**Exercise 1** (1.3.C)**.** Show that $A \to S^{-1}A$ is injective if and only if $S$ contains no zero divisors

**Lemma 1.** *A map $f$ in* Ring *is injective if an only if* $\ker(f) = \{0\}$.

Proof

Suppose $f$ is injective, then if $x \in \ker(f)$ it holds that $f(x) = 0$. As $f$ is a morphism of rings $0 = f(0)$ which means that $f(x) = f(0)$ and as $f$ is injective, we can conclude that $x = 0$ meaning that the kernel is trivial.
On the flip-side, take $f(x) = f(y)$. As $f$ is a morphism, $f(x - y) = 0$. But then $x - y \in \ker(f)$ which means that $x - y = 0$, letting us conclude that $x = y$. Thus $f$ is injective.

Answer

Call $\pi(a) = \frac{a}{1}$ the canonical map from $A$ to $S^{-1}A$. We will prove that $\ker(\pi) = \{0\}$ if and only if $S$ has no zero divisors.
To that effect suppose $s \in S$ is a zero divisor. This means that

$$\exists t(t \neq 0 \wedge st = 0).$$

Now $\pi(t) = \frac{t}{1}$. But inside $S^{-1}A$ we have $\frac{t}{1} = \frac{0}{1}$ because

$$\frac{t}{1} = \frac{0}{1} \iff \exists t' \in S(t'(t \cdot 1 - 0 \cdot 1) = 0).$$

Namely, such a $t'$ would be $s$. So $\pi(t) = 0$ and, as $t \neq 0$, we have that $\ker(\pi)$ is not trivial.
On the other hand suppose $s, t \in S$ are elements that satisfy $st = 0$. For the sake of argument suppose $t \neq 0$, we are set to prove that $s = 0$ and to do this, we'll show that $s \in \ker(\pi)$. Notice that

$$\pi(s) = \frac{s}{1} = \frac{0}{1} \iff \exists s' \in S(s'(s \cdot 1 - 1 \cdot 0) = 0).$$

The $s'$ we are looking for is $s' = t$. So, it follows that $s \in \ker(\pi)$. But as $\pi$ has trivial kernel, $s = 0$ which is what we wanted.

**Exercise 2** (1.3.Q)**.** Describe the colimit of the diagram $F : J \to$ Set given by $* \leftarrow * \to *$.If the two squares in the following commutative diagram are Cartesian diagrams, show that the "outside rectangle" (involving $U, V, Y$, and $Z$) is also a Cartesian diagram.

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

**Answer**

Suppose that there's an object $T$ with maps to $U$ and $Y$ as in the following diagram:

$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left] \arrow[dddr, bend right] & & \\ & U \arrow[r] \arrow[d] & V \arrow[d] \\ & W \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd}$$

We wish to show that there's a unique morphism $T \to U$ such that $T \to V$ and $T \to Y$ factor through $T \to U$.

Our first step is to construct a unique morphism from $T \to W$. This is done because we have a morphism $T \to X$ (the composition of $T \to V$, $V \to X$) and a morphism $T \to Y$ whose compositions to $Z$ agree. By universal property of the pullback ($W$ as pullback) we have that there's a unique morphism $T \to W$ through which the respective morphisms factor.

$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, dashed, bend right, "\exists!"'] & & \\ & U \arrow[r] \arrow[d] & V \arrow[d] \\ & W \arrow[r] & X \end{tikzcd}$$

With this in hand, we now have $T \to V$ and $T \to W$ whose compositions to $X$ agree. So by universality of $U$, there exists a unique morphism $T \to U$ through which the corresponding morphisms factor.

**Exercise 3** (1.3.T)**.** Show that coproduct for Set is disjoint union.

**Answer**

Recall that the disjoint union of $A_1$ and $A_2$ is defined as a set as

$$A_1 \mathbin{\dot{\cup}} A_2 \{ (a_i, i) : \ a_i \in A_i, \ I = 1, 2 \}.$$

This allows us to define maps $\iota_i : A_i \to A_1 \mathbin{\dot{\cup}} A_2, \ x \mapsto (x, 1)$ which are morphisms in Set because they're defined everywhere. We are to show that this set satisfies the universal property of coproducts.
Suppose $B$ is a set such that $f_i : \ A_i \to B$ are well defined. We must define a unique function $g : \ A_1 \mathbin{\dot{\cup}} A_2 \to B$ such that $f_i = g\iota_i$, this is done as follows:

$$g(a, i) = \begin{cases} f_1(a), \ i = 1, \\ f_2(a), \ i = 2. \end{cases}$$

We verify the factoring property:

$$g \circ \iota_1(a_1) = g(a_1, 1) = f_1(a_1), \quad g \circ \iota_2(a_2) = g(a_2, 2) = f_2(a_2).$$

By construction, we have defined $g$ uniquely.

