MATH519 — Complex Analysis

Based on the lectures by Jeff Achter

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Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This course is an introduction to analytic functions of a single complex variable. The subject is beautiful.– it turns out that a function with a complex derivative is highly structured – and enjoys a give and take with many other areas of mathematics.

Requirements

Knowledge of convergence of sequences, series: limits, continuity, differentiation, integration of one-variable functions is required.

Contents

Chapter 1

First Midterm

1.1 Interim | HW1

Exercise 1.1.1 (1.1 Stein & Shakarchi)**.** Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$.
(b) $1/z = \overline{z}$.
(c) $\mathrm{Re}(z) = 3$
(d) $\mathrm{Re}(z) > c$, (resp.,$\geqslant c$) where $c \in \mathbb{R}$.
(e) $\mathrm{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$.
(f) $|z| = \mathrm{Re}(z) + 1$.
(g) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$.

Answer

i) The first set is the set of points at the same distance from z_1 and z_2. If we consider the line segment z_1z_2, then the set in question is the bisector of that line segment.
ii) Note that

$$1/z = \overline{z} \iff 1 = \overline{z}z \iff 1 = |z|^2 \iff 1 = |z|,$$

thus the set is the unit circle.
iii) The set is a perpendicular line to the real axis at $z = 3$.
iv) This infinite set is an infinite half plane to the right (but not including) of the line $z = c$. In the other case, we do include the line in question.

v) Let us rephrase this inequality in terms of real numbers. Call $a = a_1 + ia_2$, $b = b_1 + ib_2$ and $z = x + iy$. Then

$$\mathrm{Re}(az + b) = \mathrm{Re}[a_1 x - a_2 y + b_1 + i(a_2 x + a_1 y + b_2)],$$

thus our desired inequality is true whenever $a_1 x - a_2 y + b_1 > 0$. Solving for y we get $y > (a_1 x + b_1)/a_2$, which is the half plane located above the line $y = (a_1 x + b_1)/a_2$.

vi) The equation in question is equivalent to

$$\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 = (\mathrm{Re}(z) + 1)^2.$$

To ease the notation, assume $z = x + iy$. Then the equation reads

$$x^2 + y^2 = x^2 + 2x + 1 \iff y^2 = 2x + 1 \iff x = (y^2 - 1)/2.$$

It holds the the parabola in question contains the points which satisfy the equation.

vii) This set is a line parallel to the real axis at $z = c$

Exercise 1.1.2. Do the following:

i) Show that the complex conjugation map $\kappa : \mathbb{C} \to \mathbb{C},\ z \mapsto \overline{z}$ is an involution, i.e., a ring homomorphism such that $\kappa \circ \kappa = \mathrm{id}$.

ii) Suppose $a \in \mathbb{R},\ z \in \mathbb{C}$. Show that

$$\mathrm{Re}(az) = a\,\mathrm{Re}(z), \quad \text{and} \quad \mathrm{Im}(az) = a\,\mathrm{Im}(z).$$

Answer

Let us take $z = x + iy$ with $x, y \in \mathbb{R}$.

i) We have $\overline{z} = x + i(-y) = x - iy$. Once more we get $\overline{\overline{z}} = x - i(-y) = x + iy = z$. Thus $\overline{\overline{z}} = z$ for any $z \in \mathbb{C}$. In conclusion $\overline{\overline{\cdot}} = \mathrm{id}$.

ii) It holds that

$$\begin{aligned}\mathrm{Re}(az) &= \mathrm{Re}(ax + aiy) = ax = a\,\mathrm{Re}(z),\\ \mathrm{Im}(az) &= \mathrm{Im}(ax + aiy) = ay = a\,\mathrm{Im}(z).\end{aligned}$$

Exercise 1.1.3. Do the following:

i) Prove that $|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w})$.
ii) Use this to prove the parallelogram rule: $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$.

Answer

i) Note that

$$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + w\overline{z} + z\overline{w} + w\overline{w}.$$

The number $w\overline{z}$ is the conjugate of $z\overline{w}$, and summing a number and its conjugate returns twice its real part. Thus we get the desired identity.

ii) As the past identity holds for all complex numbers, it holds when $w = -w$. This means that $|z-w|^2 = |z|^2 + |-w|^2 + 2\operatorname{Re}(z(\overline{-w})) = |z|^2 + |w|^2 - 2\operatorname{Re}(z\overline{w})$ and summing this together with the first identity gives us the parallelogram law.

