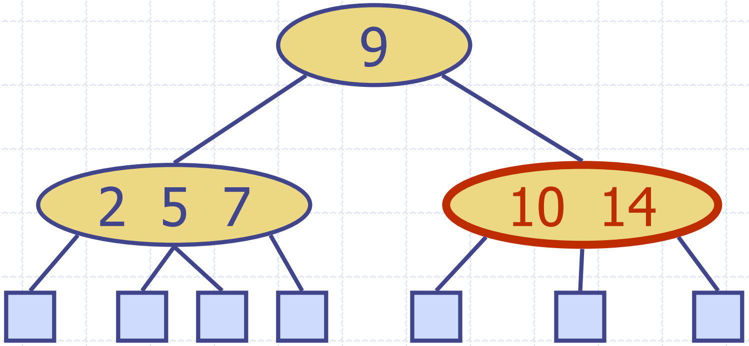
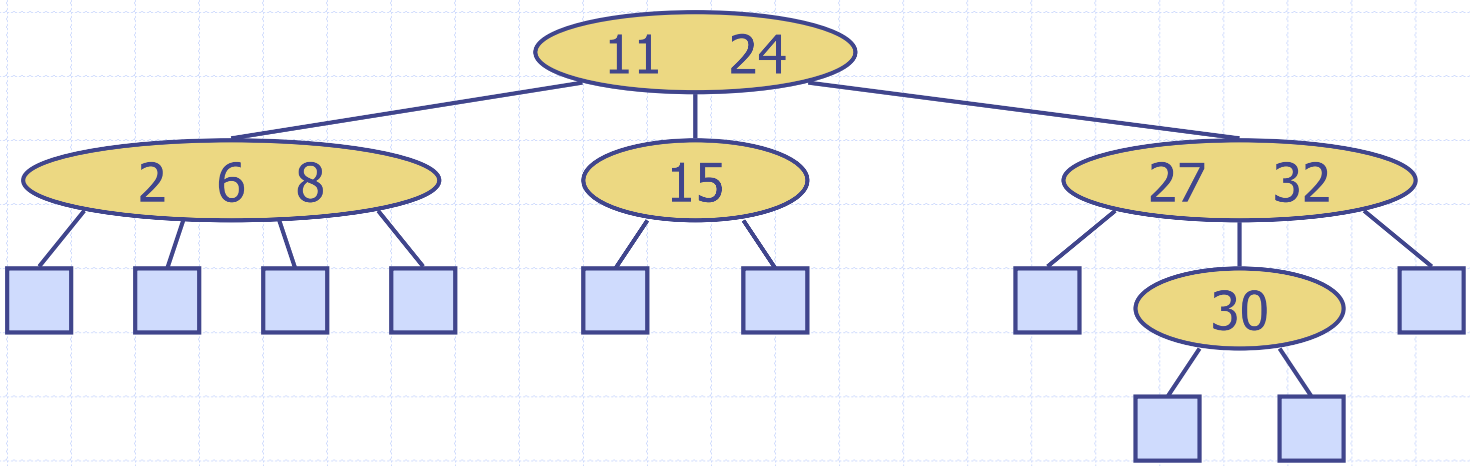
Modified version of the presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

(2,4) Trees



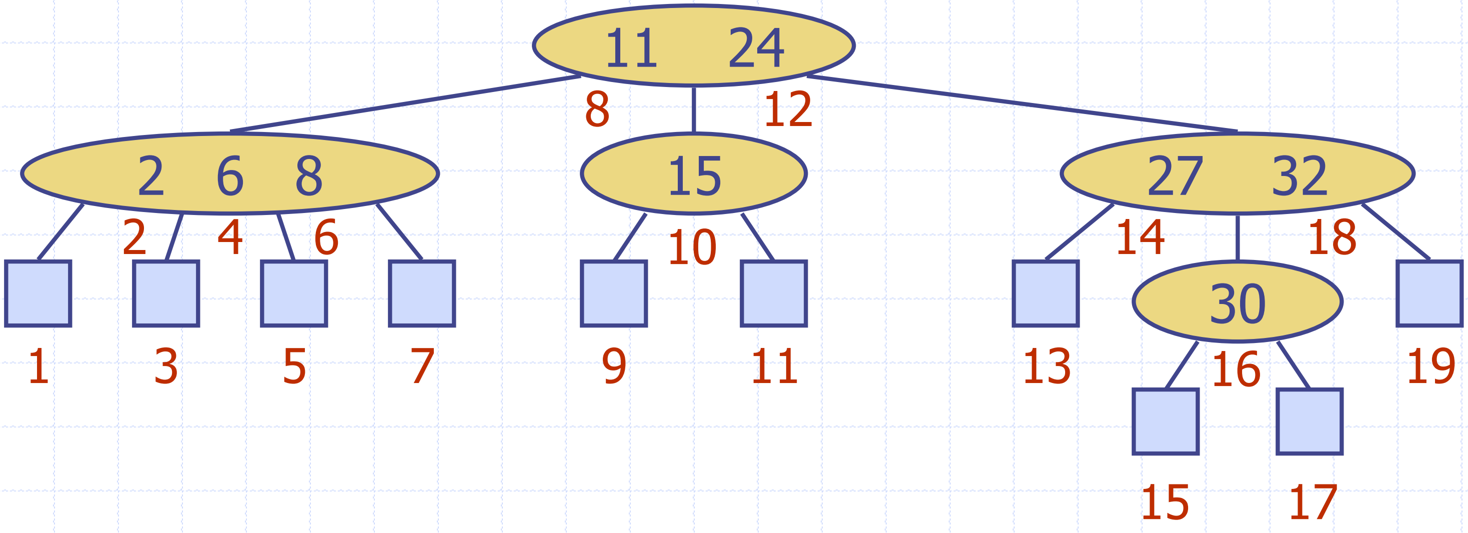
Multi-Way Search Tree

- A multi-way search tree is an ordered tree such that
  - Each internal node has at least two children and stores $d-1$ key-element items (k_i, o_i), where d is the number of children
  - For a node with children $v_1\ v_2\ \ldots\ v_d$ storing keys $k_1\ k_2\ \ldots\ k_{d-1}$
    - keys in the subtree of v_1 are less than k_1
    - keys in the subtree of v_i are between k_{i-1} and k_i $(i = 2, \ldots, d-1)$
    - keys in the subtree of v_d are greater than k_{d-1}
  - The leaves store no items and serve as placeholders



