Exercise 1 (Exercise 1)**.** Prove that the tensor product of two Hadamard matrices is a Hadamard matrix.

Answer

Suppose H, K are two Hadamard matrices. We have $HH^\intercal = mI$ and $KK^\intercal = nI$ and we must show that $(H \otimes K)(H \otimes K)^\intercal = mnI$ where the last identity matrix has size $(mn) \times (mn)$.
The product enjoys two properties which are essential for our purpose:

- ⋄ Transposition distributes over the product: $(A \otimes B)^\intercal = A^\intercal \otimes B^\intercal$.
- ⋄ The *mixed-product property*: If A, B, C, D are matrices, then

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

With this in hand we see that

$$(H \otimes K)(H \otimes K)^\intercal = HH^\intercal \otimes KK^\intercal = mnI_m \otimes I_n = mnI_{mn}$$

and thus $H \otimes K$ is Hadamard as desired.

Lemma 1. *Transposition is distributive with respect to the product.*

Proof

Assume A has size $k \times \ell$, then

$$(A \otimes B)^\intercal = \begin{pmatrix} a_{11}B & \dots & a_{1\ell}B \\ \vdots & \ddots & \vdots \\ a_{k1}B & \dots & a_{k\ell}B \end{pmatrix}^\intercal = \begin{pmatrix} a_{11}B^\intercal & \dots & a_{k1}B^\intercal \\ \vdots & \ddots & \vdots \\ a_{1\ell}B^\intercal & \dots & a_{k\ell}B^\intercal \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{k1} \\ \vdots & \ddots & \vdots \\ a_{1\ell} & \dots & a_{k\ell} \end{pmatrix} \otimes B^\intercal.$$

And we may recognize $A^\intercal$ as the last matrix. So the transposition property holds.

Lemma 2. *The mixed product property holds.*

Proof

Assume A has size $k \times \ell$ and C has size $\ell \times m$, then

$$(A \otimes B)(C \otimes D) = \begin{pmatrix} a_{11}B & \dots & a_{1\ell}B \\ \vdots & \ddots & \vdots \\ a_{k1}B & \dots & a_{k\ell}B \end{pmatrix} \begin{pmatrix} c_{11}D & \dots & c_{1m}D \\ \vdots & \ddots & \vdots \\ c_{\ell 1}D & \dots & c_{\ell m}D \end{pmatrix}$$

and multiplying this two matrices we obtain entries of the form

$$\left(\sum_{r=1}^{\ell} a_{ir} c_{rj} \right) BD = (AC)_{ij} BD.$$

Thus we have

$$(A \otimes B)(C \otimes D) = \begin{pmatrix} (AC)_{11}BD & \dots & (AC)_{1m}BD \\ \vdots & \ddots & \vdots \\ (AC)_{k1}BD & \dots & (AC)_{km}BD \end{pmatrix} = (AC) \otimes (BD).$$

Exercise 2 (Exercise 2)**.** Prove that there's only one $2-(7,3,1)$ design up to isomorphism.

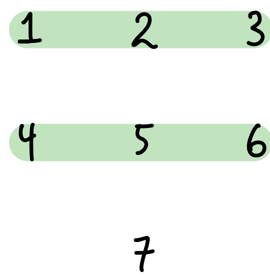
Answer

We know that the Fano plane is an example of a $2-(7,3,1)$ design. Given another $2-(7,3,1)$ design, $(X, \mathcal{B})$, we will find an isomorphism between the Fano plane and our design.
First note that this is a square design. Take 2 blocks B, B', then it must happen that $|B \cap B'| = 1$.

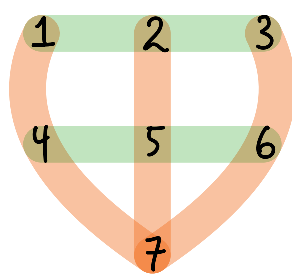
- The intersection can't be larger than 1 because every pair of points is contained in precisely 1 block, not more than 1.
- If two blocks are disjoint then we name their elements

$$B = \{1, 2, 3\}, \quad \text{and} \quad B' = \{4, 5, 6\}$$

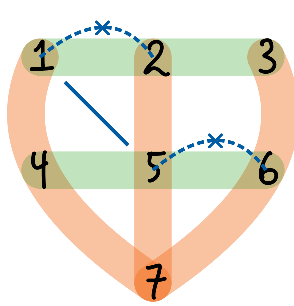
  which leaves 7 out of the mix.



