Exercise 1 (Exercise 1, Bonus)**.** What is your favorite bijection, and why? (This problem does not count towards your 10 point total problem limit; it will be an extra fun point added on top to whatever you hand in for real below.)

Answer

I have a whole set of bijections which I may call my favorite. Consider (f_d) where $d \in \texttt{DATES}$, the set `DATES` consists of all the days where I went to class. For each d, we have

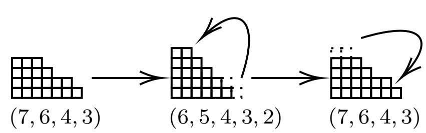
$$f_d : \{\,\texttt{students in 502 class on day } d\,\} \to \{\,\text{seats in room E204}\,\},$$
$$\texttt{student} \mapsto \text{preferred seat on day } d.$$

The function f_d is a bijection when restricted to its image because no two `grad students` can sit in the same chair.
In particular, amongst the f_d's, the ones I'm most partial to, are the ones in `DATES`$'$ where this is the subset of days in which I did not fall asleep in class.
These are my favorite bijections because they are unique and different. I'm almost sure[a] no one in class has a bijection like this as their favorite. Also, the idea of writing this made me chuckle, it's not everyday that you can have fun like this.[b]

[a] the set where I'm not sure has measure zero.
[b] **Everyone** in class collaborated with me on this problem. If it weren't for them, I wouldn't have been able to do this problem quite literally.

Exercise 2 (Exercise 2)**.** Apply Franklin's involution to $(7, 6, 4, 3)$ and check that applying it again returns the original partition.

Answer



For the partition $(7, 6, 4, 3)$ the smallest part is $r = 3$ and we have

$$7 = 1 + 6 \neq 2 + 4$$

so in this case $s = 2$. As $s < r$ we move 2 boxes from the staircase to the new top row. This gives us $(6,5,4,3,2)$. Now the smallest part is $r=2$ and we have that

$$6 = 1+5 = 2+4 = 3+3 = 4+2 \neq 5+0$$

so in this case $s=5$. Therefore we move the r's row and place each of the blocks on the first r rows. This gives us $(7,6,4,3)$ back.

Exercise 3 (Exercise 3, Sagan 3.16(c))**.** Show that the generating function for the number of partitions of n with $\lambda_1 = k$ equals the generating function for the number of partitions of n with exactly k parts and that both are equal to the product $x^k/[(1-x)\dots(1-x^k)]$.

Answer

Let us call $p(n,k)$ the number of partitions of n with exactly k parts and $\widehat{p}(n,k)$, the number of partitions of n with $\lambda_1 = k$. To show that their generating functions are equal, it suffices to prove that $p = \widehat{p}$ for all n, k.
To that effect, take $\lambda \vdash n$ with k parts. When conjugating λ we see that λ^* now has largest part equal to k. In terms of their Young tableaux, λ has dimensions $k \times \lambda_1$ while λ^* has dimensions $\lambda_1 \times k$.
Conjugation of partitions is a bijective map, which means that every partition with k parts is uniquely associated to a partition with largest part k and back. In conclusion it must hold that $p = \widehat{p}$ and therefore the generating functions are equal.
To see the product formula let us consider instead the number of partitions with largest part *at most* k. The generating product function for that sequence is

$$\begin{aligned}&\big((1+x+x^2+\dots)((x^2)^0+(x^2)^1+(x^2)^2+\dots)\dots((x^k)^0+(x^k)^1+(x^k)^2+\dots)\big)\\ =&\frac{1}{(1-x)(1-x^2)\dots(1-x^k)}.\end{aligned}$$

The last infinite sum counts the number parts in λ equal to k. If we wish to guarantee that there is one part equal to k we must multiply that factor by x^k. Therefore we have that the product which counts $\widehat{p}$ is

$$\begin{aligned}&\big((1+x+x^2+\dots)((x^2)^0+(x^2)^1+(x^2)^2+\dots)\dots((x^k)^1+(x^k)^2+(x^k)^3+\dots)\big)\\ =&\frac{x^k}{(1-x)(1-x^2)\dots(1-x^k)}.\end{aligned}$$

And since $p = \widehat{p}$ it holds that both functions are equal to the product.