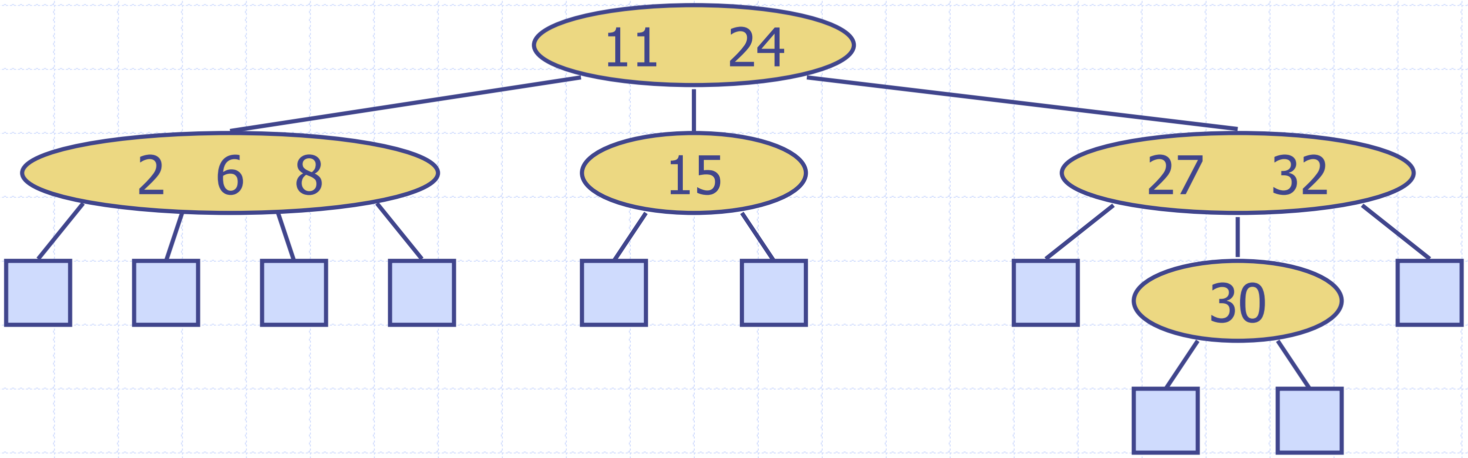
Multi-Way Inorder Traversal

- We can extend the notion of inorder traversal from binary trees to multi-way search trees
- Namely, we visit item (k_i, o_i) of node v between the recursive traversals of the subtrees of v rooted at children v_i and v_{i+1}
- An inorder traversal of a multi-way search tree visits the keys in increasing order



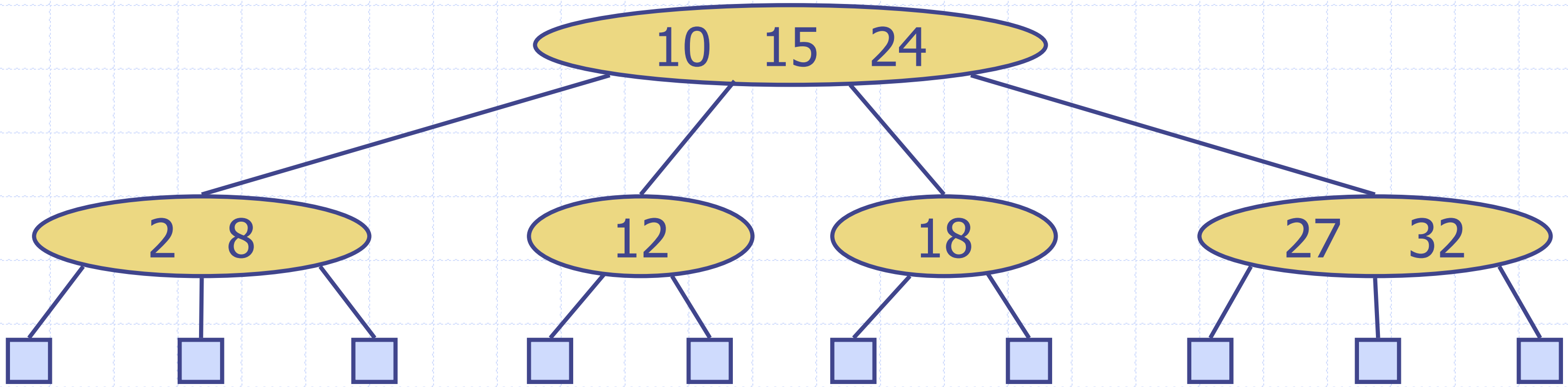
Multi-Way Searching

- Similar to search in a binary search tree
- A each internal node with children $v_1\, v_2\, \ldots\, v_d$ and keys $k_1\, k_2\, \ldots\, k_{d-1}$
  - $k = k_i$ ($i = 1, \ldots, d-1$): the search terminates successfully
  - $k < k_1$: we continue the search in child v_1
  - $k_{i-1} < k < k_i$ ($i = 2, \ldots, d-1$): we continue the search in child v_i
  - $k > k_{d-1}$: we continue the search in child v_d
- Reaching an external node terminates the search unsuccessfully
- Example: search for 30



(2,4) Trees

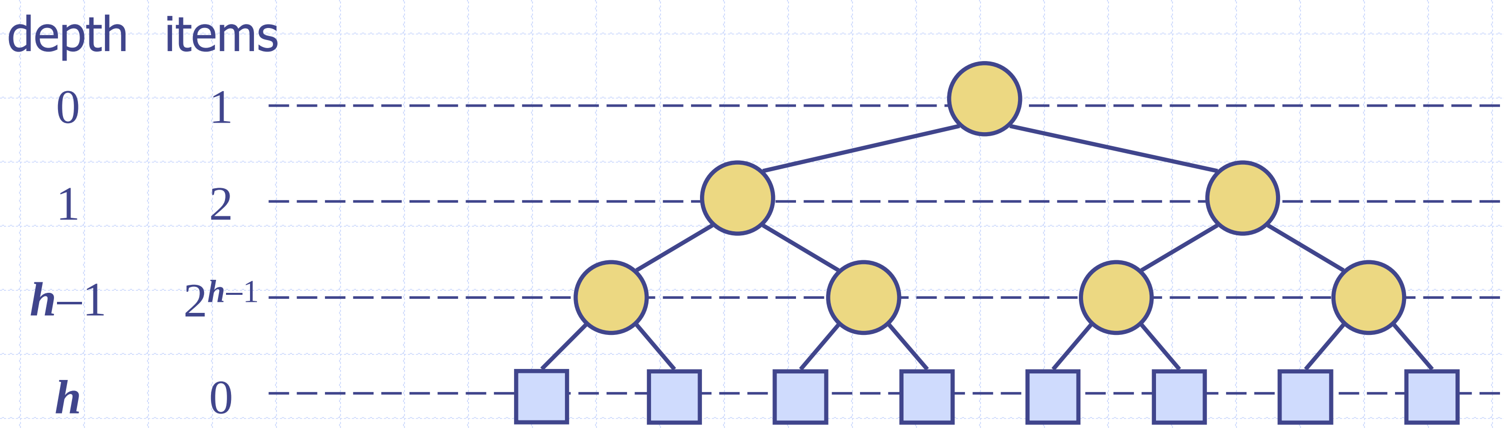
- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search with the following properties
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node



