Exercise 1 (Mandatory)**.** *stuff*

Answer

Behold:

8	10							
	5	5						
		4	6	6				
		1	2	2	3			
				1	1	2	7	9

Exercise 2 (Exercise 3)**.** Suppose λ/μ is a horizontal strip of size $n = |\lambda| - |\mu|$ consisting of rows of lengths $\alpha_1, \alpha_2, \ldots, \alpha_k$. How many *standard* Young tableaux are there of shape λ/μ?

Answer

Recall that any horizontal strip has no blocks on top of each other. This means that the only order that matters is left-to-right.

Exercise 3. Given an (undirected, labeled) graph G, a proper coloring of G is an assignment of a positive integer "color" to each vertex such that no two adjacent vertices have the same color.

If the colors assigned to the vertices are $c_1, c_2, \ldots, c_n$ (with some c_i's possibly being equal to each other), define the monomial of the coloring C to be

$$x^c = x_{c_1} x_{c_2} \cdots x_{c_n}.$$

Finally, define the chromatic symmetric function of G to be

$$X_G(\underline{x}) = \sum_c x^c$$

where the sum ranges over all proper colorings c of G.

i) Show that X_G is indeed a symmetric function for any graph G.
ii) Prove that if K_n is the complete graph on n vertices, then $X_{K_n} = e_n$.
iii) Compute X_{P_3}, X_{P_4} and X_{P_5} and express them in terms of elementary and Schur bases.

Answer

Let's begin by talking again about the chromatic symmetric function. Suppose $c : G \to [r]$ is an r-coloring of G, so for that particular coloring the monomial will be

$$x_1^{|c^{-1}(1)|} x_2^{|c^{-1}(2)|} \dots x_r^{|c^{-1}(r)|},$$

where $c^{-1}(i)$ is the inverse image of i, the vertices which are colored i. ATTACH EXAMPLE

i) To show that X_G is symmetric, we must show that

$$X_G(\underline{x}) = X_G(\sigma(\underline{x})) = X_G(x_{\sigma(1)}, x_{\sigma(2)}, \dots).$$

The permutation acts on the indices which represent the colors, so a permutation of the variables is a permutation on the colors used to paint the graph. Let us see that after permuting the colors, we still get a proper coloring.

With our coloring $c : G \to [r]$, let $\sigma \in S_r$. Pick $u, v \in G$ with $uv \in E$ such that $c(u) \neq c(v)$. So, as σ is a permutation we

$$c(u) \neq c(v) \Rightarrow \sigma(c(u)) \neq \sigma(c(v)) \Rightarrow \tilde{c}(u) \neq \tilde{c}(v)$$

where we have defined $\tilde{c} = \sigma \circ c$. The function $\tilde{c}$ is also a proper coloring of G and so, as σ was arbitrary, we see that a permutation of the colors gives us another proper coloring.

In other words, a particular vertex colored i is colored j after applying σ. As the vertex had no neighbors colored i, it won't have j neighbors so the coloring is proper.

Finally, as X_G runs through all possible colorings, after permuting we get the same sum but in a different order by the previous argument. We conclude that $X_G(\underline{x}) = X_G(\sigma(\underline{x}))$ and therefore X_G is symmetric.

ii) In the complete graph, all the vertices are connected which means proper colorings of K_n use n colors. So expanding X_{K_n} by monomials, we see that each monomial contains n different variables where each one is related to each vertex on K_n. Such expansion can be written as

$$X_{K_n} = \sum_{(*)} x_{i_1} x_{i_2} \dots x_{i_n}$$

where $(*) : i_1 \neq i_2, i_1, \neq i_3, \dots, i_2 \neq i_3 \dots$ and so on. By ordering our vertices we get that $(*)$ becomes $i_1 < i_2 < \cdots < i_n$ which brings us to e_n. So $X_{K_n} = e_n$.

iii) The chromatic number of a path graph is 2, however the sum runs over all proper colorings so we may use more than 2.

- Beginning with P_3 we may color by alternating the colors corresponding to monomials of the form $x_i^2 x_j$ or by painting all of the vertices differently ($x_i x_j x_k$). This means that

$$X_{P_3} = m_{(2,1)} + m_{(1,1,1)}.$$

Using CoCalc to convert to the elementary and Schur basis we get

$$X_{P_3} = e_{(2,1)} - 2e_3 = s_{(2,1)} - s_{(1,1,1)}.$$

- For P_4 we have the following monomials

$$\begin{cases} x_i^2 x_j^2 \leftarrow \text{alternating 2 colors.} \\ x_i^2 x_j x_k \leftarrow \text{alternating } P_3 \text{ and one more color.} \\ x_i x_j x_k x_\ell \leftarrow \text{4 colors.} \end{cases}$$

This means that

$$\begin{aligned} X_{P_4} &= m_{(2,2)} + m_{(2,1,1)} + m_{(1,1,1,1)} \\ &= e_{(2,2)} - e_{(3,1)} - e_4 \\ &= s_{(2,2)} - s_{(1,1,1,1)}. \end{aligned}$$

- Finally for P_5 we can add another vertex of another color to all the previous colorings to get a coloring of P_5. This amounts to monomials of the form

$$\begin{cases} x_i^2 x_j^2 x_k \leftarrow \text{alternating 2 colors plus one more.} \\ x_i^2 x_j x_k x_\ell \leftarrow \text{alternating } P_4 \text{ and one more color.} \\ x_i x_j x_k x_\ell x_m \leftarrow \text{4 colors plus one more.} \end{cases}$$

With this, we are only missing a 2-coloring of P_5 which is related to the monomial $x_i^3 x_j^2$. We may consider a coloring of the form

$$1 - 2 - 3 - 2 - 1$$

but this type of coloring is considered within the monomial $x_i^2 x_j^2 x_k$ for example. Recall that the number of variables is the number of colors used. With this we obtain:

$$\begin{aligned} X_{P_5} &= m_{(2,2,1)} + m_{(2,1,1,1)} + m_{(1,1,1,1,1)} + m_{(3,2)} \\ &= e_{(2,2,1)} - 2e_{(3,1,1)} + 3e_{(4,1)} - 4e_5 \\ &= s_{(3,2)} - s_{(3,1,1)} + s_{(2,1,1,1)} - 2s_{(1,1,1,1,1)}. \end{aligned}$$