Fuzzy Logic : Introduction

What is Fuzzy logic?

- Fuzzy logic is a mathematical language to express something.

  This means it has grammar, syntax, semantic like a language for communication.

- There are some other mathematical languages also known
  - **Relational algebra** (operations on sets)
  - **Boolean algebra** (operations on Boolean variables)
  - **Predicate logic** (operations on well formed formulae (wff), also called predicate propositions)

- **Fuzzy logic deals with Fuzzy set.**

First time introduced by Lotfi Abdelli Zadeh (1965), University of California, Berkley, USA (1965).

What is fuzzy?

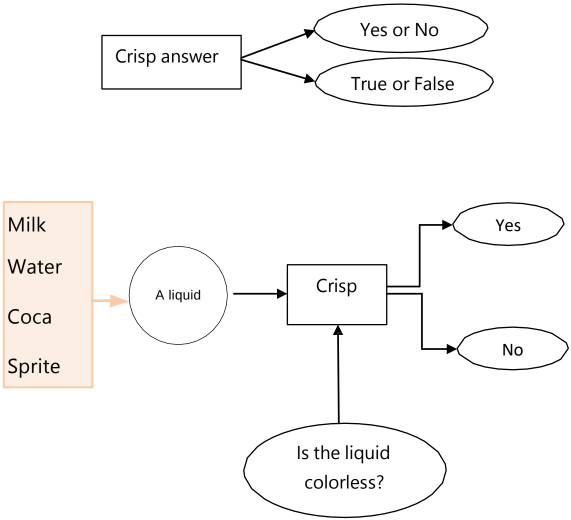
1. Dictionary meaning of **fuzzy** is not clear, noisy etc.

   Example: Is the picture on this slide is fuzzy?

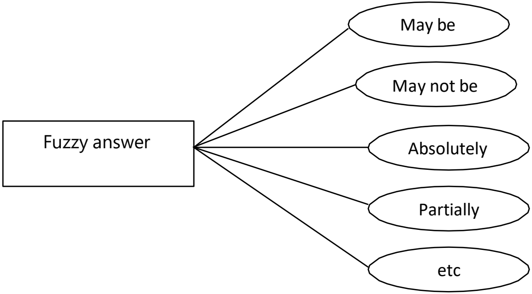
2. Antonym of fuzzy is **crisp**

   Example: Are the chips crisp?

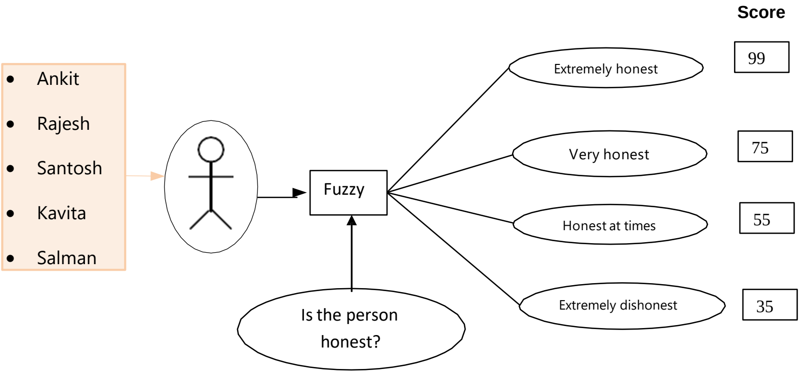
Example : Fuzzy logic vs. Crisp logic



Example : Fuzzy logic vs. Crisp logic



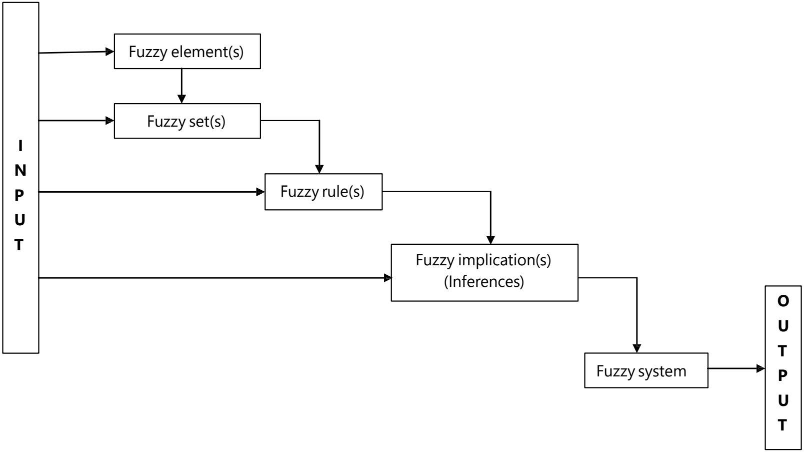
Example : Fuzzy logic vs. Crisp logic



World is fuzzy!

