

**Exercise 1.** Let  $A : V \rightarrow W$  be a linear map between vector spaces.

- (a) Show that the induced map  $\bigwedge^k(V) \rightarrow \bigwedge^k(W)$  is well-defined by

$$v_1 \wedge \dots \wedge v_k \mapsto Av_1 \wedge \dots \wedge Av_k$$

(extending linearly to sums).

- (b) Show that the map  $A^* : W^* \rightarrow V^*$  defined by  $(A^*(\eta))(v) := \eta(A(v))$  determines a map  $\bigwedge^k(W^*) \rightarrow \bigwedge^k(V^*)$ .
- (c) Show that, if  $V$  is an  $n$ -dimensional vector space, then the map  $\bigwedge^n(V) \rightarrow \bigwedge^n(V)$  is multiplication by  $\det A$ .

### Answer

To prove well-definedness of a map, it suffices to take two representatives of the same class and see that they map to the same place.

- (a) Consider then, without loss of generality,

$$v_1 \wedge v_2 \wedge \dots \wedge v_k = -(v_2 \wedge v_1 \wedge \dots \wedge v_k).$$

This second element we can reinterpret as

$$(-v_2) \wedge v_1 \wedge \dots \wedge v_k.$$

Applying  $\bigwedge^k(A)$  to this we get

$$\begin{cases} Av_1 \wedge Av_2 \wedge \dots \wedge Av_k \\ A(-v_2) \wedge Av_1 \wedge \dots \wedge Av_k \end{cases}$$

and using the fact that  $A$  is linear we get

$$\begin{aligned} A(-v_2) \wedge Av_1 \wedge \dots \wedge Av_k &= -(Av_2 \wedge Av_1 \wedge \dots \wedge Av_k) \\ &= Av_1 \wedge Av_2 \wedge \dots \wedge Av_k \end{aligned}$$

and this is the desired representation of the image. **FINISH MULTILINEAR**

This allows to see that  $\bigwedge^k(A)$  is well-defined.

- (b) The map  $A^*$  does indeed define a map from the exterior powers, namely  $\bigwedge^k(A^*)$ . **FINISH**

- (c) **NO IDEA**

**Exercise 2.** Show that the vectors  $v_1, \dots, v_k \in V$  are linearly independent if and only if  $v_1 \wedge \dots \wedge v_k \neq 0$  as an element of  $\bigwedge^k(V)$ .

### Answer

Assume that  $\{v_1, \dots, v_k\}$  is linearly dependent, then if  $\{v_1, \dots, v_\ell\}$  is a maximally independent set, we may write any  $v_i$  with  $\ell < i \leq k$  as a linear combination of  $\{v_1, \dots, v_\ell\}$ .

This means that

$$\begin{aligned} v_1 \wedge \dots \wedge v_k &= v_1 \wedge \dots \wedge v_{\ell+1} \wedge \dots \wedge v_k \\ &= v_1 \wedge \dots \wedge \sum_{i=1}^k c_i v_i \wedge \dots \wedge v_k \\ &= \sum_{i=1}^k c_i (v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_k) \end{aligned}$$

and all the summands will be zero as we will find repeated  $v_i$ 's in each term.

**Exercise 3.** We say that an element of  $\bigwedge^k(V)$  is *decomposable* if it can be written as  $v_1 \wedge \dots \wedge v_k$ .

- (a) Suppose  $v, w, x, y \in V$ . Find necessary and sufficient conditions for  $v \wedge w + x \wedge y \in \bigwedge^2(V)$  to be decomposable.
- (b) Show that  $\omega \in \bigwedge^2(\mathbb{R}^4)$  is decomposable if and only if  $\omega \wedge \omega = 0$ .

### Answer

- (a) If it was the case that the element is decomposable, then there exist  $a, b \in V$  such that

$$v \wedge w + x \wedge y = a \wedge b.$$

Observe now that

$$(a \wedge b)^{\wedge 2} = a \wedge b \wedge a \wedge b = 0$$

so that

$$(v \wedge w + x \wedge y)^{\wedge 2} = 0.$$

Expanding out this quantity we obtain

$$\begin{aligned} 0 &= (v \wedge w)^{\wedge 2} + (v \wedge u) \wedge (x \wedge y) + (x \wedge y) \wedge (v \wedge w) + (x \wedge y)^{\wedge 2} \\ &= v \wedge u \wedge x \wedge y + (-1)^2 (v \wedge w \wedge x \wedge y) \\ &= 2(v \wedge w \wedge x \wedge y) \end{aligned}$$

so it must occur that

$$v \wedge w \wedge x \wedge y = 0$$

or in other words, the vectors are linearly dependent.

