MATH519 — Complex Analysis

Based on the lectures by Jeff Achter

Notes written by Ignacio Rojas

Spring 2023

Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This course is an introduction to analytic functions of a single complex variable. The subject is beautiful.– it turns out that a function with a complex derivative is highly structured – and enjoys a give and take with many other areas of mathematics.

Requirements

Knowledge of convergence of sequences, series: limits, continuity, differentiation, integration of one-variable functions is required.

Contents

Chapter 1

First Midterm

1.1 Interim | HW1

Exercise 1.1.1 (1.1 Stein & Shakarchi)**.** Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$.
(b) $1/z = \overline{z}$.
(c) $\mathrm{Re}(z) = 3$
(d) $\mathrm{Re}(z) > c$, (resp.,$\geqslant c$) where $c \in \mathbb{R}$.
(e) $\mathrm{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$.
(f) $|z| = \mathrm{Re}(z) + 1$.
(g) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$.

Answer

i) The first set is the set of points at the same distance from z_1 and z_2. If we consider the line segment $z_1 z_2$, then the set in question is the bisector of that line segment.

ii) Note that

$$1/z = \overline{z} \iff 1 = \overline{z}z \iff 1 = |z|^2 \iff 1 = |z|,$$

thus the set is the unit circle.

iii) The set is a perpendicular line to the real axis at $z = 3$.

iv) This infinite set is an infinite half plane to the right (but not including) of the line $z = c$. In the other case, we do include the line in question.

v) Let us rephrase this inequality in terms of real numbers. Call $a = a_1 + ia_2$, $b = b_1 + ib_2$ and $z = x + iy$. Then

$$\mathrm{Re}(az + b) = \mathrm{Re}[a_1x - a_2y + b_1 + i(a_2x + a_1y + b_2)],$$

thus our desired inequality is true whenever $a_1x - a_2y + b_1 > 0$. Solving for y we get $y > (a_1x + b_1)/a_2$, which is the half plane located above the line $y = (a_1x + b_1)/a_2$.

vi) The equation in question is equivalent to

$$\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 = (\mathrm{Re}(z) + 1)^2.$$

To ease the notation, assume $z = x + iy$. Then the equation reads

$$x^2 + y^2 = x^2 + 2x + 1 \iff y^2 = 2x + 1 \iff x = (y^2 - 1)/2.$$

It holds the the parabola in question contains the points which satisfy the equation.

vii) This set is a line parallel to the real axis at $z = c$

Exercise 1.1.2. Do the following:

i) Show that the complex conjugation map $\kappa : \mathbb{C} \to \mathbb{C},\ z \mapsto \overline{z}$ is an involution, i.e., a ring homomorphism such that $\kappa \circ \kappa = \mathrm{id}$.
ii) Suppose $a \in \mathbb{R},\ z \in \mathbb{C}$. Show that

$$\mathrm{Re}(az) = a\,\mathrm{Re}(z), \quad \text{and} \quad \mathrm{Im}(az) = a\,\mathrm{Im}(z).$$

Answer

Let us take $z = x + iy$ with $x, y \in \mathbb{R}$.

i) We have $\overline{z} = x + i(-y) = x - iy$. Once more we get $\overline{\overline{z}} = x - i(-y) = x + iy = z$. Thus $\overline{\overline{z}} = z$ for any $z \in \mathbb{C}$. In conclusion $\overline{\overline{\cdot}} = \mathrm{id}$.
ii) It holds that

$$\begin{aligned} \mathrm{Re}(az) &= \mathrm{Re}(ax + aiy) = ax = a\,\mathrm{Re}(z), \\ \mathrm{Im}(az) &= \mathrm{Im}(ax + aiy) = ay = a\,\mathrm{Im}(z). \end{aligned}$$

Exercise 1.1.3. Do the following:

i) Prove that $|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w})$.
ii) Use this to prove the parallelogram rule: $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$.

Answer

i) Note that

$$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + w\overline{z} + z\overline{w} + w\overline{w}.$$

The number $w\overline{z}$ is the conjugate of $z\overline{w}$, and summing a number and its conjugate returns twice its real part. Thus we get the desired identity.

ii) As the past identity holds for all complex numbers, it holds when $w = -w$. This means that $|z-w|^2 = |z|^2+|-w|^2+2\operatorname{Re}(z(\overline{-w})) = |z|^2+|w|^2-2\operatorname{Re}(z\overline{w})$ and summing this together with the first identity gives us the parallelogram law.

Exercise 1.1.4 (1.5 Stein & Shakarchi)**.** A set Ω is said to be pathwise connected if any two points in Ω can be joined by a (piecewise-smooth) curve entirely contained in Ω. The purpose of this exercise is to prove that an open set Ω is pathwise connected if and only if Ω is connected.

i) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a parametrization $z : [0,1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let

$$t_* = \sup_{0 \leqslant t \leqslant 1} \{\, t : \forall s[(0 \leqslant s < t) \Rightarrow (z(s) \in \Omega_1)] \,\}.$$

Arrive at a contradiction by considering the point $z(t_*)$.

ii) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subseteq \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subseteq \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega 1$ as desired.