Our world is better described with fuzzily!

Concept of fuzzy system



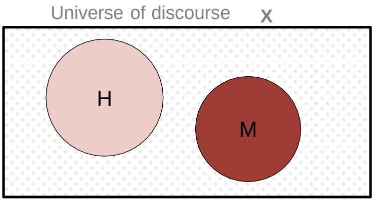
Concept of fuzzy set

To understand the concept of **fuzzy set** it is better, if we first clear our idea of **crisp set**.

X = The entire population of India.

H = All Hindu population = $\{ h_1, h_2, h_3, \ldots, h_L \}$

M = All Muslim population = $\{ m_1, m_2, m_3, \ldots, m_N \}$



Here, All are the sets of finite numbers of individuals.

Such a set is called crisp set.

Example of fuzzy set

Let us discuss about fuzzy set

X = All students in **116U01E514**

S = All **Good students**.
S = { (s, g) | s $\in$ X } and g(s) is a measurement of goodness of the student *s*.

Example:

S = { (Rajat, 0.8), (Kavita, 0.7), (Salman, 0.1), (Ankit, 0.9) } etc.

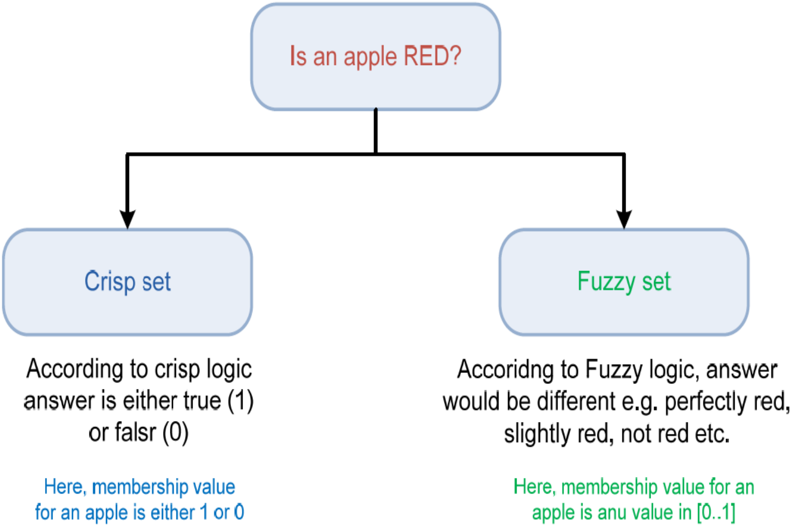


Figure 1: Crisp vs. Fuzzy sets

Fuzzy set vs. Crisp set

Crisp Set	Fuzzy Set
1. S = { s | s $\in$ X }	1. F = (s, μ) | s $\in$ X and μ(s) is the degree of s.
2. It is a collection of elements.	2. It is collection of ordered pairs.
3. Inclusion of an element s $\in$ X into S is crisp, that is, has strict boundary **yes** or **no**.	3. Inclusion of an element s $\in$ X into F is fuzzy, that is, if present, then with a degree of **membership**.

Fuzzy set vs. Crisp set

Note: A crisp set is a fuzzy set, but, a fuzzy set is not necessarily a crisp set.

Example:
H = { $(h_1, 1), (h_2, 1), \ldots, (h_L, 1)$ }

Person = { $(p_1, 1), (p_2, 0), \ldots, (p_N, 1)$ }

In case of a crisp set, the elements are with extreme values of degree of membership namely either 1 or 0.

How to decide the degree of memberships of elements in a fuzzy set?

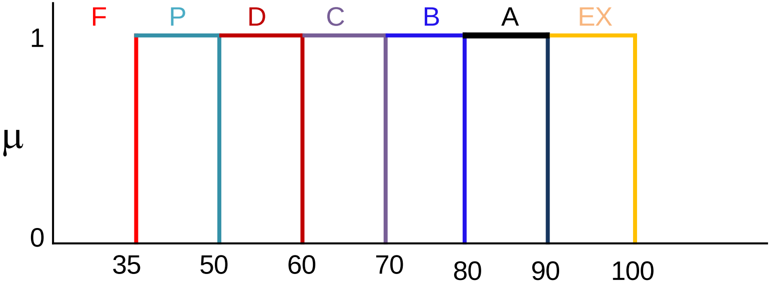
City	Bangaluru	Mumbai	Hyderabad	Kharagpur	Chennai	Delhi
DoM	0.95	0.90	0.80	0.01	0.65	0.75

How the cities of comfort can be judged?

