

Exercise 1 (Exercise 3). Prove that a word w has highest weight (i.e., $E_i(w) = 0$ for all i) if and only if w is Yamanouchi

Answer

First suppose $w \in [n]^k$ is a word of length k on the alphabet $[n]$. Now suppose additionally that w is Yamanouchi. This means that for every $s \leq k$, the suffix

$$w_{k-s+1} \dots w_{k-1} w_k$$

contains at least as many i 's after the $(i+1)$'s. In particular this holds when $s = k$. So when applying the raising E_i operator we pair $(i+1)$ with an i to its right as a parenthesis. There are as much i 's as $(i+1)$'s so every $(i+1)$ is paired and so the E_i operator can't convert any $(i+1)$ to an i .

As i is arbitrary, we can't apply any E_i to w which means that w has highest weight.

On the other hand^a suppose $w \in [n]^k$ has highest weight. Then for all i , we can't apply E_i to w . This means that in w , it is possible to match all the $(i+1)$'s with i 's that succeed them.

The previous fact lets us see that when reading w from right-to-left we will find at least as many i 's as we find $(i+1)$'s. In other words, this means that w is Yamanouchi.

^aOnce again, **Kelsey, Trent**, and myself have talked about this problem in order to under the idea.

Exercise 2 (Exercise 4). Formulate and prove a Yamanouchi-type condition for w to be lowest weight, that is, $F_i(w) = 0$ for all $i \in [n]$. Such a word is called anti-ballot.

Answer

We will define anti-ballot by remembering the definition of ballot. Recall a ballot word w in $[n]^k$ has the property that for every $p \leq k$, the prefix

$$w_1 w_2 \dots w_p$$

contains at least as many i 's before the $(i+1)$'s. So an anti-ballot word will contain as many $(i+1)$'s before the i 's. The claim is as follows:

A word w has lowest weight if and only if w is anti-ballot.

To prove this we first assume a word w is anti-ballot. Before applying F_i we notice that all the latter i 's must already be paired with a previous $(i + 1)$ because the word is anti-ballot.

This property holds for all i so it happens that we can't apply F_i to w , thus w has lowest weight.

On the flip-side, if we assume w has lowest weight, it means that all i 's in w are matched with $(i + 1)$ that precede them. So it must hold that there are at least as many $(i + 1)$'s as i 's in w . This condition holds for every i which means that w is anti-ballot.

Remark 1. It is important to notice that an anti-ballot word is not necessarily Yamanouchi even if it contains all the possible letters of the alphabet in question. The word

$$33 \dots 321$$

is anti-ballot but not Yamanouchi.

The examples from the crystal graphs contain Yamanouchi words on the top and anti-ballot words on the bottom.