Question 1

A.)

$$\frac{}{a_1 \cong a_2}\ a_1, a_2 \in \mathbb{Q} \qquad \frac{t_1 \cong t_2 \quad t_3 \cong t_4}{(a_1, t_1, t_3) \cong (a_2, t_2, t_4)}\ \begin{matrix} a_1, a_2 \in \mathbb{Q} \\ t_1, t_2, t_3, t_4 \in \text{Tree} \end{matrix}$$

B.) A has leaves $\underset{6\ \ 6}{\overset{18}{\wedge}}$ with $6+6 \neq 18$ and $\frac{6 \cdot 6}{6} \neq 18$ so $A \notin \text{SixTree}$

C is not full so $C \notin \text{SixTree}$

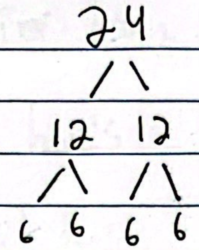
D has a leaf of 12 so $D \notin \text{SixTree}$

For B: We proceed by inversion:

$$a = 6 = \frac{\text{root}(6)\,\text{root}(6)}{6} = 6$$

$$\text{SumNode}\ \frac{\text{ProdNode}\ \dfrac{(\text{Leaf}\ \dfrac{}{6 \in \text{SixTree}} \quad \text{6Leaf}\ \dfrac{}{6 \in \text{SixTree}}}{(6,6,6) \in \text{SixTree}} \qquad \text{Leaf18}\ \dfrac{}{18 \in \text{SixTree}}}{(24,(6,6,6),18) \in \text{SixTree}}\ \begin{matrix} 24 \\ = \text{root}(6,6,6) \\ + \\ \text{root}(18) \\ = 6 + 18 = 24 \end{matrix}$$

1(c.)



$(24, (12,6,6), (12,6,6))$

$a = root(6) + root(6) = 12$

$$\dfrac{\dfrac{\dfrac{}{6 \in SixTree}\,\text{Leaf 6} \quad \dfrac{}{6 \in SixTree}\,\text{Leaf 6}}{(12,6,6) \in SixTree}\,\text{SumNode} \qquad \dfrac{\dfrac{}{6 \in SixTree}\,\text{Leaf 6} \quad \dfrac{}{6 \in SixTree}\,\text{Leaf 6}}{(12,6,6) \in SixTree}\,\text{SumNode}}{(24(12,6,6),(12,6,6))}\,\text{Sumnode}$$

$a = root(6) + root(6) = 12$

$a = root(12,6,6) + root(12,6,6)$
$= 12 + 12$
$= 24$

OR

$a = 12 = root(6) + root(6) = 12$

$$\dfrac{\dfrac{\dfrac{}{6 \in SixTree}\,\text{Leaf 6} \quad \dfrac{}{6 \in SixTree}\,\text{Leaf 6}}{(12,6,6) \in SixTree}\,\text{Sumnode} \qquad \dfrac{\dfrac{}{6 \in SixTree}\,\text{Leaf 6} \quad \dfrac{}{6 \in SixTree}\,\text{Leaf 6}}{(12,6,6) \in SixTree}\,\text{Sumnode}}{(24, (12,6,6), (12,6,6))}\,\text{Prodnode}$$

$a = 12 = root(6) + root(6) = 12$

$a = 24 = \dfrac{root(12,6,6)\, root(12,6,6)}{6}$
$= 144/6$
$= 24$

1D.) i For any property P, If $ST = SixTree$

- $P(6)$ holds
- $P(18)$ holds
- For all $t_1 \in ST$, $t_2 \in ST$ if $P(t_1), P(t_2)$ holds then $P(a, t_1, t_2)$ holds where $a = root(t_1) + root(t_2)$
- For all $t_1 \in ST$, $t_2 \in ST$, if $P(t_1), P(t_2)$ holds then $P(a, t_1, t_2)$ holds where $a = \frac{root(t_1) root(t_2)}{6}$

Then

- for all $t \in ST$, $P(t)$ holds

ii) For all $t \in SixTree$, We will prove the property $P(t) =$ "$\exists i \in \mathbb{Z}.\ root(t) = 6i$"

Note that if $root(t)$ is divisible by 6, it is also divisible by 2, so this implies the given property.