Answer

i) We will proceed using a topological argument instead of a metric one. As the function γ is continuous, it pulls back Ω_1 and Ω_2 into $[0,1]$ as open sets. However, as the sets are disjoint, their inverse images are disjoint as well.

In other words, we have found two open disjoint sets which separate $[0, 1]$:

$$[0, 1] = \gamma^{-1}[\Omega_1] \cupdot \gamma^{-1}[\Omega_2].$$

But this is impossible because $[0, 1]$ is a connected set. Thus, our assumption that Ω was disconnected must be false. We conclude that path-connectedness implies connectedness.

ii) Take Ω_1, Ω_2 as in the statement. Then Ω_1 is non-empty as $w \in \Omega_1$ because it's connected to itself through a trivial path. Suppose now that $z \in \Omega_1$ and that $d(z, \partial\Omega_1) > r > 0$. Take $x \in B(z, r)$, then there exists a line-segment between z and x and there's a smooth curve which connects $z \in \Omega_1$ with w. Thus the piecewise-continuous path from x to z and from z to w is a path which connects x and w. As x is arbitrary, it follows that $B(z, r) \subseteq \Omega_1$, and thus Ω_1 is open.

Formally, if $\gamma : [0, 1] \to \Omega_1$ is the map which parametrizes the curve between z and w and $r : [0, 1] \to B(z, r)$ is the map $t \mapsto tz + (1 - t)x$, then the curve from x to w is parametrized by the function

$$f = \begin{cases} 2tz + (1 - 2t)x, \ t \in [0, 1/2], \\ \gamma(2t - 1), \ t \in [1/2, 1]. \end{cases}$$

On the other hand take a point $z \in \Omega_2$ and let $d(z, \partial\Omega_2) > r > 0$. Consider a point $x \in B(z, r)$ and assume by way of contradiction that such x can be connected to w by a curve which can be parametrized by a smooth function γ. As the ball is convex, we can connect z to x and then to w creating a path between z and w. This is impossible as z cannot be connected to w by a path, thus our assumption must be false. It holds that x cannot be connected to w by a path and thus $x \in \Omega_2$. Therefore Ω_2 is also open. We conclude that $\Omega = \Omega_1 \cup \Omega_2$ is a union of two disjoint open sets, and since Ω is connected, it must hold that Ω_2 is empty. We conclude that Ω is path-connected.

Exercise 1.1.5 (1.7 Stein & Shakarchi)**.** The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

i) Let $z, w \in \mathbb{C}$ such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1$$

if $|z| < 1$ and $|w| < 1$, and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1$$

if $|z| = 1$ or $|w| = 1$. ⟦Hint: Why can one assume that z is real? I then suffices to prove that $(r-w)(r-\overline{w}) \leqslant (1-rw)(1-r\overline{w})$ with equality for appropriate r and $|w|$.⟧⟦Here is an alternate approach, which you may use if you like. Fix $w \in \mathbb{C}$ with $w < 1$, and consider the function $z \mapsto \frac{w-z}{1-\overline{w}z}$. What is $\overline{f(z)}$? By computing $f(z)\overline{f(z)}$, show that $|z| = 1$ implies $|f(z)| = 1$. Find a point z with $|z| < 1$ such that $|f(z)| < 1$. Since f is continuous, this shows that f takes the unit disc to itself. (Why?)⟧

ii) Prove that for a fixed $w \in \mathbb{D}$, the mapping $F : z \mapsto \frac{w-z}{1-\overline{w}z}$ satisfies the following:

a) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.
b) F interchanges 0 and w.
c) $|F(z)| = 1$ if $|z| = 1$.
d) F is bijective. ⟦Hint: Calculate $F \circ F$.⟧

Answer

i) The inequality in question is equivalent to

$$0 \leqslant |w-z| < |1-\overline{w}z|.$$

Since the quantities are positive, we can square them and preserve the order. It holds that

$$0 \leqslant |w-z|^2 < |1-\overline{w}z|^2 \iff 0 \leqslant (w-z)\overline{(w-z)} < (1-\overline{w}z)\overline{(1-\overline{w}z)},$$

Simplifying this expression we get

$$\begin{aligned}
&(w-z)(\overline{w}-\overline{z}) < (1-\overline{w}z)(1-w\overline{z}) \\
\iff &w\overline{w} - w\overline{z} - z\overline{w} + z\overline{z} < 1 - w\overline{z} - \overline{w}z + \overline{w}zw\overline{z} \\
\iff &|w|^2 + |z|^2 < 1 + |w|^2|z|^2 \\
\iff &0 < (1-|w|^2)(1-|z|^2).
\end{aligned}$$

The inequality is true whenever both moduli are less than one, and whenever either is one equality is achieved.

ii) Now we suppose $w \in \mathbb{D}$ which means that $|w| < 1$. Taking $z \in \mathbb{D}$ and applying F gives us the quantity $\frac{w-z}{1-\overline{w}z}$ which by the previous argument, has modulus less than 1 whenever w, z do.

The function F is holomorphic because it is a quotient of holomorphic functions. The denominator is never zero inside the domain because that would mean that $1 = \overline{w}z$. And taking moduli in both sides of the equation gives us

$$1 = |1| = |w||z| < 1$$

which is impossible.

