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Chapter 1

Introduction and background

Our goal is to understand the calculation of Gromov-Witten invariants of the space $\overline{M}_{g,n}(\mathbb{P}^r, d)$, the moduli space of degree d maps to $\mathbb{P}^r$, using techniques from Atiyah-Bott localization. To begin this endeavor, we must first study the moduli space $\overline{M}_{g,n}$ and its intersection theory. This space, which parametrizes genus g Riemann surfaces with n marked points, was originally introduced and studied by Deligne and Mumford.

Subsequently, we will introduce the concept of equivariant cohomology and the Atiyah-Bott localization theorem. We will demonstrate the theorem's usefulness through several illustrative examples, culminating in its application to the calculation of Gromov-Witten invariants via localization on the moduli space of maps.

1.1 A motivating example

The question that initially motivated my study of this topic appears deceptively simple:

Which are the quadratic curves which pass through 4 points in $\mathbb{R}^2$ and no three of them are collinear?

This question might be a bit tough to tackle right now, but let us simplify by giving ourselves four particular points.

Example 1.1.1. Let us find all the conics passing through the points

$$(1, 1), (1, -1), (-1, -1) \quad \text{and} \quad (-1, 1).$$

At first *a circle* should be our best guess for a conic passing through this four points. As seen in the following figure, $x^2 + y^2 = 2$ is a circle containing the aforementioned points.
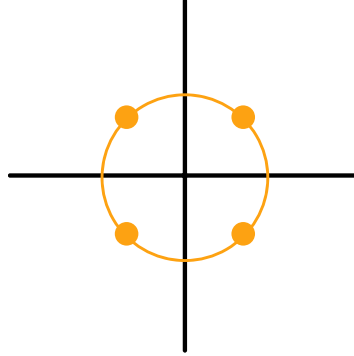
Figure 1.1: One of the quadratic curves passing through our points: $x^2 + y^2 = 2$.

Ideally we would like to stretch and shrink the circle in order to make it an ellipse. We know ellipses have equations of the form

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

but in order to use our circle equation we will instead add coefficients to the equation in the following fashion

$$Ax^2 + By^2 = 2.$$

These coefficients are determined by the points on the curve, we may derive the relation by plugging in the coordinates of a point into the equation:

$$A(1)^2 + B(1)^2 = 2 \Rightarrow B = 2 - A \Rightarrow tx^2 + (2 - t)y^2 = 2$$

where we take $t = A$ to get the last equation. The following curves are the ones we obtain given different values of t:

- $(t = 1)$: A circle.
- $(1 < t < 2)$: An ellipse.
- $(t = 2)$: The pair of lines $x^2 = 1$.
- $(t > 2)$: A hyperbola.

However we are left with one curve which passes through the points in question. To find it we will assume t is non-zero. From our parametric equation we obtain

$$tx^2 + (2 - t)y^2 = 2 \Rightarrow x^2 + o(t) + y^2 = \frac{2}{t} \xrightarrow[t \to \infty]{} x^2 = y^2$$

which is the pair of lines $y = \pm x$. Observe that this behavior is independent of the sign of the infinity we are going to. As such, these are all the conics passing through this four points. We omit the complex case in this example but this can be done in full generality.
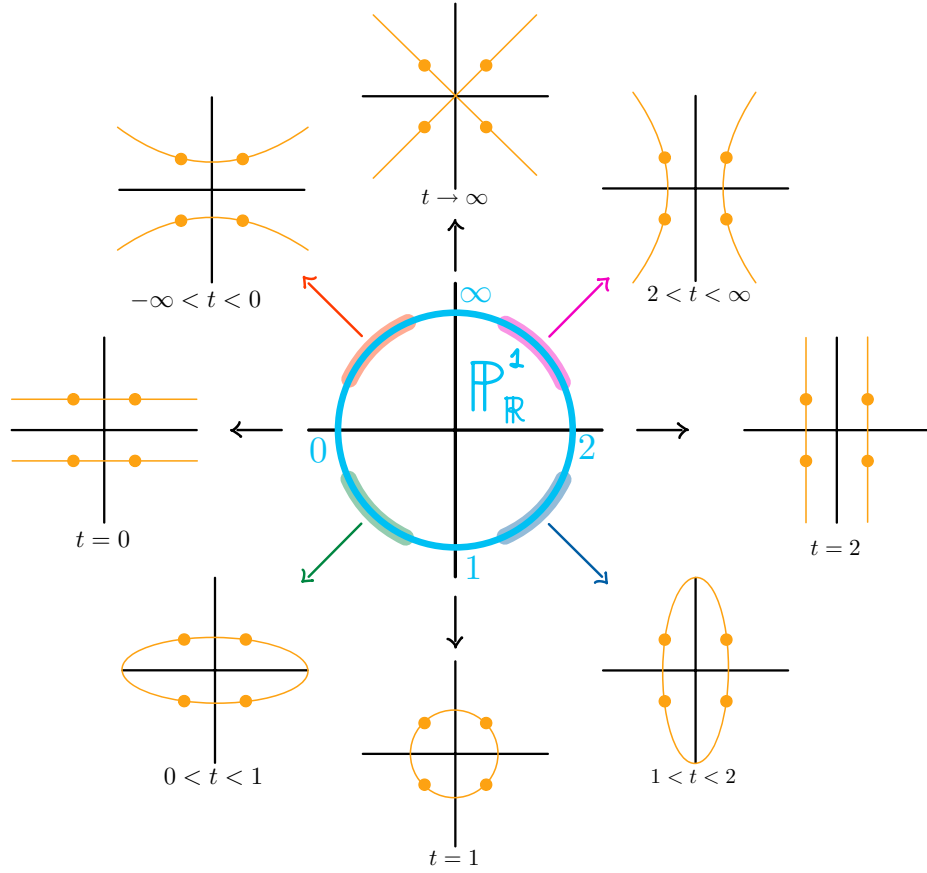
Figure 1.2: The projective line seen as the moduli space $\overline{M}_{0,4}$.

This example suggests a strategy for solving the original problem for any choice of four points.

Example 1.1.2. Given four distinct points A, B, C, and D in general position, we look for all the possible conics passing through them.

Define F to be the reducible conic formed by the union of the lines through A and B, and C and D. Similarly, let G be the conic formed by the lines through A and C, and B and D. Then the family of conics

$$\lambda F + \mu G = 0, \quad [\lambda, \mu] \in \mathbb{P}^1$$

describes all conics passing through the four given points.

In other words, the space of conics passing through four points in general position is naturally parametrized by $\mathbb{P}^1$. This is an example of a moduli space: a geometric space whose points parametrize certain types of objects.

1.2 A different viewpoint on the same problem

Kapranov [11] studied spaces which parametrized curves of certain degree passing through a number of points. The problem of classifying conics with marked points is one of such cases, and this transforms into the problem of classifying four-pointed copies of $\mathbb{P}^1$. There is an equivalence between pointed curves and marked Riemann surfaces. Following this, we will emphasize the viewpoint of marked Riemann surfaces, as it better aligns with the study I have realized.

Proposition 1.2.1. *Any smooth conic in the projective plane is isomorphic to the projective line.*

For our four pointed conic, we get an isomorphic $(\mathbb{P}^1, p_1, \ldots, p_4)$.
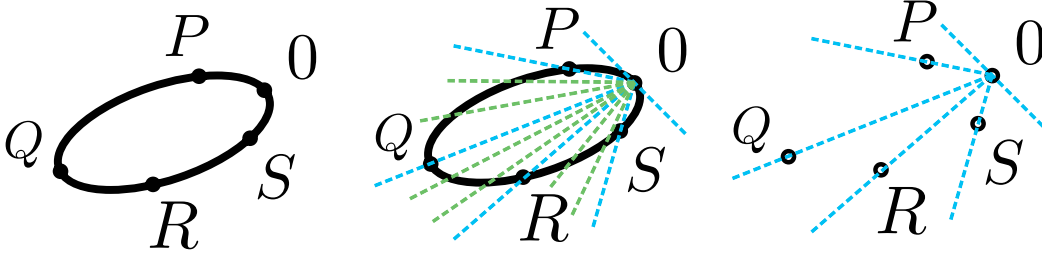

Figure 1.3: Projection from 0 turns conic into $\mathbb{P}^1$

The idea of the proof consists in taking a point in the conic from which we project lines and then create a bijection between this lines and $\mathbb{P}^1$.

As the conic is smooth, such lines will only have one other point of intersection with the curve meaning that we can find a bijection between points in the conic and points in the projective line.

Remark 1.2.2. Observe that for any given four points, we will recover $\mathbb{P}^1$ along different tuples of points. When we start varying the points through which the conics pass, we will obtain a different tuple. Our question is then, how do we classify this projective lines along with their points?

Theorem 1.2.3. *The automorphism group of $\mathbb{P}^1$ is the group of Möbius transformations, PGL_2. Given any two ordered triple of points*

$$p_1, p_2, p_3 \quad \textit{and} \quad q_1, q_2, q_3$$

of the projective line, there exists a unique automorphism T of the projective line such that $T(p_i) = q_i$.

We will assume the first part of this result and quickly we mention the proof for the second part.

Proof

Any such automorphism is of the form

$$z \mapsto \frac{az + b}{cz + d}$$

so let us build a map which takes p_1, p_2, p_3 to $0, 1, \infty$ (in affine coordinates). Such a map is

$$z \mapsto \frac{z - p_1}{z - p_3} \left(\frac{p_2 - p_3}{p_2 - p_1} \right).$$

This maps p_1, p_2, p_3 to $0, 1, \infty$ respectively. Call this map T_p and then create a similar T_q, the desired function T is $T_q^{-1} T_p$.

Definition 1.2.4. Call the image of the fourth point p_4 under the aforementioned map T_p, the <u>cross-ratio</u> of the tuple $(p_1, \ldots, p_4)$.

For our intents and purposes, the problem changed from

      When are two conics passing through 4 points in general position isomorphic?

to the projective variant

      When are two 4-marked projective lines isomorphic?

And with the previous result we can give the answer.

Theorem 1.2.5. *The isomorphism class of $(\mathbb{P}^1, 0, 1, \infty, t)$ is determined by the value $t \in \mathbb{P}^1 \backslash \{ 0, 1, \infty \}$.*

Each smooth conic then corresponds to a $(\mathbb{P}^1, 0, 1, \infty, t)$ where t varies in $\mathbb{P}^1 \backslash \{ 0, 1, \infty \}$. Intuitively for now we will call it the moduli space of 4-pointed $\mathbb{P}^1$'s, $M_{0,4}$.

But observe we are missing two things, only one of which we have talked about before. The new missing aspect is the lack of compactness of this space. $\mathbb{P}^1$ as a whole is compact, but removing three points also removes that property.

The other missing objects are the *singular* conics. Asking where they went is like asking "what if we let t go to any of the three special points?" We will defer the interested reader to Kock and Vainsencher's *Green Book* [13] to fill out the details of the universal family business. In our treatment it suffices to know that whenever

two points collide, we will blowup $\mathbb{P}^1$ at that point adding an exceptional divisor isomorphic to $\mathbb{P}^1$ and then adding the two marked points there.

This treatment guarantees that $\overline{M}_{0,4}$ is now compact and so we have a nicer moduli space. Let us now delve deeper into this notion and throughly explain how we added the singular conics.

1.3 Moduli of curves

As a quick refresher, let us clarify the objects which are parametrized by the moduli spaces of our interest.

Definition 1.3.1. A <u>Riemann surface</u> is a complex analytic manifold of dimension 1.