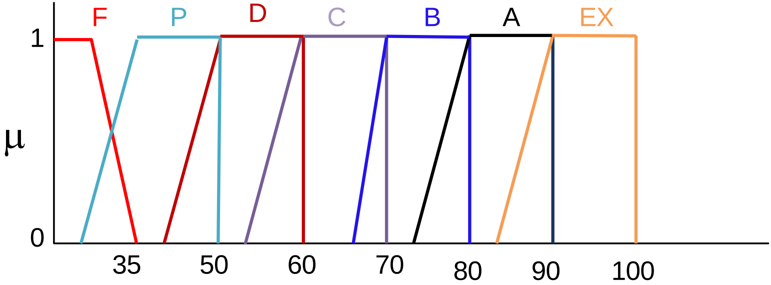
Example: Course evaluation in a crisp way

1. EX = Marks ≥ 90
2. A = $80 \leq$ Marks < 90
3. B = $70 \leq$ Marks < 80
4. C = $60 \leq$ Marks < 70
5. D = $50 \leq$ Marks < 60
6. P = $35 \leq$ Marks < 50
7. F = Marks < 35

Example: Course evaluation in a crisp way



Example: Course evaluation in a fuzzy way



Few examples of fuzzy set

- High Temperature
- Low Pressure
- Color of Apple
- Sweetness of Orange
- Weight of Mango

Note: Degree of membership values lie in the range [0...1].

Some basic terminologies and notations

Definition 1: Membership function (and Fuzzy set)

If X is a universe of discourse and x $\in$ X, then a fuzzy set A in X is defined as a set of ordered pairs, that is

$A = \{(x, \mu_A(x)) | x \in X\}$ where μ_A(x) is called the membership function for the fuzzy set A.

Note:
μ_A(x) map each element of X onto a membership grade (or membership value) between 0 and 1 (both inclusive).

Question:
How (and who) decides μ_A(x) for a Fuzzy set A in X?

Some basic terminologies and notations

Example:

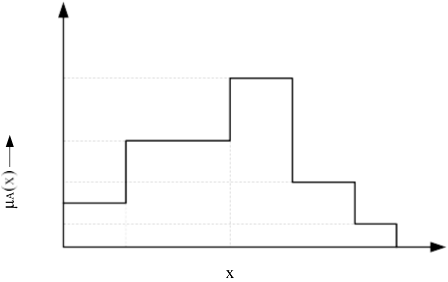
X = All cities in India

A = City of comfort

A={(New Delhi, 0.7), (Bangalore, 0.9), (Chennai, 0.8), (Hyderabad, 0.6), (Kolkata, 0.3), (Kharagpur, 0)}

Membership function with discrete membership values

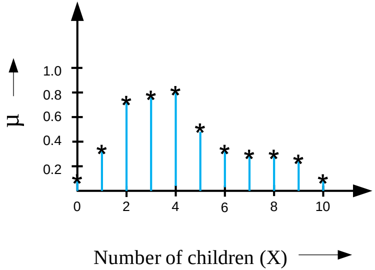
The membership values may be of discrete values.



A fuzzy set with discrete values of μ

Membership function with discrete membership values

Either elements or their membership values (or both) also may be of discrete values.



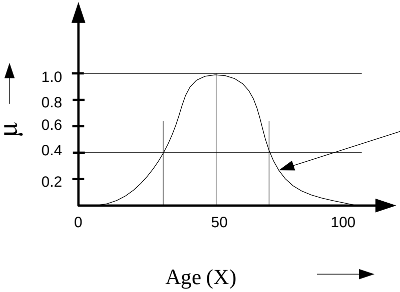
A = "Happy family"

A ={(0,0.1),(1,0.30),(2,0.78)……(10,0.1)}

Note : X = discrete value

How you measure happiness ??

Membership function with continuous membership values



$$\mu_B(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^4}$$

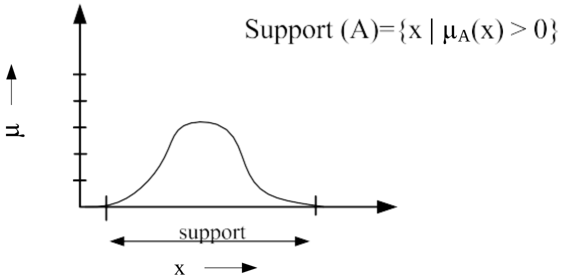
$$B=\{(x,\mu_B(x))\}$$

B = "Middle aged"

Note : x = real value
= R^+

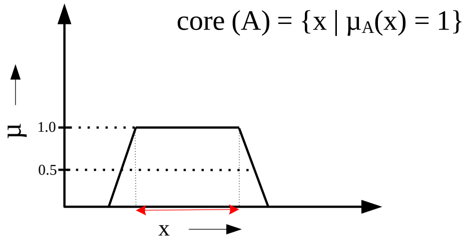
Fuzzy terminologies: Support

Support: The support of a fuzzy set A is the set of all points $x \in X$ such that $\mu_A(x) > 0$



