Exercise 1 (1.3.T)**.** Show that coproduct for Set is disjoint union.

Answer

Recall that the disjoint union of A_1 and A_2 is defined as a set as

$$A_1 \mathbin{\dot\cup} A_2\{\, (a_i, i) : \; a_i \in A_i, \; I = 1, 2\,\}.$$

This allows us to define maps $\iota_i : A_i \to A_1 \mathbin{\dot\cup} A_2, \; x \mapsto (x, 1)$ which are morphisms in Set because they're defined everywhere. We are to show that this set satisfies the universal property of coproducts.
Suppose B is a set such that $f_i : \; A_i \to B$ are well defined. We must define a unique function $g : \; A_1 \mathbin{\dot\cup} A_2 \to B$ such that $f_i = g\iota_i$, this is done as follows:

$$g(a, i) = \begin{cases} f_1(a), \; i = 1, \\ f_2(a), \; i = 2. \end{cases}$$

We verify the factoring property:

$$g \circ \iota_1(a_1) = g(a_1, 1) = f_1(a_1), \quad g \circ \iota_2(a_2) = g(a_2, 2) = f_2(a_2).$$

By construction, we have defined g uniquely.

Exercise 2 (1.3.U)**.** Suppose $A \to B$ and $A \to C$ are two ring morphisms, so in particular B and C are A-modules. Recall that $B \otimes_A C$ has a ring structure.

i) Show that there is a natural morphism $\iota_B : B \to B \otimes_A C, \; b \mapsto b \otimes 1$. Similarly for C.

ii) Show that this gives a pushout on rings. In other words, the following diagram satisfies the universal property of the pushout.

$$\begin{tikzcd}
B \otimes_A C & C \arrow[l, "\iota_C"'] \\
B \arrow[u, "\iota_B"] & A \arrow[l, "\beta"] \arrow[u, "\alpha"']
\end{tikzcd}$$

Answer

i) The map $\iota_B(b) = b \otimes 1$ is a homomorphism in virtue that $B \otimes_A C$ is a tensor product. By construction, all bilinear maps factor through the tensor product as linear maps. This map is one of the factors which should be linear. The same holds for C.

ii) Let us now take M an A-module with morphisms $f_B : B \to M$ and $f_C : C \to M$. This can described by the following diagram:

$$\begin{tikzcd}
M & & \\
& B \otimes_A C & C \arrow[l, "\iota_C"'] \arrow[ull, bend right, "f_C"'] \\
& B \arrow[u, "\iota_B"] \arrow[uul, bend left, "f_B"] & A \arrow[l, "\beta"] \arrow[u, "\alpha"']
\end{tikzcd}$$

However, let us take advantage of the tensor product, *gatekeeper of bilinear maps*. This morphisms can be combined into a bilinear map from $B \times C \to M$. We define

$$f : B \times C \to M,\ (b, c) \mapsto f_B(b) f_C(c)$$

and by universal property of the tensor product, there exists a unique map $\tilde{f} : B \otimes_A C \to M$ through which f factors. Finally f_B and f_C factor through $\tilde{f}$ by diagram chasing and thus by universality of the tensor product we have that it satisfies the pushout universal property in this case.

Exercise 3. Describe the colimit of the diagram $F : J \to \mathsf{Set}$ given by $* \leftarrow * \to *$.

Answer

Recall that the colimit of a diagram $F : J \to \mathsf{C}$ is an object $\mathrm{colim} A_i \in \mathrm{Obj}\mathsf{C}$ with morphisms $f_j : A_j \to \mathrm{colim} A_i$ such that if $m : k \to j$ is a morphism in J, then the following diagram commutes

$$\begin{tikzcd}
\mathrm{colim} A_i & \\
A_j \arrow[u, "f_j"] & A_k \arrow[l, "F(m)"] \arrow[ul, "f_k"']
\end{tikzcd}$$

In our case, since we only have three objects the diagram looks like this

$$\begin{tikzcd} CL & C \arrow[l] \\ B \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

where CL is the colimit object. In this particular case the colimit coincides with the pushout by universality.