Toolkit

- An eigenvalue of a matrix A is a real number λ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v}$. Matrices of odd size always have at least one real eigenvalue, and 0 being an eigenvalue guarantees that A is non-invertible.
- Any subset of a linearly independent set is itself linearly independent.
- The symbol $\|\cdot\|$ denotes the Euclidean norm. If $\mathbf{x} = (x_1, x_2, \dots, x_d)$, then
$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$
- (Matrix-determinant lemma). For an invertible matrix A and vectors $\mathbf{u}, \mathbf{v}$,
$$\det\bigl(A + \mathbf{u}\mathbf{v}^{\mathsf{T}}\bigr) = (1 + \mathbf{v}^{\mathsf{T}} A^{-1}\mathbf{u}) \det(A).$$
- For a linear operator $T : \mathbb{R}^d \to \mathbb{R}^d$, its operator norm is defined as
$$\inf\{c \geqslant 0 : \ \|T\mathbf{v}\| \leqslant c\|\mathbf{v}\|, \ \forall \mathbf{v} \in \mathbb{R}^d\}.$$
Equivalently,
$$\|T\| = \sup_{\mathbf{v} \neq 0} \frac{\|T\mathbf{v}\|}{\|\mathbf{v}\|}.$$
For a linear operator given by a $d \times d$ matrix, one can compute
$$\|T\| = \max_j \sum_{i=1}^{d} |T_{ij}|.$$

Exercise 1. In *Determinant Tic-Tac-Toe*, Player 1 places a 1 in an empty 3×3 matrix. Player 0 responds by placing a 0 in another empty position. Play continues alternately until the matrix is filled with five 1's and four 0's. Player 0 wins if the determinant is 0, and Player 1 wins otherwise. Assuming optimal play, who wins, and what is the strategy?

Exercise 2. Generalize the game to a $d \times d$ matrix. If Players 1 and 0 alternate filling entries with 1's and 0's, what can be said about optimal strategies and outcomes?

Exercise 3 (Bulgarian National Olympiad, 2023, Problem 5)**.** For a fixed $n \in \mathbb{N}$, find the minimum value of
$$|x_1| + |x_1 - x_2| + |x_1 + x_2 - x_3| + \cdots + |x_1 + x_2 + \cdots + x_{n-1} - x_n|,$$
where $x_1, x_2, \dots, x_n \in \mathbb{R}$ satisfy $|x_1| + |x_2| + \cdots + |x_n| = 1$.

Exercise 4 (Problems Seminar, UNAM)**.** Let $n = abcd$ be a four-digit number with $a \neq 0$. Define

$$d(n) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Compute

$$\sum_{n=1000}^{9999} d(n).$$

Exercise 5 (Adapted from a similar problem)**.** Let $a_1, a_2, \ldots, a_{2n} \in \mathbb{R}$, and define the matrices

$$A = \begin{pmatrix} a_1^2+1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2^2+1 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & \cdots & a_n^2+1 \end{pmatrix}, \quad B = \begin{pmatrix} a_1a_{n+1} & a_1a_{n+2} & \cdots & a_1a_{2n} \\ a_2a_{n+1} & a_2a_{n+2} & \cdots & a_2a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_na_{n+1} & a_na_{n+2} & \cdots & a_na_{2n} \end{pmatrix},$$

$$C = \begin{pmatrix} a_{n+1}^2+1 & a_{n+1}a_{n+2} & \cdots & a_{n+1}a_{2n} \\ a_{n+2}a_{n+1} & a_{n+2}^2+1 & \cdots & a_{n+2}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2n}a_{n+1} & a_{2n}a_{n+2} & \cdots & a_{2n}^2+1 \end{pmatrix}.$$

Construct the block matrix

$$M = \begin{pmatrix} A & \mathbf{0} & B \\ \mathbf{0}^\intercal & 1 & \mathbf{0}^\intercal \\ B^\intercal & \mathbf{0} & C \end{pmatrix},$$

and compute $\det(M)$.

Exercise 6 (Costa Rican Problems Seminar)**.** Suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^d$ satisfy

$$\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k = 0.$$

Show that there exists a permutation σ of $\{1, 2, \ldots, k\}$ such that for each $n \in \{1, 2, \ldots, k\}$,

$$\left\| \sum_{i=1}^{n} \mathbf{x}_{\sigma(i)} \right\| \leqslant \left(\sum_{i=1}^{n} \left\| \mathbf{x}_{\sigma(i)} \right\|^2 \right)^{1/2}.$$

Exercise 7 (Costa Rican TST 2022)**.** Let $\mathcal{M}$ be the set of 5×5 real matrices of rank 3. For $A \in \mathcal{M}$, consider the $2^5 - 1$ nonempty subsets of its columns, and let k_A denote the number of these subsets that are linearly independent.

Determine the minimum and maximum possible values of k_A as A varies in $\mathcal{M}$.

Exercise 8 (Costa Rican TST 2022)**.** Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation. We say T is *tangential* if $\mathbf{v} \cdot T\mathbf{v} = 0$ for all $\mathbf{v} \in \mathbb{R}^d$.

(a) For even d, give an example of an invertible tangential transformation.

(b) Show that for odd d, no tangential transformation is invertible.