Exercise 1 (2.1.1)**.** Suppose R is a ring, prove the following:

(a) Every maximal ideal is prime.
(b) Every prime ideal is radical.
(c) If $I \trianglelefteq R$, then $\sqrt{I} \trianglelefteq R$.

Answer

i) Let $\mathfrak{m} \trianglelefteq R$, maximal, and $r, s \in R$ with $rs \in \mathfrak{m}$. Aiming for a contradiction let us suppose that neither r nor s lie inside of $\mathfrak{m}$.

Since $r \notin \mathfrak{m}$, then the ideal generated by r and $\mathfrak{m}$ is the whole ring R. This means that $ar + m_1 = 1$ for $a \in R$ and $m_1 \in \mathfrak{m}$. Likewise for some $b \in R$ and $m_2 \in \mathfrak{m}$ it follows that $bs + m_2 = 1$. Now

$$1 = (ar + m_1)(bs + m_2) = (ab)rs + arm_2 + bsm_1 + m_1m_2$$

and this last expression is a combination of elements in $\mathfrak{m}$. It follows that $1 \in \mathfrak{m}$, but this means that $R = \mathfrak{m}$. This is impossible, so it must follow that $r \in \mathfrak{m}$ or $s \in \mathfrak{m}$.

ii) Suppose $\mathfrak{p} \trianglelefteq R$ is prime. We want to show that for any $p \in \sqrt{\mathfrak{p}}$, $p \in \mathfrak{p}$. As $p \in \sqrt{\mathfrak{p}}$, then $p^n \in \mathfrak{p}$. We will prove that $p^n \in \mathfrak{p} \Rightarrow p \in \mathfrak{p}$.

By induction, our base case is $n = 1$, but there is nothing to prove there. So let us suppose that $p^{n+1} \in \mathfrak{p}$. Then

$$p^{n+1} = pp^n \in \mathfrak{p} \Rightarrow (p \in \mathfrak{p}) \vee (p^n \in \mathfrak{p}).$$

If the first statement holds, we are done. If the second one holds, we are done by induction hypothesis.

iii) First, let us show that the radical is non-empty. Since $0 \in I$, then $0 = 0^n \in I$ and it follows that $0 \in I$.

Suppose now that $x, y \in \sqrt{I}$, we will show that $x + y, xy \in \sqrt{I}$, thus proving that $\sqrt{I}$ is a subring. Our hypothesis tells us that $x^m, y^n \in I$ for some $m, n \in \mathbb{N}$. In this case we have that

$$(x + y)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k y^{m+n-k},$$

and $x^k y^{m+n-k} \in \sqrt{I}$ because

$$k < n \iff k + (m + n) < n + (m + n) \iff (m + n) < m + 2n - k$$

which means that in case that one of our elements is not inside, then the other one will surely be.

Now $(x^m)^n, (y^n)^m \in I$ and then $(xy)^{mn} \in I$ and so $xy \in \sqrt{I}$.

Finally if $r \in R$ and $x \in \sqrt{I}$, then $rx^m \in I$. It follows that $(rx)^m \in I$ and so $rx \in \sqrt{I}$, proving that $\sqrt{I}$ is absorbent. We conclude that $\sqrt{I}$ is an ideal of R whenever I is.

Exercise 2 (2.1.2)**.** Suppose R is a ring, prove the following:

(a) $\mathfrak{m}$ is maximal $\iff$ ${}^{R}/_{\mathfrak{m}}$ is a field.
(b) $\mathfrak{p}$ is prime $\iff$ ${}^{R}/_{\mathfrak{p}}$ is an integral domain.

