Exercise 1 (5.10(a) Stein& Shakarchi)**.** Find the Hadamard product for $e^z - 1$.

Answer

Recall Hadamard's theorem states that if f is an entire function with order of growth ρ and $k = \lfloor \rho \rfloor$ then

$$f(z) = e^{p(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$$

where (a_n) is the collection of non-null zeroes of f, p has degree at most k and $m = \operatorname{ord}(f, 0)$.
In our case $e^z - 1$ has order of growth 1 and it has simple zeroes at $z = 2\pi i n$ for $n \in \mathbb{Z}$. In particular the order of zero is one. This means that

$$e^z - 1 = e^{a_1 z + a_0} z \prod_{n \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{2\pi i n}\right) e^{z/2\pi i n}.$$

To simplify this product we multiply opposites across the origin:

$$\begin{aligned}\left[\left(1 - \frac{z}{2\pi i n}\right) e^{z/2\pi i n}\right]\left[\left(1 - \frac{z}{2\pi i(-n)}\right) e^{z/2\pi i(-n)}\right] &= \left(1 + \left(\frac{z}{2\pi i n}\right)^2\right) e^{z/2\pi i n} e^{-z/2\pi i n} \\ &= 1 + \frac{z^2}{4\pi^2 n^2}\end{aligned}$$

So we get

$$e^z - 1 = e^{a_1 z + a_0} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Dividing both sides by z we get

$$\frac{e^z - 1}{z} = e^{a_1 z + a_0} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$$

and as z approaches 0 we get that

$$1 = e^{a_0}(1) \Rightarrow a_0 = 0.$$

Expanding the exponential function as a Taylor series and comparing coefficients we get the following:

$$z + \frac{z^2}{2} + O(z^3) = (1 + a_1 z + \frac{(a_1 z)^2}{2} + O(z^3)) z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$$

Thus we obtain

$$z + \frac{z^2}{2} + O(z^3) = z + a_1 z^2 + O(z^3) \Rightarrow a_1 = \frac{1}{2}.$$

In conclusion we have

$$e^z - 1 = e^{z/2} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Exercise 2 (5.11 Stein& Shakarchi)**.** Show that if f is an entire function of finite order that omits two values, then f is constant. This result remains true for any entire function and is known as Picard's little theorem. ⟦Hint: If f misses a, then $f(z) - a$ is of the form $e^{p(z)}$ where p is a polynomial.⟧

Answer

Assume f omits two values a, b which means that

$$f(z) - a = e^{p(z)}, \quad \text{and} \quad f(z) - b = e^{q(z)} \quad \text{for some} \quad p, q \quad \text{polynomials} \quad .$$

From this, we may subtract one equation from the other to get

$$b - a = e^{p(z)} - eq(z)$$

and now we may differentiate both sides of the equation to obtain

$$0 = p'(z)e^{p(z)} - q'(z)e^{q(z)}.$$

As this equation holds for *all* $z \in \mathbb{C}$ it must happen that $p'(z)$ and $q'(z)$ have the same zeroes with the same multiplicities. Thus $q'(z) = cp'(z)$ for some non-zero $c \in \mathbb{C}$. Returning to our equation we have

$$e^{p(z)}p'(z) = cp'(z)e^{q(z)} \Rightarrow e^{p(z)} = ce^{q(z)} \Rightarrow ce^{q(z)} - e^{q(z)} = b - a.$$

Differentiating this equation we obtain

$$(c-1)q'(z)e^{q(z)} = 0 \Rightarrow q'(z) = 0 \Rightarrow q \quad \text{is constant} \quad .$$

This allows us to conclude that f is constant as $f = e^q + b$.
If it occured that $c = 1$, then $p' = q'$ and so $q(z) = p(z) + d$ for some $d \in \mathbb{C}$. Replacing this in the equation we have

$$e^{p(z)} - e^{p(z)+d} = b - a \Rightarrow (1 - e^d)e^{p(z)}p'(z) = 0 \Rightarrow p'(z) = 0 \Rightarrow p \quad \text{is constant}$$

and once again we deduce f is constant. Finally if it was the case that $d = 0$, then $p = q$ but this means that

$$f(z) - e^{p(z)} = a = b$$

and this can't happen as a, b are different values of $\mathbb{C}$. In conclusion we have that f is constant.

