MATH676 — Tropical Geometry

Based on the lectures by Renzo Cavalieri

Notes written by Ignacio Rojas

Fall 2023

This is a topics course on this stuff

Requirements

Knowledge on stuff

TO DO:

- ⋄ Write info on course description and requirements.
- ⋄ Polish info from day 1
- ⋄ Polish last part of day 2
- ⋄ Continue adding info from Renzo's digital notes

Contents

Chapter 1

Combinatorial Shadow of Algebraic Geometry

1.1 Day 1 | 20230821

Think of an algorithm where the input is an algebraic variety and the output is a combinatorial object, a piecewise linear object.

Example 1.1.1. Consider as an input a line in the plane. Say $V(x + y - 1)$, then an output would be a tropical line. If we remain in the plane and consider a higher degree polynomial, say an elliptic curve, as an output we obtain a tropical cubic.

Leaving the plane behind and thinking of abstract nodal curves, we can think of a sphere attached to a torus which is attached to a genus 2 torus, then the corresponding object is what we call the dual graph.

Right now we do not know the specific algorithm, but we can observe that the outputs are *more simple* than the inputs. So the important question is:

What algebraic information does the simplified object remember? How do we extract the information the object remembers? And once we know how to work with this objects, can we return to algebraic geometry from any kind of these objects?

Observe that the number of ends which go to infinity corresponds with the degree.

1.2 Day 2 | 20230823

Algebraic Geometry on $\mathbb{T}$

Let us talk about ways to get into tropical geometry. We will first define the tropical semifield which the base set over which we will do algebraic geometry.

Definition 1.2.1. The tropical semifield is the set $(\mathbb{R} \cup \{-\infty\})$ equipped with tropical addition and multiplication:

$$\begin{cases} x \oplus y = \max(x, y) \\ x \odot y = x + y \end{cases}$$

With this set we can make multivariable polynomials

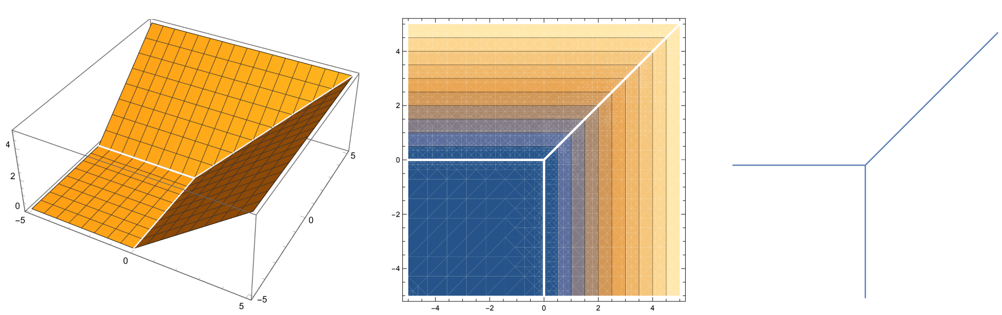
$$p(\underline{x}) : (\mathbb{R} \cup \{-\infty\})^n \to \mathbb{R} \cup \{-\infty\}$$

which gives rise to their *tropicalization*, a piecewise linear function $\mathrm{Trop}(p) : \mathbb{R}^n \to \mathbb{R}$.

Example 1.2.2. Consider the polynomial

$$p(x, y) = x \oplus y \oplus 0,$$

its tropicalization is $\mathrm{Trop}(p)(x, y) = \max(x, y, 0)$ which indeed is a piecewise linear function from $\mathbb{R}^2$ to $\mathbb{R}$. Observe that the surface is not smooth where the planes meet,



(a) $x \oplus y \oplus 0$ (b) Projection onto xy-plane (c) Corner locus

this is what we will call the *corner locus* or *tropical hypersurface*.

Definition 1.2.3. The tropical hypersurface $V(\mathrm{Trop}(p))$ is the codimension 1 locus in $\mathbb{R}^n$ where the function is non-linear (corner locus).

Example 1.2.4. If we consider higher degree tropical polynomials, they will become linear in the usual sense. Consider

$$p(x) = 3x^2 = 3 \odot x \odot x = 3 + x + x = 3 + 2x$$

which is indeed linear.

Valued fields

Definition 1.2.5. The field of <u>Puiseux series</u> or rational functions over $\mathbb{C}$ is $\mathbb{C}(t)$ where the elements are of the form

$$f(t) = \sum_{i=k_0}^{\infty} a_i t^{i/n}.$$

The lower bound k_0 could be negative and the exponents, are rational with bounded denominators.

