Exercise 1. Let us introduce two notions:

- ◇ Let $(\mathbb{R}, \leqslant)$ denote the category whose objects are real numbers and there exists a morphism $f : x \to y$ if and only if $x \leqslant y$.
- ◇ The category $(\mathbb{Z}, \leqslant)$ is the same but for $\mathbb{Z}$.

The inclusion $\iota : \mathbb{Z} \hookrightarrow \mathbb{R}$ induces a fully faithful functor between these categories.

Show that $(\iota, \lfloor * \rfloor)$ and $(\lceil * \rceil, \iota)$ are pairs of adjoint functors.

Answer

Let us observe first that the Hom-sets in these categories are either empty or singletons. This is because $x \leqslant y$ or not. In the positive case $\mathrm{Hom}(x, y)$ is a singleton, on the other one, it's empty.
In order to organize, x, y will be elements of $\mathbb{Z}$, and $\alpha, \beta \in \mathbb{R}$.
To show that $(\iota, \lfloor * \rfloor)$ are a pair of adjoint functors, we must show that

$$\mathrm{Hom}(\iota(x), \alpha) \to \mathrm{Hom}(x, \lfloor \alpha \rfloor), \ x \in \mathbb{Z}, \ \alpha \in \mathbb{R}$$

is a bijection and for $x \leqslant y$ (in other words $f : x \to y$), the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(\iota(y), \alpha) \arrow[r] \arrow[d] & \mathrm{Hom}(\iota(x), \alpha) \arrow[d] \\
\mathrm{Hom}(y, \lfloor \alpha \rfloor) \arrow[r] & \mathrm{Hom}(x, \lfloor \alpha \rfloor)
\end{tikzcd}$$

We prove that $\mathrm{Hom}(\iota(x), \alpha) \to \mathrm{Hom}(x, \lfloor \alpha \rfloor)$ is a bijection by considering two cases:

- ◇ Either $x \leqslant \alpha$, and this means that $x \leqslant \lfloor \alpha \rfloor$ which means that both sets are singletons and therefore there exists a bijection between them.
- ◇ Or $x > \alpha \geqslant \lfloor \alpha \rfloor$ and both sets are empty and the empty function satisfies what we ask.

The following diagrams exhibit the possibilities of what the previous diagram converts to:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\
\varnothing \arrow[r] & \varnothing
\end{tikzcd}
\qquad
\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & * \arrow[d] \\
\varnothing \arrow[r] & *
\end{tikzcd}
\qquad
\begin{tikzcd}
* \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & *
\end{tikzcd}$$

- The first case exhibits the case $\alpha \leqslant x \leqslant y$, then $\lfloor\alpha\rfloor \leqslant x \leqslant y$ which means that all of the sets are empty and therefore the empty function commutes all the way around.
- In the second case we have $x \leqslant \alpha \leqslant y$. Still $\lfloor\alpha\rfloor \leqslant y$ but the least the $\lfloor\alpha\rfloor$ can be is x so the Hom-sets on the right are non-empty. Composition with the empty function results in the empty function results in the empty function so our diagram commutes.
- In the last case $x \leqslant y \leqslant \alpha$ and so $x \leqslant y \leqslant \lfloor\alpha\rfloor$. All sets are non-empty and since they are singletons, the diagram commutes.

This lets us conclude that there is a natural bijection between our Hom-sets and therefore $(\iota, \lfloor * \rfloor)$ forms an adjoint pair.
With a similar argument we can show that

$$\mathrm{Hom}(\lceil\alpha\rceil, x) \to \mathrm{Hom}(\alpha, \iota(x))$$

is a bijection and for $\alpha \leqslant \beta$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(\lceil\beta\rceil, x) \arrow[r] \arrow[d] & \mathrm{Hom}(\lceil\alpha\rceil, x) \arrow[d] \\
\mathrm{Hom}(\beta, \iota(x)) \arrow[r] & \mathrm{Hom}(\alpha, \iota(x))
\end{tikzcd}$$

Exercise 2 (1.6.D Vakil)**.** Show that a map of complexes induces a map of homology $H^i(A^\bullet) \to H^i(B^\bullet)$ and furthermore, H^i is a covariant functor from $\mathsf{Com}_{\mathsf{C}} \to \mathsf{C}$. $\llbracket$Feel free to deal with the special case Mod_A.$\rrbracket$

Answer

We will work inside the category of modules in this case. Consider two complexes $A^\bullet, B^\bullet$ with a map of complexes $\varphi : A^\bullet \to B^\bullet$ where $\varphi^i : A^i \to B^i$. To define a map between homology, we will first show that the chain map preserves cycles and boundaries.

