Exercise 1 (Exercise 4). Prove that the standard representation of S_n, defined as the orthogonal subspace to the all ones vector under the permutation representation, is isomorphic to $V_{(n-1,1)}$ and is therefore irreducible.

> **Answer**
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> Let us call $\mathbf{o} = (1, 1, \ldots, 1) \in \mathbb{R}^n$, we know that
>
> $$\dim(\mathrm{gen}(\mathbf{o})) + \dim(\mathrm{gen}(\mathbf{o})^\perp) = n \Rightarrow \dim(\mathrm{gen}(\mathbf{o})^\perp) = n - 1.$$
>
> Now the Specht module $V_{(n-1,1)}$ has dimension
>
> $$\dim(V_{(n-1,1)}) = \#\,\mathrm{SYT}((n-1,1)),$$
>
> and any standard Young tableau must begin with a 1 on the lower-left corner. We have $n - 1$ possibilities for the block on top and the rest are determined because there's only one way to order. Thus, there are $n - 1$ standard Young tableaux with shape $(n - 1, 1)$. This means that
>
> $$\dim(V_{(n-1,1)}) = \#\,\mathrm{SYT}((n-1,1)) = n - 1,$$
>
> and thus $V_{(n-1,1)} \simeq \mathrm{gen}(\mathbf{o})^\perp$ as vector spaces. Now pick bases for each of these spaces, by showing that the group action commutes with the isomorphism we can extend linearly to conclude that they are isomorphic as representations[a]. We pick the basis
>
> $$\mathrm{gen}(\mathbf{o})^\perp = \mathrm{gen}(\mathbf{e}_2 - \mathbf{e}_1, \mathbf{e}_3 - \mathbf{e}_1, \ldots, \mathbf{e}_n - \mathbf{e}_1,),\ V_{(n-1,1)} = \mathrm{gen}(x_2 - x_1, \ldots, x_n - x_1)$$
>
> and if φ is the isomorphism which sends $e_i - e_1$ to $x_i - x_1$ then take $\sigma \in S_n$. We have
>
> $$\sigma(\varphi(e_i - e_1)) = \sigma(x_i - x_1) = x_{\sigma(i)} - x_{\sigma(1)} = \varphi(e_{\sigma(i)} - e_{\sigma(1)}) = \varphi(\sigma(e_i - e_1))$$
>
> and this holds for all basis elements. As the isomorphism is compatible with the action, we have that this is an isomorphism of representations.
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> [a]I had only verified that they were isomorphic as vector spaces when I was discussing this with **Ian**. Then he reminded me that I should also verify the part about the group action.

Exercise 2 (Exercise 5). The **regular representation** of S_3 is the action by left multiplication on the vector space of formal linear combinations of elements of S_3. In matrix form, one can think of labeling the rows and columns of a 6×6 matrix by the elements of S_3, and sending each element of S_3 to the 6×6 permutation matrix given by its action by left multiplication.

Decompose the regular representation of S_3 completely into irreducibles.

Answer

We begin by recalling
$$S_3 = \langle (12), (123) \rangle$$
so it suffices to write out the matrices for the two generators to find the matrices for the other elements:

$$(12) \mapsto M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (123) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

All the other matrices can be obtained by multiplying these two. The representation is thus $\{\,\mathrm{id}, M, N, N^2, MN, MN^2\,\}$[a].

Now notice that the vectors
$$\mathbf{u} = (1,1,1,1,1,1)^\mathsf{T} \quad \text{and} \quad \mathbf{v} = (1,-1,-1,-1,1,1)^\mathsf{T}$$
are common eigenvectors of the previous matrices[b].

The vectors $\mathbf{u}, \mathbf{v}$ generate dimension 1 subspaces of $\mathbb{R}^6$ which corresponds to a dimension 1 representation of S_3. In particular $\mathrm{gen}(u)$ corresponds to the trivial representation while $\mathbf{v}$, to the *sign* representation.

There are no more simultaneous eigenvectors which means that there are no more irreducible components of dimension 1. Thus the remaining representations are contained in
$$\mathrm{gen}(\mathbf{u}, \mathbf{v})^\perp \quad \text{and} \quad \dim(\mathrm{gen}(\mathbf{u}, \mathbf{v})^\perp) = 6 - 2 = 4$$

As it has dimension 4, it must be a direct sum of a pair 2-dimensional representations of S_3. Thus it holds that
$$\mathrm{regular} = \mathrm{trivial} \oplus \mathrm{sign} \oplus \mathrm{standard} \oplus \mathrm{standard}.$$

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[a]The appendix contains expressions for the remaining matrices.

[b]This cannot be justified with the simultaneous diagonalization theorem because M and N don't commute as S_3 is non-abelian. However by inspection we can find that they are eigenvectors.