Consider the valuation

$$\operatorname{val}_0 : \mathbb{C}(t) \to \mathbb{R} \cup \{\infty\}, \begin{cases} 0 \mapsto \infty \\ f \mapsto \text{order of vanishing at } 0. \end{cases}$$

This order of vanishing is the value α such that f/t^α approaches a finite non-zero value. The corresponding coefficient in the series expansion of f for this value is called the valuation coefficient.

Example 1.2.6. What happens to the order of vanishing when you add two functions? Consider $f = t^2,\ g = t^3$, then $f + g = t^2 + t^3$ which has order of vanishing 2. Observe that $2 = \min(2, 3)$.

In general what happens is that

$$\operatorname{val}_0(f_1 + f_2) \geqslant \min(\operatorname{val}_0 f_1, \operatorname{val}_0 f_2), \quad \text{and} \quad \operatorname{val}_0(f_1 f_2) = \operatorname{val}_0(f_1) + \operatorname{val}_0(f_2).$$

We can do algebraic geometry over this field! Let K be the field of rational functions, if $p(\underline{x}) \in K[\underline{x}]$ then we consider the algebraic variety

$$X = V(p) = \{\mathbf{x} : p(\mathbf{x}) = 0\} \subseteq K^n.$$

Taking the image through the n-fold valuation, we will obtain a set in $(\mathbb{R} \cup \{\infty\})^n$. The tropicalization of X is the image via this map:
and here $\operatorname{Trop}(V(p))$ is the tropical hypersurface for p.

$$\begin{array}{ccc}
\mathrm{val}_0 : K^n & \longrightarrow & (\mathbb{R} \cup \{\, \infty \,\})^n \\
\cup\!\mid & & \cup\!\mid \\
V(p) & \longmapsto & \overline{\mathrm{val}_0(V(p))} \\
\| & & \| \\
\{\mathbf{x} : p(\mathbf{x}) = 0\} & & \mathrm{Trop}(V(p))
\end{array}$$

Example 1.2.7. Consider the polynomial in $K[x, y]$

$$p(x, y) = tx + y + t^2,$$

then the variety is $X = \{\, (x, y) : tx + y + t^2 = 0 \,\}$ which we can solve to $y = -tx - t^2$.

If we choose $x = 0$ then y becomes $-t^2$. Now we take the valuation of $(0, -t^2)$ and so $(\infty, 2) \in \mathrm{Trop}(X)$.

Amoebas

Let us return to the usual stage and consider $p \in \mathbb{C}[\underline{x}]$ which defines an algebraic variety $X = V(p) \subseteq \mathbb{C}^n$. Now consider the map which sends every coordinate's modulus to its logarithm in base t:

$$\mathbb{C}^n \to (\mathbb{R} \cup \{\, -\infty \,\})^n, \quad (z_1, \ldots, z_n) \to (\log_t |z_1|, \ldots, \log_t |z_n|).$$

The image of X under this map, $\log_t(X)$, is the t-amoeba of X. If we take the limit as $t \to \infty$ then we get the *spine* of the amoeba.

Example 1.2.8. When $p(x, y) = x + y - 1$ then we can describe $V(p)$ via the parametrization $(x, 1 - x)$. So the corresponding t-amoeba in the real case is

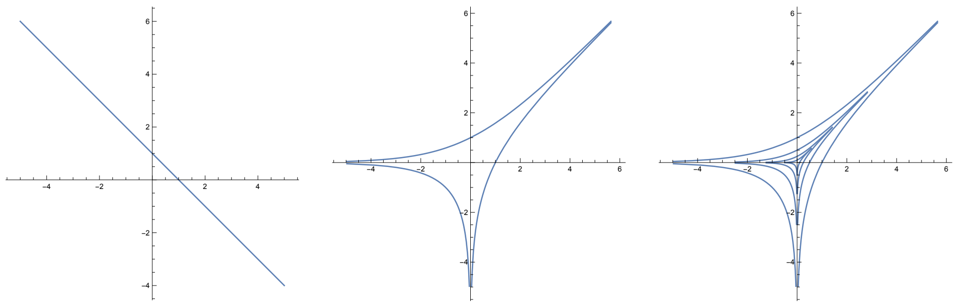
$$\{\, (\log_t |x|, \log_t |1 - x|) : x \in \mathbb{R} \,\}$$

and we ordinarily take the limit, we see that the functions converge to zero point-by-point. But the set is actually approaching the spine!

