**Exercise 1** (Exercise 8, Stanley 1.44.a)**.** Show that the total number of cycles of all even permutations of $[n]$ and the total number of cycles of all odd permutations of $[n]$ differ by $(-1)^n(n-2)!$. Use generating functions.

I must start by making a review of group theory which has helped me throughout the solution of this problem.

**Definition 1.** Suppose $\pi$ is a permutation in $S_n$.

The order of a permutation is the amount of times we need to compose it with itself to obtain the identity permutation.

The parity of a permutation depends on the number of transpositions which compose it. A permutation is even when it is a product of an *even* number of transpositions. Likewise for odd permutations.

The sign of a permutation is $1$ when $\pi$ is even. When $\pi$ is odd, the sign is $-1$.

For example, $\operatorname{ord}((123) = 3$ since $(123)(123)(123) = (123)(132) = \mathrm{id}$. Also $(12)(34)$ is an even permutation since it's a product of two transpositions.

**Proposition 1.** *For any cycle $c = (x_1x_2 \dots x_\ell)$, $\operatorname{ord}(c) = \ell$ the length of the cycle.*

*The sign of the cycle $c$ can be computed as $(-1)^{\operatorname{ord}(c)-1}$.*

*The sign function is multiplicative.*

This is because one can decompose a cycle of order $\ell$ can be decomposed into $\ell - 1$ transpositions.

**Theorem 1.** *Suppose $\pi \in S_n$ can be decomposed into a product of $k$ cycles, $c_1c_2 \dots c_k$. Then the sign of $\pi$ is the product of the signs of $c_i$'s. The following formula holds:*[1]

$$\operatorname{sgn}(\pi) = (-1)^{\sum_{j=1}^{k} \operatorname{ord}(c_j)-k}.$$

This is because

$$\operatorname{sgn}(\pi) = \operatorname{sgn}(c_1 \dots c_k) = \operatorname{sgn}(c_1) \dots \operatorname{sgn}(c_k) = (1)^{\operatorname{ord}(c_1)-1} \dots (-1)^{\operatorname{ord}(c_k)-1}$$

and by summing up the exponents we obtain the desired formula.

*Remark* 2. The formula holds *even when the decomposition includes 1-cycles*. This is because the identity permutation has order 1.

The count of $\sum \operatorname{ord}(c)$ goes up by one, and the $k$ count (amount of cycles) also goes up by one. Therefore parity is preserved.[2]

With this in hand let us proceed.

[1] **Sam** helped me out when verifying that this formula holds.

[2] This observation is key when recognizing the generating function. **Ian** was the one who pointed me out the fact that I could use length 1 cycles to fill out some missing spaces.

**Answer**

Let us call $E_n$ to be the amount of cycles across all of the even permutations in $S_n$. Likewise for $O_n$, the number of cycles across odd permutations.
The quantity we are interested in is $D_n = E_n - O_n$. Suppose $\pi = c_1 \ldots c_k$ is an even permutation, this means that $\pi$ adds $k$ cycles to the count of $E_n$. Likewise if $\pi$ were odd, it adds $k$ cycles to $O_n$.
Since at the end we are subtracting $O_n$ from $E_n$, then we should take into account the sign when adding. This is our first key point.
In general, $\pi$ contributes with $\operatorname{sgn}(\pi)k$ cycles to $D_n$. Counting[a] across all the permutations with $k$ cycles we get

$$D_n = \sum_{k=1}^{n} \operatorname{sgn}(\pi) k c(n,k) = \sum_{k=1}^{n} (-1)^{\sum \operatorname{ord}(c_j) - k} k c(n,k)$$

where the $c_j$'s are the decomposition in disjoint cycles of each permutation and $c(n,k)$ is the unsigned Stirling number of the first kind which counts the amount of permutations of $S_n$ with $k$ cycles in their decomposition.
This formula looks *oddly similar* to the Pochammer symbol's generating function[b]

$$(x)_n = \sum_{k=1}^{n} s(n,k) x^k$$

evaluated at $x = 1$. This is because $s(n,k) = (-1)^{n-k} c(n,k)$.

[a] The idea to count across all permutations given their cycle length using the Stirling numbers comes from stackexchange: `math.se/113202`.

[b] I wrote this solution yesterday (0908) and I had not realized that this identity is part of the homework. I just tunnel-visioned the last problem since it was the 8 point one.

We reach a conundrum at this stage because in general $\sum \operatorname{ord}(c_j) \neq n$. For example consider the transposition $(12)$, but in $S_{10^{10}}$. In this case, the sum of the orders is $2$. Because we are only counting the transposition. However $n = 10^{10}$, which most definitely is not equal to 2.

**Ian**'s key observation comes at play here, we can count the $1$-cycles which are being multiplied tacitly to $(12)$. We have $(12) = (12)(3)(4)\ldots(10^{10})$. All of this transpositions have order 1, save for the first one. Adding up all of the orders, we do indeed get $10^{10}$! Now, recall that adding the $1$-cycles to our representation does not alter the parity, so the theorem about the parity still holds.

