Exercise 1 (2.3.C. Vakil)**.** Suppose $\mathcal{F}$ is a presheaf and $\mathcal{G}$ is a sheaf, both of sets, on X. Let $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Show that this is a sheaf of sets on X.

Answer

We first need to show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a presheaf, this requires a sensible notion of restriction mapping which satisfies the following:

i) $\mathrm{res}_{U,U} = \mathrm{id}_{(*)}$ where the identity map is over the object $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$.
ii) If $U \subseteq V \subseteq W$ then $\mathrm{res}_{W,U} = \mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}$.

Let us consider two objects $\mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$ and $\mathrm{Mor}(\mathcal{F}|_V, \mathcal{G}|_V)$ with $U \subseteq V$. A restriction mapping acts on sections, and sections on these sets are morphisms of sheaves. Our restriction mapping takes $\varphi \in \mathrm{Mor}(\mathcal{F}|_V, \mathcal{G}|_V)$ to $\mathrm{res}_{V,U}(\varphi) \in \mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$, but recall φ is a collection of maps of objects of the form

$$\varphi(W) : \mathcal{F}(W) \to \mathcal{G}(W), \quad \text{with} \quad W \subseteq V.$$

In this sense, it suffices to only consider the open sets contained in U. We declare that $\mathrm{res}_{V,U}(\varphi)$ is the collection of maps

$$\varphi(W) : \mathcal{F}(W) \to \mathcal{G}(W), \quad \text{with} \quad \underline{W \subseteq U}.$$

i) The map $\mathrm{res}_{U,U}(\varphi)$ acts as follows, every map of the form $\varphi(W)$ with $W \subseteq U$ is sent to the map $\varphi(W)$ between the same objects because $W \subseteq U$ is still itself.

   This means that $\mathrm{res}_{U,U}$ is the identity map in $\mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$.
ii) Now suppose $U \subseteq V \subseteq W$ are open sets, then $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}$ acts on φ first by restricting from open sets in W to open sets in V and next by passing from open sets in V to only considering the open sets in U.

   This is the same as starting with the open sets in W and then only considering the open sets in U. The last action is the same as what $\mathrm{res}_{W,U}$ does to φ.

This allows us to conclude that the sheaf-Hom is indeed a presheaf. We now have to verify the two sheaf axioms:

i) Take (U_i) an cover of $U \subseteq X$ with $\varphi, \psi : \mathcal{F}|_U \to \mathcal{G}|_U$ sections which coincide in every covering set. This means that

$$\operatorname*{res}_{U,U_i}(\varphi) = \operatorname*{res}_{U,U_i}(\psi) \iff \forall i\,[\varphi(V) = \psi(V),\ V \subseteq U_i]\,.$$

Where $\varphi(V), \psi(V)$ are maps of objects from $\mathcal{F}(V)$ to $\mathcal{G}(V)$. We wish to show that they coincide on all of U, which means that for any $V \subseteq U$ and $f \in \mathcal{F}(V)$, it holds that

$$\varphi(V)(f) = \psi(V)(f).$$

Even though we may not talk about these sections directly, we can talk about them after restricting from V to $V \cap U_i$ where U_i is any covering set. To do so, let us introduce the following diagrams:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "{\varphi(V),\psi(V)}"] \arrow[d, "\mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}^{\mathcal{G}}_{V,V\cap U_i}"] \\ \mathcal{F}(V\cap U_i) \arrow[r] & \mathcal{G}(V\cap U_i) \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r] \arrow[d] & (*) \arrow[d] \\ \mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}(f) \arrow[r] & (**) \end{tikzcd}$$

The lower arrow in the left diagram is either of the two morphisms $\varphi(V \cap U_i), \psi(V \cap U_i)$. The right diagram is the same but section-wise:

- ⋄ The upper right corner is the image of the section f inside $\mathcal{G}(V)$ through $\varphi(V)$ or $\psi(V)$.
- ⋄ The lower right corner can be interpreted in two ways which coincide:

$$\varphi(V \cap U_i)\left[\operatorname*{res}^{\mathcal{F}}_{V,V\cap U_i}(f)\right] = \operatorname*{res}^{\mathcal{G}}_{V,V\cap U_i}(\varphi(V)(f))$$

  and the same expression for ψ when that's the case. This equality is due to the fact that φ, ψ are morphisms of sheaves and therefore commute with restrictions.

