Exercise 1 (Exercise 3). Prove that a word w has highest weight (i.e., $E_i(w) = 0$ for all i) if and only if w is Yamanouchi

Answer

First suppose $w \in [n]^k$ is a word of length k on the alphabet $[n]$. Now suppose additionally that w is Yamanouchi. This means that for every $s \leqslant k$, the suffix

$$w_{k-s+1} \dots w_{k-1} w_k$$

contains at least as many i's after the $(i+1)$'s. In particular this holds when $s = k$. So when applying the raising E_i operator we pair $(i+1)$ with an i to its right as a parenthesis. There are as much i's as $(i+1)$'s so every $(i+1)$ is paired and so the E_i operator can't convert any $(i+1)$ to an i.
As i is arbitrary, we can't apply any E_i to w which means that w has highest weight.
On the other hand[a] suppose $w \in [n]^k$ has highest weight. Then for all i, we can't apply E_i to w. This means that in w, it is possible to match all the $(i+1)$'s with i's that succeed them.
The previous fact lets us see that when reading w from right-to-left we will find at least as many i's as we find $(i+1)$'s. In other words, this means that w is Yamanouchi.

[a] Once again, **Kelsey**, **Trent**, and myself have talked about this problem in order to under the idea.

Exercise 2 (Exercise 4). Formulate and prove a Yamanouchi-type condition for w to be lowest weight, that is, $F_i(w) = 0$ for all $i \in [n]$. Such a word is called anti-ballot.

Answer

We will define anti-ballot by remembering the definition of ballot. Recall a ballot word w in $[n]^k$ has the property that for every $p \leqslant k$, the prefix

$$w_1 w_2 \dots w_p$$

contains at least as many i's *before* the $(i+1)$'s. So an anti-ballot word will contain as many $\underline{(i+1)}$'s before the $\underline{i}$'s. The claim is as follows:

A word w has lowest weight if and only if w is anti-ballot.

To prove this we first assume a word w is anti-ballot. Before applying F_i we notice that all the latter i's must already be paired with a previous $(i+1)$ because the word is anti-ballot.
This property holds for all i so it happens that we can't apply F_i to w, thus w has lowest weight.
On the flip-side, if we assume w has lowest weight, it means that all i's in w are matched with $(i+1)$ that precede them. So it must hold that there are at least as many $(i+1)$'s as i's in w. This condition holds for every i which means that w is anti-ballot.

Remark 1. It is important to notice that an anti-ballot word is not necessarily Yamanouchi even if it contains all the possible letters of the alphabet in question. The word

$$33\dots321$$

is anti-ballot but not Yamanouchi.

The examples from the crystal graphs contain Yamanouchi words on the top and anti-ballot words on the bottom.

Exercise 3 (Exercise 7)**.** How many ballot words of length n have only 1's and 2's?

Answer

First let us begin by noticing that any Yamanouchi word can be associated to an SYT T with $\text{sh}(T) = \lambda$, $|\lambda| = n$ and whose height is at most two.
Reading the Yamanouchi word from right-to-left, the entry w_{k-i+1} tells us the row in which we will insert i into the table. This insertion process is different from RSK insertion in the sense that we only "let the number fall along the row".
As an **example**, consider the the Yamanouchi word 212211121. This word is associated to the tableau

$$\begin{array}{|c|c|c|c|c}\hline 2 & 6 & 7 & 9 & \\ \hline 1 & 3 & 4 & 5 & \multicolumn{1}{c|}{8} \\ \hline\end{array}$$

because reading from right-to-left, the first 1 tells us to put the 1 in the first row, the 2 means that the 2 goes on the second row. The coming string 111 tells us that $3, 4$ and 5 all go in the first row after the one and so on.
This algorithm produces a standard Young tableau of height 2 as the word is Yamanouchi.