Moduli Spaces of Stable Curves with Marked Points: Examples and Connections to Trees.

Ignacio Rojas

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Abstract

This work explores the concept of moduli spaces of stable curves with marked points, which are sets of parameters describing families of objects. These spaces can be used to solve problems in enumerative geometry, such as determining the number of curves passing through a given number of points. The common principle underlying these solutions is the association of the objects with a moduli space, which provides a different perspective on the problem. We illustrate this connection with examples.

Keywords: moduli space, curves enumerative geometry, parametrization.
MSC classes: Primary `14D22`; Secondary `05C05, 14H10`.

1 Introduction

Let us begin with a simple question:

Which are the quadratic curves which pass through 4 *points in* $\mathbb{R}^2$ *and no three of them are collinear?*

This question might be a bit tough to tackle right now, but let us simplify. How about if the points are $(1,1),(1,-1),(-1,-1)$ and $(-1,1)$? At once the following idea should pop-in in our heads: *a circle*! The circle which passes through these points is described by the equation $x^2+y^2=2$ as seen in Figure 1. Ideally we would like to stretch and shrink the circle in order to make it an ellipse. We

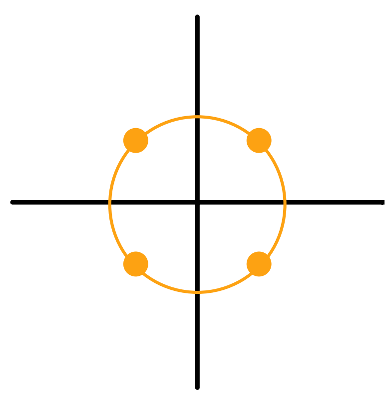

Figure 1: One of the quadratic curves passing through our points: $x^2+y^2=2$.

know ellipses have equations of the form $x^2/a^2 + y^2/b^2 = 1$, but to begin from our circle equation we will instead add coefficients to the equation

$$Ax^2 + By^2 = 2.$$

These coefficients are determined by the points on the curve, we may derive the relation by plugging in a point into the equation:

$$A(1)^2 + B(1)^2 = 2 \Rightarrow B = 2 - A \Rightarrow tx^2 + (2-t)y^2 = 2$$

where we take $t = A$ to get the last equation. We annotate the curves we obtain given different values of t:

- $(t = 1)$: A circle.
- $(1 < t < 2)$: An ellipse.
- $(t = 2)$: The pair of lines $x^2 = 1$.
- $(t > 2)$: A hyperbola.

However we are left with one curve which passes through the points in question. To find it we will assume t is non-zero. From our parametric equation we obtain

$$tx^2 + (2-t)y^2 = 2 \Rightarrow x^2 + o(t^2) + y^2 = \frac{2}{t^2} \xrightarrow[t\to\infty]{} x^2 = y^2$$

which is the pair of lines $y = \pm x$. Observe that this behavior is independent of the sign of the infinity we are going to. In essence what we have seen is that all the quadratic curves passing through our set of points can be parametrized by $\mathbb{R} \cup \{\infty\}$. Formally:

Proposition 1. *The moduli space $\overline{\mathcal{M}}_{0,4}$ can be identified with $\mathbb{P}^1_{\mathbb{R}}$.*

Intuitively the moduli space is a set of parameters. When the points vary *continuously*, the objects they parametrize deform *continuously* as well. What we have done here is not a proof of the previous proposition but it may serve as evidence that it is true.

To study this space and other spaces which may arise in this fashion, we may ask a question like *how many such curves can we find?* In order to do this, we will address this problem by connecting it with graphs.

2 Connection with trees

As a first approach we could consider an incidence graph where our vertices are the marked points and they are connected if they are in the same component of our curve. However that might produce undesirable results as it could lead to disconnected graphs.

Definition 1. For a point in $\overline{\mathcal{M}}_{0,X}$ (which represents a curve), we define the dual tree to that curve as:

- $V = X \cup I$ where I is the set of irreducible components in our curve. The set X attaches **labels** to our vertices while the curves are unlabeled.
- Vertices in X are not connected between themselves, but $u \in X$ is adjacent to $v \in I$ if u lies in the irreducible component associated to v.

