Exercise 1. Suppose $\mathcal{F}$ is a presheaf and $\mathcal{G}$ is a sheaf, both of sets, on X. Let $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Show that this is a sheaf of sets on X.

Answer

We first need to show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a presheaf, this requires a sensible notion of restriction mapping which satisfies the following:

i) $\mathrm{res}_{U,U} = \mathrm{id}_{(*)}$ where the identity map is over the object $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$.
ii) If $U \subseteq V \subseteq W$ then $\mathrm{res}_{W,U} = \mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}$.

Let us consider two objects $\mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$ and $\mathrm{Mor}(\mathcal{F}|_V, \mathcal{G}|_V)$ with $U \subseteq V$. A restriction mapping acts on sections, and sections on these sets are morphisms of sheaves. Our restriction mapping takes $\varphi \in \mathrm{Mor}(\mathcal{F}|_V, \mathcal{G}|_V)$ to $\mathrm{res}_{V,U}(\varphi) \in \mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$, but recall φ is a collection of maps of objects of the form

$$\varphi(W) : \mathcal{F}(W) \to \mathcal{G}(W), \quad \text{with} \quad W \subseteq V.$$

In this sense, it suffices to only consider the open sets contained in U. We declare that $\mathrm{res}_{V,U}(\varphi)$ is the collection of maps

$$\varphi(W) : \mathcal{F}(W) \to \mathcal{G}(W), \quad \text{with} \quad \underline{W \subseteq U}.$$

i) The map $\mathrm{res}_{U,U}(\varphi)$ acts as follows, every map of the form $\varphi(W)$ with $W \subseteq U$ is sent to the map $\varphi(W)$ between the same objects because $W \subseteq U$ is still itself.

   This means that $\mathrm{res}_{U,U}$ is the identity map in $\mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$.
ii) Now suppose $U \subseteq V \subseteq W$ are open sets, then $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}$ acts on φ first by restricting from open sets in W to open sets in V and next by passing from open sets in V to only considering the open sets in U.

   This is the same as starting with the open sets in W and then only considering the open sets in U. The last action is the same as what $\mathrm{res}_{W,U}$ does to φ.

This allows us to conclude that the sheaf-Hom is indeed a presheaf. We now have to verify the two sheaf axioms:

i) Take (U_i) an cover of $U \subseteq X$ with $\varphi, \psi : \mathcal{F}|_U \to \mathcal{G}|_U$ sections which coincide in every covering set. This means that

$$\operatorname*{res}_{U,U_i}(\varphi) = \operatorname*{res}_{U,U_i}(\psi) \iff \forall i \left[\varphi(V) = \psi(V),\ V \subseteq U_i\right].$$

Where $\varphi(V), \psi(V)$ are maps of objects from $\mathcal{F}(V)$ to $\mathcal{G}(V)$. We wish to show that they coincide on all of U, which means that for any $V \subseteq U$ and $f \in \mathcal{F}(V)$, it holds that

$$\varphi(V)(f) = \psi(V)(f).$$

Even though we may not talk about these sections directly, we can talk about them after restricting from V to $V \cap U_i$ where U_i is any covering set. To do so, let us introduce the following diagrams:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\varphi(V)\text{,}\psi(V)"] \arrow[d, "\mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}^{\mathcal{G}}_{V,V\cap U_i}"] \\ \mathcal{F}(V \cap U_i) \arrow[r] & \mathcal{G}(V \cap U_i) \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r] \arrow[d] & (*) \arrow[d] \\ \mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}(f) \arrow[r] & (**) \end{tikzcd}$$

The lower arrow in the left diagram is either of the two morphisms $\varphi(V \cap U_i), \psi(V \cap U_i)$. The right diagram is the same but section-wise:

- ◇ The upper right corner is the image of the section f inside $\mathcal{G}(V)$ through $\varphi(V)$ or $\psi(V)$.
- ◇ The lower right corner can be interpreted in two ways which coincide:

$$\varphi(V \cap U_i)\left[\operatorname*{res}^{\mathcal{F}}_{V,V\cap U_i}(f)\right] = \operatorname*{res}^{\mathcal{G}}_{V,V\cap U_i}(\varphi(V)(f))$$

  and the same expression for ψ when that's the case. This equality is due to the fact that φ, ψ are morphisms of sheaves and therefore commute with restrictions.