Continuing on with the assumption that we are counting every permutation together with its 1-cycles, our formula for $D_n$ becomes

$$D_n = \sum_{k=1}^{n} (-1)^{n-k} k c(n,k) = \sum_{k=1}^{n} k s(n,k)$$

which we recognize as the derivative of the Pochammer symbol's generating function evaluated at $x = 1$.
The derivative in question is precisely

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=1} (x)_n = \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=1} [(x)_{n-1}(x-(n-1))] \\
\Rightarrow &\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=1} (x)_n = \left(\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=1} (x)_{n-1}\right)(x-(n-1)) \mid_{x=1} + (x)_{n-1} \mid_{x=1} \\
\Rightarrow &D_n = D_{n-1}(2-n) + \delta_{n1}.
\end{aligned}$$

This recurrence relation allows us to find $D_n$ given the initial condition that $D_1 = 1$, because $E_1 = 1$ (the identity) and $O_1 = 0$. For $n \geqslant 1$ we have $\delta_{n1} = 0$, so

$$D_n = D_{n-1}(2-n) = [D_{n-2}(2-(n-1))](2-n) = D_{n-2}(3-n)(2-n).$$

Inductively we can see that this quantity is

$$D_1 \ldots (4-n)(3-n)(2-n) = (-1)^{n-2}(n-2)! = (-1)^n (n-2)!$$

and therefore $E_n - O_n = (-1)^n (n-2)!$ as desired.

**Exercise 2** (Exercise 9, Stanley 1.44.b)**.** Give a bijection proof of the previous fact.

Answer

[a] Let us proceed inductively and create a sufficient number of bijections.
Our base case is $S_4$[b] in which $E_4 = 14$, and $O_4 = 12$. It holds that $D_4 = 2$, and according to the formula $D_4 = (-1)^4(4-2)! = 2$.
Without losing generality, let us assume that $n$ is even. In that case our inductive hypothesis tells us that the difference in the number of cycles from evens to odds

is negative. So if we have $A$ cycles among the even permutations, we have that:

$$\begin{cases} E_{n-1} = A \\ O_{n-1} = A + (n-3)! \end{cases}$$

Now, let us come up with a function from $S_{n-1}$ to $S_n$ which adds the element $n$ to each permutation. Consider

$$\varphi_j : S_{n-1} \to S_n,\ \pi \mapsto \pi(jn)^c,$$

this functions switches the parity of $\pi$. It is also an injective function since we can cancel the products inside of $S_n$ by right-multiplying the inverse of the transposition on the right. Thus $\varphi_j$ is a bijection between $S_{n-1} \to \text{Im}[\varphi_j]$ and we can decompose $S_n$ in the following way

$$S_n = \text{Im}[\varphi_1] \cup \text{Im}[\varphi_2] \cup \cdots \cup \text{Im}[\varphi_{n-1}] \cup \text{Stab}(n)$$

where the last set is the set of permutations which *fix* $n$. We make a final map

$$\varphi_n : S_{n-1} \to \text{Stab}(n),\ \pi \mapsto \pi(n)$$

which doesn't switch the parity of $\pi$ at all. Each of the sets inside the decomposition is disjoint from one another. If

$$\pi \in \text{Im}[\varphi_i] \cap \text{Im}[\varphi_j] \Rightarrow \exists \pi_1, \pi_2 \in S_{n-1}(\pi_1(in) = \pi_2(jn)),$$

and this can't happen[d]. So we are not missing nor over-counting anyone.
Now that have our setup, let us count. Since the $\varphi_j$ switch parities, we get the following

$$\begin{cases} E_n = \underbrace{(A + (n-3)!) + (A + (n-3)!) + \cdots + (A + (n-3)!)}_{\text{One from each Im}[\varphi_j]} + A \\ O_{n-1} = \underbrace{A + A + \cdots + A}_{\text{One from each Im}[\varphi_j]} + (A + (n-3)!) \end{cases}$$

which we can summarize into the following system:

$$\begin{cases} E_n = (n-1)O_{n-1} + E_{n-1} \\ O_n = (n-1)E_{n-1} + O_{n-1} \end{cases}$$

Finally we find that the difference of this quantities is

$$D_n = (n-1)(-D_{n-1}) + D_{n-1} = (2-n)D_{n-1}.$$

This is the same recurrence we arrived at on the last exercise. The result follows.

[a] This proof is in most part due to **Andrew**. He was the one who came up with each of the functions. I just had a vague idea on how to construct a parity-switching bijection, but he was the one who refined it and made it work.

[b] I literally went to `Groupprops Subwiki - Element Structure of symmetric group:`$S_4$ and counted them.

[c] When initially discussing this problem together with **Ian** on Wednesday, we also came up with a function which did this to every permutation in order to build a recurrence. The thing is that we were'nt clear on how to use it.

[d] I still need to prove this.