Exercise 1. If $V = \mathbb{V}(F_1, \ldots, F_r)$ is an affine variety in $\mathbb{A}^n$, then the tangent bundle TV is a subvariety of $\mathbb{A}^n \times \mathbb{A}^n$. Find the equations defining TV in $\mathbb{A}^n \times \mathbb{A}^n$. You should label your coordinates of $\mathbb{A}^n \times \mathbb{A}^n$ as $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. Do the case $r = 1$ first.

Answer

The tangent bundle can be described as the collection of points on our variety along their corresponding points in tangent space. This is

$$TV = \{\, (p, v) : \ p \in V, \ v \in T_pV \,\} \subseteq V \times T_pV.$$

In our particular case, suppose $r = 1$ first and in that case $V = \mathbb{V}(F)$, a hypersurface. For $\mathbf{x} = (x_1, \ldots, x_n) \in V$ we have that points on $T_{\mathbf{x}}V$ are points $\mathbf{y}$ which satisfy the equation

$$L(\mathbf{y}) = 0 \iff \langle \nabla F(\mathbf{x}) | \mathbf{y} - \mathbf{x} \rangle = 0.$$

So in total, the tangent bundle in this case is

$$\{\, (\mathbf{x}, \mathbf{y}) \in \mathbb{A}^n \times \mathbb{A}^n : \ F(\mathbf{x}) = 0, \ \langle \nabla F(\mathbf{x}) | \mathbf{y} - \mathbf{x} \rangle = 0 \,\}.$$

In the more general case, the idea is similar but with r polynomials. We have that the tangent bundle is

$$TV = \{\, (\mathbf{x}, \mathbf{y}) \in \mathbb{A}^n \times \mathbb{A}^n : \ \forall j \, (F_j(\mathbf{x}) = 0 \ \wedge \ \langle \nabla F_j(\mathbf{x}) | \mathbf{y} - \mathbf{x} \rangle = 0) \,\}.$$

Exercise 2. Let Γ be the graph of a rational map $X \dashrightarrow Y$. Prove that the projection $\Gamma \to X$ is a birational equivalence.

Answer

Suppose $F : X \dashrightarrow Y$ is our rational map and pick $\varphi : U \to Y$ a representative of F with $U \subseteq X$ dense and open. The graph Γ is

$$\Gamma = \overline{\Gamma}_\varphi = \overline{\{\, (x, \varphi(x)) : \ x \in U \,\}}.$$

The projection $\pi_x : \Gamma \to X$ is a birational equivalence because we can restrict ourselves to $\Gamma_\varphi{}^a$. Then

$$\pi_x \mid_{\Gamma_\varphi} : \Gamma_\varphi \to X, \ (x, \varphi(x)) \mapsto x$$

is a representative. An inverse can be found as easily by reverting the direction of the arrow. Call

$$\varepsilon : X \to \Gamma_\varphi,\ x \mapsto (x, \varphi(x))$$

a representative of the inverse rational map. By construction π and ε are inverses of each other.

[a] Is this set Zariski open? I'm not sure. I know it's dense in Γ by definition but not if it's open.

Exercise 3. Recall that the Cremona transform is the rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined as

$$[x_0 : \cdots : x_n] \mapsto [1/x_0 : \cdots : 1/x_n].$$

Find equations defining the graph of ϕ as a subvariety of $\mathbb{P}^n \times \mathbb{P}^n$.

Answer

Let us begin by considering a low-dimensional case. Suppose we are working in $\mathbb{P}^2$ so that the Cremona transform is $[x : y : z] \mapsto [a : b : c] = [\frac{1}{x} : \frac{1}{y} : \frac{1}{z}]$. The graph of ϕ in this case is

$$\{\, ([x : y : z], [a : b : c]) :\ a = 1/x,\ b = 1/y,\ c = 1/z \,\} \subseteq \mathbb{P}^2 \times \mathbb{P}^2.$$

We can see that there is the following relation between the variables: $ax = 1$, $by = 1$ and $cz = 1$. If we multiply ay or az we don't get any of the following expressions $\{\, bx, bz, cx, cy \,\}$. So the only linear relations that hold are the variables with the *corresponding* inverse. There's no higher degree relation between the variables that crosses terms so we intuit that the relations in question are the only relations possible. In this case

$$\Gamma_\phi = \mathbb{V}(ax - 1, by - 1, cz - 1),$$

and in the general case we have $\mathbf{x} = [x_0 : \cdots : x_n]$ and $\mathbf{y} = [y_0 : \cdots : y_n] = [1/x_0 : \cdots : 1/x_n]$. The equations defining the graph in this case are $x_i y_i = 1$ for all i. It holds that

$$\Gamma_\phi = \mathbb{V}(x_i y_i - 1)_{i=0}^n.$$

Exercise 4. Let B be the blowup of $\mathbb{P}^2$ at $[0:0:1]$. Find equations defining B as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$. Show that there is a morphism defined everywhere from B to $\mathbb{P}^1 \times \mathbb{P}^1$.

Answer

The blowup of $\mathbb{P}^2$ at a point p is defined as

$$\mathcal{BL}_p = \{\, (x, \ell) : \; p, x \in \ell \,\} \subseteq \mathbb{P}^2 \times \mathbb{P}^1,$$

in particular the blowup at $[0:0:1]$ is the collection of (x, ℓ) where $x \in \ell$ and ℓ goes through $[0:0:1]$. So if $([x:y:z],[u:v]) \in \mathbb{P}^2 \times \mathbb{P}^1$ then

$$B = \mathcal{BL}_{[0:0:1]}(\mathbb{P}^2) = \mathbb{V}(xv - yu).$$

The morphism in question is the *forgetful map* which forgets the z coordinate:

$$([x:y:z],[u:v]) \mapsto ([x:y],[u:v]),$$

this map is a morphism as term-by-term it's homogenous.

Exercise 5. An algebraic variety is *rational* if it's birationally equivalent to projective space (of some dimension). Show that the nodal plane curve defined by the equation $y^2 - x^2 - x^3 = 0$ is rational. ⟦Hint: Project from the node.⟧

Answer

The initial idea is that since the nodal curve is indeed a curve, then it should be equivalent to something which looks like a curve. Our idea takes us to think about the projective line $\mathbb{P}^1$.
According to the hint, by projecting from the origin we can associate a point $(x, y) \in V = \mathbb{V}(y^2 - x^3 - x^2)$ to the slope of the line from that point to the node. A line from the node to (x, y) follows the equation $y = tx$. However the point $(0, 0)$ can't be mapped through this process, so we declare it's sent to the point at infinity. The inverse map is the parametrization of V in terms of t given by $t \mapsto (t^2 - 1 : t^3 - t^2)$.
The rational map is

$$\varphi : V \to \mathbb{P}^1, \; (x, y) \mapsto \left[\frac{y}{x} : 1\right] = [y : x], \; \varphi^{-1}([t:1]) = (t^2 - 1 : t^3 - t^2)$$

and the points at infinity of V are mapped to the lines with slope 1 and -1.