Homework

Due: Friday, March 10

1. [SS]3.14.
2. As in class, consider the unit sphere
$$X = \left\{(a,b,c) : a^2+b^2+c^2 = 1\right\} \subset \mathbb{R}^3.$$
Let $N = (0,0,1)$, $S = (0,0,-1)$, $U_N = X \smallsetminus N$, $U_S = X \smallsetminus S$. Consider the following three charts on X:
$$\begin{aligned} U_N &\xrightarrow{\phi_N} \mathbb{C} \\ (a,b,c) &\longmapsto \tfrac{a+ib}{1-c} \end{aligned}$$
$$\begin{aligned} U_S &\xrightarrow{\phi_S} \mathbb{C} \\ (a,b,c) &\longmapsto \tfrac{a+ib}{1+c} \end{aligned}$$
$$\begin{aligned} U_S &\xrightarrow{\psi_S} \mathbb{C} \\ (a,b,c) &\longmapsto \tfrac{a-ib}{1+c} \end{aligned}$$
   (a) The inverse of ϕ_N is
$$\phi_N^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$
   Calculate $\phi_S^{-1}(z)$ and $\psi_S^{-1}(z)$.

   (b) Among the three charts $\{(U_N,\phi_N),(U_S,\phi_S),(U_S,\psi_S)\}$, one pair is compatible (i.e., $\alpha \circ \beta^{-1}$ is holomorphic) and the other two are not. Which is which? Why? (HINT: *Remember that a function f is holomorphic if and only if $\partial_{\overline{z}} f = 0$; colloquially, a function is holomorphic if it doesn't involve any $\overline{z}$'s.*)
3. If f is meromorphic on Ω and $z_0 \in \Omega$, we define the order of f by
$$\mathrm{ord}_{z_0}(f) = \begin{cases} 0 & f \text{ holomorphic at } z_0, f(z_0) \neq 0 \\ m & f \text{ has a zero of order } m \text{ at } z_0 \\ -m & f \text{ has a pole of order } -m \text{ at } z_0 \end{cases};$$

see also HW5.

(a) Let $p(z)$ be a polynomial of degree d, thought of as a meromorphic function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Use the definition of a pole at infinity ([SS, p. 87]) to show that $\operatorname{ord}_\infty p(z) = -\deg d$.

(b) Show that if $p(z)$ is a polynomial, then

$$\sum_{z_0 \in \hat{\mathbb{C}}} \operatorname{ord}_{z_0}(f) = 0.$$

(HINT: *Use the fundamental theorem of algebra.*)

(c) Show that if $f(z) = \frac{p(z)}{q(z)}$ is a rational function (i.e., a quotient of polynomials), then

$$\sum_{z_0} \operatorname{ord}_{z_0}(f) = 0.$$

(HINT: *Use the previous step and HW5#4Aa.*)