For every point in the surface, there's a neighborhood which is isomorphic to $\mathbb{C}$ and transition functions are linear isomorphisms of $\mathbb{C}$. We will interchangeably say Riemann surface or *smooth compact complex curve*.

Example 1.3.2. The following classes define Riemann surfaces.

(a) $\mathbb{C}$ itself is a Riemann surface with one chart.

(b) Any open set of $\mathbb{C}$ is a Riemann surface.

(c) A holomorphic function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ defines a Riemann surface by considering $\Gamma_f \subseteq \mathbb{C}^2$. There's only one chart determined by the projection and the inclusion i_{Γ_f} is its inverse.

(d) Take another holomorphic function f, then $\{ f(x,y) = 0 \}$ is a Riemann surface such that
$$\mathrm{Sing}(f) = \{ \partial_x f = \partial_y f = f = 0 \} = \varnothing.$$

This means that at every point the gradient identifies a normal direction to the level set $f = 0$. In particular, there's a well defined tangent line. The inverse function theorem guarantees that this is a complex manifold.

(e) The first compact example is $\mathbb{P}^1$.

Definition 1.3.3. The <u>moduli space</u> $M_{g,n}$ is the set of isomorphism classes of genus g, n-pointed Riemann surfaces.

Recalling our motivating example, we called the space which classified the conics $M_{0,4}$. According to this definition, we are classifying 4-pointed genus 0 Riemann surfaces. It is indeed the case that $\mathbb{P}^1$ is the unique up-to-isomorphism Riemann surface of genus 0. Let us explore what happens in further genus higher up.

Example 1.3.4. The space $M_{1,1}$ parametrizes 1-pointed genus 1 Riemann surfaces, or 1-marked *elliptic curves*.

Any such curve is isomorphic to

$$\mathbb{C}\Big/_L, \quad L = \mathbb{Z}u + \mathbb{Z}v, \quad \text{where} \quad u, v \in \mathbb{Z},$$

and the image of the origin under the quotient map is the natural choice for the marked point. We have that two lattices L_1, L_2 determine the same elliptic curve whenever

$$\exists \alpha \in \mathbb{C}^\times (L_2 = \alpha L_1).$$

So that

$$M_{1,1} = \{\,\text{lattices}\,\}\Big/_{\mathbb{C}^\times}$$

but we can be more precise!

Explicitly, a lattice $L = \text{gen}_{\mathbb{Z}}(u, v)$ can be rescaled to

$$\tilde{L} = \frac{1}{u}L = \text{gen}_{\mathbb{Z}}(1, \tau).$$

This quantity τ always lies in the upper half plane when

$$\arg(v) > \arg(u) \bmod [-\pi, \pi]$$

which means that $\tau \in \mathbb{H}$ parametrizes $\left[\mathbb{C}/_L\right]$. Let us apply two $\text{SL}_2(\mathbb{Z}) = \text{gen}(S, T)$ actions on τ which will leave the quotient unchanged:

$$\begin{cases} T : \tau \mapsto \tau + 1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \tau = \dfrac{\tau + 1}{0 + 1}, \\[4mm] S : \tau \mapsto -\dfrac{1}{\tau} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ \tau = \dfrac{0 - 1}{\tau + 0}. \end{cases}$$

Then observe that the lattices

$$\text{gen}_{\mathbb{Z}}(1, T \cdot \tau) \quad \text{and} \quad \text{gen}_{\mathbb{Z}}(1, S \cdot \tau)$$

give us the same quotient. From this we can be more specific and say

$$M_{1,1} = \mathbb{H}\Big/_{\text{SL}_2(\mathbb{Z})}.$$

Once again, we arrive at the same issue that this space is not compact. Last time we found a way to compactify the space by adding some *evidently* missing points. But it is unclear here how to deal with this conundrum. We address the problem by introducing the notion of stability.

Stable curves

Definition 1.3.5 ([17], pg. 16). A genus g, n-pointed <u>stable curve</u> $(C, p_1, \ldots, p_n)$ is a compact complex algebraic curve satisfying:

(a) The only singularities of C are simple nodes.

(b) Marked points and nodes are all distinct. Marked points and nodes do not coincide.

(c) $(C, p_1, \ldots, p_n)$ has a finite number of automorphisms.

Throughout, we assume that stable curves are connected. The *genus* of C is the arithmetic genus, or equivalently, the genus of the curve obtained when *smoothing the nodes*.

Remark 1.3.6. Observe that the definition calls the object an *algebraic curve* and not a manifold. It's almost a manifold, but it's not because it can lack smoothness.

From here on, we can talk about $\overline{M}_{g,n}$, the moduli space of stable, genus g, n-pointed curves. The technicalities of whether this space is actually an orbifold or a stack will not concern us in this work.

Proposition 1.3.7. *The dimension of $\overline{M}_{g,n}$ is $3g - 3 + n$.*

Originally, Riemann showed that the dimension of the space without marks is $3g - 3$. The proof goes along the lines of finding the dimensions of a Picard group, a Hurwitz space and a particular Grassmannian. The moduli space can be located in a diagram involving all of these spaces and thus its dimension can be found. There's modern approaches to this count but this is the core of Riemann's original idea. Adding marks just adds a degree of freedom per mark so that we get the extra $+n$ in dimensions.

Let us condense the criterion of finite number of automorphisms into a single, easy-to-spell-out idea which will be of use.

Theorem 1.3.8 ([17], pg. 17). *A stable curve admits a finite number of automorphisms (as in condition c) if and only if every connected component C_i of its normalization with genus g_i and n_i special points satisfies*
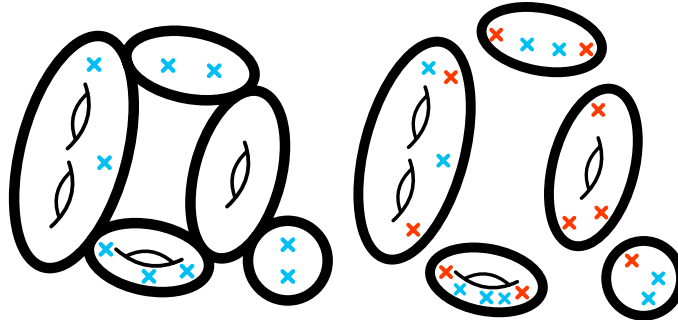
$$2 - 2g_i - n_i < 0.$$

Figure 1.4: Nodal curve and its normalization

Remark 1.3.9. Normalization can be intuitively understood as the process of "ungluing" a variety at its singularities.

Formally, the normalization of a variety X is a non-singular (possibly disconnected) variety $\widetilde{X}$ equipped with a finite birational morphism

$$\nu : \widetilde{X} \to X.$$

This map is an isomorphism over the smooth locus of X but may identify several points in $\widetilde{X}$ over a singular point of X. Thus, we may think of $\widetilde{X}$ as a version of X in which the undue gluings of subvarieties. In the case of curves, normalization replaces each node with two distinct smooth points, resulting in a smooth curve (or collection of curves) marked by the preimages of the nodes.

> We will show that whenever the inequality holds, we have the components of the curve have finitely many automorphisms.
> Starting by considering the $g = 0$ case, we may see that whenever $n \geqslant 3$ we have $\mathbb{P}^1$'s with at least 3 marks. Thus there's no automorphisms besides the identity as any such map should fix three points.
> For the case of $g = 1$, the inequality holds when n is at least 1 as well. DISCUSS Elliptic involution?
> And finally for $g > 2$, look at mse/1680144 Consider hyperelliptic involution.
> These three conditions are summarized in the inequality $2 - 2g_i - n_i < 0$.

Example 1.3.10. Observe that following our motivational example, we get to the three stable curves in $\overline{M}_{0,4}$. Each one of these is a nodal curve where the marks and the node differ. And applying the theorem, we see that each component of the normalization satisfies

$$2 - 0 - 3 = -1 < 0$$

so that we do have the stability condition.

Example 1.3.11. For the case of $M_{1,1}$, we compactify by adding a point representing singular stable curve. This is a one-point $\mathbb{P}^1$ where we attach two points together. The resulting curve has arithmetic genus one. It can be also imagined as "pinching a loop around the torus which doesn't go around the hole". This is once again a stable curve as we have the stability condition:

$$2 - 2(1) - 1 = -1 < 0.$$

Once equipped with the idea of stability we can now calmly talk about $\overline{M}_{g,n}$ as the set of isomorphism classes of *stable curves*. This space can be treated as an orbifold [17], as a Deligne-Mumford stack, but we will not deal with such intricacies here. Instead, we will now delve into the cohomology of this space and talk about the combinatorial tools that will help us further along the road.

1.4 Cohomological classes of the moduli space

The Chow ring

Intuitively, for a non-singular variety V, we define the *Chow ring $A^*(V)$*, whose elements *correspond* to subvarieties of V, and the product reflects the intersection of these subvarieties. The ring is graded by codimension:

$$A^*(V) = \bigoplus_i A^i(V)$$

where $A^i(V)$ consists of classes of subvarieties with codimension i. Ideally, the intersection of a codimension m subvariety X and a codimension n subvariety Y would yield a subvariety of codimension $m + n$.

However, this does not always hold. Imagine a hyperplane H intersected with itself. So there's a complication in defining this product.

Definition 1.4.1. The i-th Chow group $A^i(V)$ of a non-singular variety V consists of equivalence classes of codimension i cycles, where two cycles are equivalent if their difference is a principal divisor, i.e., the zero set of a rational function.

The Chow ring is the direct sum over all Chow groups:

$$A^*(V) = \bigoplus_i A^i(V)$$

The intersection product on the Chow ring is well-defined:

$$[X] \cap [Y] = \sum_{[Z]} i(X, Y; Z)[Z]$$

where $[X]$, $[Y]$, $[Z]$ denote rational equivalence classes of cycles, and $i(X, Y; Z)$ is an *intersection number*, representing the multiplicity of the intersection at Z.

Remark 1.4.2. In the cases when we do have a transversal intersection between X and Y, it holds that

$$\begin{cases} [X] \cap [Y] = [X \cap Y], \\ \operatorname{codim}(X \cap Y) = \operatorname{codim}(X) + \operatorname{codim}(Y). \end{cases}$$

Remark 1.4.3. The Chow ring is related to the cohomology ring via a homeomorphism

$$A^i(V) \to H_{2n-2i}(V) \to H^{2i}(V)$$

where the first map is is the cycle map, and then we apply Poincaré duality. Further exploration of the question as to where the Chow group lies inside the cohomology leads to the Hodge conjecture.

Given the previous, we will indistinguishably call subvarieties *cohomology classes of their respective codimension*. Only the basic idea of what the Chow ring as a set of isomorphism classes of subvarieties is and the idea that intersection of subvarieties corresponds to the product inside the Chow ring is essential.

Chern classes

Similar to the basics of the Chow ring, we will be reviewing the following properties about Chern classes. These are a particular case of characteristic classes which are elements of the cohomology of the base of a vector bundle $E \xrightarrow{\pi} B$. The precise definition of what a Chern class is is not necessary.

Theorem 1.4.4 (Chern classes cheat sheet). *Suppose $E \xrightarrow{\pi} B$ is a rank r vector bundle. We have the following*:

$\diamond$ $c_i(E) \neq 0$ *whenever $0 \leqslant i \leqslant r$.*

$\diamond$ $c_i(E)$ *has degree i in the Chow ring. This means $c_i(E) \in A^i(B)$.*

$\diamond$ $c_0(E) = 1$ *is the fundamental class. It's usually the case that we rescale in order for this to be exactly 1.*

◇ *If we define*

$$c_{tot} = c_0 + c_1 + \cdots + c_r$$

and we have a short exact sequence of vector bundles

$$0 \to F \to E \to Q \to 0,$$

then $c_{tot}(E) = c_{tot}(F) \cdot c_{tot}(Q)$. In particular we have

$$c_1(E) = c_1(F) + c_1(Q).$$

◇ *If E is a line bundle L, i.e. rank 1, then*

$$c_1(L) = [\operatorname{div}(s)],$$

this is the class of the divisor of a meromorphic section.