Answer

(a) $(\Rightarrow)$ If $\mathfrak{m}$ is a proper maximal ideal, take $r \in R \backslash \mathfrak{m}$. Then the ideal generated by r and $\mathfrak{m}$ is the whole ring R. It follows that for some $a \in R$ and $m \in \mathfrak{m}$, $ar + m = 1$. If we translate this expression to the quotient ring we obtain

$$ar + m \equiv 1 \bmod \mathfrak{m} \Rightarrow ar \equiv 1 \bmod \mathfrak{m}.$$

Since r was arbitrary, we have found an inverse a for any element r inside the quotient ring. Since ${}^{R}/_{\mathfrak{m}}$ is already a commutative ring with identity, and now we have inverses, it follows that ${}^{R}/_{\mathfrak{m}}$ is a field.

$(\Leftarrow)$ On the other hand suppose ${}^{R}/_{\mathfrak{m}}$ is a field. Let I be an ideal of R which properly contains $\mathfrak{m}$. If $r \in I \backslash \mathfrak{m}$, then there exists $s \in R$ such that $rs \equiv 1 \bmod \mathfrak{m}$. This means that

$$rs - 1 \in \mathfrak{m} \subsetneq I \Rightarrow rs - (rs - 1) \in I \Rightarrow 1 \in I \Rightarrow I = R.$$

Since I is arbitrary, it follows that no proper ideal besides the whole ring contains $\mathfrak{m}$. This means that $\mathfrak{m}$ is maximal.

(b) $(\Rightarrow)$ If $\mathfrak{p}$ is a prime ideal, suppose $rs \equiv 0 \bmod \mathfrak{p}$. This means that $rs \in \mathfrak{p}$, from which follows that $r \in \mathfrak{p}$ or $s \in \mathfrak{p}$ because $\mathfrak{p}$ is a prime ideal. In any case this means that

$$r \equiv 0 \bmod \mathfrak{p} \quad \vee\ s \equiv 0 \bmod \mathfrak{p}$$

and therefore ${}^{R}/_{\mathfrak{p}}$ has no zero divisors.

($\Leftarrow$) The opposite direction is quite similar. Consider $rs \in \mathfrak{p}$ for some $r, s \in R$. Then

$$rs \in \mathfrak{p} \Rightarrow rs \equiv 0 \bmod \mathfrak{p} \Rightarrow (r \equiv 0 \bmod \mathfrak{p}) \vee (s \equiv 0 \bmod \mathfrak{p}) \Rightarrow (r \in \mathfrak{p}) \vee (s \in \mathfrak{p})$$

where the second implication follows from the fact that ${}^{R}/_{\mathfrak{p}}$ has no zero divisors. We conclude that $\mathfrak{p}$ is prime.

Exercise 3. Suppose $f : R \to S$ is a ring homomorphism and $\mathfrak{q} \trianglelefteq S$ is a prime ideal. Show that $f^{-1}[\mathfrak{q}] \trianglelefteq R$ is a prime ideal.

Answer

We will first show that ring homomorphisms take ideals back into ideals, and then that they take primes back into primes.
Suppose $r_1, r_2 \in f^{-1}[\mathfrak{q}]$. We want to see that this an absorbent subring. Our hypothesis tells us that

$$\begin{aligned} f(r_1), f(r_2) \in \mathfrak{q} \Rightarrow & f(r_1) + f(r_2), f(r_1)f(r_2) \in \mathfrak{q} \\ \Rightarrow & f(r_1 + r_2), f(r_1 r_2) \in \mathfrak{q} \\ \Rightarrow & r_1 + r_2, r_1 r_2 \in f^{-1}[\mathfrak{q}]. \end{aligned}$$

This lets us conclude that $f^{-1}[\mathfrak{q}]$ is a subring of R. To prove it is absorbent, suppose that $r \in R$ and $p \in f^{-1}[\mathfrak{q}]$. This means that $f(r) \in S$ and $f(p) \in \mathfrak{q}$, and since $\mathfrak{q}$ is a prime ideal in S, it follows that $f(r)f(p) \in \mathfrak{q}$. We conclude that $rp \in f^{-1}[\mathfrak{q}]$, and thus, this set is an ideal.
Let us now consider r_1, r_2 as before, but now with the hypothesis that $r_1 r_2 \in f^{-1}[\mathfrak{q}]$. This means that

$$\begin{aligned} f(r_1)f(r_2) = f(r_1 r_2) \in \mathfrak{q} \Rightarrow & (f(r_1) \in \mathfrak{q}) \vee (f(r_2) \in \mathfrak{q}) \\ \Rightarrow & (r_1 \in f^{-1}[\mathfrak{q}]) \vee (r_1 \in f^{-1}[\mathfrak{q}]). \end{aligned}$$

We conclude that $f^{-1}[\mathfrak{q}]$ is also prime.