Now $F(0) = \frac{w-0}{1-0} = w$ and $F(w) = \frac{w-w}{1-|w|^2} = 0$. The denominator in the last expression is never zero because $|w| < 1$.

By the second part of the previous argument it holds that $|z| = 1$ immediately gives us $|F(z)| = 1$. And finally we will see that F is an involution:

$$F(F(z)) = F\left(\frac{w-z}{1-\overline{w}z}\right) = \frac{w - \left(\frac{w-z}{1-\overline{w}z}\right)}{1-\overline{w}\left(\frac{w-z}{1-\overline{w}z}\right)}.$$

Homogenizing and clearing denominators we get

$$\frac{w(1-\overline{w}z) - w + z}{1 - \overline{w}z - \overline{w}(w-z)} = \frac{-w\overline{w}z + z}{1-\overline{w}w} = \frac{(-w\overline{w}+1)z}{1-\overline{w}w} = z.$$

This means that F is it's own inverse and therefore, F is bijective.

1.2 Day 1 | 20230120

The Complex Numbers

To construct the complex numbers we take the real numbers, adjoin a variable and mod out by $\langle x^2+1 \rangle$. We can also define $\mathbb{C}$ as $\{ a+bi : a, b \in \mathbb{R} \}$ with the property $i^2 = -1$. This means that we can multiply complex numbers in the following way:

$$(a+bi)(c+di) = ac + (bc+ad)i + bdi^2 = (ac-bd) + (ad+bc)i.$$

Also as x^2+1 is irreducible in $\mathbb{R}[x]$, $\mathbb{C}$ is a finite field extension of $\mathbb{R}$ of degree 2. As a 2-dimensional vector space $\{ 1, i \}$ is a basis for $\mathbb{C}$.

The map $a + bi \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$ is not a ring homomorphism, it's a bijection with a bit of structure. The map $z \mapsto \alpha z$, when $\alpha = a + bi$, is a linear map with the following action over the basis

$$\alpha \cdot 1 = \alpha \Rightarrow [\alpha] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\alpha \cdot i = -b + ai \Rightarrow [\alpha] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

which means that $[\alpha] = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. The converse, if we have a $\mathbb{R}$-linear transformation, then it's $\mathbb{C}$-linear if and only if it looks like $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Definition 1.2.1. The complex conjugation map is $a + bi \mapsto a - bi$, or $z \mapsto \overline{z}$.

This map is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

Example 1.2.2. For $\alpha = a + bi$, we have

$$\overline{2\alpha} = \overline{2(a + bi)} = \overline{2a + 2bi} = 2a - 2bi = 2\overline{al}.$$

Whereas if instead

$$\overline{i\alpha} = \overline{ai - b} = -b - ai \neq i\overline{\alpha} = b + ai.$$

As a $\mathbb{R}$-linear map, we can identify with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By looking at the shape of this matrix we can see that it is not $\mathbb{C}$-linear.

Lemma 1.2.3. *The map $z \mapsto \overline{z}$ is a ring homomorphism*

Proof

$\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{zw}$.

With the complex conjugation we can pick out the real and imaginary parts of $\alpha = a + bi$.

$$\alpha + \overline{\alpha} = 2\,\mathrm{Re}(\alpha), \quad \alpha - \overline{\alpha} = 2i\,\mathrm{Im}(\alpha)$$

A Notion of Size

Can't do geometry without one. Notice that for $z = a + bi$

$$z\overline{z} = a^2 + b^2 > 0.$$

From a complex number we have extracted a positive quantity.

Definition 1.2.4. The complex modulus of z is $|z| = \sqrt{z\overline{z}}$.

The fact that every number has n roots is very important in complex analysis.
As a vector in the plane, the norm of z is $|z|$

INC FIG

This means that $a + bi \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$ is an isometry. In this sense the distance between two complex numbers is $d(z, w) = |z - w|$.

Polar Coordinates (*ad hoc*)

For $\theta \in \mathbb{R}$, define

$$\exp(i\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta) \Rightarrow |\exp(i\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

Every point in the unit circle is of the form $e^{i\theta}$ and vice-versa.

INC FIG

For non-zero complex numbers, $z = |z|e^{i\theta}$ for some θ.

Definition 1.2.5. For a complex number $z = re^{i\theta}$, an argument of z is θ.

To have a well defined function, we mod out by multiples of 2π:

$$\arg : \mathbb{C}\backslash\{0\} \to \mathbb{R}/2\pi\mathbb{Z},$$

and we obtain a group isomorphism. In general, "lengths multiply, angles add."
For inverses if $z = re^{i\theta}$, then $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$.

Definition 1.2.6. The upper-half plane is $\mathbb{H} = \{\operatorname{Im}(z) > 0\}$.

Lemma 1.2.7. *If H is a half plane* $\operatorname{Im}(z - \beta/\gamma) > 0$

1.3 Day 2 | 20230123

Recall the complex conjugation map and the modulus of a complex number. This gives us an isometry between $\mathbb{R}^2$ and $\mathbb{C}$. Let us prove the lemma from last time.

