MATH601 — Advanced Combinatorics

Based on the lectures by Maria Gillespie

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Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This course will focus on the combinatorics of Young tableaux, crystal bases, root systems, Dynkin diagrams, and symmetric functions arising in representation theory of matrix groups and Lie algebras.

Requirements

Familiarity with the basics of group theory and symmetric functions is helpful.

Contents

Chapter 1

1.1 Day 1 | 20240819

We will start by reviewing the representation theory of finite groups and the Lie group and Lie algebra representations. The objective is to classify semi-simple Lie algebras and groups. This classification is quite combinatorial.

Review of representation theory of finite groups

Recall groups are sets G endowed with a binary operation $\circ$ such that

(a) There is an identity element e: $g \circ e = e \circ g = g$.

(b) Every element possesses an inverse. For each g, there is an h such that $g \circ h = e = h \circ g$.

(c) The operation $\circ$ is associative.

Example 1.1.1. The symmetric group is the set of permutations of $[n]$. We denote it $(S_n, \circ)$ where our operation is composition. We will use this group quite a lot.

Example 1.1.2. We will be working with $\mathrm{GL}_n(\mathbb{C})$ where $\mathbb{C}$ will come in as more useful than $\mathbb{R}$. The general linear group is characterized by the property that $\det(A) \neq 0$ for $A \in \mathrm{GL}_n(\mathbb{C})$.

Example 1.1.3. Given two groups we can construct $G \times H$ by doing operations pointwise. We can also take subgroups and quotient groups.

Example 1.1.4. Take the special linear group $\mathrm{SL}_n(\mathbb{C})$ which is the set of matrices A with $\det(A) = 1$. This is a subgroup of $\mathrm{GL}_n(\mathbb{C})$.

There's a lot more of matrix groups such as $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$ and unitary groups $\mathrm{SU}_n(\mathbb{C})$.

Groups which are representations of themselves

Symmetry groups are groups of linear transformations of $\mathbb{C}^n$ (some Euclidean space) that fix some shape. Any such group is a subgroup of $\mathrm{GL}_n(\mathbb{C})$. Matrices here don't collapse points nor anything.

Example 1.1.5. The symmetry group of a diamond in the plane can be found by analyzing the symmetries of the figure.HMMM The group in question is the Klein-4 group which can be seen as

$$\{\, \mathrm{id}, r_x, r_y, r_x r_y \,\}.$$

Similarly we can see it as

$$\{\, \mathrm{id}, (24), (13), (13)(24) \,\}$$

Fell asleep

1.2 Day 2 | 20240821

We were looking at direct sums of representations. Recall representations are maps which take group elements to matrices.

$$\rho \oplus \sigma : G \to \mathrm{GL}_{n+m}(\mathbb{C})$$

and this map will send g to a block matrix. A central question in representation theory is to classify the irreducible representations of some object. This is a central question because for finite groups, irreducible is the same as indecomposable.

Definition 1.2.1. A representation is indecomposable when it can't be written as a direct sum of smaller representations.

Irreducible means that it has no non-trivial proper representations. This is analogous to the idea of prime and irreducible numbers. In the most general case where groups may be infinite, irreducible implies indecomposable.

Alternative definitions for representations

We may define it as a vector space V with an action $G \times V \to V$ so that

$$g(hv) = (gh)v$$

and it should be a linear action in the sense that $v \mapsto gv$ is a linear transformation.

This is equivalent to the previous definition because V can be seen as $\mathbb{C}^n$. So the definition gives rise to a map

$$G \to \mathrm{Aut}(V),\ g \mapsto g\cdot$$

Even more *objecty* is the next definition. We can see a representation as a module over a group ring $\mathbb{C}G$. This set is made up of formal linear combinations of elements of G.

We endow it with a module structure, for any element $g \in G$ in particular in $\mathbb{C}G$ we can make it a coefficient $gv \in V$ as a $\mathbb{C}G$-module.

Subrepresentations

Now that we have all the algebraic structure we can use it to define subrepresentations. Because a subrepresentation will be a subspace which inherits the action for example.

Definition 1.2.2. $W \subseteq V$ is a subrepresentation of G (when V represents G) if

- W is a subspace of V, and
- W is G-invariant in the sense that the image of $G \times W \to V$ is contained in W.

We will also say that V is irreducible if there's no proper nonzero subrepresentation $W \subseteq V$.