◇ *In general, <u>Euler class</u> is*

$$e(E) := c_r(E) = [Z(s)] - [P(s)].$$

Here s is a section which in local coordinates may be expressed as

$$s = (s_1, ..., s_r)$$

and so Z represents the class of zeroes, while P is the class of poles of the section.

◇ *Chern classes commute with pullbacks.*

For more, check out this math.se post[1] or here[2].

Dual graphs

We will be describing the cohomology classes of $\overline{M}_{g,n}$ via their combinatorial data. Each curve has a naturally associated dual graph which we will get to following the ideas of Schmitt and van Zelm [14]. The idea of stable graph was initially formalized by Graber and Pandhariphande [9] and then developed further by Yang [16].

Definition 1.4.5. A <u>simple graph</u> G is a pair of sets (V, E) such that

$$G = (V, E), \quad E \subseteq \binom{V}{2} = \{ X \subseteq V : |X| = 2 \}.$$

The set V is called the set of vertices and elements of E are edges.

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[1] math.stackexchange.com/q/989147/
[2] https://rigtriv.wordpress.com/2009/11/03/chern-classes-part-1/

This is equivalent to the following:

◇ E is the graph of a symmetric relation in V.

◇ Schmitt and van Zelm take this characterization to be a map

$$s : E \to \mathrm{Sym}^2 V = {V \otimes V} \Big/ {u \otimes v - v \otimes u}.$$

More generally we can look for graphs which have multiple edges between vertices and loops around one vertex.

Definition 1.4.6. A multigraph is a graph G in which E and its elements can be multisets. This allows for multiple edges and loops respectively.

The Schmitt and van Zelm condition allows this automatically.

Definition 1.4.7. A vertex/edge-labeling by integers is a function from V or E to $\mathbb{Z}_{\geqslant 0}$ respectively.

As our curves will be marked, we would like to denote that in a graph. The way to do this is via a set of *half-edges* or *legs*. Graphs can be defined by only specifying a set of half-edges and declaring certain properties, but we won't pursue that avenue[3]. Following Schmitt and van Zelm we take the idea of half-edges to define stable graphs.

Definition 1.4.8. A stable graph Γ is a tuple of sets and maps:

$$\Gamma = (V, H, L, g, \iota, a, \zeta)$$

where V is the set of vertices, H the set of half-edges and L is the set of legs. We have the following conditions:

(a) The map $g : V \to \mathbb{Z}_{\geqslant 0}$ is a labeling of the vertices to denote their genus.

(b) $\iota : H \to H$ is a fixed-point free involution, intuitively switching ends of a edge.

(c) The map $a : H \to V$ assigns half-edges to vertices.

(d) The map $\zeta : L \to V$ assigns the legs to the vertices.

(e) The orbits of ι form the edges via the induced function by a on E.

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[3]See more in `math.se/2678435`.

(f) And each vertex v satisfies the condition

$$2 - 2g(v) - |a^{-1}(v)| - |\zeta^{-1}(v)| < 0.$$

We may call $n(v) := |a^{-1}(v)| + |\zeta^{-1}(v)|$ the valence[4] of the vertex v. This is number of half-edges and legs incident in v.

Observe immediately the similarity between the last condition in the stable graph definition and the stability condition for curves in theorem 1.3.8. If the definition sounds too technical it's because it is! We should carry in our minds that a stable graph is labeled multigraph which satisfies the stability condition. Let's now associate a graph to a curve.

Definition 1.4.9. The <u>dual graph</u> G of the curve $(C, p_1, \ldots, p_n)$ is built from its normalization N.

◇ The vertices of G correspond to each connected component of N:

$$V(G) = \{ X_i : X_i \quad \text{is a connected component of} \quad N \}.$$

Each vertex has an associated genus g_v equal to the genus of the component X_i.

◇ The half-edges correspond to points in each component mapping to the same node in C.

◇ L corresponds to the marked points.

◇ The genus function g is the genus of X_i.

◇ ι transposes points mapping to the same node.

◇ a and ζ identify to-be nodes and marks with component they lie on.

Example 1.4.10. Consider the following curve in $\overline{M}_{4,9}$ along its normalization. We have omitted the labeling on the markings for now but we have distinguished the special points on each component by coloring nodes red and markings blue. We now build the corresponding dual graph following the aforementioned procedure. In the first graph we have color-coded the information by coloring the edges red and the half-edges blue. More usually than not, we will draw the graphs as the one on the right possibly including labels for the marks.

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[4]Usually in graph theory this would be called the degree of a vertex, but to lose ambiguity when talking about maps, we will call it valence.
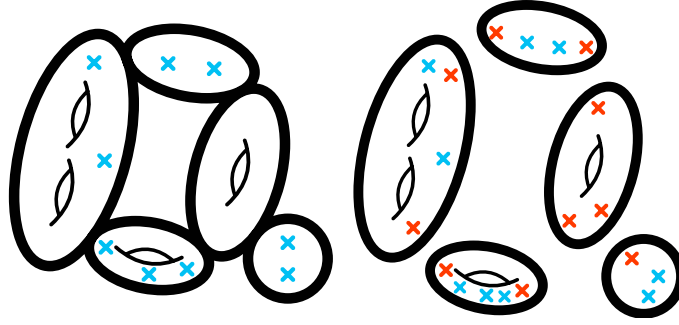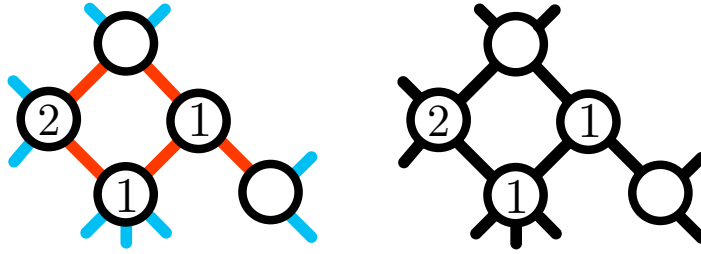
Figure 1.5: Unlabeled nodal curve in $\overline{M}_{4,9}$

Figure 1.6: Dual graph corresponding to the curve in figure 1.5

The following result will come in handy to categorize where in the Chow ring of $\overline{M}_{g,n}$ does a particular graph live.

Theorem 1.4.11. *The codimension of a marked curve C is the number of nodes it has. This is the number of edges that its dual graph has.*

After pondering this result and looking at the previous example we may think of the question about what happens when two curves have the same dual graph. For that, we consolidate in the following definition.

Definition 1.4.12. A <u>stratum</u> in $\overline{M}_{g,n}$ is the closure of the set of all curves with the same topological type. Strata are classified by dual graphs.

These strata are what we will use to organize data inside the Chow ring. For a curve C, we will talk about the class $[C] \in A^*(\overline{M}_{g,n})$ as the cohomology element representing it.

Example 1.4.13. So taking into account the previous curve from example 1.4.10, consider also the next curve. This curve also belongs in the same stratum as the one before.
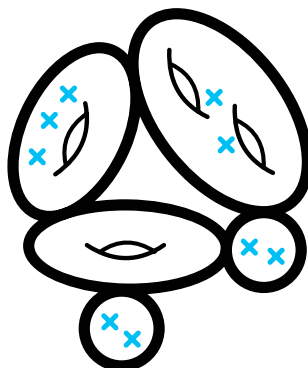
Figure 1.7: Differnt nodal curve in $\overline{M}_{4,9}$

The tautological classes, ring and morphisms

Definition 1.4.14. The boundary of $\overline{M}_{g,n}$ is

$$\partial M_{g,n} = \overline{M}_{g,n} \backslash M_{g,n}$$

and it is the set parametrizing *singular* stable curves.

Cohomology classes in the boundary are called <u>boundary cycles</u> and those of codimension one are called <u>boundary divisors</u>.

Let us explore some of the classes inside $A^*(\overline{M}_{g,n})$.

(a) The fundamental class 1 is the class representing the smooth locus $M_{g,n}$. The dual graph corresponds to a single vertex with genus g and n half-edges. It is also the multiplicative identity of the Chow ring.

(b) The class $\delta_{a,A}$ is the codimension 1 class corresponding to graphs with two vertices, one with genus a and marks labeled by A and another vertex with genus $g - a$ and marks $[n] \backslash A$. Observe that

$$\delta_{a,A} = \delta_{g-a,A^c}.$$

(c) The class δ_{irr} is also a codimension 1 class corresponding to graphs with one unique vertex with all the marks and a loop formed by an edge. The vertex has genus $g - 1$ and the rest of the arithmetic genus is encoded in the loop.

Let us view some examples of such classes.
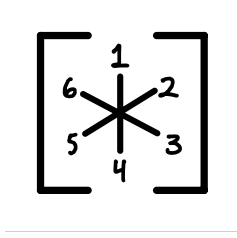
Figure 1.8: Fundamental class representative of $\overline{M}_{0,6}$

Example 1.4.15. In $\overline{M}_{0,6}$, the fundamental class $1 \in A^0(M_{0,6})$ is the class of all curves homeomorphic to a 6-pointed $\mathbb{P}^1$. It can be represented by the dual graph:

The top stratum in $\overline{M}_{0,6}$ contains only one topological type of curve as all other 6-pointed $\mathbb{P}^1$'s will be homeomorphic to this one.

Example 1.4.16. Going into the boundary of $\overline{M}_{0,6}$ we can find two different types of strata of the form $\delta_{0,A}$, for example From the stratum on the left, we can find 9 other
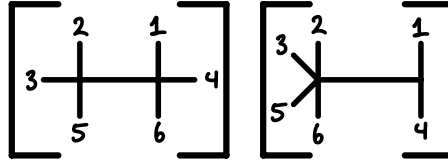
Figure 1.9: Boundary divisors $\delta_{0,\{1,4,6\}}$ and $\delta_{0,\{1,4\}}$ of $\overline{M}_{0,6}$

strata with the same graph design but different labelings. This is because there are

$$\frac{1}{2}\binom{6}{3,3} = \frac{6!}{2\cdot 3!\cdot 3!} = 10$$

ways to label the tree. The factor of $\frac{1}{2}$ accounts for the symmetry in the tree.

For the one on the right we can find $\binom{6}{4,2} = 15$ labelings. Observe that there is no class δ_{irr} in $\overline{M}_{0,6}$.

In fact, there's not an irreducible divisor class until we start climbing up in genus. Before proceeding in this account, let's illustrate a bit of the combinatorics at hand. As we mentioned, the ability of representing a curve via a graph is of great help when enumerating this objects.

Proposition 1.4.17. *There are*

$$\frac{1}{2}\sum_{k=2}^{n-2}\binom{n}{k} = 2^{n-1} - n - 1$$

irreducible boundary divisors in $\overline{M}_{0,n}$.

> **Proof**
>
> A partition into two sets of $[n]$ of size k and $n-k$ can be done in $\binom{n}{k,n-k}$ ways. Observe that this coefficient is precisely $\binom{n}{k}$. We need to divide by 2 because $\binom{n}{k} = \binom{n}{n-k}$. And as we need $2 \leqslant k \leqslant n-2$ for the curve to be stable, we must sum over all of those possibilities.