Height of a (2,4) Tree

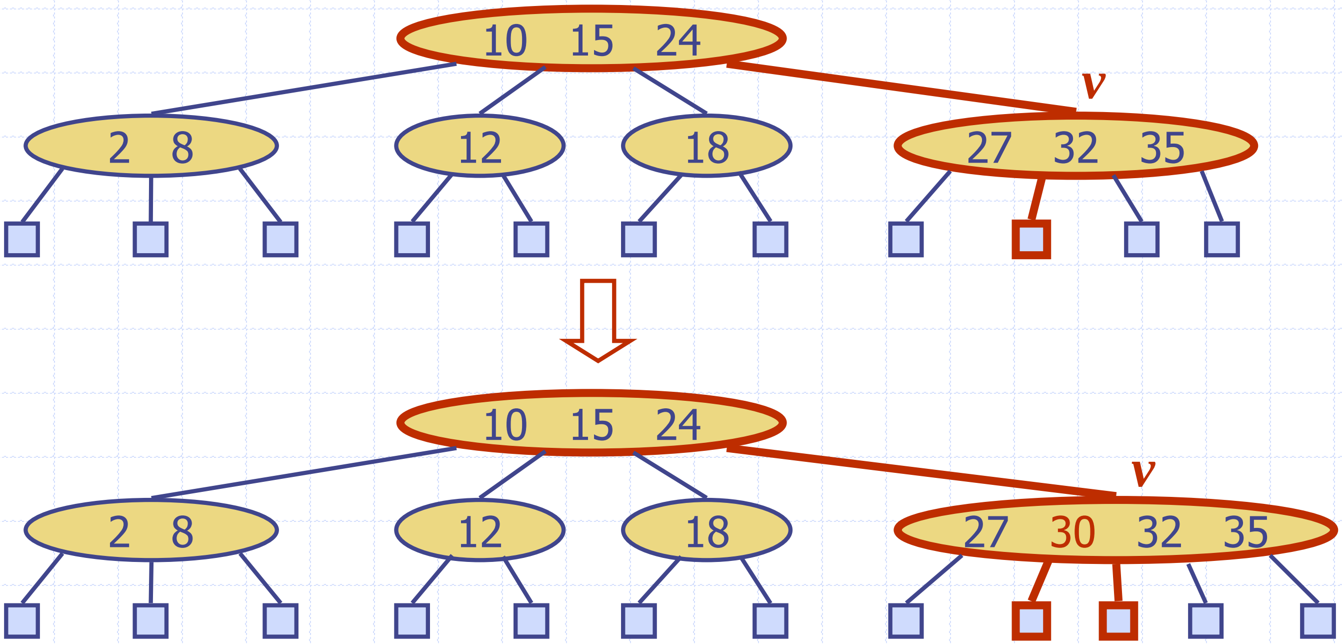
- **Theorem:** A (2,4) tree storing n items has height $O(\log n)$

  Proof:
  - Let h be the height of a (2,4) tree with n items
  - Since there are at least 2^i items at depth $i = 0, \ldots, h-1$ and no items at depth h, we have
    $$n \geq 1 + 2 + 4 + \ldots + 2^{h-1} = 2^h - 1$$
  - Thus, $h \leq \log(n + 1)$
- Searching in a (2,4) tree with n items takes $O(\log n)$ time



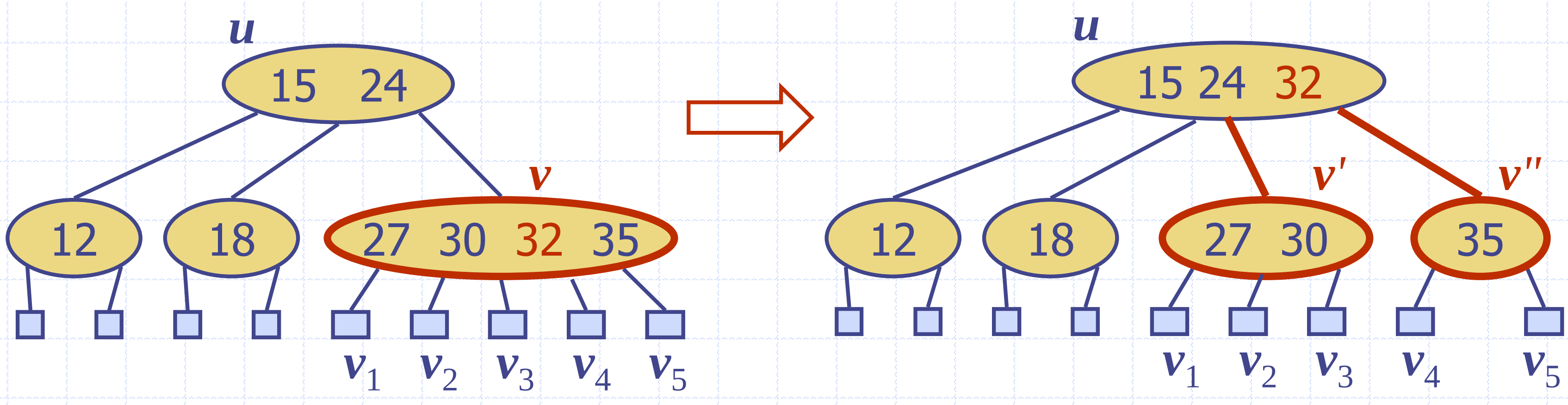
Insertion

- We insert a new item (k, o) at the parent v of the leaf reached by searching for k
  - We preserve the depth property but
  - We may cause an overflow (i.e., node v may become a 5-node)
- Example: inserting key 30 causes an overflow



Overflow and Split

- We handle an overflow at a 5-node v with a split operation:
  - let $v_1 \ldots v_5$ be the children of v and $k_1 \ldots k_4$ be the keys of v
  - node v is replaced by nodes v' and v''
    - v' is a 3-node with keys $k_1\ k_2$ and children $v_1\ v_2\ v_3$
    - v'' is a 2-node with key k_4 and children $v_4\ v_5$
  - key k_3 is inserted into the parent u of v (a new root may be created)
- The overflow may propagate to the parent node u



