Exercise 1. (Exercise 3.12.11) Show that

$$\mathcal{F}\ell(d_1, \ldots, d_k) \cong \mathrm{O}(n)/(\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k)),$$

where $n_1 = d_1$ and $n_i = d_i - d_{i-1}$ for $i = 2, \ldots, k$. (In other words, the n_i are the jumps in dimension as we go up the flag.)

Exercise 2. Let M be a manifold with an affine connection ∇. Suppose $\alpha : I \to M$ is a constant curve; that is, $\alpha(t) = p$ for all $t \in I$. Let V be a vector field along α, meaning that $V(t) \in T_{\alpha(t)}M = T_pM$ just gives a curve in the tangent space T_pM. Show that $\frac{DV}{dt} = V'(t)$; that is, the covariant derivative agrees with the usual derivative in this case, regardless of what ∇ is.

Exercise 3. (Exercise 4.3.4) Show that an affine connection ∇ is compatible with a Riemannian metric g on M if and only if, for any vector fields V and W along a smooth curve $\alpha : I \to M$, we have

$$\left.\frac{d}{dt}\right|_{t=t_0} g_{\alpha(t)}(V(t), W(t)) = g_{\alpha(t_0)}\left(\frac{DV}{dt}, W\right) + g_{\alpha(t_0)}\left(V, \frac{DW}{dt}\right).$$

In other words, for compatible connections we can use the usual product rule to differentiate the inner product.