Exercise 1 (Exercise 1)**.** Prove that the tensor product of two Hadamard matrices is a Hadamard matrix.

Answer

Suppose H, K are two Hadamard matrices. We have $HH^\intercal = mI$ and $KK^\intercal = nI$ and we must show that $(H \otimes K)(H \otimes K)^\intercal = mnI$ where the last identity matrix has size $(mn) \times (mn)$.
The product enjoys two properties which are essential for our purpose:

- Transposition distributes over the product: $(A \otimes B)^\intercal = A^\intercal \otimes B^\intercal$.
- The *mixed-product property*: If A, B, C, D are matrices, then

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

With this in hand we see that

$$(H \otimes K)(H \otimes K)^\intercal = HH^\intercal \otimes KK^\intercal = mnI_m \otimes I_n = mnI_{mn}$$

and thus $H \otimes K$ is Hadamard as desired.

Exercise 2 (Exercise 4)**.** The **complementary design** to a design $\mathcal{D} = (X, \mathcal{B})$ is the pair $\mathcal{D}^c = (X, \mathcal{B}^c)$ where $\mathcal{B}^c = \{ X \backslash B : B \in B \}$. Show that if $\mathcal{D}$ is a $1 - (v, k, \lambda)$ design then $\mathcal{D}^c$ is a $1 - (v, v - k, v\lambda/k - \lambda)$ design.

Answer

We know that in $\mathcal{D}^c$ we have v vertices. Any block is of the form $X \backslash B$ with B having size k, so all the blocks in $\mathcal{D}^c$ have size $v - k$ as desired.
It remains to show that every point is in exactly $\frac{v\lambda}{k} - \lambda$ blocks. Now, let us manipulate this quantity:
Recall r is the number of blocks containing a point, in this case as we have a 1-design, we have that $r = \lambda$, so

$$\frac{v\lambda}{k} = \frac{vr}{k} = \frac{bk}{k} = b, \quad \text{the number of blocks.}$$

So we must show that every point is in $b - r$ blocks, but now this is immediate because any point already on r blocks, is not inside the remaining $b - r$ blocks. But that is what it means to be inside a block in the complementary design. We conclude that $\mathcal{D}^c$ is indeed a $1 - (v, v - k, b - r)$ design.

Exercise 3 (Exercise 6)**.** Prove that the edge-complement of a strongly regular graph is strongly regular, and find the new parameters in terms of the previous.

Answer

Suppose G is strongly regular with parameters (n, k, λ, μ). Pick a vertex v and look at its neighbors, there are k of them. Out of the remaining $n-1$ vertices, v is not connected to $n-1-k$ of them.
Now pick another vertex w, if they are not connected then they share μ common neighbors. From the remaining $n-2$ vertices, u, v are only connected to their neighborhoods. Removing all the vertices in the neighborhoods doubly counts the intersection, and we know that intersection has size μ. So in total, v, w are not connected to

$$(n-2) - 2k + \mu \quad \text{vertices}$$