Exercise 1 (7.3.G Vakil)**.** Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. ⟦Hint: Show that this is true when X is affine. Use the fact that morphisms glue, Exercise 7.3.B. (This is even true in the category of locally ringed spaces. You are free to prove it in this generality, but it is easier in the category of schemes.)⟧

Answer

If this was true in the case X was affine, we could use the affine covering of X, (U_i) and apply the same argument on each U_i and glue the isomorphisms together into the big isomorphism.

Exercise 2 (8.1.D Vakil)**.** Verify that the class of open embeddings satisfies property (ii) of 8.1.

More specifically: suppose $i : U \hookrightarrow Z$ is an open embedding, and $\rho : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists and $U \times_Z Y \to Y$ is an open embedding ⟦Hint: I'll even tell you what $U \times_Z Y$ is: $(\rho^{-1}(U),)$⟧

In particular, if $U \hookrightarrow Z$ and $V \hookrightarrow Z$ are open embeddings, $U \times_Z V \simeq U \cap V$: "intersection of open embeddings is fiber product".

Exercise 3 (10.1.A)**.** Use Exercise 7.3.G ($\mathrm{Hom}_{\mathsf{Sch}}(W, \mathrm{Spec}\, A) = \mathrm{Hom}_{\mathsf{Rings}}(A, \mathcal{O}_W(W))$) to show that given ring maps $C \to A$ and $C \to B$,

$$\mathrm{Spec}(A \otimes_C B) \simeq \mathrm{Spec}(A) \times_{\mathrm{Spec}(C)} \mathrm{Spec}(B)$$

(Interpret tensor product as the "fibered coproduct" in the category of rings.) Hence the fibered product of affine schemes exists (in the category of schemes). (This generalizes the fact that the product of affine lines exist, Exercise 7.6.E(a).)

Exercise 4 (7.6.E)**.** In this exercise, $\mathbb{Z}$ may be replaced by any ring.

i) (*Affine n-space represents the functor of n functions.*) Show that the contravariant functor from ($\mathbb{Z}$-)schemes to Sets

$$X \mapsto \{\, (f_1, \ldots, f_n) : \ f_i \in \mathcal{O}_X(X) \,\}$$

is represented by $\mathbb{A}^n_{\mathbb{Z}}$. Show that $\mathbb{A}^1_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \simeq \mathbb{A}^2_{\mathbb{Z}}$, in other words that $\mathbb{A}^2$ satisfies the universal property of $\mathbb{A}^1 \times \mathbb{A}^1$.

ii) (*The functor of invertible functions is representable.*) Show that the contravariant functor from ($\mathbb{Z}$-)schemes to Sets taking X to invertible functions on X is representable by $\operatorname{Spec} \mathbb{Z}[t, \frac{1}{t}]$.