**Exercise 1** (5.10(a) Stein& Shakarchi). Find the Hadamard product for  $e^z - 1$ .

## **Answer**

Recall Hadamard's theorem states that if f is an entire function with order of growth  $\rho$  and  $k = |\rho|$  then

$$f(z) = e^{p(z)} z^m \prod_{n=1}^{\infty} E_k \left(\frac{z}{a_n}\right)$$

where  $(a_n)$  is the collection of non-null zeroes of f, p has degree at most k and  $m = \operatorname{ord}(f, 0)$ .

In our case  $e^z-1$  has order of growth 1 and it has simple zeroes at  $z=2\pi in$  for  $n \in \mathbb{Z}$ . In particular the order of zero is one. This means that

$$e^{z} - 1 = e^{a_1 z + a_0} z \prod_{n \in \mathbb{Z} \setminus \{0\}} \left( 1 - \frac{z}{2\pi i n} \right) e^{z/2\pi i n}.$$

To simplify this product we multiply opposites across the origin:

$$\left[ \left( 1 - \frac{z}{2\pi i n} \right) e^{z/2\pi i n} \right] \left[ \left( 1 - \frac{z}{2\pi i (-n)} \right) e^{z/2\pi i (-n)} \right] = \left( 1 + \left( \frac{z}{2\pi i n} \right)^2 \right) e^{z/2\pi i n} e^{-z/2\pi i n}$$

$$= 1 + \frac{z^2}{4\pi^2 n^2}$$

So we get

$$e^{z} - 1 = e^{a_1 z + a_0} z \prod_{n=1}^{\infty} \left( 1 + \frac{z^2}{4\pi^2 n^2} \right)$$

Dividing both sides by z we get

$$\frac{e^z - 1}{z} = e^{a_1 z + a_0} \prod_{n=1}^{\infty} \left( 1 + \frac{z^2}{4\pi^2 n^2} \right)$$

and as z approaches 0 we get that

$$1 = e^{a_0}(1) \Rightarrow a_0 = 0.$$

Expanding the exponential function as a Taylor series and comparing coefficients we get the following:

$$z + \frac{z^2}{2} + O(z^3) = (1 + a_1 z + \frac{(a_1 z)^2}{2} + O(z^3)) z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$$

Thus we obtain

$$z + \frac{z^2}{2} + O(z^3) = z + a_1 z^2 + O(z^3) \Rightarrow a_1 = \frac{1}{2}.$$

In conclusion we have

$$e^{z} - 1 = e^{z/2} z \prod_{n=1}^{\infty} \left( 1 + \frac{z^{2}}{4\pi^{2} n^{2}} \right).$$

**Exercise 2** (5.11 Stein& Shakarchi). Show that if f is an entire function of finite order that omits two values, then f is constant. This result remains true for any entire function and is known as Picard's little theorem.  $\llbracket \text{Hint: If } f \text{ misses } a$ , then f(z) - a is of the form  $e^{p(z)}$  where p is a polynomial.  $\rrbracket$ 

## Answer

Assume f omits two values a, b which means that

$$f(z) - a = e^{p(z)}$$
, and  $f(z) - b = e^{q(z)}$  for some  $p, q$  polynomials

From this, we may subtract one equation from the other to get

$$b - a = e^{p(z)} - eq(z)$$

and now we may differentiate both sides of the equation to obtain

$$0 = p'(z)e^{p(z)} - q'(z)e^{q(z)}.$$

As this equation holds for *all*  $z \in \mathbb{C}$  it must happen that p'(z) and q'(z) have the same zeroes with the same multiplicities. Thus q'(z) = cp'(z) for some non-zero  $c \in \mathbb{C}$ . Returning to our equation we have

$$e^{p(z)}p'(z) = cp'(z)e^{q(z)} \Rightarrow e^{p(z)} = ce^{q(z)} \Rightarrow ce^{q(z)} - e^{q(z)} = b - a.$$

Differentiating this equation we obtain

$$(c-1)q'(z)e^{q(z)} = 0 \Rightarrow q'(z) = 0 \Rightarrow q$$
 is constant

This allows us to conclude that f is constant as  $f = e^q + b$ .

If it occured that c=1, then p'=q' and so q(z)=p(z)+d for some  $d\in\mathbb{C}$ . Replacing this in the equation we have

$$e^{p(z)} - e^{p(z)+d} = b - a \Rightarrow (1 - e^d)e^{p(z)}p'(z) = 0 \Rightarrow p'(z) = 0 \Rightarrow p$$
 is constant

and once again we deduce f is constant. Finally if it was the case that d = 0, then p = q but this means that

$$f(z) - e^{p(z)} = a = b$$

and this can't happen as a, b are different values of  $\mathbb{C}$ . In conclusion we have that *f* is constant.

