Exercise 1. A smooth manifold M is called *orientable* if there exists a collection of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ so that, for every α, β such that $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \varnothing$, the differential of the change of coordinates $\phi_\beta^{-1} \circ \phi_\alpha$ has positive determinant.

(a) Show that for any n, the sphere S^n is orientable.

(b) Prove that, if M and N are smooth manifolds and $f : M \to N$ is a local diffeomorphism at all points of M, then N being orientable implies that M is orientable. Is the converse true?

Answer

(a) Consider the sphere without its north and south pole:

$$U = S^n \backslash \{ \mathbf{e}_{n+1} \}, \quad \text{and} \quad V = S^n \backslash \{ -\mathbf{e}_{n+1} \}.$$

These two sets form an atlas of S^n along with the stereographic projections

$$\phi : U \to \mathbb{R}^n, \ \mathbf{u} \mapsto \frac{1}{1 - u_{n+1}}(u_1, \ldots, u_n),$$
$$\psi : V \to \mathbb{R}^n, \ \mathbf{u} \mapsto \frac{1}{1 + u_{n+1}}(u_1, \ldots, u_n).$$

For $\mathbf{u} \in S^n$, call $\mathbf{x} = \phi(\mathbf{u})$ and $\mathbf{y} = \psi(\mathbf{u})$. In order to find the transition function $\psi\phi^{-1}$, we first make the observation that

$$\|\mathbf{x}\|^2 = \|\phi(\mathbf{u})\|^2 = \frac{1}{(1 - u_{n+1})^2}(u_1^2 + \cdots + u_n^2) = \frac{1 - u_{n+1}^2}{(1 - u_{n+1})^2} = \frac{1 + u_{n+1}}{1 - u_{n+1}},$$

and from this we can see that

$$\phi^{-1}(\mathbf{x}) = \frac{1}{1 + \|\mathbf{x}\|^2}(2x_1, \ldots, 2x_n, \|\mathbf{x}\|^2 - 1).$$

Applying ψ we get the transition function to be

$$\begin{aligned}
\mathbf{y} = \psi\phi^{-1}(\mathbf{x}) &= \frac{1}{1 + \left(\frac{\|\mathbf{x}\|^2 - 1}{1 + \|\mathbf{x}^2\|}\right)} \left(\frac{2x_1}{1 + \|\mathbf{x}\|^2}, \ldots, \frac{2x_n}{1 + \|\mathbf{x}\|^2} \right) \\
&= \frac{1}{\frac{1 + \|\mathbf{x}\|^2 + \|\mathbf{x}\|^2 - 1}{\|\mathbf{x}\|^2 + 1}} \frac{2}{1 + \|\mathbf{x}\|^2} \mathbf{x} \\
&= \frac{1 + \|\mathbf{x}\|^2}{2\|\mathbf{x}\|^2} \frac{2}{1 + \|\mathbf{x}\|^2} \mathbf{x} = \frac{\mathbf{x}}{\|\mathbf{x}\|^2}
\end{aligned}$$

The differential of this map can be calculated using the product rule. Call $f = \frac{1}{\|\mathbf{x}\|^2}, G = \text{id}$, then

$$J(fG) = \nabla f \otimes G + fJG = \left(\frac{-1}{\left(\|\mathbf{x}\|^2\right)^2} 2\mathbf{x} \right) \otimes \mathbf{x} + \frac{1}{\|\mathbf{x}\|^2} \text{Id}.$$

Using the matrix determinant lemma we may see that

$$\begin{aligned}
\det(JfG) &= \left(1 + \frac{-2}{\|\mathbf{x}\|^4} \mathbf{x}^\mathsf{T} \left(\|\mathbf{x}\|^2 \text{Id}\right) \mathbf{x}\right) \det\left(\frac{1}{\|\mathbf{x}\|^2} \text{Id}\right) \\
&= (1 - \frac{2}{\|\mathbf{x}\|^2} \mathbf{x}^\mathsf{T} \mathbf{x}) \frac{1}{\|\mathbf{x}\|^{2n}} \\
&= (1 - 2) \frac{1}{\|\mathbf{x}\|^{2n}} = \frac{-1}{\|\mathbf{x}\|^{2n}}
\end{aligned}$$

This doesn't mean that the sphere is non-orientable, but that my choice of atlas was a poor choice. I will revisit this problem in a bit.

(b) We will build an atlas on M whose transitions functions' differential have positive determinant. To that effect, let $\{(V_\alpha, \psi_\alpha)\}$ be an atlas of N which makes N orientable, this means that we have $\det J(\psi_\beta \psi_\alpha^{-1}) > 0$ for α, β.

