MATH502 — Combinatorics 2

Based on the lectures by Maria Gillespie

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Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This is the second semester of an introductory graduate-level course on combinatorics. We will be covering symmetric function theory, Young tableaux, counting with group actions, designs, matroids, finite geometries, and not-so-finite geometries.

The goal of this class is to give an overview of the wide variety of topics and techniques in both classical and modern combinatorial theory.

Requirements

Knowledge on theory of enumeration, generating functions, combinatorial species, the basics of graph theory, posets, partitions and tableaux, and basic symmetric function theory is required.

Contents

Chapter 1

Symmetric functions

1.1 Day 1 | 20230120

Definition 1.1.1. $f(x_1, x_2, \dots)$ is symmetric if it's fixed under permutations of variables. For a permutation σ this is,

$$f(x_{\sigma(1), x_{\sigma(2)}}, \dots) = f(x_1, x_2, \dots).$$

Example 1.1.2. The function

$$f(x_1, \dots, x_4) = x_1^5 + \dots + x_4^5$$

is known as p_5 or $m_{(5)}$, where p is the power-sum symmetric function and m, the monomial symmetric function.

We can have the function defined on infinitely many variables. Consider the function g defined as

$$g = x_1^4 x_2 + x_1^4 x_3 + \dots + x_i^4 x_j + \dots + 3x_1 + \dots + 3x_i + \dots = m_{(4,1)} + 3m_{(1)}.$$

Let us recall some **notation**,

$$\begin{cases} \Lambda_R(x_1, \dots, x_n) \to \text{symmetric functions on } n \text{ variables over } R, \\ \Lambda_R(\underline{x}) \to \text{symmetric functions on } \textit{infinitely} \text{ many variables over } R. \end{cases}$$

In our case $R = \mathbb{Q}$, so the object of study is $\Lambda_{\mathbb{Q}}$.

Proposition 1.1.3. *The space $\Lambda_{\mathbb{Q}}^n$ is the space of symmetric functions of degree n. Its dimension is $p(n)$, the number of partitions of n.*

This is because, for every such function we can decompose it into monomials and the monomial symmetric functions form a basis.

Bases of Λ_Q

Suppose $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ with $\lambda_1 \geqslant \ldots \geqslant \lambda_k$.

Monomial Symmetric Functions

The function $m_\lambda(\underline{x})$ is the smallest symmetric function which contains the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_k^{\lambda_k}$ as a term. In general

$$m_\lambda = \sum_{i_1 \neq \cdots \neq i_k} x_{i_1}^{\lambda_1} \ldots x_{i_k}^{\lambda_k}.$$

Example 1.1.4. Consider the partition $(5,3) \vdash 8$. The function $m_{(5,3)}$ will be different depending on the number of variables:

- ◇ In one variable we can't have monomials of the form $x_i x_j$, so $m_{(5,3)} = 0$.
- ◇ In two variables we have $m_{(5,3)}(x,y) = x^5 y^3 + y^5 x^3$.
- ◇ In three variables the function is

$$m_{(5,3)}(x,y,z) = x^5y^3 + y^5z^3 + z^5x^3 + y^5x^3 + z^5y^3 + x^5z^3.$$

Considering some special cases, take the partition $(1,1,1,1) \vdash 4$, then

$$\begin{aligned} m_{(1,1,1,1)}(u,v,x,y,z) &= uvxy + vxyz + xyzu + yzuv + zuvx \\ &= uvxy + uxyz + uvyz + uvxz + vxyz. \end{aligned}$$

For cases with less than 4 variables the function is zero and in exactly four, it has 1 term. The partition $(4) \vdash 4$ returns the function

$$m_{(4)}(x) = x^4,\ m_{(4)}(x,y) = x^4 + y^4,\ m_{(4)}(x,y,z) = x^4 + y^4 + z^4,$$

and so on with any number of variables.

Remark 1.1.5. The number of terms in $m_\lambda(x_1, \ldots, x_d)$ is I actually don't know, while the degree of m_λ is $|\lambda| = n$.

Elementary Symmetric Functions

Definition 1.1.6. For any $r \in \mathbb{N}$, the elementary symmetric function e_r is $m_{(1,1,\ldots,1)}$ (r ones). For λ, a partition, $e_\lambda = \prod e_{\lambda_i}$. As an alternative for $m_{(1,1,\ldots,1)}$ we can also write

$$e_r(x_1, \ldots, x_d) = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} x_{i_1} \ldots x_{i_r}.$$

Example 1.1.7. Let us calculate $e_{(2,1)}$ for 1 through 3 variables. When we have $e_{(2,1)}(x) = e_2(x)e_1(x)$, we can't compute $e_2(x)$ because there are no two-term monomials with only one variable. On two variables we have the following

$$e_{(2,1)}(x,y) = e_2(x,y)e_1(x,y) = (xy)(x+y) = x^2y + y^2x$$

and when talking about 3 variables the following happens:

$$\begin{aligned} e_{(2,1)}(x,y,z) &= e_2(x,y,z)e_1(x,y,z) \\ &= (xy+yz+zx)(x+y+z) \\ &= x^2y + y^2z + z^2x + y^2x + z^2y + x^2z + 2xyz. \end{aligned}$$

Consider now the partitions $(2,2,2,2)$ and (5). Then

$$e_{(2,2,2,2)} = e_2^4 \Rightarrow e_{(r,r,\ldots,r)} = e_r^{m_r(\lambda)}$$

where $m_i(\lambda)$ is number of parts of λ equal to i. For the partition (5) we have that $e_{(5)} = e_5$ and in general $e_{(n)} = e_n$.

Remark 1.1.8. As before we don't know how many terms per function, but knowing m implies knowing e. As for the degree, it holds that $\deg(e_\lambda) = |\lambda|$.

Homogenous Symmetric Functions

- Homogenous: $h_\lambda = \prod h_{\lambda_i}$ and $h_d = x_1^d + \cdots + x_1^{d-1}x_2 + \cdots + x_1^{d-2}x_2^2 + x_1^{d-2}x_2x_3 + \ldots$. In general $h_d = \sum_{\lambda \vdash d} m_\lambda$.

- Power sum: $p_\lambda = \prod p_{\lambda_i}$ and $p_d = \sum x_i^d$.

For Schur basis recall SSYT

Example 1.1.9. Consider $\lambda = (5,4,1)$, rows $\leqslant\rightarrow$ and columns $<$, we associate the monomial $x_1^2x_2^3x_3^3x_4^2 := x^T$.

- Schur: $s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$ but also $\sum K_{\lambda\mu}m_\mu$ where the sum is over SSYT of shape λ, content μ.

Schur function motivation (preview)

The first place they showed up is in the representation theory of Lie group. The function $s_\lambda(x_1, \ldots, x_n)$ is a character of irreducible polynomial representations of GL_n. In theoretical physics we have matrix groups acting on particles, representations are smaller matrix groups of things that they are mapping to. We want to take tensor product and direct sums of representations, the tensor product is related to multiplication of Schur function while direct sum into sum of Schur functions.

There's also the Schur-Weyl duality which takes representations into the Weyl group. Under the *Frobenius map*, s_λ corresponds to irreducible representations of S_n.

A more modern application of Schur function goes into geometry, s_λ correspond to Schubert varieties in Grassmannians. Multiplication corresponds to interesections and sum to unions.

There's also context in Probability Theory. But in the end, Schur positivity is important because of this connections.

Definition 1.1.10. $f \in \Lambda$ is Schur-positive if $f = \sum c_\lambda s_\lambda$, $c_\lambda \geqslant 0$.

Example 1.1.11. $3s_{(2,1)} + 2s_{(3)}$ schur pos but change 2 to $-\frac{1}{2}$ then not.

1.2 day 2

Alg defn Schur fncs

Definition 1.2.1. A function is antisymmetric if for $\pi \in S_n$,

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \operatorname{sgn}(\pi) f(x_1, \ldots, x_n).$$

Example 1.2.2. The following functions are antisymmetric:

(a) $f(x, y) = x - y$ then $f(y, x) = -f(x, y)$.
(b) $g(x, y) = (x - y)(x + y)$.
(c) $h(x, y) = x^2 y - y^2 x$.

Notice that the last function can factor as $h = -xy(x - y)$. We claim that this is always the case.

Lemma 1.2.3. *Every antisymmetric polynomial f in two variables x, y can factor as $f(x, y) = (x - y)g(x, y)$ where g is symmetric.*

Proof

Suppose f is antisymmetric, then $f(x,x) = 0$ by taking $y = x$. This means that $(x-y) \mid f$. Thus $f(x,y) = (x-y)g(x,y)$ and we now need to show that g is symmetric.

$$g(y,x) = \frac{f(y,x)}{y-x} = \frac{-f(x,y)}{-(x-y)} = \frac{f(x,y)}{x-y} = g(x,y).$$

Monomial Antisymmetric Functions

Definition 1.2.4. Given a strict partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 > \cdots > \lambda_k$, we define

$$a_\lambda(x_1, \ldots, x_n) = x_1^{\lambda_1} \cdots x_k^{\lambda_k} \pm \text{similar terms} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_k x_{\pi(k)}^{\lambda_k}.$$

This a_λ can be zero.

