Exercise 1 (Exercise 3)**.** Prove that in the lattice of flats of a matroid, if X, Y are flats, then $X \vee Y = \mathrm{cl}(X \cup Y)$ and $X \wedge Y = X \cap Y$. You may use any of the matroid axioms.

We will use the following lemma.

Lemma 1. *For a set* X*,*

$$\mathrm{cl}(X) = \bigcap_{\substack{F \text{ is flat} \\ F \supseteq X}} F.$$

Proof

If F is flat and $F \supseteq X$ then $\mathrm{cl}(X) \subseteq \mathrm{cl}(F) = F$ because F is already closed. This is true for all flats which contain X so $\mathrm{cl}(X) \subseteq \bigcap F$. On the other hand $\mathrm{cl}(X)$ is a flat which contains X. Now $\bigcap F$ is the intersection of *all* the flat sets which contain X, in particular, $\mathrm{cl}(X)$. This means that $\bigcap F \subseteq \mathrm{cl}(X)$ and so they are equal.

The importance of this lemma is that it gives meaning to the phrase, *the closure of a set is the smallest flat which contains it.*

Answer

Observe that

$$X, Y \subseteq X \cup Y \subseteq \mathrm{cl}(X \cup Y)$$

which means $\mathrm{cl}(X \cup Y)$ is an upper bound for X, Y. Thus $X \vee Y \subseteq \mathrm{cl}(X \cup Y)$. [a]On the other hand,

$$X, Y \subseteq X \vee Y \Rightarrow X \cup Y \subseteq X \vee Y.$$

As $X \vee Y$ is a flat and $\mathrm{cl}(X \cup Y)$ is the smallest flat which contains $X \cup Y$, it must occur that

$$\mathrm{cl}(X \cup Y) \subseteq X \vee Y \quad \text{and so} \quad \mathrm{cl}(X \cup Y) = X \vee Y.$$

We will now prove that $X \cap Y$ is flat, observe that

$$X \cap Y \subseteq \mathrm{cl}(X \cap Y) \subseteq \mathrm{cl}(X), \mathrm{cl}(Y) = X, Y$$

where the middle containment follows from the subset property of closure. Now $\mathrm{cl}(X \cap Y)$ being a subset of both X, Y is also a subset of $X \cap Y$. So we conclude that $X \cap Y = \mathrm{cl}(X \cap Y)$ and so $X \cap Y$ is flat.

Finally we have

$$X \wedge Y \subseteq X, Y \Rightarrow X \wedge Y \subseteq X \cap Y,$$

and the other direction follows from greatest lower bound property. As $X \cap Y$ is a flat contained in X, Y then it is smaller than the greatest lower bound of X, Y which is $X \wedge Y$. In this context, *smaller* means *contained in*. Therefore $X \cap Y = X \wedge Y$.

[a] **Ian** just entered the house and I wanted to discuss the problem with him. After a brief moment of thinking we got it.

Exercise 2 (Exercise 5)**.** Suppose $\{x, y\}, \{y, z\}$ are circuits of a matroid and none of x, y, z are loops. Show that $\{x, z\}$ is a circuit.

Answer

Applying the circuit exchange axiom to $\{x, y\}$ and $\{y, z\}$ we get a circuit C such that

$$C \subseteq (\{x, y\} \cup \{y, z\}) \backslash \{y\} = \{x, z\}.$$

Out of the subsets of $\{x, z\}$, none of $\varnothing, \{x\}, \{z\}$ are circuits, so $C = \{x, z\}$ and therefore $\{x, z\}$ is a circuit.

Exercise 3 (Exercise 6)**.** Given a matroid $M = (E, \mathcal{B})$ in terms of basis axioms, define the associated simple matroid $\overline{M}$ by:

- Removing loops from E.
- Removing one element of each 2-circuit from E arbitrarily (i.e. choose one element from each *parallel class* - equivalence class of elements in 2-circuits with each other - to keep).
- Defining $\overline{\mathcal{B}}$ to be the set of basis formed by replacing any element of a parallel class that was in a basis B by the corresponding element in that parallel class.

Take the remaining edges to be $\overline{E}$. Define $\overline{M} = (\overline{E}, \overline{\mathcal{B}})$. Show that $\overline{M}$ is a simple matroid. ⟦Hint: Use the previous problem.⟧

Answer

We[a] wish to show that the collection $\overline{\mathcal{B}}$ satisfies the basis axioms. As the original $\mathcal{B}$ was non-empty, there is a $B \in \mathcal{B}$ which was a basis. It couldn't consist entirely of loops as it's a basis so when taking representatives we don't make B empty.