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1D iii.)

Rule 1: WTS $\exists i \in \mathbb{Z}$ s.t. $root(6) = 6i$.
Take $i = 1$.

Rule 2: WTS $\exists i \in \mathbb{Z}$ s.t. $root(18) = 6i$.
Take $i = 3$.

Rule 3: WTS for $t_1 \in ST$, $t_2 \in ST$ if

$a = root(t_1) + root(t_2)$ then $\exists i \in \mathbb{Z}$ s.t.

$root(a, t_1, t_2) = 6i$. By induction hypothesis we

have $\exists i', i''$ s.t. $root(t_1) = 6i'$ $root(t_2) = 6i''$. Then

$a = root(t_1) + root(t_2) = 6i' + 6i'' = 6(i' + i'')$. So

$i = i' + i''$ is a solution since $\mathbb{Z}$ is closed under

addition.

Rule 4: WTS for $t_1, t_2 \in ST$ where

$a = \dfrac{root(t_1)\, root(t_2)}{6}$ then $\exists i \in \mathbb{Z}$ s.t.

$root(a, t_1, t_2) = 6i$. By induction hypothesis, we

assume property holds for $t_1, t_2 \Rightarrow \exists i', i''$ s.t.

$root(t_1) = 6i'$, $root(t_2) = i''$. Now, since $\mathbb{Z}$

closed under multiplication let $i = i'i''$. Then

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$$\begin{aligned} root(a, t_1, t_2) = a &= \frac{root(t_1)\,root(t_2)}{6} \\ &= \frac{6i'\,6i''}{6} \\ &= 6i'i'' \\ &= 6i \quad \text{for } i = i'i'' \end{aligned}$$

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2a.) $\langle foo := 0;\ (foo := foo + 1;\ foo)^5, \sigma_0 \rangle$

$\Rightarrow 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$

$\Rightarrow 120$

$\langle (120), \sigma \rangle$ where $\sigma(foo) = 5$

2B.)

$$Exp_1 \quad \frac{}{\langle e_2^{\,n}, \sigma \rangle \rightarrow \langle 1, \sigma' \rangle} \quad n \leq 0$$

$$Exp_2 \quad \frac{}{\langle e_2^{\,n}, \sigma \rangle \rightarrow \langle e_2 \times e_2^{\,n-1}, \sigma \rangle} \quad n > 0$$

$$Exp_3 \quad \frac{\langle e_1, \sigma \rangle \rightarrow \langle e_1', \sigma' \rangle}{\langle e_2^{\,e_1}, \sigma \rangle \rightarrow \langle e_2^{\,e_1'}, \sigma' \rangle}$$

3A.) For any property P,

If

Var: For all variables x, stores σ and integers n such that $\sigma(x) = n$, $P(\langle x, \sigma\rangle \to \langle n, \sigma\rangle)$ holds

Add: For all integers n, m, p such that $p = n + m$, and stores σ, $P(\langle n+m, \sigma\rangle \to \langle p, \sigma\rangle)$ holds

Mul: For all integers n, m, p such that $p = n \times m$ and stores σ, $P(\langle n \times m, \sigma\rangle \to \langle p, \sigma\rangle)$ holds

Asg: For all variables x, integers n and expressions $e \in Exp$, $P(\langle x := n; e, \sigma\rangle \to \langle e, \sigma[x \mapsto n]\rangle)$

LAdd: For all expressions $e_1, e_2, e_1' \in Exp$ and stores σ and σ', if $P(\langle e_1, \sigma\rangle \to \langle e_1', \sigma'\rangle)$ holds then $P(\langle e_1 + e_2, \sigma\rangle \to \langle e_1' + e_2, \sigma'\rangle)$ holds

