Fuzzy Relations, Rules and Inferences

Fuzzy Relations

Crisp relations

To understand the fuzzy relations, it is better to discuss first crisp relation.

Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$

Note :

(1) $A \times B \neq B \times A$

(2) $|A \times B| = |A| \times |B|$

(3)$A \times B$ provides a mapping from $a \in A$ to $b \in B$.

The mapping so mentioned is called a relation.

Crisp relations

Example 1:
Consider the two crisp sets A and B as given below. $A = \{1, 2, 3, 4\}$
$B = \{3, 5, 7\}$.

Then, $A \times B = \{(1,3), (1,5), (1,7), (2,3), (2,5), (2,7), (3,3), (3,5), (3,7), (4,3), (4,5), (4,7)\}$

Let us define a relation R as $R = \{(a,b) | b = a + 1, (a,b) \in A \times B\}$

Then, $R = \{(2,3), (4,5)\}$ in this case.

We can represent the relation R in a matrix form as follows.

$$R = \begin{array}{c c} & \begin{array}{ccc} 3 & 5 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

Union:

$$R(x, y) \cup S(x, y) = max(R(x, y), S(x, y));$$

Intersection:

$$R(x, y) \cap S(x, y) = min(R(x, y), S(x, y));$$

Complement:

$$\overline{R(x, y)} = 1 - R(x, y)$$

Example: Operations on crisp relations

Example:
Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

1. $R \cup S$
2. $R \cap S$
3. $\overline{R}$

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y, Z. Then $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S = \{(x, z) | (x, y) \in R \text{ and } (y, z) \in S \text{ and } \forall y \in Y\}$$

Max-Min Composition

Given the two relation matrices R and S, the max-min composition is defined as $T = R \circ S$;

$$T(x, z) = max\{min\{R(x, y), S(y, z) \text{ and } \forall y \in Y\}\}$$

Composition: Composition

Example:

Given

$X = \{1, 3, 5\}; Y = \{1, 3, 5\}; R = \{(x, y) | y = x + 2\}; S = \{(x, y) | x < y\}$

Here, R and S is on $X \times Y$.

Thus, we have

$R = \{(1, 3), (3, 5)\}$

$S = \{(1, 3), (1, 5), (3, 5)\}$

R= $\begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ and S= $\begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

Using max-min composition $R \circ S$= $\begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set $X_1, X_2, ..., X_n$
- Here, n-tuples $(x_1, x_2, ..., x_n)$ may have varying degree of memberships within the relationship.
- The membership values indicate the strength of the relation between the tuples.

Example:

$X = \{$ typhoid, viral, cold $\}$ and $Y = \{$ running nose, high temp, shivering $\}$

The fuzzy relation R is defined as

$$\begin{array}{cc} & \begin{array}{ccc} runningnose & hightemperature & shivering \end{array} \\ \begin{array}{c} typhoid \\ viral \\ cold \end{array} & \begin{bmatrix} 0.1 & 0.9 & 0.8 \\ 0.2 & 0.9 & 0.7 \\ 0.9 & 0.4 & 0.6 \end{bmatrix} \end{array}$$

Fuzzy Cartesian product

Suppose
A is a fuzzy set on the universe of discourse X with $\mu_A(x)|x \in X$

B is a fuzzy set on the universe of discourse Y with $\mu_B(y)|y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given by $\mu_R(x, y) = \mu_{A\times B}(x, y) = min\{\mu_A(x), \mu_B(y)\}$

Example :

$A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$and $B = \{(b_1, 0.5), (b_2, 0.6)\}$

$$R = A \times B = \begin{array}{c c} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

Union:

$$\mu_{R \cup S}(a, b) = max\{\mu_R(a, b), \mu_S(a, b)\}$$

Intersection:

$$\mu_{R \cap S}(a, b) = min\{\mu_R(a, b), \mu_S(a, b)\}$$

Complement:

$$\mu_{\overline{R}}(a, b) = 1 - \mu_R(a, b)$$

Composition

$$T = R \circ S$$
$$\mu_{R \circ S} = max_{y \in Y}\{min(\mu_R(x, y), \mu_S(y, z))\}$$

