

Exercise 1 (7.8 Stein& Shakarchi). The function ζ has infinitely many zeros in the critical strip. This can be seen as follows.

i) Let

$$F(s) = \xi(1/2 + s), \quad \text{where} \quad \xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s).$$

Show that $F(s)$ is an even function of s and as a result, there exists G such that $G(s^2) = F(s)$.

ii) Show that the function $(s-1)\zeta(s)$ is an entire function of growth order 1, that is

$$|(s-1)\zeta(s)| \leq A_\epsilon e^{a_\epsilon |s|^{1+\epsilon}}.$$

As a consequence $G(s)$ is of growth order $1/2$.

iii) Deduce from the above that ζ has infinitely many zeros in the critical strip.

[[Hint: To prove the first two parts use the functional equation for $\zeta(s)$. For the last one, use a result of Hadamard, which states that an entire function with fractional order has infinitely many zeros (Exercise 14 in Chapter 5).]]

Answer

i) Observe that

$$\begin{aligned} F(-s) &= \xi(1/2 - s) = \xi\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - s\right) = \xi\left(1 - \frac{1}{2} - s\right) \\ &= \xi\left(1 - \left(\frac{1}{2} + s\right)\right) = \xi\left(\frac{1}{2} + s\right) \end{aligned}$$

where the last equality comes from the identity $\xi(s) = \xi(1-s)$ for all $s \in \mathbb{C}$.

ii) We know that $\zeta(s)$ has a pole of order 1 at $s = 1$ and that's its only pole. So the function $(s-1)\zeta(s)$ is holomorphic on the whole plane which means it's entire.

Exercise 2 (7.6 Stein& Shakarchi). Read [SS]7.6, assume its result, and proceed as follows. Let δ be the function defined in [SS]7.6:

$$\delta(a) = \begin{cases} 1 & 1 < a \\ \frac{1}{2} & a = 1 \\ 0 & 0 \leq a < 1 \end{cases}$$

Fix a positive real number X which is not an integer.

- i) Show that $\Psi(X) = \sum_{n \geq 1} \Lambda(n) \delta\left(\frac{X}{n}\right)$.
- ii) Consider $G(s) = \frac{X^s}{s} \left(\frac{-\zeta'(s)}{\zeta(s)} \right)$, show that

$$\Psi(X) = \frac{1}{2\pi i} \int_{\{\operatorname{Re}(s)=c\}} G(s) ds.$$

[[Hint: Assume you can exchange summation and integration; you will need to use our formula from class for $L(\zeta(s))$, which is also in [SS] Chapter 7, section 2.]]

Answer

Exercise 3. One uses the results of the previous problems in the following way.

- i) Show that $\operatorname{res}(G, 1) = X$. [[Hint: Use the fact that $\zeta(s)$ has a pole at $s = 1$ of order 1.]]
- ii) Show that $\operatorname{res}(G, 0) = \lim_{s \rightarrow 0} \frac{-\zeta'(s)}{\zeta(s)}$. *It turns out that this is $-\log(2\pi)$.*
- iii) Show that $\sum \operatorname{res}(G, \rho) = -\frac{1}{2} \log(1 - X^{-2})$, where the sum is over the trivial zeros of $\zeta(s)$.

From here, moving c “all the way to the left” means that we pick up all the residues of $G(s)$, and we are left with von Mangoldt’s explicit formula:

$$\psi(X) = X - \sum \frac{X^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - X^{-2})$$

where the sum is over all critical zeroes of $\zeta(s)$.

Answer

- i) Observe that

$$\operatorname{res}(G, 1) = \lim_{s \rightarrow 1} (s-1) \frac{X^s}{s} \left(\frac{-\zeta'(s)}{\zeta(s)} \right) = \lim_{s \rightarrow 1} \frac{X^s}{s} \lim_{s \rightarrow 1} (s-1) (L(\zeta(s))).$$

The limit on the right is the residue at $s = 1$ of the logarithmic derivative of ζ , it is known that this residue is the order of the point in question of the

function. This means that

$$\text{res}(G, 1) = X \cdot -\text{ord}(\zeta, 1) = X \cdot 1 = X.$$

ii) In this case, we have that

$$\text{res}(G, 0) = \lim_{s \rightarrow 0} (s) \frac{X^s}{s} \left(\frac{-\zeta'(s)}{\zeta(s)} \right) = \left(\lim_{s \rightarrow 0} X^s \right) \left(\lim_{s \rightarrow 0} \frac{-\zeta'(s)}{\zeta(s)} \right)$$

and the left limit turns to 1 so we obtain the desired result.

iii) It is a subtle observation that

$$-\frac{1}{2} \log(1 - X^{-2}) = \frac{1}{2} \sum_{n \geq 1} \frac{\left(\frac{1}{X^2}\right)^n}{n} = \sum_{n \geq 1} \frac{X^{-2n}}{2n}.$$

Now, the trivial zeroes of the zeta function are at $s = -2n$, so

$$\text{res}(G, -2n) = \lim_{s \rightarrow -2n} (s + 2n) \frac{X^s}{s} \left(\frac{-\zeta'(s)}{\zeta(s)} \right) = \left(\lim_{s \rightarrow -2n} \frac{X^s}{s} \right) (-\text{ord}(\zeta, -2n))$$

where the limit evaluates to $\frac{X^{-2n}}{-2n}$ and the order is 1 so we obtain $\frac{X^{-2n}}{2n}$ and summing through all trivial zeroes we obtain the desired result.