

MATH676 — Tropical Geometry

Based on the lectures by Renzo Cavalieri

Notes written by Ignacio Rojas

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Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This is a topics course on this stuff

Requirements

Knowledge on stuff

TO DO:

- ◇ Write info on course description and requirements.
- ◇ Polish notes from TG11 about primitive vectors and pages 6-8
- ◇ Write Interim about grobner complexes
- ◇ Check lines 2274 and 2279

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Chapter 1

Combinatorial Shadow of Algebraic Geometry

1.1 Day 1 | 20230821

Think of an algorithm where the input is an algebraic variety and the output is a combinatorial object, a piecewise linear object.

Example 1.1.1. Consider as an input a line in the plane. Say $V(x + y - 1)$, then an output would be a tropical line. If we remain in the plane and consider a higher degree polynomial, say an elliptic curve, as an output we obtain a tropical cubic.

Leaving the plane behind and thinking of abstract nodal curves, we can think of a sphere attached to a torus which is attached to a genus 2 torus, then the corresponding object is what we call the dual graph.

Right now we do not know the specific algorithm, but we can observe that the outputs are *more simple* than the inputs. So the important question is:

What algebraic information does the simplified object remember? How do we extract the information the object remembers? And once we know how to work with this objects, can we return to algebraic geometry from any kind of these objects?

Observe that the number of ends which go to infinity corresponds with the degree.

1.2 Day 2 | 20230823

Algebraic Geometry on \mathbb{T}

Let us talk about ways to get into tropical geometry. We will first define the tropical semifield which the base set over which we will do algebraic geometry.

Definition 1.2.1. The tropical semifield is the set $(\mathbb{R} \cup \{-\infty\})$ equipped with tropical addition and multiplication:

$$\begin{cases} x \oplus y = \max(x, y) \\ x \odot y = x + y \end{cases}$$

With this set we can make multivariable polynomials

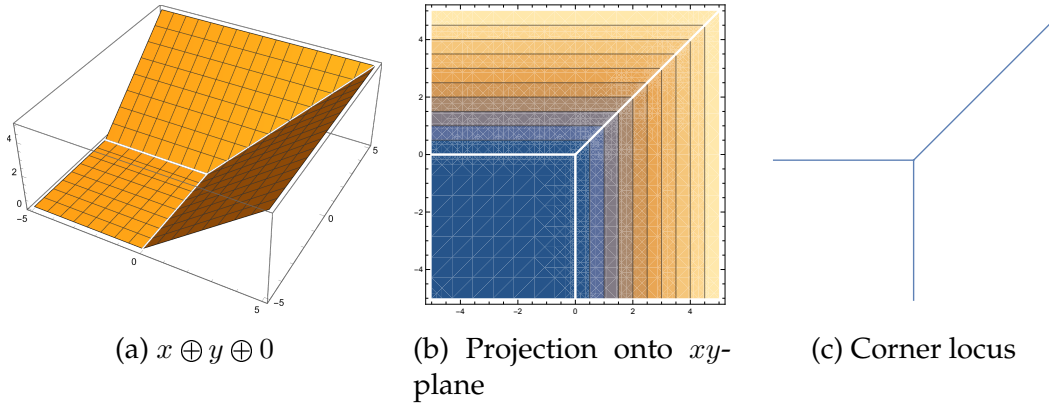
$$p(x_1, \dots, x_n) : (\mathbb{R} \cup \{-\infty\})^n \rightarrow \mathbb{R} \cup \{-\infty\}$$

which gives rise to their *tropicalization*, a piecewise linear function $\text{Trop}(p) : \mathbb{R}^n \rightarrow \mathbb{R}$.

Example 1.2.2. Consider the polynomial

$$p(x, y) = x \oplus y \oplus 0 \in (\mathbb{R} \cup \{-\infty\})[x, y].$$

When tropicalizing we get the piecewise linear function $\text{Trop}(p)(x, y) = \max(x, y, 0)$ which goes from \mathbb{R}^2 to \mathbb{R} . Observe that the surface is not smooth where the planes meet, this is



what we will call the *corner locus* or *tropical hypersurface*.

Definition 1.2.3. The tropical hypersurface $V(\text{Trop}(p))$ is the codimension 1 locus in \mathbb{R}^n where the function is non-linear (corner locus).

Example 1.2.4. If we consider higher degree tropical polynomials, they will become linear in the usual sense. Consider

$$p(x) = 3x^2 = 3 \odot x \odot x = 3 + x + x = 3 + 2x$$

which is indeed linear with respect to usual sum and product.

Valued fields

Definition 1.2.5. The field of Puiseux series or rational functions over \mathbb{C} is $\mathbb{C}(t)$ where the elements are of the form

$$f(t) = \sum_{i=k_0}^{\infty} a_i t^{i/n}.$$

The lower bound k_0 could be negative and the exponents, are rational with bounded denominators.

If we have not seen Puiseux series before it would be useful to consider a quick example:

Example 1.2.6. Entirely by definition, consider the Puiseux series

$$f(t) = \sum_{i=-12}^{\infty} \frac{-i}{12} t^{i/6}.$$

Expanding out the first few terms we get

$$\frac{12}{12} t^{-12/6} + \frac{11}{12} t^{-11/6} + \frac{10}{12} t^{-10/6} + \frac{9}{12} t^{-9/6} + \dots = t^{-2} + \frac{11}{12} t^{-11/6} + \frac{5}{6} t^{-5/3} + \frac{3}{4} t^{-3/2} + \dots$$

This field can be equipped with a valuation

$$\text{val}_0 : \mathbb{C}(t) \rightarrow \mathbb{R} \cup \{\infty\}, \begin{cases} 0 \mapsto \infty \\ f \mapsto \text{order of vanishing at } 0. \end{cases}$$

This order of vanishing is the value α such that f/t^α approaches a finite non-zero value. Formally we can express this as

$$\text{val}_0(f) = \min \left\{ \alpha \leq \infty : \lim_{t \rightarrow 0} \frac{f}{t^\alpha} \in]0, \infty[\right\}.$$

The corresponding coefficient in the series expansion of f for this value is called the valuation coefficient.

Example 1.2.7. In our original example, we can see that if we divide f by t^{-2} (or similarly multiply by t^2) we get

$$t^2 f(t) = 1 + \frac{11}{12} t^{1/6} + \frac{5}{6} t^{1/3} + \frac{3}{4} t^{1/2} + \dots \xrightarrow{t \rightarrow 0} 1 + 0 + 0 + \dots$$

If we were to divide by a lower power of t , say t^{-3} , we would get

$$t^3 f(t) = t + \frac{11}{12} t^{7/6} + \frac{5}{6} t^{4/3} + \frac{3}{4} t^{3/2} + \dots \xrightarrow{t \rightarrow 0} 0 + 0 + \dots$$

and even if zero is a finite value, we have stated that the order of vanishing makes f/t^α approach a finite **non-zero** value.

While on the other hand if we divide by a higher value than -2 , say -1 , then we get

$$tf(t) = t^{-1} + \frac{11}{12}t^{-5/6} + \frac{5}{6}t^{-2/3} + \frac{3}{4}t^{-1/2} + \dots \xrightarrow{t \rightarrow 0} \infty.$$

So in this case, we get a **non-finite** value. At least with this example, it gives us the intuition that the order of vanishing is a unique value.

We now ask, how does the order of vanishing behave when operating functions algebraically? What happens to the order of vanishing when you add two functions? Or when we multiply them?

Example 1.2.8. Consider two small functions $f(t) = t^2$ and $g(t) = t^3$, then $f + g = t^2 + t^3$ which has order of vanishing 2. Observe that $2 = \min(2, 3)$.

In general what happens is that

$$\text{val}_0(f + g) \geq \min(\text{val}_0 f, \text{val}_0 g), \quad \text{and} \quad \text{val}_0(fg) = \text{val}_0(f) + \text{val}_0(g).$$

We can do algebraic geometry over this field! Let \mathbb{K} be the field of Puiseux series, if $p(x_1, \dots, x_n) \in \mathbb{K}[x_1, \dots, x_n]$ then we consider the algebraic variety

$$X = V(p) = \{ \mathbf{x} \in \mathbb{K}^n : p(\mathbf{x}) = 0 \} \subseteq \mathbb{K}^n.$$

Each entry of \mathbf{x} is a Puiseux series we can take the valuation of. So the image through the n -fold valuation of X will be a set in $(\mathbb{R} \cup \{ \infty \})^n$. We will call the tropicalization of X the image through this map. This is the tropical hypersurface for p .

$$\begin{array}{ccc} \text{val}_0 : \mathbb{K}^n & \longrightarrow & (\mathbb{R} \cup \{ \infty \})^n \\ \text{UI} & & \text{UI} \\ V(p) & \longmapsto & \overline{\text{val}_0(V(p))} \\ \parallel & & \parallel \\ \{ \mathbf{x} : p(\mathbf{x}) = 0 \} & & \text{Trop}(V(p)) \end{array}$$

Figure 1.2: Diagram on the tropical hypersurface

Example 1.2.9. Consider the polynomial in $\mathbb{K}[x, y]$

$$p(x, y) = tx + y + t^2,$$

then the variety is $X = \{ (x, y) : tx + y + t^2 = 0 \}$. We can solve the equation to $y = -tx - t^2$.

If we choose $x = 0$ then y becomes $-t^2$. Now we take the valuation of $(0, -t^2)$ and so $(\infty, 2)$ is a point in $\text{Trop}(X)$.

Amoebas

Let us return to the usual stage and consider $p \in \mathbb{C}[x_1, \dots, x_n]$ which defines an algebraic variety $X = V(p) \subseteq \mathbb{C}^n$. Now consider the map which sends every coordinate's modulus to its logarithm in base t :

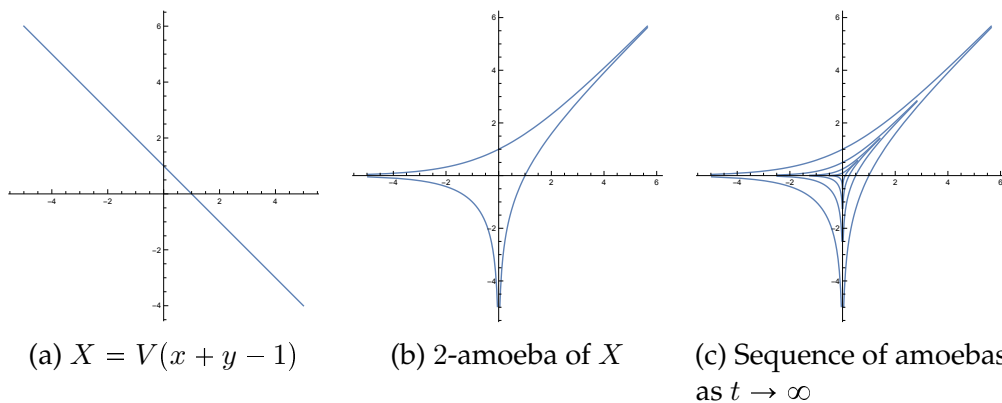
$$\mathbb{C}^n \rightarrow (\mathbb{R} \cup \{-\infty\})^n, \quad (z_1, \dots, z_n) \rightarrow (\log_t |z_1|, \dots, \log_t |z_n|).$$

The image of X under this map, $\log_t(X)$, is the t -amoeba of X . If we take the limit as $t \rightarrow \infty$ then we get the *spine* of the amoeba.

Example 1.2.10. When $p(x, y) = x + y - 1$ then we can describe $V(p)$ via the parametrization $(x, 1 - x)$. So the corresponding t -amoeba in the real case is

$$\{ (\log_t |x|, \log_t |1 - x|) : x \in \mathbb{R} \}$$

and we ordinarily take the limit $\lim_{t \rightarrow \infty} \frac{\log |x|}{\log t}$, we see that the functions converge to zero point-by-point. But the set is actually approaching the spine!



Observe that the spine approaches the tropical hypersurface associated to p . In other words we have that the tropical hypersurface is $\lim_{t \rightarrow \infty} \log_t(V(p))$.

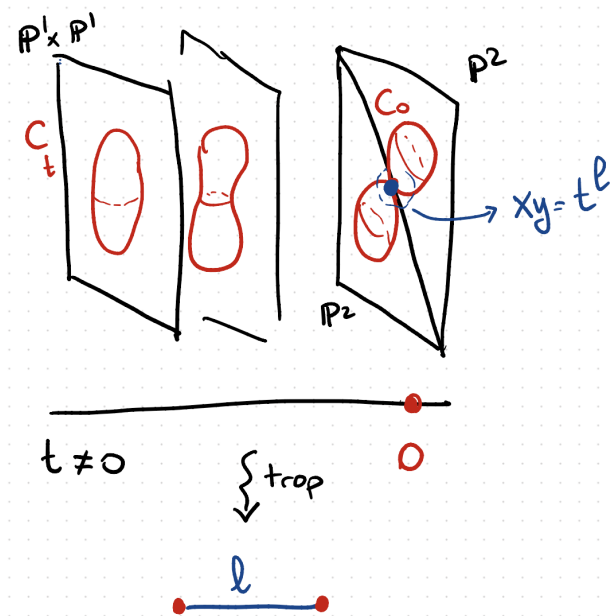
Degenerations

We may parametrize any algebraic variety with a time variable, then converting the information to a graph, edges code the information about how fast the node forms related to the length.

Consider a family of **of what, what is this family of?! Stuff? Curve in $P_1 \times P_1$ which eventually becomes P_2 ?**

It is too early to understand this point of view. We will set everything up to get to it.

In general, the big idea will be to explore and understand these perspectives in the case of plane curves. We want to show how they are equivalent and then recover classical algebraic geometry results in terms of tropical geometry.