See Figure 11.18 in the book for a parent overflow example

Analysis of Insertion

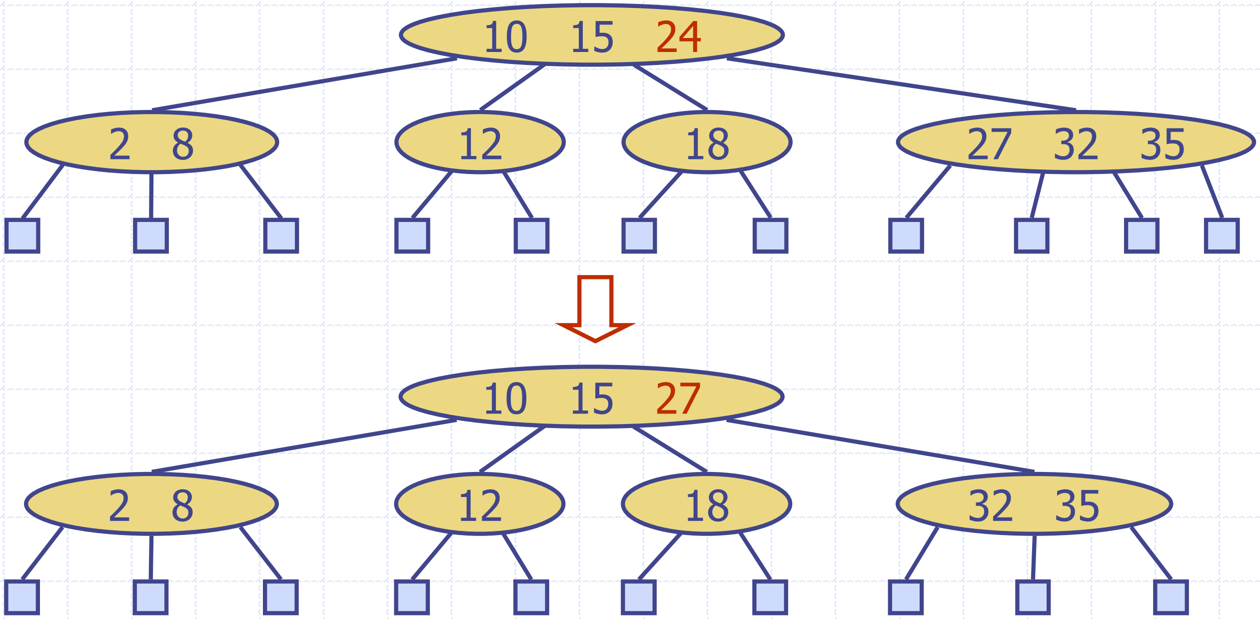
Algorithm *put*(k, o)

1. We search for key k to locate the insertion node v
2. We add the new entry (k, o) at node v
3. **while** *overflow*(v)
    if *isRoot*(v)
        create a new empty root above v
    $v \leftarrow$ *split*(v)

- Let T be a (2,4) tree with n items
    - Tree T has $O(\log n)$ height
    - Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
    - Step 2 takes $O(1)$ time
    - Step 3 takes $O(\log n)$ time because each split takes $O(1)$ time and we perform $O(\log n)$ splits
- Thus, an insertion in a (2,4) tree takes $O(\log n)$ time

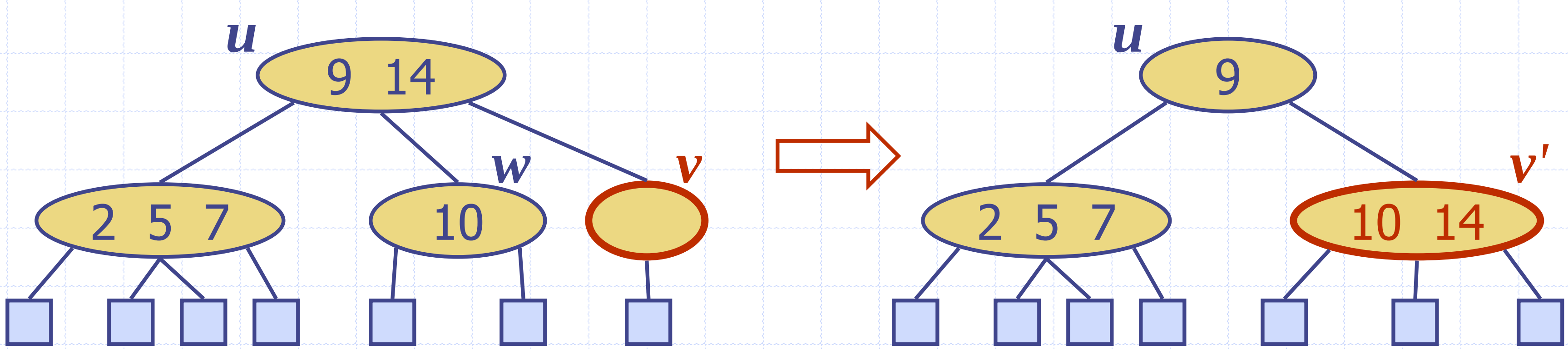
Deletion

- We reduce deletion of an entry to the case where the item is at the node with leaf children
- We replace the entry with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter entry
- Example: to delete key 24, we replace it with 27 (inorder successor)



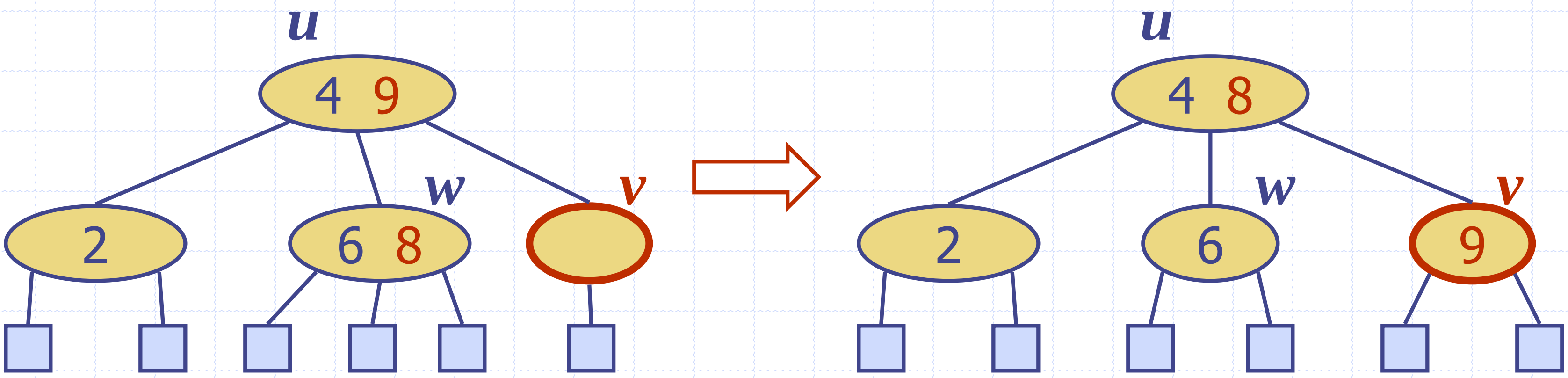
Underflow and Fusion

- Deleting an entry from a node v may cause an underflow, where node v becomes a 1-node with one child and no keys
- To handle an underflow at node v with parent u, we consider two cases
- Case 1: the adjacent siblings of v are 2-nodes
  - Fusion operation: we merge v with an adjacent sibling w and move an entry from u to the merged node v'
  - After a fusion, the underflow may propagate to the parent u