Exercise 4 (Exercise 3, Sagan 3.16(d))**.** The Durfee square of λ is the largest square partition (d^d) such that $(d^d) \subseteq \lambda$. Use this concept to prove

$$\sum_{n=0}^{\infty} p(n)x^n = \sum_{d=0}^{\infty} \frac{x^{d^2}}{(1-x)^2(1-x^2)^2 \dots (1-x^d)^2}.$$

Answer

What the problem is telling us is to count the total number of partitions in a different way than the one we know.
For that effect consider a partition λ with its corresponding Young diagram. Such a partition can be decomposed into a Durfee square, a *right* component and an *up* component. Still viewing such components as Young diagrams, we can see a characteristic which defines them:

- The *right* component is a partition of $n-d$ with at most d parts.
- The *up* component is a partition (of some number) with largest part at most d.

By the previous problem we know that these types of partitions are in correspondence and each is counted by a product generating function. So for all sizes of Durfee squares we can count all partitions these way: fix a Durfee square size, then attach the *right* and *up* components to the Young diagram. Since each partition necessarily contains a Durfee square, we add a factor of x^{d^2} while each of the two components are each counted by $1/((1-x)\dots(1-x^d))$. Multiplying these together gives us the desired identity

$$\sum_{n=0}^{\infty} p(n)x^n = \sum_{d=0}^{\infty} \frac{x^{d^2}}{(1-x)^2(1-x^2)^2 \dots (1-x^d)^2}.$$

Exercise 5 (Exercise 3, Sagan 3.17). Let a_n be the number of partitions of n such that any part i is repeated fewer than i times and let b_n be the number of partitions such that no part is a square. Using generating functions, show that $a_n = b_n$.

Answer

We can expand the product generating function for a_n in the following way

$$\begin{aligned} a_n &\xrightarrow{ogf} (1)((x^2)^0 + (x^2)^1)((x^3)^0 + (x^3)^1 + (x^3)^2)((x^4)^0 + (x^4)^1 + (x^4)^2 + (x^4)^3)\dots \\ &= \prod_{i=1}^{\infty} \sum_{j=0}^{i-1} x^{ij}. \end{aligned}$$

The sum inside the product is a geometric sum equal to $\frac{1-x^{i^2}}{1-x^i}$. So it holds that

$$a_n \xrightarrow{ogf} \prod_{i=1}^{\infty} \frac{1-x^{i^2}}{1-x^i} = \frac{\displaystyle\prod_{i=1}^{\infty} \frac{1}{1-x^i}}{\displaystyle\prod_{i=1}^{\infty} \frac{1}{1-x^{i^2}}}.$$

The last product is the generating function which counts all partitions divided by the function which counts partitions of square numbers. In total, the quotient counts the number of partitions of numbers which are not squares. Thus

$$b_n \xrightarrow{ogf} \frac{\displaystyle\prod_{i=1}^{\infty} \frac{1}{1-x^i}}{\displaystyle\prod_{i=1}^{\infty} \frac{1}{1-x^{i^2}}}$$

and as a_n and b_n have the same generating function, it must hold that $a_n = b_n$.

Exercise 6 (Exercise 3, Sagan 3.18)**.** If $m \geqslant 2$, use generating functions to prove that the number of partitions where each part is repeated fewer than m times equals the number of partitions of n into parts not divisible by m.

Answer

Call a_n the number of partitions of n where each part is repeated fewer than m times and b_n the number of partitions of n into parts not divisible by m. By a similar reasoning to the last problem we have that

$$\begin{aligned} a_n &\xrightarrow{ogf} ((x^1)^0 + \cdots + (x^1)^{m-1})((x^2)^0 + \cdots + (x^2)^{m-1})((x^3)^0 + \cdots + (x^3)^{m-1})\ldots \\ &= \prod_{i=1}^{\infty} \sum_{j=0}^{m-1} x^{ij} = \prod_{i=1}^{\infty} \frac{1 - x^{im}}{1 - x^i}. \end{aligned}$$

This last product can be seen to be equal to

$$\frac{\displaystyle\prod_{i=1}^{\infty} \frac{1}{1 - x^i}}{\displaystyle\prod_{i=1}^{\infty} \frac{1}{1 - x^{im}}}$$

where the product on the top counts the number of total partitions and the one on the bottom counts partitions of numbers whose parts are multiples of m. Removing such factors from the top leaves us with the partitions of numbers whose parts are not divisible by m. This means that

$$b_n \xrightarrow{ogf} \frac{\displaystyle\prod_{i=1}^{\infty} \frac{1}{1 - x^i}}{\displaystyle\prod_{i=1}^{\infty} \frac{1}{1 - x^{im}}} \Rightarrow a_n = b_n.$$