Observe that the spine approaches the tropical hypersurface associated to p. In other words we have that the tropical hypersurface is $\lim_{t \to \infty} \log_t(V(p))$.

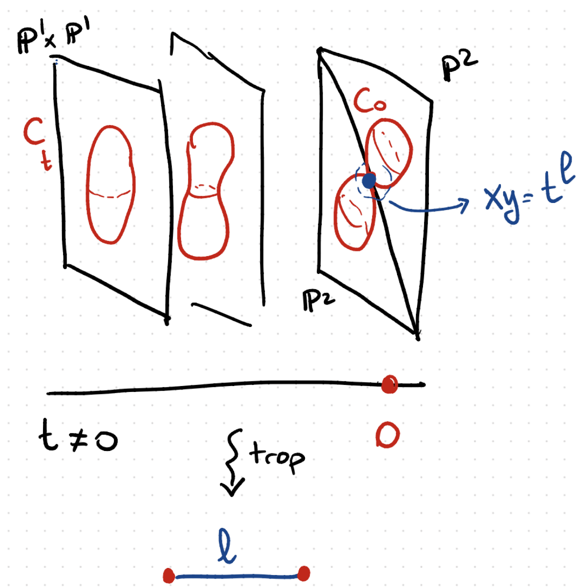
Degenerations

We may parametrize any algebraic variety with a time variable, then converting the information to a graph, edges code the information about how fast the node forms related to the length.



(a) $X = V(x + y - 1)$ (b) 2-amoeba of X (c) Sequence of amoebas as $t \to \infty$

Consider a family of of what, what is this family of?! Stuff? Curve in P1xP1 which eventually becomes P2?



It is too early to understand this point of view. We will set everything up to get to it.

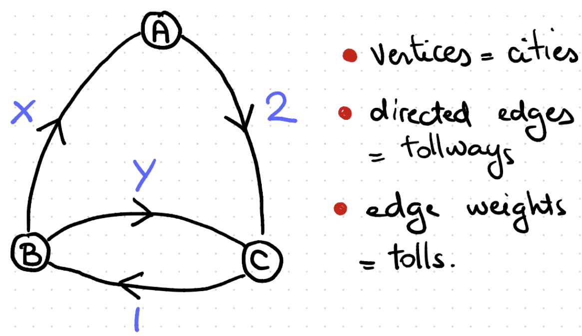
In general, the big idea will be to explore and understand these perspectives in the case of plane curves. We want to show how they are equivalent and then recover classical algebraic geometry results in terms of tropical geometry.

1.3 Interim

Tropical Arithmetics

Minimizing Tolls

Consider a set of cities connected by a network of toll-ways: If we only care about



minimizing toll expenses, what would be the cheapest way to go from one given city to another? Let us record the information as an incidence matrix.

$$M = \begin{pmatrix} 0 & \infty & 2 \\ x & 0 & y \\ \infty & 1 & 0 \end{pmatrix}$$

In this matrix, the rows determine the outbound city, while the columns are the destination. Each entry records the cost of a toll and tolls are considered to be infinite when the road does not exist. We can also think of M as recording the cheapest toll to go from one city to another with at most one move.

But if we wanted to find the cheapest way from one city to another in **two** moves, we could use M^2 with standard matrix multiplication. However we must redefine our basic operations as follows:

$$+ = \min, \quad \cdot = +$$

$$\begin{pmatrix} 0 & \infty & 2 \\ x & 0 & y \\ \infty & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} \min\begin{pmatrix} 0+0 \\ \infty + x \\ 2+\infty \end{pmatrix} & \min\begin{pmatrix} 0+\infty \\ \infty+0 \\ 2+1 \end{pmatrix} & \min\begin{pmatrix} 0+2 \\ \infty+y \\ 2+0 \end{pmatrix} \\ \min\begin{pmatrix} x+0 \\ 0+x \\ y+\infty \end{pmatrix} & \min\begin{pmatrix} x+\infty \\ 0+0 \\ y+1 \end{pmatrix} & \min\begin{pmatrix} x+2 \\ 0+y \\ y+0 \end{pmatrix} \\ \min\begin{pmatrix} \infty+0 \\ 1+x \\ 0+\infty \end{pmatrix} & \min\begin{pmatrix} \infty+\infty \\ 1+0 \\ 0+1 \end{pmatrix} & \min\begin{pmatrix} \infty+2 \\ 1+y \\ 0+0 \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 3 & 2 \\ x & \min(0, y+1) & \min(x+2, y) \\ 1+x & 1 & \min(0, 1+y) \end{pmatrix}.$$