Recall now that $\varphi(V) = \psi(V)$ for $V \subseteq U_i$, in particular we have $\varphi(V \cap U_i) = \psi(V \cap U_i)$. So mapping f from the upper left to the lower right gives us

$$\begin{aligned}\underset{V,V\cap U_i}{\overset{\mathcal{G}}{\mathrm{res}}}(\varphi(V)(f)) &= \varphi(V \cap U_i)\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\mathrm{res}}}(f)\right] \\ &= \psi(V \cap U_i)\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\mathrm{res}}}(f)\right] \\ &= \underset{V,V\cap U_i}{\overset{\mathcal{G}}{\mathrm{res}}}(\psi(V)(f))\end{aligned}$$

where the first and last equalities occur because φ and ψ are morphisms of sheaves and the middle one because of the hypothesis.

By the identity axiom on $\mathcal{G}$, as $\mathcal{G}$ is a sheaf, we can conclude that $\varphi(V)(f) = \psi(V)(f)$. This means that $\varphi(V) = \psi(V)$, but as $V \subseteq U$ is arbitrary, we conclude that $\varphi = \psi$ and therefore we get the identity axiom.

ii) Once again let us take (U_i) to be an open cover of $U \subseteq X$ along with $\varphi_i \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i)$ for each i. These are morphisms of sheaves, which means that for all open subsets $V \subseteq U_i$ they are maps between objects:

$$\varphi_i(V) : \mathcal{F}(V) \to \mathcal{G}(V), \ V \subseteq U_i.$$

Assume now that the condition $\mathrm{res}_{U_i,U_i\cap U_j}(\varphi_i) = \mathrm{res}_{U_j,U_i\cap U_j}(\varphi_j)$ holds for all i, j. We must show that there exists a section $\varphi \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$, which is

$$\varphi(V) : \mathcal{F}(V) \to \mathcal{G}(V), \ V \subseteq U$$

that satisfies $\mathrm{res}_{U,U_i}(\varphi) = \varphi_i$. This means that for open sets $V \subseteq U_i$, it must hold that

$$\underset{U,U_i}{\mathrm{res}}(\varphi)(V) = \varphi_i(V), \ V \subseteq U_i.$$

For this purpose, we will use the gluing axiom on the sheaf $\mathcal{G}$. Let us now proceed by taking a section $f \in \mathcal{F}(V)$ with $V \subseteq U$ and map it through the following diagram:

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[dr, "\mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}"'] & & \\
& \mathcal{F}(V\cap U_i) \arrow[r, "\varphi_i(V\cap U_i)"] \arrow[d, "\mathrm{res}^{\mathcal{F}}_{V\cap U_i,V\cap U_i\cap U_j}"'] & \mathcal{G}(V\cap U_i) \arrow[d, "\mathrm{res}^{\mathcal{G}}_{V\cap U_i,V\cap U_i\cap U_j}"] \\
& \mathcal{F}(V\cap U_i\cap U_j) \arrow[r] & \mathcal{G}(V\cap U_i\cap U_j)
\end{tikzcd}$$

where the lower arrow is the map $\varphi_i(V\cap U_i\cap U_j)$. We can construct a similar diagram of φ_j. A section $f\in\mathcal{F}(V)$ maps through that diagram as follows:

$$\begin{tikzcd}
f \arrow[dr] & & \\
& \mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}(f) \arrow[r] \arrow[d] & \varphi_i(V\cap U_i)\left[\mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}(f)\right] \arrow[d] \\
& \mathrm{res}^{\mathcal{F}}_{V\cap U_i,V\cap U_i\cap U_j}\left[\mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}(f)\right] \arrow[r] & (*)
\end{tikzcd}$$

and the lower right corner is either the restriction of the upper right corner, or the image of the lower left which by $\varphi_i(V\cap U_i\cap U_j)$. As the vf_i are morphisms of sheaves, both elements are equal. This can be expressed as follows

$$\begin{aligned}
&\varphi_i(V\cap U_i\cap U_j)\left(\underset{V\cap U_i,V\cap U_i\cap U_j}{\overset{\mathcal{F}}{\mathrm{res}}}\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\mathrm{res}}}(f)\right]\right)\\
&=\underset{V\cap U_i,V\cap U_i\cap U_j}{\overset{\mathcal{G}}{\mathrm{res}}}\left(\varphi_i(V\cap U_i)\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\mathrm{res}}}(f)\right]\right)
\end{aligned}$$