Operations on Fuzzy relations: Examples

Example:

$X = (x_1, x_2, x_3)$; $Y = (y_1, y_2)$; $Z = (z_1, z_2, z_3)$;

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$\mu_{R \circ S}(x_1, y_1) = max\{min(x_1, y_1), min(y_1, z_1), min(x_1, y_2), min(y_2, z_1)\}$
$= max\{min(0.5, 0.6), min(0.1, 0.5)\} = max\{0.5, 0.1\} = 0.5$ and so on.

Fuzzy relation : An example

Consider the following two sets P and D, which represent a set of paddy plants and a set of plant diseases. More precisely

$P = \{P_1, P_2, P_3, P_4\}$ a set of four varieties of paddy plants
$D = \{D_1, D_2, D_3, D_4\}$ of the four various diseases affecting the plants

In addition to these, also consider another set $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let, R be a relation on $P \times D$, representing which plant is susceptible to which diseases, then R can be stated as

$$R = \begin{array}{c c} & \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} & \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

Fuzzy relation : An example

Also, consider T be the another relation on $D \times S$, which is given by

$$S = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Obtain the association of plants with the different symptoms of the disease using **max-min composition**.

$$R \circ S = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

Fuzzy Propositions

Two-valued logic vs. Multi-valued logic

- The basic assumption upon which crisp logic is based - that every proposition is either TRUE or FALSE.
- The classical two-valued logic can be extended to multi-valued logic.
- As an example, three valued logic to denote true(1), false(0) and indeterminacy ($\frac{1}{2}$).

Two-valued logic vs. Multi-valued logic

Different operations with three-valued logic can be extended as shown in the following truth table:

a	b	$\wedge$	$\vee$	$\neg$a	$\Longrightarrow$	$=$
0	0	0	0	1	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	0	1	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	0	1	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1

Fuzzy connectives used in the above table are:

AND ($\wedge$), OR ($\vee$), NOT ($\neg$), IMPLICATION ($\Longrightarrow$) and EQUAL ($=$).

Three-valued logic

Fuzzy connectives defined for such a three-valued logic better can be stated as follows:

Symbol	Connective	Usage	Definition
$\neg$	NOT	$\neg$P	$1 - T(P)$
$\vee$	OR	$P \vee Q$	max{T(P), T(Q) }
$\wedge$	AND	$P \wedge Q$	min{ T(P),T(Q) }
$\Longrightarrow$	IMPLICATION	$(P \Longrightarrow Q)$ or $(\neg P \vee Q)$	max{(1 - T(P)), T(Q) }
$=$	EQUALITY	$(P = Q)$ or $[(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)]$	$1 - \lvert T(P) - T(Q) \rvert$

Fuzzy proposition

Example 1:

P : Ram is honest

1. T(P) = 0.0 : Absolutely false
2. T(P) = 0.2 : Partially false
3. T(P) = 0.4 : May be false or not false
4. T(P) = 0.6 : May be true or not true
5. T(P) = 0.8 : Partially true
6. T(P) = 1.0 : Absolutely true.

Example 2 :Fuzzy proposition

P : Mary is efficient ; T(P) = 0.8;

Q : Ram is efficient ; T(Q) = 0.6

1. **Mary is not efficient.**
   $T(\neg P) = 1 - T(P) = 0.2$
2. **Mary is efficient and so is Ram.**
   $T(P \wedge Q) = min\{T(P), T(Q)\} = 0.6$
3. **Either Mary or Ram is efficient**
   $T(P \vee Q) = max T(P), T(Q) = 0.8$
4. **If Mary is efficient then so is Ram**
   $T(P \Rightarrow Q) = max\{1 - T(P), T(Q)\} = 0.6$

Fuzzy proposition vs. Crisp proposition

- The fundamental difference between crisp (classical) proposition and fuzzy propositions is in the range of their truth values.
- While each classical proposition is required to be either true or false, the truth or falsity of fuzzy proposition is a matter of degree.
- The degree of truth of each fuzzy proposition is expressed by a value in the interval [0,1] both inclusive.