Sometimes it is possible to decompose a representation into a direct sum of subrepresentations.

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Definition 1.2.3. A character of a representation is the trace map $g \mapsto \mathrm{tr}(\rho(g))$.

Properties

(a) $\chi_{V \oplus W} = \chi_V + \chi_W$.

(b) $\chi_{V \otimes W} = \chi_V \chi_W$.

(c) χ_V uniquely determines the representation.

1.3 Day 3 | 20240823

Lie groups

Definition 1.3.1. A Lie group is a real smooth manifold G with a group structure such that

$$(g, h) \mapsto gh^{-1}$$

is differentiable.

A manifold is a set such that around each point there's a local neighborhood that's topologically equivalent to $\mathbb{R}^n$. Elliptic curves are examples of manifolds.

Definition 1.3.2. An algebraic group is an algebraic variety with a group structure. In this case the multiplication map should be algebraic.

In certain specializations these two are the same object. In the case of complex Lie groups, we talk about smooth complex manifolds.

Example 1.3.3.
- $(\mathbb{C}^n, +)$ is a Lie group. But it's not compact. sleepy sleepy
- GL_n

Lemma 1.3.4. *(Zariski-)Closed subgroups of a Lie group are also Lie groups.*

Example 1.3.5. In particular B_n, the set of upper triangular matrices in GL_n, forms a Lie group. The torus T_n, the group of diagonal matrices, is also a Lie group.

It is called the torus because it's isomorphic to $(\mathbb{C}\backslash 0)^n$ and $\mathbb{C}\backslash 0$ looks like a circle while $(\mathbb{C}\backslash 0)^2$ is the product of two circles which is the torus.

The Classical Groups

The special linear group SL_n consists of matrices whose determinant is 1. The classical groups are called clasiccal because they have very nice properties. In particular type A is what we call SL_n.

To talk about the special orthogonal group SO_n we should first fix a symmetric bilinear form $(\cdot, \cdot)$ which is positive-definite. The orthogonal group O_n consists of matrices which preserve this form. The special orthogonal group in particular is the subgroup of matrices with determinant 1.

Remark 1.3.6. Over $\mathbb{R}$, O_n is actually the group of rigid transformations which is generated by reflections and rotations. For SO_n, it's only the rotations group.

We can also alternatively define O_n as
$$\{\, A : A^{\mathsf{T}} A = I \,\}$$
because
$$\langle Av|Aw\rangle = \langle v|w\rangle$$
and from this
$$v^{\mathsf{T}} A^{\mathsf{T}} A w = v^{\mathsf{T}} w.$$
Comparing entry by entry we get the desired property.

It's also a fact that O_n is disconnected, one component is SO_n and the other is the set of matrices with determinant -1. Finally type B means SO_{odd} while D means $\mathrm{SO}_{\text{even}}$. The type C groups are the symplectic groups.

1.4 Day 4 | 20240826

Continuing on with the classical groups, we will be talking about the Symplectic group of even dimension. We will be fixing a symplectic form which is a non-degenerate, skew-symmetric, bilinear form.

Example 1.4.1. The dot product is not symplectic because it's symmetric.

Example 1.4.2. Consider the form
$$v_1 w_{2n} + v_2 w_{2n-1} + \cdots + v_n w_{n+1} - v_{n-1} w_n - v_{n+1} w_n - v_{n+2} w_{n-1} - \cdots - v_{2n} w_1.$$
If Ω is such a matrix of a form, for example when $2n = 6$ we have
$$\Omega := \begin{pmatrix} &&&&&1\\ &&&&1&\\ &&&1&&\\ &&-1&&&\\ &-1&&&&\\ -1&&&&& \end{pmatrix} \Rightarrow (v, w) = v^{\mathsf{T}} \Omega w$$

From this our first definition of the symplectic group is matrices which preserve this product.

Definition 1.4.3. The symplectic group Sp_{2n} is
$$\{\, M : (Mv, Mw) = (v, w) \,\}$$
or equivalently
$$\{\, M : M^{\mathsf{T}} \Omega M = \Omega \,\}.$$
We will simplify the notation to type C.

Representation of Lie groups

Definition 1.4.4. A representation of a Lie group is a map which is also differentiable and a group homomorphism.

1.5 Day 5 | 20240828

For a partition $\lambda \vdash n$, we call $S^\lambda V$

$$\Lambda^{\mu_1} V \otimes \Lambda^{\mu_2} V \otimes \cdots \otimes \Lambda^{\mu_k} V$$

where μ is the conjugate partition.

