

# MATH601 — Advanced Combinatorics

Based on the lectures by Maria Gillespie

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Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This course will focus on the combinatorics of Young tableaux, crystal bases, root systems, Dynkin diagrams, and symmetric functions arising in representation theory of matrix groups and Lie algebras.

## **Requirements**

Familiarity with the basics of group theory and symmetric functions is helpful.

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# Chapter 1

## 1.1 Day 1 | 20240819

We will start by reviewing the representation theory of finite groups and the Lie group and Lie algebra representations. The objective is to classify semi-simple Lie algebras and groups. This classification is quite combinatorial.

### Review of representation theory of finite groups

Recall groups are sets  $G$  endowed with a binary operation  $\circ$  such that

- (a) There is an identity element  $e$ :  $g \circ e = e \circ g = g$ .
- (b) Every element possesses an inverse. For each  $g$ , there is an  $h$  such that  $g \circ h = e = h \circ g$ .
- (c) The operation  $\circ$  is associative.

**Example 1.1.1.** The symmetric group is the set of permutations of  $[n]$ . We denote it  $(S_n, \circ)$  where our operation is composition. We will use this group quite a lot.

**Example 1.1.2.** We will be working with  $GL_n(\mathbb{C})$  where  $\mathbb{C}$  will come in as more useful than  $\mathbb{R}$ . The general linear group is characterized by the property that  $\det(A) \neq 0$  for  $A \in GL_n(\mathbb{C})$ .

**Example 1.1.3.** Given two groups we can construct  $G \times H$  by doing operations point-wise. We can also take subgroups and quotient groups.

**Example 1.1.4.** Take the special linear group  $SL_n(\mathbb{C})$  which is the set of matrices  $A$  with  $\det(A) = 1$ . This is a subgroup of  $GL_n(\mathbb{C})$ .

There's a lot more of matrix groups such as  $SO_n(\mathbb{C})$ ,  $Sp_{2n}(\mathbb{C})$  and unitary groups  $SU_n(\mathbb{C})$ .

## Groups which are representations of themselves

Symmetry groups are groups of linear transformations of  $\mathbb{C}^n$  (some Euclidean space) that fix some shape. Any such group is a subgroup of  $GL_n(\mathbb{C})$ . Matrices here don't collapse points nor anything.

**Example 1.1.5.** The symmetry group of a diamond in the plane can be found by analyzing the symmetries of the figure. **HMMM** The group in question is the Klein-4 group which can be seen as

$$\{ \text{id}, r_x, r_y, r_x r_y \}.$$

Similarly we can see it as

$$\{ \text{id}, (24), (13), (13)(24) \}$$

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## 1.2 Day 2 | 20240821

We were looking at direct sums of representations. Recall representations are maps which take group elements to matrices.

$$\rho \oplus \sigma : G \rightarrow GL_{n+m}(\mathbb{C})$$

and this map will send  $g$  to a block matrix. A central question in representation theory is to classify the irreducible representations of some object. This is a central question because for finite groups, irreducible is the same as indecomposable.

**Definition 1.2.1.** A representation is indecomposable when it can't be written as a direct sum of smaller representations.

Irreducible means that it has no non-trivial proper representations. This is analogous to the idea of prime and irreducible numbers. In the most general case where groups may be infinite, irreducible implies indecomposable.

## Alternative definitions for representations

We may define it as a vector space  $V$  with an action  $G \times V \rightarrow V$  so that

$$g(hv) = (gh)v$$

and it should be a linear action in the sense that  $v \mapsto gv$  is a linear transformation.

This is equivalent to the previous definition because  $V$  can be seen as  $\mathbb{C}^n$ . So the definition gives rise to a map

$$G \rightarrow \text{Aut}(V), g \mapsto g \cdot$$

Even more *objecty* is the next definition. We can see a representation as a module over a group ring  $\mathbb{C}G$ . This set is made up of formal linear combinations of elements of  $G$ .