- Suppose $z \in A^i$ is a cycle, then $f^i(z) = 0$. Composing with φ^{i+1} we still get 0. However, by commutativity we have

$$0 = \varphi^{i+1}(f^i(z)) = g(\varphi^i(z)) \Rightarrow g(\varphi^i(z)) = 0$$

which means that $\varphi^i(z)$ is a cycle in B^i. The following diagram represents the previous situation:

$$\begin{tikzcd}
z \in A^i \arrow[r, "f^i"] \arrow[d, "\varphi^i"'] & 0 \in A^{i+1} \arrow[d, "\varphi^{i+1}"] \\
\varphi^i(z) \in B^i \arrow[r, "g^i"'] & 0 \in B^{i+1}
\end{tikzcd}$$

- ⋄ On the other hand suppose $y \in A^i$ is a boundary. Then

$$\exists x(x \in A^{i-1} \wedge f^{i-1}(x) = y).$$

We wish to find an $\widetilde{x} \in B^{i-1}$ such that $g^{i-1}(\tilde{x}) = \varphi^i(y)$, so we claim that such $\widetilde{x}$ is $\varphi^{i-1}(x)$. By diagram commutativity we have that

$$g^{i-1}(\varphi^{i-1}(x)) = \varphi^i(f(x)) = \varphi^i(y)$$

which means that $\varphi^i(y)$ is a boundary. Diagrammatically we have

$$\begin{tikzcd}
\exists x \in A^{i-1} \arrow[r, "f^{i-1}"] \arrow[d, "\varphi^{i-1}"'] & y \in A^i \arrow[d, "\varphi^i"] \\
\exists ? \tilde{x} \in B^{i-1} \arrow[r, "g^{i-1}"'] & \varphi^i(y) \in B^i
\end{tikzcd}$$

Now recall that the homology groups are defined as ${}^{\ker(f^i)}/_{\mathrm{Im}(f^{i-1})}$ which means that there is a projection map $\pi_A^i : \ker(f^i) \to H^i(A^\bullet)$.
Composing this with our chain map[a] we get

$$\pi_B^i \circ \varphi^i : \ker(f^i) \to H^i(B^\bullet).$$

As φ^i preserves boundaries, it holds that elements in $\mathrm{Im}(f^{i+1}) \subseteq \ker(f^i)$ are sent to $\mathrm{Im}(g^{i+1})$ which is the identity element in $H^i(B^\bullet)$. So by universality $H^i(A^\bullet)$ as a quotient, there exists a unique morphism $H^i(A^\bullet) \to H^i(B^\bullet)$. This is interpreted as a diagram as follows:

$$\begin{tikzcd}
\ker(f^i) \arrow[r, "\pi_B^i \circ \varphi^i"] \arrow[d, "\pi_A^i"'] & H^i(B^\bullet) \\
{}^{\ker(f^i)}/_{\mathrm{Im}(f^{i-1})} = H^i(A^\bullet) \arrow[ur, dashed, "\exists ! \varphi^{\bullet i}"'] &
\end{tikzcd}$$

From the relation $\pi_B^i \circ \varphi^i = \varphi^{\bullet i} \circ \pi_A^i$ we can define $\varphi^{\bullet i}$ concretely as

$$\varphi^{\bullet i}([z]) = [\varphi^i(z)].$$

This also shows that H^i acts as a covariant functor because we began with a map of complexes $\varphi : A^\bullet \to B^\bullet$ and obtained $\varphi^{\bullet i} : H^i(A^\bullet) \to H^i(B^\bullet)$ which follows the direction of our original map.

[a]Restricted to the kernel since cycles get sent to cycles.

Exercise 3. Let C be an abelian category and let $C \in \mathsf{Obj}(\mathsf{C})$. Show that $\mathrm{Hom}_{\mathsf{C}}(C, *) : \mathsf{C} \to \mathsf{Ab}$ is a left-exact covariant functor.

