Exercise 1. Find an example of two curves in $\mathbb{P}^2$ that have the same degree but are not isomorphic.

Answer

Let us consider the curves $V_1 = \mathbb{V}(xy)$ and $V_2 = \mathbb{V}(xy - z^2)$. To find the degrees of these curves we will calculate their Hilbert polynomials. To that effect let us decompose $\mathbb{C}[x, y, z]$ into equally graded parts and then use the relations in our ideals:

$$\mathbb{C}[x, y, z] = \mathbb{C} \oplus \mathrm{gen}(x, y, z) \oplus \mathrm{gen}(x^2, y^2, z^2, xy, xz, yz) \oplus \ldots$$

And so, applying the relation $xy = 0$ we lose an xy in the R_2 component. Looking at the degree 3 component we get

$$\mathrm{gen}(x^3, y^3, z^3, \underline{x^2y}, x^2z, \underline{y^2x}, y^2z, z^2x, z^2y, \underline{xyz}),$$

where the underlined elements are the generators we lose. We can see that the elements we have lost are the degree 1 generators multiplied by xy. Likewise in the case of R_2 we lost the xy when we multiplied 1 by it. Therefore, the amount of generators of R_m in $\mathbb{C}[V_1]$ will be $\left(\!\binom{2+1}{m}\!\right) - \left(\!\binom{2+1}{m-2}\!\right)$. This quantity is

$$\begin{aligned}\binom{2+m}{m} - \binom{2+m-2}{m-2} &= \frac{(m+2)!}{2m!} - \frac{m!}{2(m-2)!} \\ &= \frac{(m+2)(m+1)}{2} - \frac{m(m-1)}{2} \\ &= 2m + 1,\end{aligned}$$

and so if the degree of the Hilbert polynomial is k, then $\deg(V) = k!a_k$. It holds that the degree of $\mathbb{V}(xy)$ is 2. This can also be seen by intersecting a *general line* through the variety.

On the other hand, when taking the quotient by $\mathrm{gen}(xy - z^2)$ and doing the same process we are losing[a] the same amount (albeit different ones) of generators on each step. Thus the Hilbert polynomial for V_2 is also $2m + 1$.

Finally, notice that V_1 is a reducible variety as $V_1 = \mathbb{V}(x) \cup \mathbb{V}(y)$ and V_2 is irreducible. Should there be an isomorphism between these varieties, it should preserve reducibility. This is impossible so it holds that V_1 and V_2 are not isomorphic, but they have the same degrees.

[a]Not exactly losing, I think a better word or description would be *adding a trivial generator to our set*.

Exercise 2. Do the following:

i) Find the Hilbert polynomial P of a k-dimensional linear subvariety of $\mathbb{P}^n$.
ii) Describe the Hilbert scheme of varieties in $\mathbb{P}^n$ with Hilbert polynomial P.

Answer

i) Let us begin by considering a dimensional argument. Recall adding any equation reduces our dimension by 1, from dimension n to dimension k we have lost $n-k$ dimensions so our variety V is

$$V = \mathbb{V}(L_1, \ldots, L_{n-k}).$$

Each of the equations $L_j = 0$ is of the form $\langle \mathbf{a}_j | x \rangle = 0$, so without losing generality, let us suppose that the j^{th} entry of $\mathbf{a}_j$ is non-zero. We can solve for x_j as a linear combination of the other variables. Adding this relations to $\mathbb{C}[x_0, \ldots, x_n]$ we see that at degree one we already lose $n-k$ variables due to the relations. In total the dimension of R_m, the m^{th} graded part is $\left(\!\binom{k+1}{m}\!\right) = \binom{k+m}{m}$. This is the Hilbert function of V.

ii) The Hilbert scheme of varieties with polynomial $\left(\!\binom{k+1}{m}\!\right) = \frac{1}{k!}m^k + \ldots$ contains all varieties of dimension k and degree $k!\frac{1}{k!} = 1$. So all the k-dimensional linear subvarieties are in the Hilbert scheme.

An *educated guess* would lead us to think that the other inclusion is also true, however I don't know how to go about this. (How can I proceed?)

Exercise 3. Assume that the variety $V \subseteq \mathbb{P}^n$ has the Hilbert polynomial $P(n)$. Calculate the Hilbert polynomial of the image variety $\nu_d(V) \subseteq \mathbb{P}^{\binom{n+d}{d}-1}$ of the Veronese map. ⟦Hint: Do the case of $V = \mathbb{P}^1$ first.⟧