We endow it with a module structure, for any element  $g \in G$  in particular in  $\mathbb{C}G$  we can make it a coefficient  $gv \in V$  as a  $\mathbb{C}G$ -module.

### Subrepresentations

Now that we have all the algebraic structure we can use it to define subrepresentations. Because a subrepresentation will be a subspace which inherits the action for example.

**Definition 1.2.2.**  $W \subseteq V$  is a subrepresentation of  $G$  (when  $V$  represents  $G$ ) if

- ◊  $W$  is a subspace of  $V$ , and
- ◊  $W$  is  $G$ -invariant in the sense that the image of  $G \times W \rightarrow V$  is contained in  $W$ .

We will also say that  $V$  is irreducible if there's no proper nonzero subrepresentation  $W \subseteq V$ .

Sometimes it is possible to decompose a representation into a direct sum of subrepresentations.

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**Definition 1.2.3.** A character of a representation is the trace map  $g \mapsto \text{tr}(\rho(g))$ .

### Properties

- (a)  $\chi_{V \oplus W} = \chi_V + \chi_W$ .
- (b)  $\chi_{V \otimes W} = \chi_V \chi_W$ .
- (c)  $\chi_V$  uniquely determines the representation.

### 1.3 Day 3 | 20240823

#### Lie groups

**Definition 1.3.1.** A Lie group is a real smooth manifold  $G$  with a group structure such that

$$(g, h) \mapsto gh^{-1}$$

is differentiable.

A manifold is a set such that around each point there's a local neighborhood that's topologically equivalent to  $\mathbb{R}^n$ . Elliptic curves are examples of manifolds.

**Definition 1.3.2.** An algebraic group is an algebraic variety with a group structure. In this case the multiplication map should be algebraic.

In certain specializations these two are the same object. In the case of complex Lie groups, we talk about smooth complex manifolds.

**Example 1.3.3.**  $\diamond (\mathbb{C}^n, +)$  is a Lie group. But it's not compact. **sleepy sleepy**

$\diamond \text{GL}_n$

**Lemma 1.3.4.** (Zariski-)Closed subgroups of a Lie group are also Lie groups.

**Example 1.3.5.** In particular  $B_n$ , the set of upper triangular matrices in  $\text{GL}_n$ , forms a Lie group. The torus  $T_n$ , the group of diagonal matrices, is also a Lie group.

It is called the torus because it's isomorphic to  $(\mathbb{C} \setminus 0)^n$  and  $\mathbb{C} \setminus 0$  looks like a circle while  $(\mathbb{C} \setminus 0)^2$  is the product of two circles which is the torus.

#### The Classical Groups

The special linear group  $\text{SL}_n$  consists of matrices whose determinant is 1. The classical groups are called classical because they have very nice properties. In particular type A is what we call  $\text{SL}_n$ .

To talk about the special orthogonal group  $\text{SO}_n$  we should first fix a symmetric bilinear form  $(\cdot, \cdot)$  which is positive-definite. The orthogonal group  $\text{O}_n$  consists of matrices which preserve this form. The special orthogonal group in particular is the subgroup of matrices with determinant 1.

*Remark 1.3.6.* Over  $\mathbb{R}$ ,  $\text{O}_n$  is actually the group of rigid transformations which is generated by reflections and rotations. For  $\text{SO}_n$ , it's only the rotations group.

We can also alternatively define  $O_n$  as

$$\{ A : A^T A = I \}$$

because

$$\langle Av | Aw \rangle = \langle v | w \rangle$$

and from this

$$v^T A^T A w = v^T w.$$

Comparing entry by entry we get the desired property.

It's also a fact that  $O_n$  is disconnected, one component is  $SO_n$  and the other is the set of matrices with determinant  $-1$ . Finally type B means  $SO_{\text{odd}}$  while  $D$  means  $SO_{\text{even}}$ . The type  $C$  groups are the symplectic groups.

## 1.4 Day 4 | 20240826

Continuing on with the classical groups, we will be talking about the Symplectic group of even dimension. We will be fixing a symplectic form which is a non-degenerate, skew-symmetric, bilinear form.

