Exercise 1. Prove that all entire functions that are also injective take the form $f(z) = az + b$with $a, b \in \mathbb{C}$, and $a \neq 0$. ⟦Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.⟧

Answer

The function $g(z) = f(1/z)$ has a singularity at $z = 0$. If it were removable, then g is bounded on $B(0, R)$ for some $R > 0$.
This means that f is bounded outside $B(0, R)$, but as f is entire, it's continuous and so it's bounded *inside* $B(0, R)$.
By Liouville's theorem f is constant. But that contradicts the fact that f is injective.
Now assume g has an essential singularity at $z = 0$. By the Casorati-Weierstrass theorem, we have a neighborhood of the origin $B(0, R)$ with $R > 0$, such that $g[B(0, R)]$ is dense in $\mathbb{C}$. This means that $f[\{\, |z| > R \,\}]$ is dense in $\mathbb{C}$

Exercise 2. As in class, consider the unit sphere

$$X = \{\, (a, b, c) : \ a^2 + b^2 + c^2 = 1 \,\} \subseteq \mathbb{R}^3$$

Let $N = (0, 0, 1), S = (0, 0, -1), U_N = X \backslash N, U_S = X \backslash S$. Consider the following three charts on X:

- ◇ $\phi_N : U_N \to \mathbb{C}, \ (a, b, c) \mapsto \frac{a+ib}{1-c}$.
- ◇ $\phi_S : U_S \to \mathbb{C}, \ (a, b, c) \mapsto \frac{a+ib}{1+c}$.
- ◇ $\psi_S : U_S \to \mathbb{C}, \ (a, b, c) \mapsto \frac{a-ib}{1+c}$.

Do the following:

i) The inverse of ϕ_N is

$$\phi_N^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2 + 1}, \frac{2\,\mathrm{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

Calculate ϕ_S^{-1} and ψ_S^{-1}.

ii) Among the three charts $\{\, (U_N, \phi_N), (U_S, \phi_S), (U_S, \psi_S) \,\}$, one pair is compatible and the other two are not. Which is which? Why?

⟦Hint: Remember a function is holomorphic if and only if $\partial_{\overline{z}} f = 0$.⟧

Answer

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