Exercise 1 (3.2.E Vakil)**.** Show that we have identified all the prime ideals of $\mathbb{C}[x, y]$.

⟦Hint: Suppose $\mathfrak{p}$ is a prime ideal that is not principal. Show you can find $f, g \in \mathfrak{p}$ with no common factor. By considering the Euclidean algorithm in the Euclidean domain $\mathbb{C}(x)[y]$, show that you can find a nonzero $h \in \operatorname{gen}(f, g) \subseteq \mathfrak{p}$. Using primality, show that one of the linear factors of h, say $(x - a)$, is in $\mathfrak{p}$. Similarly show there is some $(y - b) \in \mathfrak{p}$. ⟧

The example in the book before the exercise describes $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$. The example shows that

- 0 is a prime ideal.
- Ideals of the form $\operatorname{gen}(x - a, y - b)$ with $a, b \in \mathbb{C}$ are prime. Even more, that they are maximal.
- And finally ideals of the form $\operatorname{gen}(f)$ with an irreducible f are also prime.

The hint tells us to take a prime ideal and assume it is not of the form $\operatorname{gen}(f)$ with an irreducible f. Then we will conclude that it is of the form $\operatorname{gen}(x - a, y - b)$ which is the other only non-zero possibility.

Answer

Take a non-principal ideal $\mathfrak{p} \in \operatorname{Spec} \mathbb{C}[x, y]$, we begin by wanting to find such f, g with $\gcd(f, g) = 1$.
If this were not the case, then all polynomials in $\mathfrak{p}$ would have a common factor. Let $p = \gcd(f)_{f \in \mathfrak{p}}$, then p is a generator for $\mathfrak{p}$. As it was the case that $\mathfrak{p}$ wasn't principal, our assumption that no such f, g exist must be false.
Assume that g's degree in y is lower than f's we may apply the division algorithm on $\mathbb{C}(x)[y]$ to obtain

$$f = qg + r, \quad q, r \in \mathbb{C}(x)[y] \quad \text{and} \quad \deg_y(r) \leqslant \deg_y(g).$$

We may iterate this process and continue dividing with the residues in order to obtain

$$g = q_2 r + r_2 \Rightarrow r = q_3 r_2 + r_3 \Rightarrow \ldots$$

until we reach a point where the remainder has degree zero in y. Retracing the equalities from the last point to the first equation, let us write

$$f(x, y) = \frac{q_1(x, y)}{q_2(x)} g(x, y) + \frac{r_1(x)}{r_2(x)}$$

where $\frac{r_1}{r_2}$ is the last remainder. Homogenizing we obtain an equation of the form

$$q_2 r_2 f = q_1 r_2 g + r_1 q_2 \Rightarrow r_1 q_2 \in \mathrm{gen}(f, g)$$

and we may also see that $r_1 q_2$ is a polynomial depending only on x. Thus we may factor it into

$$r_1 q_2(x) = \prod_{i=1}^{d}(x - a_i) \Rightarrow \exists j((x - a_j) \in \mathfrak{p}).$$

The same argument may be repeated but this time we obtain a polynomial $(y - b) \in \mathfrak{p}$. With this we have

$$\mathrm{gen}(x - a, y - b) \subseteq \mathfrak{p}$$

and as $\mathfrak{p}$ is a proper prime ideal, it must occur that $\mathfrak{p}$ is this maximal ideal.

Exercise 2 (3.2.K Vakil)**.** Suppose S is a multiplicative subset of A. Describe an order-preserving bijection of the prime ideals of $S^{-1}A$ with the prime ideals of A that don't meet the multiplicative set S.

Answer

Exercise 3 (3.2.Q Vakil)**.** Consider the map of sets $\pi : \mathbb{A}^n_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ given by the ring map $\mathbb{Z} \to \mathbb{Z}[x_1, \ldots, x_n]$. If $p \in \mathbb{Z}$ is prime, describe a bijection between the fiber $\pi^{-1}([\mathrm{gen}](p))$ and $\mathbb{A}^n_{\mathbb{F}_p}$. (You won't need to describe either set! Which is good because you can't.) This exercise may give you a sense of how to picture maps (see Figure 3.7), and in particular why you can think of $\mathbb{A}^n_{\mathbb{Z}}$ as an "$\mathbb{A}^n$-bundle" over $\mathrm{Spec}\,\mathbb{Z}$. (Can you interpret the fiber over $[(0)]$ as $\mathbb{A}^n_k$ for some field k?)

Answer

Missing from last HW

Exercise 4. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of sets on a topological space X. Show that the following are equivalent:

(a) ϕ is an epimorphism in the category of sheaves.

(b) ϕ is surjective on the level of stalks: $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for $p \in X$.