1.3 Day 3 | 20230825

Recall that the last time we discussed the classical (25 to 30 years old) ways to get to tropical geometry. We now would like to answer the question

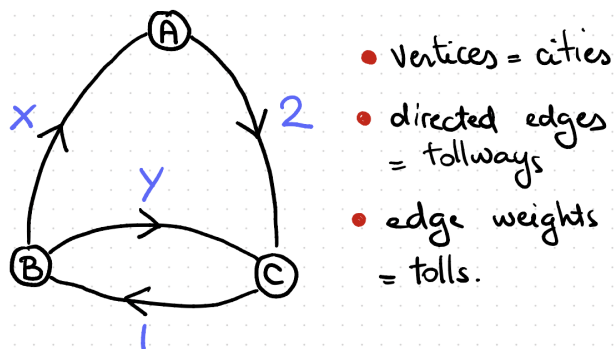
Where do tropical numbers come from?

So let us begin with an applications problem and see how the tropical numbers arise from the context of the problem.

Tropical Arithmetics

Minimizing Tolls

Consider a set of cities connected by a network of toll-ways: If we only care about minimizing



toll expenses when traveling, what would be the cheapest way to go from one given city to

another? Let us record the information as an incidence matrix:

$$M_{ij} = \text{price of going from city } i \text{ to city } j \text{ in at most one trip} \Rightarrow M = \begin{pmatrix} 0 & \infty & 2 \\ x & 0 & y \\ \infty & 1 & 0 \end{pmatrix}$$

In this matrix, the rows determine the outbound city, while the columns are the destination. Each entry records the cost of a toll and tolls are considered to be infinite when the road does not exist. We can also think of M as recording the cheapest toll to go from one city to another with at most one move.

How would we compute the best strategy of going from city i to j in *at most two trips*? If for example we want to find trips from A to B in two steps then we have three choices:

$$AAB, \quad ABB, \quad ACB.$$

The costs of each one are

$$(0, \infty), \quad (\infty, 0), \quad (2, 1)$$

so we sum them and take the minimum. That will be the optimal route from A to B in two steps. In fact, if we relate this to the entries of the matrix M , we could use M^2 . However we must redefine our basic operations as follows:

$$+ = \min, \quad \cdot = +$$

So we have the identification

$$(1, 2) \text{ entry of } M^2 = \sum_{j=1}^3 M_{1j} M_{j2} = \min(M_{11} + M_{12}, M_{12} + M_{22}, M_{13} + M_{32}).$$

In general:

$$\begin{aligned} \begin{pmatrix} 0 & \infty & 2 \\ x & 0 & y \\ \infty & 1 & 0 \end{pmatrix}^2 &= \begin{pmatrix} \min \begin{pmatrix} 0+0 \\ \infty+x \\ 2+\infty \end{pmatrix} & \min \begin{pmatrix} 0+\infty \\ \infty+0 \\ 2+1 \end{pmatrix} & \min \begin{pmatrix} 0+2 \\ \infty+y \\ 2+0 \end{pmatrix} \\ \min \begin{pmatrix} x+0 \\ 0+x \\ y+\infty \end{pmatrix} & \min \begin{pmatrix} x+\infty \\ 0+0 \\ y+1 \end{pmatrix} & \min \begin{pmatrix} x+2 \\ 0+y \\ y+0 \end{pmatrix} \\ \min \begin{pmatrix} \infty+0 \\ 1+x \\ 0+\infty \end{pmatrix} & \min \begin{pmatrix} \infty+\infty \\ 1+0 \\ 0+1 \end{pmatrix} & \min \begin{pmatrix} \infty+2 \\ 1+y \\ 0+0 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 3 & 2 \\ x & \min(0, y+1) & \min(x+2, y) \\ 1+x & 1 & \min(0, 1+y) \end{pmatrix}. \end{aligned}$$

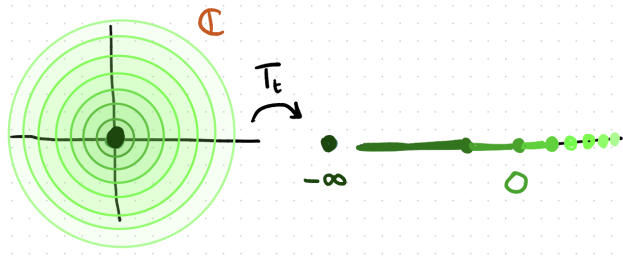
Observe that $1+y$ can be the minimum in the diagonal when we allow *negative tolls*.

Remark 1.3.1. If we disallow negative tolls, the products M^n eventually stabilize to a matrix whose entries record the cheapest way to get from one city to another in n steps.

This gives us an intuition that minimization problems correspond to linear algebra problems over $(\mathbb{T}, +, \cdot)$ which is precisely $(\mathbb{R} \cup \{\infty\}, \min, +)$.

Forgetting phases

Consider the map $T_t : \mathbb{C} \rightarrow \{-\infty\} \cup \mathbb{R}$, $z \mapsto \log_t(|z|)$. This map is surjective, and this we



can see by checking it. Any $x \in \mathbb{R}$ is

$$T_t(t^x e^{i\theta}) = \log_t |t^x e^{i\theta}| = \log_t t^x = x, \quad \theta \in \mathbb{R}.$$

This means that the inverse image of a point contains a plethora of points, in fact:

$$\begin{cases} T_t^{-1}(x) = \{t^x e^{i\theta}\} \subseteq \mathbb{C}, & \text{for } x \in \mathbb{R}, \\ T_t^{-1}(-\infty) = 0. \end{cases}$$

With this in hand, we wish to define an exotic addition and multiplication on $\{-\infty\} \cup \mathbb{R}$ using T_t . We will dequantize!

We begin with **hyper-addition**, the output will be a subset of $\{-\infty\} \cup \mathbb{R}$ so it's not a binary operation by itself.

$$x \oplus_t y := T_t(T_t^{-1}(x) + T_t^{-1}(y)) = [\log_t(|t^x - t^y|), \log_t(t^x + t^y)].$$

This is an interval in $\{-\infty\} \cup \mathbb{R}$, in order to make \oplus_t into an operation we take a limit:

$$\begin{array}{ccc} x \oplus_t y & \xrightarrow{\lim_{t \rightarrow \infty}} & x \oplus y = \lim_{t \rightarrow \infty} x \oplus_t y \\ \downarrow \max & & \downarrow \max \\ x +_t y & \xrightarrow{\lim_{t \rightarrow \infty}} & x + y = \max(x, y) \end{array}$$

Remark 1.3.2. Note that \diamond is still a hyperoperation. Its output is not a singleton *only* when adding a number to itself:

$$x \diamond y = \begin{cases} \max(x, y), & x \neq y \\ [-\infty, x], & x = y \end{cases}$$

Formally this process, taking a limit of a family of operations, is known as *dequantization*. In the case of multiplication, the process goes a lot smoother when defining it:

$$x \cdot y = T_t [T^{-1}(x) \cdot T^{-1}(y)] = \log_t [(t^x e^{i\theta})(t^y e^{i\varphi})] = \log_t t^{x+y} = x + y$$

Example 1.3.3. Let us consider a small example like summing 2 and 4. Observe that

$$4 \diamond_t 2 = T_t(T_t^{-1}(4) + T_t^{-1}(2)) = T_t |t^4 e^{i\theta} + t^2 e^{i\varphi}|$$

and the term on the inside can be simplified to $t^4(e^{i\theta} + t^{-2}e^{i\varphi})$. T_t takes that expression to

$$4 + \log_t |e^{i\theta} + t^{-2}e^{i\varphi}| = 4 + \frac{\log |e^{i\theta} + t^{-2}e^{i\varphi}|}{\log t}.$$

What happens if we take the limit as $t \rightarrow \infty$? We get an independent from t result! The term on the right vanishes and we are left with $4 = \max(4, 2)$. So it got a tad bit better, but it's still a hyperoperation!

Exercise 1.3.4. Check how the definition of $+$ and \cdot extend to the *number* $-\infty$.

The point of this exercise is to operate $-\infty$ with finite numbers and itself.

For a finite x we will find $x + (-\infty)$. This is the limit of the previous hyperoperation:

$$x \diamond_t (-\infty) = T_t(T_t^{-1}(x) + T_t^{-1}(-\infty)) = T_t(T_t^{-1}(x) + 0) = T_t(T_t^{-1}(x)) = x.$$

If we let t grow, the result doesn't change and so this goes according to $\max(x, -\infty) = x$.

On the other hand when taking the product:

$$x \cdot (-\infty) = T_t [T^{-1}(x) \cdot T^{-1}(-\infty)] = T_t [T^{-1}(x) \cdot 0] = T_t(0) = \log_t(0) \rightarrow -\infty$$

which is also similar to the notion of $x + (-\infty) = -\infty$.

We can now proceed to operate $-\infty$ with itself:

$$(-\infty) \diamond_t (-\infty) = T_t(0) = \log_t(0) = -\infty = \max(-\infty, -\infty),$$

and when taking the product:

$$(-\infty) \cdot (-\infty) = T_t(0) \log_t(0) = -\infty = (-\infty) + (-\infty)$$

where the last sum is a sum in the usual sense.

So, summarizing this process:

- ◊ We forgot about the phase of the complex numbers and only looked at them radially.
- ◊ The modulus of these numbers was scaled logarithmically.
- ◊ Finally we took the limit of these operations and obtained the desired (somewhat) result.

This is known as Maslov¹ dequantization and with this we can see $(\mathbb{T}, +, \cdot)$ as $(\{-\infty\} \cup \mathbb{R}, \max, +)$. Also, we will abbreviate $\lim_{t \rightarrow \infty} T_t$ with $T_{t \rightarrow \infty}$

1.4 Interim 1 | Valuations

Valuations

In essence a valuation provides a measure of the *size* (or multiplicity) of elements in a field.

Definition 1.4.1. If K is a field, a valuation on K is a mapping

$$\text{val}: K \rightarrow \mathbb{R} \cup \{\infty\}$$

with the properties:

- i) $\text{val}(x) = \infty \iff x = 0$,
- ii) $\text{val}(xy) = \text{val}(x) + \text{val}(y)$,
- iii) $\text{val}(x + y) \geq \min(\text{val}(x), \text{val}(y))$, with equality if $\text{val}(x) \neq \text{val}(y)$.

In this case, we say that K is a valued field.

Previous discussion has shown us that the order of vanishing, val_0 is a valuation of the field of Puiseux series \mathbb{K} . The properties can be shown to be true by writing out two Puiseux series and showing that indeed they obey the properties.

Exercise 1.4.2. Verify that the field of Puiseux series, is indeed a valued field with valuation val_0 .

There's also a another common example coming from number theory which is the p -adic valuation.

Example 1.4.3. We first define the valuation on \mathbb{Z} as

$$v_p(a) = \max\{k \in \mathbb{Z} : p^k \mid a\},$$

where p is a prime number. For the rational numbers the valuation is defined as $v_p(m/n) = v_p(m) - v_p(n)$.

¹Viktor Pavlovich Maslov (1930615-20230803)

This valuation can be used to study the field of p -adic numbers which is the completion of \mathbb{Q} with respect to the p -adic absolute value $|r|_p = p^{-v_p(r)}$.

Exercise 1.4.4. In a similar fashion, verify that v_p is a valuation over \mathbb{Q} .

Chapter 2

The Tropical Numbers

2.1 Day 4 | 20230828

We have seen where our ideas come from. Certain kinds of minimization problems give rise to our tropical numbers. Also by expressing complex numbers in a logarithmic scale without phase then when inducing a sum we actually get a hypersum. The way we converted into an operation is by taking a limit. Then the algebraic structure we obtained was once again the tropical numbers. Let us talk about the perspective of valued fields.

Puiseux series

Recall from our times in Calculus 1 that when resolving indeterminate limits, the relevant information is contained in the order of vanishing of the function.

Example 2.1.1. Consider the limit $\lim_{t \rightarrow 0} \frac{\sin(x)}{x} = 1$. Near $t = 0$ we have

$$\sin(t) = t + o(t) \sim t^1 \quad \text{and} \quad \frac{1}{t} = t^{-1} \quad \text{so} \quad t^1 t^{-1} = t^0 = 1.$$

From this, we care to study the orders of zeroes and poles of Laurent series. In order to extend the class of functions to an algebraically closed field, we consider Puiseux series, or rational functions. We can identify Puiseux series as

$$\mathbb{C}\{\{t\}\} = \bigcup_{n \in \mathbb{N}} \mathbb{C}(t^{1/n}).$$

Concretely, elements here are Laurent series with rational exponents and the exponents of terms with non-zero coefficients have a common denominator.

Example 2.1.2. The series $\sum_{k=-37}^{\infty} t^{k/42}$ is a Puiseux series while $\sum_{k=1}^{\infty} t^{1/k}$ is not because the exponents keep getting smaller and smaller.

This is the most natural algebraically closed field with a *canonical* valuation. This is the function:

$$\text{val} : \mathbb{C}\{\{t\}\} \rightarrow \mathbb{R} \cup \{\infty\}, \begin{cases} 0 \mapsto \infty \\ t^{p/q} + \text{higher order} \mapsto p/q \end{cases}$$

In other words the valuation sends $\sum_{k=k_0}^{\infty} a_k t^{q_k}$ to q_{k_0} .

Proposition 2.1.3. For $\alpha, \beta \in \mathbb{C}\{\{t\}\}$, the valuation enjoys the following properties:

- i. $\text{val}(\alpha \cdot \beta) = \text{val}(\alpha) + \text{val}(\beta)$.
- ii. $\text{val}(\alpha + \beta) \geq \min(\text{val}(\alpha), \text{val}(\beta))$.

