La función de Cantor

1

Sea $D_n = [0,1] \setminus C_n$. Entonces

$$D_1 = \left(\tfrac{1}{3}, \tfrac{2}{3}\right)$$

$$D_2 = D_1 \cup \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right)$$

En general

$$D_n = \bigcup_{j=1}^{2^n - 1} [\alpha_j^n, \beta_j^n] \quad , \quad \begin{matrix} \alpha_j < \beta_j \\ \beta_j < \alpha_{j+1} \end{matrix}$$

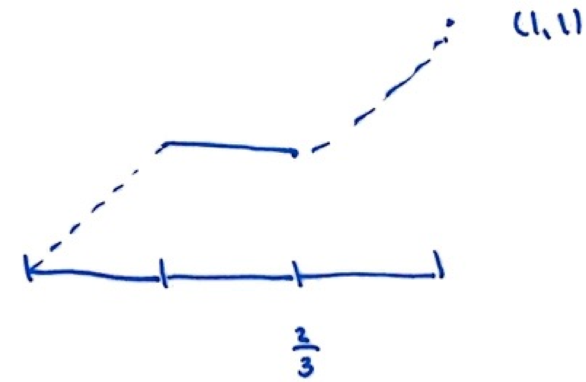
Tome $I_j^n = [\alpha_j^n, \beta_j^n]$, y

$$l_j^n(x) = \frac{1}{2^n} \frac{(x - \beta_j^n)}{\alpha_{j+1}^n - \beta_j^n} + \frac{j}{2^n}$$

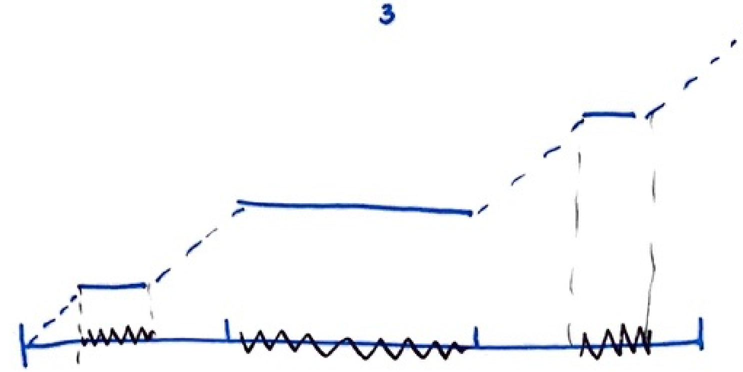
Definimos

$$f_n(x) = \begin{cases} \dfrac{j}{2^n} & \text{si} \quad \alpha_j^n \le x \le \beta_j^n \\[2ex] \dfrac{x}{\alpha_1 2^n} & \text{si} \quad 0 \le x \le \alpha_1^n \\[2ex] l_j^n(x) & \text{si} \quad \beta_j^n \le x \le \alpha_{j+1}^n \end{cases}$$

D_1



D_2



Es fácil probar que $f_n(x)$ es creciente. Ahora

$$D_{n+1} = \bigcup_{j=1}^{2^n-1} [\alpha_j^n, \beta_j^n]$$

$$\cup$$

$$\bigcup_{m=1}^{2^n} [\tilde{\alpha}_m^n, \tilde{\beta}_m^n]$$

con

$$0 < \tilde{\alpha}_1^n < \tilde{\beta}_1^n < \alpha_1$$

$$\beta_i^n < \tilde{\alpha}_{i+1}^n < \tilde{\beta}_{i+1}^n < \alpha_{i+1}^n.$$

Entonces

$$f_{n+1}(x) = f_n(x) \quad \text{si } x \in [\alpha_j^n, \beta_j^n]$$

Si $x \in [\hat{\alpha}_j^n, \hat{\beta}_j^n]$, ent

$$f_{n+1}(x) = \frac{j-1}{2^{n+1}}$$

Luego $|f_n(x) - f_{n+1}(x)| \leq \frac{1}{2^n}$ para todo $x \in [0,1]$. Por el M-test de Weierstrass tenemos que

$$\sum_{n=1}^{\infty} [f_{n+1} - f_n]$$

converge uniformemente. Ent

$$f_n \xrightarrow[n\to\infty]{} f$$

uniformemente

③

La función resultante es la función de Cantor. Ent f es continua y creciente, $f(0)=0$, $f(1)=1$.

Además si $x \in I_j^u$ ent

$$f_u(x) = f_j(x) = \frac{j}{2^u}$$

si $j \geq u$. Ent $f(x) = \frac{j}{2^u}$.

$\therefore f'(x) = 0$ en I_j^u

$\therefore f'(x) = 0$ en $[0,1] \setminus C$.