Example 1.2.5. For two variables we've seen some antisymmetric polynomials. Let us calculate

$$a_{(3,1)}(x,y) = x^3y - y^3x.$$

The smallest possible example in 3 variables is

$$a_{(2,1,0)}(x,y,z) = x^2y + y^2z + z^2x - y^2x - z^2y - x^2z.$$

This can be factored as $(x-y)(y-z)(x-z)$. A similar construction gives us

$$a_{(4,2,0)}(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - y^4x^2 - z^4y^2 - x^4z^2,$$

but how does this factor? We get

$$a_{(4,2,0)}(x,y,z) = (x^2-y^2)(y^2-z^2)(x^2-z^2) = a_{(2,1,0)}(x,y,z)(x+y)(y+z)(x+z).$$

Lemma 1.2.6. *The set $\{ a_\lambda \}_{\lambda \text{ strict}}$ is a basis of the antisymmetric polynomials over $\mathbb{Q}$, $A_{\mathbb{Q}}$. Even more any a_λ is divisible by a_ρ where $\rho = (n-1, n-2, \ldots, 2, 1, 0)$.*

As an algebra generator, a_ρ is a generator.

Proof

WRITE

Proposition 1.2.7. *The a_ρ antisymmetric function is also the Vandermonde determinant:*

$$a_\rho = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \dots & x_1^2 & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \dots & x_2^2 & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \dots & x_n^2 & x_n & 1 \end{pmatrix}$$

Schur Polynomials

Definition 1.2.8. The Schur polynomial of $\lambda \in \mathrm{Par}$ is

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})}.$$

Here $\lambda + \rho$ is the pointwise sum as arrays.

Remark 1.2.9. This is the Weyl character proof.

The following proof is due to Proctor(1987) find ref

Lemma 1.2.10. *Any a_λ can be seen as a determinant in the following way:*

$$a_\lambda(\underline{x}) = \det \begin{pmatrix} x_1^{\lambda_1} & x_1^{\lambda_2} & \dots & x_1^{\lambda_n} \\ x_2^{\lambda_1} & x_2^{\lambda_2} & \dots & x_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \dots & x_n \end{pmatrix}$$

Proof

We want to see that

$$\frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})} = \sum x^T$$

where the sum ranges through T's which are SSYT(la) with max entry n.

(a) We will show a recursion for the combinatorial definition that the character formula will also satisfy. It holds that

$$s_\lambda(\underline{x}) = \sum s_\mu(\underline{x}) x_n^{|\lambda|-|\mu|}$$

where μ has $n-1$ parts with $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \dots$.

(b) We also show that the ratio of determinants satisfies the same recursion.

Example 1.2.11. Consider $\lambda = (8, 8, 4, 1, 1)$ and $\mu = (8, 5, 2, 1)$, then $\lambda \backslash \mu$ is a skew-table in which we can fill in n's

Corollary 1.2.12. *The Schur polynomials are a basis of* $\Lambda_{\mathbb{Q}}$.

1.3 Day 3 | 20230125

Recall $\Lambda = \mathbb{Q}[e_1, e_2, \dots]$ where the e_j's are the elementary symmetric functions. So the e_j's are algebraic generators of Λ and they're algebraically independent. Equivalently, as a vector space, $\{ e_\lambda : \lambda \in \text{Par} \}$ is a basis.

Proposition 1.3.1. *A homomorphism* $f : \Lambda \to \Lambda$ ($f(a + b) = f(a) + f(b),\ f(ab)f(a)f(b)$ *for* $a, b \in \Lambda$) *is fully determined by where it sends the* $e_i's$.

Definition 1.3.2. The map $\omega \in \text{End}(\Lambda)$ will send e_j to h_j.

Example 1.3.3. Consider $f = 3e_{(2,1)} + 2e_3$, then applying ω we get

$$\omega(f) = \omega(3e_{(2,1)} + 2e_3) = 3h_{(2,1)} + 2h_3.$$

For p_2, we can decompose to $e_1^2 - 2e_2$. So

$$\omega(p_2) = \omega(e_1^2 - 2e_2) = h_1^2 - 2h_2$$

and we can expand this last expression into

$$(x_1 + x_2 + \dots)^2 - 2(x_1^2 + x_2^2 + \dots + x_1x_2 + x_1x_3 + \dots) = -x_1^2 - x_2^2 - \dots$$

and we recognize this last term as $-p_2$. *This is not a coincidence.*

Theorem 1.3.4. *The map* ω *is involutive.*

Proof

It suffices to prove that $\omega(h_j) = e_j$. We will use power expansions and generating functions. We have

$$H(t) = \frac{1}{1 - x_1 t}\frac{1}{1 - x_2 t} \cdots = \sum h_n(\underline{x})t^n,$$

and this comes from expanding the $1/(1 - y)$'s as geometric series. When collecting the coefficients of t^n we get exactly $h_n(\underline{x})$. Similarly, for the elementary symmetric functions,

$$E(t) = (1 + x_1 t)(1 + x_2 t) \cdots = \sum e_n t^n.$$

When multiplying to obtain the coefficient of t^n we get a plethora of different x_j's which form the e_j's. Now from this expressions we have $H(t)E(-t) = 1$ which means that

$$\left(\sum h_n(\underline{x})t^n\right)\left(\sum e_n(\underline{x})(-t)^n\right) \Rightarrow \sum_{k=0}^{n}(-1)^k e_k h_{n-k} = 0,\ n \geqslant 1.$$

Now applying the map to the equation we get

$$\omega\left(\sum_{k=0}^{n}(-1)^k e_k h_{n-k}\right) = \sum_{k=0}^{n}(-1)^k h_k \omega(h_{n-k}) = 0.$$

After reindexing, we get that both e_j's and $\omega(h_j)$'s are determined recursively by the h_j's in the same way. Thus we conclude that $\omega(h_j) = e_j$.

Lemma 1.3.5. *The following equation holds for the power-sum symmetric functions*:

$$\exp\left(\sum \frac{1}{n} p_n(\underline{x})p_n(\underline{y})\right) = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} =: \Omega(\underline{x}, \underline{y}).$$

It also holds that

$$\Omega(\underline{x}, \underline{y}) = \sum_{l} a \frac{1}{z^\lambda} p_\lambda(\underline{x}) p_\lambda(\underline{y})$$

where $z_\lambda = \prod k^{m_k} m_k!$ *where* m_k *is the number of parts of* λ *equal to* k.

Proof

We will prove both parts separately. For the first equation we will take the logarithm on both sides:

$$\sum \frac{1}{n} p_n(\underline{x})p_n(\underline{y}) = \log\left(\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j}\right)$$

and after manipulating the logarithm we get

$$\sum_{i,j=1}^{\infty}(\log(1) - \log(1 - x_i y_j)) = \sum_{i,j=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n} x_i^n y_j^n.$$

We can separate[a] into

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_i x_i^n \right) \left(\sum_j y_j^n \right).$$

Now taking $\exp$ on both sides we get equality.
By not removing the exponential we get the following expression

$$\exp\left(\sum \frac{1}{n} p_n(\underline{x}) p_n(\underline{y}) \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum \frac{1}{n} p_n(\underline{x}) p_n(\underline{y}) \right)^k.$$

To get a term of the form $p_\lambda(\underline{x})p_\lambda(\underline{y})$ we have to choose which parts of the λ come from each of the factors in $\sum \frac{1}{n} p_n(\underline{x})p_n(\underline{y})$. If $\ell(\lambda) = k$ then it comes from the k^{th} term in the exponential sum. If $\lambda = (\lambda_1, dots, \lambda_1, \dots, 2, \dots, 2, 1, \dots, 1)$ with m_{λ_1} λ_1's, m_1 1's, then out of k elements we have to choose m_1 1's and so on. Thus there are $\binom{k}{m_{\lambda_1}, \dots, m_1}$ choices and each i in λ comes with a $\frac{1}{i}$. Therefore the coefficient of $p_\lambda(\underline{x})p_\lambda(\underline{y})$ is

$$\frac{1}{k!} \frac{k!}{m_1! m_2! \dots} \frac{1}{1^{m_1}} \frac{1}{2^{m_2}} \cdots = \frac{1}{z_\lambda}.$$

[a] Are we using Fubini-Tonelli here?

Lemma 1.3.6. *We have the following identities*

$$\exp\left(\sum \frac{(-1)^{n-1}}{n} p_n(\underline{x}) p_n(\underline{y}) \right) = \prod_{i,j=1}^{\infty} \frac{1}{1 + x_i y_j} = \sum_\lambda \frac{(-1)^{n-\ell(\lambda)}}{z_\lambda} p_\lambda(\underline{x}) p_\lambda(\underline{y}).$$

Lemma 1.3.7. *Another equality for $\Omega(\underline{x}, \underline{y})$ is*

$$\Omega(\underline{x}, \underline{y}) = \sum_\lambda m_\lambda(\underline{x}) h_\lambda(\underline{y})$$

Theorem 1.3.8. *It holds that $\omega(p_\lambda) = (-1)^{n-k} p_\lambda$ where k is the number of parts of λ.*

Proof

Applying ω to Ω, but *only working with $\underline{y}$ variables* we get

$$\omega(\Omega) = \omega\left(\sum_\lambda m_\lambda(\underline{x}) h_\lambda(\underline{y}) \right) = \sum_\lambda m_\lambda(\underline{x}) e_\lambda(\underline{y}) = \prod_{i,j=1}^{\infty} (1 + x_i y_j) = \sum_\lambda \frac{1}{z_\lambda} (-1)^{n-k_\lambda} p_\lambda(\underline{x}) p_\lambda(\underline{y}).$$

Comparing coefficients with

$$\omega\left(\sum_l a\frac{1}{z^\lambda}p_\lambda(\underline{x})p_\lambda(\underline{y})\right)$$

we get the result.

1.4 Day 4 | 20230127

To continue exploring the ring of symmetric functions we need a couple of tools. One of them is the involution which we have already seen. But the other one is a scalar product which is compatible with the multiplication.

