Exercise 1 (Exercise 2)**.** Let $d > 1$ be a positive integer. A *d-ary De Bruijn sequence* of degree n is a sequence of length d^n containing every length n sequence in $([d-1]^*)^n$ exactly once as a circular factor.

i) Show that there always exists a d-ary De Bruijn sequence of degree n for any n.
ii) Find the number of d-ary De Bruijn sequences that begin with n zeroes. ⟦Hint: You may want to consult the computation for $d = 2$ given at the end of chapter 5.⟧

Remark. For this exercise, we will shift notation up by one and instead of considering the alphabet set as $[d-1]^* = \{0, 1, \ldots, d-1\}$, we will consider $[d] = \{1, \ldots, d\}$. We will also call $B(d, n)$ the set of De Bruijn sequences of length d^n with alphabet $[d]$.

Also, we will define the Bruijn graph $G_{d,n}$ as follows:

$$\begin{cases} V = [d]^n = \{ (s_1, \ldots, s_n) : \forall i (s_i \in [d]) \}, \\ E = \{ ((s_1, \ldots, s_n), (t_1, \ldots, t_n)) : \forall i (1 \leqslant i \leqslant n-1 \Rightarrow t_i = s_{i+1}) \wedge t_n \in [d] \}. \end{cases}$$

That is, the edge set is formed by pairs of strings of the form $(s_1, s_2, \ldots, s_n)$ and $(s_2, s_3, \ldots, s_n, t)$ where $t \in [d]$. We are shifting all indices in our string and adding a new admissible character.

Answer

i) First, let us prove that $G_{d,n}$ is Eulerian. Consider any vertex $v = (s_1, \ldots, s_n) \in G_{d,n}$, it holds that $d_{\text{out}}(v) = d_{\text{in}}(v) = d$.

First, consider the edges out of v: any out-neighbor of v is a vertex $(s_2, \ldots, s_n, t)$ with $t \in [d]$. Since there are d options for the character t, v sends an edge to each one of them.

Similarly, every edge which connects to v comes from a vertex of the form $(t, s_1, \ldots, s_{n-1})$. There are d vertices of that form in $G_{d,n}$. We conclude that $d_{\text{out}}(v) = d_{\text{in}}(v) = d$.

The De-Bruijn graph is also strongly connected: any two vertices u, v can be reached from one another after deleting and appending sufficient characters. It follows that $G_{d,n}$ is Eulerian for any d and any n.

Take an Eulerian cycle in a De Bruijn graph $G_{d,n-1}$. Such cycle traverses all the edges of our graph, and given that $d_{\text{out}}(v) = d$ for all v, it holds that

$$|E(G_{d,n-1})| = d \cdot |G_{d,n-1}| = d \cdot d^{n-1} = d^n.$$

Labeling the edges by the character it appends to each vertex, we get a string of length d^n which contains all possible substrings of length n in d characters. A d-ary de Bruijn sequence is minimal with respect to this property so it must hold that the sequence generated is a De Bruijn sequence.

ii) This problem is asking us to find all Eulerian walks which begin on any edge out of the vertex $(1, 1, \dots, 1)$[a]

Ideas:Recursion, BEST theorem, ch9 aigner

[a]Once again we remind ourselves of the preference of notation.

Exercise 2. An *undirected Eulerian tour* is a tour on the edges of an undirected graph using every undirected edge exactly once (in just one direction). Derive necessary and sufficient conditions for the existence of an undirected Eulerian tour in an undirected graph. Prove your result

Answer

An Eulerian tour must begin and end somewhere. All the other vertices are part of the walk, so we must *enter and exit* every one of those vertices. We should at least be able to *exit* the first vertex and *enter* the last one. Thus the result should be G has Eulerian tour if and only if

- ◇ Either all the vertices have even degree.
- ◇ Or exactly two vertices have odd degree.

In the first case we have an Eulerian *circuit* which is also a tour. On the second case, we can only find a tour from u to v. Let us prove that

A connected graph has an Eulerian circuit if and only iff all the vertices have even degree.

From which we will deduce

A connected graph has an Eulerian tour if and only if two vertices have odd degree.

Suppose G has an Eulerian circuit labeled by the vertices $x_1x_2 \dots x_m x_1$ where it is possible that there are repetitions among this sequence. Then, any vertex that appears k times in this sequence will have degree $2k$ because it will be entered and existed according to the circuit.

To prove the other direction we proceed by induction on the number of edges. Suppose G is a graph with m edges whose vertices all have even degrees. In the base case, when $m = 0$

Exercise 3 (Exercise 4)**.** The adjacency matrix of a directed graph D is A such that $a_{ij} = [\![(i, j) \in E]\!]$.

i) Show that the $(i, j)^{\text{th}}$ entry of A^k is the number of directed paths of length k from v_i to v_j in D.

ii) Verify that the following equality holds, where we consider both sides as formal power series in x with coefficients in the ring $M_n(\mathbb{Q})$:

$$(I - xA)^{-1} = 1 + xA + x^2A^2 + x^3A^3 + \dots$$

iii) Using the previous part along with the explicit formula for the inverse of a matrix (in terms of cofactors), show that the generating function for the number of paths $p_{i,j}(n)$ of length n from v_i to v_j is

$$\sum_{n=0}^{\infty} p_{i,j}(n)x^n = \frac{(-1)^{i+j}\det\left[(I - xA)^{(j,i)}\right]}{\det(I - xA)},$$

where $A^{(i,j)}$ is the $(i, j)^{\text{th}}$ minor of A.