Equality holds when $\text{val}(\alpha) \neq \text{val}(\beta)$.

So if we decide to define operations on $\mathbb{R} \cup \{\infty\}$ by inducing them from the operations on $\mathbb{C}\{\{t\}\}$, then we obtain

$$x \diamond y = \text{val}(\text{val}^{-1}(x) + \text{val}^{-1}(y)), \quad x \cdot y = \text{val}(\text{val}^{-1}(x) \cdot \text{val}^{-1}(y)).$$

Now \cdot coincides with usual addition and $+$ is the hyperoperation

$$x \diamond y = \begin{cases} \min(x, y) & \text{when } x \neq y, \\ [\min(x, y), \infty] & \text{when } x = y. \end{cases}$$

Example 2.1.4. If we try to sum 0 with itself, we get

$$0 \diamond 0 = \text{val}((a_0 + a_1 t^{q_1} + \dots) + (-a_0 + b_1 t^{r_1} + \dots))$$

and this could be either q_1 or r_1 because the constant terms cancel!

The only natural way to turn this into an operation is to define $x + y = \min(x, y)$. In conclusion, the field of Puiseux series with the order of vanishing and poles is congruent to $(\mathbb{T}, +, \cdot)$ which in this case is $(\mathbb{R} \cup \{\infty\}, \min, +)$.

The Tropical Semifield

Definition 2.1.5. The tropical semifield is $(\mathbb{T}, \oplus, \odot)$ where we can choose:

- ◇ $\mathbb{T} = \mathbb{R} \cup \infty$, \oplus to be \min and \odot is $+$, the min convention.
- ◇ $\mathbb{T} = \{-\infty\} \cup \mathbb{R}$, $\oplus = \max$ and $\odot = +$, the max convention.

There is a natural isomorphism between the two choices given by $x \mapsto -x$. As we have mentioned, different contexts may be more natural than the other when using certain conventions. We will typically use the max convention.

Proposition 2.1.6. *The following algebraic properties hold for $(\mathbb{T}, +, \cdot)$:*

- i) $0_{\mathbb{T}} = -\infty$.
- ii) $1_{\mathbb{T}} = 0$.
- iii) $x \oplus y = 0_{\mathbb{T}}$ only has the solution $x = y = 0_{\mathbb{T}}$. This means that only $-\infty$ has an additive inverse.
- iv) Addition is idempotent: $x \oplus x = x$.
- v) Every non-zero element has a multiplicative inverse: $1/x = -x$.

Proof

- i) Observe that $x \oplus 0_{\mathbb{T}} = \max(x, -\infty) = x$.
- ii) $x \odot 1_{\mathbb{T}} = x + 0 = x$.
- iii) $x \oplus y = 0_{\mathbb{T}} \iff \max(x, y) = -\infty \Rightarrow x = y = -\infty$.
- iv) $x \oplus x = \max(x, x) = x$.
- v) $x \cdot (1/x) = x + (-x) = 0 = 1_{\mathbb{T}}$.

Observe that it is not possible to adjoin formal additive inverses. Suppose that for $x \in \mathbb{T}$ there exists a y such that $x + y = 0_{\mathbb{T}}$, then

$$(x \oplus x) \oplus y = x \oplus y = 0_{\mathbb{T}} \quad \text{and} \quad x \oplus (x \oplus y) = x \oplus 0_{\mathbb{T}} = x \quad \text{but} \quad x \neq 0_{\mathbb{T}}.$$

This means that any invertible element necessarily has to be $-\infty$.

Exercise 2.1.7 (2-). Which other algebraic properties do these operations enjoy? We have claimed for example that \oplus is associative. Prove this.

Are the operations commutative? Do they distribute with respect to each other?

Let us assume without losing generality that

$$x < y < z$$

and then

$$\begin{aligned} x \oplus (y \oplus z) &= \max(x, \max(y, z)) = \max(x, z) = z \\ (x \oplus y) \oplus z &= \max(\max(x, y), z) = \max(y, z) = z \end{aligned}$$

which shows us that \oplus is associative. We can see that it's commutative by realizing that \max is also commutative.

Finally we ask if distribution occurs, but let us observe first that

$$y < z \Rightarrow x + y < x + z,$$

now we can see that

$$x \odot (y \oplus z) = x \odot y \oplus x \odot z$$

$$\begin{aligned} &\iff x + \max(y, z) = \max(x + y, x + z) \\ &\iff x + z = x + z \end{aligned}$$

which proves distributivity. Also it is not necessary to check distribution on the other side as our operations are commutative.

Proposition 2.1.8 (Weird Fun Facts). *Recall that the usual Pascal Triangle is built by adding the previous two elements to get the next one. In the tropical case we have*

$$\begin{array}{ccccccc} & & 1_{\mathbb{T}} & & & & 0 \\ & & & & & & \\ 1_{\mathbb{T}} & & 1_{\mathbb{T}} & & = & & 0 & 0 \\ & & & & & & \\ 1_{\mathbb{T}} & & 1_{\mathbb{T}} & & 1_{\mathbb{T}} & & 0 & 0 & 0 \end{array}$$

and this extends downwards with the same pattern.

In the case of the tropical binomial theorem, we have a “freshman’s dream” type of identity:

$$(x \oplus y)^n = x^n \oplus y^n.$$

Observe that in other words, what the binomial theorem is telling us is a restatement of the identity $n \max(x, y) = \max(nx, ny)$.

Exercise 2.1.9 (2). Recall that the coefficients in the expansion for the binomial theorem are the corresponding elements in the rows of the Pascal Triangle. Verify if the coefficients agree in the tropical case for the binomial theorem.

We must verify that the coefficient of every monomial in the expansion of $(x \oplus y)^n$ does match the n^{th} row of Pascal’s triangle. To that effect note that

$$\begin{aligned} (x \oplus y)^n &= (x \oplus y) \odot (x \oplus y) \odot \cdots \odot (x \oplus y) \\ &= x^n \oplus x^{n-1}y \oplus \cdots \oplus xy^{n-1} + y^n \\ &= x^n \oplus y^n \end{aligned}$$

and in the second line, remember we would usually have $\binom{n}{k}$ terms of the form $x^k y^{n-k}$. However, as addition is idempotent here, all those terms become just one term.

Also, observe that the coefficient (tropically) multiplying each term is $1_{\mathbb{T}}$. This is because multiplication by one is just adding by zero. So it is indeed the case that all coefficients in the binomial expansion are $1_{\mathbb{T}}$.

Finally observe that for any k , $kx + (n - k)y \leq \max(nx, ny)$ which means that the only terms that survive are the power n monomials in the expansion.

The Optimal Assignment Problem

Suppose we have n jobs for n workers. Each worker can only work one job and once the job is taken, no one else can do it. We wish to assign a job to each worker in order to maximize our company's profit.

Example 2.1.10. As a little example consider Alice and Bob's hydroponics farm. When working with the weeds Alice produces 5 credits while working with the water she produces 6. On the other hand Bob produces 3 and 5 respectively.

It is easy to see that Alice should be assigned to the weed and Bob to the water in order to maximize. But let us apply what we know with tropical arithmetics.

Call

M_{ij} = amount of credits work i produces when doing job j .

Then we can summarize the previous information in a matrix

$$M = \begin{pmatrix} 5 & 6 \\ 3 & 5 \end{pmatrix}$$

and if we take the tropical determinant (which is really a permanent since we lack subtraction) we get

$$\text{Trop det } M = 5 \cdot 5 + 6 \cdot 3 = \max(5 + 5, 6 + 3) = 10$$

which is the maximal profit we can make by assigning our workers.

Exercise 2.1.11. Do the following:

- (1-) Construct a 3×3 matrix with non-permuted entries such that there's more than one possible assignment for the optimal jobs.
- (1) Use the combinatorial definition of permanent to show that the tropical determinant of M is indeed the maximal profit. [Hint: The definition of permanent is the same as the determinant but without the $(-1)^{\text{sgn } \sigma}$.]
- (1) Assuming you know the tropical determinant of a matrix, devise a way to identify one job combination which reaches the optimum value. [Actually! It is not necessary to know the value of the determinant.]

i) Consider a matrix $A \in \mathbb{R}^{3 \times 3}$. For a given $n \in \mathbb{N}$, the profit, we may build an infinite family of matrices which satisfy the required conditions.

The conditions our matrix must satisfy are sums of permuted entries.

In this case the solution is given by 5 parameters including the profit:

$$(n - f - h, n - g - g, n - f - g - h + i, f + g - i, f + h - i, f, g, h, i)$$

so a valid matrix could be

$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & 5 & 3 \\ 2 & 4 & 2 \end{pmatrix}.$$

ii) The permanent by definition is

$$\bigoplus_{\sigma \in S_n} \bigodot_{i=1}^n M_{i\sigma_i} = \max_{\sigma \in S_n} \left(\sum_{i=1}^n M_{i\sigma_i} \right).$$

What the last expression says is, out all the possible permutations, which is the highest sum over all possible job assignments. So the permanent will indeed find the maximal profit.

iii) We can proceed with a greedy algorithm. Row by row, choose the largest element. Then eliminate the column the found element was in and repeat the process.

For example, pick $A_{1k} = \max(\text{row } 1)$, then throw out column k and repeat the process with the $(1, k)$ minor of A . [This doesn't actually prove that the greedy algorithm works.]

Remark 2.1.12. Observe that the first problem can be solved in any dimension d , because in total we have $2d$ equations while having d^2 indeterminates. As $2d < d^2$ for $d \geq 2$, we have that the problem will always be under-determined. So there's always more than one possible optimal assignment.

We now have another question,

Is there an instance where the greedy algorithm fails to find an optimal assignment for the jobs?

Exercise 2.1.13 (5). Prove or disprove, the greedy algorithm will find an optimal assignment for the jobs given the conditions above. You may assume to know the value of the permanent of the matrix.

2.2 Day 5 | 20230830

The last time we talked about the algebraic structure of the value group of the Puiseux series. We now have plenty of motivation of why would we define the tropical numbers.

Tropical Polynomials and Roots

An univariate, tropical, (Laurent) monomial is equivalent to an affine linear function with integer coefficients. Such a monomial is an expression of the form

$$a \odot x^{\odot m}, \quad a \in \mathbb{T}, \quad m \in \mathbb{Z}.$$

Example 2.2.1. We have for example:

$$5x^2 \leftrightarrow 5 + 2x, \quad 2x^{-3} \leftrightarrow 2 - 3x.$$

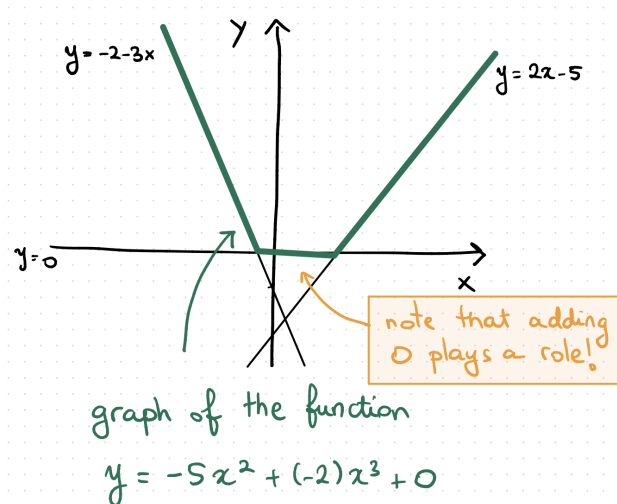
The second one is a Laurent monomial because of the negative power. Also consider $\sqrt{5} \odot x^{\odot 3}$ which corresponds to $y = \sqrt{5} + 3x$. Notice how the slope is always an integer, meanwhile the translation can be any number.

An univariate tropical (Laurent) polynomial is a finite sum of monomials which give rise to a *convex*, continuous, piecewise, affine linear function with integer slopes.

Example 2.2.2. Consider the function $-5 \odot x^{\odot 2} \oplus (-2) \odot x^{\odot -3} \oplus 0$ which corresponds to

$$\max(-5 + 2x, -2 - 3x, 0).$$

If we graph this functions we obtain Observe that this function is indeed convex, and fulfills



all of the previous properties from before.

In fact the map from $\mathbb{T}[x]$ to convex, affine piecewise linear functions with *finitely* many distinct regions of linearity is surjective. If we don't want to take the finiteness condition into consideration, we have to amplify the domain to tropical Laurent series.

A small measure of care should be taken because there are multiple tropical polynomials which map to the same function.

Example 2.2.3. Consider the functions

$$p_1 = x + \frac{1}{x} + 0, \quad p_2 = x + \frac{1}{x} - 2.$$

When converting we get

$$\max(x, -x, 0), \quad \max(x, -x, -2)$$

which produce $|x|$ in both cases. Adding something which is smaller than the minimum

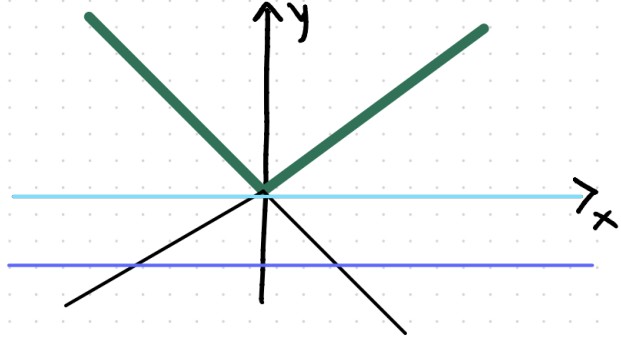


Figure 2.1: Failure of injectivity as both functions map to $|x|$ with $y = 0$ and $y = -2$ shown.

value of the function doesn't change it in general. It also doesn't have to be a constant in general. In the previous example, the the monomial $(-4) \odot x^{\odot 1}$ is smaller than any of the linear functions, so adding it changes nothing.