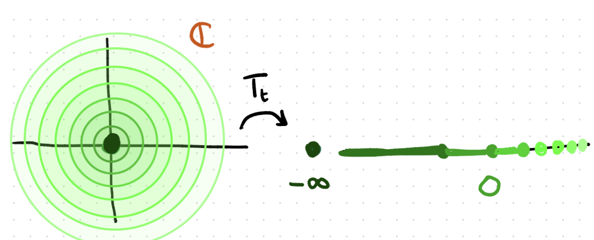
Observe that $1 + y$ can be the minimum in the diagonal when we allow *negative tolls*.

Remark 1.3.1. If we disallow negative tolls, the products M^n eventually stabilize to a matrix whose entries record the cheapest way to get from one city to another in n steps.

This gives us an intuition that minimization problems correspond to linear algebra problems over $(\mathbb{T}, +, \cdot)$ which is precisely $(\mathbb{R} \cup \{ \infty \}, \min, +)$.

Forgetting phases

Recall that any complex number can be written as $z = re^{i\theta}$ where $r \geqslant 0$. Consider the map $T_t : \mathbb{C} \to \{ -\infty \} \cup \mathbb{R}, \quad z \mapsto \log_t(r)$. This map is surjective, and this we can see by



checking it is right-invertible. Observe that:

$$\begin{cases} T_t^{-1}(x) = \{ t^x e^{i\theta} \} \subseteq \mathbb{C}, & \text{for} \quad x \in \mathbb{R}, \\ T_t^{-1}(-\infty) = 0. \end{cases}$$

With this in hand, we wish to define an exotic addition and multiplication on $\{ -\infty \} \cup \mathbb{R}$ using T_t. We will dequantize!

We begin with **hyper-addition**, the output will be a subset of $\{-\infty\}$ so it's not a binary operation by itself.

$$x \diamondplus_t y\colon = T_t(T_t^{-1}(x) + T_t^{-1}(y)) = \left[\log_t(|t^x - t^y|), \log_t(t^x + t^y)\right].$$

This is an interval in $\{-\infty\} \cup \mathbb{R}$, in order to make $\diamondplus_t$ into an operation we take a limit:

$$\begin{tikzcd}
x \diamondplus_t y \arrow[r, "\lim\limits_{t\to\infty}"] \arrow[d, "\max"] & x \diamondplus y = \lim_{t\to\infty} x \diamondplus_t y \arrow[d, "\max"] \\
x +_t y \arrow[r, "\lim\limits_{t\to\infty}"] & x + y = \max(x, y)
\end{tikzcd}$$

Remark 1.3.2. Note that $\diamondplus$ is still a hyperoperation. Its output is not a singleton *only* when a dding a number to itself:

$$x \diamondplus y = \begin{cases} \max(x, y), & x \neq y \\ [-\infty, x], & x = y \end{cases}$$

Formally this process, taking a limit of a family of operations, is known as *dequantization*.

In the case of multiplication, things go a lot smoother when defining it:

$$x \cdot y = T_t\left[T^{-1}(x) \cdot T^{-1}(y)\right] = \log_t\left[(t^x e^{i\theta})(t^y e^{i\varphi})\right] = \log_t\left(t^{x+y} e^{i(\theta+\varphi)}\right)$$

Separating the logarithm we get $(x + y) + \log\left(e^{i(\theta+\varphi)}\right)/\log(t)$, then letting t grow without bound we see that the operation converges to $x + y$.

Exercise 1.3.3. Check how the definition of $+$ and $\cdot$ extend to the *number* $-\infty$.

Since I don't understand exactly what to do, I'll do my best

For a finite x we will find $x + (-\infty)$. This is the limit of the previous hyperoperation:

$$x \diamondplus_t (-\infty) = T_t(T_t^{-1}(x) + T_t^{-1}(-\infty)) = T_t(T_t^{-1}(x) + 0) = T_t(T_t^{-1}(x)) = x.$$

If we let t grow, the result doesn't change and so this goes according to $\max(x, -\infty) = x$.

