Sheaf Cohomology of Line Bundles over $\mathbb{P}^1$: Intuition and Examples.

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Abstract

We explore the concept of line bundles, starting with an intuitive understanding and formal definition. From this, we delve into the examples of line bundles of the projective line. In this case, all line bundles are all of a particular form $\mathcal{O}(d)$. Transitioning into cohomology, we will investigate $H^\bullet(\mathbb{P}^1, \mathcal{O}(d))$ and calculate it after motivating it by examples.

Keywords:
Line bundle, projective space, sheaf cohomology, Čech cohomology.

1 Introduction

Intuitively a line bundle is a collection of lines with additional properties. Take a point b in a base variety B and place a copy of $\mathbb{C}$ above it. Over each point of the variety we have a copy of the complex line, so that's where we get the line bundle. Specifically:

Definition 1. A line bundle over a base B is a map $\pi : L \to B$ with the following properties:

◇ There's an open cover $(U_i)_{i\in I}$ of B such that
$$\pi^{-1}(U_i) \simeq U_i \times \mathbb{C}$$
where we call $\phi_i : \pi^{-1}(U) \to U \times \mathbb{C}$ the isomorphism.

◇ For $b \in U_i \cap U_j$, the composition
$$\{b\} \times \mathbb{C} \xrightarrow{\phi_i^{-1}} \pi^{-1}(b) \xrightarrow{\phi_j} \{b\} \times \mathbb{C}$$
is a linear isomorphism. This map, $\phi_j \phi_i^{-1}$, is multiplication by a nonzero scalar λ_b.

A fiber over $b \in B$ of the line bundle is $\pi^{-1}(b)$.

Remark 1. Despite the global definition of line bundles, they can also be defined locally through charts. This is achieved by specifying the open cover of the base variety and providing the transition functions between the charts.

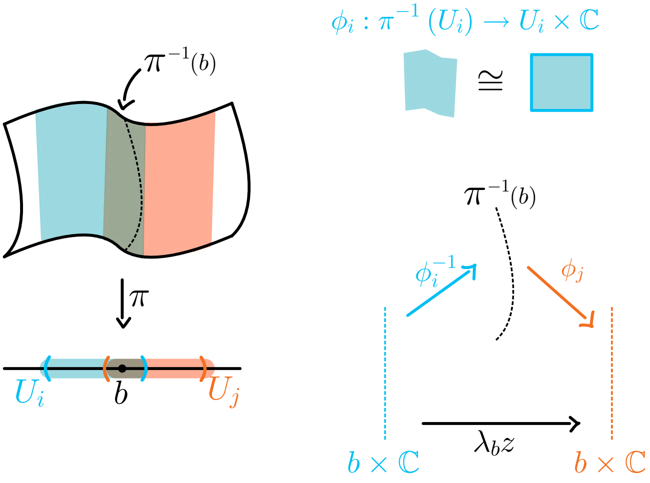

Figure 1: Line Bundle with the two properties

Remark 2. The first property describes the *local triviality* of a line bundle, indicating that around each point, the fibers are isomorphic to $\mathbb{C}$. This may not be true when considered globally. The second property ensures the compatibility of these local trivializations, maintaining a consistent linear structure. This allows different parts of the bundle to exhibit varying linear behaviors while maintaining a coherent structure.

Example 1. Over any base B we have the trivial line bundle which is

$$\pi : B \times \mathbb{C} \to B, \quad (b,z) \mapsto b.$$

The transition functions are all identity maps.

Example 2. Consider the tangent bundle to the unit circle S^1. The points on the circle can be parametrized as

$$(\cos(t),\sin(t)), \quad t \in [0,2\pi[.$$

The tangent vector at any point on the circle must be perpendicular to the radius at that point. This tangent vector can be represented as:

$$\lambda(-\sin(t),\cos(t)), \quad \lambda \in \mathbb{R}.$$

Consequently, the tangent bundle to the circle is isomorphic to $S^1 \times \mathbb{R}$. The transition functions can be defined through the Jacobian!

