Exercise 1 (Exercise 1)**.** Prove that all three definitions of representations of finite groups given in the lecture notes are equivalent. Then, for the examples of the groups G and H from Examples 2.1 and 2.2 in the lecture notes, express these representations as a vector space with an action, and as a module.

The definitions in question are:

Definition 1. A representation of a group G over a field $\mathbb{F}$ is a homomorphism

$$\rho : G \to \mathrm{GL}_n(\mathbb{F})$$

where $\mathrm{GL}_n(\mathbb{F})$ is the group of invertible $n \times n$ matrices over $\mathbb{F}$.

Definition 2. A representation of a group G over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space V along with an action $G \rhd V$ by linear transformations, i.e. a homomorphism $\rho : G \to \mathrm{GL}(V)$.

Definition 3. A representation of a group G over a field $\mathbb{F}$ is an $\mathbb{F}G$-module V. (Here $\mathbb{F}G$ is the group ring consisting of formal linear combinations of elements of G over $\mathbb{F}$. A module is essentially a "vector space over a ring".)

Answer

Observe that the first definition gives rise to a map

$$G \to \mathrm{Aut}(V), g \mapsto \rho(g)$$

where $V = \mathbb{F}^n$ and $\rho(g)$ acts as a linear transformation on $v \in V$. This is an action because ρ is a homomorphism, namely:

- $\rho(e) = I_{n\times n}$.
- And $\rho(gh) = \rho(g)\rho(h)$ once again because ρ is a homomorphism.

So naturally we get the definition of representation as a vector space.
On the flipside any representation map can extend its domain to the group ring via linearity:

$$\widehat{\rho}\left(\sum_{g\in G} c_g g\right) := \sum_{g\in G} c_g \rho(g)$$