On the other hand when taking the product:

$$x \cdot (-\infty) = T_t\left[T^{-1}(x) \cdot T^{-1}(-\infty)\right] = T_t\left[T^{-1}(x) \cdot 0\right] = T_t(0) = \log_t(0) \to -\infty$$

which is also similar to the notion of $x + (-\infty) = -\infty$.
We can now proceed to operate $-\infty$ with itself:

$$(-\infty) \bigoplus_t (-\infty) = T_t(0) = \log_t(0) = -\infty = \max(-\infty, -\infty),$$

and when taking the product:

$$(-\infty) \cdot (-\infty) = T_t(0) \log_t(0) = -\infty = (-\infty) + (-\infty)$$

where the last sum is a sum in the usual sense.

So, summarizing this process:

- ◇ We forgot about the phase of the complex numbers and only looked at them radially.
- ◇ The modulus of these numbers was scaled logarithmically.
- ◇ Finally we took the limit of these operations and obtained the desired (somewhat) result.

This is known as Maslov dequantization and with this we can see $(\mathbb{T}, +, \cdot)$ as $(\{-\infty\} \cup \mathbb{R}, \max, +)$. Also, we will abbreviate $\lim_{t\to\infty} T_t$ with $T_{t\to\infty}$

Puiseux series

Recall from our times in Calculus 1 that when resolving indeterminate limits, the relevant information is contained in the order of vanishing of the function.

Example 1.3.4. Consider the limit $\lim_{t\to 0} \frac{\sin(x)}{x} = 1$. Near $t = 0$ we have

$$\sin(t) = t + o(t) \sim t^1 \quad \text{and} \quad \frac{1}{t} = t^{-1} \quad \text{so} \quad t^1 t^{-1} = t^0 = 1.$$

From this, we care to study the orders of zeroes and poles of Laurent series. In order to extend the class of functions to an algebraically closed field, we consider Puiseux series, or rational functions. We can identify Puiseux series as

$$\mathbb{C}\{\{t\}\} = \bigcup_{n\in\mathbb{N}} \mathbb{C}(t^{1/n}).$$

Concretely, elements here are Laurent series with coefficients having a common denominator. This field can be equipped with the valuation

$$\mathrm{val} : \mathbb{C}\{\{t\}\} \to \{-\infty\} \cup \mathbb{R}, \begin{cases} 0 \mapsto -\infty \\ t^{p/q} + \text{higher order} \mapsto p/q \end{cases}$$

Proposition 1.3.5. *The previous valuation enjoys the following properties*:

i) $\mathrm{val}(\alpha \cdot \beta) = \mathrm{val}(\alpha) + \mathrm{val}(\beta)$.

ii) $\mathrm{val}(\alpha + \beta) \leqslant \min(\mathrm{val}(\alpha), \mathrm{val}(\beta))$.

Equality holds when $\mathrm{val}(\alpha) \neq \mathrm{val}(\beta)$. So if we decide to define operations on $\{-\infty\} \cup \mathbb{R}$ by inducing them from the operations on $\mathbb{C}\{\{t\}\}$, then we obtain

$$x \oplus y = \mathrm{val}\left(\mathrm{val}^{-1}(x) + \mathrm{val}^{-1}(y)\right), \qquad x \cdot y = \mathrm{val}\left(\mathrm{val}^{-1}(x) \cdot \mathrm{val}^{-1}(y)\right).$$

Now $\cdot$ coincides with usual addition and $+$ is the hyperoperation

$$x \oplus y = \begin{cases} \min(x, y) & \text{when} \quad x \neq y, \\ [-\infty, \min(x, y)] & \text{when} \quad x = y. \end{cases}$$

The only natural way to turn this into an operation is to define $x + y = \min(x, y)$. In conclusion, the field of Puiseux series with the order of vanishing and poles is congruent to $(\mathbb{T}, +, \cdot)$ which in this case is $(\{-\infty\} \cup \mathbb{R}, \min, +)$.

The Tropical Semifield

Definition 1.3.6. The <u>tropical semifield</u> is $(\mathbb{T}, +, \cdot)$ where we can choose:

- $\mathbb{T} = \mathbb{R} \cup \infty$, $+ = \min$ and $\cdot = +$, the min convention.
- $\mathbb{T} = \{-\infty\} \cup \mathbb{R}$, $+ = \max$ and $\cdot = +$, the max convention.

There is a natural isomorphism between the two choices given by $x \mapsto -x$. As we have mentioned, different contexts may be more natural than the other when using certain conventions. We will tipically use the $\max$ convention.

