Applications of Integrals

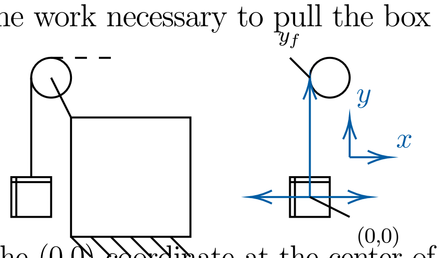
Work

When a force is applied to an object, energy is transferred from or into it. *Work* is transferred energy and *doing work* is the act of transferring that energy.

Definition. If $F = F(x)$ is a force moves an object moved along a path with endpoints x_i and x_f, then the work done by F is

$$\int_{x_i}^{x_f} F(x)\mathrm{d}x.$$

Example 1. Suppose a box weighs 19 N. It is hanging by an infinitesimally thin rope which is 6 m long. We will find the work necessary to pull the box up.



We place the (0,0) coordinate at the center of box assuming its mass is concentrated at this point. Displacement goes from $y_i = 0$ m to $y_f = 6$ m.

As only the gravitational force acts of the box, the work required to move the box from y_i to y_f is equal to:

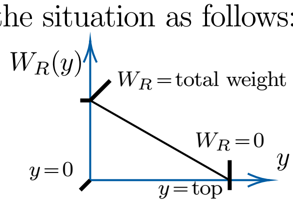
$$W = \int_{y_i=0}^{y_f=6} (19\ \text{N})(\mathrm{d}y\ \text{m}) = \int_0^6 19\mathrm{d}y\ \ \text{Nm} = \underline{114\ \text{J}}.$$

Remark. We will not worry about units, in this example all units were included to illustrate.

Example 2. Suppose the rope is **not weightless**. The *linear density* of the rope is $\rho_R = 5\ \text{Nm}^{-1}$, and it's pulled upwards at a speed of $1\ \text{ms}^{-1}$. **As it's pulled up, mass decreases. Therefore less work is required to pull the rope**. We can summarize:

$$\begin{cases} \text{Weight box} = F_B = 7\ \text{N} \\ \text{Length rope} = \ell_R = 10\ \text{m} \\ \text{Weight rope} = F_R = ??? \end{cases}$$

At the starting point, the whole weight of the rope contributes to the work. And at the end, there's no more rope. Since we are pulling with constant velocity we can model the situation as follows:



The total weight of the rope is

$$W_{R,i} = \rho_R \cdot \ell_R = 50\ \text{N}.$$

And at the end $W_{R,f} = 0$ N. The speed at which the rope moves is $1\ \text{ms}^{-1}$ so the rate of change in speed is the same as in distance.

We can model the weight at any height y using the linear equation

$$W_R(y) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) y + b = \left(\frac{0-50}{10-0}\right) y + 50 = -5y + 50.$$

Notice that this can also be modeled by the equation

$$W_R(y) = \rho_R(\ell_R - y) = 5(10-y).$$

Thus the work is

$$W = \underbrace{\int_0^{10} 7\mathrm{d}y}_{\text{box}} + \underbrace{\int_0^{10} (-5y+50)\mathrm{d}y}_{\text{rope}} = 70 + 250 = \underline{320\ \text{J}}.$$

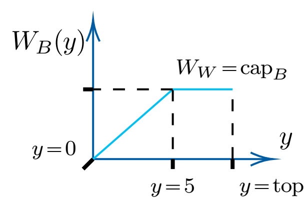
Example 3. Suppose it's now raining at a rate of $2\ \text{Ns}^{-1}$, we are now raising a **open** box at $1\ \text{ms}^{-1}$ with the following conditions:

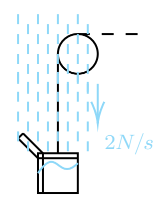
$$\begin{cases} \text{Weight box} = F_B = 12\ \text{N},\ \text{Box capacity} = \text{cap}_B = 10\ \text{N} \\ \text{Length rope} = \ell_R = 8\ \text{m},\ \text{Density rope} = \rho_R = 4\ \text{Nm}^{-1} \end{cases}$$

Since it's **raining**, the box is getting filled with water whose weight will contribute to the work. We have to model this as with the weight of the rope.

Initially, $W_{W,0} = 0$ N since the box is empty, and the box can't hold more than 10 N, then that's $W_{W,f}$.

However the box takes $t_{\text{fill}} = \frac{\text{cap}_B}{v_{\text{fill}}} = \frac{10}{2} = 5$ s to fill. And by that time, the box has been raised just 5 m. The final 3 m will be with the full box of water. We can model as follows:





By using a linear relation we can see that the water's weight is

$$\begin{cases} W_W(y) = \dfrac{10-0}{5-0} y + b = 2y,\ 0 \leqslant y \leqslant 5 \\ W_W(y) = 10,\ 5 \leqslant y \leqslant 8 \end{cases}$$

Finally the work is

$$W = \underbrace{\int_0^8 12\mathrm{d}y}_{\text{box}} + \underbrace{\int_0^8 4(8-y)\mathrm{d}y}_{\text{rope}} + \underbrace{\int_0^5 2y\mathrm{d}y + \int_5^8 10\mathrm{d}y}_{\text{water}} = \underline{279\ \text{J}}.$$

Practice

If the box is full of water and it's leaky ($2\ \text{Ns}^{-1}$) and it wasn't raining, what is the work required to raise it?

Hydrostatics

Mass