Exercise 1. (Exercise 3.12.11) Show that

$$\mathcal{F}\ell(d_1, \ldots, d_k) \cong \mathrm{O}(n)/(\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k)),$$

where $n_1 = d_1$ and $n_i = d_i - d_{i-1}$ for $i = 2, \ldots, k$. (In other words, the n_i are the jumps in dimension as we go up the flag.)

Exercise 2. Let M be a manifold with an affine connection ∇. Suppose $\alpha : I \to M$ is a constant curve; that is, $\alpha(t) = p$ for all $t \in I$. Let V be a vector field along α, meaning that $V(t) \in T_{\alpha(t)}M = T_pM$ just gives a curve in the tangent space T_pM. Show that $\frac{DV}{dt} = V'(t)$; that is, the covariant derivative agrees with the usual derivative in this case, regardless of what ∇ is.

Exercise 3. (Exercise 4.3.4) Show that an affine connection ∇ is compatible with a Riemannian metric g on M if and only if, for any vector fields V and W along a smooth curve $\alpha : I \to M$, we have

$$\left.\frac{d}{dt}\right|_{t=t_0} g_{\alpha(t)}(V(t), W(t)) = g_{\alpha(t_0)}\left(\frac{DV}{dt}, W\right) + g_{\alpha(t_0)}\left(V, \frac{DW}{dt}\right).$$

In other words, for compatible connections we can use the usual product rule to differentiate the inner product.

Answer

Let us suppose first that ∇ is compatible with g. If α is a curve, we may take an orthonormal basis of $T_{\alpha(t_0)}M$:

$$\{\, u_1(t_0), \ldots, u_n(t_0) \,\}.$$

As ∇ is compatible with g, we may parallel-transport this basis throughout all the curve α. This means that for any $t \in I$,

$$\langle u_1(t), \ldots, u_n(t) \rangle = T_{\alpha(t)}M.$$

Now, our vector fields V, W may be expressed as linear combinations of these basic elements in the following way:

$$\begin{cases} V(t) = \displaystyle\sum_{k=1}^{n} \alpha_k u_k(t) \\ W(t) = \displaystyle\sum_{k=1}^{n} \beta_k u_k(t) \end{cases} \Rightarrow \begin{cases} \dfrac{DV}{dt} = \displaystyle\sum_{k=1}^{n} \alpha_k' u_k(t) \\ \dfrac{DW}{dt} = \displaystyle\sum_{k=1}^{n} \beta_k' u_k(t) \end{cases}$$

where α_k, β_k are smooth functions. Now if we compute the quantity of the left, we have that

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=t_0} g_{\alpha(t)}(V(t), W(t)) \\
&= \sum_{k=1}^{n}\sum_{\ell=1}^{n} \left.\frac{d}{dt}\right|_{t=t_0} \alpha_k\beta_\ell g_{\alpha(t)}(u_k(t), u_\ell(t)) \\
&= \sum_{k=1}^{n} \left.\frac{d}{dt}\right|_{t=t_0} \alpha_k\beta_k \\
&= \sum_{k=1}^{n} \left.\frac{\mathrm{d}\alpha_k}{\mathrm{d}t}\right|_{t=t_0} \beta_k + \sum_{k=1}^{n} \alpha_k \left.\frac{\mathrm{d}\beta_k}{\mathrm{d}t}\right|_{t=t_0}
\end{aligned}$$

and then readding indices by multiplying $\delta_{k\ell}$ and a sum through ℓ we recover the final expression:

$$\sum_{\ell=1}^{n}\sum_{k=1}^{n} \left.\frac{\mathrm{d}\alpha_k}{\mathrm{d}t}\right|_{t=t_0} \beta_k g_{\alpha(t_0)}(u_k(t), u_\ell(t)) + \sum_{\ell=1}^{n}\sum_{k=1}^{n} \alpha_k \left.\frac{\mathrm{d}\beta_k}{\mathrm{d}t}\right|_{t=t_0} g_{\alpha(t_0)}(u_k(t), u_\ell(t)).$$

Condensing everything by linearity we recover

$$g_{\alpha(t_0)}\left(\frac{DV}{dt}, W\right) + g_{\alpha(t_0)}\left(V, \frac{DW}{dt}\right).$$