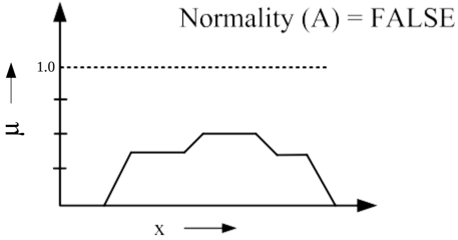
Fuzzy terminologies: Core

Core: The core of a fuzzy set A is the set of all points x in X such that $\mu_A(x) = 1$



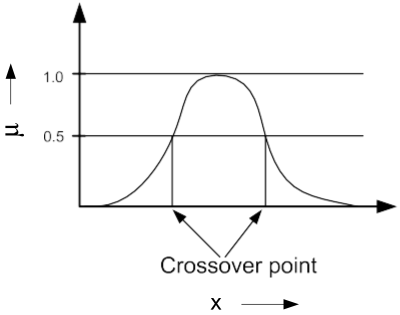
Fuzzy terminologies: Normality

Normality : A fuzzy set A is a normal if its core is non-empty. In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$.



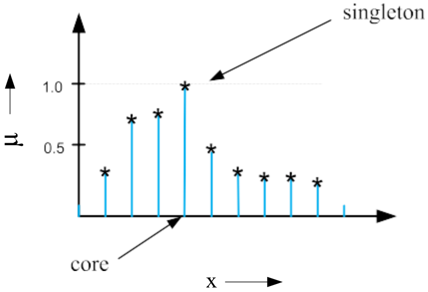
Fuzzy terminologies: Crossover points

Crossover point : A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$. That is
Crossover $(A) = \{x \,|\mu_A(x) = 0.5\}$.



Fuzzy terminologies: Fuzzy Singleton

Fuzzy Singleton : A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = |\{ x \mid \mu_A(x) = 1\}| = 1$. Following fuzzy set is not a fuzzy singleton.



Fuzzy terminologies: α-cut and strong α-cut

α-cut and strong α-cut :

The α-cut of a fuzzy set A is a crisp set defined by

$$A_\alpha = \{ x \mid \mu_A(x) \geq \alpha \}$$

Strong α-cut is defined similarly :

$$A_\alpha' = \{ x \mid \mu_A(x) > \alpha \}$$

Note : Support(A) = A_0' and Core(A) = A_1.

Fuzzy terminologies: Bandwidth

Bandwidth :

For a normal and convex fuzzy set, the bandwidth (or width) is defined as the distance the two unique crossover points:

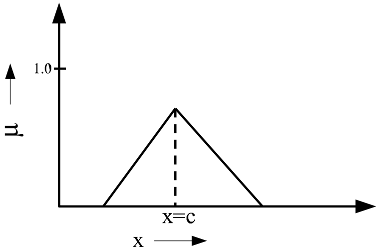
Bandwidth(A) = $| x_1 - x_2 |$

where $\mu_A(x_1) = \mu_A(x_2) = 0.5$

Fuzzy terminologies: Symmetry

Symmetry :

A fuzzy set A is symmetric if its membership function around a certain point $x = c$, namely μ_A(x + c) = μ_A(x - c) for all x $\in$ X.



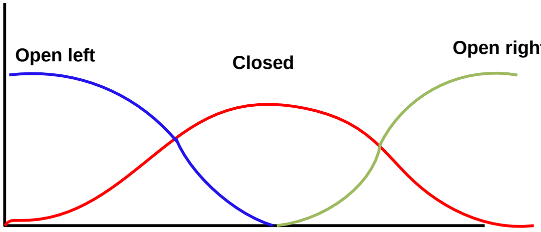
Fuzzy terminologies: Open and Closed

A fuzzy set A is

Open left
If $\lim_{x \to -\infty} \mu_A(x) = 1$ and $\lim_{x \to +\infty} \mu_A(x) = 0$

Open right:
If $\lim_{x \to -\infty} \mu_A(x) = 0$ and $\lim_{x \to +\infty} \mu_A(x) = 1$

Closed
If : $\lim_{x \to -\infty} \mu_A(x) = \lim_{x \to +\infty} \mu_A(x) = 0$



Fuzzy Membership Functions

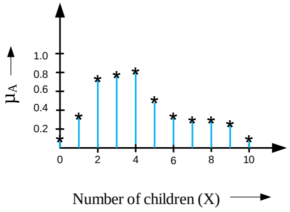
Fuzzy membership functions

A fuzzy set is completely characterized by its membership function (sometimes abbreviated as *MF* and denoted as μ). So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

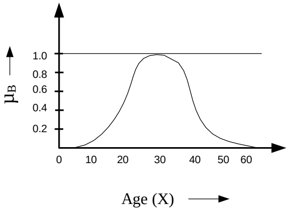
Note: A membership function can be on

(a) a discrete universe of discourse and
(b) a continuous universe of discourse.

