Exercise 1. Consider the parallel lines $L = \mathbb{V}(x)$ and $L' = \mathbb{V}(x-1)$ in $\mathbb{A}^2$ Let $\overline{L}$ and $\overline{L}'$ denote their projective closures in $\mathbb{P}^2$ (embed $\mathbb{A}^2$ into $\mathbb{P}^2$ via the map $(x, y) \mapsto [x : y : 1]$).

Recall that $\mathbb{P}^2$ is covered by the open charts $U_x = \{ x \neq 0 \}$, $U_y = \{ y \neq 0 \}$ and $U_z = \{ z \neq 0 \}$. Show that when restricted to the affine chart $U_y \simeq \mathbb{A}^2$, the lines $\overline{L}$ and $\overline{L'}$ are no longer parallel.

Answer

Recall that the projective closure of an affine variety V is $V \cup \{ \text{limit points} \}$. In our case we first embed into $\mathbb{P}^2$:

$$\begin{cases} L = \{ [0 : y : 1] : \ y \neq 0 \} = \left\{ \left[0 : 1 : \dfrac{1}{y} \right] : \ y \neq 0 \right\}, \\ L' = \{ [1 : y : 1] : \ y \neq 0 \} = \left\{ \left[\dfrac{1}{y} : 1 : \dfrac{1}{y} \right] : \ y \neq 0 \right\}. \end{cases}$$

By taking limits of y to infinity we get that

$$\begin{cases} \overline{L} = L \cup \{ [0 : 1 : 0] \}, \\ \overline{L}' = L' \cup \{ [0 : 1 : 0] \}. \end{cases}$$

When restricting this varieties to U_y we get the $\mathbb{V}(x)$ and $\mathbb{V}(x - z)$. These two lines will intersect at the origin and at $[0 : 1 : 0]$, it follows that they are not parallel.

Exercise 2. Let $f_i(\mathbf{x}), g_i(\mathbf{x})$ be polynomials for $1 \leqslant i \leqslant m$. Consider the open subset $U \subseteq \mathbb{A}^n$ defined by

$$U = \mathbb{A}^n \backslash \mathbb{V}(g(\mathbf{x})), \ g = \prod_{i=1}^{m} g_i.$$

Define $F(\mathbf{x}) = \left(\frac{f_i(\mathbf{x})}{g_i(\mathbf{x})} \right)_{1 \leqslant i \leqslant m}$. Prove that F is a *morphism of quasi-projective varieties*. (As defined in section 4.1)

Before beginning, let us recall the definition of morphism between quasi-projective varieties:

Definition 1. If $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ are quasi-projective varieties, then a function $F : V \to W$ is a morphism if

$$\forall p \in V \exists (F_j)_{j=0}^m (F_j \in \mathbb{C}[x_0, \ldots, x_n])[(q \mapsto [F_j(q)]_{j=0}^m) \mid_{U_p} \equiv F].$$

In other words, a function $F : V \to W$ is a quasi-projective morphism If

i) V and W are quasi-projective varieties.

ii) For any point $p \in V$, there exists an open neighborhood U_p of p and homogenous polynomials $F_j \in \mathbb{C}[x_0, \ldots, x_n]$ such that the function $\widetilde{F} : V \to \mathbb{P}^m,\ q \mapsto [F_0(q) : \cdots : F_m(q)]$ coincides with F in U_p. This is $\widetilde{F}\big|_{U_p} \equiv F$.

Answer

Let us verify the conditions in the definition, first the function F is defined as a map between quasi-projective varieties. We now want to construct a function whose components are homogenous polynomials that fulfills the second condition on our definition.

For that effect embed F from $\mathbb{A}^n$ into $\mathbb{P}^n$ and multiply everything by g, we get

$$g \cdot F = [g_1 g_2 \cdots g_n : f_1 g_2 \cdots g_n : g_1 f_2 \cdots g_n : \cdots : g_1 g_2 \cdots f_n] = (g) \cup \left(\frac{g f_i}{g_i}\right)_{i=1}^{m}.$$

Now let us homogenize each entry to degree d is sufficiently big (bigger than all the degrees of the entries of $g \cdot F$). We will homogenize by taking using the map

$$\varepsilon^{-1}(F) = {}^hF,\ {}^hF(\mathbf{x}, z) = z^d F\left(\frac{1}{z}\mathbf{x}\right).$$

Now call $\widetilde{F} = \left[{}^hg : {}^h\left(\frac{gf_1}{g_1}\right) : \cdots : {}^h\left(\frac{gf_m}{g_m}\right)\right]$, we claim that inside $\mathbb{A}^n \backslash \mathbb{V}(g)$ the action of F and $\widetilde{F}$ is the same.

Note that inside this set, we have can project the image through $z = 1$ and since we are not in the zero locus of g, we can cancel each one to recover our original function F. So, by construction, F coincides with a function $\widetilde{F}$ whose entries are homogenous polynomials. It follows that F is a morphism of quasi-projective varieties.

Exercise 3. Show that $\mathrm{GL}_n(\mathbb{C})$. the set of invertible $[n \times n]$ matrices has the structure of an affine algebraic variety.

Answer

We can initially see that $\mathrm{GL}_n(\mathbb{C}) = \mathbb{A}^{n^2} \backslash \mathbb{V}(\det)$. This means that it is not an affine algebraic variety in our initial sense. However, recall that now an affine variety is a quasi-projective variety which is isomorphic to a closed subset of affine space.

For that effect let us go one dimension higher, then the function

$$F : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{A}^{n^2+1},\ A \mapsto \left(A, \frac{1}{\det(A)}\right)$$

is well defined on $\mathrm{GL}_n(\mathbb{C})$. If z is our new coordinate, then the polynomial $z\det(A) - 1$ vanishes on $(\mathrm{GL}_n(\mathbb{C})) \subseteq \mathbb{A}^{n^2+1}$. This set is precisely the affine algebraic variety $\mathbb{V}(z\det - 1)$ and so $\mathrm{GL}_n(\mathbb{C}) \simeq \mathbb{V}(z\det - 1)$. We conclude that $\mathrm{GL}_n(\mathbb{C})$ is an affine algebraic variety under the new sense.

Exercise 4. If $U = \mathbb{A}^2 \backslash \{ (0,0), (1,1) \}$, find a basis of U consisting of affine varieties.

Answer

Originally my thought was to take the varieties $\mathbb{V}(x-y)$, $\mathbb{V}(x+y)$ and $\mathbb{V}(x+y-1)$ and then, their complements. However we reach an impasse because $\mathbb{A}^2\backslash\mathbb{V}(x+y)$ contains $(1,1)$, and $\mathbb{A}^2\backslash V(x+y-1)$ contains the origin.
To fix this issue we will take another polynomial instead of the last two. Let us consider $\mathbb{V}(y-x^2)$, then

$$\{ (0,0), (1,1) \} = \mathbb{V}(x-y) \cup \mathbb{V}(y-x^2).$$

We claim that a basis for U made up of affine varieties is $\mathbb{A}^2\backslash\mathbb{V}(x-y)$ and $\mathbb{A}^2\backslash\mathbb{V}(y-x^2)$. FINISH

Exercise 5. Couldn't get to it :(