Euler
Characteristics of Toric Varieties via Localization

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Spring, 2025

Abstract

The Euler characteristic is an invariant of manifolds which can be computed as the alternating sum of its Betti numbers. In this project, we approach this calculation by integrating the manifold's Euler class. Atiyah-Bott localization will help us to refine the process.

Our varieties come equipped with a torus action so we would like a cohomology which remembers this structure. This leads to equivariant cohomology, and in our cases, there will loci of our varieties which will remain fixed. Through this analysis, we will achieve our objective to demonstrate that the Euler characteristic of toric varieties depends solely on the number of torus-fixed points they contain.

Keywords: Euler characteristic, Euler class, Betti numbers, toric variety, fixed loci, equivariant cohomology, Atiyah-Bott localization. *MSC classes*: Primary 57S12; Secondary 14F43, 55N91.

1 Premier

This project arises from my interest in localization techniques and equivariant cohomology, particularly in relation to my research on the moduli space of stable maps. Developing a deeper intuition for these concepts through concrete examples will be valuable for my broader studies.

The structure of this project is as follows:

- ◇ Define the Euler characteristic and realize it as the integral of the Euler class of a manifold.
- ◇ Introduce equivariant cohomology and the Atiyah-Bott localization theorem.
- ◇ Apply this theorem to compute the Euler characteristic of toric varieties, including $\mathbb{P}^n$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathrm{Hilb}^n(\mathbb{C}^2)$.

This project aligns with the course by offering an alternative perspective on manifolds, by viewing group actions as another part of their study. Through this approach, we gain a new way to calculate invariants and insight into algebraic geometry.

2 Manifolds and Euler characteristic

Definition 1. For a manifold M, call its i^{th} Betti number

$$b_i = \dim H_i(M),$$

the rank of M's i^{th} homology group. The Euler characteristic of the manifold M is defined as

$$\chi(M) = \sum_{i=0}^{\infty} (-1)^i b_i.$$

Observe that this definition generalizes the usual definition of Euler characteristic for graphs:

Example 1. Consider a planar graph G. We may construct a 2-dimensional CW complex by taking:

◇ 0-cells as vertices,

◇ 1-cells as edges, and

◇ 2-cells as faces. We must also consider the *exterior face to the graph*.

In this case we have that

$$b_0 = |V|, \quad b_1 = |E|, \quad b_2 = |F|, \quad \text{and} \quad b_i = 0, i \geqslant 3.$$

Adding up the Betti numbers as in the characteristic computation we obtain

$$\chi(G) = |V| - |E| + |F|$$

which corresponds to Euler's polyhedron formula. This quantity is 2 and aligns with $\chi(S^2) = 2$ as homology is homotopy-invariant.

Another way to compute the Euler characteristic is via Chern's generalization of the Gauss-Bonnet theorem which is the main tool we intend to use in this exploration.

Theorem 1. *Suppose M is a compact and oriented manifold without boundary of real dimension $2n$. Then*

$$\int_M e(TM) = \chi(M),$$

where TM is the tangent bundle of M and $e(TM) \in H^{2n}(M)$ is its Euler class.

Chern's original proof goes along the following lines:

◇ First show that $\pi^*(e(TM))$ is an exact form. The map π is the projection $\pi : TM \to M$. Then there is a form $\varphi \in H^{2n-1}(TM)$ such that

$$d\varphi = \pi^*(e(TM)).$$

◇ Then is X is a vector field (a section of the tangent bundle) on M, it has only isolated zeroes and singularities. If $S \subseteq M$ is its set of singularities we may further realize as a section

$$X : M \backslash S \to TM.$$

Chern proved that $\partial X(M \backslash S) \in H_{2n-1}(TM)$.

◇ Then the integral of the Euler class can be manipulated into
$$\int_M e(TM) = \int_{M\setminus S} X^*(d\varphi) = \int_{X(M\setminus S)} \mathrm{d}\varphi = \int_{\partial X(M\setminus S)} \varphi$$
where Stokes is applied in the last step.

◇ Finally, this last integral can be realized as the sum of indices of X, which by Poincaré-Hopf is precisely the Euler characteristic.

Really quickly: Undefined to defined

Vector bundles $E \xrightarrow{\pi} B$ carry certain information through their Chern classes. These are elements in $A^i(B)$, the i^{th} Chow group of B, which we may interpret via
$$A^i(B) \to H_{2n-2i}(B) \to H^{2i}(B)$$
where the first map takes cycles to cycles and then we're applying Poincaré duality.

