Math 601: Advanced Combinatorics I
Homework 6 - Due Nov 8

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 15. The maximum possible score on this homework is 15 points. See the syllabus for details.

Problems

1. (1 point) Determine the dimension of the adjoint representation of $\mathfrak{gl}_n$.
2. (4 points) Show that $\mathfrak{gl}_n$ has the same root system as $\mathfrak{sl}_n$, but embedded in one higher dimensional weight space $\mathfrak{h}^*$. Explan why this implies that $\mathfrak{gl}_n$ is not semisimple.
3. (1 point) Determine the dimension of the adjoint representation of $\mathfrak{so}_{2n+1}$.
4. (3 points) Write out a basis for the adjoint representation of $\mathfrak{so}_5$ and show how it corresponds to the root system in type B.
5. (3 points) Generalize the computation we did in class to show that the Killing form for $\mathfrak{sl}_n$, when restricted to the Cartan subalgebra $\mathfrak{h}$, satisfies

$$\langle X, Y\rangle = 2n\mathrm{tr}(XY).$$

   (You may not use the next problem unless you solve that one as well.)

6. (5 points) Show that Killing form for $\mathfrak{sl}_n$ satisfies $\langle X, Y\rangle = 2n\mathrm{tr}(XY)$ in general for any elements X, Y of $\mathfrak{sl}_n$. An outline of one approach to follow can be found here: https://www.math.stonybrook.edu/∼jstarr/M552s24/M552s22ps8.pdf
7. (3 points) Do your best to draw the root system for $\mathfrak{sl}_4$ in three dimensional space. You may use a computer to plot it if it aids the visual. Make sure you list out what all the vectors are and explain how you made the plot, so that if it comes out not entirely clear I can see what your process was and grade based on that. If you'd rather 3D print it or make it out of toothpicks or pipe cleaners or something, that is also acceptable if you bring it to class on the due date.
8. (4 points) Show that the longest word $s_1s_2s_1$ in S_3, when applied to $\mathfrak{sl}_3$ crystals, gives the "up-down symmetry" of the crystal diagrams (in other words, the highest weight element gets sent to the lowest weight, and if an element x can be obtained from the highest weight by a sequence of f_i operators, then its image under $s_1s_2s_1$ can be obtained from the lowest weight by the corresponding sequence of e_i operators).
9. (3 points) What is the size of the Weyl group of type G_2? Write out its elements as reduced words in the two simple reflections s_1, s_2 corresponding to the two simple roots of G_2.