Proposition 1.3.7. *The following algebraic properties hold for* $(\mathbb{T}, +, \cdot)$:

i) $0_{\mathbb{T}} = -\infty$.

ii) $1_{\mathbb{T}} = 0$.

iii) $x + y = 0_{\mathbb{T}}$ *only has the solution* $x = y = 0_{\mathbb{T}}$. *This means that only* $-\infty$ *has an additive inverse.*

iv) *Addition is idempotent*: $x + x = x$.

v) *Every non-zero element has a multiplicative inverse*: $1/x = -x$.

Observe that it is not possible to adjoin formal additive inverses. If $x + y = 0_{\mathbb{T}}$, then

$$(x + x) + y = x + y = 0_{\mathbb{T}} \quad \text{and} \quad x + (x + y) = x + 0_{\mathbb{T}} = x \quad \text{but} \quad x \neq 0\_\mathbb{T}.$$

Proposition 1.3.8 (Weird Fun Facts)**.** *Recall that the usual Pascal Triangle is built by adding the* previous two *elements to get the next one. In the tropical case we have*

$$\begin{array}{ccccc} & & 1_{\mathbb{T}} & & \\ & 1_{\mathbb{T}} & & 1_{\mathbb{T}} & \\ 1_{\mathbb{T}} & & 1_{\mathbb{T}} & & 1_{\mathbb{T}} \end{array} \quad = \quad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{array}$$

and this extends downwards with the same pattern.

In the case of the tropical binomial theorem, the identity is

$$(x + y)^n = x^n + y^n \quad \textit{and} \quad n \max(x, y) = \max(nx, ny).$$

Exercise 1.3.9. Recall that the coefficients in the expansion for the binomial theorem are the corresponding elements in the rows of the Pascal Triangle. Verify if the coefficients agree in the tropical case for the binomial theorem.

The Optimal Assignment Problem

Suppose we have n jobs for n workers. Each worker can only work one job and once the job is taken, no one else can do it. We wish to assign a job to each worker in order to maximize our company's profit.

Example 1.3.10. As a little example consider Alice and Bob's hydroponics farm. When working with the weeds Alice produces 5 credits while working with the water she produces 6. On the other hand Bob produces 3 and 5 respectively.

It is easy to see that Alice should be assigned to to the weed and Bob to the water in order to maximize. But let us apply what we know with tropical arithmetics.

Call

$$M_{ij} = \text{amount of credits work } i \text{ produces when doing job } j.$$

Then we can summarize the previous information in a matrix

$$M = \begin{pmatrix} 5 & 6 \\ 3 & 5 \end{pmatrix}$$

and if we take the tropical determinant (which is really a permanent since we lack subtraction) we get

$$\operatorname{Trop}\det M = 5 \cdot 5 + 6 \cdot 3 = \max(5+5, 6+3) = 10$$

which is the maximal profit we can make by assigning our workers.

Exercise 1.3.11. Do the following:

(1-) Construct a 3×3 matrix with non-permuted entries such that there's more than one possible assignment for the optimal jobs.

(1) Use the combinatorial definition of permanent to show that the tropical determinant of M is indeed the maximal profit. ⟦ Hint: The definition of permanent is the same as the determinant but without the $(-1)^{\operatorname{sgn}\sigma}$. ⟧

(5) Assuming you know the tropical determinant of a matrix, devise a way to identify one job combination which reaches the optimum value.

Tropical Polynomials and Roots

An univariate,tropical, (Laurent) monomial is equivalent to an affine linear function with integer coefficients.

Example 1.3.12. We have for example:

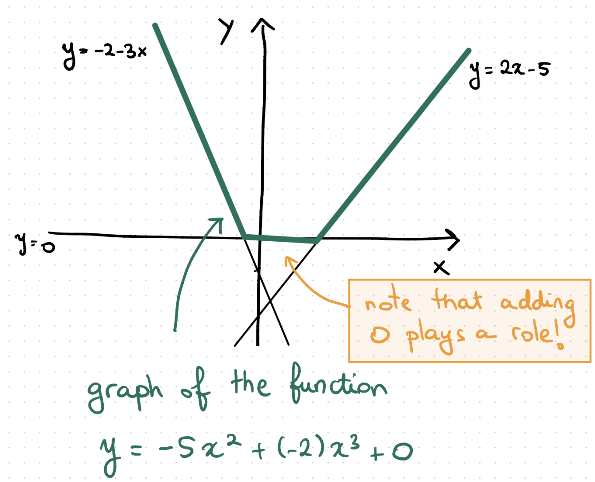
$$5x^2 \leftrightarrow 5 + 2x, \quad 2x^{-3} \leftrightarrow 2 - 3x \text{ (Laurent)}.$$

An univariate tropical (Laurent) polynomial is a finite sum of monomials which give rise to a *convex*, continuous, piecewise affine, linear function with integer slopes.