**Exercise 4.** Let  $V$  be an  $n$ -dimensional inner product space. We can extend the inner product from  $V$  to all of  $\bigwedge(V)$  by setting the inner product of homogeneous elements of different degrees equal to zero and by letting

$$\langle w_1 \wedge \dots \wedge w_k, v_1 \wedge \dots \wedge v_k \rangle = \det (\langle w_i, v_j \rangle)_{i,j}$$

and extending bilinearly.

Since  $\bigwedge^n(V)$  is a one-dimensional real vector space,  $\bigwedge^n(V) - \{0\}$  has two components. An *orientation* on  $V$  is a choice of component of  $\bigwedge^n(V) - \{0\}$ . If  $V$  is an oriented inner product space, then there is a linear map  $\star : \bigwedge(V) \rightarrow \bigwedge(V)$  called the star map, which is defined by requiring that for any orthonormal basis  $e_1, \dots, e_n$  for  $V$ ,

$$\begin{aligned} \star(1) &= \pm e_1 \wedge \dots \wedge e_n, & \star(e_1 \wedge \dots \wedge e_n) &= \pm 1, \\ \star(e_1 \wedge \dots \wedge e_k) &= \pm e_{k+1} \wedge \dots \wedge e_n, \end{aligned}$$

where in each case we take “+” if  $e_1 \wedge \dots \wedge e_n$  is in the preferred component of  $\bigwedge^n(V)$  and we take “−” otherwise. Notice that  $\star : \bigwedge^k(V) \rightarrow \bigwedge^{n-k}(V)$ .

- Prove that if  $e_1, \dots, e_n$  is an orthonormal basis for  $V$ , then the  $e_{i_1} \wedge \dots \wedge e_{i_k}$  with  $1 \leq i_1 < \dots < i_k \leq n$  and  $1 \leq k \leq n$  give an orthonormal basis for  $\bigwedge(V)$ .
- Prove that, as a map  $\bigwedge^k(V) \rightarrow \bigwedge^k(V)$ ,  $\star\star = (-1)^{k(n-k)}$ .
- Prove that, for  $\omega, \eta \in \bigwedge^k(V)$ , their inner product is given by

$$\langle \omega, \eta \rangle = \star(\omega \wedge \star\eta) = \star(\eta \wedge \star\omega).$$

## Answer

(a) Indeed, if  $\{e_i\}_{i \in [n]}$  forms an orthonormal basis of  $V$ , then each collection

$$\{e_I : |I| = k, I \subseteq [n]\}, \quad e_I = \bigwedge_{i \in I} e_i,$$

spans the  $k^{\text{th}}$  exterior power of  $V$ . In consequence the whole exterior algebra is spanned by the same collection letting  $|I|$  range up to  $n$ .

To see linear independence it suffices to see orthogonality. Between different index sizes it is clear via hypothesis. So let  $I, J \in \binom{[n]}{k}$  and consider  $\langle e_I | e_J \rangle$ . The obtained quantity is the determinant of the Gram matrix formed by the bases  $\{e_i\}_{i \in I}$  and  $\{e_j\}_{j \in J}$ . **What if we get a permutation matrix? Is it even possible? like e1e3e4 and e4e1e3.**

(b) Proving this fact for a basic element suffices, so take  $e_I$

**Exercise 5.** Let  $M^n$  be a closed manifold (i.e., a compact manifold without boundary) and let  $\omega \in \Omega^1(M)$  so that  $\omega_p \neq 0$  for all  $p \in M$  (i.e., for all  $p$ , there exists  $v \in T_p M$  so that  $\omega_p(v) \neq 0$ ). Show that  $\omega$  is not exact.

## Answer

It is equivalent to show that if  $\omega$  is exact, then there exists  $p \in M$  so that  $\omega_p = 0$ . So to our effect, assume  $\omega$  is exact. Then there is an  $f : M \rightarrow \mathbb{R}$  such that  $df = \omega$ . As  $M$  is compact, then there exists a point  $p \in M$  at which  $f$  attains a minimum (otherwise, we'll just have to switch signs). **FINISH**