This result is called the **transfer-matrix method**, as it gives a method of proving a sequence has a rational generating function, by showing that the sequence counts paths in a certain directed graph.

iv) Let b_n be the number of sequences of length $n + 1$ with entries from $[3]$ that start with 1, end with 3, and do not contain the subsequences 22 or 23. Find a closed formula for the generating function of b_n using the transfer-matrix method, by constructing a directed graph in which certain paths are counted by b_n. You may use a computer to calculate the determinants, but you must write out the directed graph and the corresponding matrices.

Answer

i) Suppose D is a directed graph with n vertices, we will proceed by induction and use A^2 as a base case. The $(i, j)^{\text{th}}$ entry of A^2 is given by

$$\sum_{k=1}^{n} a_{ik}a_{kj} = a_{i1}a_{1j} + a_{i2}a_{2j} + \cdots + a_{in}a_{nj},$$

and every term $a_{ik}a_{kj}$ counts the number of edges from v_i to v_k times the number of edges from v_k to v_j. But a length 2 path from v_i to v_j can go through any other vertex $u \in D$. Since each path must be different, we sum each possibility to get the complete number.

Suppose now that the $(i,j)^{\text{th}}$ entry of A^m is the number of paths of length m from v_i to v_j for $m \leqslant k-1$. Now

$$A_k = (A^{k-1})A \Rightarrow (A^k)_{ij} = \sum_{\ell=1}^{n} (A^{k-1})_{il} A_{\ell j}.$$

We can thus decompose the $(i,j)^{\text{th}}$ entry of A^k into a sum of terms of the form $(A^{k-1})_{il}A_{\ell j}$. By induction hypothesis $(A^{k-1})_{il}$ is the number of paths from v_i to v_ℓ and adding the edge which could go from v_ℓ to v_j we get a new path. However v_ℓ can be any vertex in D, so summing the possibilities we get the complete number of length k walks.

Thus we conclude that the $(i,j)^{\text{th}}$ entry of A^k is the number of paths of length k from v_i to v_j in D.

ii) We need to prove that the inverse of $I - xA$ is the matrix $1 + xA + x^2A^2 + x^3A^3 + \dots$ and to do that we will multiply them:

$$\begin{aligned}&(I - xA)(1 + xA + x^2A^2 + x^3A^3 + \dots)\\ =&(1 + xA + x^2A^2 + x^3A^3 + \dots) - (xA - x^2A^2 - x^3A^3 - x^4A^4 - \dots) = I.\end{aligned}$$

We do not worry about convergence issues because this is a formal power series.

iii) The $(i,j)^{\text{th}}$ entry of $\sum_{n=0}^{\infty} A^n x^n$ is $\sum_{n=0}^{\infty} p_{i,j}(n)x^n$ and recalling that this matrix is actually $(I - xA)^{-1}$, we must find the entries of this matrix. By the inverse formula for a matrix we get

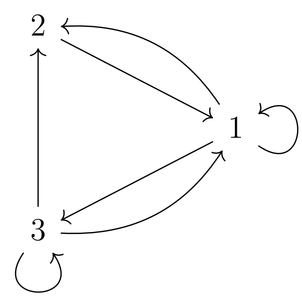
$$(I - xA)^{-1} = \frac{1}{\det(I - xA)} \operatorname{adj}(I - xA) = \frac{1}{\det(I - xA)} \operatorname{cof}(I - xA)^{\mathsf{T}}.$$

The $(i,j)^{\text{th}}$ entry of the cofactors matrix is $(-1)^{i+j}\det\big[(I - xA)^{(i,j)}\big]$, the determinant of the minor matrix obtained by deleting the i^{th} row and j^{th} column. Thus the $(i,j)^{\text{th}}$ of the transpose is obtained by switching the indices.

We conclude that the $(i,j)^{\text{th}}$ entry of $\sum_{n=0}^{\infty} A^n x^n$ is

$$\left(\frac{1}{\det(I - xA)} \operatorname{cof}(I - xA)^{\mathsf{T}}\right)_{i,j} = \frac{1}{\det(I - xA)} (-1)^{i+j} \det\big[(I - xA)^{(j,i)}\big].$$

iv) Consider the following graph which encodes the construction of our string:



Finding the amount of strings of length $n+1$ beginning with 1, ending in 3 is the same as counting walks from vertex 1 to 3 of length n. This means that $b_n = p_{1,3}(n)$ and we know that that sequence's generating function is

$$\frac{1}{\det(I-xA)}(-1)^{1+3}\det\big[(I-xA)^{(3,1)}\big].$$

To find a closed form we start by noting that the adjacency matrix of our graph is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow I - xA = \begin{pmatrix} 1-x & -x & -x \\ -x & 1 & 0 \\ -x & -x & 1-x \end{pmatrix}.$$

We expand by minors using the third row so that we can calculate the minor determinant at once:

$$\begin{aligned} &\det(I-xA) \\ =&(-x)\det\begin{pmatrix} -x & -x \\ 1 & 0 \end{pmatrix} - (-x)\det\begin{pmatrix} 1-x & -x \\ -x & 0 \end{pmatrix} + (1-x)\det\begin{pmatrix} 1-x & -x \\ -x & 1 \end{pmatrix} \\ =&(-x)(0-(-x)) + (x)(0-x^2) + (1-x)[(1-x)-x^2] \\ =&1-2x-x^2 \end{aligned}$$

We conclude that the generating function is $\frac{-x}{1-2x-x^2}$.