MATH 620: Variational Methods and Optimization I

Homework 1

Problem 1 (Lower semicontinuous functions). In class, we proved that any continuous function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ has (at least one) local minimum in D if D is compact. We also convinced ourselves that all three conditions – boundedness and closedness of D (which together constitute compactness in finite dimensional spaces) and continuity of f – were in fact necessary to guarantee the existence of a minimum.

a) Show one example each of domains D and functions f, for each of the three conditions that violate that one condition and that do not have a minimum. In other words, show that omitting any of the conditions does not result in a situation where existence of a minimum is guaranteed.

b) In truth, the statement above is not quite optimal. Continuity of the function is not actually necessary, even though it is easy to find discontinuous functions that do not have a minimum on a compact set D. Indeed, it is not difficult to find *discontinuous* functions that *do* have a minimum on a compact set D. Give a one and a two-dimensional example.

c) The resolution to this conundrum is that obviously the set of continuous functions is too small, and the set of potentially discontinuous functions too large. We need to seek another set of function that lies between. This set is the class of *lower semicontinuous functions*. A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is called lower semicontinuous at $x \in D$ if $f(x) \leq \lim_{k\to\infty} f(x_n)$ for all sequences $x_n \to x$; more generally, f is called lower semicontinuous if it is lower semicontinuous at all $x \in D$. (Obviously, if the statement holds with equality, then the function is *continuous*; furthermore, a function that is both lower and upper semicontinuous is of course also continuous.)

   Repeat the proof of the existence of a minimum for functions that only satisfy this weaker condition. Point out, in particular, where the proof deviates or is different from the one we have seen in class.

(40 points)

Problem 2 (Compactness in finite and infinite dimensional spaces).

a) We have sketched in class how one shows that a bounded and closed set in a finite dimensional space $\mathbb{R}^n$ is compact. (Here, let us use the "sequential compactness" we defined in class, rather than the topological one mentioned as an aside.) Work out the proof of this statement in detail and rigor. You will, in particular, need to work out the volume of the sets we consider in each step of the iteration, and how that affects the possible distance of any two points in it; then use this maximal possible distance rigorously to establish convergence. The key step in the proof is to show that if you make the volume smaller by bisecting the volume, the maximal distance must also decrease (perhaps not in each step individually, but after a fixed number of go-arounds).

b) Show in detail and rigor why this proof does not work in infinite dimensional spaces.

c) One could think of other ways of proving the statement, but fundamentally they fail because of a slightly surprising fact: *The volume of a ball of radius 1 goes to zero as the dimension goes to infinity.* In other words, ensuring that a sequence is entirely enclosed in a sequence of smaller and smaller volumes does not guarantee that it actually converges because that no longer implies that points are closer and closer to each other in large space dimensions.

   Confirm that the fact above is indeed true. You could look up the volume of the unit ball in n space dimension, but showing some kind of proof would be better :-)

(30 points)

Problem 3 (Compactness in finite and infinite dimensional spaces). You have seen two examples of sequences in infinite spaces that are enclosed in bounded and closed sets but that have no sub-sequences that converge. One of them was in the "space of sequences" ℓ_2,

$$\ell_2 := \left\{ x = (x_1, x_2, \ldots) : \|x\|_{\ell_2} = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} < \infty \right\},$$

and consisted of the sequence $x_1 = (1, 0, 0, \ldots), x_2 = (0, 1, 0, \ldots), \ldots$, which is completely enclosed in the bounded and closed unit ball.

The other in the "space of square integrable functions",

$$L_2 := \left\{ f : \|f\|_{L_2} = \left(\int_0^1 |f(x)|^2 \right)^{1/2} < \infty \right\}.$$

Here, the sequence $f_n(x) = \sin(n\pi x)$ does not converge and has no convergent subsequence. Again, this sequence is in the bounded and closed unit ball.

Try to come up with another non-convergent sequence in the set of operators defined as follows: Let $X = L_\infty$ be the space of all functions that are bounded on the interval $[0, 1]$:

$$L_\infty := \left\{ f : \|f\|_{L_\infty} = \sup_{0 \le x \le 1} |f(x)| < \infty \right\}.$$

Then consider *linear* operators $A : X \to X$ that somehow map on this space, and define the space $\mathcal{A}$ of such operators as follows:

$$\mathcal{A} := \left\{ A : X \to X \ : A \text{ is linear, } \|A\|_{\mathcal{A}} = \sup_{f \in L_\infty} \frac{\|Af\|_{L_\infty}}{\|f\|_{L\infty}} < \infty \right\}.$$

Because we have assumed that these operators are linear, we can equivalently define this set as

$$\mathcal{A} := \left\{ A : X \to X \ : A \text{ is linear, } \|A\|_{\mathcal{A}} = \sup_{f \in L_\infty, \|f\|_{L\infty} = 1} \|Af\|_{L_\infty} < \infty \right\}.$$

In other words, $\mathcal{A}$ consists of those operators that when given a bounded function returns a bounded function.

Next, we have to consider a bounded and closed subset D of this space $\mathcal{A}$. Let us choose the unit ball,

$$D := \{ A : X \to X \ : A \text{ is linear, } \|A\|_{\mathcal{A}} \le 1 \}.$$

That is, D are exactly those operators A that map a bounded function f to another function Af whose amplitude (the supremum of its absoute value) is at most as large as that of f. D is clearly closed and bounded.

Your task is to construct a sequence of operators $\{A_n\} \subset D$ for which there is no subsequence that converges – i.e., that illustrates that the set D is *not* compact.

(To this end, we need to define what it means for a sequence A_k to converge to a postulated limit operator $A \in \mathcal{A}$ that may or may not be in D. We do this in the obvious way: We say that $\lim A_k \to A$ if $\lim_{k\to\infty} \|A - A_k\|_{\mathcal{A}} = 0$, or in other words

$$\sup_{f \in L_\infty, \|f\|_{L\infty} = 1} \|Af - A_k f\|_{L_\infty} \to 0.$$

Conversely, to prove that $A_k \not\to A$, all you need is to find a particular f^* with $\|f^*\|_{L\infty} = 1$ (i.e., f^* has magnitude one) for which $\|Af^* - A_k f^*\|_{L_\infty} \not\to 0$. Of course, we really don't quite know yet whether such a limit operator A exists, so you may want to look at conditions on $A_k f$ that guarantee that the sequence converges. In particular, A_k converges if for every $\epsilon > 0$ there exists $N < \infty$ so that $\|A_m - A_n\|_{\mathcal{A}} < \epsilon$ for all $n, m \ge N$ – in other words, the Cauchy criterion.)

(30 points)