Exercise 1. Do the following:

i) Let $q = (a_1, \dots, a_n)$ be a point in $\mathbb{A}^n$. Using the fact that $I(q)$ is a maximal ideal in $\mathbb{C}[x_1, \dots, x_n]$, prove that the coordinate ring of q is isomorphic to $\mathbb{C}$.

ii) If $i : \{ q \} \to \mathbb{A}^n$ is the inclusion map, show that the pullback homomorphism

$$i^\sharp : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[q] = \mathbb{C}$$

sends a function $f(x_1, \dots, x_n)$ to the complex number $f(a_1, \dots, a_n)$ obtained by evaluating at that point.

Answer

i) The ideal $I(q)$ is in fact $\mathrm{gen}(x_1 - a_1, \dots, x_n - a_n)$, a maximal ideal in $\mathbb{C}[x_1, \dots, x_n]$. Then the coordinate ring of $\{ q \}$ is precisely

$$\mathbb{C}[q] = \mathbb{C}[x_1, \dots, x_n] \Big/ I(\{ q \}) = \mathbb{C}[x_1, \dots, x_n] \Big/ \mathrm{gen}(x_1 - a_1, \dots, x_n - a_n).$$

The evaluation homomorphism ε_q with help of the 1^{st} isomorphism theorem gives us the desired isomorphism. This is clearly a surjective map since we can get to any complex number by solving a linear equation and its kernel is the aforementioned ideal.

ii) Since the inclusion mapping is a morphism of algebraic varieties, then it induces a pullback homomorphism between the coordinate rings. By definition its action is as follows:

$$i^\sharp : \mathbb{C}[\mathbb{A}^n] \to \mathbb{C}[q], \ g \mapsto g \circ i.$$

Let us unpack the terminology. First, the inclusion homomorphism is the identity mapping restricted to $\{ q \}$. Then the pullback can be expressed as

$$i^\sharp : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}, \ g(\mathbf{z}) \mapsto g(\mathrm{id}\,|_{\{ q \}}(\mathbf{z})).$$

In this sense the action of $g \circ i$ is

$$\mathbb{A}^n \xrightarrow[\mathbf{z} \mapsto q]{\mathrm{id}|_{\{ q \}}} \mathbb{A}^n \xrightarrow[g \mapsto g(-)]{g} \mathbb{C} \Rightarrow \mathbb{A}^n \xrightarrow[g \mapsto g(q)]{g \circ i} \mathbb{C},$$

and thus, since the action of this map is the same as ε_q, we conclude that $i^\sharp = \varepsilon_q$.

Exercise 2. Prove that if $F : V \to W$ is an isomorphism of affine algebraic varieties, then the pullback homomorphism is a ring isomorphism.

Answer

The pullback homomorphism is precisely F

Exercise 3. Let $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ be affine algebraic varieties. Let $\tilde{F} : \mathbb{A}^n \to \mathbb{A}^m$ be a morphism. Show that

$$\tilde{F}(V) \subseteq W \iff \tilde{F}^\sharp : \mathbb{C}[y_1, \dots, y_m] \to \mathbb{C}[x_1, \dots, x_n] \text{ sends } I(W) \text{ to } I(V).$$

⟦Hint: W=V(I(W)).⟧

Exercise 4 (2.6.1)**.** Prove that $\operatorname{Spec}(R)$ of a commutative ring R can be given the structure of a topological space whose closed sets are of the form $V(I) = \{ \mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq I \}$ for $I \trianglelefteq R$.

Exercise 5 (2.5.(1,2))**.** Do the following:

i) Show that the pullback $\mathbb{C}[W] \xrightarrow{F^\sharp} \mathbb{C}[V]$ is injective if and only if F is *dominant*. This is, $F(V)$ is dense in W.
ii) Show that the pullback $\mathbb{C}[W] \xrightarrow{F^\sharp} \mathbb{C}[V]$ is surjective if and only if F defines an isomorphism between V and some algebraic subvariety of W.