Integration by Parts

The substitution formula (u-*sub*) works as an analog to the chain rule in differentiation. The integration by parts formula is analogous to the *product rule*.

Remark. Suppose f,g are differentiable. Then
$$\frac{\mathrm{d}}{\mathrm{d}x}(fg)=f\frac{\mathrm{d}}{\mathrm{d}x}(g)+g\frac{\mathrm{d}}{\mathrm{d}x}(f)$$
and integrating both sides we obtain:
$$fg=\int\frac{\mathrm{d}}{\mathrm{d}x}(fg)=\int f(g')\mathrm{d}x+\int(f')g\mathrm{d}x.$$
Rearranging the equality we obtain the formula
$$\int f(g')\mathrm{d}x=fg-\int(f')g\mathrm{d}x.$$

Definition. The <u>integration by parts</u> formula for two functions u and v is given by
$$\int u\mathrm{d}v=uv-\int v\mathrm{d}u,$$
where $\mathrm{d}u=\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ and likewise for v.

An easy way to remember the right hand side of this formula is with the mnemonic <u>*ultraviolet voodoo*</u>.

Example 1. If we are asked to integrate xe^x by itself, we can't do it. However with the formula we can:
$$\int xe^x\mathrm{d}x,\text{ with } u=x,\ \mathrm{d}v=e^x\mathrm{d}x.$$
The u is the function which is easy to differentiate and the v is the one which is easier to integrate.

We obtain $\mathrm{d}u=\mathrm{d}x$ by differentiating and $v=e^x$ after integrating. Thus rearranging the integral we get
$$\int xe^x\mathrm{d}x=xe^x-\int e^x\mathrm{d}x.$$
The last integral we can compute so in the end we obtain
$$\int xe^x\mathrm{d}x=xe^x-\int e^x\mathrm{d}x=\underline{xe^x-e^x}.$$

Example 2. Consider the following integral
$$\int x^2e^x\mathrm{d}x.$$
To integrate we have to apply the same formula. Here we take
$$u=x^2\text{ and }\mathrm{d}v=e^x\mathrm{d}x.$$
We differentiate u and integrate $\mathrm{d}v$ to obtain
$$\mathrm{d}u=2x\mathrm{d}x\text{ and }v=e^x.$$
Given this we can arrange the integration by parts formula as follows:
$$\int x^2e^x\mathrm{d}x=\underbrace{x^2}_{u}\underbrace{e^x}_{v}-\int\underbrace{e^x}_{v}\underbrace{2x\mathrm{d}x}_{\mathrm{d}u}.$$
We can factor out a two from the last integral, and using the result from the previous example we get
$$\int x^2e^x\mathrm{d}x=x^2e^x-2(xe^x-e^x)=\underline{x^2e^x-2xe^x+2e^x}.$$

Practice

In a group with your classmates calculate the following integral
$$\int x^3e^x\mathrm{d}x.$$
This calculation can be made easier by using the result of the previous examples.
How about calculating
$$\int x^ne^x\mathrm{d}x,\text{ for } n=4,\ldots,7?$$

Example 3. Consider the integral
$$\int x\sin(x)\mathrm{d}x.$$
Like in the previous cases, we take
$$u=x\text{ and }\mathrm{d}v=\sin(x)\mathrm{d}x$$
therefore
$$\mathrm{d}u=\mathrm{d}x\text{ and }v=-\cos(x).$$
The formula gives us
$$\int x\sin(x)\mathrm{d}x=x(-\cos(x))-\int(-\cos(x))\mathrm{d}x.$$
Integrating the cosine and taking out the minuses gives us
$$\int x\sin(x)\mathrm{d}x=\underline{-x\cos(x)+\sin(x)}.$$
However, we haven't asked ourselves what happens if we take
$$u=\sin(x)\text{ and }\mathrm{d}v=x\mathrm{d}x.$$
In this case we get
$$\mathrm{d}u=\cos(x)\mathrm{d}x\text{ and }v=\frac{x^2}{2}$$
and thus after arranging the integral with the formula we get
$$\int x\sin(x)\mathrm{d}x=\sin(x)\left(\frac{x^2}{2}\right)-\int\frac{x^2}{2}\cos(x)\mathrm{d}x.$$

We have ran into a problem! Using the formula the other way didn't make this integral simpler to calculate. *That's our objective.* There's no definitive way of choosing who's u and who's v but an easy heuristic is as follows:

Proposition 4. *The order in which to choose who's u is*

- *(**L**)ogarithms*
- *(**I**)nverse Trigonometrics*
- *(**A**)lgebraic functions*
- *(**T**)rigonometric functions*
- *(**E**)xponentials*

*The mnemonic to remember these is **LIATE**.*

Usually following this idea we will get a simpler integral to calculate after applying the formula. Lets use this idea to integrate the following:

Example 5. Let us calculate the indefinite integral
$$\int \log(x)\mathrm{d}x.$$
Since the function is a logarithm, it's a top priority to differentiate that. Thus $u=\log(x)$, but… who's $\mathrm{d}v$? There's always a hidden one (1) multiplying right there, so we take $\mathrm{d}v = 1\mathrm{d}x = \mathrm{d}x$. After differentiating and integrating we get
$$\mathrm{d}u=\frac{1}{x}\mathrm{d}x \text{ and } v=x.$$
The integral becomes
$$\int \log(x)\mathrm{d}x=\log(x)\cdot(x)-\int x\left(\frac{1}{x}\right)\mathrm{d}x=\underline{x\log(x)-x}$$
Remark. Don't forget to consider integrating 1! This works sometimes!

> **Practice**
>
> Calculate the integral of $\arcsin(x)$. *Like the one we just did, use the same idea!*

Example 6. We can now calculate the integral
$$\int x\log(x)\mathrm{d}x, \text{ let } \begin{cases} u=\log(x)\Rightarrow \mathrm{d}u=1/x\mathrm{d}x, \\ \mathrm{d}v=x\mathrm{d}x\Rightarrow v=(1/2)x^2, \end{cases}$$
and then after rearranging we get
$$\begin{aligned}\int x\log(x)\mathrm{d}x&=\frac{1}{2}x^2\log(x)-\int\left(\frac{1}{2}x^2\right)\left(\frac{1}{x}\mathrm{d}x\right)\\ &=\underline{\frac{1}{2}x^2\log(x)-\frac{1}{4}x^2}.\end{aligned}$$

We could generalize this to any integer power of x, but how about a rational, or even an *irrational* power of x?

Example 7. Let us find the indefinite integral
$$\int x^{\sqrt{5}}\log(x)\mathrm{d}x.$$
In fact, the power at which x is raised *does not matter*. The process is the same! Let
$$\begin{cases} u=\log(x)\Rightarrow \mathrm{d}u=1/x\mathrm{d}x, \\ \mathrm{d}v=x^{\sqrt{5}}\mathrm{d}x\Rightarrow v=(1/(\sqrt{5}+1))x^{\sqrt{5}+1}. \end{cases}$$
Rearranging we get
$$\int x^{\sqrt{5}}\log(x)\mathrm{d}x=\frac{(x^{\sqrt{5}+1})\log(x)}{\sqrt{5}+1}-\int\left(\frac{x^{\sqrt{5}+1}}{\sqrt{5}+1}\right)\left(\frac{1}{x}\mathrm{d}x\right).$$
The integral on the right is the integral of a power of x so in the end, the result is
$$\int x^{\sqrt{5}}\log(x)\mathrm{d}x=\frac{(x^{\sqrt{5}+1})\log(x)}{\sqrt{5}+1}-\frac{x^{\sqrt{5}+1}}{(\sqrt{5}+1)^2}.$$

Example 8. In this example we don't consider a power of x multiplying another function. Let's calculate
$$\int e^x\sin(x)\mathrm{d}x.$$
By using the **LIATE** mnemonic we choose the following
$$\begin{cases} u=\sin(x)\Rightarrow \mathrm{d}u=\cos(x)\mathrm{d}x, \\ \mathrm{d}v=e^x\mathrm{d}x\Rightarrow v=e^x. \end{cases}$$
We obtain
$$\int e^x\sin(x)\mathrm{d}x=e^x\sin(x)-\int\cos(x)e^x\mathrm{d}x.$$
Applying the formula once more with this last integral:
$$\begin{cases} u_2=\cos(x)\Rightarrow \mathrm{d}u_2=-\sin(x)\mathrm{d}x, \\ \mathrm{d}v_2=e^x\mathrm{d}x\Rightarrow v_2=e^x. \end{cases}$$
If I is the original integral we get:
$$I=e^x\sin(x)-\left(e^x\cos(x)+\int e^x\sin(x)\mathrm{d}x\right).$$
This last integral is the one we are looking for. We can rearrange this equation as follows:
$$\begin{aligned}I&=e^x\sin(x)-e^x\cos(x)-I\\ \Rightarrow I&=\underline{(1/2)(e^x\sin(x)-e^x\cos(x))}\end{aligned}$$

> **Practice**
>
> With an analogous reasoning calculate the integral
> $$\int e^{2x}\sin(3x)\mathrm{d}x.$$

Exercise 9. Compute the following integrals:

- $\int x^2\cos(x)\mathrm{d}x$ (*Hint: Use Example 3*).
- $\int \log^2(x)\mathrm{d}x$.
- $\int x^\alpha\log(x)\mathrm{d}x$, $\alpha\neq 1$ is any real number.
- $\int\sqrt{x}\log(3x)\mathrm{d}x$.