  For $u, v \in I$, uv is an edge if the components meet at a nodal singularity.

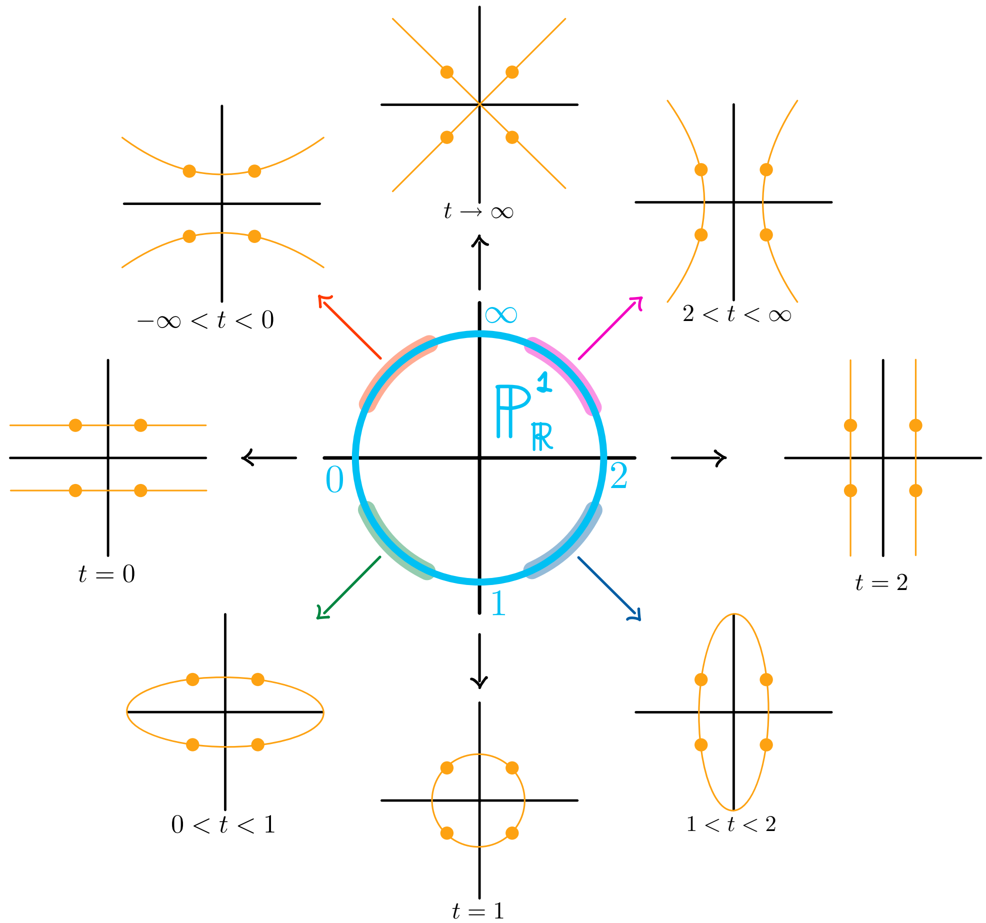

Figure 2: The projective real line as the moduli space $\overline{\mathcal{M}}_{0,4}$.

Even though we have defined the dual tree to be a tree, it may not be totally clear why this is the case: buajaja figure 1 1

Why should this process generate a tree? Why not a disconnected graph or a cycle?

This follows from the definition because we are talking about *genus* 0 curves. When we admit holes, what we are allowing in the graph is cycles.

Example 1. Let us consider the case of $\overline{\mathcal{M}}_{0,4}$, our labeled vertices will be

$$a = (1,1), \quad b = (-1,1), \quad c = (-1,-1), \quad 1 = (-1,-1).$$

We have different types of trees:

1. For ellipses and circles, the vertices are $a, b, c, 1$ or $\cdot$, and the edges are of the form $x\cdot$ for $x \in X$. This gives us a $K_{1,4}$ graph.

2. Hyperbolas have a unique component. In the projective plane, the components are connected at the point corresponding to the *slope of the asymptotes* at infinity, so the dual trees of the hyperbolas are also $K_{1,4}$ graphs.

