**Exercise 1** (4.1.A Vakil)**.** Show that the natural map $A_f \to \mathcal{O}_{\mathrm{Spec}(A)}(D(f))$ is an isomorphism. ⟦ Hint: Exercise 3.5.E Vakil. ⟧

First let us recall that Exercise 3.5.E is the following:

**Lemma 1.** *The next statements are equivalent*:

*i*) $D(f) \subseteq D(g)$.
*ii*) $\exists n (n \geqslant 1 \Rightarrow f^n \in \mathrm{gen}(g))$.
*iii*) $g$ *is an invertible element of* $A_f$.

We have proven this in class so let us make a quick recapitulation.

The first two statements are equivalent because

$$\begin{aligned} D(f) \subseteq D(g) &\iff V(g) \subseteq V(f) \\ &\iff \{\, \mathfrak{p} : \mathrm{gen}(g) \subseteq \mathfrak{p} \,\} \subseteq \{\, \mathfrak{p} : \mathrm{gen}(f) \subseteq \mathfrak{p} \,\} \end{aligned}$$

The last statement can be rephrased as *if a prime contains $g$, then it also contains $f$*. In particular this equivalent to saying

$$\begin{gathered} f \in \bigcap_{g \in \mathfrak{p}} \mathfrak{p} = \sqrt{\mathrm{gen}(g)} \\ \iff \exists n (n \geqslant 1 \Rightarrow f^n \in \mathrm{gen}(g)). \end{gathered}$$

For the last two statements, we first assume $g$ is invertible in $A_f$. This means that there exists an $n$ such that

$$\left(\frac{g}{1}\right)\left(\frac{a}{f^n}\right) = \frac{1}{1}.$$

Recall that the equality condition in the localization means that there exists and element $f^m$ with $m \geqslant 1$ which is invertible in $A_f$ such that

$$f^m(ag - f^n) = 0 \Rightarrow agf^m = f^{m+n}.$$

This last equation is in $A$ without localizing, and the term on the right, $agf^m$, is in $\mathrm{gen}(g)$. Thus the power we were searching for is $m+n$ and $f^{m+n} \in \mathrm{gen}(g)$. On the other direction, if $f^n \in \mathrm{gen}(g)$ for some $n \geqslant 1$, then there is an $a \in A$ such that

$$f^n = ag,$$

and localizing at $f$ turns this equation into $\frac{1}{g} = \frac{a}{f^n}$.

**Answer**

We begin by recalling the definition of $\mathcal{O}_{\mathrm{Spec}(A)}(D(f))$, we have

$$\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) = S^{-1}A, \quad \text{where} \quad S = \{\, g \in A : D(f) \subseteq D(g) \,\}.$$

By the lemma we can rewrite $S$ as

$$S = \{\, g \in A : \exists n(f^n \in \mathrm{gen}(g)) \,\}.$$

Now notice that when localizing at $S$ we are able to invert $f^n$ for some $n$. From this we have that $f$ is also invertible in $S^{-1}A$ because

$$f^n g = u \Rightarrow f(f^{n-1}g) = u \Rightarrow f \text{ is invertible.}$$

This means that localizing at $S$ is a further localization of $A$ at $f$ because we have already inverted all powers of $f$.
Notice however that this isn't adding anything new to $A_f$, because of the last equivalence of the lemma. Every $g$ such that $D(f) \subseteq D(g)$ is already invertible in $A_f$. We conclude that the inclusion is actually an isomorphism.

**Exercise 2** (Restrictions)**.** Do the following:

i) Explain, using Definition 4.1.1 (and not exercise 4.1.A) what the restriction map is.
ii) Explain, using exercise 4.1.A what the restriction map is.

**Answer**

i) Recall that

$$\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) = (S^f)^{-1}A, \quad \text{where} \quad S^f = \{\, h \in A : D(f) \subseteq D(h) \,\}$$

and on the same vein the set associated to $D(g)$ is the localization at $S^g = \{\, h \in A : D(g) \subseteq D(h) \,\}$. So if we take $D(f) \subseteq D(g)$, we can inject $A$ into both localizations via $a \mapsto \frac{a}{1}$ as follows:

$$\begin{tikzcd} A \arrow[r, "\varphi_f"] \arrow[dr, "\varphi_g"'] & (S^f)^{-1}A \\ & (S^g)^{-1}A \end{tikzcd}$$

Now for elements $h \in S^g$, $\varphi^f(h)$ is an invertible element in $(S^f)^{-1}A$ because $D(f) \subseteq D(g)$. So, by universality of the localization, we have that there exists a unique map

$$(S^g)^{-1}A \to (S^f)^{-1}A$$

and such map is the desired restriction map.

