Toolkit

- Quadratic forms, which are expressions of the form

$$ax^2 + bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{x}^\mathsf{T} A \mathbf{x}$$

- Relationship between quadratic forms and conics:

  Quadratic forms determine conics, these are parabolas, ellipses, and hyperbolas. The type of conic can be determined diagonalizing the matrix A.

- Minkowski's theorem for the geometry of numbers: The idea in $\mathbb{R}^2$ is that *large enough* convex and symmetric sets contain points of $\mathbb{Z}^2$ other than $(0,0)$.

Theorem (Minkowski on $\mathbb{R}^2$)**.** *Suppose X convex, bounded and symmetric set with*

$$\text{Area}(X) \geqslant 4$$

then X contains a non-zero point $(x,y) \in \mathbb{Z}^2$.

Exercise 1 (Polish Olympiad)**.** Let a, b, c be positive integers with

$$ac = b^2 + b + 1$$

Prove that the equation

$$ax^2 - (2b+1)xy + cy^2 = 1$$

has integer solutions (x, y).

Exercise 2 (Hungarian Olympiad)**.** Suppose n is a positive integer such that

$$x^2 + xy + y^2 = n$$

has rational solutions (x, y). Show that the equation also has integer solutions.