To talk about the roots, we will start with a purely combinatorial definition.

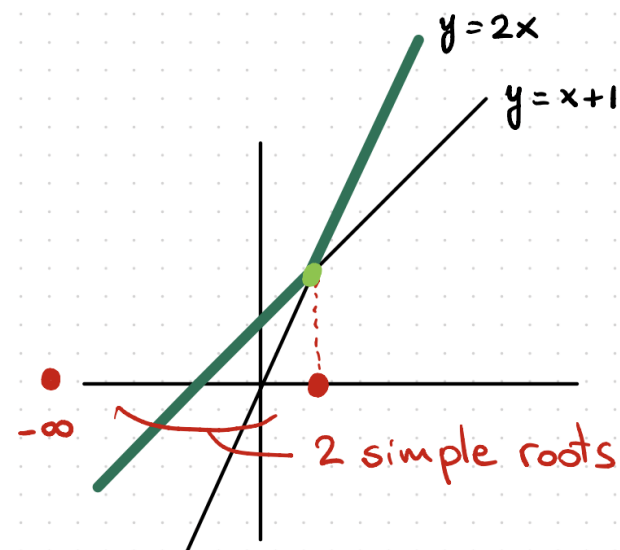
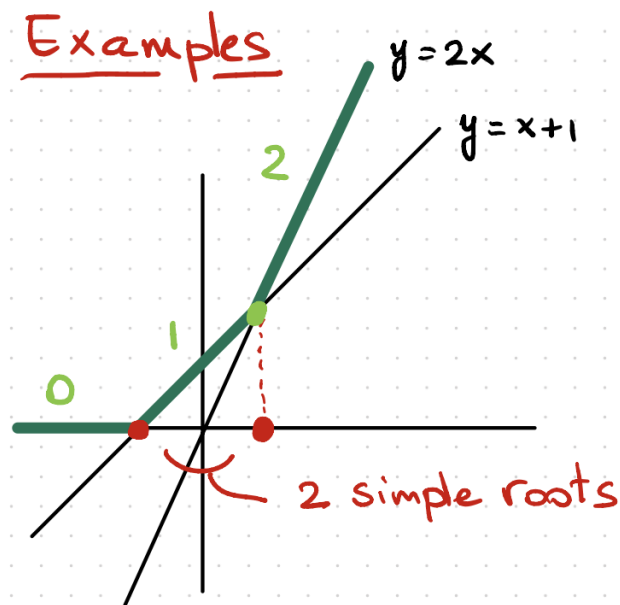
Definition 2.2.4. Given a polynomial $p \in \mathbb{T}[x]$ of degree d we say the following:

- ◇ $-\infty$ is a root of p if the slope of the piecewise linear function is non-zero for $x \ll 0$.
- ◇ $x_0 \in \mathbb{R}$ is a root of p if $p'(x_0)$ is undefined. Observe that the derivative is undefined only when there's a change in slope.

We say that the multiplicity of x_0 is the difference between slopes across x_0 . If $-\infty$ is a root, then its multiplicity is equal to the slope of the associated function for $x \ll 0$.

Example 2.2.5. Consider the polynomial $x^{\odot 2} \oplus 1 \odot x^1 \oplus 0 = \max(2x, x, 0)$. We can see that there are changes in slope at $x_1 = -1$ and $x_2 = 1$. The number of roots coincides with the degree of the polynomial as in the usual sense.

Example 2.2.6. Let's remove the zero, recall zero isn't the additive identity, so the polynomial we have is $x^{\odot 2} \oplus 1 \odot x^1 = \max(2x, x)$. Now one of the roots is still $x = 1$, but remember that if the slope is non-zero when $x \ll 0$, then $-\infty$ is a root of p . This is the case here because the slope is 1 as $x \rightarrow -\infty$. Once again there's two roots $x_1 = -\infty$ and $x_2 = 1$.

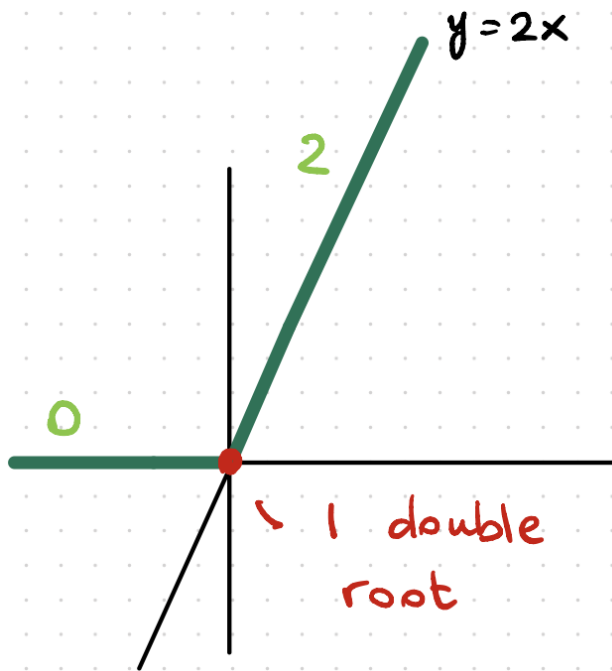
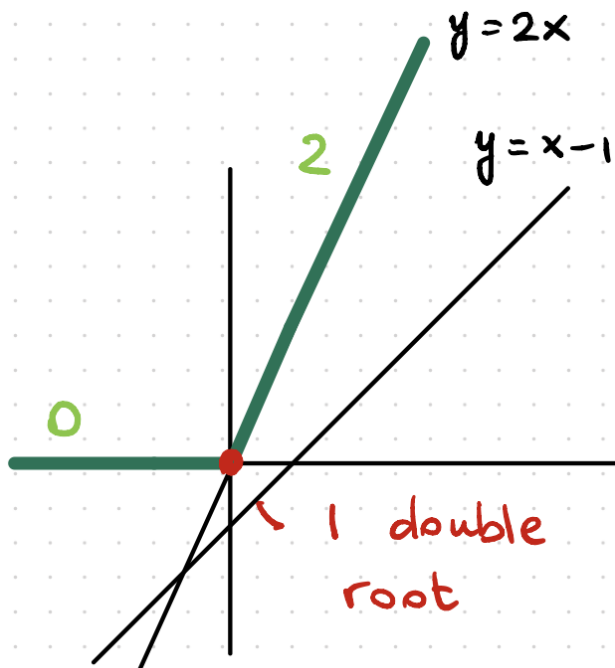


Example 2.2.7. Let us change a sign in a coefficient, take $x^2 - 1 \cdot x^1 + 0$. But what is tropical subtraction? It's not that, let's convert this slowly into what it's supposed to be:

$$x^2 - 1 \cdot x^1 + 0 = (x \cdot x) + (-1) \cdot x + 0 = (2x) + (x + (-1)) + 0 = \max(2x, x - 1, 0).$$

Observe that because the line $y = x - 1$ is below our graphs, it doesn't interfere with the calculation of zeroes. So the only place where there occurs a change in sign is $x = 0$. The slope on the right is 2 and on the left is 0 so the multiplicity is $2 - 0 = 2$.

Example 2.2.8. In a similar fashion, $x^2 + 0$ also has a double root at $x = 0$. There is only one change in slope once again at $x = 0$ and the difference in slopes is 2.



Lemma 2.2.9. For a tropical polynomial p , a finite x_0 is a root of f if and only if when we write the function as a max of linear functions, at x_0 the maximum value is obtained at least twice.

The multiplicity of the root is equal to the difference in the two extremal positions where the max is attained.

This should be more or less obvious. Being a root means that we are an intersection of

two lines which are above all the others. It's pretty useful to have this notions around.

Questions arise:

Which functions have only one simple zero at $-\infty$? What would a function with an order 2 zero at $-\infty$ look like?

Exercise 2.2.10. Do the following:

- (5) Is it possible for a function to have only a simple zero at $-\infty$? Provide an example of function with one simple zero at $-\infty$ or prove that such function cannot exist.
- (5) Do functions with zeroes at $-\infty$ have infinite order at such zero or is it arbitrarily high? If a function has a finite order zero at $-\infty$ provide an example of one with a double zero at $-\infty$. Else, prove that such functions have infinite order at that zero.

2.3 Intro

Current State of Literature

There are current books for Tropical Geometry. This includes

- ◇ *Tropical Geometry* by Maclagan-Sturmfels, which has a very algebraic take on the subject.
- ◇ *Tropical Geometry*, Which is in progress, being written by Mikhalkin-Rau, whcih has a geometric and intersection theoretic take.
- ◇ There are also various expository articles for Tropical Geometry

Tropical Geometry

Tropical Geometry is sometimes called a 'combinatorial' shadow of algebraic geometry. We take a inputs algebraic varieties, and receive a an output a piecewise linear object.

Example 2.3.1. The input can be a line in the plane \mathbb{C}^2 , i.e. $az + bw = d$. Then the output of the construction can be a tripod/tropical Y , i.e.e three lines connecting a vertex.

The input could also be an elliptic curve in \mathbb{C}^2 , and hte output can be another more complicated connection of verticles and lines (a tropical cubic)

Finally, we can consider an abstract nodal curve (a sphere and tori connected at verticies), with the corresponding piecewise lienar object being a dual graph, which has a vertex at ease component, an edge for each node, and a label for each part.

The questions that naturally arise are

- (a) What algebraic information about the initial object is carried over in the simplified object?

- (b) How do we extract the information carried by the simplified object?
- (c) Does the lifting problem have a solution?

There are four ways to tropicalize the algebraic variety.

Tropical Semi-Field

We do "algebraic geometry" over the tropical semi-field. The tropical semi-field is $(\mathbb{T}, \oplus, \odot) = (\mathbb{R} \cup \{-\infty\}, \max, +)$. We can take a polynomial $p(x_1, \dots, x_n)$. Our variety is the roots of the polynomial. If we consider the tropical $p(x_1, \dots, x_n) : (\mathbb{R} \cup \{-\infty\})^n \rightarrow \mathbb{R} \cup \{-\infty\}$, this is an affine piecewise linear function. For example, We can take $p(x_1, x_2)$.

The tropical hypersurface is the locus of non-linearity.

A tropical polynomial would be something of the form $p(x, y) : x \oplus y \oplus 0 = \max(x, y, 0)$

By the corner locus, we take the graph of our function.

Why does this work to make piecewise linear? We get expressions such as $3 \odot x^2 = 3 + 2x$.

Valued Fields

our second perspective will be of valued fields. We let K be a field with a valuation, i.e. let $K = \mathbb{C}(t)$ be the field of rational functions in 1 variables. A valuation val_0 is a function $val_0 : \mathbb{C}(t) \rightarrow \mathbb{R} \cup \{\infty\}$, where $0 \mapsto \infty$, else $f \mapsto$ order of vanishing (or pole) as you approach $t \rightarrow 0$.

We can ask what happens to the order of vanishing when you add 2 functions. In this case the order of vanishing is the minimum of the two orders. i.e. if $f_1 = t^2$, $f_2 = t^3$, then $val_0 f_1 = 2$, $val_0 f_2 = 3$, and then $val_0(f_1 + f_2) = 2$. So in this context, $val_0(f_1 + f_2) \geq \min(val_0 f_1, val_0 f_2)$, and $val_0(f_1 f_2) = val_0 f_1 + val_0 f_2$. Let $p(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$. Take the variety $X = \{(x_1, \dots, x_n) \mid p(x_1, \dots, x_n) = 0\} \subset K^n$. We can take the map $K^n \rightarrow (\mathbb{R} \cup \{\infty\})^n$ by taking the valuation at every coordinate, i.e. apply val_0 to the result. Each point of K is some rational function of t , so looking at the order of vanishing makes sense. So the tropicalization of X is the image via this map.

For example, we can take $p(x, y) = tx + y + t^2$. Then $X = \{(x, y) \mid tx + y + t^2 = 0\} = \{(x, y) \mid y = -tx - t^2\}$. So $(0, -t^2)$ is an option. So the point $(\infty, 2)$ is a point in $Trop(X)$.

Amoebas

We consider $p(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$. It defines $X = \{(z_1, \dots, z_n) \mid p(z_1, \dots, z_n) = 0\} \subset \mathbb{C}^n$. We can consider the map from \mathbb{C}^n to the image $\mathbb{C}^n \rightarrow (\mathbb{R} \cup \{-\infty\})^n$ via $(z_1, \dots, z_n) \mapsto (\log_t |z_1|, \dots, \log_t |z_n|)$. The image of X via this map $Im(X)$ is the t -Amoeba of X .

Example 2.3.2. $p(x, y) = x + y + 1$ We can take the variety, points such as $x = i, y = -1 - i$, and $p(x, y) \in X$. We then consider $(\log_t 1, \log_t \sqrt{2})$

We can take $\lim_{t \rightarrow \infty} (t - Amoeba)$, called the spine of the amoeba or $Trop(X)$.

Degenerations of Algebraic Varieties

This is a complicated way to get tropics

2.4 Motivation of Tropical Geometry

Tropical Arithmetics

Minimization Problems

An important question is where tropical numbers come from. One example is toll minimization problems. We let every city be a vertex, and every directed edge is a tollway. Let A , B , and C be the three cities/vertices. The incidence matrix would be

$$M = \begin{bmatrix} 0 & \infty & 2 \\ x & 0 & y \\ \infty & 1 & 0 \end{bmatrix}$$

where the rows represent the starting point and the columns represent the ending location, so there is a toll road from A to B which costs 2. There is no road from A to C . In other words, $M_{i,j}$ records the (minimum) price of going from city i to city j in at most one trip (one trip being moving across one toll road, i.e. traversing one edge). We can now ask how to compute the best strategy of going from i to j in at most two trips.

