Exercise 1 (5.10(a) Stein& Shakarchi)**.** Find the Hadamard product for $e^z - 1$.

Answer

Recall Hadamard's theorem states that if f is an entire function with order of growth ρ and $k = \lfloor \rho \rfloor$ then

$$f(z) = e^{p(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$$

where (a_n) is the collection of non-null zeroes of f, p has degree at most k and $m = \text{ord}(f, 0)$.

In our case $e^z - 1$ has order of growth 1 and it has simple zeroes at $z = 2\pi i n$ for $n \in \mathbb{Z}$. In particular the order of zero is one. This means that

$$e^z - 1 = e^{a_1 z + a_0} z \prod_{n \in \mathbb{Z}\setminus\{0\}} \left(1 - \frac{z}{2\pi i n}\right) e^{z/2\pi i n}.$$

To simplify this product we multiply opposites across the origin:

$$\begin{aligned}\left[\left(1 - \frac{z}{2\pi i n}\right) e^{z/2\pi i n}\right]\left[\left(1 - \frac{z}{2\pi i(-n)}\right) e^{z/2\pi i(-n)}\right] &= \left(1 + \left(\frac{z}{2\pi i n}\right)^2\right) e^{z/2\pi i n} e^{-z/2\pi i n} \\ &= 1 + \frac{z^2}{4\pi^2 n^2}\end{aligned}$$

So we get

$$e^z - 1 = e^{a_1 z + a_0} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Dividing both sides by z we get

$$\frac{e^z - 1}{z} = e^{a_1 z + a_0} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$$

and as z approaches 0 we get that

$$1 = e^{a_0}(1) \Rightarrow a_0 = 0.$$

Expanding the exponential function as a Taylor series and comparing coefficients we get the following:

$$z + \frac{z^2}{2} + O(z^3) = (1 + a_1 z + \frac{(a_1 z)^2}{2} + O(z^3))z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$$

Thus we obtain

$$z + \frac{z^2}{2} + O(z^3) = z + a_1 z^2 + O(z^3) \Rightarrow a_1 = \frac{1}{2}.$$

In conclusion we have

$$e^z - 1 = e^{z/2} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Exercise 2 (5.11 Stein& Shakarchi)**.** Show that if f is an entire function of finite order that omits two values, then f is constant. This result remains true for any entire function and is known as Picard's little theorem. ⟦Hint: If f misses a, then $f(z) - a$ is of the form $e^{p(z)}$ where p is a polynomial.⟧

Answer

Assume f omits two values a, b which means that

$$f(z) - a = e^{p(z)}, \quad \text{and} \quad f(z) - b = e^{q(z)} \quad \text{for some} \quad p, q \quad \text{polynomials} \quad .$$

From this, we may subtract one equation from the other to get

$$b - a = e^{p(z)} - eq(z)$$

and now we may differentiate both sides of the equation to obtain

$$0 = p'(z)e^{p(z)} - q'(z)e^{q(z)}.$$

As this equation holds for *all* $z \in \mathbb{C}$ it must happen that $p'(z)$ and $q'(z)$ have the same zeroes with the same multiplicities. Thus $q'(z) = cp'(z)$ for some non-zero $c \in \mathbb{C}$. Returning to our equation we have

$$e^{p(z)}p'(z) = cp'(z)e^{q(z)} \Rightarrow e^{p(z)} = ce^{q(z)} \Rightarrow ce^{q(z)} - e^{q(z)} = b - a.$$

Differentiating this equation we obtain

$$(c-1)q'(z)e^{q(z)} = 0 \Rightarrow q'(z) = 0 \Rightarrow q \quad \text{is constant} \quad .$$

This allows us to conclude that f is constant as $f = e^q + b$.
If it occured that $c = 1$, then $p' = q'$ and so $q(z) = p(z) + d$ for some $d \in \mathbb{C}$. Replacing this in the equation we have

$$e^{p(z)} - e^{p(z)+d} = b - a \Rightarrow (1 - e^d)e^{p(z)}p'(z) = 0 \Rightarrow p'(z) = 0 \Rightarrow p \quad \text{is constant}$$

and once again we deduce f is constant. Finally if it was the case that $d = 0$, then $p = q$ but this means that

$$f(z) - e^{p(z)} = a = b$$

and this can't happen as a, b are different values of $\mathbb{C}$. In conclusion we have that f is constant.

Exercise 3. Assume $\mathrm{Re}(s) = \sigma > 0$. For $n, N \in \mathbb{N}$ define

$$\delta_n(s) = \frac{1}{n^s} - \int_n^{n+1} \frac{\mathrm{d}x}{x^s} = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) \mathrm{d}x \quad \text{and} \quad F_N(s) = \sum_{n=1}^{N} \delta_n(s).$$

i) Show that $|\delta_n(s)| \leqslant \frac{|s|}{n^{\mathrm{Re}(s)+1}}$. ⟦Hint: Represent the integrand in the definition of δ_n using the observation that $\int_n^x \frac{\mathrm{d}u}{u^{s+1}} = \frac{-1}{s}(x^{-s} - n^{-s})$.⟧

ii) Show that $(F_N(s))$ converges uniformly on any half-plane of the form $\mathrm{Re}(s) \geqslant \alpha > 0$.

iii) Show that $\zeta(s) - \frac{1}{s-1}$ is bounded and holomorphic near $s = 1$. ⟦Hint: Use the fact that $\frac{1}{s-1} = \int_1^\infty x^{-s} \mathrm{d}x$.⟧