Example 1.5.1. For example if $\lambda = (5, 4, 1)$, then $\mu = (3, 2, 2, 2, 1)$ and so

$$S^{(5,4,1)} V = \Lambda^3$$

Elements can be written as a filling to the Young diagram. Such an element could be

$$(v_1 \wedge v_2 \wedge v_3) \otimes (a \wedge b) \otimes (c \wedge d) \otimes (x \wedge y) \otimes z$$

and filling the diagram we have

$$\begin{ytableau} r \\ q & b & d & y \\ p & a & c & x & z \end{ytableau}.$$

It's important to familiarize ourselves with this idea so we will itechangebly talk about

$$(e_1 \wedge e_4 \wedge e_3) \otimes (e_1 \wedge e_2) \otimes (e_5 \wedge e_3) \otimes (e_2 \wedge e_1) \otimes e_2$$

and

$$\begin{ytableau} 3 \\ 4 & 2 & 3 & 1 \\ 1 & 1 & 5 & 2 & 2 \end{ytableau} = -\begin{ytableau} 4 \\ 3 & 2 & 3 & 1 \\ 1 & 1 & 5 & 2 & 2 \end{ytableau}$$

The tableau $\begin{ytableau} 1 \\ 1 & 2 \end{ytableau}$ is zero for example.

For a basis of S^λ, we can talk about it being spanned by elementary tableau where we order each column from least to greatest. These are called <u>column-strict tableau</u>. For example

$$\begin{array}{|c|c|c|c|}
\cline{1-1}
6 \\
\cline{1-3}
4 & 5 & 3 \\
\hline
1 & 2 & 1 & 2 \\
\hline
\end{array}$$

If V is an n-dimensional vector space, then we have a largest element on our basis. This allows us to formulate the question:

How many column strict tableau are there with largest entry n? And shape λ.

From this

$$\binom{n}{\mu_1}\binom{n}{\mu_2}\cdots\binom{n}{\mu_k} = S^\lambda V.$$

Definition 1.5.2. The <u>Schur module</u> V^λ is

$$V^\lambda = {S^\lambda}\Big/\left\langle v_T - \sum_S v_S \right\rangle$$

where the sum is over S's obtained from T by

(a) Choose two columns of C_1, C_2 of T.

(b) Choose k elements from C_2.

(c) Exchange them with k elements from C_1 in all ways that preserve the order of the elements.

Example 1.5.3. Take $(4, 3, 3)$ with the filling

$$\begin{array}{|c|c|c|c|}
\cline{1-3}
5 & 7 & 6 \\
\cline{1-3}
2 & 4 & 4 \\
\hline
1 & 1 & 3 & 4 \\
\hline
\end{array}$$

so choose the first and third columns as C_1 and C_2. One relation in V_λ

Theorem 1.5.4. *The collection*

$$\{\, e_T : \, T \textit{semistandard}\, \mathrm{sh}(T) \vdash n \,\}$$

is a basis for the Schur module.

1.6 Day 6 | 20240830

Last time we defined the Schur modules. These are

$$S^\lambda V = \Lambda^{\mu_1} V \otimes \cdots \otimes \Lambda^{\mu_r}$$

where $\mu = \lambda^*$ is the conjugate or transpose. Now V^λ is S^λ modded out by column exchanges. We will show that

$$\{ e_T : T \in SSYT(\lambda), \text{largest entry} \leqslant n \}$$

is a basis for V^λ.

Example 1.6.1. Consider the tableau

$$\begin{array}{|c|cc}\hline 6 & & \\ \hline 5 & \multicolumn{1}{c|}{3} & \\ \hline 2 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{4} \\ \hline\end{array}$$

the second and third row are wrongfully ordered

Sleepy sleepy

We wil show that they are independent in the quotient.

Example 1.6.2. The idea for why D_T's are independent. We can find lex orderings and make D_T have nice leading term and then an ordering on the leading terms. E.g. $1, 1+x, 1+x+x^2, 1+x+x^2+x^3$ are independent because the leading terms are all distinct.

