Partial Fractions

Integrals involving quadratic polynomials can be solved in many different ways, not only trigonometric substitution. The partial fractions method involves separating a polynomial into its <u>irreducible</u> factors.

Example 1. Consider the integral

$$\int \frac{\mathrm{d}x}{(x-5)(x-4)}.$$

We will split the integrand into a couple of fractions whose denominators are the factors of the original denominator.

$$\frac{1}{(x-5)(x-4)} = \frac{A}{x-5} + \frac{B}{x-4},$$

where A,B are constants. To find these constants we multiply both side of the equation by the original denominator $(x-5)(x-4)$:

$$\begin{aligned} 1 = \frac{(x-5)(x-4)}{(x-5)(x-4)} &= \frac{A(x-5)(x-4)}{x-5} + \frac{B(x-5)(x-4)}{x-4} \\ &= A(x-4) + B(x-5). \end{aligned}$$

Expanding the expression in the right we obtain the equality

$$1 = Ax - 4A + Bx - 5B \Rightarrow 1 = (A+B)x + (-4A-5B)$$

and since the expression in the left is also a polynomial we can equate coefficients.

$$\begin{cases} 1 = -4A-5B \\ 0 = A+B \end{cases} \Rightarrow \begin{cases} 1 = -4A-5B \\ B = -A \end{cases}$$

We can replace the second equation into the first one

$$1 = -4A - 5(-A) \Rightarrow \underline{1 = A} \Rightarrow \underline{B = -1}.$$

Thus it follows that

$$\frac{1}{(x-5)(x-4)} = \frac{(1)}{x-5} + \frac{(-1)}{x-4},$$

and we can separate this expression using linearity of the integral:

$$\begin{aligned} \int \frac{\mathrm{d}x}{(x-5)(x-4)} &= \int \left(\frac{1}{x-5} + \frac{-1}{x-4} \right) \mathrm{d}x \\ &= \int \frac{\mathrm{d}x}{x-5} - \int \frac{\mathrm{d}x}{x-4} \\ &= \underline{\log(x-5) - \log(x-4)} \end{aligned}$$

We will not always find ourselves with a factored polynomial. Consider the following example:

Example 2. To compute the integral of $\frac{x+1}{(x-1)(x-2)}$ we will once again separate into partial fractions. Since the factors of the denominator are linear, we will separate into two fractions:

$$\begin{aligned} &\frac{x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \\ &\Rightarrow x+1 = A(x-2) + B(x-1) \\ &\Rightarrow x+1 = (A+B)x + (-2A-B). \end{aligned}$$

This time the linear coefficient on the left hand side is **not zero**. Anyways equating coefficients is analogous

$$\begin{cases} 1 = -2A-B \\ 1 = A+B \end{cases} \Rightarrow \begin{cases} 1 = -2A-B \\ B = 1-A \end{cases} \Rightarrow 1 = -2A - (1-A)$$

We solve to obtain $\underline{A = -2}$ and $\underline{B = 3}$.

Replacing in the partial fraction decomposition we obtain

$$\frac{x+1}{(x-1)(x-2)} = \frac{(-2)}{x-1} + \frac{(3)}{x-2}.$$

As we did before we can separate the integral by linearity

$$\begin{aligned} \int \frac{x+1}{(x-1)(x-2)} \mathrm{d}x &= \int \left(\frac{-2}{x-1} + \frac{3}{x-2} \right) \mathrm{d}x \\ &= \int \frac{-2}{x-1} \mathrm{d}x + \int \frac{3}{x-2} \mathrm{d}x \\ &= \underline{-2\log(x-1) + 3\log(x-2)} \end{aligned}$$

It can also occur that the polynomial in denominator is not factored. In that case, we have to factor and then decompose into partial fractions.

Example 3. Consider the rational function

$$f(x) = \frac{3x^2+2x+1}{x^3-6x^2+11x-6}$$

To separate, we have to factor the denominator into irreducibles and so we use the rational roots theorem.

The rational roots of the denominator are in the set

$$R = \{\pm 1, \pm 2, \pm 3\},$$

and after trying $x = 1$ we see that it is indeed a root. Thus

$$\begin{aligned} x^3 - 6x^2 + 11x - 6 &= (x-1)(x^2 - 5x + 6) \\ &= (x-1)(x-2)(x-3) \end{aligned}$$

We can now separate into partial fractions

$$\frac{3x^2+2x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Cross multiplying the right hand side results in

$$\begin{aligned} &A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \\ &= (A+B+C)x^2 + (-5A-4B-3C)x + (6A+3B+2C) \end{aligned}$$

We can now collect coefficients and mount our system of equations

$$\begin{cases} 3 = A+B+C \\ 2 = -5A-4B-3C \\ 1 = 6A+3B+2C \end{cases} \Rightarrow \begin{cases} A = 3 \\ B = -17 \\ C = 17 \end{cases}$$

We can now separate into our partial fractions to obtain

$$\int f(x)\mathrm{d}x = \underline{3\log(x-1) - 17\log(x-2) + 17\log(x-3)}.$$

Higher Degrees and Repeated Roots

Let us summarize the terms in the partial fraction decomposition according the possible cases:

Denominator	Partial Fraction Decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^n$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_n}{(ax+b)^n}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^n$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\cdots+\dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$