3. For $t = 0$, there are two unlabeled vertices. a and b are connected to one vertex, while c and 1 are connected to the other. At infinity, there is a nodal singularity at the point corresponding to the slope of the lines, which means they connect.

   A similar analysis can be done for $t = 2$ and $t \to \infty$, and the resulting graph is two copies of P_3 connected by their middle vertices.

The corresponding trees are shown in the following figure:

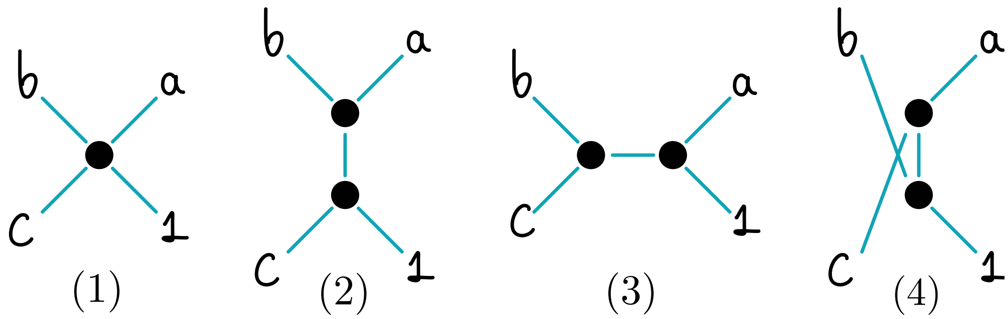

Figure 3: Trees associated to: (1) circles, ellipses and hyperbolas; (2) the curve $y^2 = 1$; (3) the curve $x^2 = 1$; and (4) the curve $x^2 = y^2$.

Remark 1. Look at the degrees of our vertices, there are no vertices of degree 2. If we remove the labels, all the trees besides (1) are isomorphic.

Also notice that when talking about the ellipses and the circle, we did not assign a particular value of t to each of the curves. We just said *an ellipse* or also *an hyperbola*, which means that the whole family of those curves is associated to the particular tree we obtained.

Definition 2. For a tree T we have that:

1. T is trivalent if all vertices of T have degree 1 or 3 and at least one vertex has degree 3.
2. T is at least trivalent if no vertex of T has degree 2 and at least one vertex has degree at least 3.

Remark 2. In our graphs, observe that the trees associated to *families* of curves like ellipses and hyperbolas, correspond to *at least trivalent* trees.

While for the particular cases $t = 0$, $t = 2$ and $t \to \infty$ we get exactly *trivalent* trees. *This is no coincidence!* The fact that at least trivalent trees correspond to a large number of curves and that the trivalent ones only to a select few.

Definition 3. The boundary stratum corresponding to a tree T is the set of curves whose dual tree is T.

Example 2. In our example, the boundary stratum of $K_{1,4}$ is

$$]-\infty, 0[\, \cup \,]0, 2[\, \cup \,]2, \infty[$$

where we identify ∞ with $-\infty$.

The remaining points $\{0\}, \{2\}$ and $\{\infty\}$ are *zero-dimensional* and these are the boundary points which correspond to the trivalent trees.

The observation that the boundary points correspond to the trivalent trees is key, because knowing this allows us to simplify the problem of counting the boundary points to counting *certain* trivalent trees. In general this result is true:

Proposition 2. *The boundary points of $\overline{\mathcal{M}}_{0,X}$ correspond to trivalent trees whose leaf set is labeled with X. If $X = \{ a, b, c, 1, 2, \ldots, n \}$, then the number of boundary points of $\overline{\mathcal{M}}_{0,X}$ is $(2n+1)!!$.*

To count the number of leaf-labeled trivalent trees L_n on $n+3$ leaves, we begin with the following small values:

- When $n = 0$, there is only one tree, $K_{1,3}$, with a unique labeling of the leaves. So $L_0 = 1$, which coincides with $(2(0) + 1)!! = 1$.
- When $n = 1$, we have two copies of P_3 joined by their middle vertices. There are 4! ways to label the four leaves without constraints. Accounting for symmetries, we have $L_1 = 4!/2^3 = 3$, as pictured in the Figure 1 above.
- For the next case we are supposed to find 15 trees. Counting by hand or considering symmetries is not the way to go. We've got to be more creative than that. A question arises:

  Is there a way to obtain the next trees from the old trees?

