Chapter 1

Introduction

It is my interest to study Riemmanian manifolds and their classification, furthermore, understanding maps between moduli spaces of this manifolds and projective space. The classification entails quite the journey, and we need to start somewhere. We begin our study with the most basic compact Riemmanian manifold, the…

1.1 The Projective Line

Everything begins as a set[1] and in particular the projective line has an underlying set.

Definition 1.1.1. As a set, the projective line is the set of lines through the origin in $\mathbb{C}^2$:

$$\mathbb{CP}^1 := \{\ell \subseteq \mathbb{C}^2,\, 0 \in \ell\},$$

where ℓ is a line.

Remark 1.1.2. At some point we will begin to abbreaviate $\mathbb{CP}^1$ to $\mathbb{P}^1_{\mathbb{C}}$ and further to $\mathbb{P}^1$ given that in the context it's sufficiently clear that we're working over the complex numbers.

We may identify this set with various other representations which may come in handy depending on the context we're talking about.

Theorem 1.1.3. *There is a bijection between $\mathbb{CP}^1$ and the following sets*:

(a) *The quotient*

$$\mathbb{C}^2 \backslash \{(0,0)\} \Big/ {}_{x \sim \lambda x}, \quad \lambda \neq 0$$

where equivalence classes are in bijection with $\mathbb{C}^*$. This bijection is non-canonical.

(b) *The quotient of the real S^3 by antipodality*:

$$S^3 \Big/ {}_{x \sim -x}.$$

[1] Unless it's a stack, stacks are not sets!

(c) *A quotient of two disjoint copies of* $\mathbb{C}$:

$$(\mathbb{C},x)\amalg(\mathbb{C},y)\Big/ y\sim\frac{1}{x}$$

(d) *The disjoint union of* $\mathbb{C}$ *and a point which we baptize "*∞*"*:

$$\mathbb{C}\amalg\{\infty\}.$$

Proof

Prove this

Observe that there are surjective maps from each of the premilinary sets onto our projective line:

$$\begin{cases} \pi_1 : \mathbb{C}^2\setminus\{(0,0)\}\to\mathbb{CP}^1, \quad \mathbf{z}\mapsto\lambda\mathbf{z}. \\ \pi_2 : S^3\to\mathbb{CP}^1, \quad \mathbf{z}\mapsto\lambda\mathbf{z}. \\ \pi_3 : (\mathbb{C},x)\amalg(\mathbb{C},y)\to\mathbb{CP}^1. \end{cases}$$

Morally, the first two maps are taking a two dimensional complex vector and assigning to it the line through the origin which passes through that vector.

However for the last map, we may imagine the copies of $\mathbb{C}$ as other lines (not through the origin) in $\mathbb{C}^2$. For example the lines $V(y-1)$ and $V(x-1)$ may represent our copies of $(\mathbb{C},x)$ and $(\mathbb{C},y)$ respectively. Then the map takes a point in $(\mathbb{C},x)$ and asks for the line through it. There's only one line not intersecting $V(y-1)$ which is the x axis in $\mathbb{C}^2$, but there's no qualms as $V(x-1)$ does intersect that line and thus we have the desired map π_3.

It is clear that if we take any line through the origin in $\mathbb{C}^2$ then we may find a preimage via any of this maps. This means that they descend as maps from the quotients to $\mathbb{CP}^1$.

Theorem 1.1.4. *The quotient topologies* τ_i *induced by the maps* π_i *on* $\mathbb{CP}^1$ *are equivalent and make* $\mathbb{CP}^1$ *a compact topological space.*

Proof

Prove this

More than just a topological space, $\mathbb{CP}^1$ is a complex analytic manifold

- Write about chaps 4,5,6 Renzo

1.2 Line bundles over $\mathbb{CP}^1$

(a) chapters 7,8,9,10 Renzo

1.3 The Picard group and the Chow ring

1.4 Higher genus Riemann surfaces

Maps between Riemann surfaces

(a) Branch and Ramification points
(b) Riemman-Hurwitz,
(c) Riemman-Roch,
(d) Serre Duality

Chapter 2

Moduli Spaces

2.1 Toy example: quadruplets of points

2.2 Quadruplets along $\mathbb{P}^1$

2.3 The moduli space of curves with n marked points

(a) Stable curves
(b) Examples $\overline{M}_{0,4}$,$\overline{M}_{0,5}$

2.4 The tautological ring inside $A^*(\overline{M})$

(a) ψ, λ classes
(b) Intersection product Examples
(c) Projection formula
(d) String and Dilaton relations
(e) Integral examples

2.5 Moduli space of maps

Chapter 3

Equivariant Cohomology and Localization

3.1 Basics of equivariant cohomology

(a) Borel Construction of Equivariant Cohomology
(b) Examples of point equivariant Cohomology
(c) Equivariant Cohomology of projective space

3.2 Atiyah-Bott Localization

(a) Example of $H_T^*(\mathbb{P}^r)$ through Localization
(b) Toric varieties Euler characteristic via Atiyah-Bott
(c) Hodge integral $\int_{\overline{M}_{0,2}(\mathbb{P}^2,1)} \mathrm{ev}_1^*([1:0:0])\mathrm{ev}_2^*([0:1:0])$ via localization.