Exercise 1. (Exercise 3.12.11) Show that

$$\mathcal{F}\ell(d_1, \ldots, d_k) \cong \mathrm{O}(n)/(\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k)),$$

where $n_1 = d_1$ and $n_i = d_i - d_{i-1}$ for $i = 2, \ldots, k$. (In other words, the n_i are the jumps in dimension as we go up the flag.)

Answer

Observe that $rO(n)$ acts on the flag variety. If $A \in \mathrm{O}(n)$, then

$$F = V_0 \subseteq \cdots \subseteq V_k \Rightarrow A \cdot F = AV_0 \subseteq \cdots \subseteq AV_k,$$

and $A \cdot F$ is still a flag of signature $(d_1, \ldots, d_k)$. This action is transitive as we can choose change-of-basis matrices in $\mathrm{O}(n)$ for our purposes to switch between two flags. This means that the unique orbit of the action is the flag variety itself. Now, observe that a block matrix in $\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k)$ fixes a flag of signature $(d_1, \ldots, d_k)$ as block-by-block, it fixes each V_k. Therefore, by the orbit-stabilizer theorem we have the desired result.

Exercise 2. Let M be a manifold with an affine connection ∇. Suppose $\alpha : I \to M$ is a constant curve; that is, $\alpha(t) = p$ for all $t \in I$. Let V be a vector field along α, meaning that $V(t) \in T_{\alpha(t)}M = T_pM$ just gives a curve in the tangent space T_pM. Show that $\frac{DV}{dt} = V'(t)$; that is, the covariant derivative agrees with the usual derivative in this case, regardless of what ∇ is.

Answer

Observe that along a curve α we have

$$\frac{DV}{\mathrm{d}t} = \nabla_{\frac{\mathrm{d}\alpha}{\mathrm{d}t}} V = \sum_{i,j,k} \left(\frac{\mathrm{d}v_k}{\mathrm{d}t} + \frac{\mathrm{d}\alpha_i}{\mathrm{d}t} v_j \Gamma_{ij}^k \right) X_k.$$

As our curve is constant, the terms on the right all cancel out so that we're left with

$$\frac{DV}{\mathrm{d}t} = \sum_k \frac{\mathrm{d}v_k}{\mathrm{d}t} X_k = V'(t).$$

Exercise 3. (Exercise 4.3.4) Show that an affine connection ∇ is compatible with a Riemannian metric g on M if and only if, for any vector fields V and W along a smooth curve $\alpha : I \to M$, we have

$$\left.\frac{d}{dt}\right|_{t=t_0} g_{\alpha(t)}(V(t), W(t)) = g_{\alpha(t_0)}\left(\frac{DV}{dt}, W\right) + g_{\alpha(t_0)}\left(V, \frac{DW}{dt}\right).$$

In other words, for compatible connections we can use the usual product rule to differentiate the inner product.

Answer

Let us suppose first that ∇ is compatible with g. If α is a curve, we may take an orthonormal basis of $T_{\alpha(t_0)}M$:

$$\{\, u_1(t_0), \dots, u_n(t_0) \,\}.$$

As ∇ is compatible with g, we may parallel-transport this basis throughout all the curve α. This means that for any $t \in I$,

$$\langle u_1(t), \dots, u_n(t) \rangle = T_{\alpha(t)}M.$$

Now, our vector fields V, W may be expressed as linear combinations of these basic elements in the following way:

$$\begin{cases} V(t) = \displaystyle\sum_{k=1}^{n} \alpha_k u_k(t) \\ W(t) = \displaystyle\sum_{k=1}^{n} \beta_k u_k(t) \end{cases} \Rightarrow \begin{cases} \dfrac{DV}{dt} = \displaystyle\sum_{k=1}^{n} \alpha_k' u_k(t) \\ \dfrac{DW}{dt} = \displaystyle\sum_{k=1}^{n} \beta_k' u_k(t) \end{cases}$$

where α_k, β_k are smooth functions. Now if we compute the quantity of the left, we have that

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=t_0} g_{\alpha(t)}(V(t), W(t)) \\
&= \sum_{k=1}^{n}\sum_{\ell=1}^{n} \left.\frac{d}{dt}\right|_{t=t_0} \alpha_k \beta_\ell g_{\alpha(t)}(u_k(t), u_\ell(t)) \\
&= \sum_{k=1}^{n} \left.\frac{d}{dt}\right|_{t=t_0} \alpha_k \beta_k \\
&= \sum_{k=1}^{n} \left.\frac{\mathrm{d}\alpha_k}{\mathrm{d}t}\right|_{t=t_0} \beta_k + \sum_{k=1}^{n} \alpha_k \left.\frac{\mathrm{d}\beta_k}{\mathrm{d}t}\right|_{t=t_0}
\end{aligned}$$

and then readding indices by multiplying $\delta_{k\ell}$ and a sum through ℓ we recover the final expression:

$$\sum_{\ell=1}^{n}\sum_{k=1}^{n}\left.\frac{\mathrm{d}\alpha_k}{\mathrm{d}t}\right|_{t=t_0}\beta_k g_{\alpha(t_0)}(u_k(t),u_\ell(t))+\sum_{\ell=1}^{n}\sum_{k=1}^{n}\alpha_k\left.\frac{\mathrm{d}\beta_k}{\mathrm{d}t}\right|_{t=t_0}g_{\alpha(t_0)}(u_k(t),u_\ell(t)).$$

Condensing everything by linearity we recover

$$g_{\alpha(t_0)}\left(\frac{DV}{dt},W\right)+g_{\alpha(t_0)}\left(V,\frac{DW}{dt}\right).$$

Now on the other hand suppose we have the identity in question. In order to show that our connection is compatible with g, we must show that for V,W parallel along α, we have that $g_\alpha(t)(V(t),W(t))$ is constant. To that effect, we will show that it has zero derivate.
Let V,W be parallel vector fields along $\alpha(t)$. Then

$$\frac{DV}{\mathrm{d}t}=\frac{DW}{\mathrm{d}t}=0,$$

and so our identity becomes

$$0=g_{\alpha(t_0)}\left(0,W\right)+g_{\alpha(t_0)}\left(V,0\right)=\left.\frac{d}{dt}\right|_{t=t_0}g_{\alpha(t)}(V(t),W(t)).$$

Thus $g_{\alpha(t)}(V(t),W(t))$ is a constant function because it has zero derivate.