Example 2.4.1. If I want to go from A to B in 2 steps, We can do $A \rightarrow A \rightarrow B$, $A \rightarrow B \rightarrow B$, or $A \rightarrow C \rightarrow B$. The associated costs are $0 + \infty$, $\infty + 0$, and $2 + 1$, respectively. The best strategy is the minimum of these three, which is $A \rightarrow C \rightarrow B$, which gives a cost of 3. However, these sums can be thought of as coming from entries of the matrix. The associated sums are $a_{11} + a_{12}$, $a_{12} + a_{22}$, and $a_{13} + a_{32}$.

Where do these values come from? The b_{12} entry of M^2 is $\sum_{j=1}^n a_{1j}a_{j2}$. So the best strategy is equal to the 12 entry of M^2 , so long as you interpret $+$ as the minimum, and $*$ as $+$. In that sense, b_{12} entry is M^2 is $\min(a_{11} + a_{12}, a_{12} + a_{22}, a_{13} + a_{32})$.

If we make the assumption that there are no negative tolls, this arithmetic for higher powers of n gives the best solution for up to n trips. Eventually we stabilize, as for a high enough N the solutions will say to stop moving for a certain number of steps. (Eventually we get something like idempotence).

The minimization problem eventually becomes a linear algebra problem over $(\mathbb{T}, \oplus, \odot) = (\mathbb{R} \cup \{\infty\}, \min, +)$.

Forget the Phase

Another context is in physics/electromagnetism. If we take a complex number, we can write it in polar coordinates, i.e. $z \in \mathbb{C}$ can be written as $z = r\theta e^{i\theta}$. Maybe we don't particularly care about the phase (θ), or our work works on a logarithmic scale. So our function would be $T_t : \mathbb{C} \rightarrow \mathbb{R} \cup \{-\infty\}$, where $z \mapsto \log_t |z|$. We add the point at negative infinity to be able to define $T_t(0)$.

We have that \mathbb{C} has its own natural operations, and we can ask if we can induce operations on $\mathbb{R} \cup \{-\infty\}$ utilizing the map T_t . If we want to define addition on $\mathbb{R} \cup \{-\infty\}$, we could define it as $x \boxplus y$. We could try $T_t(T_t^{-1}(x) + T_t^{-1}(y))$. The problem this is not well defined, as different inverse images can have different absolute values. This is a hyper operation, This is a function $X \times X \rightarrow \mathcal{P}(X)$, to the power set of X . So this function outputs an interval.

We first wish to understand $T_t^{-1}(y)$. This is $T_t^{-1}(y) = \{t^y e^{-\theta}\}$. adding and subtracting is thus given by the interval of the possible extremes, so we get the interval.

$$T_t(T_t^{-1}(x) + T_t^{-1}(y)) = [\log_t |t^x - t^y|, \log_t (t^x + t^y)]$$

But we don't want a hyperoperation, we want an operation. To pick something that always works, we can either pick the max or the minimum. We can also see if in a limiting process, we get one answer. We start with $x \boxplus_t y \rightarrow x +_t y$, with some consistent choice. we can also ask for $\lim_{t \rightarrow \infty}$ to get a new operation. We define $x \boxplus y = \lim_{t \rightarrow \infty} x \boxplus_t y$. Why is this better? If we take $t^2 e^{i\theta_1} + t^4 e^{i\theta_2}$. So what really matters at the end is that our expression is equal to $t^4(t^{-2} e^{i\theta} + e^{i\theta})$. Taking \log_t , we have $4 + \log_2 |t^{-2} e^{i\theta} + e^{i\theta}|$. Taking the limit as $t \rightarrow \infty$, we have $4 + 0 = 4$. So this is not a hyperoperation if $x = 2$ $y = 4$. But, if we have $x = 2$ and $y = 2$, we get issues. We have

$$x \boxplus y = \begin{cases} \max(x, y) & x \neq y \\ (\infty, \max(x, y)) & x = y \end{cases}$$

This suggests that the nice consistent choice to define $x \oplus y$ is max. This is fully consistent when $x \neq y$ regardless of any decision, and it makes the interval $(\infty, \max(x, y))$ work nicer. To define multiplication, we would say $x * y = T_t(T_t^{-1}(x)T_t^{-1}(y)) = T_t(t^x e^{i\theta_1} t^y e^{i\theta_2}) = x + y$. This leads to the tropical numbers, where $(\mathbb{T}, \oplus, \odot) = \mathbb{R} \cup \{-\infty\}, \max, +$.

Puiseux Series

In calc 1, we find ways to approximate. In approximation, we lose some information. we learn that $\lim_{t \rightarrow 0} \frac{\sin(t)}{t} = 1$, which essentially says $\sin(t) = t + (t)$, and $\frac{1}{t} = t^{-1} + 0$. In other words, $\lim_{t \rightarrow 0} (t + (t))t^{-1} = 1 + (1)$, where (t) goes to zero as t goes to zero. All of this is to say that we have a concept of order of vanishing of functions.

A Prototypical example of a field with a valuation is the field of Puiseux series. Denoted $\mathbb{C}\{\{t\}\}$, this is the Laurent series in t with rational exponents, and all exponents of terms with nonzero coefficients have a common denominator. For example, $\sum_{i=-37}^{\infty} t^{i/42}$ works, but $\sum_{i=1}^{\infty} t^{1/i}$ does not work. In other words, $\mathbb{C}\{\{t\}\} = \bigcup_{n \in \mathbb{N}} \text{Laur}(t^{1/n})$, that is to say it is the union of all Laurent series in the variable $t^{1/n}$. This has a valuation

$$\text{val} : \mathbb{C}\{\{t\}\} \rightarrow \mathbb{R} \cup \{\infty\} \quad (2.4.1)$$

$$(a_N \neq 0) \sum_i a_i t^{q_i} \mapsto q_N \quad (2.4.2)$$

$$0 \mapsto \infty \quad (2.4.3)$$

This map technically has as its image $\mathbb{Q} \cup \{\infty\}$.

Once again, the properties/axioms of a valuation field is that

$$\diamond \text{val}(\alpha + \beta) \geq \min\{\text{val}(\alpha) + \text{val}(\beta)\},$$

$$\diamond \text{val}(\alpha * \beta) = \text{val}(\alpha + \text{val}(\beta)).$$

If we want to induce a sum on the image, we would have a hyper operation. We define

$$x \boxplus y := \text{val}(\text{val}^{-1}(x) + \text{val}^{-1}(y)) = \begin{cases} [\min(x, y), \infty] & x = y \\ \min(x, y) & x \neq y \end{cases}. \text{ For example, we would have}$$

$0 \boxplus 0 := \text{val}(a_0 + t^{q_1} a_1 + \dots) + (-a_0 + t^{r_1} b_1 + \dots)$. Then we can let q_1 and r_1 equal whatever we want.

To avoid issues, we say $x + y := \min(x, y)$. This takes us to $(\mathbb{R} \cup \infty, \min, +)$.

2.5 The Tropical Semifield

Definition 2.5.1. The *tropical semifield* is one of $(\{-\infty\} \cup \mathbb{R}, \max, +)$, or $(\mathbb{R} \cup \{\infty\}, \min, +)$, either one of which is denoted by $(\mathbb{T}, \oplus, \odot)$. The operations are associative, and the distributive law holds.

The two definitions are isomorphic via $x \mapsto -x$. In the following writing, we will tend to utilize \max more often than \min . We have the following properties of this semifield

- (a) $\{-\infty\}$ is the additive identity,
- (b) $\{-\infty\}$ is the only element that has an additive inverse.
- (c) \oplus is an idempotent operation $x \oplus x = x$.
- (d) We cannot add inverses, not even formally.
- (e) 0 is the multiplicative identity.

(f) every element $x \neq -\infty$ has a multiplicative inverse (namely $-x$).

Example 2.5.1. Let us solve $x \oplus y = -\infty$. This is to say $\max(x, y) = -\infty$, which necessitates $x = y = -\infty$. To see that inverses are not definable, We take $x \in \mathbb{T}$. Let us attempt to construct a formal inverse y , defined by $x \oplus y = -\infty$. This will force $x = -\infty$. We see this by considering $x \oplus x \oplus y = (x \oplus x) \oplus (y) = -\infty$. As this operation is associative, we also have $x \oplus (x \oplus y) = x \oplus -\infty = x$, and so $x = -\infty$.

Weird/fun facts

Pascal's triangle in tropical arithmetic looks like all zeros. Furthermore, the freshman's dream holds $(x \oplus y)^n = x^n \oplus y^n$. This is because $(x \oplus y)^n = n * (\max(x, y)) = \max(nx, ny)$. However, the fact that $x^2 \oplus (x \odot) \oplus y^2 = x^2 \oplus y^2$ does not imply cancellation, i.e. we do not automatically have that $x \odot y = -\infty$.

The Tropical determinant of a matrix (call it permanent) gives a solution to the assignment problem: We have n -jobs to n -workers. x_{ij} is the profitability of worker i in job j , and the goal is to find the best assignment to maximize profits. This is the tropical determinant of the matrix X . We define $\text{tropdet}(X) = \sum_{\sigma \in S_d} \prod_{\sigma(i)} x_{i, \sigma(i)}$ where the sum is the tropical sum and the product is the tropical product. We do not get a signed determinant as tropical geometry does not have subtraction.

Algebraic Geometry: Tropical Univariate polynomials and their roots

Definition 2.5.2. A tropical univariate (Laurent) monomial is an expression of the form $a \odot x^{\odot m}$, where $a \in \mathbb{T}$ and $m \in \mathbb{Z}$.

In ordinary algebra, a tropical monomial corresponds to an affine linear function with integer slope.

Example 2.5.2. $\sqrt{5} \odot x^{\odot 3} \sqrt{5} + 3x = y$. This is an affine linear transformation. It is affine because we can shift via a , and it has integer slope as the slope is m . Furthermore, $\{-\infty\} \odot x^{\odot m} = -\infty + mx = -\infty$, so we maintain that multiplying by the additive identity gives the additive identity.

Definition 2.5.3. A tropical univariate (Laurent) polynomial is the finite sum of monomials.

The tropical univariate polynomial corresponds to a continuous, piecewise affine linear function with \mathbb{Z} -slopes

Example 2.5.3. $p(x) = -5 \odot x^{\odot 2} \oplus (-2) \odot x^{\odot -3} \oplus 0 = \max(-5 + 2x, -2 - 3x, 0)$

By construction, tropical polynomials give rise to convex functions. In the univariate case, the map from tropical L polynomials to convex \mathbb{Z} affine piecewise linear (with finitely

many distinct regions of linearity) functions is a surjective map. However, this map is not surjective.

Example 2.5.4. Consider $p_1(x) = x^{-1} \oplus x$, i.e.e $y = |x|$. Then consider $p_2(x) = x^{-1} \oplus x \oplus 0$. Then we once again get $y = |x|$. Furthermore, we can take $p_3 = x^{-1} \oplus x \oplus -8$, this works for any negative number.

In general, $p(x) = p(x) \oplus \{-\infty\}$. However, $p(x) \oplus$ any function which is smaller than the minimum value attained by $p(x)$ does not change the output of $p(x)$, i.e. $p(x) = -5 \odot x^{\odot 2} \oplus (-2) \odot x^{\odot -3} \oplus 0 = p(x) = -5 \odot x^{\odot 2} \oplus (-2) \odot x^{\odot -3} \oplus 0 \oplus (-4) \odot x$.

Now that we have defined polynomials, we wish to make an interpretation of roots. It does not make sense to say "values of x for which $p(x) = -\infty$, as that typically won't happen (We deal with max). Solving for 0 also doesn't help, as in this context 0 is just another number/function.

Definition 2.5.4. Let $p(x) \in \mathbb{T}[x]$ (an honest polynomial, no negative exponents, i.e. only positive slopes). Then

- ◇ $(-\infty)$ is a root of p if the slope of the corresponding affine piecewise linear function is $\neq 0$ for $x \ll 0$.
- ◇ We allow $r \in \mathbb{R}$ to be a root of p if $f'_p(r)$ (The piecewise linear function arising from p , and the derivative of THAT function). is not defined.

In other words, roots will be where the function changes slopes. Now we can discuss multiplicities.

Definition 2.5.5. If $-\infty$ is a root, its multiplicity is equal to the slope of $f_p(x)$ for $x \ll 0$. If $r \in \mathbb{R}$ is a root, its multiplicity is the difference in slopes across r .

Example 2.5.5. Take $p(x) = x^2 \oplus 1 \odot x \oplus 0 = \max(2x, 1 + x, 0)$ has two simple roots, one at $x = 0$ and one at $x = 1$. On the other hand, for $p(x) = x^2 \oplus 1 \odot x$, we get two simple roots, one at $x = 1$ and one at $-\infty$.

Now, take $p(x) \oplus (-1) \otimes x \oplus 0$. Then there is a double root at $x = 0$, as we go from slope 0 to slope 2.

Note 1. If we accept Laurent polynomials, then the multiplicity of $-\infty$ is equal to the slope towards $-\infty$.

Example 2.5.6. Take $q(x) = (-5) \otimes x^2 \oplus (-2) \otimes x^{-3} \oplus 0$, Then $-\infty$ has a pole of order 3, there is a root of order 3 at $x = -5/2$, and a root of order 2 at $x = 2$

Lemma 2.5.1. Only $-\infty$ can have a pole (root with negative multiplicity) due to concavity.