Example:



A = Fuzzy set of "Happy family"

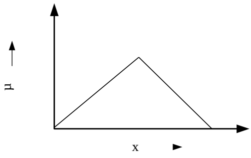


B = "Young age"

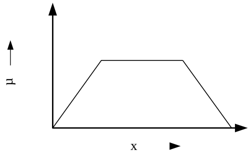
Fuzzy membership functions

So, membership function on a discrete universe of course is trivial. However, a membership function on a continuous universe of discourse needs a special attention.

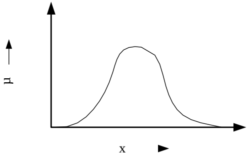
Following figures shows a typical examples of membership functions.



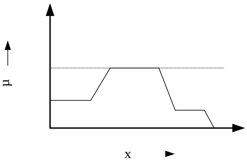
< triangular >



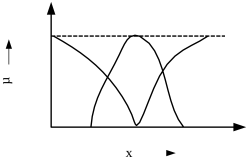
< trapezoidal >



< curve >



< non-uniform >



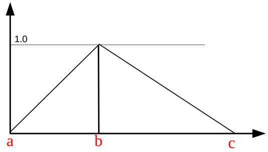
< non-uniform >

Fuzzy MFs : Formulation and parameterization

In the following, we try to parameterize the different MFs on a continuous universe of discourse.

Triangular MFs : A triangular MF is specified by three parameters $\{a, b, c\}$ and can be formulated as follows.

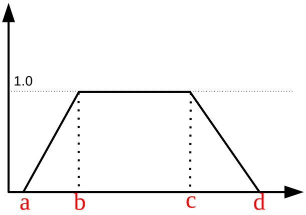
$$triangle(x; a, b, c) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } c \leq x \end{cases} \tag{1}$$



Fuzzy MFs: Trapezoidal

A trapezoidal MF is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

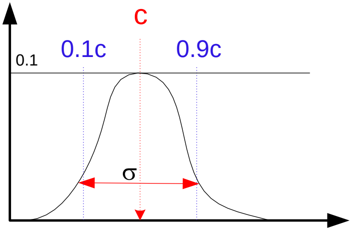
$$trapeziod(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } d \leq x \end{cases} \tag{2}$$



Fuzzy MFs: Gaussian

A Gaussian MF is specified by two parameters $\{c, \sigma\}$ and can be defined as below:

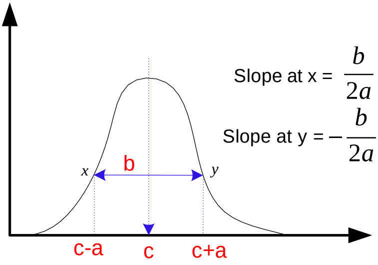
$$\text{gaussian}(x;c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}.$$



Fuzzy MFs: Generalized bell

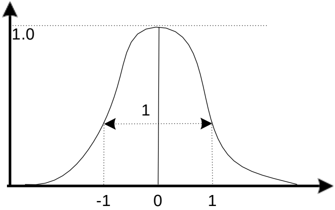
It is also called Cauchy MF. A generalized bell MF is specified by three parameters $\{a, b, c\}$ and is defined as:

$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$

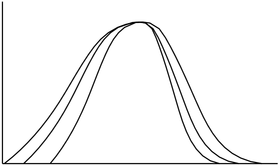


Example: Generalized bell MFs

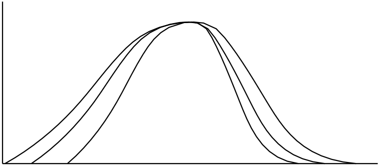
Example: $\mu(x) = \frac{1}{1+x^2}$;
$a = b = 1$ and $c = 0$;