RAdd: For all integers n, expressions $e_2, e_2' \in Exp$ and stores σ and σ', if $P(\langle e_2, \sigma\rangle \to \langle e_2', \sigma'\rangle)$ holds then $P(\langle n + e_2, \sigma\rangle \to \langle n + e_2', \sigma'\rangle)$ holds

LMul: For all expressions $e_1, e_2, e_1' \in Exp$ and stores σ and σ', if $P(\langle e_1, \sigma\rangle \to \langle e_1', \sigma'\rangle)$ holds then $P(\langle e_1 \times e_2, \sigma\rangle \to \langle e_1' \times e_2, \sigma'\rangle)$ holds

Rmul: For all integers n, expressions $e_2, e_2' \in Exp$ and stores σ and σ', if $P(\langle e_2, \sigma\rangle \to \langle e_2', \sigma'\rangle)$ holds then $P(\langle n \times e_2, \sigma\rangle \to \langle n \times e_2', \sigma'\rangle)$ holds

ASG1: For all variables x, expressions e_1, e_2, e_1' $\in$ Exp and stores σ and σ', if
$P(\langle e_1, \sigma\rangle \rightarrow \langle e_1', \sigma'\rangle)$ holds then
$P(\langle x := e_1; e_2, \sigma\rangle \rightarrow \langle x := e_1'; e_2, \sigma'\rangle)$ holds

B.) For all $\langle e,\sigma\rangle \to \langle e',\sigma'\rangle$ we will prove the property "$P(\langle e,\sigma\rangle \to \langle e',\sigma'\rangle) = \forall n,\sigma''$ if $\langle e',\sigma'\rangle \Downarrow \langle n,\sigma''\rangle$ then $\langle e,\sigma\rangle \Downarrow \langle n,\sigma''\rangle$"

C.) Var: WTS $\forall$ variables x, stores σ, integers n such that $\sigma(x)=n$, $P(\langle x,\sigma\rangle \to \langle n,\sigma\rangle)$ holds meaning $\forall n', \sigma''$ if $\langle n,\sigma\rangle \Downarrow \langle n',\sigma''\rangle$ then $\langle x,\sigma\rangle \Downarrow \langle n',\sigma''\rangle$.

Suppose $\langle n,\sigma\rangle \Downarrow \langle n',\sigma''\rangle$. By intLRG $n=n'$, $\sigma=\sigma''$. By VarLRG

$$\frac{\sigma(x)=n''}{\langle x,\sigma\rangle \Downarrow \langle n'',\sigma\rangle}$$

For some n''. By the induction principle $\sigma(x)=n \Rightarrow n=n''=n'$. Since we already have $\sigma=\sigma''$ then

$\langle x,\sigma\rangle \Downarrow \langle n',\sigma''\rangle$

Add: WTS for all integers n, m, p s.t. $p=n+m$ and stores σ that $\forall n', \sigma''$ if $\langle p,\sigma\rangle \Downarrow \langle n',\sigma''\rangle$ then

$\langle n+m,\sigma\rangle \Downarrow \langle n',\sigma''\rangle$ where $p=n+m$, $n,m,p \in \mathbb{Z}$

Suppose $\langle p,\sigma\rangle \Downarrow \langle n',\sigma''\rangle$. By INTLRG we know $p=n'$, $\sigma=\sigma''$. Using this, we know again by INTLRG that

$$\text{INTLRG}\ \frac{}{\langle n,\sigma\rangle \Downarrow \langle n,\sigma\rangle} \Rightarrow \langle n,\sigma\rangle \Downarrow \langle n,\sigma''\rangle$$

$$\text{INTLRG}\ \frac{}{\langle m,\sigma\rangle \Downarrow \langle m,\sigma''\rangle} \Rightarrow \langle m,\sigma\rangle \Downarrow \langle m,\sigma''\rangle$$

