**Exercise 1** (Exercise 4)**.** Prove the generating function identity

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \left(\!\binom{n}{k}\!\right) x^k.$$

You may either use induction on $n$, or a direct combinatorial argument about what the coefficients must be when you expand the product on the left

Answer

Differentiating both sides of the equality $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ $n$ times[a] we get

$$\begin{aligned} D^n\left(\frac{1}{1-x}\right) &= \frac{(n-1)!}{(1-x)^n}, \\ D^n\left(\sum_{k=0}^{\infty} x^k\right) &= \sum_{k=n}^{\infty} (k)(k-1)(k-2)\dots(k-n+1)x^{k-n} \\ \left(\substack{k=\ell+n \\ k\to n \\ \Rightarrow \ell\to 0}\right) &= \sum_{\ell=0}^{\infty} (\ell+n)(\ell+n-1)(\ell+n-2)\dots(\ell+1)x^{\ell}. \end{aligned}$$

We get the following equality

$$\frac{1}{(1-x)^n} = \sum_{\ell=0}^{\infty} \frac{(\ell+n)(\ell+n-1)(\ell+n-2)\dots(\ell+1)}{(n-1)!} x^{\ell},$$

and the coefficient in question is precisely

$$\frac{(\ell+n)(\ell+n-1)(\ell+n-2)\dots(\ell+1)}{(n-1)!} = \frac{(\ell+n)!}{(n-1)!\ell!} = \binom{n+\ell-1}{\ell} = \left(\!\binom{n}{\ell}\!\right).$$

[a] Implicitly I'm using induction here

This fact can also be proven using the multiplication principle:

$$\frac{1}{(1-x)^n} = \prod_{k=1}^{n} \left(\frac{1}{1-x}\right).$$

If by induction we assume that the identity holds up to $n-1$, then the product on the right becomes

$$\left[\prod_{k=1}^{n-1} \left(\frac{1}{1-x}\right)\right] \left(\frac{1}{1-x}\right) = \left(\sum_{k=0}^{\infty} \left(\!\binom{n-1}{k}\!\right) x^k\right) \left(\sum_{k=0}^{\infty} x^k\right).$$

After multiplying we obtain the sum

$$\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}\left(\!\!\binom{n-1}{j}\!\!\right)\right]x^k.$$

If we were to prove the identity $\sum_{j=0}^{k}\left(\!\binom{n-1}{j}\!\right) = \left(\!\binom{n}{k}\!\right)$, then we would be done.

**Lemma 1.** *The following identity holds for $n, k$, positive integers*:

$$\sum_{j=0}^{k}\left(\!\!\binom{n-1}{j}\!\!\right) = \left(\!\!\binom{n}{k}\!\!\right).$$

This is a type of Pascal recurrence for the multichoose coefficient. We can state the first recurrence and the inductively prove this one, or we can prove this one by a counting argument.

Initially consider the recurrence

$$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n}{k-1}\!\!\right).$$

- The quantity on the left counts the number of ways I can distribute $k$ `cookies` among $n$ `grad students`.
- For the quantity on the right, choose the $n^{\text{th}}$ `grad student`. There are two ways to give my $k$ `cookies`.
  - Either I exclude the last `grad student` and give out my $k$ `cookies` among the other $n-1$.
  - Or I give *at least* 1 `cookie` to the last one, and I give out the remaining $k-1$ among all the $n$ `grad students`.

With this recurrence it is immediate to prove the identity:

$$\begin{aligned}
\left(\!\!\binom{n}{k}\!\!\right) &= \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n}{k-1}\!\!\right) \\
&= \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n-1}{k-1}\!\!\right) + \left(\!\!\binom{n}{k-2}\!\!\right) \\
&= \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n-1}{k-1}\!\!\right) + \left(\!\!\binom{n-1}{k-2}\!\!\right) + \cdots + \left(\!\!\binom{n}{0}\!\!\right)
\end{aligned}$$

However we can prove the identity in another way:

Consider the same situation where we label the $n^{\text{th}}$ `grad student`. Giving out $k$ `cookies` to $n$ `grad students` is the same as giving $k-j$ to the last `grad student` and distribute the remaining $j$ `cookies` among the $n-1$ other `grad students`. Since this events are disjoint, the total number of ways can be obtained by summing for each $j$, thus obtaining the identity.

There are two more ways in which I'm certain that this problem can be proven:

i) Using the $n$-fold multiplication principle. The sequence $\mathbf{1} = (1)_{n\in\mathbb{N}}$'s generating function is precisely $1/(1-x)$ so

$$\left(\frac{1}{1-x}\right)^n = \left(\sum_{k=0}^{\infty} x^k\right)^n = \sum_{k=0}^{\infty} \underbrace{(\mathbf{1} * \mathbf{1} * \cdots * \mathbf{1})}_{n \text{ times}} x^k.$$

Using induction and algebraic manipulation, it is possible to prove that the convolution in question is the multichoose coefficient.

ii) The coefficient $\left(\!\binom{n}{k}\!\right)$ also counts *weak compositions* of $k$ into $n$ parts. This is in correspondence with the amount of ways one can form an $x^k$ monomial from the product

$$\left(\sum_{k=0}^{\infty} x^k\right)^n = (1 + x + x^2 + \dots)(1 + x + x^2 + \dots)\cdots(1 + x + x^2 + \dots)$$

since the exponents in the $n$ factors are the *parts* of $k$.