Now for $x \in M$, there are neighborhoods $U_x, \widetilde{V}_{f(x)}$ in M, N respectively such that f is a diffeomorphism between these sets. Pick a chart $(V_{f(x)}, \psi_{f(x)})$ from our original atlas such that $f(x) \in V_{f(x)}$. Consider then the new open sets

$$W_{f(x)} = V_{f(x)} \cap \widetilde{V}_{f(x)}$$

and restrict $\psi_{f(x)}$ into $W_{f(x)}$ by calling it $\varphi_{f(x)}$.

This defines a new atlas

$$\{\, (W_{f(x)}, \varphi_{f(x)}) \,\}$$

for N, by virtue of f being bijective, which still preserves the property that its transition functions' differential has positive determinant.

We may pullback this atlas via f into an atlas

$$\{\, (f^{-1}(W_{f(x)}), f^* \varphi_{f(x)}) \,\}$$

of M. For $x, y \in M$, we have the expression for the transition function

$$(f^* \varphi_{f(x)})(f^* \varphi_{f(y)})^{-1} = (\varphi_{f(x)} \circ f)(\varphi_{f(y)} \circ f)^{-1} = v f_{f(x)} \varphi_{f(y)}^{-1}.$$

As these are restrictions of our ψ functions, then their differential still has positive determinant. Thus, we have found an atlas of M which makes it orientable.

Quick question for the second item, the fact that M, N are locally diffeomorphic implies that pairs of neighborhoods have the same homology. In particular the relative homology groups, which are used to define local orientation, are isomorphic.

Would this be sufficient to conclude that if N is orientable then M is orientable? Why would this argument fail in the other direction? Say, why if M is orientable, could N be non-orientable?

Exercise 2. Supply the details for the proof that, if $F\colon \mathrm{Mat}_{d\times d}(\mathbb{C}) \to \mathcal{H}(d)$ is given by $F(U) = UU^*$ (where U^* is the conjugate transpose [a.k.a., Hermitian adjoint] of U), then the unitary group

$$U(d) = F^{-1}(I_{d\times d})$$

is a submanifold of $\mathrm{Mat}_{d\times d}(\mathbb{C})$ of dimension d^2. (Hint: it may be helpful to remember that a Hermitian matrix M can always be written as $M = \frac{1}{2}(M + M^*)$.)

Exercise 3. Let M be a compact manifold of dimension n and let $f : M \to \mathbb{R}^n$ be a smooth map. Prove that f must have at least one critical point.

Exercise 4. Prove that, if X, Y, and Z are smooth vector fields on a smooth manifold M and $a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$, then

(a) $[X, Y] = -[Y, X]$ (anticommutativity)

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ (linearity)

(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity)

(d) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.

Answer

We have defined the Lie bracket as the commutator of vector fields

$$[X, Y]f := X(Yf) - Y(Xf)$$

and so we have:

(a) $[Y, X]f = Y(Xf) - X(Yf) = -(X(Yf) - Y(Xf)) = -[X, Y]f$.

(b) To show linearity we have

$$\begin{aligned}&[aX + bY, Z]f \\ =&(aX + bY)(Zf) - Z((aX + bY)f) \\ =&aX(Zf) + bY(Z(f)) - Z(aXf + bYf)\end{aligned}$$

where the first equality comes by definition and the second one is the definition of the sum of linear operators. Now applygin the fact that Z is linear we get:

$$aX(Zf) + bY(Z(f)) - aZ(Xf) - bZ(Yf)$$

which we rearrenge as a sum of smooth functions now:

$$a(X(Zf) - Z(Xf)) + b(Y(Z(f)) - Z(Yf)) = (a[X, Z] + b[Y, Z])f.$$

(c) Let us take the first two terms in the sum and see that

$$\begin{aligned}&([[X, Y], Z] + [[Y, Z], X])f \\ =&[X, Y](Zf) - Z([X, Y]f) + [Y, Z](Xf) - X([Y, Z]f) \\ =&X(YZf) - Y(XZf) - Z(XYf) + Z(YXf) \\ &+ Y(ZXf) - Z(YXf) - X(YZf) + X(ZYf)\end{aligned}$$

Observe now that the 1^{st} and 7^{th}, and 4^{th} and 6^{th} terms cancel out. We are left with a term which we rearrange into...

$$\begin{aligned}&-Y(XZf) - Z(XYf) + Y(ZXf) + X(ZYf) \\ =&Y(ZXf) - Y(XZf) - (ZX(Yf) - XZ(Yf)) \\ =&Y([Z, X]f) - [Z, X](Yf) \\ =&[Y, [Z, X]]f = -[[Z, X], Y]f\end{aligned}$$

as desired.

(d) Finally if h is another smooth function on M:

$$\begin{aligned}&[fX, gY]h \\ =&fX(gYh) - gY(fXh) \\ =&fXgYh + fgXYh - gYfXh - gfYXh \\ =&fXgYh + fg(XYh - YXh) - gYfXh \\ =&(f(Xg)Y + fg[X, Y] - g(Yf)X)h\end{aligned}$$

where in the second equality we have applied the product rule for vector fields.