

Exercise 1. Draw the \mathfrak{sl}_3 crystal for weight $(3, 3, 0)$.

Exercise 2. Prove that the elements of the hyperoctahedral group, written in cycle notation as a permutation on $\{\pm 1, \dots, \pm n\}$, has all of its cycles coming in either pairs of the form $(a_1 \dots a_k)(-a_1 \dots -a_k)$, or of the form $(a_1 \dots a_k - a_1 - a_2 \dots - a_k)$.

Exercise 3. Define the Lie algebra \mathfrak{so}_{2n+1} as $\{X : X^T S + SX = 0\}$ where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0_n & I_n \\ 0 & I_n & 0_n \end{pmatrix}$$

and I_n is the $n \times n$ identity matrix and 1 is in the upper left corner. Write down what an arbitrary element X looks like, and using the fact that with respect to this setup the torus is simply the set of diagonal matrices X satisfying these conditions, explain how one obtains the type B root system.

Answer

First observe that the matrix S is a permutation matrix which acts by row permutation when applied as left multiplication. The permutation it applies is a product of disjoint transpositions of the form $(i \ n+i)$ for $i \in [n+1] \setminus \{1\}$.

Exercise 4. What is the dimension of the adjoint representation of \mathfrak{so}_7 ?

Answer

Via the isomorphism $X \mapsto [X, -]$ we have that the dimension of the adjoint representation is the same as $\dim \mathfrak{so}_7$ which is $\binom{7}{2} = 21$.

Exercise 5. Explain why the set of 5th roots of unity in the plane don't form a root system. Which axioms of root systems does it satisfy?

Answer

The axioms we should check are:

- (a) The roots span our vector space.
- (b) The reflections across hyperplanes are still roots.
- (c) Projections onto the span of a single root are an integer multiple or a half-integer multiple of the root.

(d) If α, β are roots such that $\beta = \lambda\alpha$ then $\lambda = \pm 1$.

The first axiom is satisfied as any non-zero complex number spans \mathbb{C} . The last axiom is satisfied vacuously.

The second axiom isn't satisfied as reflections across hyperplanes send the 5th roots to 10th (primitive) roots of unity. Projections are also not integer multiples nor half-integer multiples of other roots.

Exercise 6. Compute the evacuation of the Young tableau below, and then evacuate again, and show you have returned to the starting tableau.

5			
2	7	8	
1	3	4	6

Answer

We switch entries following the rule $k \mapsto n + 1 - k$ and then rotating 180°:

5			
2	7	8	
1	3	4	6

 \longrightarrow

4			
7	2	1	
8	6	5	3

 \longrightarrow

3	5	6	8
×	1	2	7
			4

Here we have already marked the first inner corner we will move. This leads us to

3	5	8	
1	2	6	7
	×		4

 \longrightarrow

3	5	8	
1	2	6	
	×	4	7

 \longrightarrow

3	8		
1	5	6	
×	2	4	7

 \longrightarrow

8			
3	5	6	
1	2	4	7

where every **green** character moved when clearing out the inner corner in the previous step. Redoing the process we obtain the skew tableau

8			
3	5	6	
1	2	4	7

 \longrightarrow

1			
6	4	3	
8	7	5	2

 \longrightarrow

2	5	7	8
×	3	4	6
			1

With the first inner corner marked, we move it out and continue the process:

5	7	8	
2	3	4	6
	×		1

 \longrightarrow

5	7	8	
2	3	4	
	×	1	6

 \longrightarrow

5	7		
2	3	8	
×	1	4	6

 \longrightarrow

5			
2	7	8	
1	3	4	6

As we have returned to our original tableau we conclude that the process is correct

Exercise 7. Compute the Hall-Littlewood polynomial $\tilde{H}_{(2,1,1)}(x; q)$.

Exercise 8. Let $w = w_1 \dots w_n$ be a word of partition content, and suppose $w_1 \neq 1$. Let $w' = w_2 \dots w_n w_1$ be formed by cycling w_1 around to the end of the word. Show that $c(w') = c(w) - 1$ where c is cocharge. This operation is called *cyclage*.

Answer

Observe that it suffices to view this on standard words. This is because we may separate a word into standard subwords and calculate cocharge^a. Consider the subword \tilde{w} of w which contains w_1 in the previous decomposition sense, as w has partition content so does \tilde{w} .

When cycling w_1 to the end of \tilde{w} , cocharge is reduced by 1 as there is a element in \tilde{w} smaller than w_1 which was to the right of w_1 . After cycling, it's to the *left* and so the cocharge labeling drops by one.

^aAh! Inadvertently **you** helped me with this problem as the decomposition idea was written on your thesis!

Exercise 9. Give a counterexample showing that the formula in the above problem does not hold in general when $w_1 = 1$.

Answer

The word 121 has cocharge labeling 000 giving it a cocharge of 0 whereas 211 has cocharge labeling 100 with cocharge 1.