**Exercise 1** (1.3.C). Show that  $A \to S^{-1}A$  is injective if and only if S contains no zero divisors

**Lemma 1.** A map f in Ring is injective if an only if  $ker(f) = \{0\}$ .

#### Proof

Suppose f is injective, then if  $x \in \ker(f)$  it holds that f(x) = 0. As f is a morphism of rings 0 = f(0) which means that f(x) = f(0) and as f is injective, we can conclude that x = 0 meaning that the kernel is trivial.

On the flip-side, take f(x) = f(y). As f is a morphism, f(x - y) = 0. But then  $x - y \in \ker(f)$  which means that x - y = 0, letting us conclude that x = y. Thus f is injective.

#### **Answer**

Call  $\pi(a) = \frac{a}{1}$  the canonical map from A to  $S^{-1}A$ . We will prove that  $\ker(\pi) = \{0\}$  if and only if S has no zero divisors.

To that effect suppose  $s \in S$  is a zero divisor. This means that

$$\exists t(t \neq 0 \land st = 0).$$

Now  $\pi(t) = \frac{t}{1}$ . But inside  $S^{-1}A$  we have  $\frac{t}{1} = \frac{0}{1}$  because

$$\frac{t}{1} = \frac{0}{1} \iff \exists t' \in S(t'(t \cdot 1 - 0 \cdot 1) = 0).$$

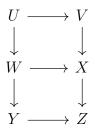
Namely, such a t' would be s. So  $\pi(t) = 0$  and, as  $t \neq 0$ , we have that  $\ker(\pi)$  is not trivial.

On the other hand suppose  $s, t \in S$  are elements that satisfy st = 0. For the sake of argument suppose  $t \neq 0$ , we are set to prove that s = 0 and to do this, we'll show that  $s \in \ker(\pi)$ . Notice that

$$\pi(s) = \frac{s}{1} = \frac{0}{1} \iff \exists s' \in S(s'(s \cdot 1 - 1 \cdot 0) = 0).$$

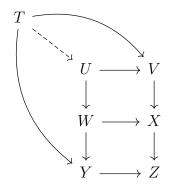
The s' we are looking for is s' = t. So, it follows that  $s \in \ker(\pi)$ . But as  $\pi$  has trivial kernel, s = 0 which is what we wanted.

**Exercise 2** (1.3.Q). Describe the colimit of the diagram  $F: J \to \mathsf{Set}$  given by  $* \leftarrow * \to *$ . If the two squares in the following commutative diagram are Cartesian diagrams, show that the "outside rectangle" (involving U, V, Y, and Z) is also a Cartesian diagram.



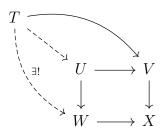
## Answer

Suppose that there's an object T with maps to U and Y as in the following diagram:



We wish to show that there's a unique morphism  $T \to U$  such that  $T \to V$  and  $T \to Y$  factor through  $T \to U$ .

Our first step is to construct a unique morphism from  $T \to W$ . This is done because we have a morphism  $T \to X$  (the composition of  $T \to V$ ,  $V \to X$ ) and a morphism  $T \to Y$  whose compositions to Z agree. By universal property of the pullback (W as pullback) we have that there's a unique morphism  $T \to W$  through which the respective morphisms factor.



With this in hand, we now have  $T \to V$  and  $T \to W$  whose compositions to X agree. So by universality of U, there exists a unique morphism  $T \to U$  through which the corresponding morphisms factor.

**Exercise 3** (1.3.T). Show that coproduct for Set is disjoint union.

## Answer

Recall that the disjoint union of  $A_1$  and  $A_2$  is defined as a set as

$$A_1 \cup A_2 \{ (a_i, i) : a_i \in A_i, I = 1, 2 \}.$$

This allows us to define maps  $\iota_i: A_i \to A_1 \odot A_2, \ x \mapsto (x,1)$  which are morphisms in Set because they're defined everywhere. We are to show that this set satisfies the universal property of coproducts.