**Exercise 4** (1.3.U)**.** Suppose $A \to B$ and $A \to C$ are two ring morphisms, so in particular $B$ and $C$ are $A$-modules. Recall that $B \otimes_A C$ has a ring structure.

i) Show that there is a natural morphism $\iota_B : B \to B \otimes_A C, \ b \mapsto b \otimes 1$. Similarly for $C$.

ii) Show that this gives a pushout on rings. In other words, the following diagram satisfies the universal property of the pushout.

$$\begin{tikzcd} B \otimes_A C & C \arrow[l, "\iota_C"'] \\ B \arrow[u, "\iota_B"] & A \arrow[l, "\beta"] \arrow[u, "\alpha"'] \end{tikzcd}$$

**Answer**

i) The map $\iota_B(b) = b \otimes 1$ is a homomorphism in virtue that $B \otimes_A C$ is a tensor product. By construction, all bilinear maps factor through the tensor product as linear maps. This map is one of the factors which should be linear. The same holds for $C$.

ii) Let us now take $M$ an $A$-module with morphisms $f_B : B \to M$ and $f_C : C \to M$. This can described by the following diagram:

$$\begin{tikzcd}
M & & \\
& B \otimes_A C & C \arrow[l, "\iota_C"'] \arrow[ull, bend right, "f_C"'] \\
& B \arrow[u, "\iota_B"] \arrow[uul, bend left, "f_B"] & A \arrow[u, "\alpha"'] \arrow[l, "\beta"]
\end{tikzcd}$$

However, let us take advantage of the tensor product, *gatekeeper of bilinear maps*. This morphisms can be combined into a bilinear map from $B \times C \to M$. We define

$$f : B \times C \to M,\ (b, c) \mapsto f_B(b) f_C(c)$$

and by universal property of the tensor product, there exists a unique map $\tilde{f} : B \otimes_A C \to M$ through which $f$ factors. Finally $f_B$ and $f_C$ factor through $\tilde{f}$ by diagram chasing and thus by universality of the tensor product we have that it satisfies the pushout universal property in this case.

**Exercise 5.** Describe the colimit of the diagram $F : J \to \mathsf{Set}$ given by $* \leftarrow * \to *$.

**Answer**

Recall that the colimit of a diagram $F : J \to \mathsf{C}$ is an object $\mathrm{colim} A_i \in \mathrm{Obj}\mathsf{C}$ with morphisms $f_j : A_j \to \mathrm{colim} A_i$ such that if $m : k \to j$ is a morphism in $J$, then the following diagram commutes

$$\begin{tikzcd}
\mathrm{colim} A_i & \\
A_j \arrow[u, "f_j"] & A_k \arrow[l, "F(m)"] \arrow[ul, "f_k"']
\end{tikzcd}$$

In our case, since we only have three objects the diagram looks like this

$$\begin{tikzcd} CL & C \arrow[l] \\ B \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

where $CL$ is the colimit object. In this particular case the colimit coincides with the pushout by universality.

**Exercise 6** (1.4.F)**.** Verify that the $A$-module described above is indeed the colimit.

The $A$-module in question is ${}^{\coprod A_i}/_{\sim}$ where $\sim$ is the relation

$$(a_i, i) \sim (a_j, j) \iff \exists(f : A_i \to A_k,\ g : A_j \to A_k)(f(a_i) = g(a_j)),$$

addition of $m_i \in M_i,\ m_j \in M_j$ is defined as

$$m_i + m_j \colon= F(u)(m_i) + F(v)(m_j)$$

where $u, v$ are arrows from $i, j$ to $\ell$. The sum lies in $M_\ell$. Multiplication is defined in an obvious[1] way and the zero element is $m_i$ such that there is an arrow $u : i \to k$ for which $F(u)(m_i) = 0$.

I must admit I wasn't able to tackle this problem.

[1]It's not obvious to me.