Exercise 1. Consider the sequence $a_n = (3n+1)!$.

i) Find a_0, a_1 and a_2.
ii) Simplify the consecutive ratio $\frac{a_{n+1}}{a_n}$ into a polynomial in terms of n.

i) $a_0 = (1)! = 1$, $a_1 = (3+1)! = 4! = 24$ and $a_2 = (6+1)! = 7! = 5040$.
ii) The consecutive ratio is

$$\frac{a_{n+1}}{a_n} = \frac{(3(n+1)+1)!}{(3n+1)!} = \frac{(3n+4)!}{(3n+1)!} = \frac{(3n+1)!(3n+2)(3n+3)(3n+4)}{(3n+1)!} = (3n+2)(3n+3)(3n+4).$$

Exercise 2. Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n^n - 2}$. Determine if the series converges or diverges.

Comparing with the series $\sum_{n=0}^{\infty} \frac{1}{n^n}$ we can see that

$$\frac{\frac{2}{n^n-2}}{\frac{1}{n^n}} = \frac{2}{1-2/n^n} \xrightarrow[n\to\infty]{} 2 > 0.$$

By the Limit Comparison Test, both series behave the same way. The series we are comparing to converges by Cauchy's Root Test because

$$\sqrt[n]{\left|\frac{1}{n^n}\right|} = \frac{1}{n} \xrightarrow[n\to\infty]{} 0 \in [0,1[.$$

We conclude that, as $\sum_{n=0}^{\infty} \frac{1}{n^n}$ converges, then $\sum_{n=0}^{\infty} \frac{2}{n^n-2}$ converges.

Exercise 3. Analyze the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ and determine if it converges or diverges.

Seeing factorials leads us to believe that we can use the ratio test. The consecutive ratios are of the form:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!(n+1)!}{(2n+2)!}}{\frac{(n!)(n!)}{(2n)!}} = \frac{(2n)!(n+1)!(n+1)!}{(2n+2)!(n!)(n!)} = \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{n^2+o(n^2)}{4n^2+o(n^2)} \xrightarrow[n\to\infty]{} \frac{1}{4}.$$

By the ratio test, as the consecutive ratios tend to $\frac{1}{4} < 1$, it follows that our series converges.