Recall now that $\varphi(V) = \psi(V)$ for $V \subseteq U_i$, in particular we have $\varphi(V \cap U_i) = \psi(V \cap U_i)$. So mapping f from the upper left to the lower right gives us

$$\begin{aligned}\overset{\mathcal{G}}{\underset{V,V\cap U_i}{\mathrm{res}}}(\varphi(V)(f)) &= \varphi(V \cap U_i)\left[\overset{\mathcal{F}}{\underset{V,V\cap U_i}{\mathrm{res}}}(f)\right] \\ &= \psi(V \cap U_i)\left[\overset{\mathcal{F}}{\underset{V,V\cap U_i}{\mathrm{res}}}(f)\right] \\ &= \overset{\mathcal{G}}{\underset{V,V\cap U_i}{\mathrm{res}}}(\psi(V)(f))\end{aligned}$$

where the first and last equalities occur because φ and ψ are morphisms of sheaves and the middle one because of the hypothesis.

By the identity axiom on $\mathcal{G}$, as $\mathcal{G}$ is a sheaf, we can conclude that $\varphi(V)(f) = \psi(V)(f)$. This means that $\varphi(V) = \psi(V)$, but as $V \subseteq U$ is arbitrary, we conclude that $\varphi = \psi$ and therefore we get the identity axiom.

ii) Once again let us take (U_i) to be an open cover of $U \subseteq X$ along with $\varphi_i \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i)$ for each i. These are morphisms of sheaves, which means that for all open subsets $V \subseteq U_i$ they are maps between objects:

$$\varphi_i(V) : \mathcal{F}(V) \to \mathcal{G}(V),\ V \subseteq U_i.$$

Assume now that the condition $\mathrm{res}_{U_i,U_i\cap U_j}(\varphi_i) = \mathrm{res}_{U_i,U_i\cap U_j}(\varphi_j)$ holds for all i, j. We must show that there exists a section $\varphi \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$, which is

$$\varphi(V) : \mathcal{F}(V) \to \mathcal{G}(V),\ V \subseteq U$$

that satisfies $\mathrm{res}_{U,U_i}(\varphi) = \varphi_i$. This means that for open sets $V \subseteq U_i$, it must hold that

$$\underset{U,U_i}{\mathrm{res}}(\varphi)(V) = \varphi_i(V),\ V \subseteq U_i.$$

For this purpose, we will use the gluing axiom on the sheaf $\mathcal{G}$. Let us now proceed by taking a section $f \in \mathcal{F}(V)$ with $V \subseteq U$ and map it through the following diagram:

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[dr, "\mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}"'] & & \\
& \mathcal{F}(V\cap U_i) \arrow[r, "\varphi_i(V\cap U_i)"] \arrow[d, "\mathrm{res}^{\mathcal{F}}_{V\cap U_i,V\cap U_i\cap U_j}"'] & \mathcal{G}(V\cap U_i) \arrow[d, "\mathrm{res}^{\mathcal{G}}_{V\cap U_i,V\cap U_i\cap U_j}"] \\
& \mathcal{F}(V\cap U_i\cap U_j) \arrow[r] & \mathcal{G}(V\cap U_i\cap U_j)
\end{tikzcd}$$

where the lower arrow is the map $\varphi_i(V \cap U_i \cap U_j)$. We can construct a similar diagram of φ_j. A section $f \in \mathcal{F}(V)$ maps through that diagram as follows:

$$\begin{tikzcd}
f \arrow[dr] & & \\
& \mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}(f) \arrow[r] \arrow[d] & \varphi_i(V\cap U_i)\left[\mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}(f)\right] \arrow[d] \\
& \mathrm{res}^{\mathcal{F}}_{V\cap U_i,V\cap U_i\cap U_j}\left[\mathrm{res}^{\mathcal{F}}_{V,V\cap U_i}(f)\right] \arrow[r] & (*)
\end{tikzcd}$$

and the lower right corner is either the restriction of the upper right corner, or the image of the lower left which by $\varphi_i(V \cap U_i \cap U_j)$. As the vf_i are morphisms of sheaves, both elements are equal. This can be expressed as follows

$$\begin{aligned}
&\varphi_i(V\cap U_i\cap U_j)\left(\underset{V\cap U_i,V\cap U_i\cap U_j}{\overset{\mathcal{F}}{\mathrm{res}}}\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\mathrm{res}}}(f)\right]\right)\\
&= \underset{V\cap U_i,V\cap U_i\cap U_j}{\overset{\mathcal{G}}{\mathrm{res}}}\left(\varphi_i(V\cap U_i)\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\mathrm{res}}}(f)\right]\right)
\end{aligned}$$

