Exercise 1. Let $A : V \to W$ be a linear map between vector spaces.

(a) Show that the induced map $\bigwedge^k(V) \to \bigwedge^k(W)$ is well-defined by

$$v_1 \wedge \ldots \wedge v_k \mapsto Av_1 \wedge \ldots \wedge Av_k$$

(extending linearly to sums).

(b) Show that the map $A^* : W^* \to V^*$ defined by $(A^*(\eta))(v) := \eta(A(v))$ determines a map $\bigwedge^k(W^*) \to \bigwedge^k(V^*)$.

(c) Show that, if V is an n-dimensional vector space, then the map $\bigwedge^n(V) \to \bigwedge^n(V)$ is multiplication by $\det A$.

Answer

To prove well-definedness of a map, it suffices to take two representatives of the same class and see that they map to the same place.

(a) Consider then, without loss of generality,

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = -(v_2 \wedge v_1 \wedge \cdots \wedge v_k).$$

This second element we can reinterpret as

$$(-v_2) \wedge v_1 \wedge \cdots \wedge v_k.$$

Applying $\bigwedge^k(A)$ to this we get

$$\begin{cases} Av_1 \wedge Av_2 \wedge \cdots \wedge Av_k \\ A(-v_2) \wedge Av_1 \wedge \cdots \wedge Av_k \end{cases}$$

and using the fact that A is linear we get

$$\begin{aligned} A(-v_2) \wedge Av_1 \wedge \cdots \wedge Av_k &= -(Av_2 \wedge Av_1 \wedge \cdots \wedge Av_k) \\ &= Av_1 \wedge Av_2 \wedge \cdots \wedge Av_k \end{aligned}$$

and this is the desired representation of the image. FINISH MULTILINEAR This allows to see that $\bigwedge^k(A)$ is well-defined.

(b) The map A^* does indeed define a map from the exterior powers, namely $\bigwedge^k(A^*)$. FINISH

(c) NO IDEA

Exercise 2. Show that the vectors $v_1, \ldots, v_k \in V$ are linearly independent if and only if $v_1 \wedge \ldots \wedge v_k \neq 0$ as an element of $\bigwedge^k(V)$.

Answer

Assume that $\{ v_1, \ldots, v_k \}$ is linearly dependent, then if $\{ v_1, \ldots, v_\ell \}$ is a maximally independent set, we may write any v_i with $\ell < i \leqslant k$ as a linear combination of $\{ v_1, \ldots, v_l \}$.
This means that

$$\begin{aligned} v_1 \wedge \cdots \wedge v_k &= v_1 \wedge \cdots \wedge v_{\ell+1} \wedge \cdots \wedge v_k \\ &= v_1 \wedge \cdots \wedge \sum_{i=1}^{k} c_i v_i \wedge \cdots \wedge v_k \\ &= \sum_{i=1}^{k} c_i (v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k) \end{aligned}$$

and all the summands will be zero as we will find repeated $v_i's$ in each term.

Exercise 3. We say that an element of $\bigwedge^k(V)$ is *decomposable* if it can be written as $v_1 \wedge \ldots \wedge v_k$.

(a) Suppose $v, w, x, y \in V$. Find necessary and sufficient conditions for $v \wedge w + x \wedge y \in \bigwedge^2(V)$ to be decomposable.

(b) Show that $\omega \in \bigwedge^2(\mathbb{R}^4)$ is decomposable if and only if $\omega \wedge \omega = 0$.

Exercise 4. Let V be an n-dimensional inner product space. We can extend the inner product from V to all of $\bigwedge(V)$ by setting the inner product of homogeneous elements of different degrees equal to zero and by letting

$$\langle w_1 \wedge \ldots \wedge w_k, v_1 \wedge \ldots \wedge v_k \rangle = \det\left(\langle w_i, v_j \rangle\right)_{i,j}$$

and extending bilinearly.

Since $\bigwedge^n(V)$ is a one-dimensional real vector space, $\bigwedge^n(V) - \{0\}$ has two components. An *orientation* on V is a choice of component of $\bigwedge^n(V) - \{0\}$. If V is an oriented inner product space, then there is a linear map $\star : \bigwedge(V) \to \bigwedge(V)$ called the star map, which is defined by requiring that for any orthonormal basis $e_1, \ldots, e_n$ for V,

$$\star(1) = \pm e_1 \wedge \ldots \wedge e_n, \qquad \star\left(e_1 \wedge \ldots \wedge e_n\right) = \pm 1,$$
$$\star(e_1 \wedge \ldots \wedge e_k) = \pm e_{k+1} \wedge \ldots \wedge e_n,$$

where in each case we take "$+$" if $e_1 \wedge \ldots \wedge e_n$ is in the preferred component of $\bigwedge^n(V)$ and we take "$-$" otherwise. Notice that $\star : \bigwedge^k(V) \to \bigwedge^{n-k}(V)$.

(a) Prove that if $e_1, \ldots, e_n$ is an orthonormal basis for V, then the $e_{i_1} \wedge \ldots \wedge e_{i_k}$ with $1 \leqslant i_1 < \ldots < i_k \leqslant n$ and $1 \leqslant k \leqslant n$ give an orthonormal basis for $\bigwedge(V)$.

(b) Prove that, as a map $\bigwedge^k(V) \to \bigwedge^k(V)$, $\star\star = (-1)^{k(n-k)}$.

(c) Prove that, for $\omega, \eta \in \bigwedge^k(V)$, their inner product is given by

$$\langle \omega, \eta \rangle = \star(\omega \wedge \star\eta) = \star(\eta \wedge \star\omega).$$

Answer

(a) Observe that each collection of wedges

$$\{\, e_{i_1} \wedge \cdots \wedge e_{i_k},\ i_j \in [n] \,\}$$

for a fixed k, spans each exterior power of V. Thus the whole collection spans the whole exterior algebra. To prove linear independence between two elements it suffices to show that they are orthogonal. Suppose I, J are ordered k-tuples so that

$$\langle \wedge e_I | \wedge e_J \rangle = \det\Big(\langle e_i | e_j \rangle_{i,j} \Big).$$

Exercise 5. Let M^n be a closed manifold (i.e., a compact manifold without boundary) and let $\omega \in \Omega^1(M)$ so that $\omega_p \neq 0$ for all $p \in M$ (i.e., for all p, there exists $v \in T_pM$ so that $\omega_p(v) \neq 0$). Show that ω is not exact.