By AddLRG

$$\frac{\langle n,\sigma\rangle \Downarrow \langle n,\sigma''\rangle \quad \langle m,\sigma\rangle \Downarrow \langle m,\sigma''\rangle}{\langle n+m,\sigma\rangle \Downarrow \langle n+m,\sigma''\rangle}$$

Since $p=n+m=n'$ then $\langle n+m,\sigma\rangle \Downarrow \langle n',\sigma''\rangle$

for all integers n, m, p s.t. $p = n \times m$ and stores σ

MUL: WTS $\forall n', \sigma''$ if $\langle p, \sigma \rangle \Downarrow \langle n', \sigma'' \rangle$ then

$\langle n \times m, \sigma \rangle \Downarrow \langle n', \sigma'' \rangle$ where $p = n \times m$, $p, n, m \in \mathbb{Z}$

Suppose $\langle p, \sigma \rangle \Downarrow \langle n', \sigma'' \rangle$. By INTLRG

we know $p = n'$, $\sigma = \sigma''$. Similarly to as in

Add, this gives us by INTLRG

$$\text{INTLRG} \; \frac{}{\langle n, \sigma \rangle \Downarrow \langle n, \sigma'' \rangle} \qquad \text{INTLRG} \; \frac{}{\langle m, \sigma \rangle \Downarrow \langle m, \sigma'' \rangle}$$

Thus by MULLRG

$$\text{MULLRG} \; \frac{\langle n, \sigma \rangle \Downarrow \langle n, \sigma'' \rangle \qquad \langle m, \sigma \rangle \Downarrow \langle m, \sigma'' \rangle}{\langle n \times m, \sigma \rangle \Downarrow \langle n''', \sigma'' \rangle}$$

Where $n''' = n \times m = p = n'$

$\Rightarrow \langle n \times m, \sigma \rangle \Downarrow \langle n', \sigma'' \rangle$

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for all variables x, integers n and expressions $e \in Exp$ that

ASG: WTS $\forall n', \sigma''$ if $\langle e, \sigma[x \mapsto n] \rangle \Downarrow \langle n', \sigma'' \rangle$

then $\langle x := n; e, \sigma \rangle \Downarrow \langle n', \sigma'' \rangle$

Suppose $\langle e, \sigma[x \mapsto n] \rangle \Downarrow \langle n', \sigma'' \rangle$.

By INT_{LRC} $\dfrac{}{\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle}$

Then by ASG_{LRC} $\dfrac{\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle \quad \langle e, \sigma[x \mapsto n] \rangle \Downarrow \langle n', \sigma'' \rangle}{\langle x := n; e, \sigma \rangle \Downarrow \langle n', \sigma'' \rangle}$

LADD WTS for all expressions $e_1, e_2, e_1' \in Exp$ and stores σ and σ' if $\forall\, n', \sigma''$ we have that if $\langle e_1', \sigma' \rangle \Downarrow \langle n', \sigma'' \rangle$ then $\langle e_1, \sigma \rangle \Downarrow \langle n', \sigma'' \rangle$ then we have that if $\langle e_1' + e_2, \sigma \rangle \Downarrow \langle n'', \sigma''' \rangle$ then $\langle e_1 + e_2, \sigma \rangle \Downarrow \langle n'', \sigma''' \rangle$ $\forall n', \sigma'''$

Suppose $\langle e_1' + e_2, \sigma \rangle \Downarrow \langle n'', \sigma''' \rangle$. Inversion

$$\text{AddLRG}\ \frac{\langle e_1', \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle \quad \langle e_2, \sigma'' \rangle \Downarrow \langle n_2, \sigma''' \rangle}{\langle e_1' + e_2, \sigma \rangle \Downarrow \langle n'', \sigma''' \rangle}$$

Where $n'' = n_1 + n_2$

By I.H. Since $\langle e_1', \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle$ then $\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle$

Then by

$$\text{AddLRG}\ \frac{\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle \quad \langle e_2, \sigma'' \rangle \Downarrow \langle n_2, \sigma''' \rangle}{\langle e_1 + e_2, \sigma \rangle \Downarrow \langle n''', \sigma''' \rangle}$$

