Exercise 1. Let G be a Lie group.

(a) Show that the set of right-invariant vector fields on G forms a Lie algebra with bracket given by the Lie bracket of vector fields. Note that the right-invariant vector fields form a vector space which is isomorphic to T_eG.

(b) Let $\mathrm{inv} : G \to G$ be given by $\mathrm{inv}(g) = g^{-1}$. Prove that if X is a left-invariant vector field on G, then $\mathrm{d}\,\mathrm{inv}(X)$ is a right-invariant vector field whose value at e is $-X_e$.

(c) Prove that the map $-d\,\mathrm{inv}$ from left-invariant vector fields to right-invariant vector fields is a Lie algebra isomorphism. (The point is that we could just have well chosen to interpret the Lie algebra of G as the right-invariant vector fields rather than the left-invariant ones.)

I will allow myself just a quick refresher. If we have a diffeomorphism $F : M \to N$, then there is a pushforward onto tangent spaces:

$$F_* \mid_x : T_xM \to T_yN, \quad y = F(x).$$

This in turn can map vector fields into vector fields via the rule

$$(F_* \mid_x)(X \mid_x) = (F_*X) \mid_y.$$

Recall a left-invariant vector field is $X \in \mathfrak{g}$ defined by

$$\mathrm{d}L_gX = X, \quad \text{equivalently} \quad (\mathrm{d}L_g)_h\,(X(h)) = X(gh), \quad h \in G.$$

In regards to the previous terms this means that

$$(L_g)_*X = X, \quad \text{or} \quad ((L_g)_* \mid_h)(X(h)) = ((L_g)_*X) \mid_{L_g(h)} = X(gh),$$

for $h \in G$. So this means that a right-invariant vector field arises from the right action of G on itself given by $R_g(h) = hg$. The right-invariant vector fields form a set $\mathfrak{g}^R$ determined by the rule

$$(R_g)_*X = X \iff ((R_g)_* \mid_h)(X(h)) = X(hg).$$

Restating finally lemma 3.3.8 for $M = N = G$ and $f = R_g$ we have the key ingredient for the first part:

$$(\mathrm{d}R_g)_h([X, Y](h)) = [X, Y](R_g(h)) = [X, Y](hg).$$

Answer

(a) To prove that $\mathfrak{g}^R$ is a Lie algebra, it suffices to see that the Lie bracket of vector fields obeys that same right-invariance. Let $g, h \in G$ and take $X^R, Y^R \in \mathfrak{g}^R$, we wish to see that

$$((R_g)_* \mid_h)([X^R, Y^R](h)) = [X^R, Y^R](hg).$$

Lemma 3.3.8 from the notes asserts that this is the case and thus, we have the desired result.

(b) The map $\mathrm{d}\,\mathrm{inv}$ is the pushforward of the map $g \mapsto g^{-1}$ in G. Assume then that $X \in \mathfrak{g}$, applying $\mathrm{d}\,\mathrm{inv}$ to our left-invariance relation we have

$$\begin{aligned} (L_g)_* X &= X \\ \Rightarrow \mathrm{inv}_* (L_g)_* X &= \mathrm{inv}_* X \\ \Rightarrow (R_{g^{-1}})_* \mathrm{inv}_* X &= \mathrm{inv}_* X \end{aligned}$$

so that $\mathrm{inv}_* X$ is a right-invariant vector field. The implication from second to third line comes from the fact that

$$\mathrm{inv} \circ L_g = R_{g^{-1}} \circ \mathrm{inv}$$

and then pushforwarding from G to $\mathfrak{g}$.

I couldn't quite piece the value at the identity. But we have the following:

- Every right-invariant vector field X^R's values at any point $g \in G$ is determined via the formula

$$X^R(g) = (\mathrm{d}R_g)_e(X^R(e)),$$

  i.e. by X^R's value at the identity.
- Same business happens for LIVF's:

$$X(g) = (\mathrm{d}L_g)_e(X(e)).$$

As $\mathrm{d}\,\mathrm{inv}\, X$ is a RIVF, we have

$$(\mathrm{inv}_* X)(g) = ((R_g)_*)_e((\mathrm{inv}_* X)(e)).$$

Via our previous relation this is

$$((R_g)_*)_e((\mathrm{inv}_* X)(e)) = \mathrm{inv}_*((L_{g^{-1}})_*)_e X(e) = \mathrm{inv}_* X(g^{-1}).$$