Example 1.4.18. Consider now the space $\overline{M}_{1,3}$ and let us identify the same kinds of classes here. The fundamental class can be immediately thought of as an elliptic curve with 3 marks. Up-to-topological type, there's two divisors of the form $\delta_{a,A}$. We have

$$\delta_{0,\{a,b\}}, \quad \text{and} \quad \delta_{0,\{a,b,c\}}.$$

These are represented by the graphs:

Figure 1.10: Boundary divisors $\delta_{0,\{a,b\}}$ and $\delta_{0,\{a,b,c\}}$ (marks unlabeled)

Whereas the irreducible divisor appears as follows:

Figure 1.11: Irreducible boundary divisor in $\overline{M}_{1,3}$ (marks unlabeled)

Remark 1.4.19. Recall the example 1.3.11 when we discussed which curve to add to compactify $M_{1,1}$. It is precisely the corresponding δ_{irr} the one which we added.

We will restrict ourselves further inside the Chow ring of $\overline{M}_{g,n}$.

Definition 1.4.20 ([17], Def. 2.6). The minimal family of subrings $R^*(\overline{M}_{g,n}) \subseteq A^*(\overline{M}_{g,n})$ stable under pushforwards by forgetful and gluing maps is called the family of tautological rings of the moduli space of stable curves.

Intuitively, the forgetting and gluing morphisms do what we expect them to do, they either "forget" a marked point or "glue" a couple of points together.

Definition 1.4.21 ([2], pg. 3). The <u>forgetful map</u> is

$$\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$$

and it assigns to a curve $(C, p_1, \ldots, p_{n+1})$ the *stabilization* of the curve $(C, p_1, \ldots, p_n)$. Whereas the <u>gluing map</u> comes in two flavors. A self-gluing

$$\xi : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$$

which takes the $(C, p_1, \ldots, p_{n+1}, p_{n+2})$ into $(C, p_1, \ldots, p_n)$ that has a node in the place where it identified p_{n+1}, p_{n+2}. This adds to the curve one arithmetic genus.

On the other hand, the map

$$\eta : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2}$$

glues two curves $(C_k, p_1, \ldots, p_{n_k+1})$ at the points p_{n_1+1}, p_{n_2+1} creating a nodal curve.

Remark 1.4.22. The forgetful map

$$\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$$

is also known as the <u>universal curve</u> over $\overline{M}_{g,n}$. It has n sections $(\sigma_i)_{i \in [n]}$. The image under the section σ_i of the curve $(C, p_1, \ldots, p_n)$ attaches to C a $\mathbb{P}^1$ at p_i and at $0 \in \mathbb{P}^1$ and relabels 1 to p_i and ∞ to p_{n+1}.

Example 1.4.23. The fundamental class of $\overline{M}_{g,n}$ is mapped via the section σ_i to the divisor $\delta_{0,\{i,n+1\}}$ for example.

Remark 1.4.24. Observe that when forgetting a mark in a $\mathbb{P}^1$-tail with 2 marks we destabilize the curve as per the condition in theorem 1.3.8. This leads into a process called <u>stabilization</u>. This only occurs for $\mathbb{P}^1$ components of our curve with less than three special points. The intuitive idea of what happens is that we collapse the component back into a point.

Example 1.4.25. Let's apply the forgetful map to a curve in $\overline{M}_{4,3}$ with marks a, b and c. Applying π_a leaves us with a destabilized $\mathbb{P}^1$ component which contracts to a node joining the two components which it was connected to. On the other hand, if we forget the mark b, then the $\mathbb{P}^1$ tail gets contracted into the node and the mark c takes the place of the node.
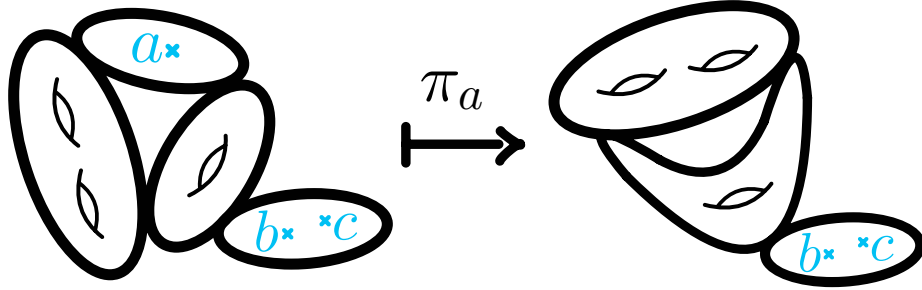
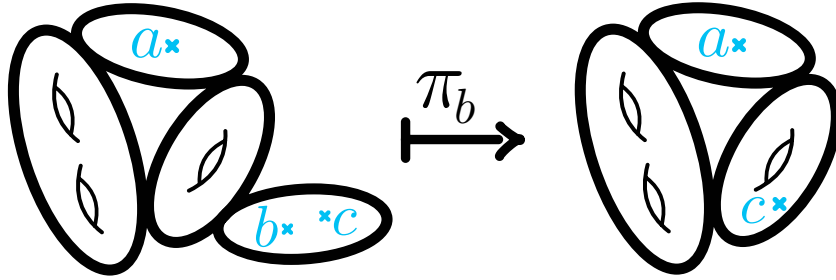Figure 1.12: Marked curve in $\overline{M}_{4,3}$ and its stabilization after π_a

Figure 1.13: Same curve in $\overline{M}_{4,3}$ but stabilized after π_b

Remark 1.4.26. One could think that an attached elliptic curve should get contracted as they have no visible marks, but they do have a special point, the node! It is in the case of pseudo-stable curves that we cannot have elliptic tails.

Further information on stabilization can be found in [13] section 1.3 p. 23. But as long as we have the intuitive idea that components get contracted, we're good to go.

Example 1.4.27. On the side of the self-gluing map observe that

$$\delta_{\text{irr}} = \frac{1}{2}\xi_* 1.$$

The half coefficient comes in for degree reasons, but intuitively this makes sense! We thus glue the fundamental class of $M_{0,5}$ to get δ_{irr} in $\overline{M}_{1,3}$. The red point in the image is just to remind us that that is the point where the two original ones mapped to.

Example 1.4.28. Finally consider gluing two different curves. The fundamental classes of $\overline{M}_{0,4}$ and $\overline{M}_{1,1}$. Formally what we're doing here is

$$\eta(1,1) = \delta_{0,\{a,b,c\}} = \delta_{1,\varnothing} \in \overline{M}_{1,3}.$$

Notice that both source curves had no codimension whereas the resulting one does have codimension!
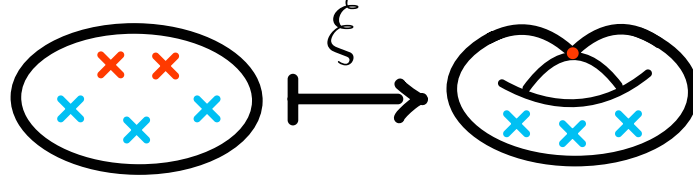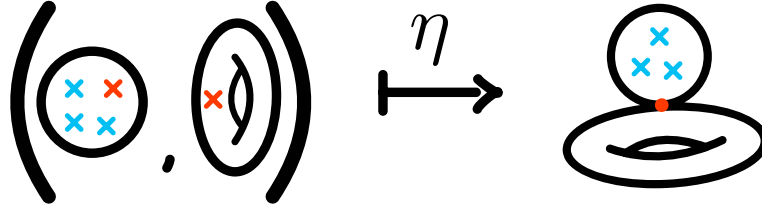
Figure 1.14: Gluing two red marks into a node

Figure 1.15: Gluing two curves into one

Remark 1.4.29. Both versions of the gluing map add one codimension because they're adding one node to a new curve.

Now that we're equipped with this relatively intuitive classes we are ready to proceed into a bit more abstract classes of the moduli space.

ψ and λ classes

The quick reference for this part is [4] section 2.2.

Definition 1.4.30. The cotangent line bundles on $\overline{M}_{g,n}$ are defined by

$$\mathbb{L}_i = \sigma_i^*(\omega_\pi)$$

where ω_π is the relative dualizing sheaf of the universal curve and σ_i is the section corresponding to the mark p_i. As the name suggests, the fiber of $\mathbb{L}_i$ over $[(C, p_1, \ldots, p_n)]$ is the cotangent line of C at p_i.

The ith psi-class ψ_i is the first Chern class of $\mathbb{L}_i$

$$\psi_i = c_1(\mathbb{L}_i) \in A^1(\overline{M}_{g,n}) \quad \text{for} \quad i \in [n].$$

Definition 1.4.31. The Hodge bundle on $\overline{M}_{g,n}$ is a rank g vector bundle and is defined as

$$\mathbb{E} = \pi_*(\omega_\pi).$$

Its fiber over a point $[(C, p_1, \ldots, p_n)]$ is $\Gamma(C, \omega_C)$is this a typo in [4], the g-dimensional space of global sections of the dualizing sheaf.

The jth lambda-class λ_j is the jth Chern class of $\mathbb{E}$

$$\lambda_j = c_j(\mathbb{E}) \in A^j(\overline{M}_{g,n}) \quad \text{for} \quad j \in [g].$$

Remark 1.4.32. Explaining a bit more on what the fibers of the Hodge bundle are, we have the following:

⋄ For a smooth curve, it's holomorphic 1-forms.

⋄ When the curve is nodal, the fibers are 1-forms at nodes such that the residues on both sides of the node are opposite. Recall that the residue theorem for complex surfaces states that sums of residues on both sides of a contour are equal.

Now returning to graphs, strata of $\overline{M}_{g,n}$ can support ψ and λ classes. In order to represent them in graphs we will add *decorations*, ψ-classes on half-edges and λ classes on vertices.

Definition 1.4.33. A decorated stable graph Γ is a stable graph $\underline{\Gamma}$ with the additional data of a monomial

$$\theta_v = \prod_{i=1}^{n(v)} \psi_i^{e_i} \prod_{j=1}^{m} \lambda_j^{f_j}$$

chosen for each vertex v.

Example 1.4.34. In order to exemplify the decorated graphs, let's view a couple of formulae which will become useful the further we delve into this subject. If $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is the map which forgets p_{n+1}, then

$$\pi^*(\psi_i) = \psi_i - \delta_{0,\{i,n+1\}} \quad \text{equivalently} \quad \psi_i = \pi^*(\psi_i) + \operatorname{Im} \sigma_i.$$

Here the ψ_i inside the pullback is supported on $\overline{M}_{g,n}$ whereas the other one in $\overline{M}_{g,n+1}$. Graphically this would be represented by two stable graphs, one decorated and the other one not.

Example 1.4.35. For lambda-classes, let's consider the Hodge bundle restricted to a divisor $\delta_{a,A}$ on $\overline{M}_{g,n}$. There exist projections

$$p_1 : \overline{M}_{g,n} \to \overline{M}_{a,|A|}, \quad \text{and} \quad p_2 : \overline{M}_{g,n} \to \overline{M}_{g-a,n-|A|}$$

$$\pi^* \left(\psi_i \right) = \left[\begin{array}{c} {}^{n}{}_{1}{}^{2}{}_{3} \\ g \\ \psi_i \end{array} \right] + \left[\begin{array}{c} {}^{i} \\ \\ {}_{n+1} \end{array} \succ\!\!-\!\!g \right]$$

Figure 1.16: Psi-class forgetful formula

$$\lambda_1 \cdot \left[a - g-a \right] = \left[\overset{\lambda_1}{a - g-a} \right] + \left[a - \overset{\lambda_1}{g-a} \right]$$

Figure 1.17: Lambda-class distribution

through which we can pullback the Hodge bundles of the smaller moduli spaces and thus get

$$\mathbb{E}|_{\delta_{a,A}} = p_1^* \left(\mathbb{E}_{a,|A|} \right) \oplus p_2^* \left(\mathbb{E}_{g-a,n-|A|} \right).$$

We can express this graphically for the first lambda-class.