Where $n''' = n_1 + n_2 = n''$

$\Rightarrow \langle e_1 + e_2, \sigma \rangle \Downarrow \langle n'', \sigma''' \rangle$

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RADD WTS $\forall$ integers n, expressions $e_2, e_2' \in Exp$ and Stores σ and σ' if $\forall n^*, \sigma^*$ $\langle e_2', \sigma' \rangle \Downarrow \langle n^*, \sigma^* \rangle$ then $\langle e_2, \sigma \rangle \Downarrow \langle n^*, \sigma^* \rangle$ then $\forall n', \sigma''$ if $\langle n + e_2', \sigma' \rangle \Downarrow \langle n', \sigma'' \rangle$ then $\langle n + e_2, \sigma \rangle \Downarrow \langle n', \sigma'' \rangle$

Suppose $\langle n + e_2', \sigma' \rangle \Downarrow \langle n', \sigma'' \rangle$. By AddLRG inversion we know that $\exists n_1, n_2, \sigma'''$ such that $\langle n, \sigma' \rangle \Downarrow \langle n_1, \sigma''' \rangle$ and $\langle e_2', \sigma''' \rangle \Downarrow \langle n_2, \sigma'' \rangle$ where $n_1 + n_2 = n'$. By IntLRG, $n = n_1$, $\sigma' = \sigma'''$. By inductive hypothesis if $\langle e_2', \sigma' \rangle \Downarrow \langle n_2, \sigma'' \rangle$ then $\langle e_2, \sigma \rangle \Downarrow \langle n_2, \sigma'' \rangle$. By IntLRG $\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle$ then

$$\text{ADD}_{LRG}\ \frac{\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle \quad \langle e_2, \sigma \rangle \Downarrow \langle n_2, \sigma'' \rangle}{\langle n + e_2, \sigma \rangle \Downarrow \langle n'', \sigma'' \rangle}$$

Where $n'' = n + n_2 = n_1 + n_2 = n'$ so

$$\langle n + e_2, \sigma \rangle \Downarrow \langle n', \sigma'' \rangle$$

n

LMUL: WTS for all expressions $e_1, e_2, e_1' \in Exp$ and Stores σ and σ' if $P(\langle e_1, \sigma\rangle \to \langle e_1', \sigma'\rangle)$ then $P(\langle e_1 \times e_2, \sigma\rangle \to \langle e_1' \times e_2, \sigma'\rangle)$ holds.

Suppose $\langle e_1' \times e_2, \sigma'\rangle \Downarrow \langle n', \sigma'''\rangle$. for arbitrary σ''', n'. By inversion we know $\exists n_1, n_2, \sigma''$

$$\text{MULT}_{LRC}\ \frac{\langle e_1', \sigma'\rangle \Downarrow \langle n_1, \sigma''\rangle \quad \langle e_2, \sigma''\rangle \Downarrow \langle n_2, \sigma'''\rangle}{\langle e_1' \times e_2, \sigma'\rangle \Downarrow \langle n', \sigma'''\rangle}$$

Where $n_1 n_2 = n'$. By induction hypothesis if $\langle e_1', \sigma'\rangle \Downarrow \langle n_1, \sigma''\rangle$ then $\langle e_1, \sigma\rangle \Downarrow \langle n_1, \sigma''\rangle$. Then

$$\text{MULT}_{LRC}\ \frac{\langle e_1, \sigma\rangle \Downarrow \langle n_1, \sigma''\rangle \quad \langle e_2, \sigma''\rangle \Downarrow \langle n_2, \sigma'''\rangle}{\langle e_1 \times e_2, \sigma\rangle \Downarrow \langle n''', \sigma'''\rangle}$$

Where $n''' = n_1 n_2 = n'$ so

$$\langle e_1 \times e_2, \sigma\rangle \Downarrow \langle n', \sigma'''\rangle$$

RMUL. WTS $\forall$ integers n, expressions $e_2, e_2' \in Exp$ and stores σ and σ' if $\forall n^*, \sigma^*$

$\langle e_2', \sigma' \rangle \Downarrow \langle n^*, \sigma^* \rangle \Rightarrow \langle e_2, \sigma \rangle \Downarrow \langle n^*, \sigma^* \rangle$

then $\forall n', \sigma''$ if $\langle n \times e_2', \sigma' \rangle \Downarrow \langle n', \sigma'' \rangle$

then $\langle n \times e_2, \sigma \rangle \Downarrow \langle n', \sigma'' \rangle$.