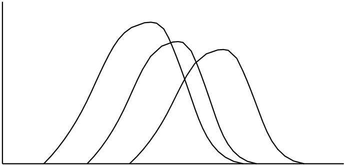
Generalized bell MFs: Different shapes



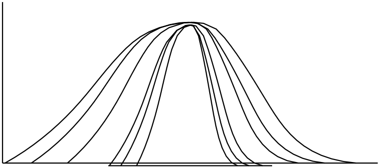
Changing a



Changing b



Changing a

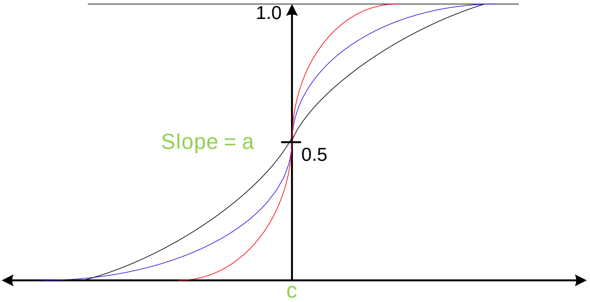


Changing a and b

Fuzzy MFs: Sigmoidal MFs

Parameters: $\{a, c\}$; where c = crossover point and a = slope at c;

$$\text{Sigmoid}(x;a,c) = \frac{1}{1+e^{-[\frac{a}{x-c}]}}$$



Fuzzy MFs : Example

Example : Consider the following grading system for a course.

Excellent = Marks $\leq$ 90

Very good = 75 $\leq$ Marks $\leq$ 90

Good = 60 $\leq$ Marks $\leq$ 75

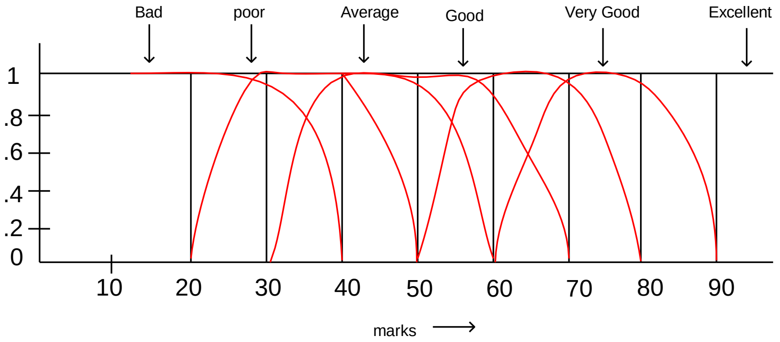
Average = 50 $\leq$ Marks $\leq$ 60

Poor = 35 $\leq$ Marks $\leq$ 50

Bad= Marks $\leq$ 35

Grading System

A fuzzy implementation will look like the following.



You can decide a standard fuzzy MF for each of the fuzzy garde.

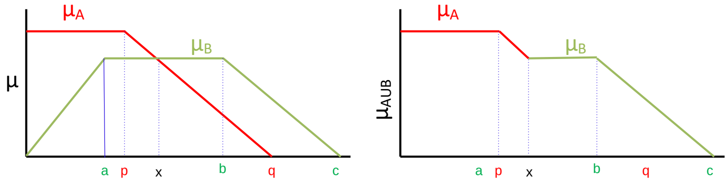
Operations on Fuzzy Sets

Basic fuzzy set operations: Union

Union ($A \cup B$):

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Example:
$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and
$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;
$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$

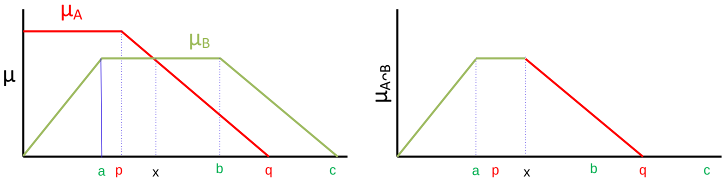


Basic fuzzy set operations: Intersection

Intersection ($A \cap B$):

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Example:
$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and
$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;
$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$



Basic fuzzy set operations: Complement

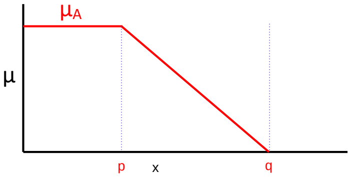
Complement (A^C):

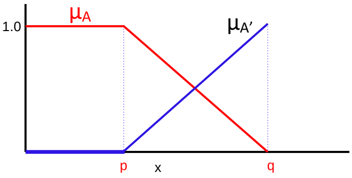
$$\mu_{A_{AC}}(x) = 1\text{-}\mu_A(x)$$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

$C = A^C = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$





Basic fuzzy set operations: Products

Algebric product or Vector product (A $\cdot$ B):

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

Scalar product $(a \times A)$:

$$\mu_{\alpha A}(x) = a \cdot \mu_A(x)$$

Basic fuzzy set operations: Sum and Difference

Sum ($A + B$):