   We haven't formally described how to compute the product of classes in $\overline{M}_{g,n}$ but this only requires knowing how to compute first Chern classes. Which we can do by using the cheat sheet theorem 1.4.4.

   As before, deep understanding or knowledge of what the relative dualizing sheaf is is not necessary to proceed. It's just important to know that these are special cohomology classes which show up when dealing with string theory as in the case of Witten's conjecture. More on psi-classes can be found in Zvonkine's [17] also in the viewpoint of differential forms which we don't touch here. Joachim Kock's notes on psi-classes [12] also contain valuable information. For now, let us put into practice the material which we've begun to describe.

1.5   The intersection product on $\overline{M}_{g,n}$

As we have discussed before, the rich combinatorial structure of the cohomology classes allows us to enumerate properties of the curves we're talking about easily. The intersection of cohomology classes corresponds to the product in the Chow ring. So if

we wish to enumerate properties of intersections, we must develop a way to multiply classes which respects the combinatorial structure.

Definition 1.5.1 ([14], pg. 11). Let Γ_1, Γ_2 be two stable graphs with the same set of legs L. A $\underline{\Gamma_2\text{-structure on }\Gamma_1}$ is a morphism

$$f : \Gamma_1 \to \Gamma_2, \quad f = (\alpha, \beta, \gamma)$$

such that

$$\alpha : V_1 \twoheadrightarrow V_2, \quad \beta : H_2 \hookrightarrow H_1, \quad \gamma : H_1 \backslash \operatorname{Im} \beta \to V_2$$

satisfying

(a) the map β commutes with involutions, i.e. $\beta \iota_2 = \iota_1 \beta$,

(b) the map α respects leg assignments, i.e. $\alpha \zeta_1 = \zeta_2$,

(c) If $h \in H_1 \backslash \operatorname{Im} \beta$ and $a_1(h) = v$ then $\alpha(v) = \gamma(h)$,

(d) and if $v \in V_2$, then

$$(\alpha^{-1}(v), \gamma^{-1}(v), \beta(a_2^{-1}(v)) \cup \zeta_2^{-1}(v), g_1, \iota_1, a_1, \zeta)$$

is a stable graph of genus $g(v)$ where g_1, a_1 and ι_1 are restricte to the appropriate subsets and ζ is defined by ζ_1 on $\zeta_2^{-1}(v)$ and by a_1 on $\beta(a_2^{-1}(v))$.

When such a map exists, we can say either

$\diamond$ Γ_1 is a $\underline{\text{specialization}}$ of Γ_2, or

$\diamond$ Γ_2 is a $\underline{\text{contraction}}$ of Γ_1.

Remark 1.5.2. In other words, when contracting Γ_1 into Γ_2, we are doing a similar process as to finding a minor Γ_1 with extra steps.

We can also extend the definition to say that $\widetilde{\Gamma}$ has a $(\Gamma_1, \dots, \Gamma_n)$ structure whenever $\widetilde{\Gamma}$ contracts into any of the Γ_i.

In order to multiply two classes via their graphs, we will look for graphs which contract into both factors of the product.

Definition 1.5.3. We call an (A, B)-structure on Γ a $\underline{\text{generic structure}}$ if every edge of Γ corresponds to an edge of A or an edge of B. In other words if

$$\beta_A(H_A) \cup \beta_B(H_B) = H_\Gamma.$$

An (A, B)-edge will be an edge which is identified with both an edge of A and one of B.

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Now we all the tools ready to define the multiplication of graphs which combinatorially explains multiplication of strata.

Definition 1.5.4 ([16], pg. 5)**.** Let G, H be decorated stable graphs and $\Gamma(G, H)$ the set of graphs with generic (G, H) structure.[5] Then the product of G and H is defined as

$$G \cdot H = \sum_{A \in \Gamma(\underline{G}, \underline{H})} \frac{1}{|\operatorname{Aut}(A)|} F_A(G, H) \cdot A$$

where we have

$$
\begin{cases}
F_A(G, H) = \left(\sum_{v \in V_G} f_A(G, v) \right) \left(\sum_{v \in V_H} f_A(H, v) \right) \prod_{e=(h_1, h_2)} (-\psi_{h_1} - \psi_{h_2}) \\
f_A(G, v) = \prod_{i=1}^{n} \psi_{\beta(i)}^{e_i} \left(\sum_{w \in \alpha^{-1}(v)} \lambda_{jw} \right)^{f_j}
\end{cases}
$$

and the last product of F is taken through the edges identified with both an edge of G and an edge of H.

As before, definitions come to life via examples.

Example 1.5.5. Let's begin with a sanity check. Consider the product in $\overline{M}_{0,4}$ of the fundamental class 1 with $\delta_{0,ac}$. Let's proceed by following the formula.

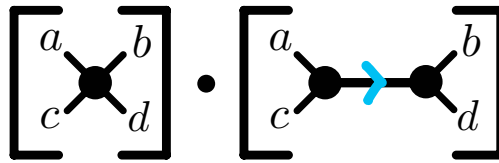
Figure 1.18: Trivial multiplication in $\overline{M}_{0,4}$

(a) We have found the graphs representing the classes in question and now we look for graphs which generically contract into both of the factors. Observe that only $\delta_{0,ac}$ contracts into both itself and the fundamental class.

(b) The edge of $\delta_{0,ac}$ is colored so that we can look at which edges get identified with it when contracting. As there's no edge in 1, the last product is empty.

---

[5]i.e. the set of all graphs which contract into G and H generically.

(c) As there's no psi- nor lambda-classes, f_A's are also trivial.

(d) Lastly, the graph $\delta_{0,ac}$ has no non-trivial automorphisms.

Therefore it happens that:

$$1 \cdot \delta_{0,ac} = (1 \cdot 1 \cdot \delta_{0,ac}) = \delta_{0,ac}$$

as expected.

Remark 1.5.6. Recall that the intersection product adds the codimensions of the factors. As 1 has codimension 0 and $\delta_{0,ac}$ has codimension 1, thus the result should have codimension 1.

Observe that this means that the resulting graphs can't have more edges than the total number edges in the factors, in this case they would also be non-generic!

Another observation is that the graphs can indeed have less edges than the expected codimension. This is because we can have multiples of psi- and lambda-classes to even out the codimension.

Figure 1.19: Divisor squared in $\overline{M}_{0,4}$

Example 1.5.7. Consider now the product $\delta_{0,ac}^2$ in $\overline{M}_{0,4}$. As this boundary divisor has codimension 1, its square should have codimension 2. As $\overline{M}_{0,4}$ has only one dimension, we expect the product to be empty. The edges of the factors are painted to distinguish them. Once again let's follow the formula with care.

Figure 1.20: Factors identified with colors

(a) We look for all the graphs contracting into $\delta_{0,ac}$ once again. There's only $\delta_{0,ac}$ once more. This class has codimension 1, so we must multiply a psi-class to level the codimension.

(b) The edge in the result is bicolored to indicate that it is identified with the edges in the factors. Here we label the half-edges as $\bullet$ and $*$ so that we must multiply a psi-class $(-\psi_\bullet - \psi_*)$. This is the only difference with the previous calculation.

(c) The graph has no non-trivial automorphisms and no decorations.

Figure 1.21: Multiplication result with psi-class

The result is then

$$\delta_{0,ac} \cdot (-\psi_\bullet - \psi_*)$$

which we can see to be zero by a codimension count or as follows. The stratum in question can be seen as the image of a gluing map:

$$\eta : \overline{M}_{0,\{a,c,\bullet\}} \times \overline{M}_{0,\{b,d,*\}} \to \overline{M}_{0,\{a,...,d\}},$$

and when pulling back the psi-classes via the gluing map to the corresponding $M_{0,3}$'s we get the same psi-classes. These are 0 in $M_{0,3}$ so that the result is also 0.

We climb up a couple of dimensions into $\overline{M}_{0,6}$ in the next example. This next classes will now have psi-classes multiplied and won't be zero!

Example 1.5.8. Consider the intersection product in $\overline{M}_{0,6}$:

$$\delta_{0,\{a,b,c\}} \delta_{0,\{ab\}}^2,$$

The expected codimension of the result is 3 as every factor contributes 1 codimension.

We begin by multiplying a copy of $\delta_{0,\{ab\}}$ with $\delta_{0,\{a,b,c\}}$. We can only find one graph with contracts into these two. It's a codimension two class which has no non-trivial automorphisms nor decorations so we now multiply by the other divisor.
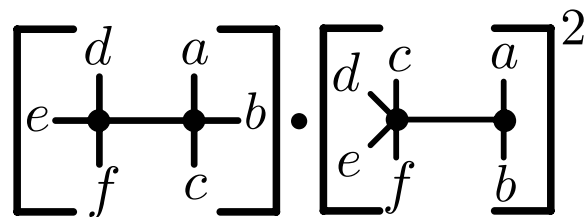
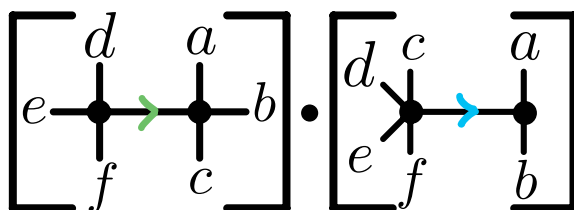Figure 1.22: Multiplication of three divisors

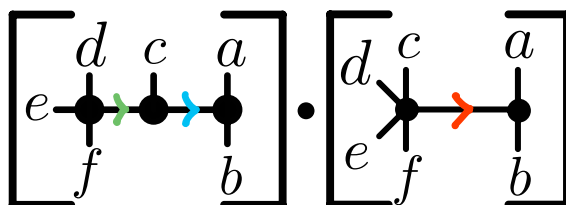Figure 1.23: First multiplication
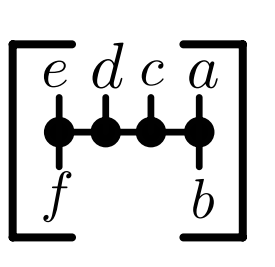

Figure 1.24: Second multiplication

Figure 1.25: Non-generic contracting graph

We expect the resulting graphs to have three edges as that would have codimension 3. Creating such a graph is impossible, as any possible graph would non-generically contract into the factors.

Such graph has three edges but *they can't be colored* to respect the contraction.

When contracting edges, the colors disappear so that we would have to leave an edge uncolored to respect the coloring.

So the only possibility is the same graph that we just obtained but multiplied by a psi-class. One of the edges is bicolored and labeling the vertices of the corresponding edge as $\bullet$ and $*$ we get

$$\delta_{0,\{ab\}}\delta_{0,\{a,b,c\}}(-\psi_\bullet - \psi_*)$$

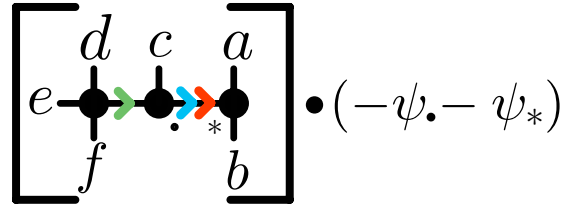This class has the correct codimension as the graph has two edges and the psi-class adds another codimension.