Underflow and Transfer

- To handle an underflow at node v with parent u, we consider two cases
- Case 2: an adjacent sibling w of v is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of w to v
    2. we move an item from u to v
    3. we move an item from w to u
  - After a transfer, no underflow occurs



Analysis of Deletion

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
- In a deletion operation:
  - We visit $O(\log n)$ nodes to locate the node from which to delete the entry
  - We handle an underflow with a series of $O(\log n)$ fusions, followed by at most one transfer
  - Each fusion and transfer takes $O(1)$ time
- Thus, deleting an item from a (2,4) tree takes $O(\log n)$ time

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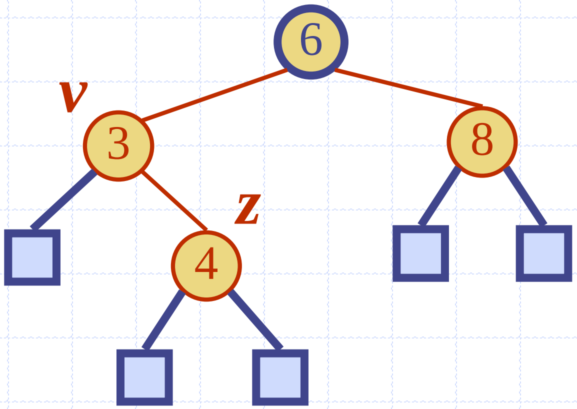
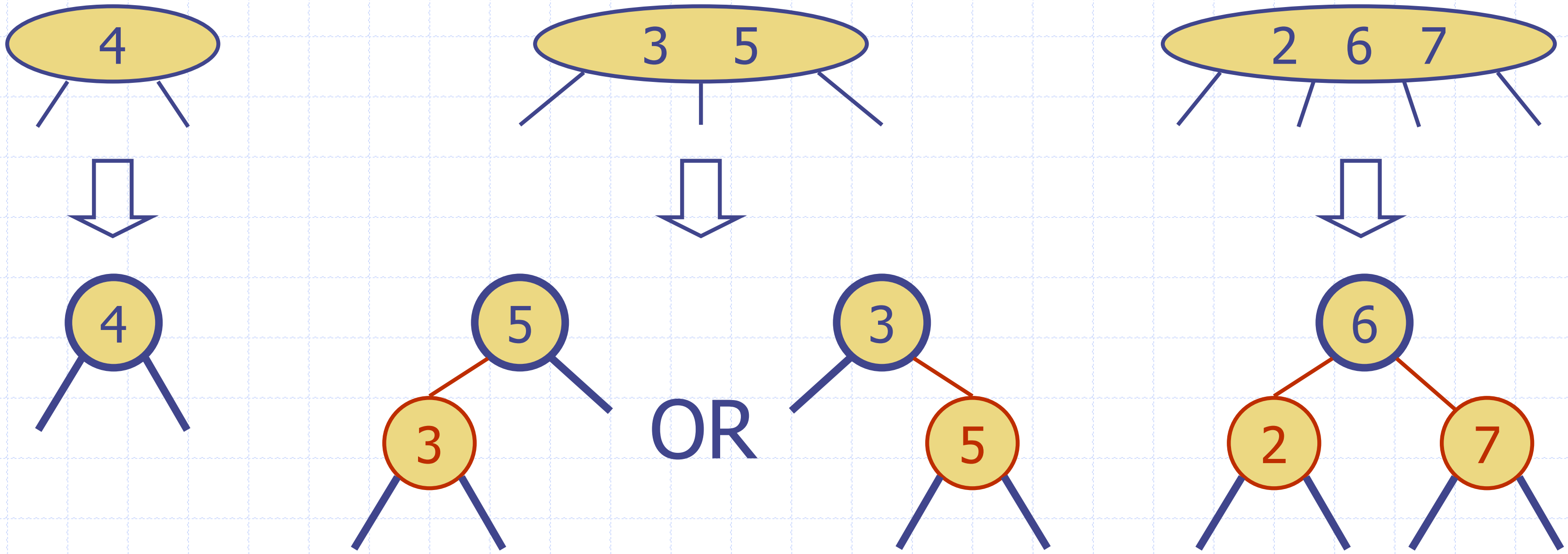
Red-Black Trees



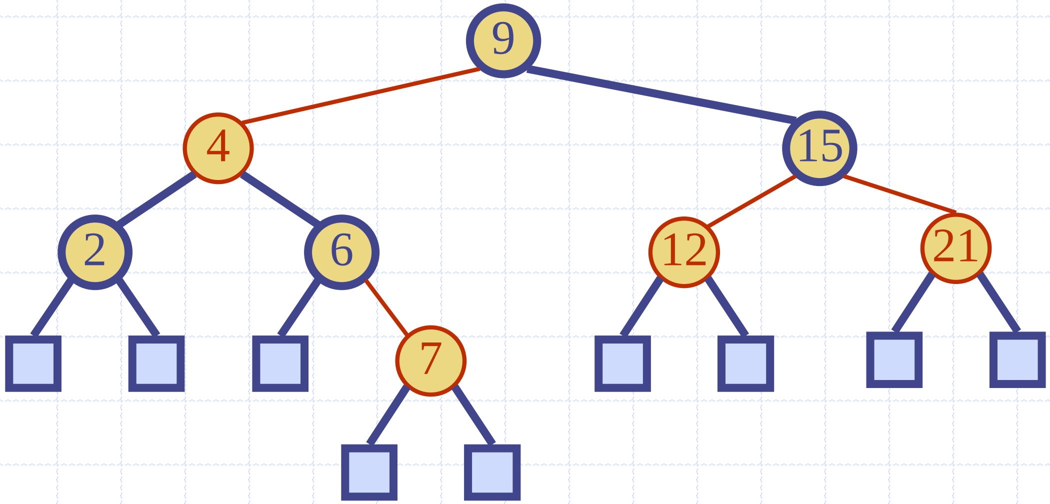
From (2,4) to Red-Black Trees

- A red-black tree is a representation of a (2,4) tree by means of a binary tree whose nodes are colored red or **black**
- In comparison with its associated (2,4) tree, a red-black tree has
  - same logarithmic time performance
  - simpler implementation with a single node type