Example 1.3.13. Consider the function $-5x^2 + (-2)x^{-3} + 0$ which corresponds to

$$\max(-5 + 2x, -2 - 3x, 0).$$

If we graph this functions we obtain Observe that this function is indeed convex, and fulfills all of the previous properties from before.



A small measure of care should be taken because any convex-PL-(etc) function corresponds to a tropical polynomial, but there are multiple tropical polynomials which map to the same function.

Example 1.3.14. Consider the functions

$$p_1 = x + \frac{1}{x} + 0, \quad p_2 = x + \frac{1}{x} - 2.$$

When converting we get

$$\max(x, -x, 0), \quad \max(x, -x, -2)$$

which produce $|x|$ in both cases.

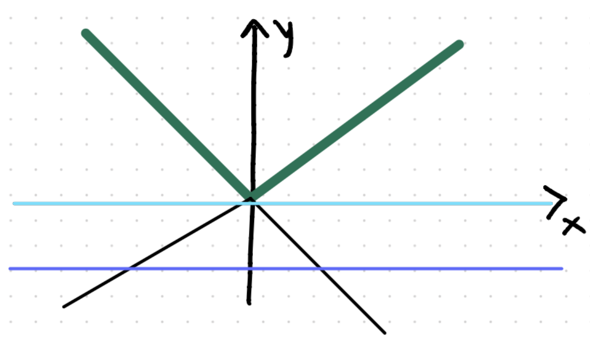


Figure 1.3: Failure of injectivity as both functions map to $|x|$ with $y = 0$ and $y = -2$ shown.

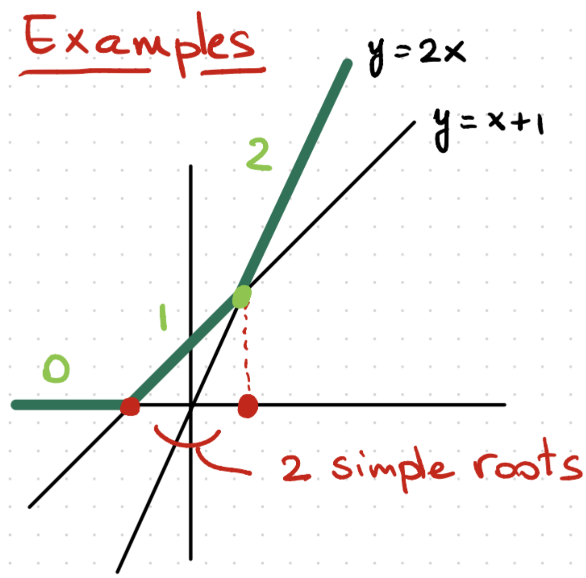
To talk about the roots, we will start with a purely combinatorial definition.

Definition 1.3.15. Given a polynomial $p \in \mathbb{T}[x]$ of degree d we say the following:

- $-\infty$ is a root of p if the slope of the piecewise linear function is non-zero for $x \ll 0$.
- $x_0 \in \mathbb{R}$ is a root of p if $p'(x_0)$ is undefined.

We say that the multiplicity is the change of slopes across the root. And of $p'(x)$, $x \ll 0$ for when the root is $-\infty$.

Example 1.3.16. Consider the polynomial $x^2 + 1 \cdot x^1 + 0 = \max(2x, x, 0)$. We can see

