DN: Doctoral Notebook

This is my doctoral notebook where I will add clean information regarding whatever I'm learning about at the moment. It should serve as a starting point for writing. ¿Writing what? You may ask, I don't know.

Contents

Chapter 1

A Study of *The Green Book* and the Moduli of Curves

1.1 Introduction and Prologue of *The Green Book*

The main objective of the green book is to prove the formula for the number N_d of rational curves of degree d passing through $3d-1$ points in general position in $\mathbb{P}^1_{\mathbb{C}}$. Let's begin by unwrapping some concepts:

Definition 1.1.1. A projective curve $\mathcal{C}$ is the zero locus of points in $\mathbb{P}^2_k$ which satisfy a homogeneous polynomial equation. Formally, for a homogeneous polynomial $f \in k[X, Y, Z]$, the projective curve determined by f is

$$V(f) = \{\, p \in \mathbb{P}^2_k : \ f(p) = 0 \,\}.$$

If f has degree d, then the curve $\mathcal{C}$ is said to be a curve of degree d.

Example 1.1.2. Consider the polynomial

$$f(X, Y, Z) = X - Y - Z.$$

Inside the affine plane $\{\, Z = 1 \,\}$, this contains all the points of the form $(X : X - 1 : 1)$. This is the line $y = x - 1$ in $\mathbb{A}^2$. But it also contains the point at infinity $(1 : 1 : 0)$. The degree 1 curve being described here is a projective line.

Example 1.1.3. The degree 2 curve described by the equation $XY - Z^2 = 0$ is an affine hyperbola containing two points at infinity $(1 : 0 : 0)$ and $(0 : 1 : 0)$.

Definition 1.1.4. A parametrization of a curve $\mathcal{C}$ is a generically injective function

$$\phi : \mathbb{P}^1_k \to \mathbb{P}^2_k,\ (S:T) \mapsto (P(S:T), Q(S:T), R(S:T)), \quad P, Q, R \in k[S,T]_h.$$

A projective plane curve admitting a parametrization is called a rational curve.

Example 1.1.5. The line $X - Y - Z = 0$ can be parametrized with $\phi(S:T) = (S, T, S - T)$. ¿Is the other curve rational?

Example 1.1.6. Degree d curves with a $d-1$-tuple point are rational. As they can be parametrized by a line passing through the singular point.

The dimension of maps from $\mathbb{P}^1$ to $\mathbb{P}^2$ of degree d

The number $3d - 1$ sounds like an arbitrary number. It certainly did to me at least; this number corresponds to the dimension of the space of maps from $\mathbb{P}^1$ to $\mathbb{P}^2$ of degree d. There's this very important question,

¿which vector space is the space of maps from $\mathbb{P}^1$ to $\mathbb{P}^2$ of degree d?

Proposition 1.1.7. *The aforementioned space has dimension $3d - 1$.*

Proof

A map $F : \mathbb{P}^1 \to \mathbb{P}^2$ is defined via homogeneous, degree d polynomials. This means that

$$F(s:t) = (X:Y:Z) = (F_1(s:t), F_2(s:t), F_3(s:t)),$$

where each F_i is a homogeneous degree d polynomial. Explicitly we may write

$$F_j(s:t) = \sum_{i=0}^{d} a_i s^{d-i} t^i = a_0 s^d + a_1 s^{d-1} t + \cdots + a_{d-1} s t^{d-1} + a_d t^d$$

which allows us to see that every F_j has $d+1$ degrees of freedom. But we have to take of changes in the input and output spaces:

- 3 dimensions off for $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$.
- 1 dimension off for projective quotients: $(X:Y:Z) = \lambda(X:Y:Z)$.

This leaves us with $3d + 3 - 3 - 1 = 3d - 1$ dimensions.

There's another way to prove this by counting the general number of degree d curves and then making sure they are rational. For this we need the genus-degree formula.

Proposition 1.1.8. *A projective curve of degree d has genus $\binom{d-1}{2}$.*

The proof of the genus-degree formula will be written down at a later point when we have to talk about Bézout's theorem. For now, the second proof of the dimension question:

Proof

Consider a general degree d curve defined by a homogeneous polynomial F. Such a polynomial can be written as a combination of monomials $X^aY^bZ^c$ where $a+b+c=d$. So to count the number of monomials, we must find the number of triples (a,b,c) of non-negative integers whose sum is d. This is precisely

$$\left(\!\!\binom{3}{d}\!\!\right) = \binom{3+d-1}{d} = \binom{d+2}{d} = \binom{d+2}{2},$$

and we have to take off 1 dimension due to projective quotients.
Hold on, how did we reduce dimension by removing arithmethic genus? But in essence what happens is that

$$\binom{d+2}{2} - 1 - \binom{d-1}{2} = (d+1) + d + (d-1) - 1 = 3d - 1.$$

Remark 1.1.9. Recall $\left(\!\binom{n}{k}\!\right)$ is the number of ways that I can distribute k cookies amongst n friends.

The whole idea is to use the moduli space of maps from $\mathbb{P}^1$ to $\mathbb{P}^r$, $\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^r, d)$, to show the formula. Isomorphism classes inside this set look like classes of bundles. And the formula is derived from intersection theory of this space.

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Bibliography

[1] Joachim Kock and Israel Vainsencher. *An Invitation to Quantum Cohomology: Kontsevich's Formula for Rational Plane Curves*. Progress in Mathematics. Birkhäuser Boston, 2006.