Lemma 1.3.1. *If $H \subseteq \mathbb{C}$ is a half plane, then there exist $\beta, \gamma \in \mathbb{C}$ such that*

$$H = \left\{ z : \operatorname{Im}\left(\frac{z-\beta}{\gamma}\right) > 0 \right\}.$$

INC FIG

Pick a point $\beta \in H$, then translate H to the origin by $z \mapsto z - \beta$. The plane is now rotated by θ at the origin so we should rotate every point. Then $z \in H - \beta$ whenever $ze^{-i\theta} \in \mathbb{H}$. REDO

Let us see an application, for a polynomial, the coefficients determine the roots. The following lemma is a technical lemma.

Lemma 1.3.2. *Suppose $p \in \mathbb{C}[z]$ and H is a half plane which contains all the roots of p. Then H contains all the roots of p'.*

Proof

We can assume p is monic, so suppose $\alpha_1, \ldots, \alpha_d$ are the roots of $\mathbb{C}$. This means that

$$p(z) = \prod_{k=1}^{d}(z-\alpha_k) \Rightarrow p'(z) = \sum_{k=1}^{d} \frac{p(z)}{z-\alpha_k} \Rightarrow \frac{p'(z)}{p(z)} = \sum_{k=1}^{d} \frac{1}{z-\alpha_k}.$$

Now suppose that H contains all α_k and suppose $z_0 \notin H$, if we show $p'(z_0) \neq 0$ we are done because all the points which make p' vanish won't be outside H. Describe H by the previous lemma, there exist β, γ such that points in H satisfy the inequality $\operatorname{Im}\left(\frac{z-\beta}{\gamma}\right) > 0$. As z_0 is not in H, then $\operatorname{Im}\left(\frac{z_0-\beta}{\gamma}\right) < 0$. For each $k \in [d]$, we have that

$$z_0 - \alpha_k = z_0 - \beta + \beta - \alpha_k = (z_0 - \beta) - (\alpha_k - \beta)$$

so by taking imaginary parts

$$\operatorname{Im}\left(\frac{z-\alpha_k}{\gamma}\right) = \operatorname{Im}\left(\frac{z-\alpha_k}{\gamma}\right) - \operatorname{Im}\left(\frac{z-\alpha_k}{\gamma}\right)$$

The quantity on the right is negative because it's a negative number minus a positive. So it holds that $\operatorname{Im}\left(\frac{\gamma}{z-\alpha_k}\right) > 0$. With this we can calculate the following:

$$\operatorname{Im}\left(\gamma \frac{p'(z_0)}{p(z_0)}\right) = \operatorname{Im}\left(\sum_{k=1}^{d} \frac{\gamma}{z_0 - \alpha_k}\right) > 0$$

so in particular this number is non-zero. Thus $p'(z_0) \neq 0$

Definition 1.3.3. A set $S \subseteq \mathbb{R}^n$ is convex if for any two points $x, y \in S$, the line segment between x and y is also contained in S. This is

$$\{\, ty + (1-t)x : \; x, y \in S \,\} \subseteq S.$$

The convex hull of S is the intersection of all convex sets containing S.

In the case of a finite set of complex numbers, the convex hull can be found by intersecting half-planes which contain them.

Corollary 1.3.4 (Gauss-Lucas). *The roots of $p'(z)$ are contained in the convex hull of the roots of $p(z)$.*

Metric Spaces

Definition 1.3.5. A metric space is a set with a distance function.

Example 1.3.6. $\mathbb{R}^n$ is a metric space with $d(x, y) = \|x - y\|$. Subsets of metric spaces with an induced distance are metric spaces.

- nbhd
- open and closed
- Cauchy

Definition 1.3.7. Cauchy sequence

1.4 Day 3 | 20230125

The defining property of $\mathbb{R}$ is that it is complete. In that sense it is possible to prove that $\mathbb{R}^n$ is also complete.

Derivatives

Recall a real function g is differentiable at x_0 if there exists a real number a such that

$$g(x) = g(x_0) + a(x - x_0) + \psi(x), \ \frac{\psi(x)}{x - x_0} \xrightarrow[x \to x_0]{0} .$$

In the same sense a multivariable function is differentiable when there exists a linear transformation such that a similar condition holds.

Definition 1.4.1. f has complex derivative iff real derivative and Cauchy-Riemann equations

Example 1.4.2. The map $z \mapsto \overline{z}$ is not complex-differentiable. First by matrix definition and second with limit.

1.5 Day 4 | 20230127

Lemma 1.5.1. *If $\sum_{n \geqslant 0} z_n$ is absolutely convergent, then it's convergent.*

Proof

If s_n is a partial sum, then

$$|s_n - s_m| = \left| \sum_{i=m+1}^{n} z_i \right| \leqslant \sum_{i=m+1}^{n} |z_i| < \varepsilon$$

because $\sum |z_n|$ is Cauchy.

Power Series

Definition 1.5.2. A <u>power series</u> (centered at 0) is an expression of the form $\sum_{n \geqslant 0} a_n z^n$.

Example 1.5.3. The power series for the exponential function is $e^z = \sum_{n \geqslant 0} \frac{z^n}{n!}$.