But, let us simplify notation a bit by remembering that the composition of restriction maps is the beginning-to-end restriction map. This means that

$$\underset{V\cap U_i,V\cap U_i\cap U_j}{\overset{\mathcal{F}}{\mathrm{res}}}\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\mathrm{res}}}(f)\right]=\underset{V,V\cap U_i\cap U_j}{\overset{\mathcal{F}}{\mathrm{res}}}(f).$$

With this in hand, and remembering the hypothesis that our φ_i's coincide on intersections of the covering sets, we have:

$$\begin{aligned}
&\underset{V\cap U_i, V\cap U_i\cap U_j}{\overset{\mathcal{G}}{\text{res}}}\left(\varphi_i(V\cap U_i)\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\text{res}}}(f)\right]\right)\\
=&\varphi_i(V\cap U_i\cap U_j)\left[\underset{V,V\cap U_i\cap U_j}{\overset{\mathcal{F}}{\text{res}}}(f)\right]\\
=&\varphi_i(V\cap U_i\cap U_j)\left[\underset{V,V\cap U_i\cap U_j}{\overset{\mathcal{F}}{\text{res}}}(f)\right]\\
=&\underset{V\cap U_j, V\cap U_i\cap U_j}{\overset{\mathcal{G}}{\text{res}}}\left(\varphi_j(V\cap U_j)\left[\underset{V,V\cap U_j}{\overset{\mathcal{F}}{\text{res}}}(f)\right]\right).
\end{aligned}$$

So by gluing the maps $\varphi_i(V\cap U_i)\left[\text{res}^{\mathcal{F}}_{V,V\cap U_i}(f)\right]$ in $\mathcal{G}$ we may construct a map $g\in\mathcal{G}(V)$ such that

$$\underset{V,V\cap U_i}{\overset{\mathcal{G}}{\text{res}}}(g)=\varphi_i(V\cap U_i)\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\text{res}}}(f)\right]$$

for each i. We finally define the glued map φ in $\mathcal{H}om$ which takes our original f to this g which we have found. It follows from our construction that

$$\underset{U,U_i}{\text{res}}(\varphi)(V)=\varphi_i(V),\ V\subseteq U_i.$$

After verifying the axioms, we may conclude that the sheaf Hom is indeed a sheaf.

Exercise 2 (2.3.F Vakil)**.** Show that the presheaf cokernel satisfies the universal property of cokernels in the category of presheaves.

Answer

Given a map of presheaves φ, we must show that for $\operatorname{coker}_{\text{pre}}\varphi$ given the following diagram:

$$\begin{tikzcd}
 & \mathcal{G} \arrow[d, "c"] \arrow[ddr, bend left, "c'"] & \\
\mathcal{F} \arrow[ur, "\varphi"] \arrow[r, "0"'] \arrow[drr, bend right, "0"'] & \operatorname{coker}_{\text{pre}} \varphi \arrow[dr, dashed] & \\
 & & \mathcal{C}
\end{tikzcd}$$

that there exists a unique morphism of presheaves $\psi : \operatorname{coker}_{\text{pre}} \varphi \to \mathcal{C}$. Taking out particular objects for any open set U we have the same diagram but in terms of objects in the underlying category which is an abelian category. Thus, there exists a unique map

$$\psi(U) : \operatorname{coker}_{\text{pre}} \varphi(U) \to \mathcal{C}(U)$$

and with this we may define the morphism of presheaves $\operatorname{coker}_{\text{pre}} \varphi \to \mathcal{C}$ by taking each of these maps into our collection of data. This immediately gives us unicity by construction and we are left to check that ψ is a morphism of presheaves. Thi means that for $U \subseteq V$, the following diagram commutes

$$\begin{tikzcd}
\operatorname{coker}_{\text{pre}} \varphi(V) \arrow[r, "\psi(V)"] \arrow[d, "\operatorname{res}^{\text{coker}}_{V,U}"'] & \mathcal{C}(V) \arrow[d, "\operatorname{res}^{\mathcal{C}}_{V,U}"] \\
\operatorname{coker}_{\text{pre}} \varphi(U) \arrow[r, "\psi(U)"'] & \mathcal{C}(U)
\end{tikzcd}$$

