Exercise 1 (Exercise 1)**.** Among a family of 20 people, 9 of them like chocolate ice cream, 7 of them like vanilla ice cream, and 2 like both chocolate and vanilla. How many don't like either flavor of ice cream? Explain how the Inclusion-Exclusion principle applies here.

Answer

Call F the set of family members, C the set of people who like chocolate ice cream and V the ones who like vanilla. The set F can be partitioned as

$$F = (C \cup V) \uplus (C \cup V)^c$$

where $(C \cup V)^c$ is the set of family members who dislike both flavors. This is the set we are looking for.
By Inclusion-Exclusion applied to the two sets C, V we have

$$|C \cup V| - |C| - |V| + |C \cap V| = 0 \Rightarrow |C \cup V| = 9 + 7 - 2 = 14.$$

As the whole family has 20 members and 14 of them like either flavor of ice cream, then 6 of them dislike both flavors.
We must use the Inclusion-Exclusion formula to find the size of the union, by summing only $|C|$ and $|V|$ we would be overcounting the the ones who like both flavors.

Exercise 2. Let P be a poset in which every interval $[x, y]$ is finite. Show that, in the incidence algebra $\mathscr{I}(P)$:

(a) f is invertible if and only if $\forall x(f(x,x) \neq 0)$.
(b) $fg = \delta \iff gf = \delta$, this is, inverses are two sided.
(c) If f is invertible then f^{-1} is unique.

Really quickly, recall that the incidence algebra is the set of *interval functions* from P to $\mathbb{C}$. In other words, we can describe $\mathscr{I}(P)$ as

$$\mathscr{I}(P) = \{ f : P^2 \to \mathbb{C} : \ x > y \Rightarrow f(x,y) = 0 \}.$$

Answer

(a) Suppose f is invertible with $fg = \delta$. If $x \in P$:

$$(f \cdot g)(x,x) = f(x,x)g(x,x) = \delta(x,x) = 1.$$

This means that, as complex numbers, $f(x,x)g(x,x) = 1$ thus none can be zero and $f(x,x) = \frac{1}{g(x,x)}$.

On the other hand, suppose $f(x,x) \neq 0$. We will construct an inverse for f inductively using the fact the every interval is finite.

Our base case is $|[x,y]| = 1$, then $x = y$ and $g(x,x) = \frac{1}{f(x,x)}$. Suppose that we have an interval $[x,y]$ of length n and for intervals of length less than n $g(x,y)$ is the inverse of $f(x,y)$. So

$$\begin{aligned}
\delta(x,y) = (fg)(x,y) &\iff 0 = \sum_{x \leqslant z \leqslant y} f(x,z)g(z,y) \\
&\iff 0 = f(x,x)g(x,y) + \sum_{x < z \leqslant y} f(x,z)g(z,y) \\
&\iff -f(x,x)g(x,y) = \sum_{x < z \leqslant y} f(x,z)g(z,y) \\
&\iff g(x,y) = \frac{-1}{f(x,x)} \sum_{x < z \leqslant y} f(x,z)g(z,y)
\end{aligned}$$

Thus it holds that when $f(x,x) \neq 0$, we can solve the previous equation to obtain an expression for the inverse of f. By induction, it follows that f is invertible.