Definition 2. For a vector bundle $E \xrightarrow{\pi} B$ of rank $r \geqslant 1$, its <u>Euler class</u> is
$$e(E) := c_r(E) = [Z(s)] - [P(s)].$$
Here s is a section which in local coordinates may be expressed as
$$s = (s_1, \ldots, s_r)$$
and so Z represents the class of zeroes, while P is the class of poles of the section.

Remark 2. The condition of e being the top Chern class allows us to see that it is a codimension r class. Observe that for a line bundle, the Euler class corresponds to the class of the divisor of a section of such a bundle.

Example 2. The tangent bundle to a manifold carries an Euler class, sections here are vector fields! So in essence, finding the value of $\int_M e(TM)$ becomes a matter of counting the zeroes and poles of a vector field over M.

Example 3. Consider the vector field F over the sphere S^2 given by
$$F(x,y,z) = (zx, zy, 1-z^2).$$
This is a section of TS^2 and
$$[Z(F)] - [P(F)] = [\boldsymbol{k}] + [-\boldsymbol{k}] - 0$$
corresponds to the zeroes and no poles of this vector field. This means that
$$e(TS^2) = 2[\text{pt.}]$$
and so
$$\chi(S^2) = \int_{S^2} 2[\text{pt.}] = 2$$
which coincides with our established notion.

Remark 3. For more examples on vector field ideas, say for the torus, check out this math.of post or this math.se post or this other one math.se. Also check this math.ov for projective plane.

My reference for the definition of the integral comes form Fulton and Pandharipande [3].

Definition 3 ([3] pg. 2)**.** For a complete variety, $c \in A^*(X)$ and $\beta \in A_k(X)$ then

$$\int_\beta c = \deg(c_k, \beta)$$

where c_k is the component of c in $A^k(X)$ and (c_k, β) is the evaluation of c_k on β giving us a zero cycle. When V is a closed, pure-dimensional subvariety of X, then we write

$$\int_V c \quad \text{instead of} \quad \int_{[V]} c.$$

It is part of my goal to concile this definition and the algebraic definition of the Euler class with our differential-geometric points of view.

More differential-geometric

We now delve into the more differential geometric side, recall we have defined the Levi-Civita connection via the following theorem:

Theorem 4. *Over a Riemmannian manifold (M, g) there exists a unique affine connection which is*

- *compatible with the metric g, and*
- *torsion-free.*

Such a connection is called the <u>*Levi-Civita connection*</u> *of M associated to the metric g.*

And recall that the notion of connection gives rise to the idea of the covariant derivative. Now, our desired *Euler form* will require a bit more, something like curvature. And for that, we first need to define a kind of second derivative for vector fields, something like

$$\nabla_X \nabla_Y Z.$$

We're close, because its local expression does include second derivatives! However it is *not tensorial*! It's possible to modify this idea in order to have a tensor.

Definition 4. For an affine connection ∇ over M, we define the <u>curvature operator</u> as

$$R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

From this, we can take information to define an Euler class alternatively in terms of differential forms.

Definition 5. For a Riemmannian manifold (M,g), we define the <u>Euler form</u> as

$$e(R) = \frac{1}{(2\pi)^n}\mathrm{Pf}(R)$$

where R is the associated curvature form of the Levi-Civita connection of M. The cohomology class of $e(R)$ is the Euler class.

Here, Pf is an operator on linear maps, which is defined in general as follows.

Definition 6. For a skew-symmetric matrix A of size $2n$, we can define its <u>Pfaffian</u> as

$$\frac{1}{2^n n!}\sum_{\sigma\in S_{2n}}\mathrm{sgn}(\sigma)\prod_{k=1}^{n} a_{\sigma(2i-1)\sigma(2i)}.$$

Remark 5. Let us observe two things:

(a) First, by virtue of the Pfaffian and the fact that the curvature form is a 2-form, the Euler class is a $2n$-form. Which means its a top form.

(b) The cohomology class of $e(R)$ is independent of ∇ or g and its a closed form.

Question. How is it that my Euler class and this Euler class are related? How can I compute the euler characteristic of the sphere with this differential form, for example?

3 Equivariant cohomology and localization

Manifolds usually don't come by themselves, like in the case of homogenous spaces, some manifolds have a lot of symmetries. These can be expressed by a group action on the manifold. We would like a cohomology theory which retains information on the group action!

Example 4 (A naïve approach)**.** Consider the S^1 action $\mathbb{CP}^1$ given by $u\cdot z = uz$. This action has two fixed points, 0 and ∞. Observe also that

$$u\cdot z = z \iff u = 1.$$

If we were to define the a cohomology which retains information on the group action (equivariant cohomology), we could say

$$H^*_{S^1}(\mathbb{CP}^1) := H^*\left({}^{\mathbb{CP}^1}\Big/_{S^1}\right).$$

However the orbit space ${}^{\mathbb{CP}^1}/_{S^1}$ is the same as a closed interval which means it has trivial cohomology.

