Exercise 1 (3.2.E Vakil)**.** Show that we have identified all the prime ideals of $\mathbb{C}[x, y]$.

⟦Hint: Suppose $\mathfrak{p}$ is a prime ideal that is not principal. Show you can find $f, g \in \mathfrak{p}$ with no common factor. By considering the Euclidean algorithm in the Euclidean domain $\mathbb{C}(x)[y]$, show that you can find a nonzero $h \in \operatorname{gen}(f, g) \subseteq \mathfrak{p}$. Using primality, show that one of the linear factors of h, say $(x - a)$, is in $\mathfrak{p}$. Similarly show there is some $(y - b) \in \mathfrak{p}$. ⟧

The example in the book before the exercise describes $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$. The example shows that

- 0 is a prime ideal.
- Ideals of the form $\operatorname{gen}(x - a, y - b)$ with $a, b \in \mathbb{C}$ are prime. Even more, that they are maximal.
- And finally ideals of the form $\operatorname{gen}(f)$ with an irreducible f are also prime.

The hint tells us to take a prime ideal and assume it is not of the form $\operatorname{gen}(f)$ with an irreducible f. Then we will conclude that it is of the form $\operatorname{gen}(x - a, y - b)$ which is the other only non-zero possibility.

Answer

Take a non-principal ideal $\mathfrak{p} \in \operatorname{Spec} \mathbb{C}[x, y]$, we begin by wanting to find such f, g with $\gcd(f, g) = 1$.
If this were not the case, then all polynomials in $\mathfrak{p}$ would have a common factor. Let $p = \gcd(f)_{f \in \mathfrak{p}}$, then p is a generator for $\mathfrak{p}$. As it was the case that $\mathfrak{p}$ wasn't principal, our assumption that no such f, g exist must be false.
Assume that g's degree in y is lower than f's we may apply the division algorithm on $\mathbb{C}(x)[y]$ to obtain

$$f = qg + r, \quad q, r \in \mathbb{C}(x)[y] \quad \text{and} \quad \deg_y(r) \leqslant \deg_y(g).$$

We may iterate this process and continue dividing with the residues in order to obtain

$$g = q_2 r + r_2 \Rightarrow r = q_3 r_2 + r_3 \Rightarrow \ldots$$

until we reach a point where the remainder has degree zero in y. Retracing the equalities from the last point to the first equation, let us write

$$f(x, y) = \frac{q_1(x, y)}{q_2(x)} g(x, y) + \frac{r_1(x)}{r_2(x)}$$

where $\frac{r_1}{r_2}$ is the last remainder. Homogenizing we obtain an equation of the form

$$q_2 r_2 f = q_1 r_2 g + r_1 q_2 \Rightarrow r_1 q_2 \in \text{gen}(f, g)$$

and we may also see that $r_1 q_2$ is a polynomial depending only on x. Thus we may factor it into

$$r_1 q_2(x) = \prod_{i=1}^{d} (x - a_i) \Rightarrow \exists j((x - a_j) \in \mathfrak{p}).$$

The same argument may be repeated but this time we obtain a polynomial $(y - b) \in \mathfrak{p}$. With this we have

$$\text{gen}(x - a, y - b) \subseteq \mathfrak{p}$$

and as $\mathfrak{p}$ is a proper prime ideal, it must occur that $\mathfrak{p}$ is this maximal ideal.

Exercise 2 (3.2.K Vakil)**.** Suppose S is a multiplicative subset of A. Describe an order-preserving bijection of the prime ideals of $S^{-1}A$ with the prime ideals of A that don't meet the multiplicative set S.

Answer

We will describe the bijection

$$\{\, \mathfrak{p} \in \operatorname{Spec} A : \; \mathfrak{p} \cap S = \varnothing \,\} \to \operatorname{Spec} S^{-1}A.$$

If $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathfrak{p} \cap S = \varnothing$ we will show that $S^{-1}\mathfrak{p}$ is a prime ideal in $S^{-1}A$. Suppose that $\frac{a_1}{s_1}\frac{a_2}{s_2} \in S^{-1}\mathfrak{p}$. Then there exist $p \in \mathfrak{p}$ and $s \in S$ such that

$$\frac{a_1}{s_1}\frac{a_2}{s_2} = \frac{p}{s} \Rightarrow u(a_1 a_2 s - s_1 s_2 p) = 0, \quad \text{for some} \quad u \in S.$$