ii) Using the previous exercise we have the isomorphism between localizing at $S^f$ and localizing at powers of $f$. So once again let us assume that $D(f) \subseteq D(g)$, then the restriction map is a function

$$\operatorname*{res}_{D(g),D(f)} A_g \to A_f.$$

In this case we have an element $\frac{a}{h^n}$ with $h \in S^g$. Recall that this means that $h$ is invertible in $A_g$ so we may write

$$\frac{1}{h} = \frac{b}{g^m}, \quad \text{where} \quad b \in A,\ m \in \mathbb{N}.$$

But now, as $D(f) \subseteq D(g)$, $g$ is invertible in $A_f$ so once again we have

$$\frac{1}{g} = \frac{c}{f^r}, \quad \text{where} \quad c \in A,\ r \in \mathbb{N}.$$

Combining these facts we have

$$\frac{a}{h^n} = a\frac{b^n}{g^{mn}} = ab^n\frac{c^{mn}}{f^{mnr}},$$

and so this element inside $A_f$ is where we map our original element to.

**Exercise 3** (4.1.D Vakil)**.** Suppose $M$ is an $A$-module. Show that the following construction describes a sheaf $\widetilde{M}$ on the distinguished base. Define $\widetilde{M}(D(f))$ to be the localization of $M$ at the multiplicative set of all functions that do not vanish outside of $V(f)$.

Define restriction maps $\mathrm{res}_{D(f),D(g)}$ in the analogous way to $\mathcal{O}_{\mathrm{Spec}(A)}$.

Show that this defines a sheaf on the distinguished base, and hence a sheaf on $\mathrm{Spec}(A)$. Then show that this is an $\mathcal{O}_{\mathrm{Spec}(A)}$-module.

Answer

We are now considering the following space $(\mathrm{Spec}(A), \widetilde{M})$, in other words we are endowing $\mathrm{Spec}(A)$ with sheaf of $A$-modules.
We first verify that $\widetilde{M}$ is a presheaf. In the same way we defined the sheaf on the basis elements we have

$$\widetilde{M}(D(f)) = (S^f)^{-1}M, \quad \text{where} \quad S^f = \{\, h \in A : D(f) \subseteq D(h)\,\}.$$

The restriction maps are defined in the same fashion by universality, so if we have the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi_f"] \arrow[dr, "\varphi_f"'] & (S^f)^{-1}M \\ & (S^f)^{-1}M \end{tikzcd}$$

then there's a restriction map from each localization to another. By universality they must be the same arrow, and as the only arrow which goes from $(S^f)^{-1}M$ to itself is the identity we have

$$\operatorname*{res}_{D(f),D(f)} = \mathrm{id}_{(S^f)^{-1}M}\,.$$

Now, we have the composition of restriction maps is the longer restriction map. Consider the following diagram:

$$\begin{tikzcd} & (S^f)^{-1}M \\ M \arrow[ur, "\varphi_f"] \arrow[r, "\varphi_g"] \arrow[dr, "\varphi_h"'] & (S^g)^{-1}M \arrow[u, dashed] \\ & (S^h)^{-1}M \arrow[u] \arrow[uu, dashed, bend right] \end{tikzcd}$$

We have that the composition exists by universality as do the smaller arrow. Universality guarantees that they are the same arrow and thus

$$\operatorname*{res}_{D(h),D(f)} = \operatorname*{res}_{D(g),D(f)} \circ \operatorname*{res}_{D(h),D(g)}\,.$$

Let us now verify the sheaf axioms. We begin by considering an open cover of $\mathrm{Spec}(A)$ which we can reduce to a finite sub-cover by quasi-compactness, this is

$$\mathrm{Spec}(A) = \bigcup_{i\in I} D(f_i) = \bigcup_{i=1}^{n} D(f_i).$$

- Recall that our sheaf sets are modules, so injectivity is equivalent to kernels being trivial. Thus to verify the identity axiom, we consider $s \in \widetilde{M}(\mathrm{Spec}(A))$ with the assumption that $\mathrm{res}_{D(f_i)}(s) = 0$. We wish to show $s = 0$.

