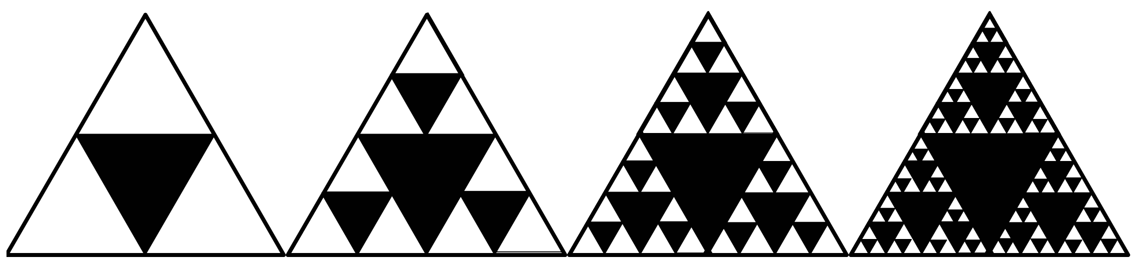
Exercise 1. Consider the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3}}{n^{5r}}$ where r is a real number and do the following:

I) Identify the general term of the series and simplify it.
II) Is our series a p-series? How can you justify that?
III) State the convergence result for p series. This means, given ANY p-series $\sum \frac{1}{n^p}$, for what values of p will that converge?
IV) Determine for which values of r will our series converge. [Hint: The answer to this item is NOT the same as the last item.]

1. The general term is $\frac{\sqrt{n^3}}{n^{5p}}$. We may simplify it to $\frac{1}{n^{5r-3/2}}$.
2. Indeed it's a p-series, as the general term looks like 1 over n to some power, in this case $5r - \frac{3}{2}$.
3. A general p series converges when $p > 1$.
4. We require that $5r - \frac{3}{2} > 1$ so solving for r we get $5r > \frac{5}{2}$ which means that $r > \frac{1}{2}$. So for values of r larger than $\frac{1}{2}$, our series converges.

Exercise 2 (Filling a triangle)**.** Consider an empty triangle of area A which we start filling with smaller triangles. The objective of this question is to determine if we can fill completely the triangle in question.



We start adding a triangle with area $a_1 = \frac{A}{4}$ in the middle, then the second step adds 3 triangles of area $\left(\frac{A/4}{4}\right) = \frac{A}{16}$. So in total we are adding an area of $a_2 = \frac{3A}{16}$.

I) In the third step, how many triangles do we add? What is the area of each of the new smaller triangles? In total how much area a_3 are we adding in the third step?
II) Derive a formula for the area added a_n at the n^{th} step by considering how many triangles are we adding and the area of each of those new triangles.
III) As a sequence, is a_n geometric? If it is, what's is initial term and common ratio?
IV) Consider the series $\sum_{n=1}^{\infty} a_n$, in terms of area, what does this series represent? What do the partial sums represent?
V) Write an expression for the partial sums of this series.
VI) Using the information above, determine if we fill up the triangle.

1. In the third step we are adding 9 triangles in total. Each triangle will have as area a fourth of the previous triangle that we added. In this case this is $\frac{A}{64}$, so the total area that we are adding is $\frac{9A}{64}$.
2. In the n^{th} step we are adding 3^{n-1} triangles each with area $\frac{A}{4^n}$. The area added is thus $a_n = \frac{3^{n-1}A}{4^n}$
3. The sequence a_n is geometric with initial term $c = \frac{A}{4}$ and common ratio $r = \frac{3}{4}$.
4. As a series, the infinite series represents the total area that is covered by the shaded triangles. The partial sums represents the total area at step m.

5. The partial sum is

$$S_m = \sum_{n=1}^{m} \frac{3^{n-1}A}{4^n} = A\left(\frac{1}{4} + \frac{3}{16} + \frac{9}{16} + \cdots + \frac{3^{m-1}}{4^m}\right)$$

from which we get

$$S_m - \frac{3}{4}S_m = A\left(\frac{1}{4} - \frac{3^m}{4^{m+1}}\right) \Rightarrow S_m = \frac{A\left(\frac{1}{4} - \frac{3^m}{4^{m+1}}\right)}{1 - \frac{3}{4}}.$$

6. The series $\sum_{n=1}^{\infty} \frac{3^{n-1}A}{4^n}$ is a convergent geometric series because $\frac{3}{4} < 1$. The limit value of this series can be found by computing the limit of the partial sums. We see that S_m tends to A as m grows. This means that in the limit, the figure fills out with the shaded triangles because the total area will be the area of the original triangle.