Exercise 1. Do the following:

I) Give a simple description of the closed sets in $\mathbb{A}^1$ (with respect to the Zariski topology).

II) Use your previous answer to prove that $\mathbb{A}^1$ is not Hausdorff.

Answer

I) If we consider $\mathbb{A}^1$ over an algebraically closed field k of characteristic zero then every closed set is of the form $V(I)$ where $I \in \operatorname{Spec}(k[x])$. Since $k[x]$ is a PID, then $I = \operatorname{gen}(p)$ for a polynomial $p \in k[x]$. Then $V(I)$ would be the set of roots inside k of p. Since p is arbitrary, every closed set $V(I)$ of $\mathbb{A}^1$ is a finite set.

This means that the open sets are the complement of the finite sets. In essence, the Zariski topology coincides with the cofinite topology over $\mathbb{A}^1$.

II) The cofinite topology is not Hausdorff, so it follows that the Zariski topology isn't Hausdorff as well.

Exercise 2. Show that the Zariski topology on $\mathbb{A}^2$ is not the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$. (Hint: Consider the diagonal.)

Answer

Recall the following topological facts:

I) If X, Y are not Hausdorff, it follows that $X \times Y$ is not Hausdorff.

II) X is Hausdorff if and only if the diagonal set is closed.

To show that the Zariski topology and the product topology are different on $\mathbb{A}^2$ we will show that the diagonal set D in $\mathbb{A}^2$ is closed. Then the argument is as follows:

$$\left(D \text{ is closed in } \mathbb{A}^2 \text{ with } \mathcal{Z}_2\right) \wedge \left(D \text{ is open in } \mathbb{A}^2 \text{ with } \mathcal{Z}_1 \times \mathcal{Z}_1\right) \Rightarrow \mathcal{Z}_2 \neq \mathcal{Z}_1 \times \mathcal{Z}_1.$$

To show that D is closed in $\mathbb{A}^2$, we can see that it is the zero locus for the polynomial $x - y$. This is $D = V(x - y)$.
On the other hand, the product topology $\mathcal{Z}_1 \times \mathcal{Z}_1$ is not Hausdorff since $\mathcal{Z}_1$ is not Hausdorff. Therefore D is open in $\mathcal{Z}_1 \times \mathcal{Z}_1$.
We conclude that both topologies are different on $\mathbb{A}^2$.

Exercise 3. Let $F : V \to W$ be a morphism of affine algebraic varieties. Prove that F is continuous in the Zariski topology.

Answer

Suppose $\mathcal{Z}$ is the Zariski topology. Recall that

$$F \text{ is continuous} \iff \forall U \in \mathcal{Z}(F^{-1}[U \backslash W] \notin \mathcal{Z}).$$

This means that F sends closed sets back to closed sets. Since a morphism is a vector of polynomials we can easily check this result.
Consider an arbitrary closed set $V_0 \subseteq W$. Then there exists an ideal $I \trianglelefteq k[\mathbf{x}]$ such that $V_0 = V(I)$. Now consider the following observation

$$\mathbf{x} \in V(I) \iff \forall p \in I(p(\mathbf{x}) = 0) \iff \forall p \in I(\mathbf{x} \in V(p)) \iff \mathbf{x} \in \bigcap_{p \in I} V(p).$$

Since the inverse image behaves well with intersections, it suffices to prove that for any polynomial p, $F^{-1}[V(p)]$ is also a closed set.
By definition

$$F^{-1}[V(p)] = \{\, \mathbf{x} \in V : F(\mathbf{x}) \in V(p) \,\},\ F(\mathbf{x}) \in V(p) \iff p(F(\mathbf{x})) = 0.$$

This means that this set is precisely $V(p \circ F)$ and therefore is a closed set. Since p was arbitrary, it holds for any closed set. Therefore F is continuous in the Zariski topology.

Exercise 4. Show that the twisted cubic V of Figure 1.5 is isomorphic to the affine line by constructing an explicit isomorphism from $\mathbb{A}^1$ to V. (Hint: See Exercise 1.2.3)

Answer

The twisted cubic is the zero locus of the polynomials $y - x^2$ and $z - x^3$. By solving the equations, this set can be seen as $\{\, (x, x^2, x^3) : x \in \mathbb{A}^1 \,\}$. We can construct a function from this! Consider the following function $\mathbf{r} : \mathbb{A}^1 \to \mathbb{A}^3,\ t \mapsto (t, t^2, t^3)$. Then $\mathbf{r}$ is an algebraic morphism since it is a vector of polynomials. We can find an inverse for $\mathbf{r}$ to prove it is an isomorphism, consider the projection map π_1. Then

$$\mathbf{r}(\pi_1(x, y, z)) = (x, x^2, x^3), \text{ and } \pi_1(\mathbf{r}(t)) = \pi_1(t, t^2, t^3) = t.$$

Exercise 5. Show that if $F : X \to Y$ is a surjective morphism of affine algebraic varieties, then the dimension of X is at least as large as the dimension of Y.

Answer

Recall that the dimension of an algebraic variety is the length of the longest chain of irreducible proper subvarieties. This is:

$$\dim X = \max_n \{ X \supsetneq V_n \supsetneq V_{n-1} \supsetneq \cdots \supsetneq V_0 = \{ \mathbf{x}_0 \} \}.$$

Suppose $d = \dim Y$, and consider a maximal proper chain of irreducible subvarieties W_j's

$$Y \supsetneq W_d \supsetneq \cdots \supsetneq W_1 \supsetneq \{ \mathbf{y}_0 \}.$$

Now let us consider the morphism F. Since it is a morphism it sends varieties back into varieties. Also the inverse image preserves subsets. This means that we can find the following chain of subvarieties inside of X:

$$X \supsetneq F^{-1}[W_d] \supsetneq \cdots \supsetneq F^{-1}[W_1] \supsetneq F^{-1}[\{ \mathbf{y}_0 \}].$$

These subvarieties are non empty since F is surjective.[a]
We are able to conclude that if at most k of those subvarieties are reducible, then $\dim(X) \geqslant d - k$[b]. FINISH

[a] If it were the case that inverse imaging preserves irreducibilty, we are done. However continuous maps from topological spaces preserve irreducibilty in the forwards direction.

[b] It'd be lovely if $k = 0$.