Red-Black Trees

- A red-black tree can also be defined as a binary search tree that satisfies the following properties:
  - Root Property: the root is black
  - External Property: every leaf is black
  - Internal Property: the children of a red node are black
  - Depth Property: all the leaves have the same black depth



Height of a Red-Black Tree

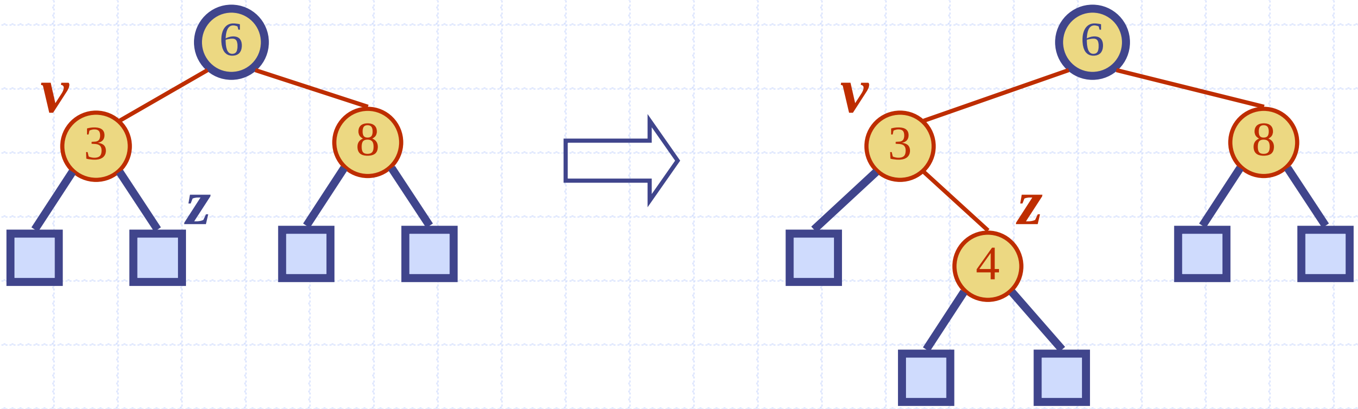
- Theorem: A red-black tree storing n items has height $O(\log n)$

  Proof:

  - The height of a red-black tree is at most twice the height of its associated (2,4) tree, which is $O(\log n)$
- The search algorithm for a binary search tree is the same as that for a binary search tree
- By the above theorem, searching in a red-black tree takes $O(\log n)$ time

Insertion

- To insert (k, o), we execute the insertion algorithm for binary search trees and color the newly inserted node red z unless it is the root
  - We preserve the root, external, and depth properties
  - If the parent v of z is black, we also preserve the internal property and we are done
  - Else (v is red) we have a double red (i.e., a violation of the internal property), which requires a reorganization of the tree
- Example where the insertion of 4 causes a double red:

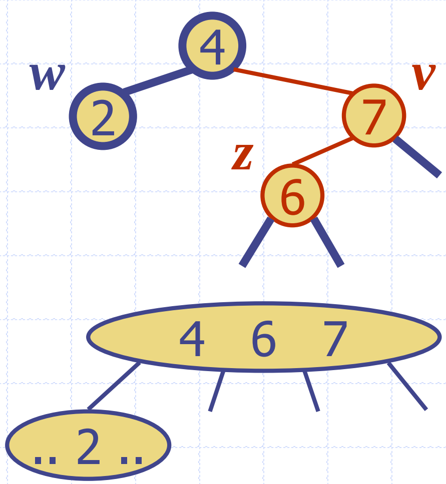


Remedying a Double Red

- Consider a double red with child z and parent v, and let w be the sibling of v

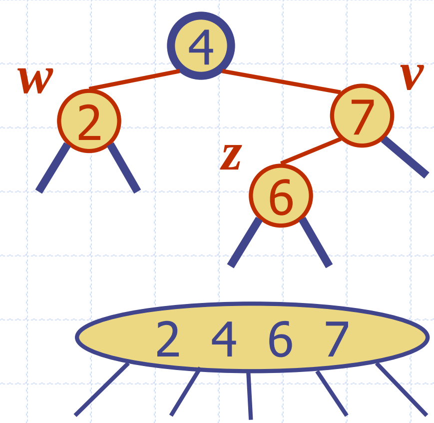
Case 1: w is black

- The double red is an incorrect replacement of a 4-node
- Restructuring: we change the 4-node replacement