  The restriction lives in one our localization, this means that

  $$\operatorname*{res}_{D(f_i)}(s) \in \widetilde{M}(D(f_i)) = (S^{f_i})^{-1}M \Rightarrow \operatorname*{res}_{D(f_i)}(s) = \frac{m}{g}, \quad \text{where} \quad g \in S^{f_i}.$$

  Now this fraction is zero, which means that there's an invertible element $u \in S^{f_i}$, which is $\frac{1}{u} = \frac{a}{f_i^s}$, that satisfies

  $$u(1 \cdot m - 0.g) = 0 \Rightarrow aum = a \cdot 0 = 0 \Rightarrow f_i^s m = 0.$$

  This last proposition holds for all $i$, and as we have $\langle f_1^s, \ldots, f_n^s \rangle = A$ we have that $1 = \sum c_i f_i^s$. As $M$ is an $A$-module, the next equation holds in $M$:

  $$m = \left( \sum_{i=1}^{n} c_i f_i^s \right) m = \sum_{i=1}^{n} c_i (f_i^s m) = \sum 0 = 0.$$

  Thus $m = 0$ and we have the identity axiom[a].

- We proceed as in the proof of the gluing axiom for the structure sheaf. First by taking a finite set of indices and then generalizing to an infinite set of indices.

  We take sections $(s_i)$ such that

  $$\operatorname*{res}_{D(f_i), D(f_i) \cap D(f_j)}(s_i) = \operatorname*{res}_{D(f_j), D(f_i) \cap D(f_j)}(s_j) \quad \text{for all} \quad i, j.$$

  The set $\tilde{M}(D(f_i) \cap D(f_j))$ is the localization of $M$ at $(S^{f_i} \cap S^{f_j})$. (Couldn't finish this one, wrapping my head around this localization was a bit jarring. Would it be possible to discuss later?)

[a] As 4.1.B mentions that it is possible to replace $D(f_i)$ by $D(f)$, thus generalizing the argument, I don't see how it is possible in the proof of the theorem nor in this exercise. This is because if don't have the finite number of $f_i$'s then we can't say that $1$ is the linear combination that we have recovered.

**Exercise 4.** Let $A = \mathbb{C}[x, y]$ and let $\mathfrak{p} = \mathrm{gen}(y)$, viewed as a point of $X = \mathrm{Spec}(A)$. What is $\mathcal{O}_{X,p}$?

Recall that $\mathcal{O}_{X,p}$ is a local ring, that is, it has a unique maximal ideal, $\mathfrak{m}_p$.
What is the residue field $\kappa_{\mathfrak{p}} = \mathcal{O}_{X,p}/\mathfrak{m}_p$?

**Answer**

The set $\mathcal{O}_{X,\mathfrak{p}}$ is the stalk of the structure sheaf at the point $\mathfrak{p} \in \operatorname{Spec} A$. Germs inside $\mathcal{O}_{X,\mathfrak{p}}$ are equivalence classes of pairs $(f, D(g))$ where $\mathfrak{p} \in D(g)$ and $f \in \mathcal{O}_X(D(g))$. Recall that in our case

$$\mathcal{O}_X(D(g)) \simeq A_g = \mathbb{C}[x,y]_g$$

so the germs are equivalence classes of rational functions with denominators $g^r$ about $\operatorname{gen}(y)$ which don't vanish.
The maximal ideal of the stalk is the germs which vanish at $\operatorname{gen}(y)$. Recall that $f$ vanishes at $\operatorname{gen}(y)$ when $f \mod \operatorname{gen}(y) = 0$. So $f$ musn't have any multiples of $y$. The only thing we are left with is rational functions on $x$ so it must hold that $\kappa_{\mathfrak{p}} \simeq \mathbb{C}(x)$.

**Exercise 5** (4.4.A Vakil)**.** Show that you can glue an arbitrary collection of schemes together. Suppose we are given:

- schemes $X_i$ (as $i$ runs over some index set $I$, not necessarily finite),
- open subschemes $X_{ij} \subseteq X_i$ with $X_{ii} = X_i$,
- isomorphisms $f_{ij} : X_{ij} \to X_{ji}$ with $f_{ii}$ the identity

such that

*the isomorphisms "agree on triple intersections", i.e.,*

$$f_{ik} \mid_{X_{ij} \cap X_{ik}} = f_{jk} \mid_{X_{ji} \cap X_{jk}} \circ f_{ij} \mid_{X_{ij} \cap X_{ik}} \circ$$

*(so implicitly, to make sense of the right side, $f_{ij}(X_{ik} \cap X_{ij}) \subseteq X_{jk}$).*

This *cocycle condition* ensures that $f_{ij}$ and $f_{ji}$ are inverses. In fact, the hypothesis that $f_{ii}$ is the identity also follows from the cocycle condition.

Show that there is a unique scheme $X$ (up to unique isomorphism) along with open subsets isomorphic to the $X_i$ respecting this gluing data in the obvious sense.⟦Hint: what is $X$ as a set? What is the topology on this set? In terms of your description of the open sets of $X$, what are the sections of this sheaf over each open set?⟧