Hall Inner Product

Recall an inner product is a function

$$\langle -|-\rangle\colon\ V\times V\to\mathbb{Q}$$

which is bilinear $\langle u+v|w\rangle = \langle u|w\rangle + \langle v|w\rangle$ and the same on the other entry. For scalars the following behavior is expected $\langle\lambda u|v\rangle = \langle u|\lambda v\rangle = \lambda\langle u|v\rangle$. Recall that if the base field is the complex numbers, then the inner product is Hermitian.

Definition 1.4.1. We say that two vectors are orthogonal when $\langle u|v\rangle = 0$.

This gives us a possible decomposition of space into several components. Suppose that $\{ u_\lambda \}_{\lambda\in\mathrm{Par}(n)}, \{ v_\lambda \}_{\lambda\in\mathrm{Par}(n)}$ are basis of Λ^n. So we would like a condition such as

$$\langle u_\lambda|v_\mu\rangle = \begin{cases} 0 & \lambda\neq\mu, \\ 1 & \lambda=\mu. \end{cases}$$

If we cap the dimension this says that $\langle u|v\rangle$ is the usual dot product. But in infinite dimensions we don't have matrices. We'll call this basis dual to one another. If miraculously we have the same basis, then this basis is orthonormal.

Definition 1.4.2 (Phillip Hall)**.** The Hall inner product is defined so that $\langle m_\lambda|h_\mu\rangle = \delta_{\lambda\mu}$.

By defining the product on two basis, we have defined it for all other elements by bilinearity.

Lemma 1.4.3. *The Hall inner product is symmetric.*

Theorem 1.4.4. *The Hall inner product is positive definite, this is $\langle f|f\rangle \geqslant 0$ and equality is achieved when $f = 0$.*

It's important to note that this statement is symmetric. However we are talking about an asymmetric definition. Last, before proving the statement we need a criteria for dual bases. But importantly, recall the result from last lecture: 1.3.7

Theorem 1.4.5. *If $u_\lambda, \{ v_\mu \}$ are dual, then $\sum_\lambda u_\lambda v_\lambda = \Omega$.*

Proof

Fix a partition of n, then

$$\delta_{\lambda\mu} = \langle m_\lambda|h_\mu\rangle = \left\langle \sum_{\rho\vdash n} \alpha_{\lambda_\rho} u_\rho \middle| \sum_{\tau\vdash n} \beta_{\mu_\tau} v_\tau \right\rangle = \sum_{\rho,\tau} \alpha_{\lambda_\rho}\beta_{\mu_\tau} \langle u_\rho|v_\tau\rangle .$$

We want $\langle u_\rho|v_\tau\rangle = \delta_{\rho\tau}$, to that effect name $A_{\rho\tau}$ the matrix whose entries are $\langle u_\rho|v_\tau\rangle$.
As u and v are dual bases, we have that $A = \mathrm{id}$. Thus $I = \alpha\beta^\mathsf{T}$ and now $\delta_{\rho\tau} = \sum \alpha_{\lambda_\rho}\beta_{\lambda_\tau}$. We are now going to use the hypothesis and the interpretation of m, h in the u, v basis. We have

$$\Omega = \sum \left(\sum \alpha u\right) \left(\sum \beta v\right) = \sum \left(\sum \alpha\beta\right) uv = \sum uv$$

so the inner sum must be one and thus we are done.

Corollary 1.4.6. *For the Hall inner product it holds that $\langle p_\lambda|p_\mu\rangle = z_\lambda\delta_{\lambda\mu}$.*

The key is to recall that p_λ is an eigenfunction of ω. Also 1.3.5. By using a power-sum decomposition it is possible to prove that the Hall inner product is positive definite.

Corollary 1.4.7. *The ω involution is orthogonal with respect to $\langle -|-\rangle$. This is $\langle \omega f|\omega g\rangle = \langle f|g\rangle$.*

Once again, the idea is to transfer to power-sum and use the fact that it's an eigenfunction.

1.5 Interim 1

Theorem 1.5.1 (Fundamental Theorem of Sym. Fnc. Thry.)**.** *Every symmetric function can be written uniquely in the form $\sum_\lambda c_\lambda e_\lambda$ with $c_\lambda \in \mathbb{Q}$.*

There are at least two proofs if not more of this fact. The first comes from Maria Gillespie's blog which Mark Haiman presented to her.

Proof

It suffices to prove the transition matrix between m and e is invertible.

For proof 2 read [10] pg. 290. Proof 3 in another Maria post

1.6 Day 5 | 20230130

Exercise 1.6.1. Compute $\omega(s_{(3,1)})$.

Answer

We have that By Jacobi-Trudi

$$s_{(3,1)} = \det \begin{pmatrix} h_3 & h_4 \\ 1 & h_1 \end{pmatrix} = h_{(3,1)} - h_4.$$

Using the omega involution, we get

Recall that $\omega : h_n \leftrightarrow e_n$, $\omega p_k = (-1)^{k-1} p_k$. We have the following questions, where do m and s map to? Also

$$\langle m|h\rangle = \delta, \ \langle p|p/z\rangle = \delta,$$

but what are e and s dual to?

Definition 1.6.2. We call $\omega m_\lambda = f_\lambda$ the <u>forgotten basis</u>.

There's not much we could say about them, they are not Schur positive and there's no patterns.

Dual to e

Recall ω is an isometry, so $\langle \omega f|\omega g\rangle = \langle f|g\rangle$, so

$$\langle e_\lambda|?\rangle = \langle h_\lambda|\omega ?\rangle = \delta_{\lambda\mu}.$$

Since $\langle h|m\rangle = \delta$, then applying ω again we get that $\langle e_\lambda|f_\mu\rangle = \delta_{\lambda\mu}$.

RSK algorithm

We want to show two things:

$$\omega s_\lambda = s_{\lambda^\intercal},\ \langle s_\lambda | s_\mu \rangle = \delta_{\lambda\mu}.$$

Proposition 1.6.3. *It holds that*

$$\sum_\lambda s_\lambda(\underline{x}) s_\lambda(\underline{y}) = \Omega = \sum_\lambda m_\lambda(\underline{x}) h_\lambda(\underline{y})$$

Proof

The sum on the left is

$$\sum_{(S,T) SSYT} x^S y^T$$

so we will study pairs (S, T) of SSYT of the same shape to show that they're equal to the sum on the right.

algorithm: process of doing the bijection.

The RSK bijection takes a pair (S, T) of SSYT of the same shape and it maps it to "two-line arrays" of length n.

Definition 1.6.4. A <u>two-line array</u> is a matrix in $\mathcal{M}_{2\times n}(\mathbb{Z}_{\geqslant 0})$ such that

i) The bottom row is weakly increasing.
ii) If $b_i = b_{i+1}$, then $a_i \leqslant a_{i+1}$, where a's are the top row and b's the bottom row.

Example 1.6.5. Consider the matrix

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{pmatrix}$$

Within "blocks", there is a weak increment. From right-to-left we will find a pair of SSYT. We will "insert" top row letters from left-to-right.

(a) Place 1st letter $\boxed{1}$

(b) For each letter, if it can go at the end of last row, put it there

$$\boxed{1}\boxed{1} \leftarrow 2,\ \boxed{1}\boxed{1}\boxed{2} \leftarrow 1$$

but one can't go after 2.

(c) Otherwise if inserting b_1, let c be the leftmost $> b$, "bump c", then insert c into the next row.

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}$$

For the bottom row, place in a new square at each step to form a "recording tableau". The recording tableau always matches the shape of the insertion one. The first three steps lead to $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}$ in the recording one. But in the fourth step we get $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}$. The next step leads us to

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}\ ,\quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}$$

then in insertion, 2 bumps 4 and 4 doesn't bump 2 on next row, so we get

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}\ ,\quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$$

The three is no problem so

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}\ ,\quad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$$

then the next one bumps out the 2, the 2 bumps the 4 on the second row to get

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}\ ,\quad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

Finally

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}\ ,\quad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}\ .$$

Why do we get SSYT. The insertion tableau gives us the question, can we make a column non-increasing? No, we are always bumping something bigger. Imagine we bump $c > b$ with b, then c replaces something that goes to the left.

$$\begin{array}{|c|c|c|}\hline \leqslant & b & c \\ \hline & \\ \cline{1-2} \\ \cline{1-1} \end{array} \Rightarrow \begin{array}{|c|c|c|}\hline & b & \\ \hline & d \\ \cline{1-2} \\ \cline{1-1} \end{array}$$

and $d > c$ so it bumps something else. The recording tableau is also a SSYT. Let us prove it.

Lemma 1.6.6 (Key Lemma 1)**.** *The insertion path (sequence of squares that are bumped) moves up and weakly left.*

Lemma 1.6.7 (Key Lemma 2)**.** *If $a \leqslant b$ and T is a SSYT, computing*

$$T \leftarrow \boxed{a} \leftarrow \boxed{b},$$

the intersection path of a in T lies strictly left of the intersection path of b in $T \leftarrow \boxed{a}$.

Proof

We will do induction on the rows with an example.

Example 1.6.8. Consider

$$\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 2 & 2 & 3 \\
\hline
2 & 2 & 3 & 3 & 4 \\
\cline{1-5}
3 & 3 & 5 & 5 \\
\cline{1-4}
4 & 4 \\
\cline{1-2}
\end{array}$$

Inserting 1 we bump the 2, then the 3 and finally the 5. We get

$$\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & o & 2 & 3 \\
\hline
2 & 2 & t & 3 & 4 \\
\cline{1-5}
3 & 3 & t & 5 & 5 \\
\cline{1-5}
4 & 4 & f \\
\cline{1-3}
\end{array}$$

so inserting the 2 we bump 3,4,5. And they will be to the side of the last sequence.

