Exercise 1 (Stein&Shakarchi 3.15(c))**.** Let $w_1, \ldots, w_n$ be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that the product of the distances from z to the points $w_j,\ 1 \leqslant j \leqslant n$, is at least 1.

Conclude that there exists a point w on the unit circle such that the product of the distances from w to the points $w_j,\ 1 \leqslant j \leqslant n$, is exactly equal to 1.

Answer

Consider the function

$$g(z) = \prod_{k=1}^{n} (z - w_k)$$

which is holomorphic. Notice that

$$|g(0)| = \prod_{k=1}^{n} |w_k| = 1$$

and by the maximum modulus principle

$$1 = |g(0)| \leqslant \sup_{z \in \partial B(0,1)} |g(z)|.$$

As g is continuous in the boundary, this supremum is reached. This means that there exists z_0 with $|z_0| = 1$ such that

$$1 \leqslant \prod_{k=1}^{n} |z_0 - w_k|.$$

On the other hand notice that $g(w_k) = 0$ for all k. So, as $|g|$ is continuous we may apply the intermediate value theorem to conclude that there are z such that $|z| = 1$ with the property

$$0 = |g(w_k)| \leqslant |g(z)| \leqslant |g(z_0)|$$

for all values in between. In particular, there must exist a point such that $|g(z)| = 1$.

Exercise 2 (Stein&Shakarchi 3.15(d))**.** Show that if the real part of an entire function f is bounded, then f is constant. ⟦ Hint: Instead of using the hint in the book, you can also proceed by considering the function $\exp(f(z))$. ⟧

Answer

Suppose f is entire and bounded, then

$$|e^f| = e^{\mathrm{Re}(f)} < \infty$$

as $\mathrm{Re}(f)$ is bounded. Then by Liouville's theorem, e^f is constant. Finally differentiating we get

$$(e^f)(f') = 0 \Rightarrow f' = 0 \Rightarrow f \quad \text{is constant} \quad .$$

Here we have used the fact that e^f is never zero.

Exercise 3. Use Rouché's theorem to give another proof of the fundamental theorem of algebra, as follows:

- Let $p(z) = \sum_{j=0}^{d} a_j z^j$ be a polynomial, where $d \geqslant 1$ and $a_d \neq 0$.
- In class, we showed that there exist constants $C > 0$ and R_0 such that, if $|z| > R_0$, then $C|z^d| > |p(z)|$.

Show that, for each $R > R_0$, $p(z)$ has exactly d roots (counted with multiplicity) of size less than R.

Answer

Let us consider $f = a_d z^d$ and $g = a_{d-1}z^{d-1} + \cdots + a_1 z + a_0$. For any $R > 0$ we have that inside the contour $\partial B(0, R)$, f has d roots.
Now consider the modulus of g, we have

$$|g(z)| = |a_{d-1}z^{d-1} + \cdots + a_1 z + a_0 \leqslant |a_{d-1}||z^{d-1}| + \cdots + |a_1||z| + |a_0|$$

and working in our contour we may bound g by

$$|a_{d-1}|R^{d-1} + \cdots + |a_1|R + |a_0| \leqslant (|a_{d-1}| + \cdots + |a_0|)R^{d-1}.$$

On the other hand for f we have $|f| = |a_d|R^d$ so

$$\frac{|g|}{|f|} \leqslant \frac{(|a_{d-1}| + \cdots + |a_0|)R^{d-1}}{|a_d|R^d} = \frac{|a_{d-1}| + \cdots + |a_0|}{|a_d|R}.$$

If we wanted $|g| \leqslant |f|$, we require

$$\frac{|a_{d-1}| + \cdots + |a_0|}{|a_d|R} \leqslant 1 \iff \frac{|a_{d-1}| + \cdots + |a_0|}{|a_d|} \leqslant R.$$

With this information in hand we may apply Rouché's theorem, in a contour with such a radius we have that f, g are holomorphic and $|f| \geqslant |g|$ so f and $f + g = p$ have the same number of zeroes inside our contour.
In conclusion p has d zeroes all inside the contour which means that they have modulus less than R.

Exercise 4. Let f be non-constant and holomorphic in an open set containing $\overline{\mathbb{D}}$, the closed unit disk. Further suppose that if $|z| = 1$, then $|f(z)| = 1$.

(a) Show that $f(z) = 0$ has a root, i.e., that the image of f contains 0. ⟦Hint: Use the maximum modulus principle.⟧

(b) Show that if $w_0 \in \mathbb{D}$, then there exists some $z_0 \in D$ such that $f(z_0) = w_0$. ⟦Hint: Apply the result of the first part to the composition of f with a suitable Blaschke factor, as in [SS] 1.7⟧

Answer

(a) Assume on the contrary that f doesn't have a root. Then in the same fashion that $|z| = 1 \Rightarrow |f(z)| = 1$ we also have that $\frac{1}{|f(z)|} = 1$.

By the maximum modulus principle, inside the ball we have that $|f|(z)| \leqslant 1$ and in the same vein we have $|f(z)| \geqslant 1$. Therefore f has constant modulus on the ball and with this we can deduce f is constant. But this is a contradiction as f is non-constant.

Our assumption that f doesn't have a root must therefore be false and with that we have that f does have a root.

(b) Now let $w_0 \in \mathbb{D}$ and consider the function $g(z) = -w_0$. For $|z| = 1$ we have

$$|f(z)| = 1 \geqslant |w_0| = |g(z)|$$

and thus by Rouché's theorem we have that f and $f + g$ have the same number of roots in $B(0, 1)$. As f has at least one root, then there is at least one z_0 such that $f(z_0) - w_0 = 0$ which means that $f(z_0) = w_0$.