Exercise 1 (Exercise 1.a)**.** Suppose $b > a$ and one computes $T \leftarrow b \leftarrow a$, in other words, inserting b into T with the RSK insertion algorithm and then inserting a into the result. Show that the insertion path of a lies weakly left of the insertion path of b.

Answer

We will divide the proof into two cases and in both of them we will show that $\texttt{IP}(a) \leqslant \texttt{IP}(b)$ where $\texttt{IP}$ is the element's insertion path. The cases are:

Either b was inserted at the end of the first row, or it wasn't.

- ⋄ Assume b is the last element of the first row. Then, as $a < b$, a will bump $c < b$ in the first row to the second row.

Exercise 2 (Exercise 3)**.** Let π be a permutation in S_n, and let ℓ be the length of the longest increasing subsequence of π and d the length of the longest decreasing subsequence of π. Show that $\ell \cdot d \geqslant n$.

Answer

Via the RSK bijection we can identify π with a SYT T of shape $\lambda \vdash n$. Also, we have that $\ell = \lambda_1$ and $d = \ell(\lambda)$. Now note that $\lambda_1 \cdot \ell(\lambda) \geqslant n$.
By considering T we can see that it is a subset of the tableau $(\lambda_1, \ldots, \lambda_1)$ ($\ell(\lambda)$ times) which represents a partition of $\lambda_1 \cdot \ell(\lambda)$. This implies the desired inequality as the size of the table T is n and the next one has size $\lambda_1 \cdot \ell(\lambda)$.Finally the RSK bijection gives us the result we are searching for.

As an example, consider the partitions $(6, 4, 2, 1)$ and $(6, 6, 6, 6)$:

$$T = \begin{array}{cccccc}
\cline{1-1}
\multicolumn{1}{|c|}{\ } & & & & & \\ \cline{1-2}
\multicolumn{1}{|c|}{\ } & \multicolumn{1}{c|}{\ } & & & & \\ \cline{1-4}
\multicolumn{1}{|c|}{\ } & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } & & \\ \cline{1-6}
\multicolumn{1}{|c|}{\ } & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } \\ \cline{1-6}
\end{array}, \quad \widetilde{T} = \begin{array}{|c|c|c|c|c|c|}
\hline
 & x & x & x & x & x \\ \hline
 & & x & x & x & x \\ \hline
 & & & & x & x \\ \hline
 & & & & & \\ \hline
\end{array}.$$

In this case $n = 13$ and by doing the process we get a partition of 24 which is larger than 13.