Exercise 1. Consider the complex number $w = 2i$. Do the following:

i) Convert w to polar form.
ii) Compute w^2, you may leave your answer in either polar or cartesian form.
iii) Now compute the number $z = \frac{w}{1+w^2}$ and leave it in cartesian form.
iv) What is the real part of z?
v) What is the complex absolute value of z?

i) In polar form w is $2e^{i\pi/2}$. This can be seen by finding $r^2 = 0^2 + 2^2$ which gives us $r = 2$ and the angle of i is the angle of (0,1) which is $\pi/2$.
ii) For w^2 we have either $(2i)^2 = -4$ or $4e^{i\pi} = -4$.
iii) The number z is $\frac{2i}{1-4} = \frac{-2}{3}i = 0 + (-2/3)i$.
iv) The real part of z is 0.
v) The complex absolute value of z is

$$|z|^2 = (-2/3)^2 \Rightarrow |z| = \frac{2}{3}.$$

Exercise 2. Consider the conservation law $ty(t^2 + y^2) = 2$.

i) What is the multivariable function f in the conservation law?
ii) Find the partial derivatives of f with respect to t and y.
iii) Build an exact differential equation by multiplying your results by $\mathrm{d}t$ and $\mathrm{d}y$ and then dividing by $\mathrm{d}t$ the whole equation.
iv) Suppose your exact differential equation comes with an initial condition $y(1) = 1$. Write the initial value problem in this case.
v) Solve the initial value problem you just wrote:
   (a) First multiply out $\mathrm{d}t$ throughout the whole equation.
   (b) Identify the functions multiplied by the differentials as the partial derivatives of a function.
   (c) Integrate either of the partial derivatives to obtain a function f.
   (d) Differentiate it again to obtain a C function with respect to your other variable and compare with your other derivative to find C.
   (e) Apply the initial value in question. If done correctly you should return to the equation in the statement.

i) The function in question is $f(t,y) = ty(t^2 + y^2)$.
ii) Expanding the function we get $t^3y + ty^3$. The partial derivatives are
$$f_t = 3t^2y + y^3, \quad f_y = t^3 + 3ty^2.$$
iii) Multiplying the terms by $\mathrm{d}t$ and $\mathrm{d}y$ we obtain
$$(3t^2y + y^3)\mathrm{d}t + (t^3 + 3ty^2)\mathrm{d}y = 0 \Rightarrow (3t^2y + y^3) + (t^3 + 3ty^2)y' = 0.$$
iv) The initial value problem is the differential equation plus the initial condition:
$$\begin{cases} (3t^2y + y^3) + (t^3 + 3ty^2)y' = 0 \\ y(1) = 1 \end{cases}$$

To solve that initial value problem we turn the equation back into $\mathrm{d}t$, $\mathrm{d}y$ form by multiplying $\mathrm{d}t$ all across the board. We get
$$(3t^2y + y^3)\mathrm{d}t + (t^3 + 3ty^2)\mathrm{d}y = 0.$$
We first integrate $3t^2y + y^3$ with respect to t to obtain:
$$t^3y + y^3t + C(y).$$

This is our f function, so now we differentiate with respect to y so we get:

$$t^3+3y^2t+C'(y)$$

and this must be equal to (t^3+3ty^2). This means that

$$C'(y)=0 \Rightarrow C(y)=c_1.$$

From this we have

$$f(t,y)=t^3y+y^3t+c_1=c_2 \Rightarrow f(t,y)=t^3y+y^3t=c_2-c_1.$$

Baptizing c_2-c_1 as c we may apply the initial condition $y(1)=1$. This means that $t=1$ when $y=1$. So applying this we obtain

$$(1)^3\cdot 1+1\cdot(1)^3=2=c.$$

From this we recover the equation $ty(t^2+y^2)=2$ which was the initial equation in question.