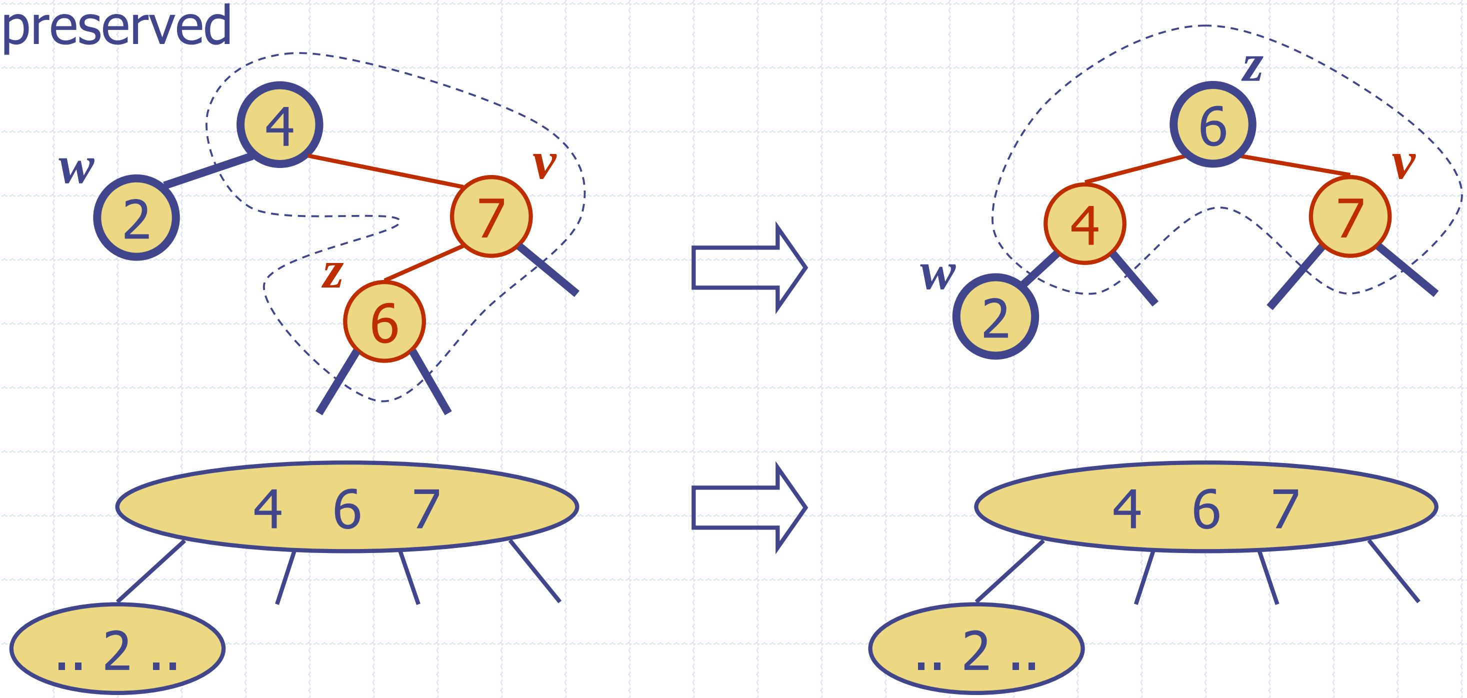
Case 2: w is red

- The double red corresponds to an overflow
- Recoloring: we perform the equivalent of a split



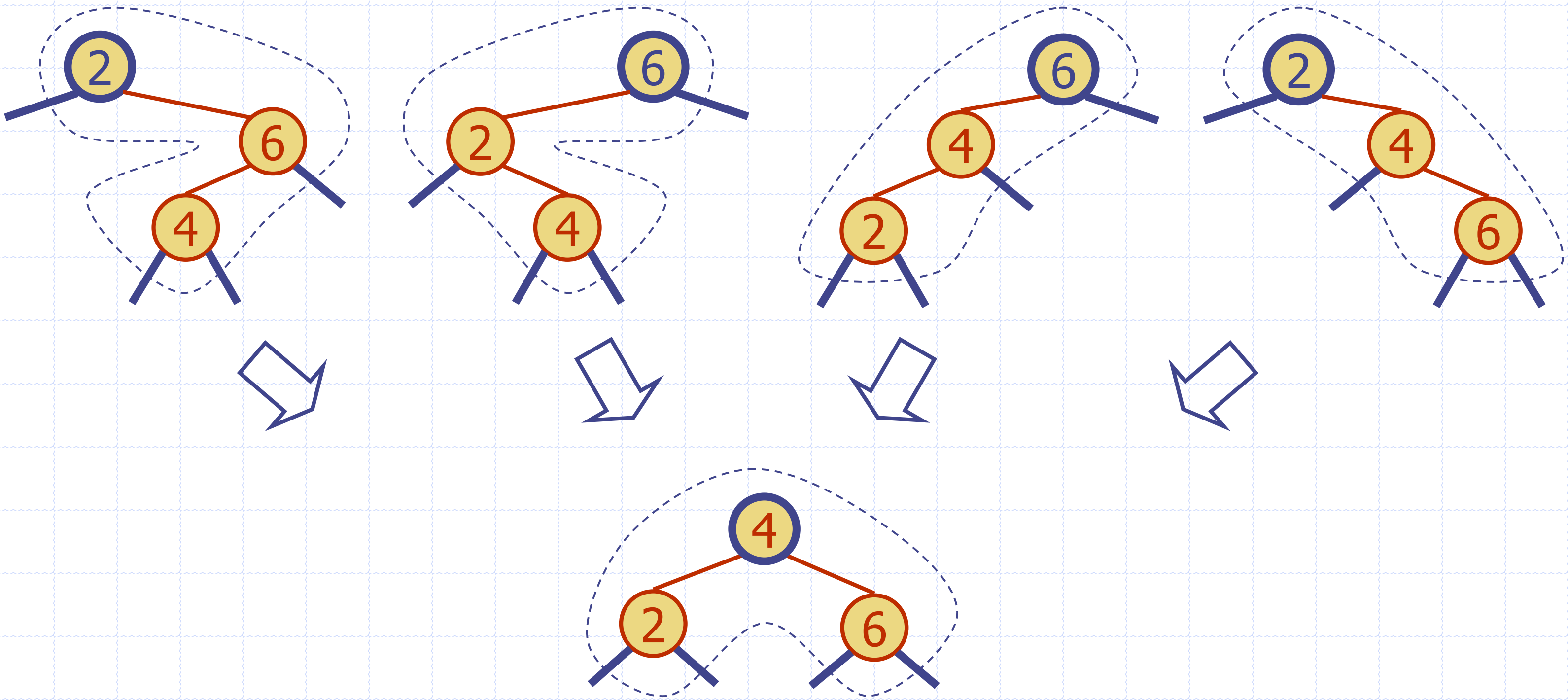
Restructuring

- A restructuring remedies a child-parent double red when the parent red node has a black sibling
- It is equivalent to restoring the correct replacement of a 4-node
- The internal property is restored and the other properties are preserved