**Exercise 3.** Assume  $Re(s) = \sigma > 0$ . For  $n, N \in \mathbb{N}$  define

$$\delta_n(s) = \frac{1}{n^s} - \int_n^{n+1} \frac{\mathrm{d}x}{x^s} = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) \mathrm{d}x \quad \text{and} \quad F_N(s) = \sum_{n=1}^N \delta_n(s).$$

- i) Show that  $|\delta_n(s)| \leq \frac{|s|}{n^{\text{Re}(s)+1}}$ . [Hint: Represent the integrand in the definition of  $\delta_n$  using the observation that  $\int_n^x \frac{du}{u^{s+1}} = \frac{-1}{s}(x^{-s} - n^{-s})$ . ]ii) Show that  $(F_N(s))$  converges uniformly on any half-plane of the form  $\text{Re}(s) \geqslant$
- $\alpha > 0$ .
- iii) Show that  $\zeta(s)-\frac{1}{s-1}$  is bounded and holomorphic near s=1. [Hint: Use the fact that  $\frac{1}{s-1}=\int_1^\infty x^{-s}\mathrm{d}x$ .]

## Answer

i) Observe that from the hint we have

$$s \int_{n}^{x} \frac{\mathrm{d}u}{u^{s+1}} = \left(\frac{1}{n^s} - \frac{1}{x^s}\right).$$

Then replacing this expression inside  $\delta_n$  we get

$$|\delta_n(s)| = \left| \int_{n}^{n+1} s \int_{n}^{x} \frac{\mathrm{d}u}{u^{s+1}} \mathrm{d}x \right| \leqslant \int_{n}^{n+1} |s| \left| \int_{n}^{x} \frac{\mathrm{d}u}{u^{s+1}} \right| \mathrm{d}x$$

and we can estimate the inner integral by taking the integrand's sup-norm. We have

$$\left| \int_{x}^{x} \frac{\mathrm{d}u}{u^{s+1}} \right| \le \sup_{n \le u \le x} \left| \frac{1}{u^{s+1}} \right| (x-n)$$

and for positive real numbers u we can estimate  $|u^z|$  as

$$|u^x||u^{iy}| = u^x|e^{iy\log(u)}| = u^x$$

so this means that

$$\sup_{n \leqslant u \leqslant x} \left| \frac{1}{u^{s+1}} \right| (x-n) = \sup_{n \leqslant u \leqslant x} \frac{1}{u^{\sigma+1}} (x-n) \leqslant \frac{x-n}{n^{\sigma+1}}$$

where the last inequality holds because  $\frac{1}{u^{\sigma+1}}$  is a decreasing function. Returning to our  $\delta_n$  estimate we have

$$\int_{n}^{n+1} |s| \left| \int_{n}^{x} \frac{\mathrm{d}u}{u^{s+1}} \right| \mathrm{d}x \leqslant \int_{n}^{n+1} |s| \frac{x-n}{n^{\sigma+1}} \mathrm{d}x = \frac{|s|}{n^{\sigma+1}} \frac{1}{2} \leqslant \frac{|s|}{n^{\sigma+1}}.$$

ii) We now consider the partial sums  $F_N = \sum_{n=1}^N \delta_n$ . Observe that we may bound  $F_N$  with the countable triangle inequality:

$$|F_N(s)| \le \sum_{n=1}^N |\delta_n(s)| \le \sum_{n=1}^N \frac{|s|}{n^{\sigma+1}}$$

and this is a series of real numbers which converges when  $\sigma > 0$ . This implies that  $F_N$  converges uniformly for  $\sigma \ge \alpha > 0$ .

iii) Observe that from our initial identity

$$\delta_n(s) = \frac{1}{n^s} - \int_{n}^{n+1} \frac{\mathrm{d}x}{x^s}$$

we can sum up to N to obtain

$$F_N(s) = \sum_{n=1}^N \frac{1}{n^s} - \sum_{n=1}^N \left( \int_n^{n+1} \frac{\mathrm{d}x}{x^s} \right) = \sum_{n=1}^N \frac{1}{n^s} - \int_1^{N+1} \frac{\mathrm{d}x}{x^s}.$$

Letting N grow without bound we arrive at

$$\lim_{N \to \infty} F_N(s) = \zeta(s) - \int_1^\infty \frac{\mathrm{d}x}{x^s} = \zeta(s) - \frac{1}{s-1}.$$

As the properties of the sequence  $(F_N(s))$  are preserved through uniform limit, we have that  $F_\infty(s)$  is bounded and holomorphic for  $\sigma>0$ . So our expression for the equality is only valid wherever the improper integral converges, and this is where  $\sigma>1$ . So we obtain the desired result as  $F_\infty$  is  $\zeta(s)-\frac{1}{s-1}$ .