Answer

Let us begin by calling $W = \nu_d(\mathbb{P}^1)$. We will first find the Hilbert Polynomial for W and then work our way up. To do this, let us remember the construction of the Veronese mapping with the naming conventions for our variables. We have that

$$\mathbb{C}[W] = \mathbb{C}[z_{d,0}, z_{d-1,1}, z_{d-2,2}, \ldots, z_{0,d}] \Big/ \operatorname{gen}(z_I z_J - z_K z_L)$$

where $I, \ldots, L \in [d]^2$ and $I + J = K + L$. We can observe that the R_1 component of this ring corresponds with the R_d component of $\mathbb{C}[s,t] = \mathbb{C}[\mathbb{P}^1]$. This is because the generators of the R_1 component are the $z_I = s^{I_1}t^{I_2}$ and $I_1 + I_2 = d$. The generators of the R_d homogeneous component of $\mathbb{C}[s,t]$ are those such elements. This means that

$$P_{\mathbb{P}^1}(d) = d + 1 = P_W(1),$$

and in the same fashion we can count the generators of the R_{kd} component of $\mathbb{C}[s,t]$ and we will find that they are in correspondence with the R_k component of $\mathbb{C}[W]$. It follows that $P_W(k) = P_{\mathbb{P}^1}(kd) = kd + 1$.
In the same fashion, a variety V will introduce a certain number of relations to $\mathbb{C}[x_0, \ldots, x_n]$. Then $P_V(m)$ will count the number of generators in R_m which has been already reduced by the relations introduced by V. The same issue happens in R_{md} so it must hold that $P_V(dm) = P_{\mathrm{Im}(\nu_d(V))}(m)$.[a]

[a] I must admit that I am at a loss. I have several ideas such as working with induction through m but I feel that it is not the way to go. I'd like to discuss this problem if possible.

Exercise 4. Using the theorem describing the defining equations for T_pV in terms of the equations for V, compute the tangent spaces of the curves in examples (1), (2), and (3) at the origin.

Answer

(a) The curve in question is $\mathbb{V}(y - x^2)$, our function is $P_1(x,y) = y - x^2$ then $\nabla P_1(x,y) = (-2x, 1)$. The tangent space at the origin is the zero locus of

$$\langle \nabla P_1(0,0) | (x,y) - (0,0) \rangle = \langle (0,1) | (x,y) \rangle = y.$$

This coincides with our original finding because $\mathbb{V}(y)$ is precisely the x-axis which is tangent to the parabola at the origin.

(b) Now we are working with $\mathbb{V}(y^2 - x^2 - x^3)$, then $P_2(x,y) = y^2 - x^2 - x^3$. The differential in this case is

$$\nabla P_2(x,y) = (-2x - 3x^2, 2y) \xrightarrow{\varepsilon_0} \nabla P_2(0,0) = (0,0)$$

and so the variety in question is the zero locus of the zero function. As the whole of $\mathbb{A}^2$ is such set, we can see that this makes sense because the origin is a singular point of our variety.

(c) Finally let us consider $\mathbb{V}(y^2 - x^3)$. In this case

$$\langle \nabla P_3(0,0) | (x,y) - (0,0) \rangle = \Big\langle (-3(0)^2, 2(0)) \Big| (x,y) \Big\rangle = 0,$$

and once again our tangent space is the whole affine plane. This is agrees with what we have seen, the curve has a singular point at the origin.

Exercise 5. Let $V \subseteq \mathbb{P}^n$ be a hypersurface defined by a homogeneous irreducible polynomial F. Find an explicit description of the tangent space to V at a point p. What conditions on p ensure that the tangent space to V at p has dimension $n-1$?

Answer

Let us begin by considering an affine chart $U_i \simeq \mathbb{A}^n$ which contains p. Our projective variety V becomes an affine variety $V \cap U_i$ which is the zero locus of the de-homogenized polynomial $\widetilde{F} = F\big|_{x_i=1}$.
We can now describe the tangent space at p as

$$T_p(V \cap U_i) = \mathbb{V}\left(\Big\langle \nabla \widetilde{F}(p) \Big| \mathbf{x} - p \Big\rangle\right).$$

The projective closure of this affine algebraic variety is the *projective tangent space* of V at p. To find this, let us simplify notation a bit by calling L the linear polynomial in question.

- ◇ We can see that L is an irreducible polynomial through a degree argument. If L were reducible then $L = pq$ and $\deg(L) = \deg(p) + \deg(q)$. As the degree is an integer, p or q must be a linear polynomial and the other a constant.
- ◇ Now the polynomial ring we are working in is a UFD so irreducibles are prime, then it holds that $\operatorname{gen}(L)$ is a prime ideal and therefore radical.
- ◇ Recall, by the projective closure theorem, the ideal generated by the homogenization of *all* elements of $\sqrt{\operatorname{gen}(L)}$ is $\mathbb{I}(\overline{V})$. But as $\sqrt{\operatorname{gen}(L)} = \operatorname{gen}(L)$ we have that $\mathbb{I}(\overline{V})$ is generated by elements of the form ${}^h(p \cdot L)$ where the homogenization is taken with respect to the variable x_i.

In summary the tangent space is the zero locus of $\operatorname{gen}({}^h(p \cdot L))$ where p is any polynomial and L is the differential of F.

Now, as F is an homogeneous irreducible polynomial, the variety V has dimension $n-1$. For the tangent space to have that same dimension, it must hold that p is a *smooth point* of V. For this to happen p must not be a *singular point* and this happens when

$$p \notin \mathbb{V}(\partial_0 F, \partial_1 F, \ldots, \partial_n F).$$