**Exercise 2** (Exercise 6)**.** Find a closed form for the generating function of the sequence $b_n$ defined by $b_0 = 1$ and for all $n \geqslant 0$, $b_{n+1} = \sum_{k=0}^{n} k b_{n-k}$. Use it to find an explicit formula for $b_n$ in terms of $n$.

Answer

Let us call $B(x) = \sum_{n=0}^{\infty} b_n x^n$. Then

$$B(x) = b_0 + b_1 x + b_2 x^2 + \ldots \Rightarrow \frac{B(x) - b_0}{x} = b_1 + b_2 x + b_3 x^2 + \cdots = \sum_{n=0}^{\infty} b_{n+1} x^n.$$

However, from the recurrence we have that

$$\sum_{n=0}^{\infty} b_{n+1} x^n = \sum_{n=0}^{\infty} (n * b_n) x^n = \left( \sum_{n=0}^{\infty} n x^n \right) B(x) = xD\left( \frac{1}{1-x} \right) B(x).$$

Equating this quantities, and using the initial condition, we get

$$\begin{aligned}
\frac{B(x) - 1}{x} = \frac{xB(x)}{(1-x)^2} &\Rightarrow B(x) \left( \frac{1}{x} - \frac{x}{(1-x)^2} \right) = \frac{1}{x} \\
&\Rightarrow B(x) = \frac{(1-x)^2}{1-2x} \\
&\Rightarrow B(x) = \frac{1}{1-2x} - \frac{2x}{1-2x} + \frac{x^2}{1-2x}.
\end{aligned}$$

This is the generating function for the sequence $(b_n)$. After converting the functions into sums and rearranging the terms, the closed form of $b_n$ can be obtained. This is done as follows:

$$\begin{aligned}
B(x) &= \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} 2^{n+1} x^{n+1} + \sum_{n=0}^{\infty} 2^n x^{n+2} \\
&= \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=1}^{\infty} 2^n x^n + \sum_{n=2}^{\infty} 2^{n-2} x^n \\
&= 1 + (2-2)x + \sum_{n=2}^{\infty} 2^{n-2} x^n
\end{aligned}$$

From the last equality we extract that

$$b_n = \begin{cases} 1, \text{ when } n = 0 \\ 0, \text{ when } n = 1 \\ 2^{n-2}, \text{ when } n \geqslant 2 \end{cases}$$

**Exercise 3** (Exercise 8)**.** Let $p(n,k)$ be the number of partitions of $n$ into exactly $k$ nonzero parts. Show that

$$\sum_{n,k} p(n,k) y^k x^n = \prod_{k=1}^{\infty} \frac{1}{1 - yx^k}.$$

**Answer**

Let us first consider the case of counting the number of partitions without any restrictions. In this case it holds that

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k} = (1 + x + x^2 + \dots)\cdots\left[(x^k)^0 + (x^k)^1 + (x^k)^2 + \dots\right]\cdots$$

because each term in the $k^{\text{th}}$ infinite sum tells us how many parts (or $\lambda_i$'s) are equal to $k$, then every $x^n$ monomial corresponds to one partition of $n$ and adding them all up we get $p(n)$.

Let us now add a dummy variable $y$ which will count the number of parts in each $\lambda$. We can expand one of the factors of the product on the right as follows

$$\frac{1}{1 - yx^j} = \left[(yx^j)^0 + (yx^j)^1 + (yx^j)^2 + \dots\right].^{a}$$

Now this dummy variable appears in every other factor. It doesn't care about what *type* of part we are counting, it only counts the total number of parts. So, in every monomial $x^n y^k$ the $x$'s exponent breaks down into the parts, for each $\lambda_i = j$, $y$'s exponent is the sum of the amount of parts we have collected. Collecting all the monomials of this type gives us a coefficient of $p(n,k)$. Thus the result follows.

[a] I was overthinking this problem a lot and thinking like in the first homework. Because I wanted to use a dummy variable $(y_n)_{n\in\mathbb{N}}$ for *each* different number. But that is too much information. Once again, **Kyle** saved the day by helping me clear out my doubts about counting.

**Exercise 4** (Exercise 9)**.** Use generating functions to prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Do NOT give a combinatorial proof. Instead, give a proof by comparing coefficients of two equal generating functions or polynomials.

Answer

Let us begin by considering the binomial formula:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

The square of this function can be computed in two ways:

i) Directly applying the formula

$$(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k.$$

ii) Or by using the multiplication principle

$$[(1+x)^n]^2 = \sum_{k=0}^{2n} \left[ \sum_{j=0}^{k} \binom{n}{j} \binom{n}{n-j} \right] x^k.$$

It follows that

$$\binom{2n}{k} = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{n-j}$$

since two equal polynomials must share the same coefficients. By setting $k = n$ and relabeling the counter from $j$ to $k$ we arrive at the desired identity.

The identity in question is an specific case of Vandermonde's identity which can be proven using the same strategy by multiplying $(1+x)^m$ with $(1+x)^n$.