1.7 Day 6 | 20230201

Exercise 1.7.1. Apply RSK to $\begin{pmatrix} 3 & 2 & 4 & 1 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

Answer

We get $\begin{array}{|c|c|c|}
\hline
1 & 4 & 5 \\
\hline
2 \\
\cline{1-1}
3 \\
\cline{1-1}
\end{array}\ , \quad \begin{array}{|c|c|c|}
\hline
1 & 3 & 5 \\
\hline
2 \\
\cline{1-1}
4 \\
\cline{1-1}
\end{array}\ .$

Notice that we got STANDARD Young tableau. So to prove it's a bijection we will begin with all different numbers.

Lemma 1.7.2. *The RSK bijection is a bijection between pairs of standard Young tableaux of the same shape and "permutations"* ($2 \times n$ *matrices whose rows are permutations.*)

To prove it's a bijection we will find an inverse by reversing the process. Look at the recording tableau, we will bump out the largest number. We will take S as the recording tableau. Then we start with the spot on S, T which corresponds to largest label in S.

- ⋄ If b is the item in such a square we "un-bump" it.
  - If in bottom row, just remove.
  - Else, let c be the rightmost entry in row below b that is less than b. Then replace b with c and repeat the process with c until the letter that is removed is done by the just removing it.

  Then we add the two letters to the matrix from right-t-left.

With the original tableau we remove the 5 and the 5 to get

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}\ , \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

then the 4 indicates that in T we must "un-bump" the 3. The three un-bumps the 2, the 2 to the 1 so that we get

$$\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}\ , \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}\ .$$

Now we get the matrix $\begin{pmatrix} x & x & x & 1 & 5 \\ x & x & x & 4 & 5 \end{pmatrix}$ and removing the 3 from S just removes the 4 from T as it is in the bottom row.

Now as this two sets are in bijection, this means that they have the same size.

Corollary 1.7.3. *Let f^λ be the number of standard Young tableau of shape λ. Then*

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!.$$

We will generalize one step at a time. Let us now assume that T is semi-standard. On the matrix, we will have that the top row is now random, but the bottom row is still from 1 to n.

Lemma 1.7.4 (Schensted)**.** *There is a bijection between (S, T), S is standard, T is SSYT, and words of length n.*

Example 1.7.5. Consider the matrix $\begin{pmatrix} 2 & 1 & 3 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ which returns the two Young tableau

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}\ , \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}\ .$$

The proof of the inverse is similar but when un-bumping, we must bump the rightmost entry *strictly* smaller than b. But we don't need this, we will do it more creatively.

Definition 1.7.6. Suppose T is a Young tableau. Then

i) The reading word of T $\mathrm{rw}(T)$ is the concatenation of rows from top to bottom.
ii) The standarization of an SSYT T, $\mathrm{std}(T)$, is the unique SYT with same relative order of entries, ties broken with "reading order".
iii) The standarization of a word is similar

In the previous example, the reading word is

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} \to 23113.$$

The standarization are as follows:

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} \to \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}\ , \quad 23113 \to 34125.$$

We can standarize the matrix

$$\begin{pmatrix} 2 & 1 & 3 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 3 & 4 & 1 & 2 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

and this matrix corresponds to the pair (S, T) where T is $\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}$. In essence, the following diagram commutes

$$\begin{tikzcd} (S,T) \arrow[d, "std"'] & 21313 \arrow[l, "RSK"'] \arrow[d] \\ (S,T') & 31425 \arrow[l, "RSK"] \end{tikzcd}$$

Definition 1.7.7. Given a content $\mu = (\mu_1, \ldots, \mu_k)$ with $\sum \mu_k = n$ (not nec. partition). Then the de-standarization with respect to μ of a SYT T is a $SSYT$ T' such that std$(T') = T$.

In this case

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \xrightarrow[\text{std}^{-1}(2,1,2)]{} \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}.$$

Recall now lemma 1.6.7 about consecutive insertions.

The Full RSK

We are now going to prove that there is an inverse to the original RSK function. Consider the following example

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} \to \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 7 \\ \hline 4 & 6 & 9 \\ \cline{1-3} 8 \\ \cline{1-1}\end{array}$$

The matrix $\begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{pmatrix}$ can be standarized to our word matrix. Then the table $\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$ also standarizes to the word table.

1.8 Day 7| 20230203

Exercise 1.8.1. Expand $h_{(3,2)}$ in Schur basis.

Answer

This is $s_{(3,2)} + s_{(4,1)} + s_{(5)}$.

Recall that (s_λ) form an orthonormal basis and m and h are dual basis. This means that if f is a symmetric function then

$$f = \sum_\lambda c_\lambda s_\lambda \Rightarrow c_\lambda = \langle f | s_\lambda \rangle, \; f = \sum_\lambda a_\lambda m_\lambda \Rightarrow a_\lambda = \langle f | h_\lambda \rangle.$$

Lets suppose now that f is any homogenous symmetric function. We will calculate the coefficient of s_λ in an h_μ expansion:

$$\langle h_\mu | s_\lambda \rangle = \langle s_\lambda | h_\mu \rangle$$

and we can interpret this as the coefficient of m_μ in s_λ. This amount is precisely the Kostka coefficient $K_{\lambda\mu}$. Thus we have the formula $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$.

Properties of the Schur functions

We wish to show that $\langle s_\lambda | s_\mu \rangle = \delta_{\lambda\mu}$ and $\omega s_\lambda = s_{\lambda^\intercal}$.

Proposition 1.8.2. $\sum_\lambda s_\lambda(\underline{x}) s_\lambda(\underline{y}) = \sum_\lambda m_\lambda(\underline{x}) h_\lambda(\underline{y})$

Proof

Expanding the sum on the left we obtain

$$\sum_\lambda s_\lambda(\underline{x}) s_\lambda(\underline{y}) = \sum \lambda \left(\sum_{T \in SSYT(\lambda)} x^T \right) \left(\sum_{S \in SSYT(\lambda)} y^S \right) = \sum_{(T,S),\ SSYT \text{same shape}} x^T y^S$$

This is basically an RSK pair and this correspond to two-line arrays, so this sum could be the same as summing over them. Thus this is

$$\sum_{\text{2line arrays}} x_{a_1} \dots x_{a_n} y_{b_1} \dots y_{b_n}.$$

We will now find the coefficient of $m_\lambda(\underline{y})$ in this expansion and show that it is $h_\lambda(\underline{x})$

What are all the ways to obtain $y_1^{\lambda_1} \dots y_k^{\lambda_k}$?

$$\begin{pmatrix} a_1^{(1)} & \dots & a_k^{(1)} & a_1^{(2)} & \dots & a_k^{(2)} & \dots \\ 1 & 1 & 1 & 2 & 2 & 2 & \dots \end{pmatrix}$$

And note that $a_1^{(i)} \leqslant \dots \leqslant a_{\lambda_i}^{(i)}$ for all i, so the coefficient is

$$\sum_{(a^{(i)}) valid tuples} x_{a_1^{(1)}} \dots x_{a_{\lambda_k}^{(k)}}$$

but this factors as

$$\prod_{i=1}^k \sum_{a_1^(i) \leqslant \dots \leqslant a_{\lambda_k}^{(i)}} x_{a_1^{(i)}} \dots x_{a_{\lambda_k}^{(i)}}.$$

We can split this because the choices are independent of the blocks and then multiply the functions together. The last term is h_{λ_i} and the product is h_λ.

If (T, S) RSKs inverse to $\begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ then $x^T y^S$ is $x_1 x_3 x_2 y_1 y_1 y_2$.

Corollary 1.8.3. $\omega s_\lambda = s_{\lambda^\intercal}$.

Proof

It suffices to show $\langle s_{\lambda^\intercal} | e_\mu \rangle = K_{\lambda\mu}$ because $\langle s_\lambda | h_\mu \rangle = K_{\lambda\mu}$ which implies that $\langle \omega s_\lambda | e_\mu \rangle = K_{\lambda\mu}$.
In other words, we wish to show that the coefficient of s_λ in e_μ is $K_{\lambda^\intercal \mu}$, the number of $SSYT$ shape $\lambda^\intercal$, content μ.
CONT

Pieri Rule

Definition 1.8.4. A <u>skew shape</u> is a diagram formed by subtracting a smaller Young diagram from a larger one.

A <u>horizontal strip</u> is a skew shape where no two boxes are in the same column. Similar a <u>vertical strip</u> doesn't have boxes in the same row.

Example 1.8.5. Suppose $\lambda = (5, 4, 4, 1)$ and $\mu = (4, 2, 2)$. Then

$$\lambda = \ydiagram{1,4,4,5}, \quad \mu = \ydiagram{2,2,4}$$

so la/μ is INSERT DIAG. Not horizontal nor vertical.

In a Young tableau, the biggest number forms a horizontal strip, so in general Young tableaux are made up of horizontal strips.

$$\begin{ytableau} 4 \\ 3 & 4 & 4 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 & 3 & 4 \end{ytableau}.$$

Theorem 1.8.6 (Pieri). *Let $r \in \mathbb{N}$, then*

$$e_r s_\lambda = \sum_{\rho/\lambda \text{vert. strip size } r} s_\rho$$

$$h_r s_\lambda = \sum_{\rho/\lambda \text{horiz. strip size } r} s_\rho$$

This is basically all the ways to fill up the shapes.

Proof

$$h_r s_l a = s_{(r)} s_\lambda = \left(\sum_{T \in SSYT((r))} x^T \right) \left(\sum_{S \in SSYT(\lambda)} x^S \right)$$

Example 1.8.7. $h_3 s_{(3,1)}$ is $x^T x^S$ is inserting the boxes of T one at a time in S.

$$\begin{ytableau} 3 \\ 1 & 2 & 3 \end{ytableau} \leftarrow \begin{ytableau} 1 & 1 & 2 \end{ytableau} = \begin{ytableau} 3 \\ 2 & 3 \\ 1 & 1 & 1 & 2 \end{ytableau}$$

so by 1.6.6 about insertion path, the new squares are a horizontal strip which is the s_ρ in the Pieri rule. Unbumping we recover something.