  In essence, we wish to add a new leaf to our graph. Intuitive ways in which we could proceed are:

  - Adding the leaf to a leaf vertex. But this actually doesn't work. We add one leaf but we lose one and even worse, now one vertex **has degree 2**. This means our tree is **no longer trivalent**.
  - Adding the leaf to a non-leaf vertex. Indeed we now have a new leaf, but the vertex we added to now has **degree 4**. So our tree is **no longer trivalent**.

  Apparently our original ideas won't work. So with a boost of creativity we will instead pop the leaf *out of an edge*:

  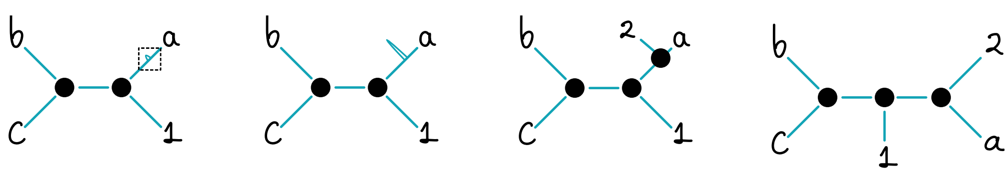

  Figure 4: *Popping* a leaf out of an edge to form a new trivalent tree.

  Note that adding a leaf to an edge creates a new vertex with degree 3 and adds a new leaf to the tree. There is no constraint on the number of vertices in a trivalent tree, so we can add as many new leaves as we want.

  Back to our three trees, each one has 5 edges which means there are 5 possible ways to add a labeled leaf. This gives us a total of 15 ways to form a 5-leaved labeled trivalent tree from the previous ones. Thus, $L_2 = 15$, which is equal to $(2{\cdot}2 + 1)!! = 1{\cdot}3{\cdot}5$.

- For $n = 3$, we count leaf-labeled trivalent trees with 6 leaves. Each of our 15 previous trees has 7 edges to which we can adjoin a new labeled leaf. For each of the trees, these are different possibilities. So in total we have $7{\cdot}L_2 = 105$ new trees.

We formalize this strategy using a couple of lemmas:

Lemma 1. *The number of edges E_n on a trivalent tree with $n+3$ leaves satisfies the recursion:*

$$E_n = E_{n-1} + 2, \quad E_0 = 3$$

which means that $E_n = 2n+3$.

Proof. We will proceed using induction. The base cases have been discussed earlier, so now we will use an $(n-1)+3 = n+2$ leaved trivalent tree as a starting point.

To add a new leaf while preserving the trivalent property, we add a new vertex to an existing edge and attach the leaf to that vertex. This process creates two new edges:

- One edge which was split in two by the addition of the new vertex in the middle.
- Another one created by attaching the leaf to the new vertex.

This means that the number of edges increased by two and the degree of the new vertex is 3, so $E_n = E_{n-1} + 2$ as desired. The recursion can be solved using the initial condition to obtain $E_n = (2n+3)$ for all n. □

Lemma 2. *The number of leaf-labeled trivalent trees with $n+3$ leaves, L_n, satisfies the recursion*

$$L_n = E_{n-1}L_{n-1}, \quad L_0 = 1.$$

Proof. The base cases have been proven in the previous discussion. So for an $(n+2)$-leaved tree, we have $a, b, c, 1, 2, \ldots, n-1$ as the labels of our leaves.

Adding the leaf labeled n can be done in E_{n-1} ways because we may attach it to any of the existing edges. Each of these new trees has a unique set of labels, and there are L_{n-1} such trees. Therefore, there are $E_{n-1}L_{n-1}$ new leaf-labeled trivalent trees.

Solving the recursion we have $L_n = (2n+1)L_{n-1}$ which means that

$$L_n = (2n+1)(2n-1)(2n-3)\cdots = (2n+1)!!.$$

□

With these results in hand the proposition is immediately true. The fact the boundary points correspond to the trivalent trees is a consequence of the fact that automorphisms of $\mathbb{P}^1$ are determined by 3 points.

3 But what really is $\overline{\mathcal{M}}_{0,n}$?

References

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