Take two elements from B and B', without losing generality there's one block which contains them $\{1, 4, x\}$. The element x can't come from B as 1 is already paired with 2 and 3 there. It can't also come from B' because there will be two blocks which contain two common elements. It must happen that the new block is $\{1, 4, 7\}$. In the same fashion we can construct blocks $2, 5, 7$ and $3, 6, 7$.



However we've reached an impasse, because we must somehow pair 1 and 5 in a block. We can't add an element from B as $1, 5, x$ will intersect B in two elements. In the same fashion, if $y \in B'$, then $|\{1, 5, y\} \cap B'| = 2$. Finally we can't have $\{1, 5, 7\}$ because $1, 7$ and $5, 7$ will be repeated in two blocks.

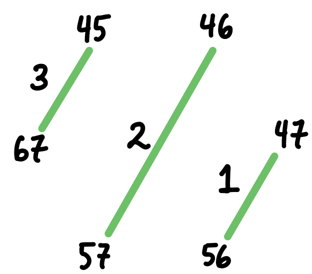


This means that we can't form more than 5 blocks given our constraint. But this contradicts Fisher's inequality as we must have at least as many blocks as vertices and $b = 5 < v = 7$ in this 2-design.

Our assumption that there are two disjoint blocks is false, so it must occur that any two blocks intersect in exactly 1 vertex. Immediately this tells us that $b = v$ and $r = k$ which means that:

- ◇ There are 7 blocks.
- ◇ Every point is contained in exactly 3 blocks.

Consider the block $B = \{1, 2, 3\}$ and then the following graph:



We will build the remaining blocks as follows, take the remaining elements $\{4, 5, 6, 7\}$ and consider the pairs of elements in the set $\{45, 46, 47, 56, 57, 67\}$. By matching disjoint pairs we form the graph on top.
The edges may be named by any arbitrary choice[a] of our elements in B. The blocks will be an edge and one of the vertex endpoints. We get the blocks

$$B, \{1, 4, 7\}, \{1, 5, 6\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 5\}, \quad \text{and} \quad \{3, 6, 7\}.$$

This is a square $2-(7, 3, 1)$ design and we may associate the lines of the Fano plane in the obvious way to the blocks of our design while vertices are also mapped to vertices.

[a] This means that we could also consider the blocks $\{1, 4, 5\}$ and $\{1, 6, 7\}$ for example.

Exercise 3 (Exercise 4)**.** The **complementary design** to a design $\mathcal{D} = (X, \mathcal{B})$ is the pair $\mathcal{D}^c = (X, \mathcal{B}^c)$ where $\mathcal{B}^c = \{X \backslash B : B \in B\}$. Show that if $\mathcal{D}$ is a $1-(v, k, \lambda)$ design then $\mathcal{D}^c$ is a $1-(v, v-k, v\lambda/k - \lambda)$ design.

Answer

We know that in $\mathcal{D}^c$ we have v vertices. Any block is of the form $X\backslash B$ with B having size k, so all the blocks in $\mathcal{D}^c$ have size $v-k$ as desired.
It remains to show that every point is in exactly $\frac{v\lambda}{k} - \lambda$ blocks. Now, let us manipulate this quantity:
Recall r is the number of blocks containing a point, in this case as we have a 1-design, we have that $r = \lambda$, so

$$\frac{v\lambda}{k} = \frac{vr}{k} = \frac{bk}{k} = b, \quad \text{the number of blocks.}$$

So we must show that every point is in $b-r$ blocks, but now this is immediate because any point already on r blocks, is not inside the remaining $b-r$ blocks. But that is what it means to be inside a block in the complementary design. We conclude that $\mathcal{D}^c$ is indeed a $1-(v, v-k, b-r)$ design.

Exercise 4 (Exercise 6)**.** Prove that the edge-complement of a strongly regular graph is strongly regular, and find the new parameters in terms of the previous.

Answer

Suppose G is strongly regular with parameters (n, k, λ, μ). Pick a vertex v and look at its neighbors, there are k of them. Out of the remaining $n-1$ vertices, v is not connected to $n-1-k$ of them.
Now pick another vertex w, if they are not connected then they share μ common neighbors. From the remaining $n-2$ vertices, u, v are only connected to their neighborhoods. Removing all the vertices in the neighborhoods doubly counts the intersection, and we know that intersection has size μ. So in total, v, w are not connected to

$$(n-2) - 2k + \mu \quad \text{vertices together.}$$

In a same fashion if they were connected, the number of shared neighbors is λ so together they wouldn't be connected to $(n-2) - 2k + \lambda$ vertices.
We conclude that if G is strongly regular with parameters (n, k, λ, μ), then G^c has parameters

$$(n, (n-1) - k, (n-2) - 2k + \mu, (n-2) - 2k + \lambda).$$