Exercise 1 (Exercise 1.a)**.** Compute $s_2(T)$ where T is the tableau below:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\cline{1-2} 3 & 4 \\
\cline{1-5} 2 & 2 & 2 & 3 & 3 \\
\hline 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
\hline
\end{array}.$$

Answer

We have

$$\mathrm{rw}(T) = 34222331111222233.$$

With this, we can pair 3 and 2's as follows:

$$\underline{342}22\overline{3\underline{311112}2}2233$$

and we can see we have 4 unpaired 2's and 2 unpaired 3's. By applying s_2 we wish to reflect this about the corresponding $\mathfrak{sl}_2$ chain to get 2 unpaired 2's and 4 3's. We have

$$34222331111222233 \xrightarrow{F_2} 34222331111222333 \xrightarrow{F_2} 34222331111223333$$

and this is the desired word. The corresponding tableau is

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\cline{1-2} 3 & 4 \\
\cline{1-5} 2 & 2 & 2 & 3 & 3 \\
\hline 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\
\hline
\end{array}$$

Exercise 2 (Exercise 1.b)**.** Explain why it suffices to show that $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ when acting on tableaux, for all i.

Answer

Checking a braid relation of the form $s_i s_j s_i$ where $|j - i| \neq 1$ is unnecessary because in those cases s_i commutes with s_j.

Exercise 3 (Exercise 1.c)**.** Show that, using the compatibility of JDT slides with crystal operators (which you may use as a fact), it suffices to show part 1 for $i = 1$, and therefore it suffices to work with $\mathfrak{sl}_3$ crystals, that is, tableaux whose letters are all $1, 2, 3$.

Answer

Consider the following diagram:

$$\begin{tikzcd} T \arrow[r, "\text{rem}"] \arrow[d, "\text{braid}"'] & \mathsf{skew}(T) \arrow[r, "\text{rect}"] & T' \arrow[d, "\text{braid}"] \\ S \arrow[r, "\text{rem}"'] & \mathsf{skew}(S) \arrow[r, "\text{rect}"'] & S' \end{tikzcd}$$

where rem is the operation which removes all letters but $i, i+1, i+2$. This diagram commutes because the braid operation is a combination of raising and lowering operators and the removal plus rectification is a JDT.
So from this, we may relabel $i, i+1, i+2$ to $1, 2, 3$ and then we can work on the tableau because the diagram commutes.

Exercise 4 (Exercise 1.d)**.** Show that one can further reduce to the case that the tableau shape has two rows.

Answer

Given the previous facts, we can reduce our case to a 3-row tableau on the alphabet $\{1, 2, 3\}$. Observe that it can't have more than 3 rows due to semistandardness. Now the third row must be comprised of 3's who should be paired with 2's below them and further those with 1's.
As s_1, s_2 act by raising/lowering, there's no way that elements on columns with more than two rows could get affected by the s_i's. So it suffices to work on tableau with only two rows as it will be the only part affected by s_i's.

Exercise 5 (Exercise 2)**.** Compute the chromatic symmetric function of the triangle graph, that is, the complete graph K_3, and express it in terms of elementary symmetric functions and in terms of Schur functions.

Answer

Observe that $\chi(K_3) = 3$ which means that there's no proper colorings with 1 or 2 colors. Thus we must color vertices $1, 2, 3$ with colors $i, j, k \in \mathbb{N}$. However there's $3!$ ways of doing this, so that each monomial $x_i x_j x_k$ is accounted $3!$ times. We thus have that

$$X_{K_3} = 3! m_{(1,1,1)} = 3! s_{(1,1,1)} = 3! e_3.$$