Example 3. The Möbius band can be seen as a line bundle over the circle. First identify S^1 as $[0,1]/0 \sim 1$ and now consider the charts

$$U = S_1 \setminus \{0\}, \quad V = S^1 \setminus \{1/2\} \quad \text{with} \quad \phi = \mathrm{id}, \quad \psi = 1 - \mathrm{id}.$$

Even though all fibers are lines, this is a non trivial line bundle *due to the transition functions*.

Example 4. The tangent bundle of $\mathbb{P}^1$ is a line bundle defined via two charts $(\mathbb{C}^2, x, \partial_x)$, $(\mathbb{C}^2, y, \partial_y)$. The chain rule gives us the relation
$$\frac{\partial}{\partial y} = \frac{\partial x}{\partial y}\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial y}\frac{1}{y}\right)\frac{\partial}{\partial x} = \frac{-1}{y^2}\frac{\partial}{\partial x} = -x^2\frac{\partial}{\partial x}$$
which lets us conclude that the transition function is
$$(x, \partial_x) \mapsto (1/y, -y^2\partial_y).$$

Sections of Line Bundles

Definition 2. A section of a line bundle L over B is a map $s: B \to L$ with $\pi s = \mathrm{id}_B$.

Sections can be defined locally on open sets $U \subseteq B$ or globally when they are defined everywhere on B. Intuitively, sections single out points in fibers. For every $b \in B$, $s(b)$ is a point on the fiber $\pi^{-1}(b)$.

Example 5. In a line bundle, every fiber is a copy of $\mathbb{C}$. And independent of the transition maps, zero is fixed. We can thus consider the zero section of the bundle as
$$s(b) = (b, 0) \in \pi^{-1}(b).$$

The space of sections of a line bundle naturally has the structure of a vector space. This is why, sometimes, elements of certain varieties are referred to as sections of certain line bundles.

2 Line Bundles over $\mathbb{P}^1$

We begin by introducing a family of complex manifolds.

Definition 3. For $d \in \mathbb{Z}$, the manifold $\mathcal{O}_{\mathbb{P}^1}(d)$ (or $\mathcal{O}(d)$) is defined by two charts and a transition function:
$$(\mathbb{C}^2, (x, u)) \xrightarrow[v = u/x^d]{y = 1/x} (\mathbb{C}^2, (y, v)).$$
This transition function is $(y, v) = \left(\frac{1}{x}, \frac{u}{x^d}\right)$ with inverse $\left(\frac{1}{y}, \frac{v}{y^d}\right)$.

We could also regard this set as $\mathbb{C}^2/\sim$ where the equivalence relation is described via the transition function. $\mathcal{O}(d)$ comes with a natural projection onto $\mathbb{P}^1$: $(x, u) \mapsto x$ which allows to see $\mathcal{O}(d)$ as a line bundle over $\mathbb{P}^1$.

Example 6. The tangent bundle over $\mathbb{P}^1$ can be realized as $\mathcal{O}(2)$.

Moreover, *any* line bundle over $\mathbb{P}^1$ is of the form $\mathcal{O}(d)$ for some d.

Proposition 1. *The space of sections of* $\mathcal{O}(d)$ *is*

- ◇ *The space of polynomials of degree d in one affine coordinate, or homogenous polynomials of degree d in both coordinates when $d \geqslant 0$.*
- ◇ *When $d < 0$ the space of sections is trivial.*

It's also possible to talk of meromorphic sections over $\mathcal{O}(d)$ as rational functions of degree d. From this we can now see…

3 $\mathcal{O}(d)$ as a sheaf

To define a sheaf we take an open set $U \subseteq \mathbb{P}^1$ and consider the space of meromorphic sections, $\mathcal{O}(d)(U)$, of $\mathcal{O}(d)$ which are *holomorphic* on U. This defines a sheaf over $\mathbb{P}^1$.

Example 7. Let's take $d = 4$ and let's consider the meromorphic sections over $U_0 = \{Y \neq 0\}$:

$$s(X:Y) = \frac{(X^2+XY+Y^2)^3}{(XY)^2}, \quad \text{and} \quad \tilde{s}(X:Y) = \frac{X^7+X^4Y^3+Y^7}{Y^3}.$$

In this chart, we can take $Y = 1$ via projective equivalence, and we get

$$\frac{(x^2+x+1)^3}{x^2}, \quad \text{and} \quad \frac{x^7+x^4+1}{1}.$$

The first section is *not* holomorphic on U_0, but the second one is!