Exercise 1.1.4 (1.5 Stein & Shakarchi)**.** A set Ω is said to be pathwise connected if any two points in Ω can be joined by a (piecewise-smooth) curve entirely contained in Ω. The purpose of this exercise is to prove that an open set Ω is pathwise connected if and only if Ω is connected.

i) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a parametrization $z : [0,1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let

$$t_* = \sup_{0 \leqslant t \leqslant 1} \{\, t : \forall s[(0 \leqslant s < t) \Rightarrow (z(s) \in \Omega_1)] \,\}.$$

Arrive at a contradiction by considering the point $z(t_*)$.

ii) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subseteq \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subseteq \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega 1$ as desired.

Answer

i) We will proceed using a topological argument instead of a metric one. As the function γ is continuous, it pulls back Ω_1 and Ω_2 into $[0,1]$ as open sets. However, as the sets are disjoint, their inverse images are disjoint as well.

In other words, we have found two open disjoint sets which separate $[0, 1]$:

$$[0, 1] = \gamma^{-1}[\Omega_1] \cup \gamma^{-1}[\Omega_2].$$

But this is impossible because $[0, 1]$ is a connected set. Thus, our assumption that Ω was disconnected must be false. We conclude that path-connectedness implies connectedness.

ii) Take Ω_1, Ω_2 as in the statement. Then Ω_1 is non-empty as $w \in \Omega_1$ because it's connected to itself through a trivial path. Suppose now that $z \in \Omega_1$ and that $d(z, \partial\Omega_1) > r > 0$. Take $x \in B(z, r)$, then there exists a line-segment between z and x and there's a smooth curve which connects $z \in \Omega_1$ with w. Thus the piecewise-continuous path from x to z and from z to w is a path which connects x and w. As x is arbitrary, it follows that $B(z, r) \subseteq \Omega_1$, and thus Ω_1 is open.

Formally, if $\gamma : [0, 1] \to \Omega_1$ is the map which parametrizes the curve between z and w and $r : [0, 1] \to B(z, r)$ is the map $t \mapsto tz + (1 - t)x$, then the curve from x to w is parametrized by the function

$$f = \begin{cases} 2tz + (1 - 2t)x, \ t \in [0, 1/2], \\ \gamma(2t - 1), \ t \in [1/2, 1]. \end{cases}$$

On the other hand take a point $z \in \Omega_2$ and let $d(z, \partial\Omega_2) > r > 0$. Consider a point $x \in B(z, r)$ and assume by way of contradiction that such x can be connected to w by a curve which can be parametrized by a smooth function γ. As the ball is convex, we can connect z to x and then to w creating a path between z and w. This is impossible as z cannot be connected to w by a path, thus our assumption must be false. It holds that x cannot be connected to w by a path and thus $x \in \Omega_2$. Therefore Ω_2 is also open. We conclude that $\Omega = \Omega_1 \cup \Omega_2$ is a union of two disjoint open sets, and since Ω is connected, it must hold that Ω_2 is empty. We conclude that Ω is path-connected.

Exercise 1.1.5 (1.7 Stein & Shakarchi)**.** The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

i) Let $z, w \in \mathbb{C}$ such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1$$

if $|z| < 1$ and $|w| < 1$, and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1$$

if $|z| = 1$ or $|w| = 1$. ⟦Hint: Why can one assume that z is real? I then suffices to prove that $(r-w)(r-\overline{w}) \leqslant (1-rw)(1-r\overline{w})$ with equality for appropriate r and $|w|$.⟧⟦Here is an alternate approach, which you may use if you like. Fix $w \in \mathbb{C}$ with $w < 1$, and consider the function $z \mapsto \frac{w-z}{1-\overline{w}z}$. What is $\overline{f(z)}$? By computing $f(z)\overline{f(z)}$, show that $|z| = 1$ implies $|f(z)| = 1$. Find a point z with $|z| < 1$ such that $|f(z)| < 1$. Since f is continuous, this shows that f takes the unit disc to itself. (Why?)⟧

ii) Prove that for a fixed $w \in \mathbb{D}$, the mapping $F : z \mapsto \frac{w-z}{1-\overline{w}z}$ satisfies the following:

a) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.
b) F interchanges 0 and w.
c) $|F(z)| = 1$ if $|z| = 1$.
d) F is bijective. ⟦Hint: Calculate $F \circ F$.⟧