1.9 Day 8 | 20230206

Exercise 1.9.1. Apply RSK to 82357146 and 62235124.

Answer

$$\begin{ytableau} 8 \\ 2 & 3 & 5 & 7 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 3 & 4 & 5 \end{ytableau}$$

then 1 bumps the 2, 2 bumps 8

$$\begin{ytableau} 8 \\ 2 & 5 & 7 \\ 1 & 3 & 4 & 6 \end{ytableau}, \begin{ytableau} 6 \\ 2 & 7 & 8 \\ 1 & 3 & 4 & 5 \end{ytableau}$$

The next one standarizes to the last string. The same recording table but we get for insertion

$$\begin{ytableau} 6 \\ 2 & 3 & 5 \\ 1 & 2 & 2 & 4 \end{ytableau}.$$

Consequences of RSK

We will talk about increasing and decreasing subsequences.

Definition 1.9.2. A longest increasing subsequence of a word $w \in \mathbb{N}^n$ is a subsequence $w_{i_1} \leqslant \ldots \leqslant w_{i_\ell}$ with $i_1 < \cdots < i_\ell$ such that ℓ is as large as possible. We will write $\ell(w)$ to be the length of the longest increasing subsequence.

A longest decreasing subsequence of a word is $w_{i_1} > \cdots > w_{i_d}$ with $i_1 < \cdots < i_d$. In this case $d(w)$ is the longest decreasing.

Example 1.9.3. In the case of 82357146, we have $2357, 2356, 146, 2346$. Notice that this is the length of ?? of the Young tableau. For decreasing we have $821, 831, \ldots$, the height of the Young tableau is the longest decreasing subsequence.

Theorem 1.9.4. *Suppose w is a word, $S = ins(w)$ is the insertion tableau through RSK and $\lambda = sh(S)$ is the shape of the table. Then $\ell(w) = \lambda_1$ and $d(w) = \lambda_1^{\mathsf{T}}$.*

To prove this we will develop some tools.

Lemma 1.9.5. *For a tableau T, $ins(rw(T)) = T$.*

The reading word of $\begin{array}{|c|c|c|c}\cline{1-1} 8 \\ \cline{1-3} 2 & 5 & 7 \\ \hline 1 & 3 & 4 & \multicolumn{1}{c|}{6} \\ \hline\end{array}$ is 82357146 which inserts to the same table precisely.

Remark 1.9.6. The column reading word also works! For this table it's 82153746. We get a bunch of decreasing subsequences. 821 creates the first column by bumping, then 53 creates the second column and so on.

Let's analyze the longest increasing subsequence of the reading word. Clearly we can get the bottom row as a longest subsequence, but looking in the reading order we need to go to the right. Going down decreases!

Lemma 1.9.7. *If $\lambda = sh(T)$ then $\ell(rw(T)) = \lambda_1$ and $d(rw(T)) = \lambda_1^{\mathsf{T}}$.*

Proof

Given an entry $a \in T$, let $b \in T$ such that $a <_{\mathrm{ro}} b$. Then b is in a column to the right of a, this means that

$$\ell(\mathrm{rw}(T)) \leqslant \#\text{columns} = \lambda_1.$$

The bottom row is an example of a subsequence where the length is achieved. So equality holds.

For decreasing it's equivalent. Now, how do we tell when two words have the same insertion tableau?

Example 1.9.8. In the case of all permutations in S_3 we have that some are equivalent FILL

Knuth equivalence

Definition 1.9.9. A Knuth move on a permutation swaps two letters a, c if $a < b < c$ (reading order) and one of consecutive subsequences acb, cab, bac, bca appears in the word.

Two words are Knuth-equivalent if they differ by a sequence of Knuth-moves.

In the first case, b is between a, c and those are always together.

Proposition 1.9.10. *Knuth equivalence defines an equivalence relation on S_n.*

Theorem 1.9.11. *Two words π, w are Knuth-equivalent iff ins$(w) =$ ins(π).*

Example 1.9.12. In size 4, 1234 is in its own class because we don't have any Knuth moves available. Same thing happens with 4321.

Consider 1243, if we apply Knuth moves we can get

- 1423
- 4123

All of these have the insertion tableau $\begin{array}{|c|cc}\hline 4 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$ whose reading word is 4123.

For the tableau $\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$, its reading word is 2413. Applying Knuth moves we get only 2143, which is the column reading word.

The tableau $\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & 1 \\ \hline \end{array}$'s equivalence class also has size 2.

Proposition 1.9.13. *If two tableau have the same shape, their equivalence classes have the same size.*

We are seeking to prove $\ell(w)$ is invariant under Knuth moves. This will imply the theorem 1.9.4 because once we know that things have the same insertion tableau and the reading word has the same longest increasing subsequence length.

1.10 Day 9 | 20230208

Exercise 1.10.1. Insert f, g and then c into

$$\begin{array}{|c|c|c|c|c|} \cline{1-1} k \\ \cline{1-3} e & i & j \\ \hline a & b & d & h & l \\ \hline \end{array}$$

and then f, c and then g.

Example 1.10.2. The Knuth equivalence class of words whose insertion tableau is

$$\begin{array}{|c|c|c|} \cline{1-2} 3 & 4 \\ \hline 1 & 2 & 5 \\ \hline \end{array}.$$

The reading word is 34125 and we can Knuth-move it. The 341 can switch into 314 (this has the form bac). From that one we can switch 2 and 5 to get 31452. Once again with 314 we get 34152 and 34512.

In total we have 5 elements.

Proposition 1.10.3. *The size of the Knuth equivalence class whose insertion tableau is T with shape λ is $\#SYT(\lambda)$.*

Proof

We have one permutation in the Knuth equivalence class for every recording tableau S that can be paired with T.

The Knuth equivalence class of $\begin{array}{|c|c|c|} \cline{1-2} 3 & 4 \\ \hline 1 & 2 & 5 \\ \hline \end{array}$ can be identified by RSK with the pairs (T, S) and S varies through all SYT of corresponding shape.

Also, recall that by that hook-length formula we have that

$$\#SYT(\lambda) = \frac{|\lambda|!}{\prod_{\text{hooks} \subseteq T} \text{size hooks}}.$$

Theorem 1.10.4. *Two permutations π, w have the same insertion tableau if and only if π is Knuth-equivalent to w.*

Proof

By induction on the length, we can assume π, w differ by a single Knuth-move on the last 3 letters. We separate into cases:

i) Want
$$T' \leftarrow b \leftarrow c \leftarrow a = T' \leftarrow b \leftarrow a \leftarrow c$$
Note that $\texttt{IP}(b) < \texttt{IP}(c)$ by lemma 1.6.7 of consecutive insertions and $\texttt{IP}(a)$ is *weakly left* of $\texttt{IP}(b)$ from which holds $\texttt{IP}(a)$ is strictly left of c's. So we can switch order.

ii) In the other case we want
$$T' \leftarrow c \leftarrow a \leftarrow b = T' \leftarrow a \leftarrow c \leftarrow b.$$
$\texttt{IP}(a)$ is *weakly left* of c's. If it's *strictly*, then we can switch, but otherwise the insertion paths of a and c collide. CHECK NOTES

Now on the other direction, we wish to show that two permutations with the same insertion tableau are Knuth-equivalent.
It suffices to show that they are Knuth-equivalent to the reading word. By induction of the size of the word, suppose $\text{ins}(w') = T'$. Then $w' \sim \text{rw}(T')$ for w' of length $n-1$.
Let $w \in S_n$ with $b = w_n$. If $T' = \text{ins}(w_1, \ldots, w_{n-1})$, by induction $w_1 \ldots w_{n-1} \sim \text{rw}(T') = (first\ row) \ldots (last\ row)$.

Example 1.10.5. For the second case consider the table

3	8	9
1	4	7

and we insert $6, 2$ then 5 but then $2, 6$ and then 5. In the first case, DUNNO

In the second case consider

$$T' = \begin{array}{|c|c|c|c|} \hline 6 \\ \cline{1-1} 4 & 7 \\ \cline{1-2} 1 & 2 & 5 & 8 \\ \hline \end{array}$$

1.11 Day 10 | 20230210

Lemma 1.11.1. *The length of the longest increasing subsequence, $\ell(w)$ is invariant under Knuth moves.*

Proof

Given an increasing subsequence, if a Knuth move changes two of its entries $a < c$, we have two cases:

i) Either b is to the right of ac so we get

$$\dots acb \dots d \dots \to \dots cab \dots \dots d \dots$$

then replacing c with b gives an increasing subsequence of same length in a new word $a < b < c$ by assumption so $\underline{a} < \underline{b} < c < \underline{d}$ where d is the next element of the subsequence after c.

ii) We have b to the left so we have

$$\dots bac \dots d \dots \to \dots bca \dots d \dots$$

and the same proof shows that replacing a and b gives s anew subsequence and a new subword of the same length.

Knuth equivalence is natural when it comes to increasing and decreasing subsequences. With the lemmas we have, we can now prove that d is the height of the insertion tableau.

Remark 1.11.2. Dual equivalence is finding 3 values acb so we can switch bca. For example

$$615342 \to 625341 \to 624351$$

and dual equivalent words have the same *recording tableau*.