Lemma 2.5.2. $r \neq -\infty$ is a root for $p(x)$ iff when you write down $f_p(x) = \max(f_{m_0}(x), \dots, f_{m_d}(x))$ at r the maximum value is obtained at least twice.

Furthermore, the multiplicity of the root is equal to the difference in the two extremal positions where the max is attained for r .

2.6 Day 6 | 20230901

How do we know that the notions of roots are natural or useful?

Factorization of Tropical Polynomials

Suppose a polynomial $p \in \mathbb{T}[x]$ has roots a_k with multiplicity m_k . Then we may factor p as a product of linear polynomials

$$p(x) = c_0 \odot (x \oplus a_k)^{m_k}.$$

This p is the affine piecewise-linear function, not the formal object. And so, in a sense, \mathbb{T} is algebraically closed. But instead of proving this, we will sketch the proof to get an idea of how things *work* with a couple of examples.

The idea of the proof is that we check that product does define a P.L. function with the right slopes and then c_0 gives the translation factor.

Example 2.6.1. First let's deal with the case where $-\infty$ is not a root. Consider the polynomial

$$p(x) = (-1) \oplus (-1) \odot x \oplus (-4) \odot x^4 = \max(-1, x - 1, 4x - 4).$$

Remember, as in the case of real polynomials, the square and cube terms are still there. The coefficient that goes along them is just $-\infty$. We can graph the polynomial in order to see the roots: The points where there is a change in slope are $a_1 = 0$ and $a_2 = 1$. Then their

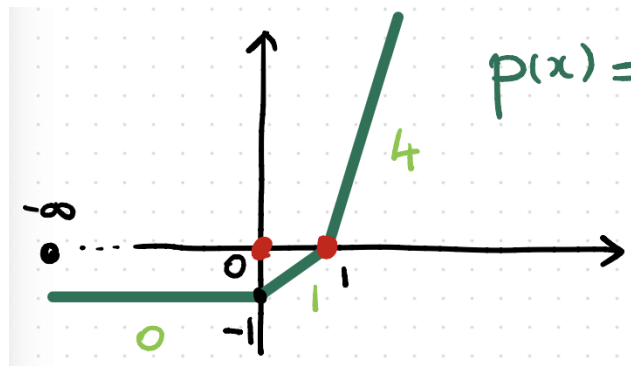


Figure 2.2: Graph of $p(x)$ with roots shown

multiplicities are $1 - 0 = 1$ and $4 - 1 = 3$ respectively. We may write p as

$$p(x) = c_0 \odot (x \oplus 0) \odot (x \oplus 1)^3 = c_0 + \max(x, 0) + \max(3x, 3).$$

Whatever function we have, we can write as the sum of three terms. So let us subdivide the tropical line in order to see which terms goes where. The constant can be determined

$x \leq 0$	$0 \leq x \leq 1$	$1 \leq x$
c_0	c_0	c_0
0	x	x
3	3	$3x$
$c_0 + 3$	$c_0 + 3 + x$	$c_0 + 4x$
Behavior of $p(x)$ across \mathbb{T}		

by plugging in $x = -\infty$. We can see that

$$\begin{aligned} p(-\infty) &= (-1) \oplus (-1) \odot (-\infty) \oplus (-4) \odot (-\infty)^4 = -1 \\ &= c_0 \odot (-\infty \oplus 0) \odot (-\infty \oplus 1)^3 = c_0 \odot 0 \odot 1^{\odot 3}. \end{aligned}$$

This gives us the equation $c_0 + 0 + 3 = -1$ which leads us to $c_0 = -4$. With this we verify that

$$p(x) = \begin{cases} -1 & x \leq 0 \\ x - 1 & 0 \leq x \leq 1 \\ 4x - 4 & 1 \leq x \end{cases}$$

So in this case $c_0 = p(-\infty) - \sum m_k a_k \in \mathbb{R}$.

Example 2.6.2. We now explore the case where $-\infty$ is a root or a pole. The argument will essentially be the same with a small modification.

Consider the function $\frac{1}{x} \oplus x$. We have $-\infty$ as a pole of order 1 and 0 is a root of order $1 - (-1) = 2$. So this can be factored as

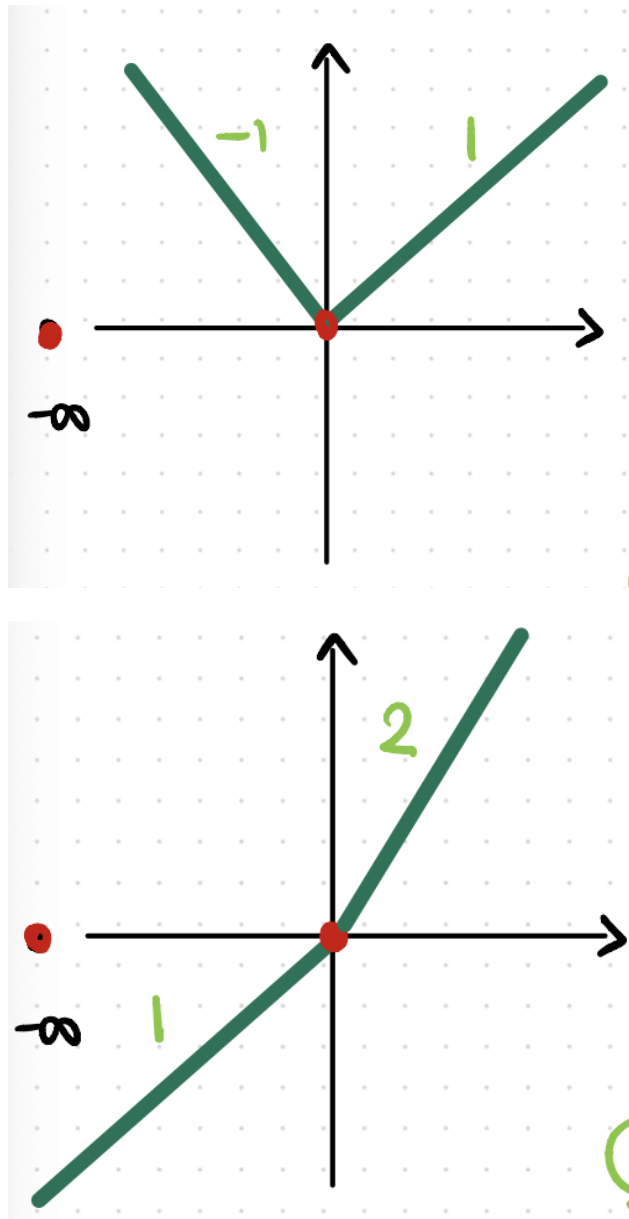
$$p(x) = c_0 \odot (x^{-1}) \odot (x \oplus 1)^2$$

and even if $-\infty$ doesn't give us a particular value for the function, we can still find $c_0 = 0$ from the equation $p(0) = 0$.

If on the other hand we have a negative slope then we have a zero at $-\infty$. Consider the function $p(x) = x + x^2$: This function has two simple roots at $-\infty$ and 0. We may factor it as

$$p(x) = c_0 \odot (x \oplus -\infty) \odot (x \oplus 0)$$

and even if $p(-\infty) = -\infty$ we can plug in 0 to get 0 back in order to get $c_0 = 1$.



Correspondence Theorems

Recall the maps

$$\begin{cases} T_t : \mathbb{C} \rightarrow \mathbb{T} & (\text{with max}), \\ \text{val} : \mathbb{C}\{\{t\}\} \rightarrow \mathbb{T} & (\text{with min}). \end{cases}$$

If we consider a polynomial

$$p(X) \in \mathbb{C}[X] \quad \text{or} \quad p(x) \in \mathbb{C}\{\{t\}\}[X]$$

then we can produce a tropical polynomial as follows:

- i. Apply T_t or val to the coefficients, and
- ii. Perform tropical operations.

We expect that if $r \in \mathbb{C}$ or $r \in \mathbb{C}\{\{t\}\}$ is a root of p , then $\lim_{t \rightarrow \infty} T_t(r)$ will be a root of the new polynomial.

Or the other way around, given $p \in \mathbb{T}[x]$, we can lift the coefficients to \mathbb{C} or the Puiseux series via the above maps. We can find the roots of the corresponding polynomials in $\mathbb{C}[x]$ or $\mathbb{C}\{\{t\}\}[x]$ and then the image of those roots via T_t or val are the tropical roots of $p(x)$.

Example 2.6.3. Consider the polynomial $p(x) = 2 \odot x \oplus 3 \in \mathbb{T}[x]$. We wish to construct a polynomial in $\mathbb{C}[x]$ which tropicalizes to p . Take the polynomial

$$q(x) = t^2 X + t^3 \in \mathbb{C}[x], \quad t > 0$$

We could certainly add phase as $e^{i\theta}$ to the t^k 's, but that won't change anything. Taking the logarithm of the coefficients we get

$$t^2 \mapsto 2 \quad \text{and} \quad t^3 \mapsto 3.$$

Then switching the operations to tropical operations we have

$$t^2 X + t^3 \xrightarrow{\text{Trop}} 2 \odot X \oplus 3$$

which was our original polynomial p .

Additionally if we solve the equation $q = 0$ we obtain the root $X = -t^3/t^2 = -t$. Now $\log_t | -t | = 1$. Lo and behold, this is the same root of $p(x)$.

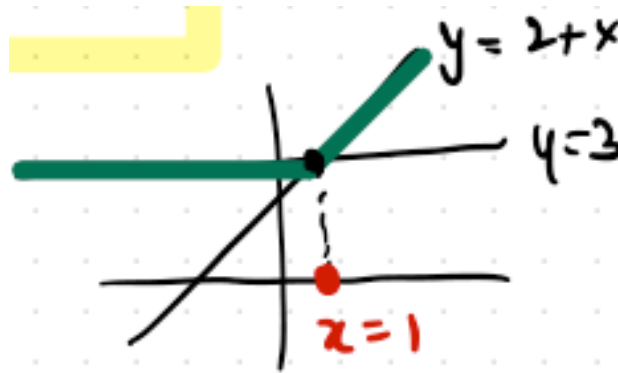


Figure 2.3: Root of $p(x)$ in correspondence with $-t$ of $q(x)$

We should be skeptical because this was only an example of a linear polynomial. Lets increase the degree and see what happens. Eventually this correspondence must be shown to hold in its entirety.

Example 2.6.4. Consider the polynomial

$$q(X) = X^2 + t^2 X + 1 \in \mathbb{C}[X] \xrightarrow{\text{Trop}} p(x) = x^2 \oplus 2 \odot x \oplus 0.$$

We can identify the roots of p as -2 and 2 . However, we may find it difficult to interpret the roots of q as roots of p . Observe that using the quadratic formula we may derive those to be:

$$X_{1,2} = \frac{-t^2}{2} \pm \frac{\sqrt{t^4 - 4}}{2} = \frac{-t^2}{2} \left(1 \pm \sqrt{1 - \frac{4}{t^4}} \right).$$

Even if taking the logarithm seems hard, we are not interested in the logarithm itself, just the limit! Observe that

$$\lim_{t \rightarrow \infty} \log_t \left| \frac{-t^2}{2} \left(1 + \sqrt{1 - \frac{4}{t^4}} \right) \right| = 2 + \lim_{t \rightarrow \infty} \frac{1}{\log(t)} \log \left| \frac{1}{2} \left(1 + \sqrt{1 - \frac{4}{t^4}} \right) \right|$$

and the quantity on the right tends to $1/\infty$ which collapses to zero and then the logarithm only has 1 as its argument. So overall we find one of our original roots: 2! The next limit has a different sign so it is not as direct. We may calculate that limit as follows:

$$\lim_{t \rightarrow \infty} \log_t \left| \frac{1}{2} \left(1 - \sqrt{1 - \frac{4}{t^4}} \right) \right| \approx \lim_{t \rightarrow \infty} \log_t \left| \frac{1}{2} \left(1 - \left(1 - \frac{4}{2t^4} \right) \right) \right| = \lim_{t \rightarrow \infty} \log_t \frac{1}{t^4} = -4.$$

So for the negative root we would actually obtain $2 - 4 = -2$ which is the other root of our polynomial.

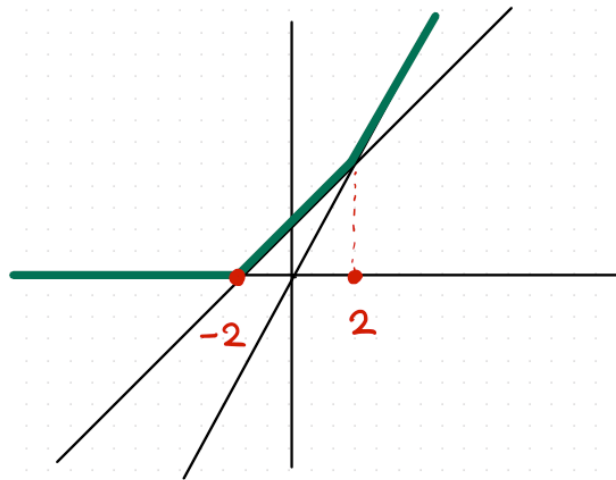


Figure 2.4: Indeed the roots of q correspond with p 's

2.7 Root finding

In previous lectures, we had the following theorem

Theorem 2.7.1. Let $q(X) = \sum A_i X^i$, $A_i \in \mathbb{C}\{\{t\}\}$, $a_i = \text{val}(A_i)$, and consider $p(x) = \text{trop}(q(X)) = \sum a_i \odot x^{\odot i}$. Then R is a root of q implies $r = \text{val}(R)$ is a root of p . Furthermore, if r is a root of p , then there exists a root R of q where $r = \text{val}(R)$.