Suppose B is a set such that  $f_i: A_i \to B$  are well defined. We must define a unique function  $g: A_1 \cup A_2 \to B$  such that  $f_i = g\iota_i$ , this is done as follows:

$$g(a,i) = \begin{cases} f_1(a), & i = 1, \\ f_2(a), & i = 2. \end{cases}$$

We verify the factoring property:

$$g \circ \iota_1(a_1) = g(a_1, 1) = f_1(a_1), \quad g \circ \iota_2(a_2) = g(a_2, 2) = f_2(a_2).$$

By construction, we have defined g uniquely.

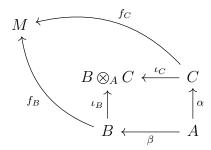
**Exercise 4** (1.3.U). Suppose  $A \to B$  and  $A \to C$  are two ring morphisms, so in particular B and C are A-modules. Recall that  $B \otimes_A C$  has a ring structure.

- i) Show that there is a natural morphism  $\iota_B: B \to B \otimes_A C, \ b \mapsto b \otimes 1.$  Similarly for C.
- ii) Show that this gives a pushout on rings. In other words, the following diagram satisfies the universal property of the pushout.

$$\begin{array}{ccc}
B \otimes_A C & \stackrel{\iota_C}{\longleftarrow} C \\
 \downarrow_B & & \uparrow^{\alpha} \\
B & \stackrel{\beta}{\longleftarrow} A
\end{array}$$

### **Answer**

- i) The map  $\iota_B(b) = b \otimes 1$  is a homomorphism in virtue that  $B \otimes_A C$  is a tensor product. By construction, all bilinear maps factor through the tensor product as linear maps. This map is one of the factors which should be linear. The same holds for C.
- ii) Let us now take M an A-module with morphisms  $f_B: B \to M$  and  $f_C: C \to M$ . This can described by the following diagram:



However, let us take advantage of the tensor product, *gatekeeper of bilinear maps*. This morphisms can be combined into a bilinear map from  $B \times C \rightarrow M$ . We define

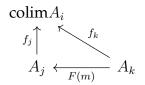
$$f: B \times C \to M, (b, c) \mapsto f_B(b) f_C(c)$$

and by universal property of the tensor product, there exists a unique map  $\tilde{f}: B \otimes_A C \to M$  through which f factors. Finally  $f_B$  and  $f_C$  factor through  $\tilde{f}$  by diagram chasing and thus by universality of the tensor product we have that it satisfies the pushout universal property in this case.

**Exercise 5.** Describe the colimit of the diagram  $F: J \to \mathsf{Set}$  given by  $* \leftarrow * \to *$ .

# Answer

Recall that the colimit of a diagram  $F: J \to \mathsf{C}$  is an object  $\mathrm{colim} A_i \in \mathsf{ObjC}$  with morphisms  $f_j: A_j \to \mathrm{colim} A_i$  such that if  $m: k \to j$  is a morphism in J, then the following diagram commutes



In our case, since we only have three objects the diagram looks like this

$$\begin{array}{ccc}
CL &\longleftarrow & C \\
\uparrow & & \uparrow \\
B &\longleftarrow & A
\end{array}$$

where CL is the colimit object. In this particular case the colimit coincides with the pushout by universality.

**Exercise 6** (1.4.F). Verify that the *A*-module described above is indeed the colimit.

The *A*-module in question is  $\coprod A_i/_{\sim}$  where  $\sim$  is the relation

$$(a_i, i) \sim (a_j, j) \iff \exists (f : A_i \to A_k, g : A_j \to A_k)(f(a_i) = g(a_j)),$$

addition of  $m_i \in M_i$ ,  $m_j \in M_j$  is defined as

$$m_i + m_j := F(u)(m_i) + F(v)(m_j)$$

where u, v are arrows from i, j to  $\ell$ . The sum lies in  $M_{\ell}$ . Multiplication is defined in an obvious<sup>1</sup> way and the zero element is  $m_i$  such that there is an arrow  $u: i \to k$  for which  $F(u)(m_i) = 0$ .

I must admit I wasn't able to tackle this problem.

<sup>&</sup>lt;sup>1</sup>It's not obvious to me.