Exercise 1 (6.8 Sagan)**.** Do the following:

i) The group of symmetries of a regular n-gon is called a dihedral group and consists of the n rotations and n reflections which map the n-gon to itself.

   Find the number of different 4-bead, r-color necklaces if necklaces are considered the same when one is a rotation or reflection of the other.

ii) Find an expression for the number of distinct n-bead, r-color necklaces if two are the same when one is a rotation or a reflection of the other.

Answer

Observe that the dihedral group which acts on a 4-bead necklace is $D_8 = \langle (1234), (13) \rangle \leqslant S_4$. We can succinctly view the elements of the group as follows:

$$\begin{aligned}&\{\,(1)(2)(3)(4), (1234), (13)(24), (1432), (13), (1234)(13), (13)(24)(13), (1432)(13)\,\}\\ =&\{\,(1)(2)(3)(4), (1234), (13)(24), (1432), (13), (14)(23), (24), (12)(24)\,\}\end{aligned}$$

With this, we may use Burnside's lemma to find the number of orbits:

$$\begin{aligned}\#\text{orbits} &= \frac{1}{|D_8|}\sum_{g\in D_8} |\operatorname{Fix}(g)| = \frac{1}{8}\sum_{g\in D_8} r^{c(g)}\\ &= \frac{1}{8}(r^4 + r^1 + r^2 + r^1 + r^3 + r^2 + r^3 + r^2) = \frac{1}{8}(r^4 + 2r^3 + 3r^2 + 2r).\end{aligned}$$