We prove this inductively.

Combinatorialization of tropical root finding

We momentarily convert to the max convention. We take $p(x) = \sum_{i=0}^d a_i \odot x^{\odot i}$. Indeed, $p(x) = \max_i \{a_i + ix\}$. There are $d + 1$ lines $y = a_i + ix$. It is a finite process to intersect every pair of these lines, and then to compare the corresponding heights of each function to find the max. To make root finding more efficient, we start from left to right. To find the left most root, consider the left most intersection. It is thus the minimum of the x -value of intersection of the line $y = a_0$ and $y = a_i + ix$, when there exist a horizontal line (otherwise start with the line with the smallest slope, or factor out powers of x). This is the $\min\{x = (a_0 - a_i)/i\} = -\max\{\frac{a_i - a_0}{i}\}$. $\frac{a_i - a_0}{i}$ is now reminiscent of slopes. We view this as rise over run, we look at all the lines through $(0, a_0)$ connecting to (i, a_i) , and we are looking for the largest such slope. This gives us a_j for our first root. Our next root is to the right of a_j . This is because the corresponding line has a larger slope than the preceding lines, and the preceding lines have a later intersection, and are thus no longer considered in our root finding. So we use that as our next starting point, and we continue on.

We get a description of how to find the roots utilizing these points. This results in the following algorithm.

- (a) The segment $[0, d] \subset \mathbb{R}$ (aka the Newton polytope of the polynomial $p(x)$) is the convex hull of the i such that $a_i \neq -\infty$ (i.e. assume degree d polynomial with constant term)s.
- (b) Find the convex hull of the points $(i, a_i) \in [0, d] \times \mathbb{T}$ for $i \in \mathbb{N} \cap [0, d]$.
- (c) We construct the line from $(0, a_0)$ to (d, a_d) , and we disregard the section of the convex hull below this line. We call the section above this line Σ^+ .
- (d) Project the vertices of Σ^+ back onto $[0, d]$. This gives a regular subdivision of the Newton polytope.

Following this algorithm, we have that

- A) The roots of $p(x)$ are in bijection with the complement of the projections, i.e. the subdivisions of the Newton polytope.

B) The value of the root corresponding to a given segment (i, j) is found by solving the equation $a_i + ix = a_j + jx$.

C) The multiplicity of the root is equal to the length of the segment.

Example 2.7.1. Let $p(x) = 0 \oplus (1 \odot x) \oplus (1 \odot x^2) \oplus x^3 \oplus (2 \odot x^4) \oplus (1 \odot x^5)$. When graphing the convex hull, the points above the line connecting a_0 and a_5 are a_1, a_2, a_4 . The vertices are a_0, a_1, a_4 , and a_5 . Thus, we expect to have two simple roots r_1 and r_3 , and one root of multiplicity of 3 r_2 . To find the root r_1 , we solve $0 = 1 + x$, $x = -1$. To find r_2 , we solve $1 + x = 2 + 4x$, which is $x = \frac{-1}{3}$. To find r_3 , we solve $2 + 4x = 1 + 5x$, which has as solution $x = 1$.

Grobner

Let \mathbb{K} be a field with a valuation, such as $\mathbb{C}\{\{t\}\}$. Then let $R_{\mathbb{K}}$ be the set of all elements with non-negative valuation, i.e. $\bigcup_{n \geq 0} \mathbb{C}[[t^{1/n}]]$. In particular, we can consider ideals and maximal ideals. and we define $M_{R_{\mathbb{K}}}$ to be the maximum ideal, the set of all elements with positive valuation. and thus $\bigcup_{n > 0} t^{1/n} \mathbb{C}[[t^{1/n}]]$. k is the residue field $R_{\mathbb{K}}/M_{R_{\mathbb{K}}} = \mathbb{C}$. The subset relation is $M_{R_{\mathbb{K}}} \subset R_{\mathbb{K}} \subset \mathbb{K}$.

Example 2.7.2. Suppose we had $q(x) = t^{-4}\sqrt{2}x + 3t^2x^2$. Suppose we picked a valuation $val(x) = -3$. We can thus ask for the valuation of the individual terms. Then $t^{-4} \rightarrow -4$, $\sqrt{2}x \rightarrow 0 + (-3)$, and $3t^2x^2 \rightarrow 2 - 6 = -4$. Now, we decide that the lowest order terms are from t^{-4} and $3t^2x^2$. Now, we no longer care about the value of t , so we only keep the coefficients. We say we have an initial form of q for a valuation of x . $In_{w=-3}(q(x)) = 1 + 3x^2$. With a polynomial in a value field, we have an equivalence relation based on the initial forms. Breaking up \mathbb{R} into initial forms gives roots of $q(x)$. The roots are where the valuations give initial forms which are not monomials.

We say \mathbb{K} is our valued field, such as $\mathbb{C}\{\{t\}\}$. $R_{\mathbb{K}}$ is the set of all series that start with $t \geq 0$, while $M_{\mathbb{K}}$ is the set of all series that start with $t > 0$. The residue field $R/M = \mathbb{C}$. If we start with a polynomial $q(X) \in \mathbb{K}[x]$ and $w = val(x) \in \mathbb{R}$, we get an initial form $In_w q$ to then get a polynomial with coefficients in the residue field \mathbb{C} . We look at the valuation of each monomial assuming $val(x) = w$, then we save only the monomials with the smallest valuation, and we only keep the coefficient in front of t^* (where t^* is the smallest term).

Example 2.7.3. Consider $q(X) = (t^{-4} + t^2) + \sqrt{2}x + 2t^2x^2$. The individual valuations are $-4, -3$, and $(2 - 6) = -4$. Since -4 is the lowest valuation, we only consider the first and third form. So we only consider $t^{-4} + t^2$, and $3t^2x^2$. Then the limit with the limit of t , we get $In_{-3}q = 1 + 3x^2$. We define $W := trop q(w)$. Then $In_w q[t^{-W}q(t^w x)]|_{t=0}$

($\sqrt{2}$ is ignored because valuations of products are added, and $\sqrt{2} = \sqrt{2}t^0$, which has valuation 0)

We now consider (\mathbb{R}, w) , with some fixed $q(X)$. We can define an equivalence relation by $w_1 \equiv w_2 \iff In_{w_1}q = In_{w_2}q$. This equivalence relation decomposes \mathbb{R} into equivalence classes of two types. The equivalence classes are either single points or open intervals. The open intervals correspond to when the initial form is a monomial. The single points are otherwise.

Example 2.7.4. Consider $q(X) = t^2 + \sqrt{2}x + 3x^2x^2$. At

Definition 2.7.1. The complement of the locus of w such that In_wq is a monomial is called the Grobner complex of $q(X)$.

The Grobner complex of $q(X)$ is equal to the roots of $trop(q(x))$.

2.8 More Variables

Let $p(x, y) = \sum a_{ij} \odot x^i \odot y^j$ be a tropical polynomial in two variables.

Definition 2.8.1. Define a tropical curve $V(p)$ to be either

- (a) The locus in the domain of piecewise linear p where p is not linear, or $(x, y) \mid \max(a_{ij} + ix + jy)$ is attained > 1 .

We have a correspondence theorem.

Theorem 2.8.1. If $q(x, y)$ is a polynomial with coefficients over a valued field $\mathbb{C}\{\{t\}\}$, and $trop(q) = p$, then the tropical curve $V(p)$ is equal to the closure of the valuation of the points $\{(val(x), val(y)) : (x, y) \in V(q)\}$.

We can then study structural properties of tropical curves. We get correspondence statements with subdivisions of Newton polygon, and we get balancing and edge weights. We will also see the tropical versions of classical plane curve theorems. In particular, we get a tropical Bezout theorem (two projective curves of degree d and e intersect in $d * e$ points) and a tropical deg/genus formula.

2.9 Interim 2

Definition 2.9.1. If $q(x) = \sum a_k x^k \in \mathbb{C}[X]$ or $\mathbb{C}\{\{t\}\}[x]$, then the tropicalization of q is

$$Trop(q) = \sum T_{t \rightarrow \infty}(a_k)x^k$$

or respectively with the valuation. In this case we omit the notation for tropical operations but the sum and product are tropical.

Theorem 2.9.2. For a polynomial q , r_k is a root of $q(x)$ with multiplicity m_k if and only if $T_{t \rightarrow \infty}(r_k)$ is a root of $\text{Trop}(q)$ of multiplicity m_k .

In the univariate case, we may prove the theorem using the following lemmas.

Lemma 2.9.3. Trop is a multiplicative function on polynomials. That is

$$\text{Trop}(pq) = \text{Trop}(p) \text{Trop}(q) \quad \text{for } p, q \in \mathbb{C}[x].$$

Lemma 2.9.4. The roots of $\text{Trop}(p) \text{Trop}(q)$ are the union of the roots of the factors. If a root is repeated then the multiplicities are added.

Exercise 2.9.5 (2). Prove the preceding lemmas and then conclude the theorem as a result.

Otherwise, we may prove the correspondence theorem in a different way. This is more conducive to a higher number of variables. This is helpful, as in higher dimensions we don't have a fundamental theorem of algebra. But, in this case, the most convenient perspective is the valued field perspective. So let us switch to that point of view and interpret

$$x \oplus y = \min(x, y).$$

Theorem 2.9.6. Let $q \in \mathbb{C}\{\{t\}\}[x]$, then $r \in \mathbb{C}\{\{t\}\}$ is a root of q if and only if $\text{val}(r) \in \mathbb{T} \cap \mathbb{Q}$ is a root of $\text{Trop}(q)$.

Proof

Let us begin by considering a root r of q , then $q(r) = 0$ which means that

$$a_0 + a_1 r + \cdots + a_d r^d = 0.$$

This is formal sum of monomials which in order to vanish, at least two of the monomials must reach a minimum order of vanishing to cancel. This is equivalent to $\text{val}(r)$ being a root of $\text{Trop}(q)$.

The other direction is substantially more difficult. This will be an instance of a realizability question. We have two cases, r is a finite root or $r = \infty$. We will assume that r is finite and do a proof by example.

Example 2.9.7. Consider the polynomial

$$q(x) = tx^3 + x^2 + x + t \Rightarrow \text{Trop } q(x) = 1 \cdot x^3 + x^2 + x + 1$$

The roots of this polynomial are $-1, 0$, and 1 . We will now find a root $r_1 \in \mathbb{C}\{\{t\}\}$ of q with $\text{val}(r_1) = 1$. For this to happen we require

$$r_1 = yt^{-1} + z \quad \text{where } y \in \mathbb{C} \quad \text{and} \quad z \in \mathbb{C}\{\{t\}\}, \text{val } z > 1.$$

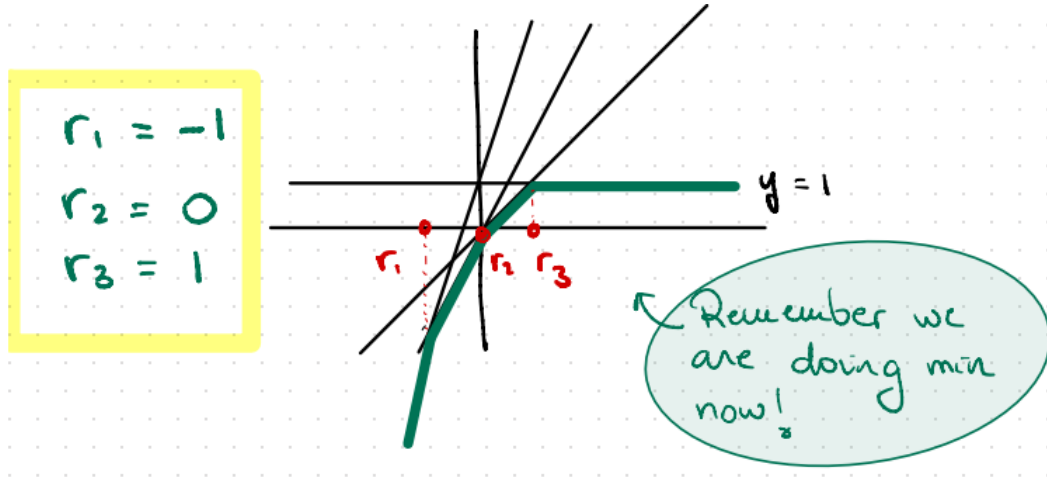


Figure 2.5: Tropicalization of q in min convention

We now plug in r_1 into q and we obtain

$$\begin{aligned} q(r_1) &= t(yt^{-1} + z)^3 + (yt^{-1} + z)^2 + (yt^{-1} + z) + t \\ &= \underline{y^3t^{-2}} + 3y^2zt^{-1} + 3yz^2 + z^3t + \underline{y^2t^{-2}} + 2yzt^{-1} + z^2 + yt^{-1} + z + t \end{aligned}$$

Extracting the coefficients we get $y^3 + y^2 = 0$ which means that $y = -1$. Plugging this back into our expression as y we get

$$3zt^{-1} - 3z^2 + z^3t - 2zt^{-1} + z^2 - t^{-1} + z + t = tz^3 - 2z^2 + (t^{-1} + 1)z + (-t^{-1} + t).$$

Tropicalizing (is it actually or is it the reverse operation?) we get

$$1 \cdot z^3 + z^2 + (-1)z + (-1)$$

which has as a root $1 > -1$ So

$$z = y + z_1 \quad \text{with} \quad y \in \mathbb{C}, \quad z_1 \in \mathbb{C}\{\{t\}\}.$$

ASK MAPLE CODE

The question now is: how do we turn this idea into a formal proof?

