Exercise 1. Suppose \mathcal{F} is a presheaf and \mathcal{G} is a sheaf, both of sets, on X. Let $\mathcal{H}om(\mathcal{F},\mathcal{G})$ be the collection of data

$$\mathcal{H}om(\mathfrak{F},\mathfrak{G})(U) := \mathrm{Mor}(\mathfrak{F}|_{U},\mathfrak{G}|_{U}).$$

Show that this is a sheaf of sets on *X*.

Answer

We first need to show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a presheaf, this requires a sensible notion of restriction mapping which satisfies the following:

- i) $res_{U,U} = id_{(*)}$ where the identity map is over the object $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$.
- ii) If $U \subseteq V \subseteq W$ then $res_{W,U} = res_{V,U} \circ res_{W,V}$.

Let us consider two objects $\operatorname{Mor}(\mathfrak{F}|_U,\mathfrak{G}|_U)$ and $\operatorname{Mor}(\mathfrak{F}|_V,\mathfrak{G}|_V)$ with $U\subseteq V$. A restriction mapping acts on sections, and sections on these sets are morphisms of sheaves. Our restriction mapping takes $\varphi\in\operatorname{Mor}(\mathfrak{F}|_V,\mathfrak{G}|_V)$ to $\operatorname{res}_{V,U}(\varphi)\in\operatorname{Mor}(\mathfrak{F}|_U,\mathfrak{G}|_U)$, but recall φ is a collection of maps of objects of the form

$$\varphi(W): \mathfrak{F}(W) \to \mathfrak{G}(W), \text{ with } W \subseteq V.$$

In this sense, it suffices to only consider the open sets contained in U. We declare that $res_{V,U}(\varphi)$ is the collection of maps

$$\varphi(W): \mathfrak{F}(W) \to \mathfrak{G}(W), \text{ with } \underline{W} \subseteq \underline{U}.$$

i) The map $\operatorname{res}_{U,U}(\varphi)$ acts as follows, every map of the form $\varphi(W)$ with $W\subseteq U$ is sent to the map $\varphi(W)$ between the same objects because $W\subseteq U$ is still itself.

This means that $res_{U,U}$ is the identity map in $Mor(\mathcal{F}|_U, \mathcal{G}|_U)$.

ii) Now suppose $U \subseteq V \subseteq W$ are open sets, then $\operatorname{res}_{V,U} \circ \operatorname{res}_{W,V}$ acts on φ first by restricting from open sets in W to open sets in V and next by passing from open sets in V to only considering the open sets in U.

This is the same as starting with the open sets in W and then only considering the open sets in U. The last action is the same as what $\operatorname{res}_{W,U}$ does to φ .

This allows us to conclude that the sheaf-Hom is indeed a presheaf. We now have to verify the two sheaf axioms:

i) Take (U_i) an cover of $U \subseteq X$ with $\varphi, \psi : \mathcal{F}|_U \to \mathcal{G}|_U$ sections which coincide in every covering set. This means that

$$\operatorname{res}_{U,U_i}(\varphi) = \operatorname{res}_{U,U_i}(\psi) \iff \forall i \left[\varphi(V) = \psi(V), \ V \subseteq U_i \right].$$

Where $\varphi(V)$, $\psi(V)$ are maps of objects from $\mathcal{F}(V)$ to $\mathcal{G}(V)$. We wish to show that they coincide on all of U, which means that for any $V \subseteq U$ and $f \in \mathcal{F}(V)$, it holds that

$$\varphi(V)(f) = \psi(V)(f).$$

Even though we may not talk about these sections directly, we can talk about them after restricting from V to $V \cap U_i$ where U_i is any covering set. To do so, let us introduce the following diagrams:

$$\begin{array}{cccc}
\mathcal{F}(V) & \xrightarrow{\varphi(V),\psi(V)} & \mathcal{G}(V) & f & \longrightarrow & (*) \\
\operatorname{res}_{V,V \cap U_i}^{\mathcal{F}} & & & \downarrow & & \downarrow \\
\mathcal{F}(V \cap U_i) & \longrightarrow & \mathcal{G}(V \cap U_i) & & \operatorname{res}_{V,V \cap U_i}^{\mathcal{F}}(f) & \longrightarrow & (**)
\end{array}$$

