MATH502 — Combinatorics 2

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Requirements

Contents

Chapter 1

Symmetric functions

1.1 Recall

Definition 1.1.1. $f(x_1, x_2, \dots)$ is symmetric if it's fixed under permutations of variables.

Example 1.1.2. $f(x_1, \dots, x_4) = x_1^5 + \cdots + x_4^5$. This is known as p_5 or $m_{(5)}$, where p is the power-sum symmetric function and m, the monomial symmetric function.

Example 1.1.3. Consider $g = x_1^4 x_2 + x_1^4 x_3 + \cdots + x_i^4 x_j + \cdots + 3x_1 + \cdots = m_{(4,1)} + 3m_{(1)}$.

Let us recall some **notation**:

i) $\Lambda_R(x_1, \dots, x_n)$ is the ring of symmetric polynomials over R. In *infinitely* many variables we have $\Lambda_R(\underline{x})$.

In the case $R = \mathbb{Q}$, then $\dim \Lambda_Q(\underline{x})_{(d)}$, where every monomial has degree d, is $p(d)$. This is the number of partitions of d. Because for every partition we can form monomials and monomials form a basis.

Bases of Λ_Q

Suppose $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geqslant \dots \geqslant \lambda_k$.

- Monomial: $m_\lambda = \sum_{i_1 \neq \dots i_k} x_{i_1}^{\lambda_1} \dots x_{i_k}^{\lambda_k}$.
- Elementary: $e_\lambda = \prod e_{\lambda_i}$ where $e_d = m_{(1,1,\dots,1)}$ (d ones).
- Homogenous: $h_\lambda = \prod h_{\lambda_i}$ and $h_d = x_1^d + \cdots + x_1^{d-1} x_2 + \cdots + x_1^{d-2} x_2^2 + x_1^{d-2} x_2 x_3 + \dots$. In general $h_d = \sum_{\lambda \vdash d} m_\lambda$.
- Power sum: $p_\lambda = \prod p_{\lambda_i}$ and $p_d = \sum x_i^d$.

For Schur basis recall SSYT

Example 1.1.4. Consider $\lambda = (5, 4, 1)$, rows $\leqslant\rightarrow$ and columns $<$, we associate the monomial $x_1^2x_2^3x_3^3x_4^2 := x^T$.

- ◇ Schur: $s_\lambda = \sum_{T\in SSYT(\lambda)} x^T$ but also $\sum K_{\lambda\mu}m_\mu$ where the sum is over SSYT of shape λ, content μ.

Schur function motivation (preview)

The first place they showed up is in the representation theory of Lie group. The function $s_\lambda(x_1, \ldots, x_n)$ is a character of irreducible polynomial representations of GL_n. In theoretical physics we have matrix groups acting on particles, representations are smaller matrix groups of things that they are mapping to. We want to take tensor product and direct sums of representations, the tensor product is related to multiplication of Schur function while direct sum into sum of Schur functions.

There's also the Schur-Weyl duality which takes representations into the Weyl group. Under the *Frobenius map*, s_λ corresponds to irreducible representations of S_n.

A more modern application of Schur function goes into geometry, s_λ correspond to Schubert varieties in Grassmannians. Multiplication corresponds to interesections and sum to unions.

There's also context in Probability Theory. But in the end, Schur positivity is important because of this connections.

Definition 1.1.5. $f \in \Lambda$ is Schur-positive if $f = \sum c_\lambda s_\lambda$, $c_\lambda \geqslant 0$.

Example 1.1.6. $3s_{(2,1)} + 2s_{(3)}$ schur pos but change 2 to $-\frac{1}{2}$ then not.

1.2 day 2

Alg defn Schur fncs

Definition 1.2.1. A function is antisymmetric if for $\pi \in S_n$,

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \operatorname{sgn}(\pi) f(x_1, \ldots, x_n).$$

Example 1.2.2. The following functions are antisymmetric:

(a) $f(x, y) = x - y$ then $f(y, x) = -f(x, y)$.
(b) $g(x, y) = (x - y)(x + y)$.
(c) $h(x, y) = x^2y - y^2x$.

Notice that the last function can factor as $h = -xy(x-y)$. We claim that this is always the case.

Lemma 1.2.3. *Every antisymmetric polynomial f in two variables x, y can factor as $f(x,y) = (x-y)g(x,y)$ where g is symmetric.*

Proof

Suppose f is antisymmetric, then $f(x,x) = 0$ by taking $y = x$. This means that $(x-y) \mid f$. Thus $f(x,y) = (x-y)g(x,y)$ and we now need to show that g is symmetric.

$$g(y,x) = \frac{f(y,x)}{y-x} = \frac{-f(x,y)}{-(x-y)} = \frac{f(x,y)}{x-y} = g(x,y).$$

Monomial Antisymmetric Functions

Definition 1.2.4. Given a strict partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 > \cdots > \lambda_k$, we define

$$a_\lambda(x_1, \ldots, x_n) = x_1^{\lambda_1} \cdots x_k^{\lambda_k} \pm \text{similar terms} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_k x_{\pi(k)}^{\lambda_k}.$$

This a_λ can be zero.