Let us take $f \in \operatorname{coker}_{\text{pre}} \varphi(V)$ and see how it maps on both sides of the diagram. However, we are not alone in this endeavor; recall that the cokernel isn't only the object, it's the object *and the epic morphism* $c(V) : \mathcal{G}(V) \to \operatorname{coker}_{\text{pre}} \varphi(V)$. By this,

$$\exists g \in \mathcal{G}(V)(c(V)(g) = f)$$

and restricting our view to the upper triangle in the cokernel diagram, we have that

$$\psi(V)(f) = \psi(V)\,[c(V)(g)] = c'(V)(g).$$

With this fact in hand, let us map f

$$\begin{aligned}\operatorname*{res}_{V,U}^{\mathcal{C}}[\psi(V)(f)] =& \operatorname*{res}_{V,U}^{\mathcal{C}}[c'(V)(g)] = c'(U)\left[\operatorname*{res}_{V,U}^{\mathcal{G}}(g)\right] \\ =& \psi(U)\left(c(U)\left[\operatorname*{res}_{V,U}^{\mathcal{G}}(g)\right]\right) = \psi(U)\left(\operatorname*{res}_{V,U}^{\text{coker}}[c(V)(g)]\right) \\ =& \psi(U)\left(\operatorname*{res}_{V,U}^{\text{coker}}(f)\right)\end{aligned}$$

where we have liberally used the fact that c, c' are maps of sheaves and thus commute with restrictions. From this chain of equalities we conclude that ψ commutes with restrctions and therefore it's a map of sheaves. This uniquely determines the map $\psi : \operatorname{coker}_{\text{pre}} \varphi \to \mathcal{C}$ and this means that $\operatorname{coker}_{\text{pre}} \varphi$ satisfies the universal property of cokernels in the category of presheaves.

Exercise 3 (2.3.H Vakil)**.** Show that a sequence of presheaves $0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$ is exact if and only if $0 \to \mathcal{F}_1(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is exact for $U \subseteq X$.

UNFINISHED

Exercise 4 (2.3.I Vakil)**.** Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of *sheaves*.

i) Show that the presheaf kernel $\ker_{\text{pre}} \varphi$ is in fact a sheaf.
ii) Show that it satisfies the universal property of kernels.

⟦Hint: The second question follows immediately from the fact that $\ker_{\text{pre}} \varphi$ satisfies the universal property in the category of *presheaves*.⟧

Answer

[a] We must show that the presheaf kernel satisfies the two sheaf axioms:

i) Let $U \subseteq X$ be an open set with (U_i) an open cover of U. Suppose $f, g \in \ker_{\text{pre}} \varphi(U)$ which coincide in every covering set. The following diagram is used[b] to define the restriction mapping on the presheaf kernel:

$$\begin{tikzcd} 0 \arrow[r] & \ker_{\text{pre}} \varphi(U) \arrow[r, "\iota"] \arrow[d, dashed, "\exists!"] & \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\operatorname{res}^{\mathcal{F}}_{U,U_i}"] & \mathcal{G}(U) \arrow[d, "\operatorname{res}^{\mathcal{G}}_{U,U_i}"] \\ 0 \arrow[r] & \ker_{\text{pre}} \varphi(U_i) \arrow[r, "\iota_i"'] & \mathcal{F}(U_i) \arrow[r, "\varphi(U_i)"'] & \mathcal{G}(U_i) \end{tikzcd}$$

So let us assume that for all i, we have $\operatorname{res}^{\ker}_{U,U_i}(f) = \operatorname{res}^{\ker}_{U,U_i}(g)$. We can include them into $\mathcal{F}(U_i)$ with ι_i show that we have

$$\iota_i \left[\operatorname*{res}^{\ker}_{U,U_i}(f) \right] = \iota_i \left[\operatorname*{res}^{\ker}_{U,U_i}(g) \right]$$

but as we have (assumed) that $\ker_{\text{pre}} \varphi$ is a presheaf, the left square commutes. So we have

$$\operatorname*{res}^{\mathcal{F}}_{U,U_i} (\iota(f)) = \operatorname*{res}^{\mathcal{F}}_{U,U_i} (\iota(g))$$

which by the identiy axiom on $\mathcal{F}$, we have that $\iota(f) = \iota(g)$. As ι is injective we have that $f = g$, verifying the identity axiom on $\ker_{\text{pre}} \varphi$.