The result we've been aiming for is

$$\ell(w) = \text{width of ins(w)} = \lambda_1$$

Proof

Suppose $T = \text{ins}(w)$ with $\lambda = \text{sh}(T)$. Then $w \sim \text{rw}(T)$, by the previous lemma we have

$$\ell(w) = \ell(\text{rw}(T)) = \lambda_1$$

and the last equality comes from Monday class. Add references

For decreasing subsequences we have the same argument. To wrap it up we have a theorem we have a theorem from Stanley [10]:

Theorem 1.11.3. *A longest i-chain of increasing subsequences of w consists of:*

- *An increasing subsequence s_1 of w.*
- *An increasing subsequence s_2 of $w \backslash s_1$.*
- *(...)An increasing subsequence s_i of $w \backslash s_1 \dots s_{i-1}$.*

Then the length of the longest i-chain is $\lambda_1 + \lambda_2 + \cdots + \lambda_i$.

Jeu de taquin

The phrase *jeu de taquin* means "teasing game". This process is equivalent to RSK and insertion. As motivation, inserting $\varnothing \leftarrow w$ and then insert ρ and finally π, then this is the same as

$$w \leftarrow \mathrm{rw}(\mathrm{ins}(\rho) \leftarrow \pi).$$

This means that there's some *associativity* in this operation.

Definition 1.11.4. A skew-SSYT is a filling of the boxes of a skew shape (skew Ferrers diagram) with $n \in \mathbb{N}$ such that rows are weakly increasing and columns are strictly increasing.

This is analogous to SSYT, so let's see an example

Example 1.11.5. $(9, 9, 5, 5, 3)/(7, 6, 3, 3)$, but the initial partition could've been $(10, 10, 6, 6, 4)$ or $(100, 9, 9, 5, 5, 3)$.

$$\begin{array}{cccccccccc}
o & & & & & & & & & \\
2 & 2 & 4 & o & & & & & & \\
 & & x & 3 & 4 & & & & & \\
 & & & 1 & 3 & o & & & & \\
 & & & & & x & 2 & 2 & 3 & \\
 & & & & & & x & 1 & 1 & o
\end{array}$$

Notice that all rows and columns are adjacent, and there are no leftover squares.

Definition 1.11.6. A corner of a Young diagram μ is a square of μ at the top of its column, right of its row.

An inner corner of λ/μ is a corner of μ. While an outer corner of λ/μ is a square outside λ that is just above its column (poss. 0) just right of a row (poss. 0).

Remark 1.11.7. Notice that adding an outer corner makes a valid shape for a partition.

The JdT game is defined through the following moves

Definition 1.11.8. An inner slide into an inner corner x of a SSYT T of skew-shape λ/μ is given by the following process:

i) Compare squares a, b in the following shape

$$\begin{array}{cc} a & \\ x & b \end{array}$$

If $a \leqslant b$ or there's no b, slide a down. Else slide b left.

ii) Repeat for the new location of x until it becomes an outer corner.

Example 1.11.9. In the previous example we can consider

$$\begin{array}{ccccc} x & 2 & 2 & 3 & \\ & x & 1 & 1 & o \end{array}$$

and slide the 1 because if we slide 2 it's no longer SSYT. But then we get an empty square, so we keep going until it's a valid shape. The end result is

$$\begin{array}{ccccc} x & 2 & 2 & o & \\ & 1 & 1 & 3 & o \end{array}$$

For a bigger example consider

$$\begin{array}{ccccc} 2 & 4 & 4 & 5 & \\ & 3 & 3 & 3 & 6 \\ & 1 & 2 & 2 & 3 \\ & x & 1 & 1 & 2 \end{array} \longrightarrow \begin{array}{ccccc} 2 & 4 & 4 & o & \\ & 3 & 3 & 5 & 6 \\ & 2 & 2 & 3 & 3 \\ & 1 & 1 & 1 & 2 \end{array}$$

Remark 1.11.10. A slide sends an SSYT to an SSYT. A case like

$$\begin{array}{ccc} & 3 & \\ 4 & x & 5 \end{array}$$

because there should be an entry there $\leqslant 3$ and > 4.

We can continue applying the process to the following inner corners to get a SSYT.

Definition 1.11.11. The rectification of a skew SSYT is the result of performing JDT slides until we don't have inner corners left. We say that the shape is straight.

Remark 1.11.12. The rw of the skew-shape is the same as the rectification.

1.12 Day 11 | 20230213

Recall we have talked about the Jeu de Taquin. We observed that JDT slides send skew SSYT to other skew SSYT. To show compatibility with Knuth-equivalence, we want will require a more general definition of reading word because for example in the middle of a JDT we may have

$$\begin{array}{|c|c|c|c|cc}\hline 3 & 4 & 7 & 8 & & \\ \hline & & 2 & 4 & & \\ \cline{3-6} & & 1 & 1 & 3 & \multicolumn{1}{c|}{3} \\ \cline{3-6}\end{array}.$$

Definition 1.12.1. A reading word of any labeled set of boxes in the 1^{st} quadrant is formed by reading the rows from top to bottom L to R with each row.

Lemma 1.12.2. *If a skew SSYT S is obtained from another skew SSYT T by a sequence of JDT slides, then rw$(S) \sim_K$ rw(T).*

Proof

Assume S, T differ by one inner-slide. There are a couple of cases, we'll show that at each step of the slide the Knuth equivalence class of the reading word is unchanged. (Even when we haven't finished the slide.)
The first case is when we do a horizontal slide

$$\begin{array}{|c|c|}\cline{1-1} a & \multicolumn{1}{c}{} \\ \hline x & b \\ \hline\end{array} \to \begin{array}{|c|c|}\cline{1-1} a & \multicolumn{1}{c}{} \\ \hline b & x \\ \hline\end{array}$$

and nothing happens here because the reading word is still ab. The interesting case is the vertical slide.
PROOF BY EXAMPLE

Example 1.12.3. Consider the tableau

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-5} s & t & u & f & f & \multicolumn{2}{c}{} \\ \cline{1-6} 1 & 2 & 5 & 6 & 7 & T & \multicolumn{1}{c}{} \\ \hline \multicolumn{2}{c|}{} & 3 & 4 & x & 8 & 9 \\ \cline{3-7}\end{array} \to \begin{array}{|c|c|c|c|c|c|c|}\cline{1-6} 1 & 2 & 5 & 6 & x & T & \multicolumn{1}{c}{} \\ \hline \multicolumn{2}{c|}{} & 3 & 4 & 7 & 8 & 9 \\ \cline{3-7}\end{array}$$

the reading word changes from $12567T3489$ to $1256T34789$. We will apply Knuth moves to the reading word of the original tableau.

We need an algorithm to guarantee that we get from one point to another. It suffices to consider the subword $567T348$ on the smaller window

$$\begin{array}{|c|c|c|c|}\hline 5 & 6 & 7 & T \\ \hline 3 & 4 & x & 8 \\ \hline\end{array}$$

Thinking in steps we want to move the 3 backwards past the 7, then 4 and finally pull the 10 back. We have

$$567T348 \to 5673T48$$

Notice that $6 > 3$ because $6 > 4 > 3$ and it's also less than 7, there's still something still past the 7 which we can use to move the 3. We get

$$5637T48 \to 56374T8$$

We use the 5 to pull the 3 out of the way, once again the 5 is there due to semi-standardness. We get

$$53674T8 \to 53647T8$$

and to move the $T = 10$ we use the 8 which is below 10 due to semi-standardness. Applying a Knuth move with $7T8$ we obtain

$$5364T78 \to 5634T78 \to 563T478 \to 56T3478$$

Remark 1.12.4. If we had more *stuff* we would wave to take a longer subword.

We said the rectification was the end goal of a JDT, with this in hand we can *well*-define it.

Definition 1.12.5. The <u>rectification</u> of a SSYT is formed by performing inner JDT slides until we have a straight shape.

This is well defined because no matter the order of slides, the reading word of the rectification is equivalent to the reading word of the original tableau.

As the Knuth equivalence determines the insertion tableau (rect(T)=ins(rw(T))) (knuth class=ins tableau thm.) So putting it all together we see it works.

Example 1.12.6. Rectify the tableau

$$\begin{array}{ccc} 2 & 4 & \\ & 1 & 5 \\ & & 3 \end{array} \to \begin{array}{ccc} 2 & & \\ 1 & 4 & 5 \\ & & 3 \end{array} \to \begin{array}{ccc} 2 & & \\ 1 & 4 & \\ & 3 & 5 \end{array} \to \begin{array}{ccc} 2 & 4 & \\ 1 & 3 & 5 \end{array}.$$

However we can also do

$$\begin{array}{ccc} 2 & 4 & \\ & 1 & 5 \\ & & 3 \end{array} \to \begin{array}{ccc} 2 & & \\ & 4 & 5 \\ & 1 & 3 \end{array} \to \begin{array}{ccc} 2 & 4 & 5 \\ & 1 & 3 \end{array} \to \begin{array}{ccc} 2 & 4 & \\ 1 & 3 & 5 \end{array}.$$

The Knuth equivalence of the reading words of all tableau is the same! The other method is to take the reading word of the original tableau 24153 and then insert it to the corresponding tableau.

Now we can completely replace insertion with rectification because we can make any word with skew-tableau.

Corollary 1.12.7. *The rectification of the skew tableau where w is any word is ins(w).*

Example 1.12.8. Consider the tableau

$$\begin{array}{ccccc} 5 & & & & \\ 4 & 6 & 7 & & \\ 1 & 2 & 4 & 5 & \\ & & & & 3 \end{array},$$

we will rectify it and see it is the same as bump-inserting the 3 into the tableau. We get

$$\begin{array}{ccccc} 5 & & & & \\ 4 & 6 & 7 & & \\ 1 & 2 & 4 & 5 & \\ & & & & 3 \end{array} \longrightarrow \begin{array}{ccccc} 5 & & & & \\ 4 & 6 & 7 & & \\ 1 & 2 & 4 & & \\ & & 3 & 5 & \end{array} \longrightarrow \begin{array}{cccc} 5 & 6 & & \\ 4 & 4 & 7 & \\ 1 & 2 & 3 & 5 \end{array}$$

DO THE SLIDING

Then doing the diagonal process is intuitively inserting by this process.

