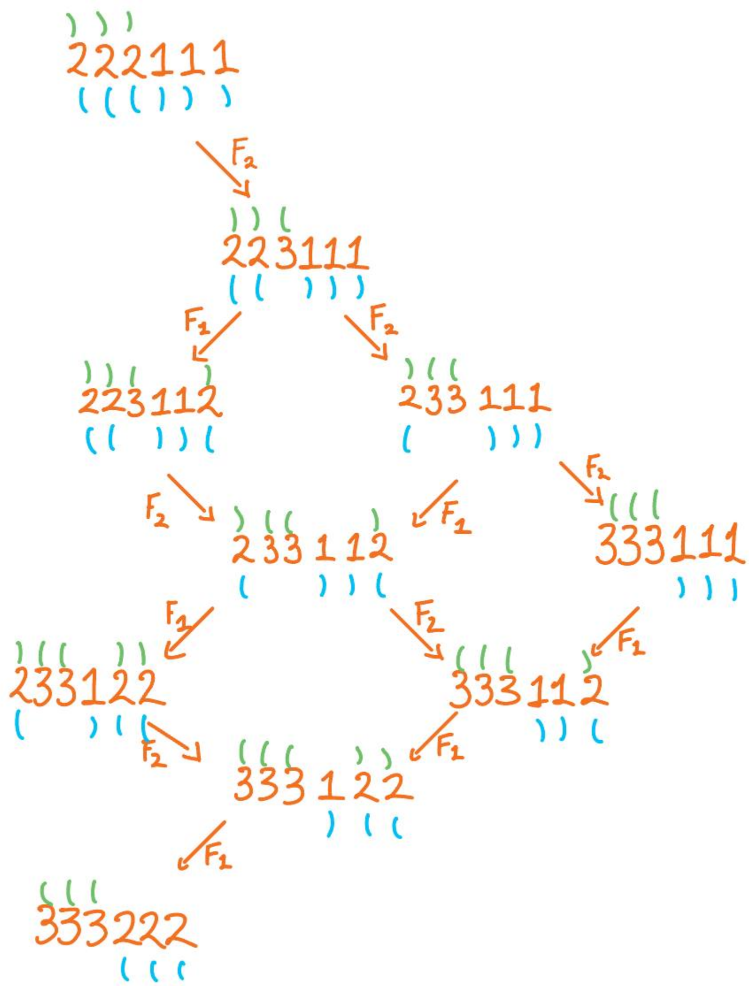
Exercise 1. Draw the $\mathfrak{sl}_3$ crystal for weight $(3, 3, 0)$.

Answer



Exercise 2. Prove that the elements of the hyperoctahedral group, written in cycle notation as a permutation on $\{\pm 1, \ldots, \pm n\}$, has all of its cycles coming in either pairs of the form $(a_1 \ldots a_k)(-a_1 \cdots - a_k)$, or of the form $(a_1 \ldots a_k - a_1 \ - a_2 \cdots - a_k)$.

I didn't give myself time for this one :(

Exercise 3. Define the Lie algebra $\mathfrak{so}_{2n+1}$ as $\{ X : X^{\mathsf{T}} S + SX = 0 \}$ where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0_n & I_n \\ 0 & I_n & 0_n \end{pmatrix}$$

and I_n is the $n \times n$ identity matrix and 1 is in the upper left corner. Write down what an arbitrary element X looks like, and using the fact that with respect to this setup the torus is simply the set of diagonal matrices X satisfying these conditions, explain how one obtains the type B root system.

Answer

First observe that the matrix S is a permutation matrix which acts by row permutation when applied as left multiplication. The permutation it applies is a product of disjoint transpositions of the form $\begin{pmatrix} i & n+i \end{pmatrix}$ for $i \in [n+1] \setminus \{ 1 \}$. Therefore, SX is a matrix with rows $i, n+i$ switched. For $X^{\mathsf{T}} S$ observe that this is $(S^{\mathsf{T}} X)^{\mathsf{T}} = (SX)^{\mathsf{T}}$ as S is symmetric. From this the equation

$$(SX)^{\mathsf{T}} + SX = 0$$

we deduce that X must be divided into 8 nonzero blocks as follows:

$$X = \begin{pmatrix} 0 & \mathbf{x}_{1,1} & \mathbf{x}_{1,2} \\ -\mathbf{x}_{1,2}^{\mathsf{T}} & A & B \\ -\mathbf{x}_{1,1}^{\mathsf{T}} & C & -A^{\mathsf{T}} \end{pmatrix}.$$

Here the vectors $\mathbf{x}_{1,:}$ represent the first row of the matrix X where the index 1 contains the first $2, \ldots, n+1$ entries whereas the other contains the ones from $n+2$ to $2n+1$. The matrices B, C have zero diagonals and A is the minor matrix of X containing information from rows 2 to $n+1$.
Now an element of the torus looks like

$$\operatorname{diag}(x_{22}, \ldots, x_{n+1\ n+1}, -x_{22}, \ldots, -x_{n+1\ n+1}),$$

and taking the elements L_i as these diagonal entries we have the relations $L_{n+i} = -L_i$ which give us the desired relationships for a type B system.

