Exercise 1 (7.8 Stein& Shakarchi)**.** The function ζ has infinitely many zeros in the critical strip. This can be seen as follows.

i) Let
$$F(s) = \xi(1/2 + 2), \quad \text{where} \quad \xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$
Show that $F(s)$ is an even function of s and as a result, there exists G such that $G(s^2) = F(s)$.

ii) Show that the function $(s-1)\zeta(s)$ is an entire function of growth order 1, that is
$$|(s-1)\zeta(s)| \leqslant A_\varepsilon e^{a_\varepsilon |s|^{1+\varepsilon}}.$$
As a consequence $G(s)$ is of growth order $1/2$.

iii) Deduce from the above that ζ has infinitely many zeros in the critical strip.

⟦Hint: To prove the first two parts use the functional equation for $\zeta(s)$. For the last one, use a result of Hadamard, which states that an entire function with fractional order has infinitely many zeros (Exercise 14 in Chapter 5).⟧

Answer

Exercise 2 (7.6 Stein& Shakarchi)**.** Read [SS]7.6, assume its result, and proceed as follows. Let δ be the function defined in [SS]7.6:

Answer

Exercise 3. One uses the results of the previous problems in the following way.

i) Show that $\text{res}(G, 1) = X$. ⟦Hint: Use the fact that $\zeta(s)$ has a pole at $s = 1$ of order 1.⟧

ii) Show that $\text{res}(G, 0) = \lim_{s\to 0} \frac{-\zeta'(s)}{\zeta(s)}$. *It turns out that this is* $-\log(2\pi)$.

iii) Show that $\sum \text{res}(G, \rho) = -\frac{1}{2}\log(1 - X^{-2})$, where the sum is over the trivial zeros of $\zeta(s)$.

From here, moving c "all the way to the left" means that we pick up all the residues of $G(s)$, and we are left with von Mangoldt's explicit formula:

$$\psi(X) = X - \sum \frac{X^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - X^{-2}\right)$$

where the sum is over all critical zeroes of $\zeta(s)$.

Answer

i) Observe that

$$\operatorname{res}(G,1) = \lim_{s\to 1}(s-1)\frac{X^s}{s}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) = \lim_{s\to 1}\frac{X^s}{s}\lim_{s\to 1}(s-1)(L(\zeta(s))).$$

The limit on the right is the residue at $s = 1$ of the logarithmic derivative of ζ, it is know that this residue is the order of the point in question of the function. This means that

$$\operatorname{res}(G,1) = X\cdot - \operatorname{ord}(\zeta,1) = X\cdot 1 = X.$$

ii) In this case, we have that

$$\operatorname{res}(G,0) = \lim_{s\to 0}(s)\frac{X^s}{s}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) = \left(\lim_{s\to 0}X^s\right)\left(\lim_{s\to 0}\frac{-\zeta'(s)}{\zeta(s)}\right)$$

and the left limit turns to 1 so we obtain the desired result.

iii) It is a subtle observation that

$$-\frac{1}{2}\log\left(1 - X^{-2}\right) = \frac{1}{2}\sum_{n\geqslant 1}\frac{\left(\frac{1}{X^2}\right)^n}{n} = \sum_{n\geqslant 1}\frac{X^{-2n}}{2n}.$$

Now, the trivial zeroes of the zeta function are at $s = -2n$, so

$$\operatorname{res}(G,-2n) = \lim_{s\to -2n}(s+2n)\frac{X^s}{s}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) = \left(\lim_{s\to -2n}\frac{X^s}{s}\right)(-\operatorname{ord}(\zeta,-2n))$$

where the limit evaluates to $\frac{X^{-2n}}{-2n}$ and the order is 1 so we obtain $\frac{X^{-2n}}{2n}$ and summing through all trivial zeroes we obtain the desired result.