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

Difference ($A - B = A \cap B^C$):

$$\mu_{A-B}(x) = \mu_{A \cap B^C}(x)$$

Disjunctive sum: $A \oplus B = (A^C \cap B) \cup (A \cap B^C)$**)**

Bounded Sum: $|\, \mathrm{A(x)} \oplus \mathrm{B(x)} \,|$

$$\mu_{|A(x) \oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$$

Bounded Difference: $|\, \mathrm{A(x)}\ \mathrm{g}\ \mathrm{B(x)} \,|$

$$\mu_{|A(x)\, \mathrm{g}\, B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Basic fuzzy set operations: Equality and Power

Equality ($A = B$):

$$\mu_A(x) = \mu_B(x)$$

Power of a fuzzy set A^α:

$$\mu_{A^\alpha}(x) = \{\mu_A(x)\}^\alpha$$

- If $a < 1$, then it is called *dilation*
- If $a > 1$, then it is called *concentration*

Caretsian Product ($A \times B$):

$$\mu_{A \times B}(x, y) = min\{\mu_A(x), \mu_B(y)$$

Example 3:

A(x) = {(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)}

B(y) = {(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)}

$$A \times B = \min\{\mu_A(x), \mu_B(y)\} = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.2 & 0.2 & 0.2 \\ x_2 & 0.3 & 0.3 & 0.3 \\ x_3 & 0.5 & 0.5 & 0.3 \\ x_4 & 0.6 & 0.6 & 0.3 \end{array}$$

Properties of fuzzy sets

Commutativity :

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Properties of fuzzy sets

Idempotence :

$$A \cup A = A$$
$$A \cap A = \mathrm{A}$$
$$A \cup \emptyset = A$$
$$A \cap \emptyset = \mathrm{A}$$

Transitivity :

$$\text{If } A \subseteq B, B \subseteq C \text{ then } A \subseteq C$$

Involution :

$$(A^c)^c = A$$

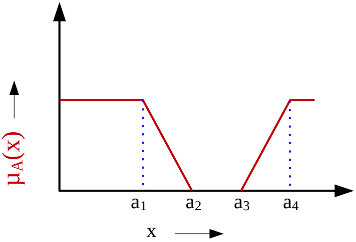
De Morgan's law :

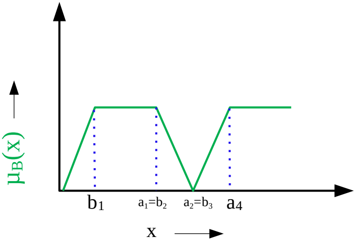
$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Few Illustrations on Fuzzy Sets

Example 1: Fuzzy Set Operations

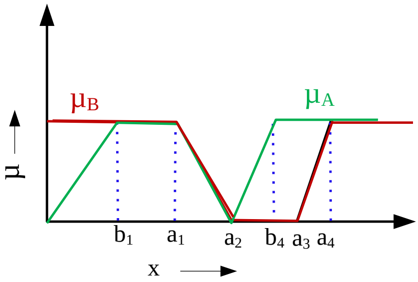
Let A and B are two fuzzy sets defined over a universe of discourse X with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. Two MFs $\mu_A(x)$ and $\mu_B(x)$ are shown graphically.





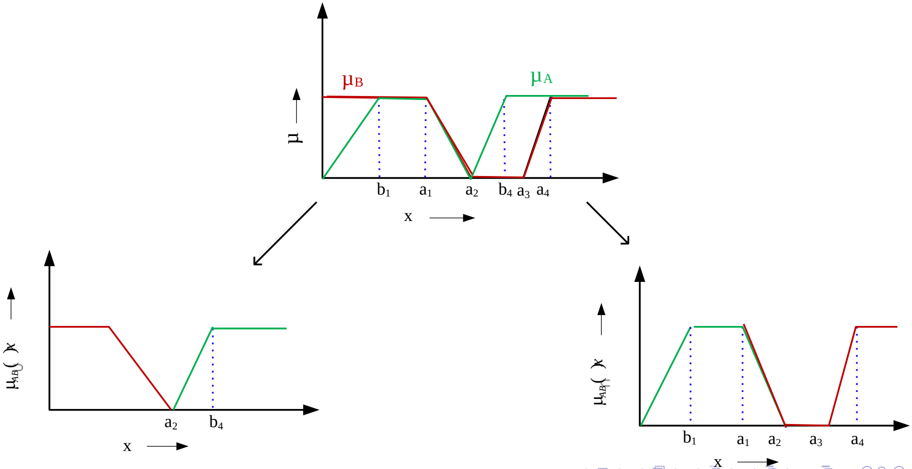
Example 1: Plotting two sets on the same graph

Let's plot the two membership functions on the same graph



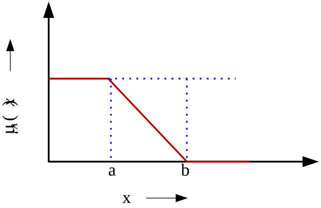
Example 1: Union and Intersection

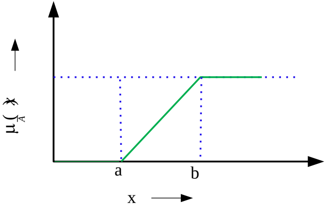
The plots of union $A \cup B$ and intersection $A \cap B$ are shown in the following.