Answer

Let us begin by considering the following diagram of C-objects:

$$\begin{tikzcd}
 & C \arrow[dl] \arrow[d, "\alpha"'] \arrow[dr, "\beta"] \arrow[drr] & & \\
0 \arrow[r] & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

where $X \to Y \to Z$ is exact, meaning that $\ker(g) = \mathrm{Im}(f)$ and f is injective. After functorising the sequence we obtain the sequence

$$0 \longrightarrow \mathrm{Hom}(C, X) \xrightarrow[f_*]{} \mathrm{Hom}(C, Y) \xrightarrow[g_*]{} \mathrm{Hom}(C, Z)$$

where $f_*(\varphi) = f \circ \varphi$. First, we show that f_* is injective and for that purpose suppose $f_*(\alpha) = 0$. This means that $f \circ \alpha$ is the zero morphism. So

$$f(\alpha(z)) = 0 \Rightarrow \alpha(z) = 0 \Rightarrow \alpha = 0, \quad z \in C,$$

which lets us conclude that f_* is injective.
To show exactness we need to see that

$$\ker(g_*) = \mathrm{Im}(f_*).$$

($\subseteq$) Suppose for that effect that $\beta \in \ker(g_*)$, then $g_*(\beta) = g \circ \beta$ is the zero map. As f is injective, by universality of the kernel, there exists $\alpha \in \mathrm{Hom}(C, X)$ such that $f_*(\alpha) = \beta$ and therefore $\beta \in \mathrm{Im}(f_*)$.

($\supseteq$) On the other hand suppose $\beta \in \mathrm{Im}(f_*)$, this means that for some $\alpha : C \to X$, $\beta = f_*(\alpha)$. Now,

$$g_*(\beta) = g_*(f_*(\alpha)) = (g \circ f) \circ \alpha = 0 \circ \alpha = 0 \Rightarrow \beta \in \ker(g_*).$$

Exercise 4 (2.2.F. Vakil)**.** Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X.

More precisely, to each open set U of X, we associate the set of continuous maps of U to Y. Show that this forms a sheaf.

Answer

The presheaf $\mathcal{F}$ of continuous functions on X consists of taking every open set U and assigning to it the set

$$\mathcal{F}(U) = \mathcal{C}(U, Y) = \{ (f : U \to Y) : \ f \text{ is continuous} \}.$$

The restriction mapping in this case is

$$\underset{V,U}{\mathrm{res}} : \mathcal{C}(V, Y) \to \mathcal{C}(U, Y), \ f \mapsto f|_U .$$

- ◇ The map $\mathrm{res}_{U,U}$ is the identity mapping because restricting to the whole set gives us the same function.
- ◇ Suppose $U \subseteq V \subseteq W$ are open sets, then we must show that

$$\underset{V,U}{\mathrm{res}} \circ \underset{W,V}{\mathrm{res}} = \underset{W,U}{\mathrm{res}} .$$

  Taking $f \in \mathcal{C}(W, Y)$ and applying $\mathrm{res}_{W,V}$ gives us $f|_V$. And when restricting again we obtain $(f|_V)|_U$. Since we have the containment of the sets, this second restriction amounts to restricting to U directly from the original set. Therefore the composition condition holds.

This shows that $\mathcal{F}$ is a presheaf. To show that this is a sheaf, we must prove that functions are determined by restrictions and that there exist *global functions*. The fact that our functions are continuous will let us demonstrate this facts.

- ◇ Suppose $f, g \in \mathcal{C}(U, Y)$ for some $U \subseteq X$ open, and that (U_i) is an open cover of U where f and g agree locally. Let $x \in U$, then as (U_i) covers U, $x \in U_i$ for some i. Thus

$$f(x) = f|_{U_i}(x) = g|_{U_i} = g(x).$$

  As x is arbitrary, we have the desired result.

- Now suppose (U_i) covers U and a collection of functions (f_i) with $f_i \in \mathcal{C}(U_i, Y)$ satisfy
$$\forall i \forall j \left(f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \right).$$
We define a global function $f : U \to Y$ by checking first where the input is. This means that
$$f(x) = f_i(x), \quad \text{when} \quad x \in U_i$$
and as f_i's coincide on intersections, this is a good definition. Finally as continuous functions are characterized by their local behavior, we have that f is a continuous function and therefore we have shown that the gluing axiom holds.

We conclude that $\mathcal{F}$ does indeed form a sheaf.