Observe that the first section can't be holomorphic because when we dehomogenize, we still get terms with x on the bottom. To fix this, we must have only a monomial in Y which dehomogenizes to 1. This allows us to see:

Proposition 2. *On the charts of $\mathbb{P}^1$, $U_0 = \{Y \neq 0\}$, $U_1 = \{X \neq 0\}$, we have*

$$\mathcal{O}(d)(U_0) \simeq \mathbb{C}[x], \quad \mathcal{O}(d)(U_1) \simeq \mathbb{C}[y], \quad \mathcal{O}(d)(U_0 \cap U_1) \simeq \mathbb{C}[x, 1/x].$$

Sheaf cohomology

Knowing that the sheaf behaves according to the previous result, we may use Čech cohomology to do our calculations. In this algebro-geometric case, with $\mathcal{O}(d)$, Čech cohomology is isomorphic to the sheaf cohomology.

We first consider the Čech complex associated to $\mathcal{O}(d)$:

$$0 \to O(d)(U_0) \oplus \mathcal{O}(d)(U_1) \xrightarrow{\mathrm{d}} O(d)(U_0 \cap U_1) \to 0$$

The action of the coboundary map is given by

$$\mathrm{d}(s, \tilde{s}) = s_{U \cap V} - \tilde{s}_{U \cap V}$$

however, this is non-descriptive. We will use the previous isomorphisms in order to see the action of the coboundary. In that case, the complex is

$$0 \to \mathbb{C}[x] \oplus \mathbb{C}[y] \xrightarrow{\mathrm{d}} \mathbb{C}[x, 1/x] \to 0.$$

We build an example in order to see what the coboundary does.

Example 8. Let's begin in $\mathcal{O}(4)$. To see the kernel of d, let's analyze what happens on the monomials and see when can it be zero. Take for example

$$\mathrm{d}(0,y)=0-y=0-x^4(1/x)=-x^3 \Rightarrow (x^3,y)\in\ker(\mathrm{d}),$$

and in the same fashion $(0,1)\mapsto -x^4$, for example. From this we can see that the kernel is generated by

$$(x^4,1),\quad (x^3,y),\quad (x^2,y^2),\quad (x,y^3),\quad \text{and}\quad (1,y^4)$$

so this is the space of degree 4 polynomials!

On the other hand, which elements does the coboundary reach? What is $\mathrm{Im}(\mathrm{d})$? For example, does the coboundary reach x^{2024} or x^{-2024}? Indeed:

$$\mathrm{d}\big(x^{2024},0\big)=x^{2024},\quad \text{and}\quad \mathrm{d}\big(0,y^{2028}\big)=x^{-2024}.$$

We can deduce that $\mathrm{Im}(\mathrm{d})$ is actually the whole of $\mathbb{C}[x,1/x]$.

Proposition 3. *The coboundary map on the previous Čech complex obeys the following*:

- ◇ *For* $d\geqslant 0$, $\ker(\mathrm{d})\simeq\mathbb{C}[x]/\mathrm{gen}(x^{d+1})$ *and* $\mathrm{Im}(\mathrm{d})\simeq\mathbb{C}[x,1/x]$.
- ◇ *For* $d<0$ *we have the opposite situation.*

From this we can see that for $d\geqslant 0$, cohomology is

$$\begin{cases} H^0(\mathbb{P}^1,\mathcal{O}(d))\simeq\mathbb{C}[x]/\mathrm{gen}(x^{d+1}) \\ H^1(\mathbb{P}^1,\mathcal{O}(d))\simeq 0 \end{cases}$$

while for $d<0$ the situation reverses, 0^{th} cohomology is trivial and 1^{st} is the degree d polynomials.

Remark 3. Observe that the 0^{th} cohomology in both cases is isomorphic to the space of global sections of $\mathcal{O}(d)$. For positive d, this is polynomials of degree d. While for d negative, there's *no* global sections.

4 Conclusion

Even though we haven't done calculations in the most formal way, the examples we have provided intuitively show the results we desired to see. The same results should be obtained when doing the cases with cellular sheaf cohomology. As of this moment we have not made this calculations but expect to do them in the coming days in order to see that everything matches up.

Bibliography

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