Instead of considering the cohomology of the orbit space M/G, which doesn't retain information on the group action, we should look for an alternive which does.

The Borel construction

The main idea for this concept is that homotopy equivalent spaces have the same cohomology. Suppose G acts on M, let us create a space EG, a *classifying space*, with the following properties:

(a) The right action $EG\cdot G$ is free. $(\forall x(\mathrm{Stab}(x)=0))$
(b) EG is contractible.
(c) There exists a unique EG up to homotopy. (EG satisfies a universal property in a category of G-spaces)

This sounds a bit risky to ask, because questions may arise. But let's avoid them for now, instead observe that

$$M\times EG\simeq M$$

as EG is contractible!

Definition 7. We call the orbit space[1] of M the quotient

$$M_G:={M\times EG}\Big/{(g\cdot x,y)\sim(x,y\cdot g)}.$$

From this we define the equivariant cohomology of M as

$$H_G^*(M):=H^*(M_G).$$

Example 5 (Cohomology of a point)**.** We know that the usual cohomology of a point is trivial, but let's check two examples to see what changes.

(a) First consider the (trivial) action of $\mathbb{Z}$ on a point. In this case we have

$$E\mathbb{Z}=\mathbb{R}\quad\text{with}\quad x\cdot n=x+n.$$

This is a free action and $\mathbb{R}$ is contractible[2]. Find the classifying space isn't very bad:

$$\boldsymbol{pt.}_{\mathbb{Z}}={\mathbb{R}}\Big/{x\sim x+n}\simeq S^1$$

so that

$$H_{\mathbb{Z}}^*(\boldsymbol{pt.})=H^*(S^1)={\mathbb{Z}[t]}\Big/{t^2}.$$

[1]This is now overloading the previous definition of orbit space M/G.

[2]You'll have to trust me on the fact that $\mathbb{R}$ is unique up to homotopy on this one.

(b) Now let's take a bigger group, say $U(1)$, but for our purposes let's call it T as in torus. The classifying space here is
$$ET = \mathbb{C}^\infty \backslash \{0\}, \quad \text{with} \quad \alpha \cdot \underline{z} = (\alpha z_i)_i.$$
The action takes a sequence of complex numbers and scalar-multiplies it by $\alpha \in T$. This action is free, and we may see that $\mathbb{C}^\infty \backslash \{0\} \simeq S^\infty$. The infinite sphere is contractible by arguments out of my scope. And certainly, this classifying space is unique. But now, the quotient in question is
$$\boldsymbol{pt.}_T = {\mathbb{C}^\infty \backslash \{0\}} \Big/ {\underline{z} \sim \alpha \underline{z}} \simeq \mathbb{P}^\infty.$$
The cohomology now is
$$H^*_T \boldsymbol{pt.} = H^* \mathbb{P}^\infty = \mathbb{C}[t].$$
From this example we can extend the calculation to see that for an n-dimensional torus T^n we have
$$H^*_{T^n} \boldsymbol{pt.} = H^*(\mathbb{P}^\infty)^n = \mathbb{C}[t_1, \dots, t_n]$$
by the Künneth formula.

Questions remain for me such as...

Question. What happens when G is a symmetric group S_n, or a finite group $\mathbb{Z}/n\mathbb{Z}$? Even more, what if G is a matrix group, or an exceptional group such as the Mathieu group[3]?

Remark 6. One can see that the idea of constructing the cohomology of the orbit space goes haywire as soon as our space is not a point. For $\mathbb{P}^1$ one has to find
$$H^* \left(T^2 \times \mathbb{P}^2 \Big/ \sim \right)$$
which becomes unsurmountably hard.

To solve this issue we ask for help with the...

Atiyah-Bott localization theorem

Theorem 7 (Atiyah and Bott, 1984). *If $G \cdot M$ is an action and $F_k \subseteq M$ are the fixed loci of the action $G \cdot F_k = F_k$, then there exists an isomorphism of cohomologies*
$$H^*_G(M) \simeq \bigoplus_k H^*_G(F_k)$$
where the inclusion maps $i_k : F_k \rightarrowtail M$ induce the morphisms:
$$\underline{i}^* : H^*_G(M) \to \bigoplus_k H^*_G(F_k),$$

[3] At the time of writing, Ignacio hasn't read Classifying Spaces of Sporadic Groups by Benson and Smith.

component-wise this is the pullback of each i_k. And on the other direction it's

$$\frac{i_*}{e(N_{\bullet|M})}:\oplus H_G^*(F_k)\to H_G^*(M),$$

where $N_{Y|X}$ is the normal bundle $Y\subseteq X$.

