Exercise 1. Let G be a Lie group.

(a) Show that the set of right-invariant vector fields on G forms a Lie algebra with bracket given by the Lie bracket of vector fields. Note that the right-invariant vector fields form a vector space which is isomorphic to T_eG.

(b) Let $\mathrm{inv} : G \to G$ be given by $\mathrm{inv}(g) = g^{-1}$. Prove that if X is a left-invariant vector field on G, then $d\,\mathrm{inv}(X)$ is a right-invariant vector field whose value at e is $-X_e$.

(c) Prove that the map $-d\,\mathrm{inv}$ from left-invariant vector fields to right-invariant vector fields is a Lie algebra isomorphism. (The point is that we could just have well chosen to interpret the Lie algebra of G as the right-invariant vector fields rather than the left-invariant ones.)

Answer

Exercise 2. Consider the special orthogonal group $\mathrm{SO}(3)$ of all 3×3 matrices B such that

$$BB^T = I \qquad \text{and} \qquad \det B = 1.$$

We saw in section 3.4 that

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

gives a basis for $T_I\,\mathrm{SO}(3)$ so that $[A_1, A_2] = A_3$, $[A_2, A_3] = A_1$, and $[A_3, A_1] = A_2$. Notice also that $A_3 = \gamma_3'(0)$, where

$$\gamma_3(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and there are similar curves whose tangents at the identity give A_1 and A_2.

Let V_1, V_2, and V_3 be the corresponding left-invariant vector fields on $\mathrm{SO}(3)$; i.e., $V_i(I) = A_i$.

Let α_i be the dual basis of left-invariant 1-forms and compute their exterior derivatives.

Answer

Exercise 3. Let G be a compact Lie group and assume $\langle \cdot, \cdot \rangle$ is an Ad-invariant inner product on $\mathfrak{g}$ (an Ad-invariant inner product on $\mathfrak{g}$ is one that satisfies $\langle X, Y \rangle = \langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y \rangle$ for all $g \in G$ and for any $X, Y \in \mathfrak{g}$).

Define $\tau_e : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by

$$\tau(X, Y, Z) = \langle [X, Y], Z \rangle.$$

(a) Show that τ_e is alternating. Since τ_e is clearly multilinear, this means τ_e can be identified with an element of $\bigwedge^3(\mathfrak{g}^*)$.

(b) Extend τ_e to a left-invariant 3-form on G in the usual way: for each $g \in G$, define $\tau_g := L_{g^{-1}}^* \tau_e$. Prove that $\tau \in \Omega^3(G)$ is bi-invariant (Hint: feel free to use the fact that a left-invariant form is bi-invariant if and only if it is conjugation-invariant). The bi-invariant 3-form τ is called the *fundamental 3-form* of the Lie group G.

(c) Explicitly compute the fundamental 3-form of $\mathrm{SO}(3)$ in terms of the α_i from the previous problem.

Answer