Figure 1.26: Final result of product

Figure 1.27: Irreducible divisor squared

Example 1.5.9 ([17], Exercise 2.4). Recall the irreducible divisor of stable curves with one non-separating node $\delta_{\text{irr}} \in \overline{M}_{1,n}$. We will show that $\delta_{\text{irr}}^2 = 0$. Following the idea from figure 1.20 in the divisor squared example for $\overline{M}_{0,4}$, we separate the product into two factors and color.

There's two kinds of graphs which contract into this one. First itself, with the edge bicolored. Due to this we have to add a psi-class because the edge is identified with both factors.

Besides this one, we have $2^n - 2$ other possible graphs representing curves with two components with the marks distributed among them. These are indexed by $A \subseteq \{n\}$ containing at least one element. On one component the curve has $|A|$ marks, and on
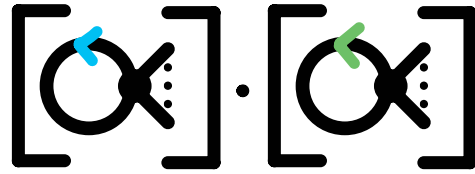
Figure 1.28: Separated product with edges colored
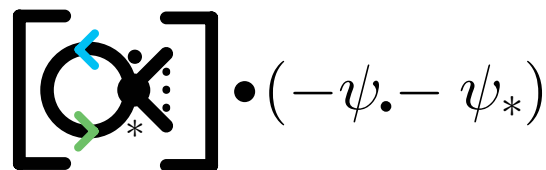
Figure 1.29: Irreducible divisor with psi-class to correct codimension

the other side, $n - |A|$. The components are joined by two nodes so it has the correct codimension. So in total, the sum of these classes gives us the result of squaring δ_{irr}.

Figure 1.30: Codimension 2 strata indexed by A

We get a factor of 2 in each of the terms accounting for the automorphisms which we turn to the other side of the equation as a half factor.check!

Figure 1.31: Final sum with all strata

Why is this zero in the end???

1.6 Integration

Fulton and Pandharipande [7] give us the definition of the integral that we will use.
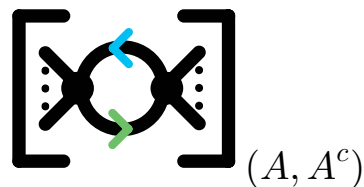
Definition 1.6.1 ([7] pg. 2). For a complete variety, $c \in A^*(X)$ and $\beta \in A_k(X)$ then

$$\int_\beta c = \deg(c_k, \beta)$$

where c_k is the component of c in $A^k(X)$ and (c_k, β) is the evaluation of c_k on β giving us a zero cycle. When V is a closed, pure-dimensional subvariety of X, then we write

$$\int_V c \quad \text{instead of} \quad \int_{[V]} c.$$

In our case, this can be seen as the pushforward of the constant map from $\overline{M}_{g,n}$ to a point.

$$A^*(\overline{M}_{g,n}) \xrightarrow{c_*} A^*(\boldsymbol{pt.}) \quad \Rightarrow \quad \int_{\overline{M}_{g,n}} := c_*.$$

The string equation

Example 1.6.2. Consider the integral

$$I = \int_{\overline{M}_{0,4}} \psi_1$$

we apply the forgetful map identity for the psi-class (as seen in example 1.4.34) to get

$$I = \int_{\overline{M}_{0,4}} \pi_2^*(\psi_1) + \operatorname{Im} s_1 = \int_{\overline{M}_{0,4}} 0 + \delta_{0,\{1,4\}}.$$

Coupling this divisor against the fundamental class gives us a zero-cycle, and as there's no coefficients we get a 1. Therefore

$$I = \int_{\overline{M}_{0,4}} \psi_1 = 1$$

We can now continue into integrals involving higher powers of psi-classes. For this, let's take into account the following diagram

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$$
\begin{array}{ccc}
\overline{M}_{g,n+1} & \quad & A^*(\overline{M}_{g,n+1}) \quad c_* = \int_{\overline{M}_{g,n+1}} \\
\pi \downarrow \quad \xrightarrow{c} \mathbf{pt.} \quad \Rightarrow \quad \pi_* \downarrow \quad A^*(\mathbf{pt.}) \\
\overline{M}_{g,n} \quad \xrightarrow{\tilde{c}} & & A^*(\overline{M}_{g,n}) \quad \tilde{c}_* = \int_{\overline{M}_{g,n}}
\end{array}
$$

Observe that via the forgetful map, we may integrate classes from the upper moduli space by pushing them forward and then integrating on the moduli space on the bottom. This means that

$$
\int_{\overline{M}_{g,n+1}} \alpha = \int_{\overline{M}_{g,n}} \pi_* \alpha, \quad \text{for} \quad \alpha \in A^*(\overline{M}_{g,n+1}).
$$

To work with pushforwards and psi-classes involving pullbacks, we must use the projection formula for the Chow ring.

Proposition 1.6.3. *Given a proper map $f : X \to Y$ between nonsingular algebraic varieties and cycle classes $\alpha \in A^*(X), \beta \in A^*(Y)$ we have*

$$
f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta.
$$

The following lemma for reduction in power of the psi-class also comes in handy.

Lemma 1.6.4. *For a forgetful map π_j and a psi-class ψ_i, we have*

$$
\psi_i^k = (\pi_j^* \psi_i)^{k-1}(\pi_j^* \psi_i + \delta_{0,\{ij\}}), \quad k \in \mathbb{Z}_{\geq 0}.
$$

Proof

We may first apply the projection formula 1.4.34 for the psi-class

$$
\psi_i^k = \psi_i(\pi_j^* \psi_i + \delta_{0,\{ij\}})^{k-1}.
$$

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Now we expand the binomial and distribute the outside psi-class as follows

$$= \psi_i \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} (\pi_j^* \psi_i)^\ell (\delta_{0,\{ij\}})^{k-1-\ell}$$

$$= \left(\psi_i \delta_{0,\{ij\}}^{k-1} + (k-1)\psi_i \delta_{0,\{ij\}}^{k-2} \pi_j^* \psi_i + \dots \right.$$

$$\left. + (k-1)\psi_i \delta_{0,\{ij\}}(\pi_j^* \psi_i)^{\ell-1} + \psi_i(\pi_j^* \psi_i)^{k-1} \right)$$

$$= \psi_i(\pi_j^* \psi_i)^{k-1}$$

$$= (\pi_j^* \psi_i)^{k-1}(\pi_j^* \psi_i + \delta_{0,\{ij\}})$$

Observe that all the terms save the last one have a multiplication $\psi_i \delta_{0,\{ij\}}^\ell$ with ℓ between 1 and $k-1$. This multiplication gives a ψ_i on $M_{0,\{*,i,j\}} \simeq M_{0,3}$ which is zero.

Observe now that the power of the psi-class has given us something of the form $\alpha \cdot f^*(\beta)$ where

$$\alpha = \pi_j^* \psi_i + \delta_{0,\{ij\}}, \quad \text{and} \quad \beta = \psi_i^{k-1}$$

and we may pull the power inside the π_j^* because its a ring homomorphism, unlike the pushforward which is just a module morphism.

This term begs to be put together with a π_* to apply the projection formula, and we have a readily accisble π_* when using the integral diagram!

Example 1.6.5. Consider the integral

$$I = \int_{\overline{M}_{0,6}} \psi_3^5 \psi_4^7.$$

Forgetting the sixth mark we may change the integral via the diagram to

$$I = \int_{\overline{M}_{0,5}} \pi_{6*}(\psi_3^5 \psi_4^7).$$

Now let's use the previous results to expand that pushforward.

$$\pi_{6*}(\psi_3^5 \psi_4^7)$$

$$= \pi_{6*}\left[(\pi_6^* \psi_3)^4 (\pi_6^* \psi_3 + \delta_{0,\{36\}})(\pi_6^* \psi_4)^6(\pi_6^* \psi_4 + \delta_{0,\{46\}}) \right]$$

$$= \pi_{6*}\left[\left((\pi_6^* \psi_3)^5 + (\pi_6^* \psi_3)^4 \delta_{0,\{36\}}\right) \left((\pi_6^* \psi_4)^7 + (\pi_6^* \psi_4)^6 \delta_{0,\{46\}}\right) \right]$$

$$= \pi_{6*}\left[\pi_6^* \left(\psi_3^5 \psi_4^7\right) + \delta_{0,\{46\}} \pi_6^* \left(\psi_3^5 \psi_4^6\right) \right.$$

$$\left. + \delta_{0,\{36\}} \pi_6^* \left(\psi_3^4 \psi_4^7\right) + \delta_{0,\{36\}} \delta_{0,\{46\}} \pi_6^* \left(\psi_3^4 \psi_4^6\right) \right]$$

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We may separate terms using the fact that the pushforward is a module homomorphism and notice that $\delta_{0,\{36\}}\delta_{0,\{46\}} = 0$ so that we may apply the projection formula.

$$\pi_{6*}\left[1 \cdot \pi_6^*\left(\psi_3^5\psi_4^7\right)\right] + \pi_{6*}\left[\delta_{0,\{46\}}\pi_6^*\left(\psi_3^5\psi_4^6\right)\right] + \pi_{6*}\left[\delta_{0,\{36\}}\pi_6^*\left(\psi_3^4\psi_4^7\right)\right]$$
$$=\pi_{6*}\left[1\right]\left(\psi_3^5\psi_4^7\right) + \pi_{6*}\left[\delta_{0,\{46\}}\right]\left(\psi_3^5\psi_4^6\right) + \pi_{6*}\left[\delta_{0,\{36\}}\right]\left(\psi_3^4\psi_4^7\right)$$

The pushforward of the fundamental class is zero and for the divisors we do get the fundamental classes in $\overline{M}_{0,5}$ after stabilizing. Thus our integral has become two over one dimension less, and the integrand has lost one power. We have

$$\int_{\overline{M}_{0,6}} \psi_3^5\psi_4^7 = \int_{\overline{M}_{0,5}} \psi_3^5\psi_4^6 + \psi_3^4\psi_4^7$$

Continuing this example leads us to the general result. This serves as a guide for the inductive step in proving the string equation formula.

Theorem 1.6.6. *Suppose I is a multi-index and $\underline{\psi}$ is an array of psi-classes, then*

$$\int_{\overline{M}_{0,n}} \underline{\psi}^I = \binom{n-3}{I}.$$

In the discussion of the previous result, we have used a couple of properties about pushforwards and pullbacks which should be enumerated.

Remark 1.6.7. If $f : X \to Y$ is a flat and proper map, then the pullback of f is a map

$$f^* : A^*(Y) \to A^*(X)$$

such that

$\diamond$ respects codimension grading,

$\diamond$ is a ring homomorphism, and

$\diamond$ generically satisfies $f^*(Z) = [f^{-1}(Z)]$.

This last property may not occur at nodes for example. Meanwhile, the pushforward is a map

$$f_* : A_*(X) \to A_*(Y)$$

which

$\diamond$ respects dimension grading,

⋄ is a module homomorphism, and

⋄ generically satisfies $f_*(Z) = [f(Z)]$. We can be more explicit here to say when things happen differently.

    (a) If $\dim f(Z) < \dim Z$, then $f_*(Z) = 0$.

    (b) If $f|_Z : Z \to f(Z)$ is finite of degree n, then $f_*(Z) = n\,[f(Z)]$.

Chapter 2

Equivariant Cohomology and Localization

2.1 Introduction to equivariant cohomology

Manifolds and varieties usually don't come by themselves, some have lots of symmetries. These can be expressed by a group action on the manifold. We would like a cohomology theory which retains information on the group action!

