

**Exercise 1.** Do the following:

- i) Give a simple description of the closed sets in  $\mathbb{A}^1$  (with respect to the Zariski topology).
- ii) Use your previous answer to prove that  $\mathbb{A}^1$  is not Hausdorff.

### Answer

- i) If we consider  $\mathbb{A}^1$  over an algebraically closed field  $k$  of characteristic zero then every closed set is of the form  $V(I)$  where  $I \in \text{Spec}(k[x])$ . Since  $k[x]$  is a PID, then  $I = \text{gen}(p)$  for a polynomial  $p \in k[x]$ . Then  $V(I)$  would be the set of roots inside  $k$  of  $p$ . Since  $p$  is arbitrary, every closed set  $V(I)$  of  $\mathbb{A}^1$  is a finite set.

This means that the open sets are the complement of the finite sets. In essence, the Zariski topology coincides with the cofinite topology over  $\mathbb{A}^1$ .

- ii) The cofinite topology is not Hausdorff, so it follows that the Zariski topology isn't Hausdorff as well.

**Exercise 2.** Show that the Zariski topology on  $\mathbb{A}^2$  is not the product topology on  $\mathbb{A}^1 \times \mathbb{A}^1$ . (Hint: Consider the diagonal.)

**Exercise 3.** Let  $F : V \rightarrow W$  be a morphism of affine algebraic varieties. Prove that  $F$  is continuous in the Zariski topology.

### Answer

Suppose  $\mathcal{Z}$  is the Zariski topology. Recall that

$$F \text{ is continuous} \iff \forall U \in \mathcal{Z}(F^{-1}[U \setminus W] \notin \mathcal{Z}).$$

This means that  $F$  sends closed sets back to closed sets. Since a morphism is a vector of polynomials we can easily check this result.

Consider an arbitrary closed set  $V_0 \subseteq W$ . Then there exists an ideal  $I \trianglelefteq k[\mathbf{x}]$  such that  $V_0 = V(I)$ .

**Exercise 4.** Show that the twisted cubic  $V$  of Figure 1.5 is isomorphic to the affine line by constructing an explicit isomorphism from  $\mathbb{A}^1$  to  $V$ . (Hint: See Exercise 1.2.3)

**Exercise 5.** Show that if  $F : X \rightarrow Y$  is a surjective morphism of affine algebraic varieties, then the dimension of  $X$  is at least as large as the dimension of  $Y$ .