Theorem 1.5.4 (Cauchy-Hadamard). *Suppose $\sum_{n \geqslant 0} a_n z^n$ has radius of convergence $\frac{1}{r} = \limsup |a_n|^{\frac{1}{n}}$. Then the series converges for $|z| < r$ and diverges for $|z| > r$.*

Proof

1.6 Day 5 | 20230130

Last time with Hadamard's criterion we learned something that we *already know*. Recall that for radii less than the radius of convergence, power series converge.

As a corollary we can prove the following:

Corollary 1.6.1. *Suppose $f(z) = \sum_{n\geqslant 0} a_n z^n$ has radius of convergence R. Then the following holds*:

i) *The formal derivative of f,*
$$g(z) = \sum_{n\geqslant 1} n a_n z^{n-1}$$
converges absolutely and uniformly on $B(0, R)$.

ii) $f'(z) = g(z)$.

Proof

Notice that
$$\lim_{n\to\infty} \sqrt[n]{n} = 1 \Rightarrow g \text{ converges.}$$

This is because $\limsup |na_n|^{1/n} = \limsup |a_n|^{1/n}$.
Call S_N the N^{th} partial sum of f. For $r < R$, suppose $|z - z_0| < r$. Then
$$\left|\frac{f(z) - f(z_0)}{z - z_0} - g(z)\right| = \left|\frac{S_n(z) - E_n(z) - S_n(z_0) + E_n(z_0) - g(z_0)(z - z_0)}{z - z_0}\right|.$$

Let us now add zero carefully and apply the triangle inequality. The previous term is less than
$$\left|\frac{S_N(z) - S_N(z_0)}{z - z_0} - S_N'(z_0)\right| + |S_N'(z_0) - g(z_0)| + \left|\frac{E_N(z) - E_N(z_0)}{z - z_0}\right|.$$

The last term which contains the errors can be written as
$$\left|\sum_{n\geqslant N} \frac{a_n(z^n - z_0^n)}{z - z_0}\right| \leqslant \sum_{n\geqslant N} n|a_n|r^{n-1}$$

and for large N, this quantity is small. With a similar reasoning we get that
$$|S_N'(z_0) - g(z_0)| \leqslant \sum_{n\geqslant N} n|a_n z^{n-1}|.$$

For z close to z_0, the first term is small as well.

Corollary 1.6.2. *A complex power series is infinitely differentiable.*

Lemma 1.6.3. *The power series of the exponential function satisfies the equality* $e^{z+w} = e^z e^w$.

Proof

$$\begin{aligned} e^z e^w &= \left(\sum_{n\geqslant 0} \frac{z^n}{n!}\right)\left(\sum_{n\geqslant 0} \frac{w^n}{n!}\right) \\ &= \sum_{n\geqslant 0} \sum_{k+\ell=n} \frac{z^k}{k!}\frac{w^\ell}{\ell!} \\ &= \sum_{n\geqslant 0} \frac{1}{n!} \sum_{k+\ell=n} \frac{n!}{k!\ell!} z^{k+\ell} \\ \sum_{n\geqslant 0} \frac{1}{n!}(z+w)^n &= e^{z+w}. \end{aligned}$$

Theorem 1.6.4. $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Lemma 1.6.5. $\theta \in \mathbb{R}$*, then* $e^{i\theta} = 1$ *iff* $\theta \in 2\pi\mathbb{Z}$.

Proposition 1.6.6. *If* $z = re^{i\alpha}$*, then* β *is an argument of* z *iff* $\alpha - \beta \in 2\pi\mathbb{Z}$.

Corollary 1.6.7. *There is a grp isom* $\mathbb{C}^\times \to \mathbb{R}_{\geqslant 0} \times \mathbb{R}/2\pi\mathbb{Z}$.

1.7 Interim 2 | HW2

Exercise 1.7.1. Suppose $S \subseteq \mathbb{C}$ is a domain and $f : S \to \mathbb{C}$ is differentiable at $z_0 \in S$.

i) Compute $f'(z_0)$ along a a trajectory $z_0 + \Delta x$ where $\Delta x \to 0$. Show that

$$f'(z_0) = u_x(z_0) + iv_y(z_0).$$

ii) Compute $f'(z_0)$ along a a trajectory $z_0 + i\Delta y$ where $\Delta y \to 0$. Show that

$$f'(z_0) = (1/i)(u_y(z_0) + iv_x(z_0)).$$

iii) Conclude that f satisfies the Cauchy-Riemann equations.

