Exercise 1 (5.10(a) Stein& Shakarchi)**.** Find the Hadamard product for $e^z - 1$.

Answer

Recall Hadamard's theorem states that if f is an entire function with order of growth ρ and $k = \lfloor \rho \rfloor$ then

$$f(z) = e^{p(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$$

where (a_n) is the collection of non-null zeroes of f, p has degree at most k and $m = \operatorname{ord}(f, 0)$.
In our case $e^z - 1$ has order of growth 1 and it has simple zeroes at $z = 2\pi i n$ for $n \in \mathbb{Z}$. In particular the order of zero is one. This means that

$$e^z - 1 = e^{a_1 z + a_0} z \prod_{n \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{2\pi i n}\right) e^{z/2\pi i n}.$$

To simplify this product we multiply opposites across the origin:

$$\begin{aligned}\left[\left(1 - \frac{z}{2\pi i n}\right) e^{z/2\pi i n}\right]\left[\left(1 - \frac{z}{2\pi i(-n)}\right) e^{z/2\pi i(-n)}\right] &= \left(1 + \left(\frac{z}{2\pi i n}\right)^2\right) e^{z/2\pi i n} e^{-z/2\pi i n} \\ &= 1 + \frac{z^2}{4\pi^2 n^2}\end{aligned}$$

So we get

$$e^z - 1 = e^{a_1 z + a_0} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Dividing both sides by z we get

$$\frac{e^z - 1}{z} = e^{a_1 z + a_0} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$$

and as z approaches 0 we get that

$$1 = e^{a_0}(1) \Rightarrow a_0 = 0.$$

Expanding the exponential function as a Taylor series and comparing coefficients we get the following:

$$z + \frac{z^2}{2} + O(z^3) = (1 + a_1 z + \frac{(a_1 z)^2}{2} + O(z^3)) z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$$

Thus we obtain

$$z + \frac{z^2}{2} + O(z^3) = z + a_1 z^2 + O(z^3) \Rightarrow a_1 = \frac{1}{2}.$$

In conclusion we have

$$e^z - 1 = e^{z/2} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Exercise 2 (5.11 Stein& Shakarchi)**.** Show that if f is an entire function of finite order that omits two values, then f is constant. This result remains true for any entire function and is known as Picard's little theorem. ⟦Hint: If f misses a, then $f(z) - a$ is of the form $e^{p(z)}$ where p is a polynomial.⟧

Answer

Assume f omits two values a, b which means that

$$f(z) - a = e^{p(z)}, \quad \text{and} \quad f(z) - b = e^{q(z)} \quad \text{for some} \quad p, q \quad \text{polynomials} \quad .$$

From this, we may subtract one equation from the other to get

$$b - a = e^{p(z)} - eq(z)$$