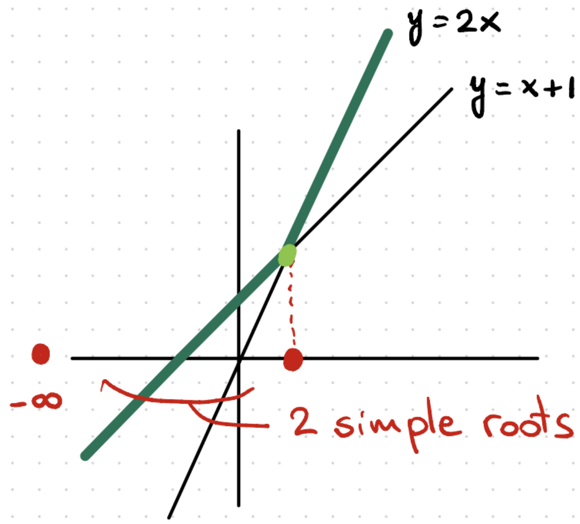


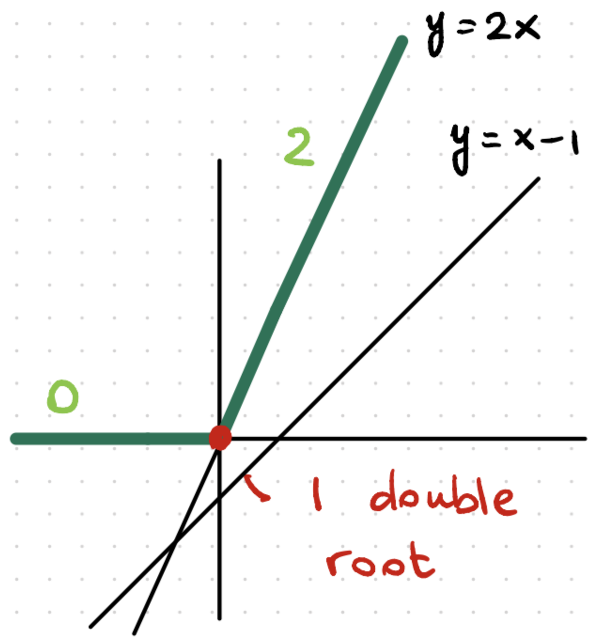
that there are changes in slope at $x_1 = -1$ and $x_2 = 1$. The number of roots coincides with the degree of the polynomial as in the usual sense.

Example 1.3.17. Let's remove the zero, recall zero isn't the additive identity, so the polynomial we have is $x^2 + 1 \cdot x^1 = \max(2x, x)$. Now one of the roots is still $x = 1$, but remember that if the slope is non-zero when $x \ll 0$, then $-\infty$ is a root of p. This is the case here because the slope is 1 as $x \to -\infty$. Once again there's two roots $x_1 = -\infty$ and $x_2 = 1$.

Example 1.3.18. Let us change a sign in a coefficient, take $x^2 - 1 \cdot x^1 + 0$. But what is tropical subtraction? It's not that, let's convert this slowly into what it's supposed to be:

$$x^2 - 1 \cdot x^1 + 0 = (x \cdot x) + (-1) \cdot x + 0 = (2x) + (x + (-1)) + 0 = \max(2x.x - 1, 0).$$

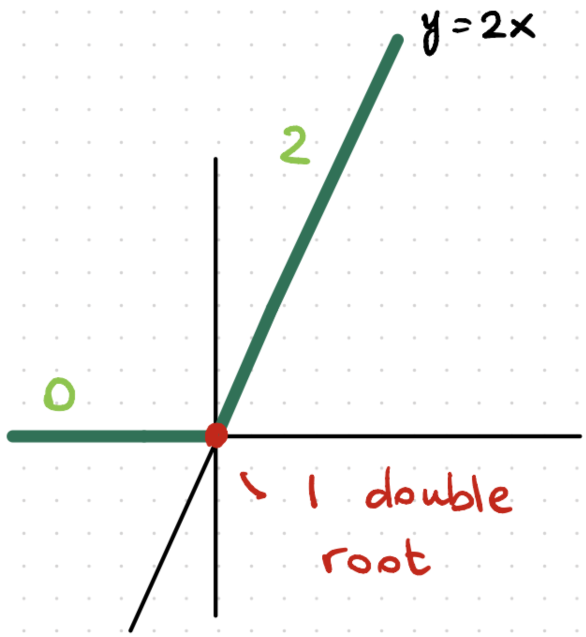




Observe that because the line $y = x - 1$ is below our graphs, it doesn't interfere with the calculation of zeroes. So the only place where there occurs a change in sign is $x = 0$. This is our unique root with multiplicity 2.

Example 1.3.19. In a similar fashion, $x^2 + 0$ also has a double root at $x = 0$. There is only one change in slope once again at $x = 0$.

Questions arise:



Which functions have only one simple zero at $-\infty$? What would a function with an order 2 zero at $-\infty$ look like?

Exercise 1.3.20. Do the following:

(5) Provide an example of function with one simple zero at $-\infty$.

(5) Now do one with a double zero at $-\infty$.

1.4 Day 3 | 20230825

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Bibliography

[1] D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry*. Graduate Studies in Mathematics. American Mathematical Society, 2021.