Answer

i) The inequality in question is equivalent to

$$0 \leqslant |w-z| < |1-\overline{w}z|.$$

Since the quantities are positive, we can square them and preserve the order. It holds that

$$0 \leqslant |w-z|^2 < |1-\overline{w}z|^2 \iff 0 \leqslant (w-z)\overline{(w-z)} < (1-\overline{w}z)\overline{(1-\overline{w}z)},$$

Simplifying this expression we get

$$\begin{aligned}
&(w-z)(\overline{w}-\overline{z}) < (1-\overline{w}z)(1-w\overline{z}) \\
\iff &w\overline{w} - w\overline{z} - z\overline{w} + z\overline{z} < 1 - w\overline{z} - \overline{w}z + \overline{w}zw\overline{z} \\
\iff &|w|^2 + |z|^2 < 1 + |w|^2|z|^2 \\
\iff &0 < (1-|w|^2)(1-|z|^2).
\end{aligned}$$

The inequality is true whenever both moduli are less than one, and whenever either is one equality is achieved.

ii) Now we suppose $w \in \mathbb{D}$ which means that $|w| < 1$. Taking $z \in \mathbb{D}$ and applying F gives us the quantity $\frac{w-z}{1-\overline{w}z}$ which by the previous argument, has modulus less than 1 whenever w, z do.

The function F is holomorphic because it is a quotient of holomorphic functions. The denominator is never zero inside the domain because that would mean that $1 = \overline{w}z$. And taking moduli in both sides of the equation gives us

$$1 = |1| = |w||z| < 1$$

which is impossible.

Now $F(0) = \frac{w-0}{1-0} = w$ and $F(w) = \frac{w-w}{1-|w|^2} = 0$. The denominator in the last expression is never zero because $|w| < 1$.

By the second part of the previous argument it holds that $|z| = 1$ immediately gives us $|F(z)| = 1$. And finally we will see that F is an involution:

$$F(F(z)) = F\left(\frac{w-z}{1-\overline{w}z}\right) = \frac{w - \left(\frac{w-z}{1-\overline{w}z}\right)}{1 - \overline{w}\left(\frac{w-z}{1-\overline{w}z}\right)}.$$

Homogenizing and clearing denominators we get

$$\frac{w(1-\overline{w}z) - w + z}{1 - \overline{w}z - \overline{w}(w-z)} = \frac{-w\overline{w}z + z}{1 - \overline{w}w} = \frac{(-w\overline{w}+1)z}{1-\overline{w}w} = z.$$

This means that F is it's own inverse and therefore, F is bijective.

1.2 Day 1 | 20230120

The Complex Numbers

To construct the complex numbers we take the real numbers, adjoin a variable and mod out by $\langle x^2+1 \rangle$. We can also define $\mathbb{C}$ as $\{ a + bi : a, b \in \mathbb{R} \}$ with the property $i^2 = -1$. This means that we can multiply complex numbers in the following way:

$$(a+bi)(c+di) = ac + (bc+ad)i + bdi^2 = (ac-bd) + (ad+bc)i.$$

Also as x^2+1 is irreducible in $\mathbb{R}[x]$, $\mathbb{C}$ is a finite field extension of $\mathbb{R}$ of degree 2. As a 2-dimensional vector space $\{ 1, i \}$ is a basis for $\mathbb{C}$.

The map $a + bi \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$ is not a ring homomorphism, it's a bijection with a bit of structure. The map $z \mapsto \alpha z$, when $\alpha = a + bi$, is a linear map with the following action over the basis

$$\alpha \cdot 1 = \alpha \Rightarrow [\alpha] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\alpha \cdot i = -b + ai \Rightarrow [\alpha] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

which means that $[\alpha] = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. The converse, if we have a $\mathbb{R}$-linear transformation, then it's $\mathbb{C}$-linear if and only if it looks like $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Definition 1.2.1. The complex conjugation map is $a + bi \mapsto a - bi$, or $z \mapsto \overline{z}$.

