Exercise 1 (Exercise 2)**.** Let P be a poset in which every interval $[x, y]$ is finite. Show that, in the incidence algebra $\mathscr{I}(P)$:

i) f is invertible if and only if $\forall x(f(x, x) \neq 0)$.
ii) $fg = \delta \iff gf = \delta$, this is, inverses are two sided.
iii) If f is invertible then f^{-1} is unique.

Really quickly, recall that the incidence algebra is the set of *interval functions* from P to $\mathbb{C}$. In other words, we can describe $\mathscr{I}(P)$ as

$$\mathscr{I}(P) = \{\, f : P^2 \to \mathbb{C} : \ x > y \Rightarrow f(x, y) = 0 \,\}.$$

Answer

i) Suppose f is invertible with $fg = \delta$. If $x \in P$:

$$(f \cdot g)(x, x) = f(x, x)g(x, x) = \delta(x, x) = 1.$$

This means that, as complex numbers, $f(x, x)g(x, x) = 1$ thus none can be zero and $f(x, x) = \frac{1}{g(x,x)}$.

On the other hand, suppose $f(x, x) \neq 0$. We will construct an inverse for f inductively using the fact the every interval is finite.

Our base case is $|[x, y]| = 1$, then $x = y$ and $g(x, x) = \frac{1}{f(x,x)}$. Suppose that we have an interval $[x, y]$ of length n and for intervals of length less than n $g(x, y)$ is the inverse of $f(x, y)$. So

$$\begin{aligned} \delta(x, y) = (fg)(x, y) &\iff 0 = \sum_{x \leqslant z \leqslant y} f(x, z)g(z, y) \\ &\iff 0 = f(x, x)g(x, y) + \sum_{x < z \leqslant y} f(x, z)g(z, y) \\ &\iff -f(x, x)g(x, y) = \sum_{x < z \leqslant y} f(x, z)g(z, y) \\ &\iff g(x, y) = \frac{-1}{f(x, x)} \sum_{x < z \leqslant y} f(x, z)g(z, y) \end{aligned}$$

Thus it holds that when $f(x, x) \neq 0$, we can solve the previous equation to obtain an expression for the inverse of f. By induction, it follows that f is invertible.

ii) First notice that the process we did in the last item is valid even if we replace *invertible* with *right-invertible*. The same process can be done in the same way to obtain a *left inverse* for f such that $hf = \delta$. This means that $hf = fg \Rightarrow h = g$.

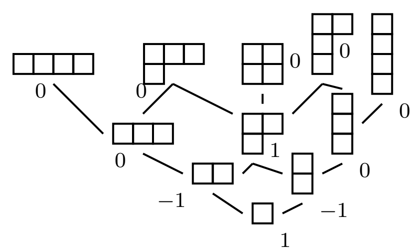
Exercise 2 (Exercise 3, Sagan 5.27)**.** Consider Young's lattice Y of all integer partitions ordered by containment of Young diagrams. Given λ, consider the interval $P_\lambda = [(1), \lambda]$ as a subposet of Y. Recall $|\lambda| = n$ when $\lambda \vdash n$.

i) Compute $\mu(P_\lambda)$ for $1 \leqslant |\lambda| \leqslant 3$.

ii) Show that $\mu(P_\lambda) = 0$ for $|\lambda| \geqslant 4$.

Answer

i) We will draw Young's lattice and add the values of the Möbius function to each of the elements right next to them:



So in essence, $\mu((1)) = \mu((2,1)) = 1$, $\mu((1,1)) = \mu((2)) = -1$ and the rest of the values are zero.

ii) We will use Rota's cross-cut theorem which states that if K is a cross-cut of a finite lattice L, then $\mu(\hat{1}) = \sum_{(*)} (-1)^{|B|}$ where the sum is taken over all $B \subseteq K$ such that $\bigwedge B = \hat{0}$ and $\bigvee B = \hat{1}$.

Recall a cross-cut is a subset $K \subseteq L$ of a lattice such that

- K is an antichain.
- K doesn't contain $\hat{1}_L$ nor $\hat{0}_L$.
- Every maximal chain $C \subseteq L$ intersects K.

The interval P_λ is indeed a finite lattice. Consider the set $\{ (1,1), (2) \}$ the atoms which cover (1). We claim that this is indeed a cross cut.

- Clearly neither $(1,1)$ can be embedded into (2) nor backwards so they are incomparable.
- $(1), \lambda$ are not in $\{ (1,1), (2) \}$.
- Now take any maximal chain[a], which must start at (1) and end at λ. Such a chain must go through $(1,1)$ or (2) to continue going up. Else, it wouldn't be maximal.

The subsets of K are in $\mathcal{P}(K) = \{\varnothing, \{(1,1)\}, \{(2)\}, K\}$. The meet of $\varnothing$ is λ and the join is 1. For the singletons, the meets and joins are themselves and for K, the meet is indeed (1) but the join is $(2,1)$.

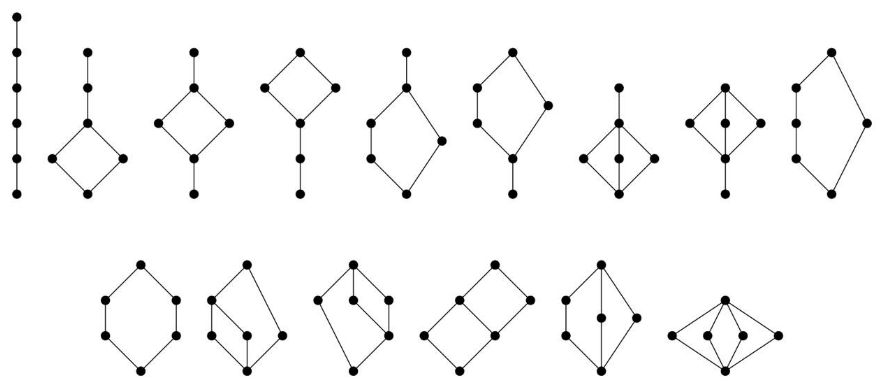
Applying Rota's cross-cut theorem we get an empty sum. So it must hold that $\mu(P_\lambda) = 0$ for any λ with $|\lambda| \geqslant 4$.

[a] **Sam** helped me with this argument.

Exercise 3 (Exercise 4)**.** Draw the Hasse diagrams of all 15 lattices on six elements. Which are upper semi-modular? Which are modular? Distributive? Atomic?

Answer

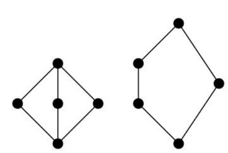
The following are the Hasse diagrams of all 15 lattices on 6 elements:



If we call the lattices $L_1, \ldots, L_{17}$ (in the most natural left-to-right, row-by-row order), then we can categorize them according to what is asked.

- The **upper semi-modular** lattices are $L_1, L_2, L_3, L_4, L_7, L_8, L_{15}$ and L_{17}.
- All the **upper semi-modular** lattices are **modular** in this case. There are no more.
- There are only two **atomic** lattices which are L_{13} and L_{17}.
- The distributive lattices are L_1 through L_4 and L_{15}.

Lattices which contain



as sublattices are not distributive. The atomic lattices were checked by hand as well as rank function computations[a].

[a] The check for this problem was done with **Sam** and **Kelsey**