Review

Limit Comparison Test

Example 1. Consider the series $\sum_{n=1}^{\infty} \frac{7}{2^n - 5\sin^m(n^3)}$, where $m > 0$ is a positive integer. We will study its convergence using the Limit Comparison Test.

i) The fastest growing term is the denominator is 2^n because sine is always bounded by its amplitude, in this case 5.

ii) Comparing the general terms we have
$$\frac{a_n}{b_n} = \frac{\frac{7}{2^n - 5\sin^m(n^3)}}{\frac{1}{2^n}} = \frac{7}{\frac{1}{2^n}(2^n - 5\sin^m(n^3))} = \frac{7}{1 - \frac{5\sin^m(n^3)}{2^n}}.$$

iii) We are interested in $\frac{5\sin^m(n^3)}{2^n}$, so we will take the absolute value and analyze:
$$\left|\frac{5\sin^m(n^3)}{2^n}\right| = \frac{5|\sin^m(n^3)|}{2^n} \leqslant \frac{5}{2^n} \to 0.$$
So the original term goes to zero as well. This means that $\frac{a_n}{b_n}$ goes to 7.

iv) As $7 > 0$, by the Limit Comparison Test, both series behave the same and thus our original series converges.

Ratio Test

Example 2. Consider the series $\sum_{n=0}^{\infty} \frac{r^n}{n!}$, $r > 0$. We will study its convergence using the ratio test:

i) The general term is $a_n = \frac{r^n}{n!}$.

ii) The next term is $a_{n+1} = \frac{r^{n+1}}{(n+1)!}$.

iii) Thus combining these two we get the consecutive ratio as:
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{r^{n+1}}{(n+1)!}}{\frac{r^n}{n!}}\right| = \left|\frac{r}{n+1}\right|.$$

iv) As r is a positive constant, the limit doesn't mind it. So when taking the limit of the consecutive ratio we get
$$\left|\frac{r}{n+1}\right| \to 0,\ n \to \infty.$$
And so, as $0 \in [0,1[$, we that our series converges after applying the ratio test.

Remark. It is important to note that this problem can also be solved using the root test.

We have that the n^{th} root of the general term is
$$\sqrt[n]{\left|\frac{r^n}{n!}\right|} = \frac{r}{\sqrt[n]{n!}}$$
and that limit will be analyzed on its own.

We will use the fact that the exponential and logarithm functions are inverses of one another:
$$\sqrt[n]{n!} = e^{\log(\sqrt[n]{n!})} = e^{\frac{1}{n}\log(n!)}.$$
The quantity $\log(n!)$ can be managed using a result called *Stirling's Approximation*. This states that for large values of n it holds that $\log(n!) \approx n\log(n)$. So using this we have that
$$\frac{1}{n}\log(n!) \approx \frac{1}{n}(n\log(n)) = \log(n).$$
As n grows, $\log(n) \to \infty$ and so $\frac{1}{n}\log(n!) \to \infty$. In this sense, $\sqrt[n]{n!} \to \infty$ and so the expression $\frac{r}{\sqrt[n]{n!}}$ goes to zero.

Remark. It is *not imperative* to know the Stirling Approximation, I've used it here for the sake of argument to prove that the root test also works.

Root Test

Example 3. Consider the series $\sum_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$, we will study its convergence using the root test:

i) The general term is $a_n = \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$.

ii) Thus the n^{th} root is
$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{n^2}{\left(2+\frac{1}{n}\right)^n}\right|} = \frac{n^{\frac{2}{n}}}{2+\frac{1}{n}}.$$

iii) We can take the limit of the n^{th} root as follows:
$$\lim_{n\to\infty} \frac{n^{\frac{2}{n}}}{2+\frac{1}{n}} = \frac{\lim_{n\to\infty}(n^{\frac{2}{n}})}{2+\lim_{n\to\infty}\frac{1}{n}} = \frac{1}{2+0} = \frac{1}{2}.$$
And so, as $\frac{1}{2} \in [0,1[$, we have that by the root test, our series converges.

Remark. Here we have used the fact that $\sqrt[n]{n} \to 1$ as n grows. This can be proven as follows:
$$\sqrt[n]{n} = e^{\log(\sqrt[n]{n})} = e^{\frac{1}{n}\log(n)}.$$
So taking the limit of the exponential now lets us analyze the limit on the inside:
$$\lim_{n\to\infty} \frac{\log(n)}{n} = \frac{\infty}{\infty}.$$
As this is an indeterminate form we pass the discrete variable to a continuous one and apply L'Hôpital's rule:
$$\frac{\log(x)}{x} \xrightarrow{L'H} \frac{\frac{1}{x}}{1} = \frac{1}{x} \to 0.$$
So this means that in discrete variables $\frac{\log(n)}{n} \to 0$. In total the whole expression evaluates to
$$\lim_{n\to\infty} e^{\frac{1}{n}\log(n)} = e^0 = 1 \Rightarrow \sqrt[n]{n} \to 1.$$

Exercises

These exercises were chosen with *malice prépensée*; two series which look quite similar might require the same test (or might not). [*Hints and procedures to this problems will be given upon request.*]

Study the convergence of the following series:

i) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(kn)}{n^2}$, where k is a positive integer.

ii) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt[3]{n^2-1}}$.

iii) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{\sqrt[3]{n^2+1}}$.

iv) $\displaystyle\sum_{n=0}^{\infty}\frac{n^k}{n!}$, where k is a positive integer..

v) $\displaystyle\sum_{n=0}^{\infty}\frac{n^n}{n!}$.

vi) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}$.

vii) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}$.

viii) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n n!}{n^n}$.

ix) $\displaystyle\sum_{n=0}^{\infty}\frac{n^n}{(n+2)^{n+2}}$.

x) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{(2n)^{2n}}$.

xi) $\displaystyle\sum_{n=1}^{\infty}\frac{n^{2n}}{(2n)^n}$.

Exercise 4. If you have successfully solved items vii) and viii) then it should be intuitive that there's a value $2<a<3$ such that the series

$$\sum_{n=1}^{\infty}\frac{a^n n!}{n^n}$$

switches from convergent to divergent.

i) Show that for $a<e$ the series converges.

ii) For $a>e$ show the opposite, the series diverges.

Studying the series $\displaystyle\sum_{n=1}^{\infty}\frac{e^n n!}{n^n}$ requires the use of *Stirling's Approximation* which we will not use. **If you wish to discuss this series let me know**.

Exercise 5. For p,q positive integers consider

$$\sum_{n=0}^{\infty}\frac{(p+1)(p+2)\cdots(p+n)}{(q+1)(q+2)\cdots(q+n)}.$$

i) Suppose $q>p+1$, show that the series converges.

ii) On the other case, $q\leqslant p+1$, show that the series diverges.

Exercise 6. The following is an example of a series for which the root test works but the ratio test fails. Consider the series

$$\sum_{n=0}^{\infty}\frac{1}{3^{n+(-1)^n}}$$

i) Find the limit obtained after applying the ratio test.

ii) Show that with the root test, this series converges.