basis

$$\tilde{\mathbf{X}}_f = B^{-1}\tilde{\mathbf{P}}_f = B^{-1}(\mathbf{P}_f + \mathbf{\Delta P}_f) = B^{-1}\mathbf{P}_f + B^{-1}\mathbf{\Delta P}_f$$

$$\tilde{\mathbf{X}}_f = X_f + B^{-1}\mathbf{\Delta P}_f$$

$$\text{being } \mathbf{\Delta P}_f = \begin{bmatrix} 0 \\ : \\ \Delta a_{kf} \\ : \\ 0 \end{bmatrix}$$

and $\mathbf{X}_f$ is the column associated with xf

$$\tilde{z}_f - c_f = z_f + c'_B B^{-1} \Delta \mathbf{P}_f - c_f$$

- If ≥ 0, the solution remains optimal;
- Otherwise, the *simplex* algorithm is applied.

B) akf is a coefficient of xf included in the optimal basis (more complex case)

Changing a column of A belonging to the identity matrix imposes the reconstitution of the same matrix which leads to a new *simplex* tableau. In this new tableau, the following situations can be verified:

1 - **Basic feasible solutions of primal and dual**

⇨ the tableau remains optimal;

2 - **Feasible basic solution of primal, but not feasible basic solution of dual**

⇨ the *simplex* algorithm is applied to obtain a new optimal solution;

3 - **Basic solution not feasible for the primal but feasible for the dual**

⇨ the dual *simplex* algorithm is applied to obtain the new optimal solution;

4 - **Basic unfeasible solutions to both primal and dual problems**

⇨ Solve the problem again (?) or

⇨ Force the output of x_f from the optimal basis (from the optimal tableau before the changes):

1' - If there is any positive element in the row of the variable xf, corresponding to a non-basic variable, take as "pivot" the one with the lowest value "z_j-cj".

Proceed to the respective iteration and apply **A)** to the new FBS

Otherwise, the process continues.

2' - Choose the variable to enter the basis:

$$\min_j \left\{(z_j\text{-}cj) : (z_j\text{-}cj) \geq 0\right\} = (z_r\text{-}cr)$$

$x_{ir} \leq 0$ choose the next variable in terms of the value of "z_j-cj".

3' - Choose the variable to leave the basis

$$Qo = \min_i \left\{ \frac{x_{io}}{x_{ir}} \;\middle|\; xir > 0 \right\} = \frac{x_{so}}{x_{sr}}$$

4' - Replace x_s by x_r at the basis and return to 1'.

Example

Consider the previous example *(page I-8)*.

Suppose that the vector of the coefficients of x_1 in the constraints was changed from $\begin{bmatrix}2\\4\\1\end{bmatrix}$ to $\begin{bmatrix}2\\3.2\\1\end{bmatrix}$.

Be the optimal tableau of the *simplex*, the following:

x_B	c_B \ c_j	6	3	0	0	0	
	x_j	x_1	x_2	x_3	x_4	x_5	**b**
x_3	0	0	0	1	-1	2	160
x_2	3	0	1	0	1/4	-1	60
x_1	6	1	0	0	0	1	160
$z_j - c_j$		0	0	0	3/4	3	1140

$x_1 = 160$
$x_2 = 60$
$x_3 = 160$
$x_4 = 0$
$x_5 = 0$
$Z = 1140$

x_1 is at the basis

$$\tilde{X}_1 = X1 + B^{-1}\Delta P_1$$

$$\tilde{X}_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}1 & -1 & 2\\0 & 1/4 & -1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\-0.8\\0\end{bmatrix} = \begin{bmatrix}4/5\\-1/5\\1\end{bmatrix}$$

x_B	c_B \ c_j	6	3	0	0	0	
	x_j	x_1	x_2	x_3	x_4	x_5	**b**
x_3	0	4/5	0	1	-1	2	160
x_2	3	-1/5	1	0	1/4	-1	60
x_1	6	1	0	0	0	1	160
x_3	0	0	0	1	-1	6/5	32
x_2	3	0	1	0	1/4	-4/5	92
x_1	6	1	0	0	0	1	160
$z_j - c_j$		0	0	0	3/4	5/18	1236

Feasible solutions for the primal and dual problems

⇨ so, they remain optimal.

Example

Consider the previous example again *(page I-8)*.

Suppose that the vector of the coefficients of the variable x_2 in the constraints was changed from $\begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Be the optimal tableau of the *simplex*, the following:

c_j / x_B c_B x_j		6	3	0	0	0	
		x_1	x_2	x_3	x_4	x_5	b
x_3	0	0	0	1	-1	2	160
x_2	3	0	1	0	1/4 *	-1	60
x_1	6	1	0	0	0	1	160
$z_j - c_j$		0	0	0	3/4	3	1140

$x_1 = 160$
$x_2 = 60$
$x_3 = 160$
$x_4 = 0$
$x_5 = 0$
$Z = 1140$

<

x_2 is at the basis

$$\tilde{X}_2 = X2 + B^{-1}\Delta P_2$$

$$\tilde{X}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1/4 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/4 \\ 0 \end{bmatrix}$$

c_j / x_B c_B x_j		6	3	0	0	0	
		x_1	x_2	x_3	x_4	x_5	b
x_3	0	0	1	1	-1	2	160
x_2	3	0	1/4	0	1/4	-1	60
x_1	6	1	0	0	0	1	160
x_3	0	0	0	1	-2	6	-80 ←
x_2	3	0	1	0	1	-4	240
x_1	6	1	0	0	0	1	160
$z_j - c_j$		0	0	0	3	-6 ←	1680

Unfeasible solutions to the primal and dual problems ⇨ We make x_2 leave the basis of the optimal tableau and then introduce the mentioned alteration.

- Positive elements on row x_2 ⇨ "1/4"
- The lowest value "z_j-c_j" corresponds to x_4 ⇨ replace x_2 with x_4

In the new obtained tableau, x2 is a NBV. For this reason, A) is applied:

x_B	c_B \ c_j, x_j	6 x_1	3 x_2	0 x_3	0 x_4	0 x_5	b
x_3	0	0	4	1	0	-2	400
x_4	0	0	4	0	1	-4	240
x_1	6	1	0	0	0	1	160
$z_j - c_j$		0	-3	0	0	+6	1140

$$\tilde{X}_2 = X2 + B^{-1}\Delta P_2$$

$$\tilde{X}_2 = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

← x_3	0	0	2 *	1	0	-2	400
x_4	0	0	1	0	1	-4	240
x_1	6	1	0	0	0	1	160
$z_j - c_j$		0	-3 ↑	0	0	+6	960

x_2	3	0	1	1/2	0	-1	200
x_4	0	0	0	-1/2	1	-3	40
x_1	6	1	0	0	0	1	160
$z_j - c_j$		0	0	+3/2	0	+3	1560

New optimal solution:

$x1^* = 160$
$x2^* = 200$
$x3^* = 0$
$x4^* = 40$
$x5^* = 0$ with $z^* = 1560$