Exercise 4 (2.3.3)**.** Show that if $I \trianglelefteq \mathbb{C}[\mathrm{x}]$ is radical, then $I = \bigcap\limits_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}$.

Answer

We have the following:

$$\bigcap_{\substack{\mathfrak{m}\supseteq I\\ \mathfrak{m}\text{ maximal}}} \mathfrak{m} = \sqrt{\bigcap_{\substack{\mathfrak{m}\supseteq I\\ \mathfrak{m}\text{ maximal}}} \mathfrak{m}} = I\left(V\left(\bigcap_{\substack{\mathfrak{m}\supseteq I\\ \mathfrak{m}\text{ maximal}}} \mathfrak{m}\right)\right) = I\left(\bigcup_{\substack{\mathfrak{m}\supseteq I\\ \mathfrak{m}\text{ maximal}}} V(\mathfrak{m})\right)$$

where we have applied the Nullstellensatz on the second-to-last equality. Since the variety associated to a maximal ideal corresponds to a point, it follows that

$$I\left(\bigcup_{\substack{\mathfrak{m}\supseteq I\\ \mathfrak{m}\text{ maximal}}} V(\mathfrak{m})\right) = I\left(\bigcup_{\mathbf{a}\in V(I)} \{a\}\right) = I[V(I)] = \sqrt{I} = I.$$

Once again in the second-to-last equality we have applied the Nullstellensatz.

We have used a couple of facts which we will prove as a lemma:

Lemma 1. *The following facts are true for any ideals I, J in a ring R:*

(*a*) $\bigcap_{\alpha\in\mathcal{A}} \mathfrak{m}_\alpha$ *is a radical ideal.*

(*b*) $V\left(\bigcap_{\alpha\in\mathcal{A}} I_\alpha\right) = \bigcup_{\alpha\in\mathcal{A}} V(I_\alpha)$.

Proof

(a) The left to right inclusion is immediate. On the other hand

$$x \in \sqrt{\bigcap_{\alpha\in\mathcal{A}} \mathfrak{m}_\alpha} \Rightarrow x^n \in \bigcap_{\alpha\in\mathcal{A}} \mathfrak{m}_\alpha \Rightarrow \forall\alpha(x^n \in \mathfrak{m}_\alpha) \Rightarrow \forall\alpha(x \in \mathfrak{m}_\alpha) \Rightarrow x \in \bigcap_{\alpha\in\mathcal{A}} \mathfrak{m}_\alpha$$

(b)

Exercise 5. Prove that the coordinate ring of an affine algebraic variety is:

i) reduced;
ii) fin. gen. as $\mathbb{C}$-algebra;
iii) Noetherian.

Answer

Recall the for a variety V, its coordinate ring is ${}^{\mathbb{C}[\mathbf{x}]}/_{I(V)}$. The ideal $I(V)$ is radical and therefore ${}^{\mathbb{C}[\mathbf{x}]}/_{I(V)}$ is reduced.
The finite set of generators are the polynomials $x_i \bmod I(V)$.
Finally any ideal in ${}^{\mathbb{C}[\mathbf{x}]}/_{I(V)}$ is finitely generated, since it has the form ${}^{I}/_{I(V)}$ with $I(V) \subseteq I \subseteq \mathbb{C}[\mathbf{x}]$ and this is a Noetherian ring.

Terminar con lema cociente por radical es reducido