Answer

By definition, for $h \in \mathbb{C}$, we have

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)$$

whenever f is differentiable at z_0.

i) Take $h = \Delta x$, a number with no imaginary part. Then separating f into its real and imaginary parts we have

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta x\to 0} \frac{f(z_0+\Delta x)-f(z_0)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0)+iv(x_0+\Delta x, y_0)-u(x_0,y_0)-iv(x_0,y_0)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0)-u(x_0,y_0)}{\Delta x} + i\lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0)-v(x_0,y_0)}{\Delta x} \\
&= \frac{\partial}{\partial x}u(x_0,y_0) + i\frac{\partial}{\partial x}v(x_0,y_0)
\end{aligned}$$

ii) On the flipside, take $h = i\Delta y$ with $\Delta y \to 0$. We once again separate f as follows:

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta y\to 0} \frac{f(z_0+i\Delta y)-f(z_0)}{i\Delta y} \\
&= \lim_{\Delta y\to 0} \frac{u(x_0, y_0+\Delta y)+iv(x_0, y_0+\Delta y)-u(x_0,y_0)-iv(x_0,y_0)}{i\Delta y} \\
&= \lim_{\Delta y\to 0} \frac{u(x_0, y_0+\Delta y)-u(x_0,y_0)}{i\Delta y} + i\lim_{\Delta y\to 0} \frac{v(x_0, y_0+\Delta y)-v(x_0,y_0)}{i\Delta y} \\
&= \frac{1}{i}\frac{\partial}{\partial y}u(x_0,y_0) + \frac{\partial}{\partial y}v(x_0,y_0) \\
&= \frac{\partial}{\partial y}v(x_0,y_0) - i\frac{\partial}{\partial y}u(x_0,y_0)
\end{aligned}$$

iii) As the derivatives along both trajectories should match, we have that

$$\frac{\partial}{\partial x}u(x_0,y_0) + i\frac{\partial}{\partial x}v(x_0,y_0) = \frac{\partial}{\partial y}v(x_0,y_0) - i\frac{\partial}{\partial y}u(x_0,y_0).$$

Two complex numbers are equal whenever both the real and imaginary parts coincide, so it must hold that

$$\frac{\partial}{\partial x}u(x_0,y_0) = \frac{\partial}{\partial y}v(x_0,y_0), \qquad \frac{\partial}{\partial x}v(x_0,y_0) = -\frac{\partial}{\partial y}u(x_0,y_0).$$

If a function is holomorphic for every z, then this translates to $u_x = v_y$ and $v_x = -u_y$.

Exercise 1.7.2 ([1] 1.13)**.** Suppose f is holomorphic in an open set Ω. Prove that in any one of the following cases:

$$\mathrm{Re}(f) \text{ is constant;} \quad \mathrm{Im}(f) \text{ is constant;} \quad |f| \text{ is constant;}$$

one can conclude that f is constant.

Answer

As f is holomorphic, f satisfies the Cauchy-Riemann equations. This means that if $f = u + iv$, then

$$u_x = v_y, \quad v_x = -u_y.$$

- If either u or v are constant, then u_x, u_y or v_x, v_y are both zero. In either of those case, by the Cauchy-Riemann equations we can conclude that the other pair of derivatives is zero respectively.
- If the complex modulus is constant, then $|f|^2 = u^2 + v^2$ is constant as well. Differentiating the expression with respect to both variables gives us

$$\begin{cases} 2uu_x + 2vv_x = 0 \\ 2uu_y + 2vv_y = 0 \end{cases} \Rightarrow \begin{cases} uu_x + vv_x = 0 \\ uu_y + vv_y = 0 \end{cases}$$

Now, for sake of argument suppose u isn't zero. Applying the Cauchy-Riemann equations we can restate the first equation as follows:

$$\begin{cases} uv_y + v(-u_y) = 0 \\ uu_y + vv_y = 0 \end{cases} \Rightarrow \begin{cases} v_y = \frac{v}{u}u_y \\ uu_y + vv_y = 0 \end{cases}$$

Substituting the first equation into the second we obtain

$$uu_y + v\left(\frac{v}{u}u_y\right) = \left(\frac{u^2 + v^2}{u}\right)u_y = 0$$

from which follows that either $u^2 + v^2 = 0$ or $u_y = 0$. In the first case, as u is a non-zero real function, it is impossible for the sum to be zero. So it must hold that $u_y = 0$.

Doing a similar process by solving for u_y on the second equation we reach the condition that $v_y = 0$ as well. From here, using the Cauchy-Riemann equations we see that all partial derivatives of u and v are zero as we wished.

In the case that $u = 0$, we refer to the first case, where u is a constant.

Finally we conclude that f is constant in any case.

Exercise 1.7.3. Prove the following:

i) The power series $\sum_{n\geqslant 0} nz^n$ doesn't converge for any point on the unit circle.
ii) The power series $\sum_{n\geqslant 0} \frac{z^n}{n^2}$ converges for *every* point in the unit circle.
iii) The power series $\sum_{n\geqslant 0} \frac{z^n}{n}$ converges for every point in the unit circle, *except* $z = 1$.

Answer

i) We will prove that the series in question isn't Cauchy. Consider S_m, the m^{th} partial sum, then
$$|S_{m+1} - S_m| = m + 1$$
because z has complex modulus 1. Recall that a sequence of complex numbers (z_n) is a Cauchy sequence whenever
$$\forall \varepsilon \exists N \left[\forall m \forall n (m \geqslant N \wedge n \geqslant N \wedge \varepsilon > 0 \Rightarrow |z_m - z_n| < \varepsilon)\right].$$
In order to prove that (S_m) isn't Cauchy we must contradict this statement. Thus we must find an $\varepsilon_0 > 0$ such that for all N, there are m, n for which $|S_m - S_n| > \varepsilon_0$.

