Exercise 1 (5.2 Stein& Shakarchi)**.** Find the order of growth of the following entire functions:

i) $p(z)$, p is a polynomial. ii) e^{bz^n}, and iii) e^{e^z}.

Answer

Recall an entire function f has order of growth at most ρ if there exist A, B such that
$$|f(z)| \leqslant Ae^{B|z|^\rho}$$

We will use the fact that if f, g have order of growth ρ_f and ρ_g, then $\operatorname{ord}(fg) \leqslant \max(\rho_f, \rho_g)$. This can be seen to be true as follows:

$$|fg(z)| \leqslant A_1e^{B_1|z|^{\rho_f}}A_2e^{B_2|z|^{\rho_g}} = A_1A_2e^{B_1|z|^{\rho_f}+B_2|z|^{\rho_g}}.$$

If it happens that $\rho_f = \rho_g$, then $\operatorname{ord}(fg) \leqslant \rho_f$. Otherwise, suppose $\rho_f > \rho_g$ then

$$B_1|z|^{\rho_f} + B_2|z|^{\rho_g} \leqslant (B_1 + B_2)|z|^{\max(\rho_f,\rho_g)}, \quad \text{for all} \quad z.$$

Then, once again, we have what we looked for:

$$|fg(z)| \leqslant A_1A_2e^{(B_1+B_2)|z|^{\max(\rho_f,\rho_g)}}.$$

In conclusion the order behaves nicely with the product.

i) For the case of the polynomials, we may factor p as $A\prod_{k=1}^{d}(z - z_k)$. So it suffices to describe the orders of the linear factors. Observe that

$$|z - z_k| \leqslant [\max(1, |z_k|)](|z| + 1).$$

In order to continue bounding this, we remember the celebrated inequality

$$e^t \geqslant 1 + t \Rightarrow e^{\frac{t}{n}} \geqslant 1 + \frac{t}{n} \Rightarrow (e^{\frac{t}{n}})^n \geqslant \left(1 + \frac{t}{n}\right)^n.$$

Now for positive t, the last quantity can be bounded below by

$$\left(1 + \frac{t}{n}\right)^n \geqslant 1 + \frac{t^n}{n^n}, \quad \text{for} \quad t \geqslant 0.$$

Summarizing we have $e^t \geqslant 1 + \frac{t^n}{n^n}$ where $t \geqslant 0$ and $n \in \mathbb{N}$.

If we had $t^n/n^n = |z|$ then $t = n|z|^{1/n}$ so the inequality becomes

$$e^{n|z|^{1/n}} \geqslant 1 + |z| \Rightarrow |z - z_k| \leqslant [\max(1, |z_k|)]e^{n|z|^{1/n}}, \quad \text{for} \quad n \in \mathbb{N}.$$

This means that the order of $z - z_k$ is at most $\frac{1}{n}$. As this holds for all $n \in \mathbb{N}$, then the order of $z - z_k$ is arbitrarily small which means it must be 0. In conclusion, by the product lemma, the order of a polynomial is zero.

ii) Note that

$$|e^{bz^n}| = \left| \sum_{k=0}^{\infty} \frac{(bz^n)^k}{k!} \right| \leqslant \sum_{k=0}^{\infty} \frac{|bz^n|^k}{k!} = e^{|b||z|^n}.$$

This immediately tells us that the order of e^{bz^n} is bounded by n. But now take $\rho < n$, then we claim that

$$Ae^{B|z|^\rho} \leqslant |e^{bz^n}|$$

But if on the contrary we assumed that there existed A, B such that $Ae^{B|z|^\rho} > |e^{bz^n}|$ then this must hold for all z. But we may assume $z = x \in \mathbb{R}$ and let $x \to \infty$. There are no A, B such that $e^{bx^n} < Ae^{B|x|^\rho}$. In conclusion n is the order of e^{bz^n}.

iii) Finally we claim that e^{e^z} as infinite order. If on the contrary we assumed that

$$|e^{e^z}| \leqslant Ae^{B|z|^n} \quad \text{for all} \quad z \quad \text{and some} \quad A, B$$

then this inequality must hold for all $z \in \mathbb{C}$. In particular, when $z = x \in \mathbb{R}$ and we let $x \gg 0$ then

$$e^{e^z} \leqslant Ae^{Bx^n}$$

But we are able to always find a larger and a larger x such that this inequality fails for all choices of A and B. As no n can bound our function, we conclude that it must have infinite order of growth.

Exercise 2. Recall if (a_j) is a sequence with $|a_j - 1| < 1$, then

$$\prod_{j \geqslant 1}(1 + a_j) \text{ converges} \iff \sum_{j \geqslant 1} \log(1 + a_j) \text{ converges}$$

where $\log$ is the principal branch of the logarithm.

i) Show that $\prod_{n \geqslant 2} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$ diverges.

ii) Show that $\prod_{n\geqslant 2}\left(1+\frac{(-1)^n}{n}\right)$ converges.