Now

$$u s_1 s_2 p \in \mathfrak{p} \Rightarrow u a_1 a_2 s \in \mathfrak{p} \Rightarrow a_1 a_2 \in \mathfrak{p} \Rightarrow a_1 \in \mathfrak{p} \vee a_2 \in \mathfrak{p}$$

from which we conclude that either $\frac{a_1}{s_1}$ or $\frac{a_2}{s_2}$ is in $S^{-1}\mathfrak{p}$, which means that $S^{-1}\mathfrak{p}$ is a prime ideal.
On the other hand if $\mathfrak{q} \in \operatorname{Spec} S^{-1}A$ we can take its preimage through the mapping:

$$\phi^{-1}[\mathfrak{q}] = \left\{ x \in A : \; \frac{x}{1} \in \mathfrak{q} \right\}$$

and we will show that this ideal doesn't intersect S.
On the contrary, if it did, if there was $s_0 \in S \cap \phi^{-1}[\mathfrak{q}]$ then $\frac{s_0}{1} \in \mathfrak{q}$. Then

$$\left(\frac{1}{s_0}\right)\left(\frac{s_0}{1}\right) = 1 \in \mathfrak{q} \quad \text{because} \quad \mathfrak{q} \quad \text{is prime} \quad .$$

This means that $\mathfrak{q}$ must be the whole ring, but if $\mathfrak{q}$ were proper, we would have a contradiction. This means that our assumption was wrong and therefore $\phi^{-1}[\mathfrak{q}] \cap S = \varnothing$.
This is the bijection in question. It preserves order because preimages of sets preserve order.

Exercise 3 (3.2.Q Vakil)**.** Consider the map of sets $\pi : \mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec}(\mathbb{Z})$ given by the ring map $\mathbb{Z} \to \mathbb{Z}[x_1, \ldots, x_n]$. If $p \in \mathbb{Z}$ is prime, describe a bijection between the fiber $\pi^{-1}([\operatorname{gen}(p)])$ and $\mathbb{A}^n_{\mathbb{F}_p}$. (You won't need to describe either set! Which is good because you can't.) This exercise may give you a sense of how to picture maps (see Figure 3.7), and in particular why you can think of $\mathbb{A}^n_{\mathbb{Z}}$ as an "$\mathbb{A}^n$-bundle" over $\operatorname{Spec}\mathbb{Z}$. (Can you interpret the fiber over $[(0)]$ as $\mathbb{A}^n_k$ for some field k?)

Answer

Exercise 4 (3.5.B Vakil)**.**

Answer

Exercise 5 (3.6.M Vakil)**.** Verify that $[\operatorname{gen}(y - x^2)] \in \mathbb{A}^2_{\mathbb{C}}$ is a generic point for $V(y - x^2)$.

Answer

Call $\mathfrak{p} = \operatorname{gen}(y - x^2)$, the closure of $\{\mathfrak{p}\}$ is

$$\overline{\{\mathfrak{p}\}} = \bigcap_{\substack{F \supseteq \{\mathfrak{p}\} \\ F \text{ closed}}} F = \bigcap_{\mathfrak{p} \in V(S)} V(S).$$

We can see that $\overline{\{\mathfrak{p}\}} \subseteq V(y - x^2)$ because $\mathfrak{p} \in V(y - x^2)$.

Exercise 6 (3.6.P Vakil)**.** Show that $\mathbb{A}^2_{\mathbb{C}}$ is a Noetherian topological space: any decreasing sequence of closed subsets of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ must eventually stabilize. Note that it can take arbitrarily long to stabilize. (The closed subsets of $\mathbb{A}^2_{\mathbb{C}}$ were described in 3.4.5.).

Show that $\mathbb{C}^2$ with the classical topology is not a Noetherian topological space.

Answer

Let (F_k) be a descending chain of closed subsets of $\mathbb{A}^2_{\mathbb{C}}$. Every F_k can be seen to be $V(I_k)$ where $I_k \lhd \mathbb{C}[x, y]$ is an ideal. So by the Nullstellensatz:

$$
\begin{aligned}
&F_1 \supseteq F_2 \supseteq \dots \\
\Rightarrow &V(I_1) \supseteq V(I_2) \supseteq \dots \\
\Rightarrow &I_1 \subseteq I_2 \subseteq \dots
\end{aligned}
$$

Now (I_k) is an ascending chain of ideals in $\mathbb{C}[x, y]$, and as this ring is Noetherian, ascending chains stabilize. Which means that there exists r with the property that

$$I_r = I_{r+1} = \dots$$

and therefore $V(I_r) = V(I_{r+1}) = \dots$ from which we extract that $\mathbb{A}^2_{\mathbb{C}}$ is Noetherian as a topological space.

On the other hand if we consider the collection of closed balls $\left(\overline{B}\left(0, \frac{1}{n}\right)\right)_{n\in\mathbb{N}}$, we see that this collection doesn't stabilize. If it did, there would exist an r such that

$$\overline{B}\left(0, \frac{1}{r}\right) = \overline{B}\left(0, \frac{1}{r+1}\right) = \dots$$

but there are no points in $\overline{B}\left(0, \frac{1}{r}\right) \setminus \overline{B}\left(0, \frac{1}{r+1}\right)$ which are limits of sequences inside $\overline{B}\left(0, \frac{1}{r+1}\right)$ because any point in the annulus has positive distance to the smaller ball.

Missing from last HW

Exercise 7. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of sets on a topological space X. Show that the following are equivalent:

(a) ϕ is an epimorphism in the category of sheaves.

(b) ϕ is surjective on the level of stalks: $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for $p \in X$.