Exercise 4. What is the dimension of the adjoint representation of $\mathfrak{so}_7$?

Answer

Via the isomorphism $X \mapsto [X, -]$ we have that the dimension of the adjoint representation is the same as $\dim \mathfrak{so}_7$ which is $\binom{7}{2} = 21$.

Exercise 5. Explain why the set of 5^{th} roots of unity in the plane don't form a root system. Which axioms of root systems does it satisfy?

Answer

The axioms we should check are:

(a) The roots span our vector space.

(b) The reflections across hyperplanes are still roots.

(c) Projections onto the span of a single root are an integer multiple or a half-integer multiple of the root.

(d) If α, β are roots such that $\beta = \lambda\alpha$ then $\lambda = \pm 1$.

The set of 5^{th} roots of unity can be described as the set of vectors of the form

$$(\zeta_k) = \left(\cos\left(\frac{2\pi i k}{5}\right), \sin\left(\frac{2\pi i k}{5}\right)\right) \quad \text{with} \quad k \in [5]\,.$$

Any two of these distinct roots generate the plane as they are all linearly independent. The reflection of ζ_k across the normal plane to ζ_h is given by

$$r_h(k) = \zeta_k - 2\frac{|\zeta_h \rangle\langle \zeta_h|}{\langle \zeta_h|\zeta_h\rangle}\zeta_k$$

$$r_h(k) = \zeta_k - 2\langle \zeta_h|\zeta_k\rangle \zeta_h$$

and we can see that

$$\langle \zeta_h|\zeta_k\rangle = \cos\left(\frac{2\pi i(h-k)}{5}\right).$$

Simplifying the above expression for the reflection we have

$$r_h(k) = \left(-\cos\left(\frac{2\pi i(2h-k)}{5}\right), -\sin\left(\frac{2\pi i(2h-k)}{5}\right)\right).$$

Observe that what this reflection is then doing is reflecting ζ_k across the line generated by ζ_h, which would indeed return another 5^{th} root of unity, *but then* it

takes its negative! This returns us a primitive 10^{th} root of unity showing that the system is not closed under reflections.
Finally the projection of ζ_k onto ζ_h is given by

$$\pi_h(k) = \frac{|\zeta_h\rangle\langle\zeta_h|}{\langle\zeta_h|\zeta_h\rangle}\zeta_k = \langle\zeta_h|\zeta_k\rangle\zeta_h$$

and it suffices to verify that the norm of this vector is not a half nor 1. However the length of this vector is $\langle\zeta_h|\zeta_k\rangle = \cos\left(\frac{2\pi i(h-k)}{5}\right)$ which is not 1 unless $h = k$ and never is $\frac{1}{2}$.

Exercise 6. Compute the evacuation of the Young tableau below, and then evacuate again, and show you have returned to the starting tableau.

$$\begin{array}{cccc} 5 & & & \\ 2 & 7 & 8 & \\ 1 & 3 & 4 & 6 \end{array}$$

Answer

We switch entries following the rule $k \mapsto n+1-k$ and then rotating $180°$:

$$\begin{array}{cccc} 5 & & & \\ 2 & 7 & 8 & \\ 1 & 3 & 4 & 6 \end{array} \longrightarrow \begin{array}{cccc} 4 & & & \\ 7 & 2 & 1 & \\ 8 & 6 & 5 & 3 \end{array} \longrightarrow \begin{array}{cccc} 3 & 5 & 6 & 8 \\ \times & 1 & 2 & 7 \\ & & & 4 \end{array}.$$