In $V^{\begin{array}{|c|cc}\hline a & & \\ \hline b & \multicolumn{1}{c|}{c} & \\ \hline\end{array}}$ we have

$$D_{\begin{array}{|c|cc}\hline 2 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \\ \hline\end{array}} = \det\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22}\end{pmatrix} z_{11} = \dots$$

And

$$D_{\begin{array}{|c|cc}\hline 2 & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \\ \hline\end{array}} = z_{12} \det$$

In the monomials $z_{11}^2 z_{22}$ is larger than $z_{11}z_{12}z_{22}$ and that's how we show that they're independent of each other. This shows the elementary symmetric functions are independent.

One exciting conclusion to look at it's characters. For a Lie group the right notion is to consider H a maximal torus in a Lie group G. This is the maximal connecated, abelian Lie sub group.

Example 1.6.3. $T_n \subseteq \mathrm{GL}_n$ in this case $\chi_V : H \to \mathbb{C}$ where $h \mapsto \mathrm{tr}(h \text{ acts on } V)$. This χ_V determines V and has nice properties with direct sum and tensor products.

$$\chi_V \operatorname{diag}(x_1, \ldots, x_n)$$

is the trace of that matrix acting on V^λ. It suffices to look at a basis. For a given e_T where T is a SSYT, X acts on each e_i by doing $x_i e_i$. see

$$\begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 2 & 3 & 4 \\ \hline 1 & 1 & 2 \\ \hline\end{array} = x_1x_1x_2x_2x_3x_3x_4 \cdot \begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 2 & 3 & 4 \\ \hline 1 & 1 & 2 \\ \hline\end{array}$$

now the trace is the sum of the eigenvalues and this is x^T. So

$$\sum_{T SSYT} x^T = s_\lambda(\underline{x}).$$

1.7 Day 7 | 20240904

Theorem 1.7.1. *A representation of* GL_n *is irreducible if and only if it has a unique highest weight vector.*

Definition 1.7.2. A <u>weight vector</u> of V is $v \in V$ such that for $x \in T_n$ (the torus),

$$x \cdot v = x_1^{\alpha_1} \ldots x_n^{\alpha_n} v$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ is weight.

Recall that being in the torus meant $x = \begin{pmatrix} x_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & x_n \end{pmatrix}$.

Definition 1.7.3. A <u>highest weight vector</u> is a weight vector such that

$$B_n \cdot v = \mathbb{C}^* \cdot v$$

where B_n is the Borel matrices compromised of upper triangular matrices.

A representation is a sum of its weights: $V = \bigoplus_\alpha V_\alpha$ where $V_\alpha = \{ v \colon x \cdot dotv = x^\alpha v \}$.

Lemma 1.7.4. *The only highest weight vector in* V^λ *is* e_{T_0} *where* T_E

1.8 Day n | 20240930

Combinatorics of $\mathfrak{sl}_3$ representations

Our goal for today is to see how all irreducible $\mathfrak{sl}_3$ representations live in $(V^{(1,0)})^{\otimes n}$. We would like to describe them. Recall $V^{(1,0)}$ means that we have $1L_1$ and no L_2.

As a shorthand we will say

$$F_1 = F_{12}, F_2 = F_{23}, E_1 = E_{12}, \ldots$$

Definition 1.8.1. The word $a_1 \ldots a_n \in \{1, 2, 3\}^n$ represents the weight space corresponding to the L-diagram calculation in $(V^{(1,0)})^{\otimes n}$ corresponding to $a_1 \otimes \cdots \otimes a_n$.

Lemma 1.8.2. *The weight of* $(a_1 \ldots a_n)$ *is* $(\#1's, \#2's, \#3's)$.

Proof

By induction on n, the base case is a diagram. ASK FOR DIAGRAM,. Then the induction step wishes to show that something is additivw across $\otimes$. So recall, ¿why are weights additive across $\otimes$? Let v_α, v_β be weight vectors. We want to show $\alpha, \beta \in \eta^* = \{H \to \mathbb{C}\}$. Then finish

Corollary 1.8.3. *The highest weight words in* $(V^{(1,0)})^{\otimes n}$ *are* $a_1 \ldots a_n$ *such that very suffix has* $(\#1's) \geqslant (\#2's) \geqslant (\#3's)$.