Exercise 3 (Exercise 7.b). Show that $\tilde{H}_\mu(\mathbf{x}; 0, 1) = h_\mu$ and $\tilde{H}_\mu(\mathbf{x}; 1, 0) = h_{\mu^\mathsf{T}}$.

Answer

◇ We will begin by proving the second identity. The Macdonald polynomial at $(1, 0)$ is

$$\tilde{H}_\mu(\mathbf{x}; 1, 0) = \sum_{\sigma \text{ fills } \mu} 1^{\mathrm{inv}(\sigma)} 0^{\mathrm{maj}(\sigma)} \mathbf{x}^\sigma = \sum_{\substack{\sigma \text{ fills } \mu \\ \mathrm{maj}(\sigma) = 0}} \mathbf{x}^\sigma$$

where we have eliminated terms without zero maj because the zero factor kills them.

Now, this sum can be done in another way: a filling of μ with $\mathrm{maj}(\sigma) = 0$ implies that there no descents in the columns from top-to-bottom. This also means that we can consider the columns to be weakly-increasing from top-to-bottom.

The ways to fill every column are independent from each other and for a particular column, by counting from top-to-bottom, we are taking

$$\sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_{\mu_k^\mathsf{T}} \leqslant n} x_{i_1} \ldots x_{i_{\mu_k^\mathsf{T}}}$$

which is precisely $h_{\mu_k^\mathsf{T}}$. As the columns are independent from one another, we have that the whole h_{μ^T} is associated to the tableau as is $\tilde{H}_\mu(\mathbf{x}; 1, 0)$. We conclude that they are equal.
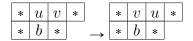
◇ [a] For the case of $\tilde{H}_\mu(\mathbf{x}; 0, 1)$, we have that this is

$$\tilde{H}_\mu(\mathbf{x}; 0, 1) = \sum_{\sigma \text{ fills } \mu} 0^{\mathrm{inv}(\sigma)} 1^{\mathrm{maj}(\sigma)} \mathbf{x}^\sigma = \sum_{\substack{\sigma \text{ fills } \mu \\ \mathrm{inv}(\sigma) = 0}} \mathbf{x}^\sigma$$

and in this case counting tableau with no relative inversions is a tad bit different because it isn't directly clear that rows will be weakly increasing.

However we may find a correspondence between *weakly-increasing on rows fillings* and *inversion-free fillings*.

Suppose we have a filling σ which is weakly-increasing on the rows, then there are no relative inversions on the first row. When considering the second row, suppose $u < v$ is the first relative inversion that appears on the row, this is because there's a $b \in \text{ROW}1$ such that $u \leqslant b < v$. We *flip* u and v and now the tableau looks like

$$\begin{array}{|c|c|c|c|}\hline * & u & v & * \\\hline * & b & * \\\cline{1-3}\end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|c|}\hline * & v & u & * \\\hline * & b & * \\\cline{1-3}\end{array}$$

which means that there's no longer a relative inversion as b *is between* u, v. If w is any element to the right of v, this process doesn't create any relative inversions because $b < v < w$.

This process can continue through the second row until we get rid of all relative inversions. Inductively we may finish the whole tableau this way until we have an inversion-less tableau.

To return to a weakly-increasing on rows tableau it suffices to order the elements on the rows of an inversion-less tableau.

Finally, the weakly-increasing on rows tableau can be paired with an h_μ by the same argument on rows than in columns as in the last item. So h_μ is a rearrangement of the sum $\tilde{H}_\mu(\mathbf{x}; 0, 1)$, we conclude that they must be equal.

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[a]**Kelsey**, **Trent**, and **yourself** helped me think through this problem.

Appendix: Matrix calculations

$$\diamond \ (132) \mapsto N^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\diamond \ (23) \mapsto MN = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\diamond \ (13) \mapsto MN^2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$\diamond \ \mathrm{id}_{S_3} \mapsto \mathrm{id}_{\mathcal{M}_6}.$

A basis of the orthogonal complement of $\mathrm{gen}(\mathbf{u}, \mathbf{v})$ is

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{x}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{x}_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 0 & -2 & 4 & -2 & 0 & 0 \\ 0 & -2 & -2 & 4 & 0 & 0 \\ -2 & 0 & 0 & 0 & 4 & -2 \\ -2 & 0 & 0 & 0 & -2 & 4 \end{pmatrix}$$

The matrix A changes basis from $\mathcal{B} = \mathbf{u}, \mathbf{v}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ to the canonical basis. Likewise A^{-1} changes basis from the canonical one to $\mathcal{B}$.

For our matrices X the matrix $A^{-1}XA$ acts as X does but on basis $\mathcal{B}$.

We have the following: When conjugating we obtain the following:

$\diamond \ (12) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

$\diamond \ (132) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$

$\diamond \ (123) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

$\diamond \ (23) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}$

$\diamond \ (13) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

$\diamond \ \mathrm{id}_{S_3} \mapsto \mathrm{id}_{\mathcal{M}_6}.$