Now, consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\text{inv}} & G \\ {\scriptstyle X}\big\downarrow & & \big\downarrow \\ \mathfrak{g} & \xrightarrow[\text{inv}_*]{} & \mathfrak{g} \end{array}$$

Assuming the diagram commutes, then

$$\text{inv}_*(X(g)) = X(\text{inv}(g)) = X(g^{-1}).$$

This is the only relation I can find between $\text{inv}_* X$ and X itself. As others only have either X's or $\text{inv}_* X$. Returning this to the relation we get

$$((R_g)_*)_e((\text{inv}_* X)(e)) = X(g)$$

as inv is involutive. Setting $g = e$ doesn't lead me anywhere reasonable, as I can't exactly see where to go from here. Where does the minus sign come from?

(c) Finally, let us assume the second item. It suffices to see that the Lie bracket is preserved. To that effect, take two vector fields X, Y and observe that

$$-\text{inv}_*[X,Y] = [X,Y]^R = [X^R, Y^R] = [-\text{inv}_* X, -\text{inv}_* Y].$$

Exercise 2. Consider the special orthogonal group $\mathrm{SO}(3)$ of all 3×3 matrices B such that

$$BB^T = I \qquad \text{and} \qquad \det B = 1.$$

We saw in section 3.4 that

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

gives a basis for $T_I \mathrm{SO}(3)$ so that $[A_1, A_2] = A_3$, $[A_2, A_3] = A_1$, and $[A_3, A_1] = A_2$. Notice also that $A_3 = \gamma_3'(0)$, where

$$\gamma_3(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and there are similar curves whose tangents at the identity give A_1 and A_2.

Let V_1, V_2, and V_3 be the corresponding left-invariant vector fields on $\mathrm{SO}(3)$; i.e., $V_i(I) = A_i$.

Let α_i be the dual basis of left-invariant 1-forms and compute their exterior derivatives.

Answer

In my attempt to answer this question I've been reminded of the fact that $\mathfrak{so}(3)' = \mathfrak{so}(3)$. I must admit I don't know exactly how to apply this fact.
On the other hand, I would like to follow example 2.7.6 but I'm completely lost on how to do it. Say I have the 1-form α_1 and I want to find

$$\mathrm{d}\alpha_1(V_2, V_3) = \cdots = \mathcal{L}_{V_2}\alpha_1(V_3).$$

But the calculation doesn't go exactly like that example. I need to discuss it.

Exercise 3. Let G be a compact Lie group and assume $\langle \cdot, \cdot \rangle$ is an Ad-invariant inner product on $\mathfrak{g}$ (an Ad-invariant inner product on $\mathfrak{g}$ is one that satisfies $\langle X, Y \rangle = \langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y \rangle$ for all $g \in G$ and for any $X, Y \in \mathfrak{g}$).

Define $\tau_e : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by

$$\tau(X, Y, Z) = \langle [X, Y], Z \rangle.$$

(a) Show that τ_e is alternating. Since τ_e is clearly multilinear, this means τ_e can be identified with an element of $\bigwedge^3(\mathfrak{g}^*)$.

(b) Extend τ_e to a left-invariant 3-form on G in the usual way: for each $g \in G$, define $\tau_g := L^*_{g^{-1}}\tau_e$. Prove that $\tau \in \Omega^3(G)$ is bi-invariant (Hint: feel free to use the fact that a left-invariant form is bi-invariant if and only if it is conjugation-invariant). The bi-invariant 3-form τ is called the *fundamental 3-form* of the Lie group G.

(c) Explicitly compute the fundamental 3-form of $\mathrm{SO}(3)$ in terms of the α_i from the previous problem.

Answer

(a) It's clear that if $X = Y$ then τ is zero, but when $Z = X$ we have

$$\langle [X, Y], X \rangle = \langle \mathrm{Ad}_g[X, Y], \mathrm{Ad}_g X \rangle.$$

Expanding the adjoint map and using bilinearity we get

$$\langle gXYg^{-1}, gXg^{-1} \rangle - \langle gYXg^{-1}, gXg^{-1} \rangle.$$

Applying the Ad invariance again doesn't lead anywhere. I can't quite figure out where to proceed.

I didn't give myself the opportunity to try the other parts and didn't understand how to apply the hint :(