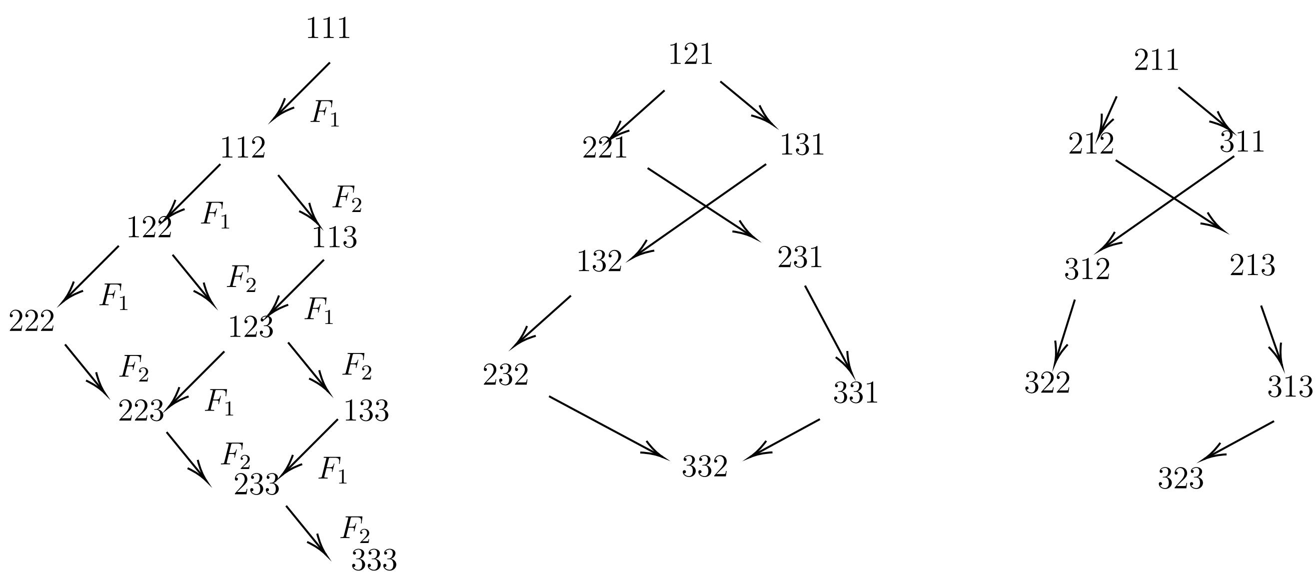
The proof is basically noting that E_1, E_2 map the word to zero when it's of highest weight.

Example 1.8.4. $(V^{(1,0)})^{\otimes 3}$ so first we find the heighest weight words: $111, 121, 211$ and 321. So this first one gives us an adjoint representation, the next one also gives us another adjoint. So this last one for 111 gives us the last 10 elements. The 321 gives us the last one. From this the representation decomposes as

$$V^{(3,0)} \oplus (V^{(2,1)})^(\oplus 2) \oplus V^{(0,0)}.$$

They all have the same weight but are independent one-dimensional weight spaces. Each word on the 111 diagram has a different weight. How many words have weight $(1, 2)$, its $122, 212$ and 221 so the dimesion of that space is $3 = \binom{3}{2} = \binom{3}{1}$.

Question. In $(V^{(1,0)})^{\otimes n}$, ¿what is the dimension of the weight space $\alpha = (a, b, c) = (a - c, b - c)$? It's however many words of length n have a $1'$s, b $2'$s and c $3'$s. This is $\binom{n}{a,b,c} = \frac{n!}{a!b!c!}$, which is counted by taking all the words and then dividing by possible rearrangements. This gives us something with the dots in the diagram.



Recall from when we talked about RSK insertion: It is compatible with $\mathfrak{sl}_2$ crystal operations on tableau reading word. In other words this is, if $\underline{a} \xrightarrow{F_i} \underline{b}$ then

- The RSK insertion tableau of $\underline{a}, \underline{b}$ matches.
- $\mathrm{rw}(\mathrm{ins}(\underline{a})) \xrightarrow{F_i} \mathrm{rw}(\mathrm{ins}(\underline{b}))$.

So in conclusion, each connected component (irreducible representation) in $(V^{(1,0)})^{\otimes n}$ corresponds to a recording tableau. Let's see how this works:

Example 1.8.5. If we take the RSK insertion of the diagram we get diagram. The bumping sequence is all the same! What that means is that we can take the reading word and apply F_1. So all the stuff we did on crystals in 502 is coming back.

If we take the other one for 211, the RSK insetion is $\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}$ but the recording tableau is $\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}$ so we're gonna count how many times an irreducible representation shows up by counting tableau.

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