Here we have already marked the first inner corner we will move. This leads us to

$$\begin{array}{cccc} 3 & 5 & {\color{green}8} & \\ {\color{green}1} & {\color{green}2} & {\color{green}6} & 7 \\ & & \times & 4 \end{array} \longrightarrow \begin{array}{cccc} 3 & 5 & 8 & \\ 1 & 2 & 6 & \\ & \times & {\color{green}4} & {\color{green}7} \end{array} \longrightarrow \begin{array}{cccc} 3 & {\color{green}8} & & \\ 1 & {\color{green}5} & 6 & \\ \times & {\color{green}2} & 4 & 7 \end{array} \longrightarrow \begin{array}{cccc} {\color{green}8} & & & \\ {\color{green}3} & 5 & 6 & \\ {\color{green}1} & 2 & 4 & 7 \end{array}$$

where every green character moved when clearing out the inner corner in the previous step. Redoing the process we obtain the skew tableau

$$\begin{array}{cccc} 8 & & & \\ 3 & 5 & 6 & \\ 1 & 2 & 4 & 7 \end{array} \longrightarrow \begin{array}{cccc} {\color{green}1} & & & \\ {\color{green}6} & {\color{green}4} & {\color{green}3} & \\ {\color{green}8} & {\color{green}7} & {\color{green}5} & {\color{green}2} \end{array} \longrightarrow \begin{array}{cccc} {\color{green}2} & {\color{green}5} & {\color{green}7} & {\color{green}8} \\ \times & {\color{green}3} & {\color{green}4} & {\color{green}6} \\ & & & {\color{green}1} \end{array}.$$

With the first inner corner marked, we move it out and continue the process:

$$\begin{array}{cccc} {\color{green}5} & {\color{green}7} & {\color{green}8} & \\ {\color{green}2} & 3 & 4 & 6 \\ & & \times & 1 \end{array} \longrightarrow \begin{array}{cccc} 5 & 7 & 8 & \\ 2 & 3 & 4 & \\ & \times & {\color{green}1} & {\color{green}6} \end{array} \longrightarrow \begin{array}{cccc} 5 & 7 & & \\ 2 & 3 & {\color{green}8} & \\ \times & {\color{green}1} & {\color{green}4} & 6 \end{array} \longrightarrow \begin{array}{cccc} 5 & & & \\ 2 & {\color{green}7} & 8 & \\ {\color{green}1} & {\color{green}3} & 4 & 6 \end{array}.$$

As we have returned to our original tableau we conclude that the process is correct

Exercise 7. Compute the Hall-Littlewood polynomial $\tilde{H}_{(2,1,1)}(x;q)$.

Answer

We first find all SSYT with content $(2,1,1)$. These are:

$$\begin{ytableau} 3 \\ 2 \\ 1 & 1 \end{ytableau}\,, \quad \begin{ytableau} 2 & 3 \\ 1 & 1 \end{ytableau}\,, \quad \begin{ytableau} 3 \\ 1 & 1 & 2 \end{ytableau}\,, \quad \begin{ytableau} 2 \\ 1 & 1 & 3 \end{ytableau}\,, \quad \begin{ytableau} 1 & 1 & 2 & 3 \end{ytableau}\,.$$

The cocharge labeling of their reading words is respectively

$$2100, \quad 1100, \quad 1000, \quad 1001, \quad 0000$$

giving us cocharges: $3, 2, 1, 2, 0$. This means that the Hall-Littlewood polynomial is

$$\begin{aligned} &q^3 s_{(2,1,1)} + q^2 s_{(2,2)} + q s_{(3,1)} + q^2 s_{(3,1)} + q^0 s_4 \\ =&q^3 s_{(2,1,1)} + q^2 s_{(2,2)} + (q + q^2) s_{(3,1)} + s_4. \end{aligned}$$

Exercise 8. Let $w = w_1 \ldots w_n$ be a word of partition content, and suppose $w_1 \neq 1$. Let $w' = w_2 \ldots w_n w_1$ be formed by cycling w_1 around to the end of the word. Show that $\boldsymbol{c}(w') = \boldsymbol{c}(w) - 1$ where $\boldsymbol{c}$ is cocharge. This operation is called *cyclage*.

Answer

Observe that it suffices to view this on standard words. This is because we may separate a word into standard subwords and calculate cocharge[a]. Consider the subword $\tilde{w}$ of w which contains w_1 in the previous decomposition sense, as w has partition content so does $\tilde{w}$.
When cycling w_1 to the end of $\tilde{w}$, cocharge is reduced by 1 as there is a element in $\tilde{w}$ smaller than w_1 which was to the right of w_1. After cycling, it's to the *left* and so the cocharge labeling drops by one.

[a] Ah! Inadvertently **you** helped me with this problem as the decomposition idea was written on your thesis!

Exercise 9. Give a counterexample showing that the formula in the above problem does not hold in general when $w_1 = 1$.

Answer

The word 121 has cocharge labeling 000 giving it a cocharge of 0 whereas 211 has cocharge labeling 100 with cocharge 1.