Definition 1.12.9. We define the product of two tableau T, U as the rectification of the skew tableau formed by connecting the lower right corner of T to the upper left of U.

Equivalently

$$T \circ U = \text{rect}(T \leftarrow \text{rw}(U)).$$

This gives us an associative operation because of the Knuth-equivalence. In consequence the set of tableaux is a monoid, as the identity element is the empty tableau.

We define it as the <u>Plactic monoid</u>, the set of SSYT's of straight shape with $\circ$.

It's interesting to look at the Plactic monoid in terms of words as well.

Definition 1.12.10. The Plactic monoid is the $\{\text{words}\}/\sim$ with the concatenation of words as the operation and $\sim$ is Knuth-equivalence.

Example 1.12.11. If $w = [2131], v = [2213]$, then

$$wv = [21312213] = [21132213].$$

For the next class we will talk about skew Schur functions. We will build up to writing skew functions in terms of the ordinary ones.

1.13 Day 12 | 20230215

Skew Schur functions

Definition 1.13.1.

$$s_{\lambda/\mu} = \sum_{*} x^T$$

where $x^T = x^{\#1's} x^{\#2's} \dots$. This is still a symmetric function by the same proof as for s_λ.

Example 1.13.2. Compute $s_{(3,2)/(1)}$ in terms of m basis PHOTO

But instead we can see that this function is also $s_{(2,2)} + s_{(3,1)}$. It turns out that skew Schur functions are Schur positive. Recall that this means that in the Schur basis expansion, all of its coefficients are positive integers.

Theorem 1.13.3. *$s_{\lambda/\mu}$ is Schur-positive,*

$$s_{\lambda/\mu} = \sum_{\nu \vdash |\lambda| - |\mu|} c^{\lambda}_{\mu,\nu} s_\nu$$

The coefficients are called the Littlewood-Richardson coefficients.

We can use the Littlewood-Richardson coefficients to count certain Young tableaux. However another way to compute coefficients which uses a similar rule is the *Knutson-Tao puzzles*.

Littlewood-Richardson rule

Definition 1.13.4. A word w of positive indices is (reverse) ballot, Yamanouchi, or lattice if every suffix $w_i w_{i+1} \dots w_n$ has partition content. This is that reading the word from right-to-left the number of $1's$ is greater the $2's$ and so on.

$$\#1's \geqslant \#2's \geqslant \#3's \geqslant \dots$$

Example 1.13.5. Consider the word 341231211, this is Yamanouchi, while 21433231211 is not Yamanouchi.

Definition 1.13.6. A skew tableau is called Littlewood-Richardson if its reading word is a Yamanouchi word.

Theorem 1.13.7. *The Littlewood-Richardson coefficient is the number of Littlewood-Richardson tableaux of shape λ/μ and content ν.*

Example 1.13.8. Let us find some Littlewood-Richarson coefficients. Consider

$$\lambda/\mu = \begin{ytableau} \; & \; & \none \\ \none & \; & \; \end{ytableau}, \ \nu = (2,2)$$

so we need to find L-R tableaux of shape λ/μ with content $(2,2)$ (2 ones and 2 twos). The only possibilities are

$$\begin{ytableau} 2 & 2 & \none \\ \none & 1 & 1 \end{ytableau}, \quad \text{and} \quad \begin{ytableau} 1 & 2 & \none \\ \none & 1 & 2 \end{ytableau}$$

and the latter's reading word is 1212 which is not Yamanouchi because it ends in a 2 and not a 1.

Let us now find the number of L-R tableaux with shape

$$\begin{ytableau} \; & \; & \none & \none & \none & \none \\ \; & \; & \none & \none & \none & \none \\ \none & \none & \; & \; & \; & \none \\ \none & \none & \none & \none & \; & \; \end{ytableau}$$

and content $(4,3,2)$. We have the following tableaux

$$\begin{ytableau} 3 & 3 & \none & \none & \none & \none \\ 1 & 2 & \none & \none & \none & \none \\ \none & \none & 1 & 2 & 2 & \none \\ \none & \none & \none & \none & 1 & 1 \end{ytableau}, \quad \text{and} \quad \begin{ytableau} 3 & 3 & \none & \none & \none & \none \\ 2 & 2 & \none & \none & \none & \none \\ \none & \none & 1 & 1 & 2 & \none \\ \none & \none & \none & \none & 1 & 1 \end{ytableau}$$

This means that

$$c^{(6,5,2,2)}_{(4,2),(4,3,2)}.$$

Notice that by proving this result, we immediately get that the skew Schur functions are Schur positive. We will use crystal-base theory. This comes from the representation theory of $\mathcal{U}_q(\mathfrak{sl}_n)$.

A crystal of tableau sorts the monomials into a graph, let us for example consider

$$\begin{ytableau} 2 \\ 1 & 1 \end{ytableau}$$

1.14 Day 13 | 20230217

Crystals on words

Operating on words means we can operate on reading words and thus on tableau.

Definition 1.14.1. For a word w of 1's and 2's, we define the raising operator E_1 and lowering operator F_1 by:

- ◇ Replacing all ones with right parenthesis and twos with left parenthesis.
- ◇ Then we pair off matching parenthesis.
- ◇ E_1 makes the first unmatched 2 to a 1 and F_1 the last unmatched 1 to a 2.

Example 1.14.2. Consider the word

$$112212111122212 \rightarrow))(()())))((()($$

Applying E_1 we get

$$112212111112212 \rightarrow 112212111111212 \rightarrow 112212111111211$$

and applying once again we get the empty word because there's no more match. We can see that applying F_1 reverses this process. While applying F_1 to our original word several times we get

$$222212112222212$$

before we get to the empty word.

Definition 1.14.3. For a word $w \in \mathbb{Z}^n_{\geqslant 0}$, then E_i is formed looking at the $(i, i+1)$, treating them as $(1,2)$ and performing E_1.

This allows us to draw graphs on words where we connect after applying the operations.

The reason they are called raising and lowering is because in terms of content, the raise and lower the content:

$$\begin{aligned} 222212112222212 &\rightarrow (4,11) \\ 112212111122212 &\rightarrow (8,7) \\ 112212111111211 &\rightarrow (11,4) \end{aligned}$$

This also correspond to weight spaces in representations of $\mathfrak{sl}_n$.

Remark 1.14.4. Recall the Lie algebra of SL_2 is

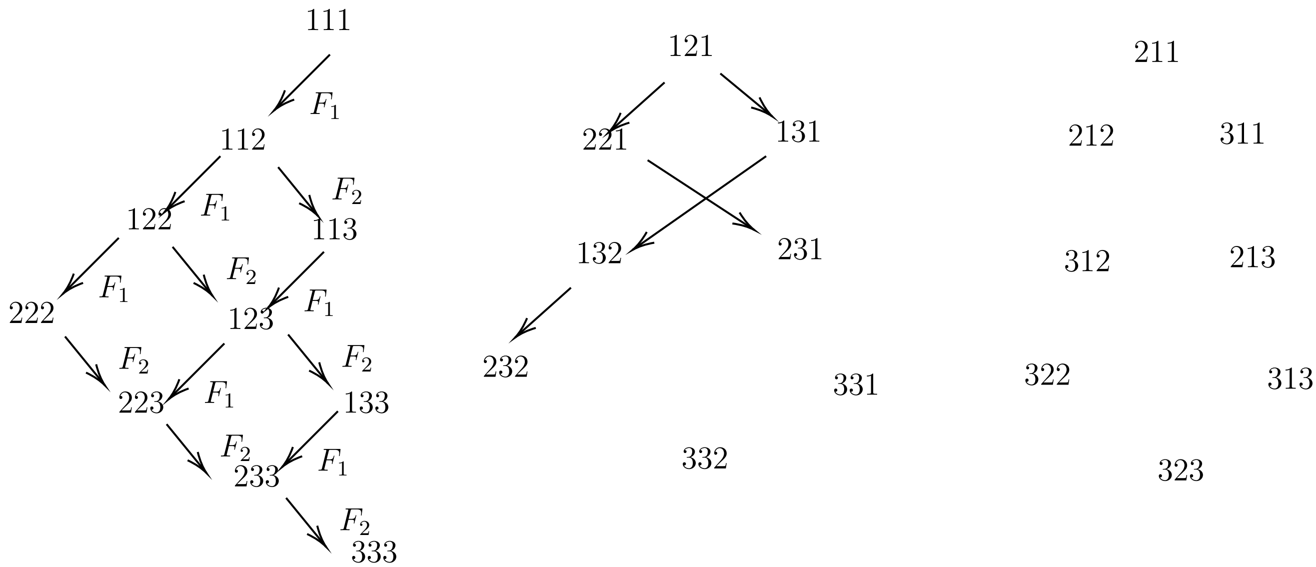
$$\mathfrak{sl}_2 = \{ A \in \mathcal{M}_2 : \operatorname{tr}(A) = 0 \}.$$

This is an additive vector space with a Lie bracket, it's not closed under multiplication. The matrices with generate this space are

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In general in $\mathfrak{sl}_n$, E_i has a 1 on top of the i^{th} position on the diagonal.

Example 1.14.5. Consider the word 111 we get the following graph



Note that the words 121 and 211 both insert to

$$\mathrm{ins}(121) = \mathrm{ins}(211) = \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}.$$

Also look at the fact that we have listed all the words of length 3. As an alternative definition, write all the words, match them by $F's$ then we get disjoint graphs. Each connected component gives us a Schur function.