Example 2.1.1 (A naïve approach)**.** Consider the S^1 action $\mathbb{P}^1$ given by $u{\cdot}z = uz$. This action has two fixed points, 0 and ∞. Observe also that

$$u{\cdot}z = z \iff u = 1.$$

If we were to define the a cohomology which retains information on the group action (equivariant cohomology), we could say

$$H^*_{S^1}(\mathbb{P}^1) := H^*\left(\left.\mathbb{P}^1\middle/S^1\right.\right).$$

However the quotient set $\left.\mathbb{P}^1\middle/S^1\right.$ is the same as a closed interval which means it has trivial cohomology.

Instead of considering the cohomology of the quotient M/G, which doesn't retain information on the group action, we should look for an alternive which does.

The Borel construction

The main idea for this concept is that homotopy equivalent spaces have the same cohomology. Suppose G acts on M, let us create a space EG, a *classifying space*, with the following properties:

(a) The right action $EG{\cdot}G$ is free. ($\forall x(\mathrm{Stab}(x) = 0)$)

(b) EG is contractible.

(c) There exists a unique EG up to homotopy. (EG satisfies a universal property in a category of G-spaces)

This sounds a bit risky to ask, because questions may arise. But let's avoid them for now, instead observe that

$$M \times EG \simeq M$$

as EG is contractible!

Definition 2.1.2. We call the <u>orbit space</u> of M the quotient

$$M_G := {}^{M \times EG}\big/_{(g{\cdot}x, y) \sim (x, y{\cdot}g)}.$$

From this we define the <u>equivariant cohomology</u> of M as

$$H_G^*(M) := H^*(M_G).$$

Example 2.1.3 (Cohomology of a point). We know that the usual cohomology of a point is trivial, but let's check two examples to see what changes.

(a) First consider the (trivial) action of $\mathbb{Z}$ on a point. In this case we have

$$E\mathbb{Z} = \mathbb{R} \quad \text{with} \quad x{\cdot}n = x + n.$$

This is a free action and $\mathbb{R}$ is contractible. Find the classifying space isn't very bad:

$$\boldsymbol{pt}{\cdot}_{\mathbb{Z}} = {}^{\mathbb{R}}\big/_{x \sim x + n} \simeq S^1$$

so that

$$H_{\mathbb{Z}}^*(\boldsymbol{pt}{.}) = H^*(S^1) = {}^{\mathbb{Z}[t]}\big/_{t^2}.$$

(b) Now let's take a bigger group, say $U(1)$, but for our purposes let's call it T as in torus. The classifying space here is

$$ET = \mathbb{C}^\infty\backslash\{0\}, \quad \text{with} \quad \alpha{\cdot}\underline{z} = (\alpha z_i)_i.$$

The action takes a sequence of complex numbers and scalar-multiplies it by $\alpha \in T$. This action is free, and we may see that $\mathbb{C}^\infty\backslash\{0\} \simeq S^\infty$. The infinite sphere is

contractible by arguments out of my scope. And certainly, this classifying space is unique. But now, the quotient in question is

$$\boldsymbol{pt.}_T = \left.\mathbb{C}^\infty\backslash\{\,0\,\}\middle/\underline{z} \sim \alpha\underline{z}\right. \simeq \mathbb{P}^\infty.$$

The cohomology now is

$$H_T^*\boldsymbol{pt.} = H^*\mathbb{P}^\infty = \mathbb{C}[t].$$

From this example we can extend the calculation to see that for an n-dimensional torus T^n we have

$$H_{T^n}^*\boldsymbol{pt.} = H^*(\mathbb{P}^\infty)^n = \mathbb{C}[t_1,\ldots,t_n]$$

by the Künneth formula.

There are questions which arose during the study of classifying spaces which we didn't have time to answer. But they surely have been studied.

Question. What happens when G is a symmetric group S_n, or a finite group $\mathbb{Z}/n\mathbb{Z}$? Even more, what if G is a matrix group, or an exceptional group such as the Mathieu group[1]?

One can see that the idea of constructing the cohomology of the orbit space goes haywire as soon as our space is not a point. For $\mathbb{P}^1$ one has to find

$$H^*\left(\left.T^2 \times \mathbb{P}^2\middle/\sim\right.\right)$$

which becomes unsurmountably hard.

We use another theorem to find the cohomology of projective space. This is Grothendieck's approach, the idea is that vector bundles

$$E \xrightarrow{\pi} B$$

can be associated to projective bundles $\mathbb{P}(E)$ whose typical fiber is isomorphic to $\mathbb{P}^{r-1}$. $\mathbb{P}(E)$ itself has a line bundle $\mathcal{O}(1)$ leading us to a Chern class

$$c_1(\mathcal{O}(1)) \in A^1(\mathbb{P}(E))$$

which is not in $A^1(B)$, we thus wish to relate $A^1(\mathbb{P}(E))$ with $A^1(B)$.

---

[1] A good reference for this is Classifying Spaces of Sporadic Groups by Benson and Smith.

Call $H = c_1(\mathcal{O}(1))$, the hyperplane class, to simplify notation. The Chow ring $A^*(P(E))$ forms a free, rank r $A^*(B)$-module with basis $\{\,1, H, \ldots, H^{r-1}\,\}$. This modularity arises from a similar reasoning to the construction of the Chow ring of $\mathbb{P}^n$, which corresponds to the case when B is a point. Now the element H^r can be expressed as a linear combination of the basic elements:

$$H^r = c_1 H^{r-1} - c_2 H^{r-2} + \cdots \pm c_r, \quad \text{for some} \quad c_i \in A^i(B).$$

These coefficients are the Chern classes in $A^*(B)$! We find the Euler class $e = c_r$ once again. It is represented by a cycle of the zeroes of a section of E. The zeroes of that section will generally have codimension r. Via this relation we can describe the cohomology of the projective bundle.

Theorem 2.1.4. *Suppose* $\mathbb{P}(E) \to B$ *is the rank* $r-1$ *projective bundle associated to* $E \xrightarrow{\pi} B$, *a vector bundle of rank* r. *Then the cohomology of* $\mathbb{P}(E)$ *is given by*

$$H^*(\mathbb{P}(E)) \simeq H^*(B)[H] \Big/ \sum_{k=0}^{r} c_k(E) H^{r-k}.$$

More importantly, this result still holds at the level of equivariant cohomology if we're talking about equivariant line bundles.

Example 2.1.5. In the case where we have $B = \boldsymbol{pt.}$, then the aforementioned projective bundle becomes $\mathbb{P}^r \to \boldsymbol{pt.}$ and the equivariant cohomology is thus

$$H_T^*(\mathbb{P}^r) \simeq \mathbb{C}[t_1, \ldots, t_{r+1}][H] \Big/ \prod_{i=1}^{r+1}(H - t_i).$$

2.2 Atiyah-Bott localization theorem

Theorem 2.2.1 (Atiyah and Bott, 1984)**.** *If* $G \cdot M$ *is an action and* $F_k \subseteq M$ *are the fixed loci of the action* $G \cdot F_k = F_k$, *then there exists an isomorphism of cohomologies*

$$H_G^*(M) \simeq \bigoplus_k H_G^*(F_k)$$

where the inclusion maps $i_k : F_k \rightarrowtail M$ *induce the morphisms*:

$$\underline{i}^* : H_G^*(M) \to \bigoplus_k H_G^*(F_k),$$

component-wise this is the pullback of each i_k. *And on the other direction it's*

$$\frac{i_*}{e(N_{\bullet \mid M})} : \oplus H_G^*(F_k) \to H_G^*(M),$$

where $N_{Y \mid X}$ *is the normal bundle* $Y \subseteq X$.

To say that we're using a localization technique to find cohomology is to apply the Atiyah-Bott theorem.

Example 2.2.2 (Projective line cohomology via localization). First, let's clearly define the action of $T^2 = (\mathbb{C}\backslash\{0\})^2$ on $\mathbb{P}^1$. For $\underline{\alpha} \in T^2$ and $[X, Y] \in \mathbb{P}^1$ we have

$$\underline{\alpha} \cdot [X, Y] := \left[\frac{X}{\alpha_1}, \frac{Y}{\alpha_2}\right].$$

Then, the only fixed points of this action are $0 = [0 : 1]$ and $\infty = [1 : 0]$:

$$\underline{\alpha} \cdot [0 : 1] = \left[0 : \frac{1}{\alpha_2}\right] = [0 : 1], \quad \text{and} \quad \underline{\alpha} \cdot [1 : 0] = \left[\frac{1}{\alpha_1} : 0\right] = [1 : 0].$$

Proving that there's no more fixed points amounts to a linear algebra exercise. Applying Atiyah-Bott we now have that

$$H^*_{T^2}(\mathbb{P}^1) \simeq H^*_{T^2}([0 : 1]) \oplus H^*_{T^2}([1 : 0])$$
$$\Rightarrow \left.\mathbb{C}[t_1, t_2, H]\middle/(H - t_1)(H - t_2)\right. \simeq \mathbb{C}[t_1, t_2] \oplus \mathbb{C}[t_1, t_2].$$

But the question is, how does this isomorphism work? It suffices to see where the generators go. On the left, we have the generators t_1, t_2 and H representing two hyperplane classes in each copy of $\mathbb{P}^\infty$ and H which represents the hyperplane class of $\mathbb{P}^1$ as a bundle over a point. Mapping these classes we get

$$\underline{i}^* \begin{cases} t_1 \mapsto (t_1, t_1), \\ t_2 \mapsto (t_2, t_2), \\ H \mapsto (t_1, t_2). \end{cases}$$

Whereas the generators on the right are the classes of the points $[0] = (1, 0)$ and $[\infty] = (0, 1)$. These points are mapped to the following classes:

$$i_* \begin{cases} [0 : 1] \mapsto H - t_2, \\ [1 : 0] \mapsto H - t_1. \end{cases}$$

And now, we are left with finding the normal bundles $N_{\boldsymbol{pt.} \mid \mathbb{P}^1}$. Observe that we may use the tangent-normal sequence for subspaces as follows:

$$0 \to T\boldsymbol{pt.} \hookrightarrow i^* T\mathbb{P}^1 \twoheadrightarrow N_{\boldsymbol{pt.} \mid \mathbb{P}^1} := \left.T\mathbb{P}^1\middle/T\boldsymbol{pt.}\right. \to 0$$

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and we have that the tangent bundle to the point is actually zero. This means that we have the isomorphism

$$i^*T\mathbb{P}^1 = T_{\boldsymbol{pt.}}\mathbb{P}^1 \simeq N_{\boldsymbol{pt.}\,|\,\mathbb{P}^1}.$$

Thus the Euler classes we are looking for are for the tangent spaces above 0 and ∞. These can be found using the equivariant Euler sequence for $T\mathbb{P}^1$, and so we get:

$$e(N_{\boldsymbol{\cdot}\,|\,\mathbb{P}^1}) \begin{cases} [0:1] \mapsto t_1 - t_2, \\ [1:0] \mapsto t_2 - t_1. \end{cases}$$

Putting this together we may see that indeed the isomorphism works as follows:

$$\begin{cases} [0:1] \mapsto \dfrac{H - t_2}{t_1 - t_2} \mapsto \left(\dfrac{t_1 - t_2}{t_1 - t_2}, \dfrac{t_2 - t_2}{t_1 - t_2} \right) = (1,0), \\ [1:0] \mapsto \dfrac{H - t_1}{t_2 - t_1} \mapsto \left(\dfrac{t_1 - t_1}{t_2 - t_1}, \dfrac{t_2 - t_1}{t_2 - t_1} \right) = (0,1). \end{cases}$$

Recall lastly that the vector $(1,0)$ represents

$$1{\cdot}[0:1] + 0{\cdot}[1:0]$$

so it's indeed the correct cohomology class.