Example 4. Let us decompose $f(x)=\frac{x^2-4x+12}{(x-2)^2(x-4)}$.

$$\frac{x^2-4x+12}{(x-2)^2(x-4)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x-4}$$

according to our table since we have a repeated root. Collecting terms on the right hand side we get the expression

$$\begin{aligned}&A(x-2)(x-4)+B(x-4)+C(x-2)^2\\ &=(A+C)x^2+(-6A+B-4C)x+(8A-4B+4C)\end{aligned}$$

Equating coefficients we get the system

$$\begin{cases}1=A+C\\ -4=-6A+B-4C\\ 12=8A-4B+4C\end{cases}\Rightarrow\begin{cases}A=-2\\ B=4\\ C=3\end{cases}$$

Thus it follows that

$$\frac{x^2-4x+12}{(x-2)^2(x-4)}=\frac{-2}{x-2}+\frac{4}{(x-2)^2}+\frac{3}{x-4}.$$

If we wanted to integrate f, we could use the linearity on the integral to get

$$\int f(x)\mathrm{d}x=\underline{-2\log(x-2)+\frac{4}{x-2}+3\log(x-4).}$$

In case we have an irreducible quadratic, it's necessary to separate according to the 3$^{\text{rd}}$ and 4$^{\text{th}}$ rows of the table.

Example 5. Let $f(x)=\frac{5x^3+x^2-x+7}{(x-2)(x^2+x+1)^2}$, we decompose into partial fractions:

$$f(x)=\frac{A}{x-2}+\frac{Bx+C}{x^2+x+1}+\frac{Dx+E}{(x^2+x+1)^2}.$$

Multiplying by $(x-2)(x^2+x+1)^2$ we get the expression

$$A(x^2+x+1)^2+(Bx+C)(x-2)(x^2+x+1)+(Dx+E)(x-2)$$

After expanding and collecting in terms of x we obtain

$$\begin{cases}0=A+B\\ 5=2A-B+C\\ 1=3A-B-C+D\\ -1=2A-2B-C-2D+E\\ 7=A-2C-2E\end{cases}\Rightarrow\begin{cases}A=1\\ B=-1\\ C=2\\ D=-1\\ E=-5\end{cases}$$

This means that

$$f(x)=\frac{1}{x-2}+\frac{-x+2}{x^2+x+1}+\frac{-x-5}{(x^2+x+1)^2}.$$

Let us integrate the quadratic fractions:

$$\begin{aligned}\int\frac{-x+2}{x^2+x+1}\mathrm{d}x&=\int\frac{-2x+4\mathbf{-5+5}}{2(x^2+x+1)}\mathrm{d}x\\ &=-\frac{1}{2}\int\frac{2x+1}{x^2+x+1}\mathrm{d}x+\frac{5}{2}\int\frac{\mathrm{d}x}{x^2+x+1}\\ &=\underline{-\frac{1}{2}\log\left(x^2+x+1\right)+\frac{5}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}\end{aligned}$$

We deal with other one in a similar manner

$$\begin{aligned}\int\frac{-x-5}{(x^2+x+1)^2}\mathrm{d}x&=\int\frac{-2x-10\mathbf{+9-9}}{2(x^2+x+1)^2}\mathrm{d}x\\ &=\frac{1}{2}\int\frac{-(2x+1)}{(x^2+x+1)^2}\mathrm{d}x-\frac{9}{2}\int\frac{\mathrm{d}x}{(x^2+x+1)^2}\\ &=\underline{-\frac{1}{2(x^2+x+1)}}-\frac{9}{2}I.\end{aligned}$$

For the last integral I we shall complete the square

$$x^2+x+1=x^2+x+\frac{1}{4}+\frac{3}{4}=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}.$$

With the trigonometric substitution $x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan(\theta)$ we can simplify the integral into

$$\begin{aligned}I&=\int\frac{(\sqrt{3}/2)\sec^2(\theta)\mathrm{d}\theta}{(9/16)\sec^4(\theta)}=\frac{8\sqrt{3}}{9}\int\cos^2(\theta)\mathrm{d}\theta\\ &=\frac{8\sqrt{3}}{9}\int\left(\frac{1+\cos(2\theta)}{2}\right)\mathrm{d}\theta=\frac{8\sqrt{3}}{9}\left(\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right)\\ &=\underline{\frac{4\sqrt{3}}{9}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{2x+1}{3(x^2+x+1)}}\end{aligned}$$

The result of this integral is the collected sum of all the underlined terms.

Exercise 6. Compute the following integrals using their partial fraction decomposition. *In some of the integrals it might be possible to simplify the fraction first.*

I) $\displaystyle\int\frac{2u^2+3u+1}{(u^2+2u+5)(u^2+3u+2)}\mathrm{d}u$

II) $\displaystyle\int\frac{8}{3x^3+7x^2+4x}\mathrm{d}x$

III) $\displaystyle\int\frac{t^2+2t-8}{t^3-6t^2+4t-24}\mathrm{d}t$

IV) $\displaystyle\int\frac{4x-3}{x^3-3x^2}\mathrm{d}x$

V) $\displaystyle\int\frac{2s-2}{s^4-1}\mathrm{d}s$

VI) $\displaystyle\int\frac{3x^5-4x^4+x^3+x^2-24x-2}{(x-1)^2(x^2+4)^2}\mathrm{d}x$