How many connected components do we have?

If we have length n, the question is how many (reverse-)ballot words of length n are there? (Why ballot words?)

Lemma 1.14.6 (Homework)**.** *For a word w*

$$E_i(w) = \varnothing \iff w \textit{ is Yamanouchi.}$$

Assuming this, on length 4 we have the following Yamanouchi words:

(a) 1111

(b) 2111

(c) 3211

(d) 3121

(e) 1121

(f) 2211

(g) 1321 (i) 1211

(h) 4321 (j) 2121

There are $n \leqslant x \leqslant n!$ components. With the lemma comes the following definition:

Definition 1.14.7. We say a word w has highest weight if $E_i(w) = \varnothing$ for all i.

Theorem 1.14.8. *By acting on the reading word we have that E_i, F_i are well defined on skew tableaux (and therefore on tableaux.)*

We can always act on the reading word, but this action doesn't necesarilly guarantee that the result is an SSYT.

Proof

Look at the $(i, i+1)$ in a tableau T. This will "look like"

$$\begin{array}{cccccccccccc} i & i & j & j & & & & & & & & \\ & & & i & i & j & j & j & & & & \\ & & & & & & i & i & & & & \\ & & & & & & & & i & i & j & j \end{array}$$ [a][b].

The columns "cancel" and some others do to, we can change the left most two $i's$ and the rightmost $i(i+1)$.

[a]We used this tableau when proving the Schur functions were symmetric.
[b]j is i+1, have to change ytableau package.

Compatibility with Knuth moves and Jeu de Taquin

Example 1.14.9. Consider a skew tableau and an inner slide:

$$\begin{array}{ccccc} 1 & 1 & 2 & 2 & \\ & & 1 & 1 & 2 \end{array} \to \begin{array}{ccccc} 1 & 2 & 2 & & \\ & 1 & 1 & 1 & 2 \end{array}$$

But applying F_1 means that we get

$$\begin{array}{ccccc} 1 & 2 & 2 & 2 & \\ & & 1 & 1 & 2 \end{array} \to \begin{array}{ccccc} 1 & 2 & 2 & & \\ & 1 & 1 & 2 & 2 \end{array}.$$

DRAW COMMDIAG

Compatibility means commutation at the level of diagrams.

1.15 Day 14 | 20230220

Today we will continue working on the Littlewood-Richardson rule. As a warm-up lets work on an example.

Exercise 1.15.1. Compute E_3 of the tableau

$$\begin{array}{|c|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-4} 3 & 3 & 4 & 6 \\ \cline{1-4} 2 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline\end{array}.$$

Answer

We can focus only on the sub skew tableau

$$\begin{array}{|c|c|c|c|}\cline{1-3} 3 & 3 & 4 \\ \hline \multicolumn{2}{c|}{} & 3 & 4 \\ \cline{3-4}\end{array}$$

with reading word 33434, the leftmost un-paired 4 is the last one so we change the word to 33433 and then replace on the tableau to get

$$\begin{array}{|c|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-4} 3 & 3 & 4 & 6 \\ \cline{1-4} 2 & 2 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline\end{array}.$$

Notice we do get a SSYT as we showed last time. We will now work on crystal operators.

Theorem 1.15.2. *E_i, F_i commute with JDT slides.*

It suffices to check this on $1-2$ tableaux and E_1, F_1. This is because E_i, F_i only affect $i, i+1$. (Look at example from last time) This example is the only case that we have to worry about because we can only have at most two rows. The only problematic case is a vertical slide like in the example.

But what about

$$\begin{array}{|c|c|c|c|c|}\cline{1-4} 1 & 2 & 2 & 2 \\ \hline \multicolumn{3}{c|}{} & 1 & 2 \\ \cline{4-5}\end{array} \to \begin{array}{|c|c|c|c|c|}\cline{1-3} 1 & 2 & 2 \\ \hline \multicolumn{2}{c|}{} & 1 & 2 & 2 \\ \cline{3-5}\end{array},$$

it slides the same way after applying F_1:

$$\begin{array}{|c|c|c|c|c|}\cline{1-4} 2 & 2 & 2 & 2 \\ \hline \multicolumn{3}{c|}{} & 1 & 2 \\ \cline{4-5}\end{array} \to \begin{array}{|c|c|c|c|c|}\cline{1-3} 2 & 2 & 2 \\ \hline \multicolumn{2}{c|}{} & 1 & 2 & 2 \\ \cline{3-5}\end{array}.$$

Theorem 1.15.3. *The operations E_i, F_i preserve the "being in the same Knuth-equivalence class". In other words*

$$E_i(w) \sim E_i(v), \quad F_i(w) \sim F_i(v).$$

Since rectification is unique and that determines the Knuth-equivalence class, we can use the previous result.

Answer

Consider the skew tableau $D_w = \operatorname{diag}(w)$ and D_v. Then

$$\operatorname{ins}(w) = \operatorname{ins}(v) \Rightarrow \operatorname{rect}(D_w) = \operatorname{rect}(D_v)$$

so

$$\begin{aligned} \operatorname{rect}(E_i(D_w)) &= E_i(\operatorname{rect}(D_w)) \\ &= E_i(\operatorname{rect}(D_v)) \\ &= \operatorname{rect}(E_i(D_w)) \end{aligned}$$

Therefore $E_i(D_w) \sim E_i(D_v)$.

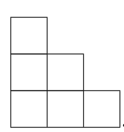
Highest Weight

We will now work with highest-weight elements.

Lemma 1.15.4. *For a straight shape λ, the SSYT's with $\operatorname{sh}(\lambda)$ all lie in a single connected component of the crystal graph.* (*i.e. connecting via E_i, F_i we get a connected component.*)

Moreover we get a unique tableau with highest weight ($E_i(T) = \varnothing$). We claim that such a tableau has row i full of i's.

Take any shape

,

as the reading word is Yamanouchi, the first entry is a 1. Then all entries to the left are 1's by semi-standardness. The next rightmost element of the next row can't be a 3 by Yamanouchi so it must be a 2 and so on.

But why is everything in the same connected component?

If we have a tableau T, then applying E_i we *must* reach T_λ. Reversing this process for any T with F_i, we get all the tableau.

Remark 1.15.5. This method works with no restriction on the letters of the word.

Corollary 1.15.6. *Every connected component of a crystal of all tableaux of a given skew shape is isomorphic as a directed graph to a straight shape crystal obtained via JDT.*

With this key step we can compute a Schur function decomposition.

Example 1.15.7. Let us compute $s_{(3,1)/(1)}$. The crystal of this SSYT is obtained by writing the highest weight fillings of the skew shape:

$$\begin{ytableau} 2 \\ \none & 1 & 1 \end{ytableau}, \qquad \begin{ytableau} 1 \\ \none & 1 & 1 \end{ytableau}.$$

We get the following crystals:

aaaaa

By rectifying with JDT we get INSERT DIAGRAMS

What this means is that adding monomials corresponding to crystal one we get $s_{(2,1)}$ and the other crystal $s_{(3)}$.

As a shortcut, we draw the Yamanouchi ones, then rectify and that's it.

Corollary 1.15.8. *The coefficient of s_ν is $s_{\lambda/\mu}$ is the number of heighest weight SSYT with* $\mathrm{sh}(T) = \lambda/\mu$ *that rectify to* $\mathrm{sh}(\nu)$.

$$\#hw\,\mathrm{sh}(\lambda/mu) \quad \textit{content} \quad \nu$$

this is exactly $c^\lambda_{\mu\nu}$.

This completes a proof of the Littlewood-Richardson rule.

Products of Schur Functions

There's another Littlewood-Richarson Rule:

Theorem 1.15.9. *Suppose μ, ν are straight shapes, then*

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda.$$

There's a proof with Knuth-equivalence and concatenation.

We know that $s_\nu \cdot s_\mu$ can be obtained by finding s_T where T is the skew shape formed by concatenating right down of μ with LEFT UPν. Then the product is

$$\sum_\lambda c^\rho_{R\lambda} s_\lambda.$$

To prove it we use Fulton's techniques [3].

We want to show $c^{\rho}_{R\lambda} = c^{\lambda}_{\mu\nu}$. This means that the number of Yamanouchi tableaux of shape ρ/R and content λ is the same as $\mathrm{sh}(\lambda/\mu)$ and content ν.

Remark 1.15.10. Consider

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
3 & 4 & & & & & & & \\
\hline
2 & 2 & 3 & & & & & & \\
\hline
1 & 1 & 2 & 4 & & & & & \\
\hline
 & & & & 3 & & & & \\
\hline
 & & & & 2 & 2 & 2 & 2 & \\
\hline
 & & & & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}.$$

the lower one has to be T_ν in any Yamanouchi filling.

We first de-RSK it to get

$$\begin{array}{|c|c|c|c|c|}
\hline
3 & 3 & 3 & & \\
\hline
2 & 2 & 2 & 2 & \\
\hline
1 & 1 & 1 & 1 & 1 \\
\hline
\end{array} \to \begin{pmatrix} 123331222111 \\ 111112222333 \end{pmatrix}$$

if r is the first row and b the second one, then we insert r into U. But when inserting we are keeping track with b, so we label the new squares with b as a skew recording tableau.

Name Q such tableau, we claim that Q is a ballot SSYT shape λ/μ content ν.

Putting this together in the example we get

$$U = \begin{array}{|c|c|c|c|}
\hline
3 & 4 & & \\
\hline
2 & 2 & 3 & \\
\hline
1 & 1 & 2 & 4 \\
\hline
\end{array} \leftarrow 123331222111$$

READ IN FULTON AND UNDERSTAND BIJECTion

Index

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