With this example, we verified that the localization theorem indeed provides an isomorphism between different cohomology rings. Now, we use localization to compute the Euler characteristic of a different variety which can be calculated using an integral of the Euler class. This theorem is due to Chern, and is a version of Gauss-Bonnet.

Theorem 2.2.3 (Chern-Gauss-Bonnet, 1944). *Suppose M is a compact and oriented manifold without boundary of real dimension $2n$. Then*

$$\int_M e(TM) = \chi(M),$$

where TM is the tangent bundle of M and $e(TM) \in H^{2n}(M)$ is its Euler class.

Example 2.2.4 (Euler characteristic of $\mathbb{P}^1 \times \mathbb{P}^1$). Consider $\mathbb{P}^1 \times \mathbb{P}^1$ and an action of T^4 via rescaling all entries as before. The fixed points under this action are

$$(0,0), \quad (\infty,0), \quad (0,\infty), \quad \text{and} \quad (\infty,\infty).$$

Let us denote by F_k, $k = 1, \ldots, 4$ the cohomology classes of the fixed points, and $i_k : F_k \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ the inclusion map. Via the Atiyah-Bott theorem, we have that

$$\int_{\mathbb{P}^1 \times \mathbb{P}^1} e(T\mathbb{P}^1 \times \mathbb{P}^1) = \int_{\mathbb{P}^1 \times \mathbb{P}^1} \sum_{k=1}^{4} \frac{i_{k*} i_k^*\big(e(T\mathbb{P}^1 \times \mathbb{P}^1)\big)}{e(N_{F_k \mid \mathbb{P}^1 \times \mathbb{P}^1})}$$

$$= \sum_{k=1}^{4} \int_{F_k} \frac{i_k^*\big(e(T\mathbb{P}^1 \times \mathbb{P}^1)\big)}{e(N_{F_k \mid \mathbb{P}^1 \times \mathbb{P}^1})}$$

$$= \sum_{k=1}^{4} \int_{F_k} \frac{e(i_k^* T\mathbb{P}^1 \times \mathbb{P}^1)}{e(N_{F_k \mid \mathbb{P}^1 \times \mathbb{P}^1})}$$

and from here we invoke the tangent-normal sequence for $F_k \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. We have that

$$0 \to TF_k \hookrightarrow i_k^* T\mathbb{P}^1 \times \mathbb{P}^1 \twoheadrightarrow N_{F_k \mid \mathbb{P}^1 \times \mathbb{P}^1} \to 0.$$

And simplifying by recalling that the tangent bundle over a point is zero, we have

$$0 \to 0 \to i_k^* T\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\cong} N_{F_k \mid \mathbb{P}^1 \times \mathbb{P}^1} \to 0.$$

This means that both Euler classes cancel out and we are left with just the fundamental class. The integral of the fundamental class over its own space gives us the value of 1 so that the whole sum is equal to 4.

This lets us conclude that $\chi(\mathbb{P}^1 \times \mathbb{P}^1) = 4$.

Remark 2.2.5. This computation did not rely on specific coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$, only on the existence of four torus-fixed points.

The variety $\mathbb{P}^1 \times \mathbb{P}^1$ is an example of a *toric variety*. One property of toric varieties is that their Euler characteristic is equal to the number of torus-fixed points.

Definition 2.2.6. A toric variety is an irreducible variety X containing a torus $T^k :=(C\backslash\{\,0\,\})^k$ as a Zariski open subset such that the action of T^k on itself extends to a morphism $T^k \times X \to X$.

In general, for toric varieties, the number of torus-fixed points equals the number of top-dimensional cones in the associated fan which is combinatorial information that can be computed easily.

Theorem 2.2.7. *For a toric variety, the Euler characteristic equals the number of torus-fixed points under the torus action.*

The proof of this theorem follows the same line as before, identifying fixed points and applying the Atiyah-Bott theorem.

2.3   Localization in the space of maps

A quick overview

Even though we have not discussed the space of maps thoroughly, it suffices to know the combinatorial information in order to calculate the desired integrals. The notion for the integral is the same as before, we will have cohomology classes of maps and when comparing them against the fundamental class what we extract is the degree.

Definition 2.3.1 ([13], pg. 52). An n-pointed map is a morphism

$$\mu : C \to \mathbb{P}^r$$

where C is a rational curve with at worst nodal singularites with n distinct marks which are smooth points of C.

   A map is called <u>Kontsevich stable</u> if any component mapped to a point is stable as a pointed curve.

   The notion of a *stable map* then, is that of map from a tree of projective lines to projective space $\mathbb{P}^r$ and then the stability condition is applied for contracted components. Recall that this means such a component must have at least 3 special points.

Remark 2.3.2. Observe that the source curve C of a stable map need not be a *stable curve* itself.

Example 2.3.3. The map

$$\mathbb{P}^1 \to \mathbb{P}^2, \quad [X : Y] \mapsto [X : Y : 0]$$

covering $\{\, Z = 0 \,\}$ is a stable map even though $\mathbb{P}^1$ is not pointed. This is because the component doesn't get contracted.

Definition 2.3.4 ([13], pg. 52). We say two maps

$$\mu : C \to \mathbb{P}^r, \quad \text{and} \quad \mu' : C' \to \mathbb{P}^r$$

are *isomorphic* whenever there is an isomorphism of the source curves which respects all structure.

   In other words, we require the existence of an isomorphism

$$\phi : C \xrightarrow{\simeq} C'$$

such that the following diagrams commute.

$$
\begin{array}{ccc}
C & \xrightarrow{\ \phi\ } & C' \\
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{\ \phi\ } & C'
\end{array}
$$

From this we can now talk about the isomorphism class of a map.

Definition 2.3.5 ([10], Def. 24.0.5, pg. 501). The <u>moduli space of stable maps</u> from n-pointed, genus g nodal curves C to X representing the class $\beta \in H_2(X, \mathbb{Z})$ as in $f_*[C] = \beta$ is denoted

$$\overline{M}_{g,n}(X, \beta).$$

The space of degree d maps to $\mathbb{P}^r$ is then obtained by replacing $X = \mathbb{P}^r$ and $\beta = d[\text{line}]$. Instead of the large notation, we will opt for $\overline{M}_{0,n}(\mathbb{P}^r, d)$

Let us recall the action of the torus $(\mathbb{C} \backslash \{\,0\,\})^2$ on $\mathbb{P}^1$ from example 2.2.2. Such action can be extended to an action on maps.

Example 2.3.6. Take the map

$$\mu : \mathbb{P}^1 \to \mathbb{P}^1, \quad [X : Y] \mapsto [X + Y : X - Y]$$

and the torus action from example 2.2.2,

$$\underline{\alpha} \cdot [X, Y] = \left[\frac{X}{\alpha_1}, \frac{Y}{\alpha_2} \right].$$

We may define the action $\underline{\alpha} \cdot \mu$ as the map

$$\underline{\alpha} \cdot \mu : \mathbb{P}^1 \to \mathbb{P}^1, \quad [X : Y] \mapsto \underline{\alpha} \cdot [X + Y : X - Y] = \left[\frac{X + Y}{\alpha_1} : \frac{X - Y}{\alpha_2} \right].$$

From this example, we may extend an action

$$G \cdot \mathbb{P}^r \quad \text{to} \quad G \cdot \overline{M}_{0,n}(\mathbb{P}^r, d)$$

via $g \cdot \mu := g \cdot (\mu(c))$ where $c \in C$ is a point on the source curve. The important part is that $\mu(c)$ is now an element of $\mathbb{P}^r$ on which we can act.

Given this, we may now give a criterion to describe the fixed loci of this action on the space of maps.

Proposition 2.3.7. *A stable map μ is fixed under an action of G if the following holds*:

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$\diamond$ *The image* $\mathrm{Im}(\mu)$ *is G-invariant. In other words*

$$G \cdot \mathrm{Im}(\mu) \subseteq \mathrm{Im}(\mu).$$

$\diamond$ *And special points* (*marks and nodes*) *are mapped to G-fixed points.*

Finally we use the result which counts the number of degree d through $3d - 1$ points.

Proposition 2.3.8 ([10], Exercise 25.1.1)**.** *The number of degree d rational plane curves passing through $3d - 1$ points in general position in $\mathbb{P}^2$ is*

$$N_d = \int_{\overline{M}_{0,3d-1}} \mathrm{ev}_1^*(P) \cup \mathrm{ev}_2^*(P) \cup \cdots \cup \mathrm{ev}_{3d-1}^*(P).$$

Two counts of N_d

Example 2.3.9. Let us begin with the example of N_1, this asks for the number of *lines* passing through 2 points in the plane. Intuitively, we know that this number is one. So let us verify it by computing the integral through localization. We will compute

$$\int_{\overline{M}_{0,2}(\mathbb{P}^2, 1)} \mathrm{ev}_1^*([1:0:0]) \, \mathrm{ev}_2^*([0:1:0]).$$

Let us begin by identifying the fixed maps in $\overline{M}_{0,2}(\mathbb{P}^2, 1)$ via the action

$$\underline{\alpha} \cdot [X : Y : Z] = \left[\frac{X}{\alpha_1} : \frac{Y}{\alpha_2} : \frac{Z}{\alpha_3} \right].$$

By proposition 2.3.7, such maps are those which send the source curve into one of the invariant lines which are $X = 0$, $Y = 0$ or $Z = 0$. The markings may be distributed either one on each end of the curve or both on one end attached to a twig contracting to one of the fixed points of $\mathbb{P}^2$.

Such fixed maps are depicted in figure 2.1. Observe for example that on the first line for loci F_1 through F_6, we have only the line $\mathbb{P}^1$ with both marks mapping onto each of the coordinate lines in $\mathbb{P}^2$. Whereas in the next line for F_7 through F_{12}, we do have curves whose contracted components are stable as pointed curves. Via the Atiyah-Bott theorem, we can now express our integral as the sum

$$\int_{\overline{M}_{0,2}(\mathbb{P}^2,1)} \mathrm{ev}_1^*([1:0:0]) \, \mathrm{ev}_2^*([0:1:0]) = \sum_{j=1}^{12} \int_{F_j} \frac{i^*(\mathrm{ev}_1^*([1:0:0]) \, \mathrm{ev}_2^*([0:1:0]))}{e(N_{F_j \,|\, \overline{M}})}.$$
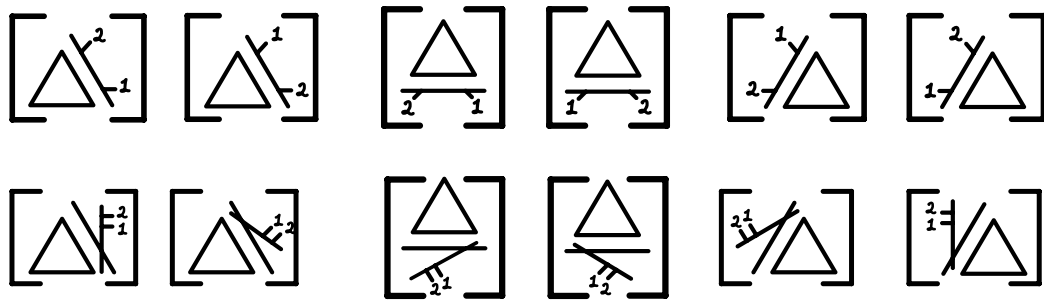
Figure 2.1: Cohomology classes of the fixed loci F_1 through F_{12} of $\overline{M}_{0,2}(\mathbb{P}^2, 1)$

◇ In order to calculate the numerator... diagram

◇ the normal bundle combes from the tangent-normal and the deformation sequence...

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