Suppose $\langle n \times e_2', \sigma' \rangle \Downarrow \langle n', \sigma'' \rangle$.

Then by MUL_{LRG} inversion we know $\exists n_1, n_2, \sigma'''$

such that $\langle n, \sigma' \rangle \Downarrow \langle n_1, \sigma''' \rangle$ and

$\langle e_2', \sigma''' \rangle \Downarrow \langle n_2, \sigma'' \rangle$ where

$n_1 n_2 = n'$. By INT_{LRG} $n = n_1$, $\sigma' = \sigma'''$.

By I.H. if $\langle e_2', \sigma' \rangle \Downarrow \langle n_2, \sigma'' \rangle$ then

$\langle e_2, \sigma \rangle \Downarrow \langle n_2, \sigma'' \rangle$. By INT_{LRG}

$\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle$. Then

$$MUL_{LRG}\ \frac{\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle \quad \langle e_2, \sigma \rangle \Downarrow \langle n_2, \sigma'' \rangle}{\langle n \times e_2, \sigma \rangle \Downarrow \langle n'', \sigma'' \rangle}$$

where $n'' = n n_2 = n_1 n_2 = n'$ so

$\langle n \times e_2, \sigma \rangle \Downarrow \langle n', \sigma'' \rangle$

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ASG1

For all variables x, expressions $e_1, e_2, e_1' \in Exp$, and stores σ and σ' suppose $P(\langle e_1, \sigma\rangle \to \langle e_1', \sigma'\rangle)$ holds.

Suppose $\langle x := e_1'; e_2, \sigma'\rangle \Downarrow \langle n', \sigma'''\rangle$ for some n', σ'''. By inversion we know there exists

$$\text{ASG}_{LRC}\ \frac{\langle e_1', \sigma'\rangle \Downarrow \langle n_1, \sigma''\rangle \quad \langle e_2, \sigma''[x \mapsto n_1]\rangle \Downarrow \langle n', \sigma'''\rangle}{\langle x := e_1'; e_2, \sigma'\rangle \Downarrow \langle n', \sigma'''\rangle}$$

By I.H. if $\langle e_1', \sigma'\rangle \Downarrow \langle n_1, \sigma''\rangle$ then $\langle e_1, \sigma\rangle \Downarrow \langle n_1, \sigma''\rangle$ then by ASG_{LRC}

$$\text{ASG LRG}\ \frac{\langle e_1, \sigma\rangle \Downarrow \langle n_1, \sigma''\rangle \quad \langle e_2, \sigma''[x \mapsto n_1]\rangle \Downarrow \langle n', \sigma'''\rangle}{\langle x := e_1; e_2, \sigma\rangle \Downarrow \langle n', \sigma'''\rangle}$$

~~4~~ Question 4

A.) Push

$$\frac{}{\langle push\ n, s\rangle \to \langle Skip, n::s\rangle}$$

Add

$$\frac{}{\langle add, n_0::n_1::s\rangle \to \langle Skip, n'::s\rangle} \quad \text{where } n' = n_0 + n_1$$

Mul

$$\frac{}{\langle mul, n_0::n_1::s\rangle \to \langle Skip, n'::s\rangle} \quad \text{where } n' = n_0 \cdot n_1$$

Dup

$$\frac{}{\langle dup, n::s\rangle \to \langle Skip, n::n::s\rangle}$$

Seq

$$\frac{}{\langle Skip; c_2, s\rangle \to \langle c_2, s\rangle}$$

Seq2

$$\frac{\langle c_1, s\rangle \to \langle c_1', s'\rangle}{\langle Skip; c_2, s\rangle \to \langle c_2, s\rangle}$$