ii) Once again consider an open cover (U_i) of $U \subseteq X$ with $f_i \in \ker_{\text{pre}} \varphi(U_i)$ for each i. Assume that for all i, j we have

$$\operatorname*{res}^{\ker}_{U_i,U_i\cap U_j}(f_i) = \operatorname*{res}^{\ker}_{U_j,U_i\cap U_j}(f_j)$$

then, using the corresponding inclusion map which $\iota_{ij} : \ker_{\text{pre}} \varphi(U_i \cap U_j) \to \mathcal{F}(U_i \cap U_j)$ we get

$$\iota_{ij} \left[\operatorname*{res}^{\ker}_{U_i,U_i\cap U_j}(f_i) \right] = \iota_{ij} \left[\operatorname*{res}^{\ker}_{U_j,U_i\cap U_j}(f_j) \right]$$

which leads us to

$$\operatorname*{res}^{\mathcal{F}}_{U_i,U_i\cap U_j} [\iota_i(f_i)] = \operatorname*{res}^{\mathcal{F}}_{U_i,U_i\cap U_j} [\iota_j(f_j)]$$

by commutitativy of the left square of the following diagram (and a similar one for j):

$$\begin{tikzcd}
0 \arrow[r] & \ker_{\text{pre}} \varphi(U_i) \arrow[r, "\iota_i"] \arrow[d, "\operatorname{res}^{\ker}_{U_i,U_i\cap U_j}"] & \mathcal{F}(U_i) \arrow[r, "\varphi(U_i)"] \arrow[d, "\operatorname{res}^{\mathcal{F}}_{U_i,U_i\cap U_j}"] & \mathcal{G}(U_i) \arrow[d, "\operatorname{res}^{\mathcal{G}}_{U_i,U_i\cap U_j}"] \\
0 \arrow[r] & \ker_{\text{pre}} \varphi(U_i \cap U_j) \arrow[r, "\iota_{ij}"'] & \mathcal{F}(U_i \cap U_j) \arrow[r, "\varphi(U_i\cap U_j)"'] & \mathcal{G}(U_i \cap U_j)
\end{tikzcd}$$

Gluing inside $\mathcal{F}$ we get $\tilde{f} \in \mathcal{F}(U)$ such that $\operatorname{res}^{\mathcal{F}}_{U,U_i}(\tilde{f}) = \iota_i(f_i)$. Mapping $\tilde{f}$ through $\varphi(U)$ we can restrict to the covering set to get

$$\operatorname*{res}^{\mathcal{G}}_{U,U_i} \left[\varphi(U)(\tilde{f}) \right] = \varphi(U_i) \left[\operatorname*{res}^{\mathcal{F}}_{U,U_i}(\tilde{f}) \right] = \varphi(U_i) \left[\iota_i(f_i) \right] = 0$$

which means that $\varphi(U)(\tilde{f})$ restricts to 0. As $\mathcal{G}$ is a sheaf, it must occur that $\varphi(U)(\tilde{f}) = 0$ and therefore by exactness of the kernel we find $f \in \ker_{\text{pre}} \varphi(U)$ such that $\iota(f) = \tilde{f}$. Such f is the desired element which satisfies the gluing axiom for $\ker_{\text{pre}} \varphi$.

[a] If time permits we will show that the presheaf kernel is also a presheaf.
[b] In the exercise to show kernel is presheaf.

Exercise 5 (2.4.C Vakil)**.** If φ, ψ are morphisms from a presheaf of sets $\mathcal{F}$ to a sheaf of sets $\mathcal{G}$ that induce the same maps on each stalk, show that $\varphi = \psi$. As a hint consider the following diagram:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d, hook] \\
\prod_{p\in U} \mathcal{F}_p \arrow[r] & \prod_{p\in U} \mathcal{G}_p
\end{tikzcd}$$

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