To say that we're using a localization technique to find cohomology is to apply the Atiyah-Bott theorem.

Example 6 (Projective line cohomology via localization)**.** First, let's clearly define the action of $T^2=(\mathbb{C}\backslash\{0\})^2$ on $\mathbb{P}^1$. For $\underline{\alpha}\in T^2$ and $[X,Y]\in\mathbb{P}^1$ we have

$$\underline{\alpha}\cdot[X,Y]:=\left[\frac{X}{\alpha_1},\frac{Y}{\alpha_2}\right]^4.$$

Then, the only fixed points of this action are $0=[0:1]$ and $\infty=[1:0]$:

$$\underline{\alpha}\cdot[0:1]=\left[0:\frac{1}{\alpha_2}\right]=[0:1],\quad\text{and}\quad\underline{\alpha}\cdot[1:0]=\left[\frac{1}{\alpha_1}:0\right]=[1:0].$$

Proving that there's no more fixed points amounts to a linear algebra exercise. Applying Atiyah-Bott we now have that

$$H_{T^2}^*(\mathbb{P}^1)\simeq H_{T^2}^*([0:1])\oplus H_{T^2}^*([1:0])$$
$$\Rightarrow \mathbb{C}[t_1,t_2,H]\Big/(H-t_1)(H-t_2)\simeq\mathbb{C}[t_1,t_2]\oplus\mathbb{C}[t_1,t_2].$$

Here, we have used a calculation-not-shown which shows what the equivariant cohomology of $\mathbb{P}^2$ is. But the question is, how does this isomorphism work? It suffices to see where the generators go. On the left, we have the generators t_1,t_2 and H representing two hyperplane classes in each copy of $\mathbb{P}^\infty$ and H which represents the hyperplane class of $\mathbb{P}^1$ as a bundle over a point. Mapping these classes we get

$$\underline{i}^*\begin{cases}t_1\mapsto(t_1,t_1),\\ t_2\mapsto(t_2,t_2),\\ H\mapsto(t_1,t_2).\end{cases}$$

Whereas the generators on the right are the classes of the points $[0]=(1,0)$ and $[\infty]=(0,1)$. These points are mapped to the following classes:

$$i_*\begin{cases}[0:1]\mapsto H-t_2,\\ [1:0]\mapsto H-t_1.\end{cases}$$

And now, we are left with finding the normal bundles $N_{\boldsymbol{pt.}|\mathbb{P}^1}$. Observe that we may use the tangent-normal sequence for subspaces as follows:

$$0\to T\boldsymbol{pt.}\hookrightarrow i^*T\mathbb{P}^1\twoheadrightarrow N_{\boldsymbol{pt.}|\mathbb{P}^1}:=T\mathbb{P}^1\Big/T\boldsymbol{pt.}\to 0$$

[4] I know this is an unorthodox choice, but it's so that the weights of a certain representation are aligned properly. I'm already traumatized enough to do it the other way around.

and we have that the tangent bundle to the point is actually zero. This means that we have the isomorphism

$$i^*T\mathbb{P}^1 = T_{\boldsymbol{pt.}}\mathbb{P}^1 \simeq N_{\boldsymbol{pt.}\,|\,\mathbb{P}^1}.$$

Thus the Euler classes we are looking for are for the tangent spaces above 0 and ∞. These can be found using the equivariant Euler sequence for $T\mathbb{P}^1$, and so we get:

$$e(N_{\bullet\,|\,\mathbb{P}^1})\begin{cases}[0:1]\mapsto t_1-t_2,\\ [1:0]\mapsto t_2-t_1.\end{cases}$$

Putting this together we may see that indeed the isomorphism works as follows:

$$\begin{cases}[0:1]\mapsto \dfrac{H-t_2}{t_1-t_2}\mapsto \left(\dfrac{t_1-t_2}{t_1-t_2},\dfrac{t_2-t_2}{t_1-t_2}\right)=(1,0),\\[2ex] [1:0]\mapsto \dfrac{H-t_1}{t_2-t_1}\mapsto \left(\dfrac{t_1-t_1}{t_2-t_1},\dfrac{t_2-t_1}{t_2-t_1}\right)=(0,1).\end{cases}$$

Recall lastly that the vector $(1,0)$ represents

$$1\cdot[0:1]+0\cdot[1:0]$$

so it's indeed the correct cohomology class.