But, let us simplify notation a bit by remembering that the composition of restriction maps is the beginning-to-end restriction map. This means that

$$\underset{V\cap U_i,V\cap U_i\cap U_j}{\overset{\mathcal{F}}{\mathrm{res}}}\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\mathrm{res}}}(f)\right] = \underset{V,V\cap U_i\cap U_j}{\overset{\mathcal{F}}{\mathrm{res}}}(f).$$

With this in hand, and remembering the hypothesis that our φ_i's coincide on intersections of the covering sets, we have:

$$\begin{aligned}
&\underset{V\cap U_i, V\cap U_i\cap U_j}{\overset{\mathcal{G}}{\operatorname{res}}}\left(\varphi_i(V\cap U_i)\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\operatorname{res}}}(f)\right]\right)\\
=&\varphi_i(V\cap U_i\cap U_j)\left[\underset{V,V\cap U_i\cap U_j}{\overset{\mathcal{F}}{\operatorname{res}}}(f)\right]\\
=&\varphi_i(V\cap U_i\cap U_j)\left[\underset{V,V\cap U_i\cap U_j}{\overset{\mathcal{F}}{\operatorname{res}}}(f)\right]\\
=&\underset{V\cap U_j, V\cap U_i\cap U_j}{\overset{\mathcal{G}}{\operatorname{res}}}\left(\varphi_j(V\cap U_j)\left[\underset{V,V\cap U_j}{\overset{\mathcal{F}}{\operatorname{res}}}(f)\right]\right).
\end{aligned}$$

So by gluing the maps $\varphi_i(V\cap U_i)\left[\operatorname{res}^{\mathcal{F}}_{V,V\cap U_i}(f)\right]$ in $\mathcal{G}$ we may construct a map $g\in\mathcal{G}(V)$ such that

$$\underset{V,V\cap U_i}{\overset{\mathcal{G}}{\operatorname{res}}}(g)=\varphi_i(V\cap U_i)\left[\underset{V,V\cap U_i}{\overset{\mathcal{F}}{\operatorname{res}}}(f)\right]$$

for each i. We finally define the glued map φ in $\mathcal{H}om$ which takes our original f to this g which we have found. It follows from our construction that

$$\underset{U,U_i}{\operatorname{res}}(\varphi)(V)=\varphi_i(V),\ V\subseteq U_i.$$

After verifying the axioms, we may conclude that the sheaf Hom is indeed a sheaf.

Exercise 2. Show that the presheaf cokernel satisfies the universal property of cokernels in the category of presheaves.

Answer

Given a map of presheaves φ, we must show that for $\operatorname{coker}_{\text{pre}}\varphi$ given the following diagram:

$$\begin{tikzcd}
 & \mathcal{G} \arrow[d, "c"] \arrow[ddr, bend left, "c'"] & \\
\mathcal{F} \arrow[ur, "\varphi"] \arrow[r, "0"'] \arrow[drr, bend right, "0"'] & \operatorname{coker}_{\text{pre}} \varphi \arrow[dr, dashed] & \\
 & & \mathcal{C}
\end{tikzcd}$$

that there exists a unique morphism of presheaves $\psi : \operatorname{coker}_{\text{pre}} \varphi \to \mathcal{C}$. Taking out particular objects for any open set U we have the same diagram but in terms of objects in the underlying category which is an abelian category. Thus, there exists a unique map

$$\psi(U) : \operatorname{coker}_{\text{pre}} \varphi(U) \to \mathcal{C}(U)$$

and with this we may define the morphism of presheaves $\operatorname{coker}_{\text{pre}} \varphi \to \mathcal{C}$ by taking each of these maps into our collection of data. This immediately gives us unicity by construction and we are left to check that ψ is a morphism of presheaves. Thi means that for $U \subseteq V$, the following diagram commutes

$$\begin{tikzcd}
\operatorname{coker}_{\text{pre}} \varphi(V) \arrow[r, "\psi(V)"] \arrow[d, "\operatorname{res}^{\text{coker}}_{V,U}"'] & \mathcal{C}(V) \arrow[d, "\operatorname{res}^{\mathcal{C}}_{V,U}"] \\
\operatorname{coker}_{\text{pre}} \varphi(U) \arrow[r, "\psi(U)"'] & \mathcal{C}(U)
\end{tikzcd}$$