⟦Hint: Use the first few term in the expansion of $\log(1+z)$⟧

Answer

Observe that it is equivalent to test the convergence of the series

- ◇ $\sum_{n\geqslant 2}\log\left(1+\frac{(-1)^n}{\sqrt{n}}\right)$, and
- ◇ $\sum_{n\geqslant 2}\log\left(1+\frac{(-1)^n}{n}\right)$.

For the first series we use the Taylor expansion of $\log(1+z)=z-\frac{z^2}{2}+O(z^3)$. In the case of our series we have

$$\sum_{n\geqslant 2}\log\left(1+\frac{(-1)^n}{\sqrt{n}}\right)\sum_{n\geqslant 2}\left[\left(\frac{(-1)^n}{\sqrt{n}}\right)-\frac{(-1)^{2n}}{2n}+O(n^{-3/2})\right].$$

Observe that the first and last term converge by Dirichlet's test and by the p-series test. This means that the behavior of our product is determined by $\sum\frac{1}{2n}$ which diverges. So our first product diverges.
In the same vein the sum can be analyzed by using Taylor's theorem:

$$\sum_{n\geqslant 2}\log\left(1+\frac{(-1)^n}{n}\right)=\sum_{n\geqslant 2}\left[\left(\frac{(-1)^n}{n}\right)+O(n^{-2})\right].$$

In this case there's no divergent term in the sum so we may apply Dirichlet's test and the p-series test to conclude that the whole series converges. Therefore the second product also converges.

Exercise 3 (Problem 5.4(a) Stein & Shakarchi)**.** Let $F(z)=\sum a_n z^n$ be entire of finite order. Then the growth order of F is intimately linked with the growth of the coefficients a_n as $n\to\infty$. In fact:

(a) Suppose $|F(z)|\leqslant Ae^{a|z|^\rho}$, then

$$\limsup_{n\to\infty}|a_n|^{1/n}n^{1/\rho}<\infty.$$

(b) Conversely, if the previous statement holds, then $|F(z)|\leqslant A_\varepsilon e^{a_\varepsilon|z|^{\rho+\varepsilon}}$ for $\varepsilon>0$.

Answer

Let $R > 0$ and notice that in the ball $B(0, R)$, F is analytic. We may write

$$F(z) = \sum \frac{1}{n!} f^{(n)}(0) z^n \Rightarrow f^{(n)}(0) = n! a_n.$$

So for $r \in]0, R[$ we may use Cauchy's inequality to obtain:

$$|n! a_n| = |f^{(n)}(0)| \leqslant \frac{n! \sup_{|z|=r} |F(z)|}{r^n} \leqslant \frac{n! A \sup_{|z|=r} e^{a|z|^\rho}}{r^n} \Rightarrow |a_n| \leqslant \frac{A e^{ar^\rho}}{r^n}.$$

We now consider the function $u^{-n} e^{u^\rho}$, differentiating, we obtain

$$(-n) u^{-n-1} e^{u^\rho} + u^{-n} e^{u^\rho} \rho u^{\rho - 1} = u^{-n-1} e^{u^\rho} (\rho u^\rho - n).$$

The minimum of this function is then achieved when $u^\rho = n/\rho$ that is, $u = (n/\rho)^{1/\rho}$. Plugging this value into our inequality we obtain

$$|a_n| \leqslant \frac{A e^{an/\rho}}{(n/\rho)^{n/\rho}} \Rightarrow |a_n|^{1/n} n^{1/\rho} \leqslant \frac{A e^{a/\rho}}{\rho^{1/\rho}}$$

which allows us to take

$$\sup_{n \geqslant k} |a_k|^{1/k} k^{1/\rho} \leqslant \frac{A e^{a/\rho}}{\rho^{1/\rho}} \Rightarrow \limsup_{n \to \infty} |a_n|^{1/n} n^{1/\rho} \leqslant \frac{A e^{a/\rho}}{\rho^{1/\rho}}.$$

Therefore we obtain the desired inequality.

Exercise 4 (Re-do of 5.2(a))**.** Prove that the order of a polynomial $p(z)$ is zero.

Answer

Using the second part of the previous problem we may see that

$$\limsup |a_n|^{1/n} < \infty$$

which occurs because all but finitely many a_n's are non zero. This means that $|p(z)| \leqslant A_\varepsilon e^{a_\varepsilon |z|^\varepsilon}$ for $\varepsilon > 0$. As ε can be arbitrarily small we have that p has order zero.