Homework 1

Due: Friday, January 27

1. [SS]1.1. *In other words, do problem 1 from Chapter 1 of Stein and Shakarchi.*

2. (a) Show that the complex conjugation map

$$\mathbb{C} \xrightarrow{\kappa} \mathbb{C}$$

$$z \longmapsto \overline{z}$$

   is an involution, i.e., a ring homomorphism such that $\kappa \circ \kappa = \mathrm{id}$.

   (b) Suppose $a \in \mathbb{R}$ and $z \in \mathbb{C}$. Show that

$$\begin{aligned}\mathrm{Re}(az) &= a\,\mathrm{Re}(z)\\ \mathrm{Im}(az) &= a\,\mathrm{Im}(z).\end{aligned}$$

3. (a) Prove that

$$|z+w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\overline{w}).$$

   (b) Use this to prove the parallelogram rule:

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

4. [SS]1.5.

5. [SS]1.7.

   Here is an alternate approach to 1.7, which you may use if you like. Fix $w \in \mathbb{C}$ with $|w| < 1$, and consider the function

$$f(z) = f_w(z) = \frac{w-z}{1-\overline{w}z}.$$

   What is $\overline{f(z)}$? By computing $f(z)\overline{f(z)}$, show that if $|z| = 1$ then $|f(z)| = 1$.

   Find a point z, $|z| < 1$, such that $|f(z)| < 1$. Since f is continuous, this shows that f takes the unit disc to itself. (Why?)