Example 1.2.5. For two variables we've seen some antisymmetric polynomials. Let us calculate

$$a_{(3,1)}(x,y) = x^3y - y^3x.$$

The smallest possible example in 3 variables is

$$a_{(2,1,0)}(x,y,z) = x^2y + y^2z + z^2x - y^2x - z^2y - x^2z.$$

This can be factored as $(x-y)(y-z)(x-z)$. A similar construction gives us

$$a_{(4,2,0)}(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - y^4x^2 - z^4y^2 - x^4z^2,$$

but how does this factor? We get

$$a_{(4,2,0)}(x,y,z) = (x^2-y^2)(y^2-z^2)(x^2-z^2) = a_{(2,1,0)}(x,y,z)(x+y)(y+z)(x+z).$$

Lemma 1.2.6. *The set $\{ a_\lambda \}_{\lambda \text{ strict}}$ is a basis of the antisymmetric polynomials over $\mathbb{Q}$, $A_{\mathbb{Q}}$. Even more any a_λ is divisible by a_ρ where $\rho = (n-1, n-2, \ldots, 2, 1, 0)$.*

As an algebra generator, a_ρ is a generator.

Proof

WRITE

Proposition 1.2.7. *The a_ρ antisymmetric function is also the Vandermonde determinant:*

$$a_\rho = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \dots & x_1^2 & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \dots & x_2^2 & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \dots & x_n^2 & x_n & 1 \end{pmatrix}$$

Schur Polynomials

Definition 1.2.8. The Schur polynomial of $\lambda \in \mathrm{Par}$ is

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})}.$$

Here $\lambda + \rho$ is the pointwise sum as arrays.

Remark 1.2.9. This is the Weyl character proof.

The following proof is due to Proctor(1987) find ref

Lemma 1.2.10. *Any a_λ can be seen as a determinant in the following way:*

$$a_\lambda(\underline{x}) = \det \begin{pmatrix} x_1^{\lambda_1} & x_1^{\lambda_2} & \dots & x_1^{\lambda_n} \\ x_2^{\lambda_1} & x_2^{\lambda_2} & \dots & x_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \dots & x_n \end{pmatrix}$$

Proof

We want to see that

$$\frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})} = \sum x^T$$

where the sum ranges through T's which are SSYT(la) with max entry n.

(a) We will show a recursion for the combinatorial definition that the character formula will also satisfy. It holds that

$$s_\lambda(\underline{x}) = \sum s_\mu(\underline{x}) x_n^{|\lambda|-|\mu|}$$

where μ has $n-1$ parts with $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \ldots$.

(b) We also show that the ratio of determinants satisfies the same recursion.

Example 1.2.11. Consider $\lambda = (8,8,4,1,1)$ and $\mu = (8,5,2,1)$, then $\lambda \backslash \mu$ is a skew-table in which we can fill in n's

Corollary 1.2.12. *The Schur polynomials are a basis of* $\Lambda_{\mathbb{Q}}$.

1.3 Day 3 | 20230125

Recall $\Lambda = \mathbb{Q}[e_1, e_2, \ldots]$ where the e_j's are the elementary symmetric functions. So the e_j's are algebraic generators of Λ and they're algebraically independent. Equivalently, as a vector space, $\{ e_\lambda : \lambda \in \text{Par} \}$ is a basis.

Proposition 1.3.1. *A homomorphism* $f : \Lambda \to \Lambda$ ($f(a+b) = f(a) + f(b),\ f(ab)f(a)f(b)$ *for* $a, b \in \Lambda$) *is fully determined by where it sends the* e_i'*s.*

Definition 1.3.2. The map $\omega \in \operatorname{End}(\Lambda)$ will send e_j to h_j.

Example 1.3.3. Consider $f = 3e_{(2,1)} + 2e_3$, then applying ω we get

$$\omega(f) = \omega(3e_{(2,1)} + 2e_3) = 3h_{(2,1)} + 2h_3.$$

For p_2, we can decompose to $e_1^2 - 2e_2$. So

$$\omega(p_2) = \omega(e_1^2 - 2e_2) = h_1^2 - 2h_2$$

and we can expand this last expression into

$$(x_1 + x_2 + \ldots)^2 - 2(x_1^2 + x_2^2 + \cdots + x_1x_2 + x_1x_3 + \ldots) = -x_1^2 - x_2^2 - \ldots$$

and we recognize this last term as $-p_2$. *This is not a coincidence.*

Theorem 1.3.4. *The map* ω *is involutive.*

Proof

It suffices to prove that $\omega(h_j) = e_j$. We will use power expansions and generating functions. We have

$$H(t) = \frac{1}{1 - x_1 t} \frac{1}{1 - x_2 t} \cdots = \sum h_n(\underline{x}) t^n,$$

and this comes from expanding the $1/(1-y)$'s as geometric series. When collecting the coefficients of t^n we get exactly $h_n(\underline{x})$. Similarly, for the elementary symmetric functions,

$$E(t) = (1+x_1t)(1+x_2t)\cdots = \sum e_n t^n.$$

When multiplying to obtain the coefficient of t^n we get a plethora of different x_j's which form the e_j's. Now from this expressions we have $H(t)E(-t) = 1$ which means that

$$\left(\sum h_n(\underline{x})t^n\right)\left(\sum e_n(\underline{x})(-t)^n\right) \Rightarrow \sum_{k=0}^{n}(-1)^k e_k h_{n-k} = 0,\ n \geqslant 1.$$