Take $\varepsilon_0 = 1$, m any sufficiently large natural number and $n = m + 1$ as we did before. Thus we have that $m + 1 > 1$ which lets us conclude that (S_m) isn't Cauchy. There are no non-Cauchy convergent sequences in $\mathbb{C}$ so it must hold that our series diverges given the condition that $|z| = 1$.

ii) Recall the Weierstrass M-test which states that if $(f_n(z))$ is a sequence of functions and there are $M_n > 0$ such that $|f_n(z)| \leqslant M_n$ and $\sum M_n$ is a convergent series, then $\sum f_n$ converges uniformly.

In this case, pick $M_n = \frac{1}{n^2}$. The series $\sum \frac{1}{n^2}$ converges as it is a p-series. Then
$$|z| = 1 \Rightarrow \left|\frac{z^n}{n^2}\right| \leqslant \frac{1}{n^2}$$
and thus we can conclude that $\sum \frac{z^n}{n^2}$ converges uniformly for points in the unit circle.

iii) The series in question is the harmonic series when $z = 1$ so it diverges. We will prove that when $|z| = 1$, but $z \neq 1$ this series is Cauchy. So let us fix z with $|z| = 1$ and call $S_m = \sum_{k=0}^{m} \frac{z^k}{k}$, then let $\varepsilon > 0$. Assume $n > m$ for sake of argument and then

$$\begin{aligned}
|S_n - S_{m-1}| &= \left| \sum_{k=m}^{n} \frac{z^k}{k} \right| \\
&= \left| \frac{1}{n} \sum_{k=1}^{n} z^k - \frac{1}{m} \sum_{k=1}^{m-1} z^k - \sum_{k=m}^{n-1} \left(\frac{1}{k+1} - \frac{1}{k} \right) \sum_{j=1}^{k} z^j \right| \\
&\leqslant \frac{1}{n} \left| \frac{z^{n+1} - z}{z-1} \right| + \frac{1}{m} \left| \frac{z^m - z}{z-1} \right| + \sum_{k=m}^{n-1} \left| \frac{z^{n+1} - z}{(z-1)(k^2+k)} \right| \\
&\leqslant \frac{2}{n} \left| \frac{1}{z-1} \right| + \frac{2}{m} \left| \frac{1}{z-1} \right| + \sum_{k=m}^{n-1} \left| \frac{1}{z-1} \right| \frac{2}{k^2+k}
\end{aligned}$$

Now let us state a couple of facts:

- ◇ $\left|\frac{1}{z-1}\right|$ might be arbitrarily large, but z is fixed. This means that $\left|\frac{1}{z-1}\right|$ is finite.
- ◇ Call $\widetilde{S}_r = \sum_{k=1}^{r} \frac{2}{k^2+k}$, it is important to note that this a sequence of *positive numbers*. $\widetilde{S}_\infty$ converges after comparing with $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

We will name $M = \left|\frac{1}{z-1}\right|$ so that the last expression can be written as follows:

$$\frac{2M}{n} + \frac{2M}{m} + M(\widetilde{S}_{n-1} - \widetilde{S}_{m-1}).$$

Now, as $\frac{1}{n}$ converges to zero, there exists $N_1 \in \mathbb{N}$ such that

$$n \geqslant N_1 \Rightarrow \frac{1}{n} < \frac{\varepsilon}{6M}, \ \varepsilon > 0$$

On the other hand as $\widetilde{S}_r$ converges, there exists an $N_2 \in \mathbb{N}$ such that

$$m, n \geqslant N_2 \Rightarrow |\widetilde{S}_m - \widetilde{S}_n| < \frac{\varepsilon}{3M}, \ \varepsilon > 0$$

Pick $N = \max N_1, N_2$ and let $\varepsilon > 0$, then whenever $m, n \geqslant N$, the following holds

$$\frac{2M}{n} + \frac{2M}{m} + M(\widetilde{S}_{n-1} - \widetilde{S}_{m-1}) \leqslant 2M\frac{\varepsilon}{6M} + 2M\frac{\varepsilon}{6M} + M\frac{\varepsilon}{3M} = \varepsilon.$$

Therefore the series in question is Cauchy and we can conclude that it converges.

Exercise 1.7.4. Let $\alpha \in \mathbb{C}$, $r > 0$ and $\gamma_r : [0, 2\pi[\to \mathbb{C}$ given by $t \mapsto re^{it} + \alpha$. Let $n \in \mathbb{N}$, calculate the integral $\int\limits_{\gamma_1(0)} z^n \mathrm{d}z$.

Answer

We can parametrize with $t \mapsto e^{it}$ with $r \in [0, 2\pi[$ so that

$$\int\limits_{\gamma_1(0)} z^n \mathrm{d}z = \int_0^{2\pi} (e^{it})^n (ie^{it}\mathrm{d}t) = \int_0^{2\pi} ie^{i(n+1)t}\mathrm{d}t = \left.\frac{e^{i(n+1)t}}{n+1}\right|_0^{2\pi} = \frac{e^{2\pi i(n+1)}}{n+1} - \frac{1}{n+1} = 0.$$

Exercise 1.7.5. Consider the following three groups:

- $\mathbb{C}^\times$ with multiplication as binary operation.
- $\mathbb{R}_{>0}$ with multiplication as binary operation.
- ${}^{\mathbb{R}}/_{2\pi\mathbb{Z}}$ with addition as binary operation.

