Exercise 1. Let Ω be a domain containing the unit circle $C_1(0)$. Show that there is no function $F(z)$ holomorphic on Ω such that $e^{F(z)} = z$ on Ω. ⟦Hint: What would $F'(z)$ be? Can you prove that $F'(z)$ admits no primitive on Ω?⟧

Answer

Suppose F is a function which satisfies the equation. Differentiating we get

$$e^{F(z)}F'(z) = 1 \to F'(z) = \frac{1}{e^{F(z)}} = \frac{1}{z}.$$

In $\mathbb{C}\setminus\,]-\infty, 0]$ we have that $\frac{1}{z}$ admits $\log(z)$ as a primitive. So let us define $G(z) = F(z) - \log(z)$, this function has derivative 0 so G is constant.
This means that $F(z) = \log(z) + C$

Exercise 2. Let Ω be a domain with $0 \notin \Omega$.

(a) Suppose f, g are continuous branches of the logarithm on Ω. Show that there is some integer n such that $g(z) = f(z) + 2\pi i n$. ⟦Hint: Ω is connected.⟧
(b) Suppose $f(z)$ is a continuous branch of the logarithm. Show that $f(z)$ is holomorphic. ⟦Hint: Ω can be covered by simply connected domains.⟧

Answer

(a) If f, g are continuous branches of the logarithm then

$$f'(z) = g'(z) = \frac{1}{z}, \quad \text{for} \quad z \in \Omega.$$

With this $f - g = c$ where c is a constant. Now

$$z = e^{g(z)} = e^(g(z) + c) = e^c z \Rightarrow e^c = 1 \Rightarrow c = 2\pi i n$$

for some $n \in \mathbb{Z}$.

(b) Let f be a continuous branch of the logarithm. This is, $e^{f(z)} = z$, so we may apply the inverse function theorem to e^z on the domain Ω. For any $w_0 \in \Omega$, let $f(w_0) = z_0$. Then we have that $\frac{\mathrm{d}e^z}{\mathrm{d}z} = e^{z_0} \neq 0$ so by the inverse function theorem there exists a neighborhood $U \ni z_0$ such that e^z is locally invertible. The theorem also guarantees that the inverse must be holomorphic, so it must happen that f is holomorphic at every point of Ω.