Canonical representation of Fuzzy proposition

- Suppose, X is a universe of discourse of five persons. Intelligent of $x \in X$ is a fuzzy set as defined below.

  Intelligent: $\{(x_1, 0.3), (x_2, 0.4), (x_3, 0.1), (x_4, 0.6), (x_5, 0.9)\}$

- We define a fuzzy proposition as follows:

  P : x is intelligent

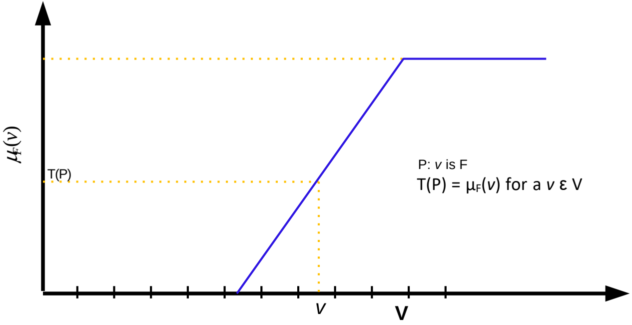
- The canonical form of fuzzy proposition of this type, P is expressed by the sentence P : v is F .

- Predicate in terms of fuzzy set.

  P : ***v* is F** ; where v is an element that takes values v from some universal set V and F is a fuzzy set on V that represents a fuzzy predicate.

- In other words, given, a particular element v , this element belongs to F with membership grade $\mu_F(v)$.

Graphical interpretation of fuzzy proposition



- For a given value *v* of variable V in proposition P, T(P) denotes the degree of truth of proposition P.

Fuzzy Implications

Fuzzy rule

- A fuzzy implication (also known as fuzzy If-Then rule, fuzzy rule, or fuzzy conditional statement) assumes the form :

  If x **is** A **then** y **is** B

  where, A and B are two linguistic variables defined by fuzzy sets A and B on the universe of discourses X and Y, respectively.

- Often, x **is** A is called the antecedent or premise, while y **is** B is called the consequence or conclusion.

Fuzzy implication : Example 1

- If pressure is High then temperature is Low
- If mango is Yellow then mango is Sweet else mango is Sour
- If road is Good then driving is Smooth else traffic is High
- The fuzzy implication is denoted as $R : A \rightarrow B$
- In essence, it represents a binary fuzzy relation R on the (Cartesian) product of $A \times B$

Fuzzy implication : Example 2

- Suppose, P and T are two universes of discourses representing pressure and temperature, respectively as follows.
- $P = \{ 1,2,3,4 \}$ and $T = \{ 10, 15, 20, 25, 30, 35, 40, 45, 50 \}$
- Let the linguistic variable High temperature and Low pressure are given as
- $T_{HIGH} = \{(20, 0.2), (25, 0.4), (30, 0.6), (35, 0.6), (40, 0.7), (45, 0.8), (50, 0.8)\}$
- $P_{LOW} = (1, 0.8), (2, 0.8), (3, 0.6), (4, 0.4)$

Fuzzy implications : Example 2

- Then the fuzzy implication **If temperature is High then pressure is Low** can be defined as

  $R : T_{HIGH} \rightarrow P_{LOW}$

$$\text{where, } R = \begin{array}{c} \\ 20 \\ 25 \\ 30 \\ 35 \\ 40 \\ 45 \\ 50 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ 0.7 & 0.7 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

Note : If temperature is 40 then what about low pressure?

Zadeh max-min Rule to compute R

With the above mentioned implications, there are a number of fuzzy implication functions that are popularly followed in fuzzy rule-based system.