With this example, we verified that the localization theorem indeed provides an isomorphism between different cohomology rings. Now, we use localization to compute the Euler characteristic of a different variety.

Example 7. Consider $\mathbb{P}^1\times\mathbb{P}^1$ and an action of T^4 via rescaling all entries as before. The fixed points under this action are

$$(0,0),\quad (\infty,0),\quad (0,\infty),\quad \text{and}\quad (\infty,\infty).$$

Let us denote by F_k, $k=1,\dots,4$ the cohomology classes of the fixed points, and $i_k : F_k \hookrightarrow \mathbb{P}^1\times\mathbb{P}^1$ the inclusion map. Via the Atiyah-Bott theorem, we have that

$$\begin{aligned}\int_{\mathbb{P}^1\times\mathbb{P}^1} e(T\mathbb{P}^1\times\mathbb{P}^1) &= \int_{\mathbb{P}^1\times\mathbb{P}^1}\sum_{k=1}^{4}\frac{i_{k*}i_k^*(e(T\mathbb{P}^1\times\mathbb{P}^1))}{e(N_{F_k\,|\,\mathbb{P}^1\times\mathbb{P}^1})}\\ &=\sum_{k=1}^{4}\int_{F_k}\frac{i_k^*(e(T\mathbb{P}^1\times\mathbb{P}^1))}{e(N_{F_k\,|\,\mathbb{P}^1\times\mathbb{P}^1})}\\ &=\sum_{k=1}^{4}\int_{F_k}\frac{e(i_k^*T\mathbb{P}^1\times\mathbb{P}^1)}{e(N_{F_k\,|\,\mathbb{P}^1\times\mathbb{P}^1})}\end{aligned}$$

and from here we invoke the tangent-normal sequence for $F_k\subseteq \mathbb{P}^1\times\mathbb{P}^1$. We have that

$$0\to TF_k\hookrightarrow i_k^*T\mathbb{P}^1\times\mathbb{P}^1\twoheadrightarrow N_{F_k\,|\,\mathbb{P}^1\times\mathbb{P}^1}\to 0.$$

And simplifying by recalling that the tangent bundle over a point is zero, we have

$$0\to 0\to i_k^*T\mathbb{P}^1\times\mathbb{P}^1\xrightarrow{\simeq} N_{F_k\,|\,\mathbb{P}^1\times\mathbb{P}^1}\to 0.$$

This means that both Euler classes cancel out and we are left with just the fundamental class. The integral of the fundamental class over its own space gives us the value of 1 so that the whole sum is equal to 4.

This lets us conclude that $\chi(\mathbb{P}^1 \times \mathbb{P}^1) = 4$.

Remark 8. This computation did not rely on specific coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$, only on the existence of four torus-fixed points.

We also implicitly used several properties not mentioned before:

(a) Chern classes commute with pullbacks, so:
$$i_k^*(e(T\mathbb{P}^1 \times \mathbb{P}^1)) = e(i_k^* T\mathbb{P}^1 \times \mathbb{P}^1).$$

(b) Integration against a pushforward restricts to the domain of the map:
$$\int_{\mathbb{P}^1 \times \mathbb{P}^1} i_{k*}(\cdot) = \int_{F_k}$$

The variety $\mathbb{P}^1 \times \mathbb{P}^1$ is an example of a *toric variety*. One property of toric varieties is that their Euler characteristic is equal to the number of torus-fixed points.

Definition 8. A toric variety is an irreducible variety X containing a torus $T^k := (C\backslash\{0\})^k$ as a Zariski open subset such that the action of T^k on itself extends to a morphism $T^k \times X \to X$.

In general, for toric varieties, the number of torus-fixed points equals the number of top-dimensional cones in the associated fan which is combinatorial information that can be computed easily.

Theorem 9. *For a toric variety, the Euler characteristic equals the number of torus-fixed points under the torus action.*

4 Conclusion

Although we have only touched briefly on toric varieties, the result we obtained is very powerful. In the world of toric geometry many invariants, like the Euler characteristic, can be computed through combinatorial data.

The geometry of toric varieties is determined by their torus action, and many of their properties are encoded in combinatorial objects like fans and polytopes. Examples include projective spaces, products of projective spaces, and even certain line bundles over them.

This project has inspired me to continue exploring the geometry of varieties, especially how global invariants like the Euler characteristic connect to local data. One question that I hope to study further is the precise relationship between the Euler form and the Euler class. I'm aware that some papers address this connection, and I look forward to studying them in more depth.

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