This map is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

Example 1.2.2. For $\alpha = a + bi$, we have

$$\overline{2\alpha} = \overline{2(a+bi)} = \overline{2a + 2bi} = 2a - 2bi = 2\overline{al}.$$

Whereas if instead

$$\overline{i\alpha} = \overline{ai - b} = -b - ai \neq i\overline{\alpha} = b + ai.$$

As a $\mathbb{R}$-linear map, we can identify with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By looking at the shape of this matrix we can see that it is not $\mathbb{C}$-linear.

Lemma 1.2.3. *The map $z \mapsto \overline{z}$ is a ring homomorphism*

Proof

$\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{zw}$.

With the complex conjugation we can pick out the real and imaginary parts of $\alpha = a + bi$.

$$\alpha + \overline{\alpha} = 2\operatorname{Re}(\alpha), \quad \alpha - \overline{\alpha} = 2i\operatorname{Im}(\alpha)$$

A Notion of Size

Can't do geometry without one. Notice that for $z = a + bi$

$$z\overline{z} = a^2 + b^2 > 0.$$

From a complex number we have extracted a positive quantity.

Definition 1.2.4. The complex modulus of z is $|z| = \sqrt{z\overline{z}}$.

The fact that every number has n roots is very important in complex analysis.
As a vector in the plane, the norm of z is $|z|$

INC FIG

This means that $a + bi \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$ is an isometry. In this sense the distance between two complex numbers is $d(z, w) = |z - w|$.

Polar Coordinates (*ad hoc*)

For $\theta \in \mathbb{R}$, define

$$\exp(i\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta) \Rightarrow |\exp(i\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

Every point in the unit circle is of the form $e^{i\theta}$ and vice-versa.

INC FIG

For non-zero complex numbers, $z = |z|e^{i\theta}$ for some θ.

Definition 1.2.5. For a complex number $z = re^{i\theta}$, an argument of z is θ.

To have a well defined function, we mod out by multiples of 2π:

$$\arg : \mathbb{C}\backslash\{0\} \to \mathbb{R}\big/ 2\pi\mathbb{Z},$$

and we obtain a group isomorphism. In general, "lengths multiply, angles add."
For inverses if $z = re^{i\theta}$, then $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$.

Definition 1.2.6. The upper-half plane is $\mathbb{H} = \{\operatorname{Im}(z) > 0\}$.

Lemma 1.2.7. *If H is a half plane* $\operatorname{Im}(z - \beta/\gamma) > 0$

1.3 Day 2 | 20230123

Recall the complex conjugation map and the modulus of a complex number. This gives us an isometry between $\mathbb{R}^2$ and $\mathbb{C}$. Let us prove the lemma from last time.

Lemma 1.3.1. *If $H \subseteq \mathbb{C}$ is a half plane, then there exist $\beta, \gamma \in \mathbb{C}$ such that*

$$H = \left\{ z : \operatorname{Im}\left(\frac{z-\beta}{\gamma}\right) > 0 \right\}.$$

INC FIG

Pick a point $\beta \in H$, then translate H to the origin by $z \mapsto z - \beta$. The plane is now rotated by θ at the origin so we should rotate every point. Then $z \in H - \beta$ whenever $ze^{-i\theta} \in \mathbb{H}$. REDO

Let us see an application, for a polynomial, the coefficients determine the roots. The following lemma is a technical lemma.

Lemma 1.3.2. *Suppose $p \in \mathbb{C}[z]$ and H is a half plane which contains all the roots of p. Then H contains all the roots of p'.*

Proof

We can assume p is monic, so suppose $\alpha_1, \ldots, \alpha_d$ are the roots of $\mathbb{C}$. This means that

$$p(z) = \prod_{k=1}^{d}(z-\alpha_k) \Rightarrow p'(z) = \sum_{k=1}^{d} \frac{p(z)}{z-\alpha_k} \Rightarrow \frac{p'(z)}{p(z)} = \sum_{k=1}^{d} \frac{1}{z-\alpha_k}.$$

