The purpose of the whole exercise 34 is to prove that the Vandermonde determinant is equal to the product of binomials. My initial thought when I saw part (a) was, *"oh, that expression is Vandermonde's determinant, I can just try to relate the tournament to something about powers of monomials."* But then I saw the last part and understood that the objective was not to use the fact that the determinant of the matrix of monomials is what it is, instead we see that *that* matrix is related to our problem and that the product is also related and therefore the matrix's determinant is what it should be. So without further ado:

Exercise 1 (Exercise 3, Stanley 2.34(a))**.** Show that $\sum_T w(T) = \prod_{i,j\in[n]}(x_j - x_i)$ where the sum is taken over all $2^{\binom{n}{2}}$ tournaments on $[n]$.

Answer

As every pair of vertices is joined by exactly one edge and there are no loops, every tournament on $[n]$ has $\binom{n}{2}$ edges. This means that the sum $\sum_T w(T)$ has $2^{\binom{n}{2}}$ terms.
On the right hand side, the product in question has $\binom{n}{2}$ factors of the form $(x_j - x_i)$ and when expanded as a sum of monomials we find $2^{\binom{n}{2}}$ summands. Each of the monomials in the expanded result of the product corresponds to a weight of a tournament.
The reasoning for that conclusion is as follows. Pick two vertices $i < j$, there are two possibilities to pick a weighted edge between these two:

- Either we have $i \to j$ with weight x_j,
- or we have $i \leftarrow j$ with weight $-x_i$.

This determines the choice of monomial in the $(x_j - x_i)$ factor. In the same way, picking either x_j or $-x_i$ in the expansion determines the orientation of the edge between i and j. No tournaments are missing nor we have more than necessary as the amount matches up. Since each of the $2^{\binom{n}{2}}$ possible weights is also counted in $\prod_{i,j\in[n]}(x_j - x_i)$, it holds that

$$\sum_T w(T) = \prod_{i,j\in[n]} (x_j - x_i).$$

Exercise 2 (Exercise 4, Stanley 2.34(b))**.** Show that a non-transitive tournament contains a 3 cycle.

Triangle-free graphs are so *fun*, the first result I saw in graph theory was Mantel's theorem. I still remember that class very vividly, my professor asked us for examples of K_3-free graphs and the best I could come up with was a path graph. Eventually other people came up with bipartite graphs and then he stated, "*The most edges that a K_3-graph can have is $n^2/4$ and the extremal graph is $K_{n/2,n/2}$.*"

It is not the same case for directed graphs but still, this problem is related to triangle-free graphs.

Answer

This statement is equivalent to T is (3-cycle)-free implies T is transitive. The transitive condition is also equivalent to $i \to j,\ j \to k \in E \Rightarrow i \to k \in E$. So suppose we have three vertices $i, j, k \in T$ such that $i \to j$ and $j \to k$ are edges of T.

If it were the case that $k \to i$ was an edge in T, this would contradict the fact that T is (3-cycle)-free, which is not the case. So there's no $k \to i$ edge.

As T is a tournament, there *must* be an edge between i and k, so the only possibility is that the edge $i \to k$ is in T. And so, we have proven that T is transitive as

$$i \to j,\ j \to k \in E \Rightarrow i \to k \in E.$$

Exercise 3 (Exercise 4, Stanley 2.34(c))**.** The relation $T \leftrightarrow T'$ means that T' was obtained by reversing a 3-cycle in T. Show that if $T \leftrightarrow T'$ then $w(T') = -w(T)$.

Answer

Suppose Δijk is the cycle we will be reversing and consider the induced subgraph generated by i, j and k. There are only two possibilities for Δijk, either

$$i \to j \to k \to i \quad \text{or} \quad i \to k \to j \to i.$$

As one of the reverse of the other, we will see that their weights differ by sign. The weight of the first one is $x_j x_k(-x_i)$ and the second one is $x_k(-x_j)(-x_i)$. FINISH