Exercise 1. Draw the $\mathfrak{sl}_3$ crystal for weight $(3, 3, 0)$.

Exercise 2. Prove that the elements of the hyperoctahedral group, written in cycle notation as a permutation on $\{\pm 1, \ldots, \pm n\}$, has all of its cycles coming in either pairs of the form $(a_1 \ldots a_k)(-a_1 \cdots - a_k)$, or of the form $(a_1 \ldots a_k - a_1 \; - a_2 \cdots - a_k)$.

Exercise 3. Define the Lie algebra $\mathfrak{so}_{2n+1}$ as $\{ X : X^{\mathsf{T}} S + SX = 0 \}$ where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0_n & I_n \\ 0 & I_n & 0_n \end{pmatrix}$$

and I_n is the $n \times n$ identity matrix and 1 is in the upper left corner. Write down what an arbitrary element X looks like, and using the fact that with respect to this setup the torus is simply the set of diagonal matrices X satisfying these conditions, explain how one obtains the type B root system.

Exercise 4. What is the dimension of the adjoint representation of $\mathfrak{so}_7$?

Exercise 5. Explain why the set of 5^{th} roots of unity in the plane don't form a root system. Which axioms of root systems does it satisfy?

Answer

Via the isomorphism $X \mapsto [X, -]$ we have that the dimension of the adjoint representation is the same as $\dim \mathfrak{so}_7$ which is $\binom{7}{2} = 21$.

Exercise 6. Compute the evacuation of the Young tableau below, and then evacuate again, and show you have returned to the starting tableau.

5			
2	7	8	
1	3	4	6

Exercise 7. Compute the Hall-Littlewood polynomial $\tilde{H}_{(2,1,1)}(x; q)$.

Exercise 8. Let $w = w_1 \ldots w_n$ be a word of partition content, and suppose $w_1 \neq 1$. Let $w' = w_2 \ldots w_n w_1$ be formed by cycling w_1 around to the end of the word. Show that $\boldsymbol{c}(w') = \boldsymbol{c}(w) - 1$ where $\boldsymbol{c}$ is cocharge. This operation is called *cyclage*.

Answer

Observe that it suffices to view this on standard words. This is because we may separate a word into standard subwords and calculate cocharge[a]. Consider the subword $\tilde{w}$ of w which contains w_1 in the previous decomposition sense, as w has partition content so does $\tilde{w}$.

When cycling w_1 to the end of $\tilde{w}$, cocharge is reduced by 1 as there is a element in $\tilde{w}$ smaller than w_1 which was to the right of w_1. After cycling, it's to the *left* and so the cocharge labeling drops by one.

[a] Ah! Inadvertently **you** helped me with this problem as the decomposition idea was written on your thesis!

Exercise 9. Give a counterexample showing that the formula in the above problem does not hold in general when $w_1 = 1$.

Answer

The word 121 has cocharge labeling 000 giving it a cocharge of 0 whereas 211 has cocharge labeling 100 with cocharge 1.