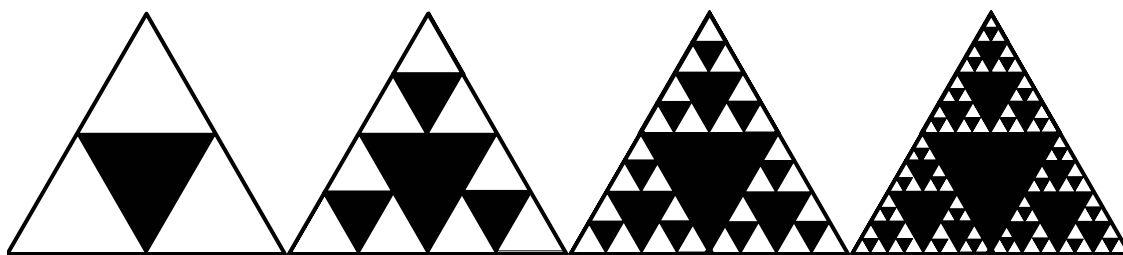


Exercise 1. Consider the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3}}{n^{5r}}$ where r is a real number and do the following:

- i) Identify the general term of the series and simplify it.
- ii) Is our series a p -series? How can you justify that?
- iii) State the convergence result for p series. This means, given ANY p -series $\sum \frac{1}{n^p}$, for what values of p will that converge?
- iv) Determine for which values of r will our series converge. [Hint: The answer to this item is NOT the same as the last item.]

1. The general term is $\frac{\sqrt{n^3}}{n^{5r}}$. We may simplify it to $\frac{1}{n^{5r-3/2}}$.
2. Indeed it's a p -series, as the general term looks like 1 over n to some power, in this case $5r - \frac{3}{2}$.
3. A general p series converges when $p > 1$.
4. We require that $5r - \frac{3}{2} > 1$ so solving for r we get $5r > \frac{5}{2}$ which means that $r > \frac{1}{2}$. So for values of r larger than $\frac{1}{2}$, our series converges.

Exercise 2 (Filling a triangle). Consider an empty triangle of area A which we start filling with smaller triangles. The objective of this question is to determine if we can fill completely the triangle in question.



We start adding a triangle with area $a_1 = \frac{A}{4}$ in the middle, then the second step adds 3 triangles of area $\left(\frac{A/4}{4}\right) = \frac{A}{16}$. So in total we are adding an area of $a_2 = \frac{3A}{16}$.

- i) In the third step, how many triangles do we add? What is the area of each of the new smaller triangles? In total how much area a_3 are we adding in the third step?
- ii) Derive a formula for the area added a_n at the n^{th} step by considering how many triangles are we adding and the area of each of those new triangles.
- iii) As a sequence, is a_n geometric? If it is, what's its initial term and common ratio?
- iv) Consider the series $\sum_{n=1}^{\infty} a_n$, in terms of area, what does this series represent? What do the partial sums represent?
- v) Write an expression for the partial sums of this series.
- vi) Using the information above, determine if we fill up the triangle.

1. In the third step we are adding 9 triangles in total. Each triangle will have as area a fourth of the previous triangle that we added. In this case this is $\frac{A}{64}$, so the total area that we are adding is $\frac{9A}{64}$.
2. In the n^{th} step we are adding 3^{n-1} triangles each with area $\frac{A}{4^n}$. The area added is thus $a_n = \frac{3^{n-1}A}{4^n}$.
3. The sequence a_n is geometric with initial term $c = \frac{A}{4}$ and common ratio $r = \frac{3}{4}$.
4. As a series, the infinite series represents the total area that is covered by the shaded triangles. The partial sums represent the total area at step m .

5. The partial sum is

$$S_m = \sum_{n=1}^m \frac{3^{n-1}A}{4^n} = A \left(\frac{1}{4} + \frac{3}{16} + \frac{9}{16} + \cdots + \frac{3^{m-1}}{4^m} \right)$$

from which we get

$$S_m - \frac{3}{4}S_m = A \left(\frac{1}{4} - \frac{3^m}{4^{m+1}} \right) \Rightarrow S_m = \frac{A \left(\frac{1}{4} - \frac{3^m}{4^{m+1}} \right)}{1 - \frac{3}{4}}.$$

6. The series $\sum_{n=1}^{\infty} \frac{3^{n-1}A}{4^n}$ is a convergent geometric series because $\frac{3}{4} < 1$. The limit value of this series can be found by computing the limit of the partial sums. We see that S_m tends to A as m grows. This means that in the limit, the figure fills out with the shaded triangles because the total area will be the area of the original triangle.