Zadeh's max-min rule :

$$R_{mm} = \bar{A} \cup (A \cap B) = \int_{X \times Y} (1 - \mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y))|_{(x,y)}$$

or

$$f_{mm}(a, b) = (1 - a) \vee (a \wedge b)$$

Example 3: Zadeh's Max-Min rule

If x is A then y is B with the implication of Zadeh's max-min rule can be written equivalently as :

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y)$$

Here, Y is the universe of discourse with membership values for all $y \in Y$ is 1, that is , $\mu_Y(y) = 1 \forall y \in Y$.

Suppose $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$

and $A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$ are two fuzzy sets.

We are to determine $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$

Example 3: Zadeh's min-max rule:

The computation of $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$ is as follows:

$$A \times B = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array} \text{ and}$$

$$\bar{A} \times Y = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 3: Zadeh's min-max rule:

Therefore,

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example 3 :

$X = \{a, b, c, d\}$

$Y = \{1, 2, 3, 4\}$

Let, $A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$

Determine the implication relation :

If x is A then y is B

Here, $A \times B =$
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example 3 :

and $\bar{A} \times Y =$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 1 & 1 & 1 & 1 \\ b & 0.2 & 0.2 & 0.2 & 0.2 \\ c & 0.4 & 0.4 & 0.4 & 0.4 \\ d & 0 & 0 & 0 & 0 \end{array}$$

$R_{mm} = (A \times B) \cup (\bar{A} \times Y) =$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 1 & 1 & 1 & 1 \\ b & 0.2 & 0.8 & 0.8 & 0.2 \\ c & 0.4 & 0.6 & 0.6 & 0.4 \\ d & 0.2 & 1.0 & 0.8 & 0 \end{array}$$

This R represents **If** x **is A then** y **is B**

Example 3 :

IF x is A THEN y is B ELSE y is C.

The relation R is equivalent to

$$R = (A \times B) \cup (\bar{A} \times C)$$

The membership function of R is given by

$\mu_R(x, y) = max[min\{\mu_A(x), \mu_B(y)\}, min\{\mu_{\bar{A}}(x), \mu_C(y)]$

Example 4:

$X = \{a, b, c, d\}$

$Y = \{1, 2, 3, 4\}$

$A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$

$C = \{(1, 0), (2, 0.4), (3, 1.0), (4, 0.8)\}$

Determine the implication relation :

If x is A then y is B else y is C

Here, $A \times B =$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example 4:

and $\bar{A} \times C =$
$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 0 & 0.4 & 1.0 & 0.8 \\ b & 0 & 0.2 & 0.2 & 0.2 \\ c & 0 & 0.4 & 0.4 & 0.4 \\ d & 0 & 0 & 0 & 0 \end{array}$$

$R =$
$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 0 & 0.4 & 1.0 & 0.8 \\ b & 0.2 & 0.8 & 0.8 & 0.2 \\ c & 0.2 & 0.6 & 0.6 & 0.4 \\ d & 0.2 & 1.0 & 0.8 & 0 \end{array}$$

Fuzzy Inferences

Fuzzy inferences

Let's start with propositional logic. We know the following in propositional logic.

1. Modus Ponens : $P, P \implies Q, \qquad \Leftrightarrow Q$
2. Modus Tollens : $P \implies Q, \neg Q \qquad \Leftrightarrow, \neg P$
3. Chain rule : $P \implies Q, Q \implies R \qquad \Leftrightarrow, P \implies R$

Inferring procedures in Fuzzy logic

Two important inferring procedures are used in fuzzy systems :

- **Generalized Modus Ponens (GMP)**

  If x is A Then y is B

  x is A'

  ---

  y is B'

- **Generalized Modus Tollens (GMT)**

  If x is A Then y is B

  y is B'

  ---

  x is A'

x is A'

Fuzzy inferring procedures

- Here, A, B, A' and B' are fuzzy sets.
- To compute the membership function A' and B' the max-min composition of fuzzy sets B' and A' ,respectively with $R(x, y)$ (which is the known implication relation) is to be used.
- Thus,

$$B' = A' \circ R(x, y) \qquad \mu_B(y) = max[min(\mu_{A'}(x), \mu_R(x, y))]$$

$$A' = B' \circ R(x, y) \qquad \mu_A(x) = max[min(\mu_{B'}(y), \mu_R(x, y))]$$

Generalized Modus Ponens (GMP)

$$P : \textbf{If } x \textbf{ is } A \textbf{ then } y \textbf{ is } B$$

Let us consider two sets of variables x and y be
$X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, respectively.