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B.) $\langle pop, [\,] \rangle$, this is not final because we still have command pop, but pop cannot be executed on empty stack so we are stuck.

c.) ⟨ push 0;
while $\neq$ 0 {
swap 1;
dup;
push -1;
add;
swap 1;
swap 2;
};
pop; ⟩

At the end of each iteration, I want something of the form $0 :: (n-i) :: (n-(i-1)) :: \cdots :: n :: []$. If I have this then in the next iteration, I move $n-i$ to the top of the stack, turn a copy of it into $(n-(i+1))$ and then reorder the top 3 elements to again have $0 :: (n-(i+1)) :: (n-i) :: (n-(i-1))$ etc.

When $i = n$, at the end of the $i-1$ iteration we have $0 :: (n-(n-1+1)) :: \cdots$
$\Rightarrow 0 :: 0 :: (n-(n-1)) :: \cdots$
So we terminate and pop the extra 0.

$n :: []$ ← Start
$0 :: n :: []$ ← end of iteration 0
$n :: 0 :: []$
$n :: n :: 0 :: []$
$-1 :: n :: n :: 0 :: []$
$(n-1) :: n :: 0 :: []$
$n :: (n-1) :: 0 :: []$
$0 :: (n-1) :: n :: []$ ← end of iteration 1
⋮
$0 :: (n-i) :: (n-(i-1)) :: \cdots n :: []$ ← end of iteration i

D.) <Push 0;

Swap 1;

Whilene {

Swap 1;
dup;
Push 1;
add;
Swap 1;
Swap 2;
}
Pop; >

$n :: []$
$0 :: n :: []$
$n :: 0 :: []$ ← iteration 0
$0 :: n :: []$
$0 :: 0 :: n :: []$
$1 :: 0 :: 0 :: n :: []$
$1 :: 0 :: n :: []$
$0 :: 1 :: n :: []$
$n :: 1 :: 0 :: []$ ← iteration 1
⋮
$n :: i :: i-1 :: \cdots :: 0 :: []$ ← iteration i

At the start of every iteration i I have something of the form $n :: i :: i-1 :: \cdots 0 :: []$. We can see this in base case iteration 0. If at the end of iteration i we have something of this form, look at iteration i+1. We first bring "i" to the top of the stack and use that to add a new element to the stack i+1. We then have top 3 elements i+1, i, n and we rearrange these to end up with $n :: i+1 :: i ::$ etc. at the end of the iteration with the rest of the stack unchanged. Thus at the start of i+1 I still have something in correct form. When $i = n-1$, we will end round i with $n :: n-1+1 :: n-1 ::$ etc. and halt. Then we pop off n and end with $n :: n-1 :: \cdots 0 :: []$.

E.) <Push 1;
Swap 1;
Push 0;
Swap 1;
Whilene {
Swap 1;
Push 1;
add;
dup;
Swap 3;
Mul;
Swap 2;
Swap 1;
};
Pop;
Pop; >

(1) Swap 1; Push 1; add;
(2) dup; Swap 3; Mul;
(3) Swap 2; Swap 1;

n::[]
1::n::[]
n::1::[]
0::n::1::[]
n::0::1::[] ← iteration 0
0::n::1::[]
1::0::n::1::[]
1::n::1::[]
1::1::n::1::[]
1::1::n::1::[]
1::n::1::[]
1::n::1::[]
n::1::1::[]

At the start of each iteration i, I have the form n::i::i!. This is true for i=0. If it's true for some i, then during iteration i+1 we bring our counter "i" to the top of the stack and increase it. We then multiply our "store" i! by the new counter i+1 giving us i+1 factorial. We then arrange our 3 elements to be back in form n::i+1::(i+1)!. When i=n-1, we end found i with n::n::n! and terminate. Then we remove n and n.