Now applying the map to the equation we get

$$\omega\left(\sum_{k=0}^{n}(-1)^k e_k h_{n-k}\right) = \sum_{k=0}^{n}(-1)^k h_k \omega(h_{n-k}) = 0.$$

After reindexing, we get that both e_j's and $\omega(h_j)$'s are determined recursively by the h_j's in the same way. Thus we conclude that $\omega(h_j) = e_j$.

Lemma 1.3.5. *The following equation holds for the power-sum symmetric functions*:

$$\exp\left(\sum \frac{1}{n}p_n(\underline{x})p_n(\underline{y})\right) = \prod_{i,j=1}^{\infty}\frac{1}{1-x_iy_j} =: \Omega(\underline{x},\underline{y}).$$

It also holds that

$$\Omega(\underline{x},\underline{y}) = \sum_{l} a\frac{1}{z^{\lambda}}p_\lambda(\underline{x})p_\lambda(\underline{y})$$

where $z_\lambda = \prod k^{m_k}k!$ *where* m_k *is the number of parts of* λ *equal to* k.

Proof

We will prove both parts separately. For the first equation we will take the logarithm on both sides:

$$\sum \frac{1}{n}p_n(\underline{x})p_n(\underline{y}) = \log\left(\prod_{i,j=1}^{\infty}\frac{1}{1-x_iy_j}\right)$$

and after manipulating the logarithm we get

$$\sum_{i,j=1}^{\infty}(\log(1)-\log(1-x_iy_j)) = \sum_{i,j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}x_i^n y_j^n.$$

We can separate[a] into

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_i x_i^n\right)\left(\sum_j y_j^n\right).$$

Now taking $\exp$ on both sides we get equality.
By not removing the exponential we get the following expression

$$\exp\left(\sum\frac{1}{n}p_n(\underline{x})p_n(\underline{y})\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum\frac{1}{n}p_n(\underline{x})p_n(\underline{y})\right)^k.$$

To get a term of the form $p_\lambda(\underline{x})p_\lambda(\underline{y})$ we have to choose which parts of the λ come from each of the factors in $\sum\frac{1}{n}p_n(\underline{x})p_n(\underline{y})$. If $\ell(\lambda)=k$ then it comes from the k^{th} term in the exponential sum. If $\lambda = (\lambda_1, dots, \lambda_1, \ldots, 2, \ldots, 2, 1, \ldots, 1)$ with m_{λ_1} λ_1's, m_1 1's, then out of k elements we have to choose m_1 1's and so on. Thus there are $\binom{k}{m_{\lambda_1},\ldots,m_1}$ choices and each i in λ comes with a $\frac{1}{i}$. Therefore the coefficient of $p_\lambda(\underline{x})p_\lambda(\underline{y})$ is

$$\frac{1}{k!}\frac{k!}{m_1!m_2!\ldots}\frac{1}{1^{m_1}}\frac{1}{2^{m_2}}\cdots = \frac{1}{z_\lambda}.$$

[a] Are we using Fubini-Tonelli here?

Lemma 1.3.6. *We have the following identities*

$$\exp\left(\sum\frac{(-1)^{n-1}}{n}p_n(\underline{x})p_n(\underline{y})\right) = \prod_{i,j=1}^{\infty}\frac{1}{1+x_iy_j} = \sum_\lambda\frac{(-1)^{n-\ell(\lambda)}}{z_\lambda}p_\lambda(\underline{x})p_\lambda(\underline{y}).$$

Lemma 1.3.7. *Another equality for $\Omega(\underline{x},\underline{y})$ is*

$$\Omega(\underline{x},\underline{y}) = \sum_\lambda m_\lambda(\underline{x})h_\lambda(\underline{y})$$

Theorem 1.3.8. *It holds that $\omega(p_\lambda) = (-1)^{n-k}p_\lambda$ where k is the number of parts of λ.*

Proof

Applying ω to Ω, but *only working with* $\underline{y}$ *variables* we get

$$\omega(\Omega) = \omega\left(\sum_\lambda m_\lambda(\underline{x})h_\lambda(\underline{y})\right) = \sum_\lambda m_\lambda(\underline{x})e_\lambda(\underline{y}) = \prod_{i,j=1}^{\infty}(1+x_iy_j) = \sum_\lambda \frac{1}{z_\lambda}(-1)^{n-k_\lambda}p_\lambda(\underline{x})p_\lambda(\underline{y}).$$

Comparing coefficients with

$$\omega\left(\sum_l a\frac{1}{z^\lambda}p_\lambda(\underline{x})p_\lambda(\underline{y})\right)$$

we get the result.