MATH 620: Variational Methods and Optimization I

Homework 2

Problem 1 (Affine subspaces, convexity). Consider the vector space $X = C^k([0,1])$ of functions that are k times continuously differentiable. We will often have to impose boundary conditions, so take the subset

$$D = \left\{ \varphi \in X : \varphi(0) = 2, \frac{\partial \varphi}{\partial x}(1) = 3 \right\} \subset X$$

of functions with a prescribed function value on the left, and prescribed derivative on the right end of the interval. (This clearly only makes sense if $k \geq 1$.)

(a) Show that this set is an affine subspace of X.

(b) Show that D is a convex set.

(c) State what the tangent space D' of D is.

(10 points)

Problem 2 (Directional derivatives). Take the function

$$f(\mathbf{x}) = \min\{|x_1|, |x_2|\}\text{sign}(x_1).$$

In class, we talked about the fact that the directional (Gateaux) derivative satisfies

$$Df(\mathbf{x}; \mathbf{v}) = (\nabla f(\mathbf{x})) \cdot \mathbf{v}.$$

For the current function, at the origin $\mathbf{x} = 0$, we have $\nabla f(0) = 0$,[1] but the directional derivative is not zero for all $\mathbf{v}$.

- Let's build intuition. Plot this function. Then use this visualization to convince yourself (and the reader of your answer) graphically that the statement about $\nabla f(0) = 0$ is true, as well as that the statement that the directional derivative is not zero in other directions is also true.
- Explain the discrepancy. What does this imply for the viability of the idea that we can look for points with $\nabla f = 0$ when searching for minima of functions?

(10 points)

[1] That's because the gradient is defined as $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)^T$ and because the function f is constant (equal to zero) along both the x_1 and x_2 axes.

Problem 3 (Directional derivatives). Take the following variation of the function of the previous problem:

$$f(\mathbf{x}) = \min\{|x_1|, |x_2|\}.$$

Visualize this function. Then show rigorously that this function still has $\nabla f(0) = 0$, but that it does not have a Gateaux derivative $Df(\mathbf{x}; \mathbf{v})$ for all $\mathbf{v}$.

(10 points)

Problem 4 (Subdifferentials). For smooth functions, we look for minima among those points $\mathbf{x}$ at which $\nabla f(\mathbf{x}) = 0$. But for functions that are not differentiable, this is clearly not the way to find candidate points.

In cases where the objective function is not differentiable, the "sub-differential" $\partial f(\mathbf{x})$ is the way out: A point is a minimizer of f if $\mathbf{0} \in \partial f(\mathbf{x})$. (You may wonder why it's actually a minimizer and could not just be a maximizer: that's already encoded in the *sub* part of *sub*differential.) Read up on how the subdifferential is defined. Then consider the functions

$$\begin{aligned} f_1(\mathbf{x}) &= |x_1| + |x_2|, \\ f_2(\mathbf{x}) &= \sqrt{|x_1|^2 + |x_2|^2}, \\ f_\infty(\mathbf{x}) &= \max\{|x_1|, |x_2|\}. \end{aligned}$$

For each, of these, (a) determine the subdifferential $\partial f(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^2$; (b) plot $\partial f(\mathbf{0})$ when evaluated at the origin $\mathbf{x} = \mathbf{0}$; (c) verify that the necessary condition for minima $\mathbf{0} \in \partial f(\mathbf{x})$ is only satisfied at the origin.

(20 points)

Bonus problem (Subdifferentials). From the results of the previous problem, one might be tempted to speculate that the subdifferential $\partial f_p(\mathbf{0})$ of $f_p(\mathbf{x}) = \|\mathbf{x}\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$ might be the unit ball $\{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\|_q \leq 1\}$, where q depends on p so that they satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Prove or disprove this. **(10 bonus points)**