Show that

$$\alpha : \mathbb{C}^\times \to \mathbb{R}_{>0} \oplus {}^{\mathbb{R}}\big/_{2\pi\mathbb{Z}}, \; z \mapsto (|z|, \arg(z))$$

is a group isomorphism as follows:

i) Show that α is a group homomorphism. [[Hint: This comes down to show that $|zw| = |z||w|$ and $\arg(zw) = \arg(z) + \arg(w)$.]]
ii) Show that α is surjective. [[Hint: For $r \in \mathbb{R}_{>0}$ and representative $\theta \in \mathbb{R}$ show that there is some $\mathbb{C}^\times$ such that $|z| = r$ and $\arg(z) = \theta$.]]
iii) Show that α is injective. [[Hint: Suppose $\alpha(z) = (1, 0)$, then show that $z = 1$.]]

Answer

i) The function α is a homomorphism because

$$\begin{aligned}\alpha(wz) &= (|wz|, \arg(wz)) = (|w||z|, \arg(w) + \arg(z)) \\ &= (|w|, \arg(w)) \circ (|z|, \arg(z)) = \alpha(w) \circ \alpha(z)\end{aligned}$$

where $\circ$ is the group operation in the direct product. To prove the equalities hold, take $wz = r_1 e^{i\theta_1}$, $w = r_2 e^{i\theta_2}$ and $z = r_3 e^{i\theta_3}$. Then

$$wz = (r_2 e^{i\theta_2})(r_3 e^{i\theta_3}) = (r_2 r_3)e^{i(\theta_2+\theta_3)} = r_1 e^{i\theta_1},$$

and as the polar representation of a complex number is unique we have that $r_1 = r_2 r_3$ and $\theta_1 = \theta_2 + \theta_3$.

ii) Take (r, θ) in the codomain of α. As $r > 0$, we can write it as $x^2 + y^2$ for $x, y \in \mathbb{R}$. Given that condition we may find the angle by the relation $\tan(\theta) = \frac{y}{x}$. Taking r and θ as given lets us construct a complex number $z = x + iy$ such that $\alpha(z) = (r, \theta)$.

iii) If it happened that $r = 1$ and $\theta = 0$, then the complex number in question could be represented as $1 \cdot e^0 = 1$. Thus $z = 1$. This means that $\ker(\alpha) = \{\,\mathrm{id}\,\}$ and thus, as α is a morphism, it's also injective.

1.8 Day 6 | 20230201

Definition 1.8.1. A parametrization of a curve is a function $z : [a, b] \to \mathbb{C}$.

It is smooth if it's differentiable and piecewise smooth if for a partition of $[a, b]$, z is smooth on the parts.

Example 1.8.2. The function $z : [0, 2\pi] \to \mathbb{C},\ t \mapsto e^{it}$ is a parametrization of the unit circle.

Definition 1.8.3. Two parametrizations w, z are equivalent if there exists a bijection $[a, b] \to [c, d]$ such that $w(s) = z(s(t))$.

Example 1.8.4. An equivalent parametrization of e^{it} is $[0, 1] \to \mathbb{C},\ t \mapsto e^{2\pi it}$.

The reverse parametrization of $z : [a, b] \to \mathbb{C}$ is $z^- : [-b, -a] \to \mathbb{C},\ t \mapsto z(-t)$. A curve is closed if it starts where it ends. Simple curves don't cross themselves.

Definition 1.8.5. The integral over a curve γ is

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t$$

where $z : [a, b] \to \gamma$ parametrizes the curve.

Example 1.8.6. Consider the integral of $\overline{z}$ over the unit circle. This is

$$\int_{\{\,|z|=1\,\}} \overline{z}\mathrm{d}z = \int_0^{2\pi} \overline{e^{it}} ie^{-it}\mathrm{d}t = \int_0^{2\pi} i\mathrm{d}t = 2\pi i.$$

Integrals over the complex numbers obey the same properties as over the real numbers. The arc length of a curve is the same as in multivariable calculus. The integral also obeys the triangle inequality.

Definition 1.8.7. A domain is a non-empty, open and connected subset of $\mathbb{C}$.

Lemma 1.8.8. *If F, f are functions defined on Ω, a domain, with $F' = f$, and $w, z \in \Omega$, then*

$$\int_\gamma f(z) \mathrm{d}z = F(z) - F(w)$$

where $\gamma \subseteq \Omega$ is a curve connecting w to z.

Corollary 1.8.9. *If the curve is closed the integral is zero.*

As a consequence, the function $\bar{z}$ has no antiderivative in any ball around the origin.

Lemma 1.8.10. *Suppose f is holomorphic on Ω and $f' = 0$ on Ω. Then f is constant.*

Proof

If $w, z \in \Omega$, then

$$0 = \int f' = f(z) - f(w) \Rightarrow f(z) = f(w)$$

so f must be constant.

Next time: Goursat's theorem.

1.9 Day 7 | 20230203

Index

Bibliography

[1] Lawrence C. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010.