Exercise 3. Assume $\mathrm{Re}(s) = \sigma > 0$. For $n, N \in \mathbb{N}$ define

$$\delta_n(s) = \frac{1}{n^s} - \int_n^{n+1} \frac{\mathrm{d}x}{x^s} = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) \mathrm{d}x \quad \text{and} \quad F_N(s) = \sum_{n=1}^{N} \delta_n(s).$$

i) Show that $|\delta_n(s)| \leqslant \frac{|s|}{n^{\mathrm{Re}(s)+1}}$. ⟦Hint: Represent the integrand in the definition of δ_n using the observation that $\int_n^x \frac{\mathrm{d}u}{u^{s+1}} = \frac{-1}{s}(x^{-s} - n^{-s})$.⟧

ii) Show that $(F_N(s))$ converges uniformly on any half-plane of the form $\mathrm{Re}(s) \geqslant \alpha > 0$.

iii) Show that $\zeta(s) - \frac{1}{s-1}$ is bounded and holomorphic near $s = 1$. ⟦Hint: Use the fact that $\frac{1}{s-1} = \int_1^\infty x^{-s}\mathrm{d}x$.⟧

Answer

i) Observe that from the hint we have

$$s\int_n^x \frac{\mathrm{d}u}{u^{s+1}} = \left(\frac{1}{n^s} - \frac{1}{x^s}\right).$$

Then replacing this expression inside δ_n we get

$$|\delta_n(s)| = \left|\int_n^{n+1} s\int_n^x \frac{\mathrm{d}u}{u^{s+1}}\mathrm{d}x\right| \leqslant \int_n^{n+1} |s| \left|\int_n^x \frac{\mathrm{d}u}{u^{s+1}}\right| \mathrm{d}x$$

and we can estimate the inner integral by taking the integrand's sup-norm. We have

$$\left|\int_n^x \frac{\mathrm{d}u}{u^{s+1}}\right| \leqslant \sup_{n\leqslant u\leqslant x} \left|\frac{1}{u^{s+1}}\right| (x-n)$$

and for positive real numbers u we can estimate $|u^z|$ as

$$|u^x||u^{iy}| = u^x|e^{iy\log(u)}| = u^x$$

so this means that

$$\sup_{n\leqslant u\leqslant x}\left|\frac{1}{u^{s+1}}\right|(x-n)=\sup_{n\leqslant u\leqslant x}\frac{1}{u^{\sigma+1}}(x-n)\leqslant\frac{x-n}{n^{\sigma+1}}$$

where the last inequality holds because $\frac{1}{u^{\sigma+1}}$ is a decreasing function. Returning to our δ_n estimate we have

$$\int_n^{n+1}|s|\left|\int_n^x\frac{\mathrm{d}u}{u^{s+1}}\right|\mathrm{d}x\leqslant\int_n^{n+1}|s|\frac{x-n}{n^{\sigma+1}}\mathrm{d}x=\frac{|s|}{n^{\sigma+1}}\frac{1}{2}\leqslant\frac{|s|}{n^{\sigma+1}}.$$

ii) We now consider the partial sums $F_N=\sum_{n=1}^N\delta_n$. Observe that we may bound F_N with the countable triangle inequality:

$$|F_N(s)|\leqslant\sum_{n=1}^N|\delta_n(s)|\leqslant\sum_{n=1}^N\frac{|s|}{n^{\sigma+1}}$$

and this is a series of real numbers which converges when $\sigma>0$. This implies that F_N converges uniformly for $\sigma\geqslant\alpha>0$.

iii) Observe that from our initial identity

$$\delta_n(s)=\frac{1}{n^s}-\int_n^{n+1}\frac{\mathrm{d}x}{x^s}$$

we can sum up to N to obtain

$$F_N(s)=\sum_{n=1}^N\frac{1}{n^s}-\sum_{n=1}^N\left(\int_n^{n+1}\frac{\mathrm{d}x}{x^s}\right)=\sum_{n=1}^N\frac{1}{n^s}-\int_1^{N+1}\frac{\mathrm{d}x}{x^s}.$$

Letting N grow without bound we arrive at

$$\lim_{N\to\infty}F_N(s)=\zeta(s)-\int_1^\infty\frac{\mathrm{d}x}{x^s}=\zeta(s)-\frac{1}{s-1}.$$

As the properties of the sequence $(F_N(s))$ are preserved through uniform limit, we have that $F_\infty(s)$ is bounded and holomorphic for $\sigma>0$. So our expression for the equality is only valid wherever the improper integral converges, and this is where $\sigma>1$. So we obtain the desired result as F_∞ is $\zeta(s)-\frac{1}{s-1}$.