Now suppose that H contains all α_k and suppose $z_0 \notin H$, if we show $p'(z_0) \neq 0$ we are done because all the points which make p' vanish won't be outside H. Describe H by the previous lemma, there exist β, γ such that points in H satisfy the inequality $\operatorname{Im}\left(\frac{z-\beta}{\gamma}\right) > 0$. As z_0 is not in H, then $\operatorname{Im}\left(\frac{z_0-\beta}{\gamma}\right) < 0$. For each $k \in [d]$, we have that

$$z_0 - \alpha_k = z_0 - \beta + \beta - \alpha_k = (z_0 - \beta) - (\alpha_k - \beta)$$

so by taking imaginary parts

$$\operatorname{Im}\left(\frac{z-\alpha_k}{\gamma}\right) = \operatorname{Im}\left(\frac{z-\alpha_k}{\gamma}\right) - \operatorname{Im}\left(\frac{z-\alpha_k}{\gamma}\right)$$

The quantity on the right is negative because it's a negative number minus a positive. So it holds that $\operatorname{Im}\left(\frac{\gamma}{z-\alpha_k}\right) > 0$. With this we can calculate the following:

$$\operatorname{Im}\left(\gamma \frac{p'(z_0)}{p(z_0)}\right) = \operatorname{Im}\left(\sum_{k=1}^{d} \frac{\gamma}{z_0 - \alpha_k}\right) > 0$$

so in particular this number is non-zero. Thus $p'(z_0) \neq 0$

Definition 1.3.3. A set $S \subseteq \mathbb{R}^n$ is convex if for any two points $x, y \in S$, the line segment between x and y is also contained in S. This is

$$\{\, ty + (1-t)x : \ x, y \in S \,\} \subseteq S.$$

The convex hull of S is the intersection of all convex sets containing S.

In the case of a finite set of complex numbers, the convex hull can be found by intersecting half-planes which contain them.

Corollary 1.3.4 (Gauss-Lucas). *The roots of $p'(z)$ are contained in the convex hull of the roots of $p(z)$.*

Metric Spaces

Definition 1.3.5. A metric space is a set with a distance function.

Example 1.3.6. $\mathbb{R}^n$ is a metric space with $d(x, y) = \|x - y\|$. Subsets of metric spaces with an induced distance are metric spaces.

- nbhd
- open and closed
- Cauchy

Definition 1.3.7. Cauchy sequence

1.4 Day 3 | 20230125

The defining property of $\mathbb{R}$ is that it is complete. In that sense it is possible to prove that $\mathbb{R}^n$ is also complete.

Derivatives

Recall a real function g is differentiable at x_0 if there exists a real number a such that

$$g(x) = g(x_0) + a(x - x_0) + \psi(x),\ \frac{\psi(x)}{x - x_0} \xrightarrow[x \to x_0]{0} .$$

In the same sense a multivariable function is differentiable when there exists a linear transformation such that a similar condition holds.

Definition 1.4.1. f has complex derivative iff real derivative and Cauchy-Riemann equations

Example 1.4.2. The map $z \mapsto \overline{z}$ is not complex-differentiable. First by matrix definition and second with limit.

1.5 Day 4 | 20230127

Lemma 1.5.1. *If $\sum_{n \geqslant 0} z_n$ is absolutely convergent, then it's convergent.*

Proof

If s_n is a partial sum, then

$$|s_n - s_m| = \left| \sum_{i=m+1}^{n} z_i \right| \leqslant \sum_{i=m+1}^{n} |z_i| < \varepsilon$$

because $\sum |z_n|$ is Cauchy.

Power Series

Definition 1.5.2. A power series (centered at 0) is an expression of the form $\sum_{n \geqslant 0} a_n z^n$.

Example 1.5.3. The power series for the exponential function is $e^z = \sum_{n \geqslant 0} \frac{z^n}{n!}$.

Theorem 1.5.4 (Cauchy-Hadamard)**.** *Suppose $\sum_{n \geqslant 0} a_n z^n$ has radius of convergence $\frac{1}{r} = \limsup |a_n|^{\frac{1}{n}}$. Then the series converges for $|z| < r$ and diverges for $|z| > r$.*

Proof

Índice Analítico

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