Exercise 1 (2.4.H. Vakil)**.** Show that $\mathcal{F}^{\mathrm{sh}}$ (using the tautological restriction maps) forms a sheaf.

Answer

We will prove both sheaf axioms:

- Suppose $U \subseteq X$ is covered by (U_i) and $f, g \in \mathcal{F}^{\mathrm{sh}}(U)$ are two sections such that $\mathrm{res}_{U,U_i}(f) = \mathrm{res}_{U,U_i}(g)$ for all i. Now in terms of germs we have that

$$(f, U_i) \sim (g, U_i)$$

because they agree on the whole set U_i. So $(f_p) = (g_p)$ for $p \in U_i$ at the level of stalks. As (U_i) is a cover of U it holds that $(f_p) = (g_p)$ for $p \in U$, so as elements of $\prod_{p \in U} \mathcal{F}_p$ they are equal.

Now notice $(f_p)_{p \in U}$ is a collection of compatible germs, because every $p \in U$ is contained in some U_i and the section $s \in \mathcal{F}(U_i)$ that agrees as germs with (f_p) is precisely the restriction of f to that particular U_i. In other words, the s we are looking for is $\mathrm{res}_{U,U_i}(f)$. By the same argument (g_p) is also a collection of compatible germs and therefore both are in $\mathcal{F}^{\mathrm{sh}}(U)$. We conclude that $f = g$ as elements of $\mathcal{F}^{\mathrm{sh}}(U)$.

- Once again take (U_i) an open cover of $U \subseteq X$ but now $f_i \in \mathcal{F}^{\mathrm{sh}}(U_i)$ such that $\mathrm{res}_{U_i, U_i \cap U_j}(f_i) = \mathrm{res}_{U_j, U_i \cap U_j}(f_j)$. Once again in terms of germs we have

$$(f_i, U_i \cap U_j) \sim (f_j, U_i \cap U_j)$$

so that $(f_{i,p}) = (f_{j,p})$ for $p \in U_i \cap U_j$. This lets us define a collection of germs $(f_p) \in \prod_{p \in U} \mathcal{F}_p$ such that for a fixed p, f_p is $f_{i,p}$ where $p \in U_i$. This is a good definition because of the gluing condition.

Finally we must show that this collection of germs is compatible. For that effect take $p \in U$, so $p \in U_i$ for some i. Then $(f_{i,p})$ is a collection of compatible germs in U_i, so

$$\exists V (p \in V) \wedge \exists s (s \in \mathcal{F}(V) \wedge s_q = f_{i,q})$$

for $q \in U_i$. But inside U_i we have that f_q coincides with $f_{i,q}$ and thus with s_q. But this argument holds for any i and V is a neighborhood of p in U so it must occur that (f_p) is a collection of compatible germs. By construction, its restriction to U_i is f_i and thus we have found the glued function.

Exercise 2 (2.4.I. Vakil)**.** Describe a natural map of presheaves $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$.

Answer

If sh were a map of presheaves then for $U \subseteq X$ open,

$$\mathrm{sh}(U) : \mathcal{F}(U) \to \mathcal{F}^{\mathrm{sh}}(U)$$

should commutes with restrictions.The most *natural* idea of where to send $f \in \mathcal{F}(U)$ is to the collection of germs $(f_p)_{p\in U} \in \prod_{p\in U} \mathcal{F}_p$ and then checking that it's a compatible collection.

We see that that is the case: any point $p \in U$ has U as an open neighborhood and the section whose germs coincide with f's on all of U is f itself. This means that (f_p) is a compatible collection and thus is in $\mathcal{F}^{\mathrm{sh}}(U)$.

MISSING: show that it commutes with restrictions and makes the natural diagram commute.

Exercise 3 (2.4.J Vakil)**.** Show that the map sh satisfies the universal property of sheafification.

Answer

This problem is not asking us to restate the definition, it's asking us to take the previous map and then see that *that* satisfies the universal property according to the definition.

We want to see that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then there exists a unique $\psi : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}$ such that $\psi \circ \mathrm{sh} = \varphi$. In other words we want the following diagram to commute:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\mathrm{sh}"] \arrow[dr, "\varphi"'] & \mathcal{F}^{\mathrm{sh}} \arrow[d, dashed, "\exists!"] \\
& \mathcal{G}
\end{tikzcd}$$

For that effect, notice that φ induces a map of stalks:

$$(\varphi_p) : \prod_{p\in U} \mathcal{F}_p \to \prod_{p\in U} \mathcal{G}_p,$$

and we can restrict this map to only collections of compatible germs in order to build a map $\mathcal{F}^{\mathrm{sh}}(U) \to \mathcal{G}(U)$. FINISH

Exercise 4 (2.4.K Vakil)**.** Show that the sheafification functor is left-adjoint to the forgetful functor from sheaves on X to presheaves on X.

Answer

We must show that $(\mathrm{sh}, \mathscr{F})$, where $\mathscr{F}$ is the forgetful functor, form an adjoint pair. This means that we must show that there is a natural bijection:

$$\mathrm{Mor}_{\text{sheaf}}(\mathcal{F}^{\mathrm{sh}}, \mathcal{G}) \to \mathrm{Mor}_{\text{presheaf}}(\mathcal{F}, \mathscr{F}(\mathcal{G})).$$

Let us describe the mappings:

- If we have $\varphi : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}$ then we get a map of presheaves by composing with $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ so that
$$(\varphi \circ \mathrm{sh}) : \mathcal{F} \to \mathscr{F}(G)$$
is our desired map of presheaves.
- On the other hand, if we have a map of presheaves
$$\psi : \mathcal{F} \to \mathscr{F}(\mathcal{G})$$
then we can extend it to all of $\mathcal{G}$ by remembering $\mathcal{G}$ is indeed a sheaf. Then, by the universal property[a] of sheafification, there exists a unique morphism of sheaves
$$\widetilde{\psi} : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G},$$
and we have that $\psi = \widetilde{\psi} \circ \mathrm{sh}$.

This processes are inverses of each other so this is the bijection in question. CHECK NATURAL

[a] I know that we are using the universal property as mentioned in the book. However I fail to completely understand that this exercise is a restatement of the universal property.

Exercise 5. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of sets on a topological space X. Show that the following are equivalent:

(a) ϕ is an epimorphism in the category of sheaves.

(b) ϕ is surjective on the level of stalks: $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for $p \in X$.