Example 1: Intersection

The plots of union $\mu_{\bar{A}}(x)$ of the fuzzy set A is shown in the following.





Fuzzy set operations: Practice

Consider the following two fuzzy sets A and B defined over a universe of discourse [0,5] of real numbers with their membership functions

$$\mu_A(x) = \frac{x}{1+x} \text{ and } \mu_B(x) = 2^{-x}$$

Determine the membership functions of the following and draw them graphically.

i. $\overline{A}$, $\overline{B}$

ii. $A \cup B$

iii. $A \cap B$

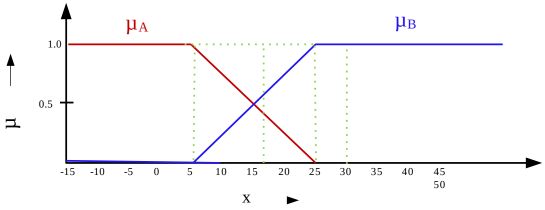
iv. $(A \cup B)^c$ [Hint: Use De' Morgan law]

Example 2: A real-life example

Two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively defined as below.

A = **Cold climate** with $\mu_A(x)$ as the MF.

B = **Hot climate** with $\mu_B(x)$ as the M.F.



Here, X being the universe of discourse representing entire range of temperatures.

Example 2: A real-life example

What are the fuzzy sets representing the following?

1. **Not cold climate**
2. **Not hold climate**
3. **Extreme climate**
4. **Pleasant climate**

Note: Note that "Not cold climate" $/=$ "Hot climate" and vice-versa.

Example 2 : A real-life example

Answer would be the following.

1. **Not cold climate**

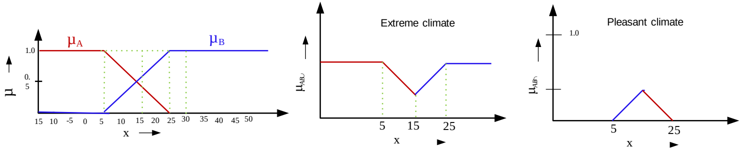
   $\overline{A}$ with $1 - \mu_A(x)$ as the MF.
2. **Not hot climate**

   $\overline{B}$ with $1 - \mu_B(x)$ as the MF.
3. **Extreme climate**

   $A \cup B$ with $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ as the MF.
4. **Pleasant climate**

   $A \cap B$ with $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ as the MF.

The plot of the MFs of $A \cup B$ and $A \cap B$ are shown in the following.



Few More on Membership Functions

Generation of MFs

Given a membership function of a fuzzy set representing a linguistic hedge, we can derive many more MFs representing several other linguistic hedges using the concept of Concentration and Dilation.

- **Concentration:**

$$A^k = [\mu_A(x)]^k \; ; k > 1$$

- **Dilation:**

$$A^k = [\mu_A(x)]^k \; ; k < 1$$

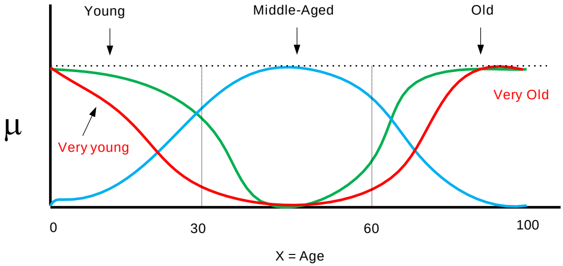
Example : Age = { Young, Middle-aged, Old }

Thus, corresponding to Young, we have : Not young, Very young, Not very young and so on.

Similarly, with Old we can have : old, very old, very very old, extremely old etc.
Thus, Extremely old = $(((old)^2)^2)^2$ and so on

Or, More or less old = $A^{0.5} = (old)^{0.5}$

Linguistic variables and values



$$\mu_{young}(x) = bell(x, 20, 2, 0) = \frac{1}{1+(\frac{x}{20})^4}$$

$$\mu_{old}(x) = bell(x, 30, 3, 100) = \frac{1}{1+(\frac{x-100}{30})^6}$$

$$\mu_{middle-aged} = bell(x, 30, 60, 50)$$

Not young = $\overline{\mu_{young}(x)} = 1 - \mu_{young}(x)$ __________

Young but not too young = $\mu_{young}(x) \cap \mu_{young}(x)$