**Example 1.4.1.** The dot product is not symplectic because it's symmetric.

**Example 1.4.2.** Consider the form

$$v_1 w_{2n} + v_2 w_{2n-1} + \cdots + v_n w_{n+1} - v_{n-1} w_n - v_{n+1} w_n - v_{n+2} w_{n-1} - \cdots - v_{2n} w_1.$$

If  $\Omega$  is such a matrix of a form, for example when  $2n = 6$  we have

$$\Omega := \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & -1 & & & \\ & -1 & & & & \\ -1 & & & & & \end{pmatrix} \Rightarrow (v, w) = v^T \Omega w$$

From this our first definition of the symplectic group is matrices which preserve this product.

**Definition 1.4.3.** The symplectic group  $Sp_{2n}$  is

$$\{ M : (Mv, Mw) = (v, w) \}$$

or equivalently

$$\{ M : M^T \Omega M = \Omega \}.$$

We will simplify the notation to type  $C$ .

## Representation of Lie groups

**Definition 1.4.4.** A representation of a Lie group is a map which is also differentiable and a group homomorphism.

### 1.5 Day 5 | 20240828

For a partition  $\lambda \vdash n$ , we call  $S^\lambda V$

$$\Lambda^{\mu_1} V \otimes \Lambda^{\mu_2} V \otimes \dots \otimes \Lambda^{\mu_k} V$$

where  $\mu$  is the conjugate partition.

**Example 1.5.1.** For example if  $\lambda = (5, 4, 1)$ , then  $\mu = (3, 2, 2, 2, 1)$  and so

$$S^{(5,4,1)} V = \Lambda^3$$

Elements can be written as a filling to the Young diagram. Such an element could be

$$(v_1 \wedge v_2 \wedge v_3) \otimes (a \wedge b) \otimes (c \wedge d) \otimes (x \wedge y) \otimes z$$

and filling the diagram we have

$r$				
$q$	$b$	$d$	$y$	
$p$	$a$	$c$	$x$	$z$

It's important to familiarize ourselves with this idea so we will interchangeably talk about

$$(e_1 \wedge e_4 \wedge e_3) \otimes (e_1 \wedge e_2) \otimes (e_5 \wedge e_3) \otimes (e_2 \wedge e_1) \otimes e_2$$

and

3					
4	2	3	1		
1	1	5	2	2	

 $= -$ 

4					
3	2	3	1		
1	1	5	2	2	

The tableau 

1	
1	2

 is zero for example.

For a basis of  $S^\lambda$ , we can talk about it being spanned by elementary tableau where we order each column from least to greatest. These are called column-strict tableau. For example



6			
4	5	3	
1	2	1	2

If  $V$  is an  $n$ -dimensional vector space, then we have a largest element on our basis. This allows us to formulate the question:

*How many column strict tableau are there with largest entry  $n$ ? And shape  $\lambda$ .*

From this

$$\binom{n}{\mu_1} \binom{n}{\mu_2} \cdots \binom{n}{\mu_k} = S^\lambda V.$$

**Definition 1.5.2.** The Schur module  $V^\lambda$  is

$$V^\lambda = S^\lambda / \left\langle v_T - \sum_S v_S \right\rangle$$

where the sum is over  $S$ 's obtained from  $T$  by

- (a) Choose two columns of  $C_1, C_2$  of  $T$ .
- (b) Choose  $k$  elements from  $C_2$ .
- (c) Exchange them with  $k$  elements from  $C_1$  in all ways that preserve the order of the elements.

**Example 1.5.3.** Take  $(4, 3, 3)$  with the filling

5	7	6	
2	4	4	
1	1	3	4

so choose the first and third columns as  $C_1$  and  $C_2$ . One relation in  $V_\lambda$

**Theorem 1.5.4.** *The collection*

$$\{ e_T : T \text{ semistandard sh}(T) \vdash n \}$$

*is a basis for the Schur module.*



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