This basis makes our collection of bases non-empty.
Regarding the other base axiom, suppose have $\overline{B}_1 \neq \overline{B}_2$ in our collection $\overline{\mathcal{B}}$ with two particular elements $x \in \overline{B}_1 \backslash \overline{B}_2$ and $y \in \overline{B}_2 \backslash \overline{B}_1$. We wish to show that

$$\overline{B}_1 \backslash \{ x \} \cup \{ y \} \in \overline{\mathcal{B}}.$$

Observe that $\overline{B}_1$ and $\overline{B}_2$ were created from bases $B_1, B_2 \in \mathcal{B}$ and x, y are the chosen representatives of their parallel class. In particular x, y still satisfy the desired condition in the original matroid. As the original matroid *is a matroid* we can say that

$$B_1 \backslash \{ x \} \cup \{ y \} \in \mathcal{B}.$$

Taking the quotient by the parallel relation we can find that it corresponds to $\overline{B}_1 \backslash \{ x \} \cup \{ y \}$ in $\mathcal{B}$.

[a]In this problem we have collaborated with **Clare** and **Andrew**.

Exercise 4 (Exercise 7)**.** Let $(E, \mathcal{I})$ be a matroid and let $w : E \to \mathbb{R}$ be any weight function.

Show that applying the greedy algorithm to find a basis of minimal weight does indeed find a basis of minimal weight.

Answer

[a]We begin by recalling Kruskal's algorithm for graphs. From this we deduce what the algorithm for matroids must be.
Given a graph G, Kruskal's algorithm produces a minimum weight spanning tree T by doing the following:

i) Begin by choosing an edge of minimum weight.
ii) The next edge to be chosen must fulfill these condition:

    - It must not have been chosen previously.
    - It doesn't form a cycle with the previously chosen edges.

    Among the edges that fulfill this condition, we pick one of minimal weight and repeat the procedure.

With this we may elaborate a similar algorithm, but instead of guaranteeing that edges don't form a cycle, we want edges to not form a circuit instead. In the same fashion we are still adding edges of minimal weight to our potential basis. The algorithm in this case is as follows:

i) We begin by choosing an edge of minimal weight. If the set is empty we are done.
ii) The next edge must satisfy the following:

   - ◇ It must not have been chosen previously.
   - ◇ It doesn't form a *circuit* with the previously chosen edges.

   Among the edges that fulfill this condition, we pick one of minimal weight and repeat the procedure until we arrive to a maximal independent set.

Suppose that the algorithm outputs a basis $B = \{ e_1, \ldots, e_r \}$ with $w(e_1) \leqslant \ldots \leqslant w(e_r)$. If $B' = \{ f_1, \ldots, f_r \}$ is another basis, we may reorder the elements so that $w(f_1) \leqslant \ldots \leqslant w(f_r)$. To show B is of minimum weight it suffices to show that $w(e_k) \leqslant w(f_k)$ for all k.
Let us assume the contrary, not all of the e_k weigh less than the f_k. Assume k_0 is the first index at which this occurs. This means that

$$w(e_{k_0}) > w(f_{k_0})$$

so let us retrace our steps with the algorithm. At step k_0 we have our partially built basis B:

$$B_{\text{inc}} = \{ e_1, e_2, \ldots, e_{k_0-1} \}.$$

Consider also the partial base of B', $B'_{\text{inc}} = \{ f_1, \ldots, f_{k_0} \}$.
At this moment we have two independent sets B_{inc} and B'_{inc} and $|B_{\text{inc}}| \leqslant |B'_{\text{inc}}|$. Invoking the augmentation axiom we may find $f_\ell \in B'_{\text{inc}} \backslash B_{\text{inc}}$ such that $B_{\text{inc}} \cup \{ f_\ell \}$ is independent. However, it must occur that

$$w(f_\ell) \leqslant w(f_{k_0}) < w(e_{k_0}) \Rightarrow w(f_\ell) < w(e_{k_0})$$

first because all the elements in B'_{inc} have lower weight than f_{k_0} and second because of what we assumed. But this contradicts the algorithm's process.
At this step, the algorithm should've chosen f_ℓ instead of e_{k_0}. But it did not, it picked e_{k_0}. This means our assumption was wrong, and thus no f_k weighs less than an e_k. In conclusion the algorithm produces a basis of minimum weight.

[a] As in the last homework, I received invaluable assistance from **Oxley**'s book.