The lower arrow in the left diagram is either of the two morphisms $\varphi(V \cap U_i)$, $\psi(V \cap U_i)$. The right diagram is the same but section-wise:

- \diamond The upper right corner is the image of the section f inside $\mathfrak{G}(V)$ through $\varphi(V)$ or $\psi(V)$.
- ♦ The lower right corner can be interpreted in two ways which coincide:

$$\varphi(V \cap U_i) \begin{bmatrix} \mathfrak{F} \\ \operatorname{res} \\ V, V \cap U_i \end{bmatrix} = \underset{V, V \cap U_i}{\operatorname{res}} (\varphi(V)(f))$$

and the same expression for ψ when that's the case. This equality is due to the fact that φ , ψ are morphisms of sheaves and therefore commute with restrictions.

Recall now that $\varphi(V) = \psi(V)$ for $V \subseteq U_i$, in particular we have $\varphi(V \cap U_i) = \psi(V \cap U_i)$. So mapping f from the upper left to the lower right gives us

$$\operatorname{res}_{V,V \cap U_{i}}^{\mathfrak{G}}(\varphi(V)(f)) = \varphi(V \cap U_{i}) \left[\operatorname{res}_{V,V \cap U_{i}}^{\mathfrak{F}}(f) \right] \\
= \psi(V \cap U_{i}) \left[\operatorname{res}_{V,V \cap U_{i}}^{\mathfrak{F}}(f) \right] \\
= \operatorname{res}_{V,V \cap U_{i}}^{\mathfrak{G}}(\psi(V)(f))$$

where the first and last equalities occur because φ and ψ are morphisms of sheaves and the middle one because of the hypothesis.

By the identity axiom on \mathcal{G} , as \mathcal{G} is a sheaf, we can conclude that $\varphi(V)(f)=\psi(V)(f)$. This means that $\varphi(V)=\psi(V)$, but as $V\subseteq U$ is arbitrary, we conclude that $\varphi=\psi$ and therefore we get the identity axiom.

ii) Once again let us take (U_i) to be an open cover of $U \subseteq X$ along with $\varphi_i \in \mathcal{H}om(\mathcal{F},\mathcal{G})(U_i)$ for each i. These are morphisms of sheaves, which means that for all open subsets $V \subseteq U_i$ they are maps between objects:

$$\varphi_i(V): \mathfrak{F}(V) \to \mathfrak{G}(V), \ V \subseteq U_i.$$

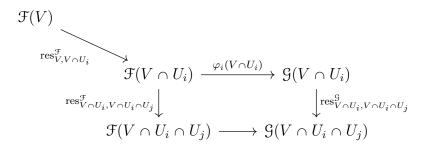
Assume now that the condition $\operatorname{res}_{U_i,U_i\cap U_j}(\varphi_i)=\operatorname{res}_{U_i,U_i\cap U_j}(\varphi_j)$ holds for all i,j. We must show that there exists a section $\varphi\in\mathcal{H}om(\mathfrak{F},\mathfrak{G})(U)$, which is

$$\varphi(V): \mathfrak{F}(V) \to \mathfrak{G}(V), \ V \subseteq U$$

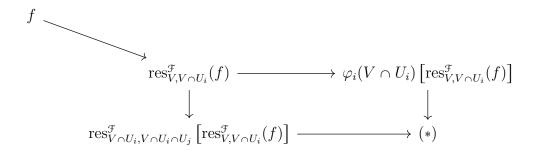
that satisfies $\operatorname{res}_{U,U_i}(\varphi) = \varphi_i$. This means that for open sets $V \subseteq U_i$, it must hold that

$$\operatorname{res}_{U,U_i}(\varphi)(V) = \varphi_i(V), \ V \subseteq U_i.$$

For this purpose, we will use the gluing axiom on the sheaf \mathfrak{G} . Let us now proceed by taking a section $f \in \mathfrak{F}(V)$ with $V \subseteq U$ and map it through the following diagram:



where the lower arrow is the map $\varphi_i(V \cap U_i \cap U_j)$. We can construct a similar diagram of φ_j . A section $f \in \mathcal{F}(V)$ maps through that diagram as follows:



and the lower right corner is either the restriction of the upper right corner, or the image of the lower left which by $\varphi_i(V \cap U_i \cap U_j)$. As the vf_i are morphisms of sheaves, both elements are equal. This can be expressed as follows

$$\varphi_{i}(V \cap U_{i} \cap U_{j}) \begin{pmatrix} \mathfrak{F} & \mathfrak{F} \\ \operatorname{res} & \operatorname{res} \\ V \cap U_{i}, V \cap U_{i} \cap U_{j} \end{pmatrix} = \underset{V \cap U_{i}, V \cap U_{i} \cap U_{j}}{\operatorname{res}} \left(\varphi_{i}(V \cap U_{i}) \begin{bmatrix} \mathfrak{F} \\ \operatorname{res} \\ V, V \cap U_{i} \end{pmatrix} \right)$$

But, let us simplify notation a bit by remembering that the composition of restriction maps is the beginning-to-end restriction map. This means that

$$\operatorname{res}_{V \cap U_i, V \cap U_i \cap U_j} \left[\operatorname{res}_{V, V \cap U_i} (f) \right] = \operatorname{res}_{V, V \cap U_i \cap U_j} (f).$$

With this in hand, and remembering the hypothesis that our φ_i 's coincide on intersections of the covering sets, we have:

$$\operatorname{res}_{V \cap U_{i}, V \cap U_{i} \cap U_{j}} \left(\varphi_{i}(V \cap U_{i}) \begin{bmatrix} \mathfrak{F} \\ \operatorname{res} \\ V, V \cap U_{i} \end{bmatrix} \right) \\
= \varphi_{i}(V \cap U_{i} \cap U_{j}) \begin{bmatrix} \mathfrak{F} \\ \operatorname{res} \\ V, V \cap U_{i} \cap U_{j} \end{bmatrix} \\
= \varphi_{i}(V \cap U_{i} \cap U_{j}) \begin{bmatrix} \mathfrak{F} \\ \operatorname{res} \\ V, V \cap U_{i} \cap U_{j} \end{bmatrix} \\
= \operatorname{res}_{V \cap U_{j}, V \cap U_{i} \cap U_{j}} \left(\varphi_{j}(V \cap U_{j}) \begin{bmatrix} \mathfrak{F} \\ \operatorname{res} \\ V, V \cap U_{j} \end{bmatrix} \right).$$

So by gluing the maps $\varphi_i(V \cap U_i) \left[\operatorname{res}_{V,V \cap U_i}^{\mathfrak{F}}(f) \right]$ in \mathfrak{G} we may construct a map $g \in \mathfrak{G}(V)$ such that

$$\operatorname{res}_{V,V \cap U_i}^{\mathfrak{G}}(g) = \varphi_i(V \cap U_i) \left[\operatorname{res}_{V,V \cap U_i}^{\mathfrak{F}}(f) \right]$$

for each i. We finally define the glued map φ in $\mathcal{H}om$ which takes our original f to this g which we have found. It follows from our construction that

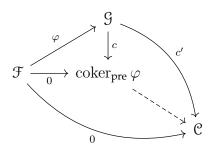
$$\operatorname{res}_{U,U_i}(\varphi)(V) = \varphi_i(V), \ V \subseteq U_i.$$

After verifying the axioms, we may conclude that the sheaf Hom is indeed a sheaf.

Exercise 2. Show that the presheaf cokernel satisfies the universal property of cokernels in the category of presheaves.

Answer

Given a map of presheaves φ , we must show that for $\operatorname{coker}_{\operatorname{pre}} \varphi$ given the following diagram:



that there exists a unique morphism of presheaves $\psi : \operatorname{coker}_{\operatorname{pre}} \varphi \to \mathfrak{C}$. Taking out particular objects for any open set U we have the same diagram but in terms of objects in the underlying category which is an abelian category. Thus, there exists a unique map

$$\psi(U) : \operatorname{coker}_{\mathsf{pre}} \varphi(U) \to \mathfrak{C}(U)$$

and with this we may define the morphism of presheaves $\operatorname{coker_{pre}} \varphi \to \mathfrak{C}$ by taking each of these maps into our collection of data. This immediately gives us unicity by construction and we are left to check that ψ is a morphism of presheaves. Thi means that for $U \subseteq V$, the following diagram commutes