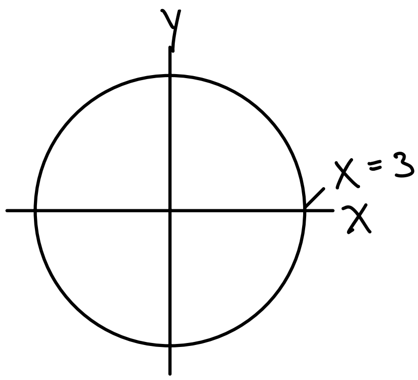
Exercise 1. We will explore a surface through contour lines.

- The vertical contour lines are all of the form $\underline{x^2+y^2=9}$.
- Horizontal contour lines across the y direction are of the form $\underline{x^2=9-a^2}$ with $-3 \leqslant a \leqslant 3$. These lines live in planes parallel to the xz-plane.

A key fact is that this shape is *symmetric across the origin*. Do the following:

1. Draw an example of the vertical contour lines on a plane. What figure are they?
2. Draw the horizontal contour lines at the values of $a=-3$ and $a=0$. Describe the figures and mention what happens for values $-3<a<0$.
3. Draw a diagram in 3 dimensions joining both sets of contour lines. What is the surface that we obtain?

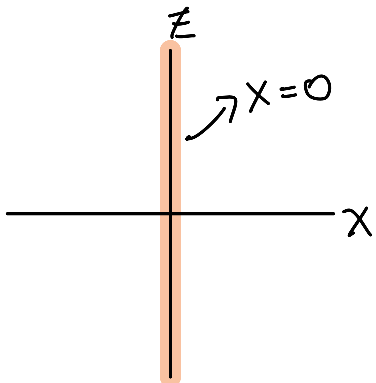
1.



circles of radius 3

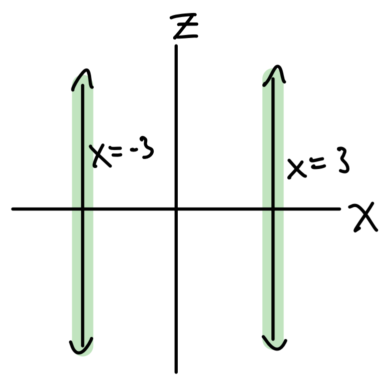
2. At $a=-3$:

$x^2 = 9-(-3)^2 = 9-9=0$



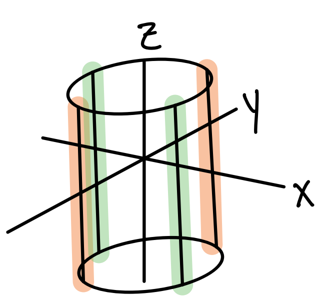
At $a=0$:

$x^2 = 9-(0)^2 = 9 \longrightarrow x^2 = 9 \longrightarrow x = \pm 3$



* Between -3 and 0 it's lines coming apart from one another.

3.



¡It's a cylinder!

Exercise 2. Consider the expressions

$$z=(x+y)(x-y), \quad \begin{cases} x=r\cos(\theta) \\ y=r\sin(\theta) \end{cases}$$

Calculate the partial derivatives $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

We may write this expressions as two functions:

$$z=f(x,y)=(x+y)(x-y), \quad \text{and} \quad G(r,\theta)=(r\cos(\theta),r\sin(\theta)).$$

The task is then to find $D(F(G))_x$ which by the chain rule is

$$D(F(G))_{(r,\theta)}=D(f)_{G(r,\theta)}\cdot DG_{(r,\theta)}=\nabla f_{G(r,\theta)}\cdot JG_{(r,\theta)}.$$

The derivatives of our functions are

$$\nabla f_{(x,y)}=(f_x,f_y)=(2x,-2y) \quad \text{and} \quad JG=\begin{pmatrix} G_{1x} & G_{1y} \\ G_{2x} & G_{2y} \end{pmatrix}=\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}.$$

Evaluating the gradient at $G(r,\theta)$ we get

$$\nabla f_{G(r,\theta)}=(2(r\cos(\theta)),-2(r\sin(\theta))).$$

The product of this gradient with the Jacobian will give us the desired result:

$$\begin{aligned}
\nabla f_{G(r,\theta)}\cdot JG &= (2r\cos(\theta),-2r\sin(\theta))\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} (2r\cos(\theta),-2r\sin(\theta))\cdot(\cos(\theta),\sin(\theta)) \\ (2r\cos(\theta),-2r\sin(\theta))\cdot(-r\sin(\theta),r\cos(\theta)) \end{pmatrix} \\
&= \begin{pmatrix} 2r\cos^2(\theta)-2r\sin^2(\theta) \\ -2r^2\sin(\theta)\cos(\theta)-2r^2\sin(\theta)\cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} 2r\cos(2\theta) \\ -2r^2\sin(2\theta) \end{pmatrix}
\end{aligned}$$

So from this $\frac{\partial z}{\partial r}=2r\cos(2\theta)$ and $\frac{\partial z}{\partial \theta}=-2r^2\sin(2\theta)$. The answers could've also been left as the second-to-last line.