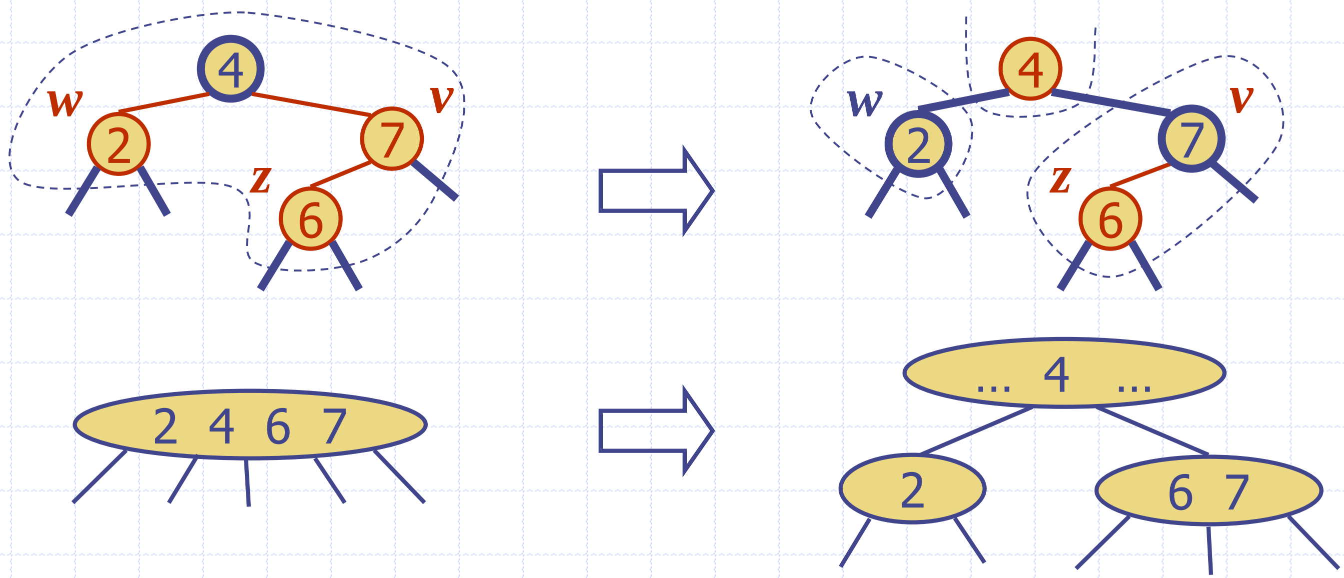
Restructuring

- There are four restructuring configurations depending on whether the double red nodes are left or right children



Recoloring

- A recoloring remedies a child-parent double red when the parent red node has a red sibling
- The parent *v* and its sibling *w* become black and the grandparent *u* becomes red, unless it is the root
- It is equivalent to performing a split on a 5-node
- The double red violation may propagate to the grandparent *u*



There is no need to re-color the root since a black node can have red and black children. Furthermore, the root is traversed in any case so the depth property is preserved

Analysis of Insertion

Algorithm *insert*(*k*, *o*)

1. We search for key ***k*** to locate the insertion node ***z***

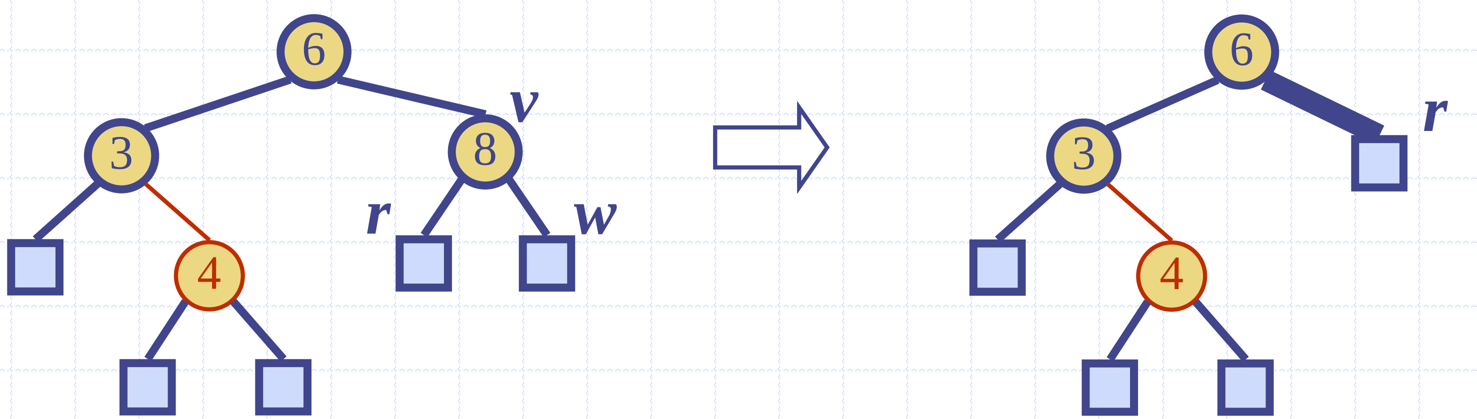
2. We add the new entry (***k***, ***o***) at node ***z*** and color ***z*** red

3. **while** ***doubleRed***(***z***)
   if ***isBlack***(***sibling***(***parent***(***z***)))
   z ← ***restructure***(***z***)
   return
   else { ***sibling***(***parent***(***z***) is red }
   z ← ***recolor***(***z***)

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
- Step 2 takes $O(1)$ time
- Step 3 takes $O(\log n)$ time because we perform
  - $O(\log n)$ recolorings, each taking $O(1)$ time, and
  - at most one restructuring taking $O(1)$ time
- Thus, an insertion in a red-black tree takes $O(\log n)$ time

Deletion

- To perform operation remove(k), we first execute the deletion algorithm for binary search trees
- Let v be the internal node removed, w the external node removed, and r the sibling of w
  - If either v of r was red, we color r black and we are done
  - Else (v and r were both black) we color r ***double black***, which is a violation of the depth property requiring a reorganization of the tree
- Example where the deletion of 8 causes a double black:



Remedying a Double Black

- The algorithm for remedying a double black node w with sibling y considers three cases

  Case 1: y is black and has a red child
  - We perform a restructuring, equivalent to a transfer, and we are done (figure 11.29)

  Case 2: y is black and its children are both black
  - We perform a recoloring, equivalent to a fusion, which may propagate up the double black violation (figure 11.30)

  Case 3: y is red
  - We perform an adjustment, equivalent to choosing a different representation of a 3-node, after which either Case 1 or Case 2 applies (figure 11.31)
- Deletion in a red-black tree takes $O(\log n)$ time

Red-Black Tree Reorganization

Insertion	remedy double red	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	change of 4-node representation	double red removed
recoloring	split	double red removed or propagated up

Deletion	remedy double black	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	transfer	double black removed
recoloring	fusion	double black removed or propagated up
adjustment	change of 3-node representation	restructuring or recoloring follows

Notes

- RB Tree Visualization:
http://www.cs.usfca.edu/~galles/visualization/RedBlack.html
- 2-4 Trees and RB Trees go hand in hand
- I suggest you go over chapters 11.4 and 11.5 together
  - Figures are especially helpful!
  - The book also has source code
- In practice, most of the time RB trees are used instead of 2-4 trees.
- But why use RB Trees if we have AVL Trees?

AVL vs RB Trees

	Height	Search	Insert	Delete
AVL	~1.44log n	O(log n)	Search + Rebalance O(log n) + O(1)	Search + Rebalance O(log n) + O(log n)
RB	~2log n	O(log n)	Search + Recolor* + Restructure O(log n) + O(log n) + O(1)	Search + Recolor* + Restructure O(log n) + O(log n) + O(1)

- The worst case recoloring is O(log n) but it is actually **amortized O(1)!** Even though deletion is still O(log n) in both cases, RB trees run faster. Furthermore, recoloring has a much smaller constant than tree rotations. Thus RB trees are preferred for removal intensive applications (which also implies insertion intensive since you cannot delete non-existent nodes)
- Even though both heights are O(log n), AVL trees are shorter making searching faster thus they should be used for lookup intensive applications OR when n gets very large to such that searching dominates the complexity