Also, let us consider the following.

$A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$

$B = \{(y_1, 1), (y_2, 0.4)\}$

Then, given a fact expressed by the proposition x is A' ,
where $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$
derive a conclusion in the form y **is** B' (using generalized modus ponens (GMP)).

Example: Generalized Modus Ponens

If x is A Then y is B

x is A'

---

y is B'

We are to find $B' = A' \circ R(x,y)$ where $R(x,y) = max\{A \times B, \overline{A} \times Y\}$

$$A \times B = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array} \text{ and } \overline{A} \times Y = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Note: For $A \times B$, $\mu_{A \times B}(x,y) = min(\mu_A x, \mu_B(y))$

Example: Generalized Modus Ponens

$$R(x,y) = (A \times B) \cup (\overline{A} \times y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

Now, $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

Therefore, $B' = A' \circ R(x,y) =$

$$\begin{bmatrix} 0.6 & 0.9 & 0.7 \end{bmatrix} \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \end{bmatrix}$$

Thus we derive that y is B' where $B' = \{(y_1, 0.9), (y_2, 0.5)\}$

Example: Generalized Modus Tollens

Generalized Modus Tollens (GMT)

P: If x is A Then y is B

Q: y is B'

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x is A'

Example: Generalized Modus Tollens

- Let sets of variables x and y be $X = \{x_1, x_2, x_3\}$ and $y = \{y_1, y_2\}$, respectively.
- Assume that a proposition **If** x **is** A **Then** y **is** B given where $A = \{(x_1, 0.5), (x_2, 1.0), (x_3, 0.6)\}$ and $B = \{(y_1, 0.6), (y_2, 0.4)\}$
- Assume now that a fact expressed by a proposition y **is** B is given where $B' = \{(y_1, 0.9), (y_2, 0.7)\}$.
- From the above, we are to conclude that x **is** A'. That is, we are to determine A'

Example: Generalized Modus Tollens

- We first calculate $R(x, y) = (A \times B) \cup (\overline{A} \times y)$

$$R(x,y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

- Next, we calculate $A' = B' \circ R(x, y)$

$$A' = \begin{bmatrix} 0.9 & 0.7 \end{bmatrix} \circ \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array} = \begin{bmatrix} 0.5 & 0.9 & 0.6 \end{bmatrix}$$

- Hence, we calculate that x is A' where
  $A' = [(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)]$

Practice

Apply the fuzzy GMP rule to deduce **Rotation is quite slow**

Given that :

- If temperature is High then rotation is Slow.
- temperature is Very High

Let,

$X = \{30, 40, 50, 60, 70, 80, 90, 100\}$ be the set of temperatures.

$Y = \{10, 20, 30, 40, 50, 60\}$ be the set of rotations per minute.

Practice

The fuzzy set High(H), Very High (VH), Slow(S) and Quite Slow (QS) are given below.

$H = \{(70, 1), (80, 1), (90, 0.3)\}$

$VH = \{(90, 0.9), (100, 1)\}$

$S = \{(30, 0.8), (40, 1.0), (50, 0.6)\}$

$QS = \{(10, 1), (20, 0.8)\}$

1. If temperature is High then the rotation is Slow.

$$R = (H \times S) \cup (\overline{H} \times Y)$$

2. temperature is Very High

Thus, to deduce ”rotation is Quite Slow”, we make use the composition rule $QS = VH \circ R(x, y)$