- i. We do one root at a time, starting with the rightmost one.
- ii. Observe that if r is a tropical root and $\alpha = yt^r$ with y chosen so cancellation happens, then denoting \tilde{q} , q without the x^0 term:

$$\text{Trop}(q(x + \alpha)) > \text{Trop}(\tilde{q}) \oplus \text{Trop}(q(\alpha)).$$

- iii. Finally we iterate and check that the sequence of r_i 's goes to ∞ .

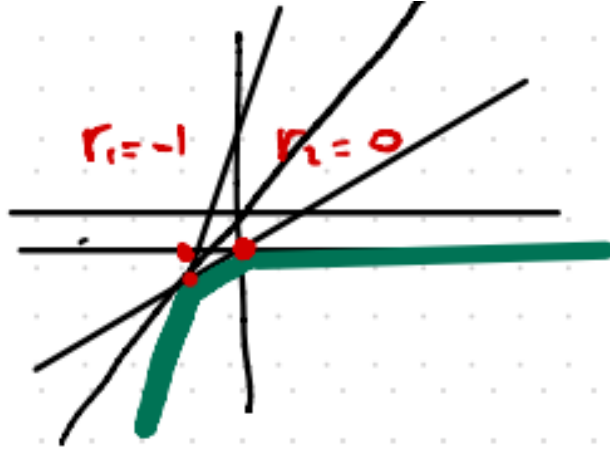


Figure 2.6: I don't know what this is

Combinatorialization of Root Finding

We will be using the max convention now. So let us consider $p(x) = \sum_{k=0}^d a_k x^k$. Can we have a systematic and simple way to say how many roots, with what multiplicity, and what equations to solve?

The left-most root can be found via

$$\min \left(\frac{a_0 - a_k}{k} \right) = \text{achieved by } k \text{ such that } \frac{a_0 - a_k}{k} \text{ is maximized.}$$

In other words we are looking for the largest slope: We may repeat this argument for the

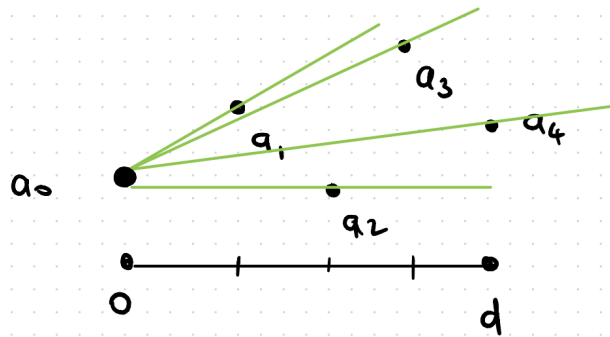


Figure 2.7: Difference of coefficients as slopes

following roots to get the following algorithm:

- i. Let $p_k = (k, a_k) \in [0, d] \times \{-\infty\} \cup \mathbb{R}$.
- ii. Now Σ is the convex hull of the points $\{p_k : k \in [d]\}$. We may divide the region into Σ^+ and Σ^- .

iii. Call $q_i = \pi(p_i)$ for p_i 's that for the vertices of Σ^+ .

The roots will be in bijection with the connected components of $[0, d] \setminus \{q_i\}_{i \in I}$ and the multiplicity is the length of the segment.

Example 2.9.8. Take for example the polynomial

$$p(x) = 0 + 1 \cdot x + 1 \cdot x^2 + x^3 + 2 \cdot x^4 + 1 \cdot x^5.$$

We now place the points in our diagram and project: From this we deduce that there are 2

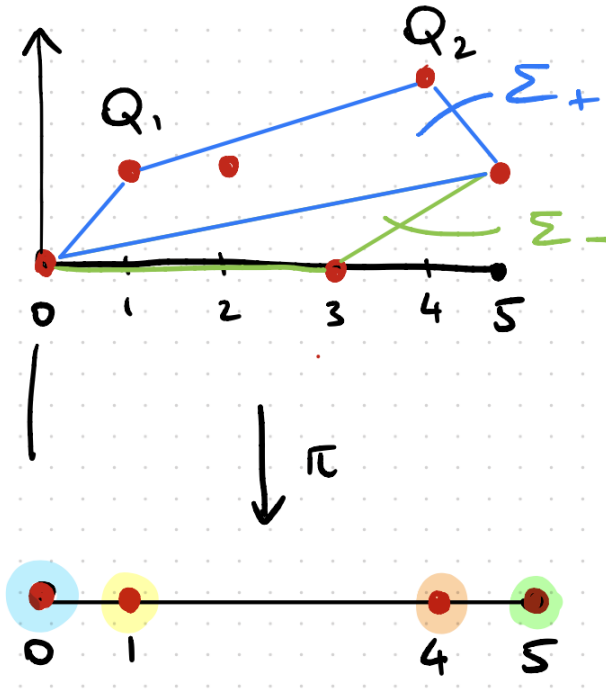


Figure 2.8: Root finding for $p(x)$

simple roots and 1 triple root. This come from the equations

$$\begin{cases} 0 = x + 1 & \Rightarrow x = -1 \\ x + 1 = 2 + 4x & \Rightarrow x = -1/3 \\ 2 + 4x = 1 + 5x & \Rightarrow x = 1 \end{cases}$$

Gröbner Complexes

If K is a field with a valuation, then call

$$\begin{cases} R_K \subseteq K = \text{elements with non-negative valuation} \\ \mathfrak{m} \subseteq R_K = \text{elements with positive valuation} \end{cases}$$

so R_K/\mathfrak{m} is a residue field. In the case of tropical polynomials, they form a Gröbner complex¹.

$$\mathfrak{m} = \bigcup_n t^{1/n} \mathbb{C}[[t^{1/n}]] \subseteq R_K = \bigcup_n \mathbb{C}[[t^{1/n}]] \subseteq \mathbb{C}\{\{t\}\}, \quad \text{and} \quad R_K/\mathfrak{m} = \mathbb{C}.$$

Definition 2.9.9. Given $q \in K[x]$ and $w \in \mathbb{T}$, the initial form of $q(x)$ with respect to w is a polynomial in $k[x]$ that records the part of q that has lowest order when $\text{val}(x) = w$.

Example 2.9.10. Let us consider the polynomial

$$q(x) = t^{-4} + \sqrt{2}x + 3t^2x^2,$$

Here²

$$t^{-4} \rightarrow -4, \quad \sqrt{2}x \rightarrow -3, \quad 3t^2x^2 \rightarrow -4, \quad \text{so} \quad w = -3^3.$$

We may construct the initial form as $I_w q = 1 + 3x^2$ but formally this is $[t^4(q(t^{-3}x))]_{t=0}$ and in general if $W = \text{Trop } q(w)$ then

$$I_w q = [t^{-W}(q(t^w x))]_{t=0}.$$

Gröbner Complex of $q(x)$

Polyhedral decomposition of \mathbb{R} (in the case of a valuation space if we want, we can also add in ∞ but it usually is left out.) induced by the equivalence relation

$$w_1 \sim w_2 \iff In_{w_1} q = In_{w_2} q^4$$

Example 2.9.11. Consider the polynomial

$$t^2 + \sqrt{2}x + 3t^2x^2$$

and each monomial maps⁵ to $2, w$ and $2 + 2w$ respectively. So the tropical roots are the

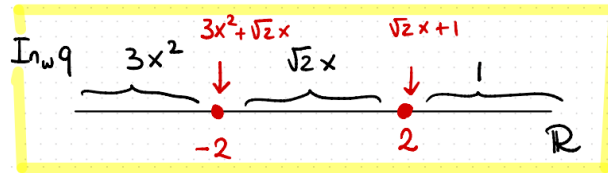


Figure 2.9: Initial form determination and roots

locus where the initial form is not a monomial.

¹What are Gröbner complexes? To see in interim.

²What does this mean?

³I srsly don't understand

⁴Does this refer to initial form?

⁵Through what? The valuation?

2.10 Day 7 | 20230906

Our First Correspondence Theorem

Definition 2.10.1. Given a family of polynomials

$$q_t = \sum A_k(t)x^k \in \mathbb{C}[x] \quad \text{with} \quad t > 1$$

then the tropicalization of q_t is

$$\text{Trop}(q_t)(x) = \sum a_k \odot x^{\odot k}, \quad \text{where} \quad a_k = \lim_{t \rightarrow \infty} T_t(A_k).$$

We may also use the min convention by exchanging the field to Puiseux series and T_t by the valuation.

Theorem 2.10.2 (Correspondence). *For a polynomial q_t , R_t is a root of q_t if and only if $\text{Trop}(R_t) = \lim_{t \rightarrow \infty} T_t(R_t)$ is a root of $\text{Trop}(q_t)$.*

This is saying that we have an object in algebraic geometry, a polynomial. Tropical geometry will somehow know about its roots by degenerating it. Then it's easy to find the tropical roots and then there must be certain algebraic roots which should map to them. It may not be easy to understand this last map but at least we have some qualitative information.

We will use the fundamental theorem of algebra to reduce to the linear case. So the first step is to prove the theorem for the case of linear polynomials. We have a couple of lemmas to finish the proof and expand it to the general case:

Lemma 2.10.3. *Trop is a multiplicative function on polynomials. That is*

$$\text{Trop}(pq) = \text{Trop}(p) \odot \text{Trop}(q) \quad \text{for} \quad p, q \in \mathbb{C}[x].$$

This first lemma doesn't add anything weird because the tropical product is just the usual addition.

Lemma 2.10.4. *The roots of $\text{Trop}(p) \odot \text{Trop}(q)$ are the union of the roots of the factors. If a root is repeated then the multiplicities are added.*

Essentially what this is saying is that if we have two piecewise linear functions which change slope at the same place, then the sum will also change slope at the same place. As the functions are convex, a root can never be cancelled. Except possibly $-\infty$.

Higher Dimension

We will go back to the Puiseux series convention now:

$$P(X) \in \mathbb{C}\{\{t\}\}[X], \quad P(R) = 0 \iff \text{Trop}(P)(\text{val}(R)) = 0.$$

The easy direction is to begin with a root of our Puiseux polynomial. Let

$$P(X) = \sum A_i(t)X^i, \quad \text{and} \quad \text{Trop}(P)(X) = \sum a_i \odot x^i$$

where $a_i = \text{val}(A_i)$. Let $R = R(t)$ be a root of $P(X)$.

We know $\text{val}(P(R)) = \infty$ because $P(R) = 0$. Formally $\text{val}(P(R))$ should be greater or equal than the minimum of the valuation of each of the monomials evaluated at R . In other words

$$\min(\text{val}(A_i(t)R^i)) = \min_i(a_i + i \text{val}(R)) = \text{Trop}(P)(R).$$

Since we know that strict inequality holds, the terms in the formal evaluation with lowest order must cancel, in other words, the minimum is attained at least twice by two different monomials.

Last week we mentioned attaining the minimum twice is the same as being a root.

Example 2.10.5. Consider the polynomial $(t^2 + 7t^3)X + (t^5 + t^{27}) = Q(X)$. The root here is $R = -\frac{t^5+t^{27}}{t^2+7t^3}$ and its valuation is $5 - 2 = 3$. If we plug in something of this form instead of X we get

$$Q(-t^3 + O(t^4)) = (t^2 + 7t^3)(-t^3 + O(t^4)) + (t^5 + t^{27}) = (-t^5) + t^5 + O(t^6)$$

In particular the first thing that will cancel is the lowest order term: t^5 . So *two* monomials must have lower order term.

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The next question is if this process makes sense if we instead begin with a Puiseux series polynomial. If the process ends up being the same, does this mean that tropical geometry over a trivial valued field is uninteresting? That's not the case, it's only because we are in dimension zero.

Gröbner Complexes

These types of complexes arise in commutative algebra. The setup begins with a valuated field, in our case Puiseux series $\mathbb{C}\{\{t\}\}$. We can find the ring of integers, the positive valuated elements, in our field. These types of functions are regular at $t = 0$. Inside this ring we have the maximal ideal of functions which vanish at zero. If we wish we can take a quotient to find the residue field which is a copy of \mathbb{C} .

Everytime we are given the data of polynomial q in $\mathbb{C}\{\{t\}\}[x]$ plus a choice of a valuation, we can recover the initial form of q which is a polynomial with coefficients in the residue field.

The way to find it is to look at the valuation of each monomial assuming $\text{val}(x) = w$ and then save only the monomials with the smallest valuation and only keep the coefficient in front of the smallest term.

Example 2.11.1. Consider the polynomial

$$q(x) = t^{-4} + t^2 + \sqrt{2}x + 3t^2x^2$$

and take $w = -3$. This means that $\text{val}(x) = -3$. Let us now consider the valuation monomial by monomial:

The term $(t^{-4} + t^2)$ has valuation -4 because there's no x , next for $\sqrt{2}x$ we have

$$\text{val}(\sqrt{2}x) = \text{val}(\sqrt{2}) + \text{val}(x) = 0 + (-3) = -3$$

so it has valuation -3 and $3t^2x^2$ has valuation $2 - 6 = -4$. We now consider only the first and last terms as they have the smallest valuation and extract the coefficients of the smallest terms. In the case of $t^{-4} + t^2$ its the 1 accompanying the t^{-4} and a 3 accompanying the last term. So the initial form is

$$\text{In}_{-3}(q) = 1 + 3x^2.$$

FORMULA for initial form

We now define an equivalence relation over (\mathbb{R}, w) :

$$w_1 \sim w_2 \iff \text{In}_{w_1}q = \text{In}_{w_2}q$$

which separates \mathbb{R} into two types of equivalence classes:

- ◇ Single points in which the initial form is not a monomial.
- ◇ Open intervals where the initial is a monomial.

Example 2.11.2. Consider the polynomial

$$t^2 + \sqrt{2}x + 3t^2x^2$$

and each monomial maps⁶ to $2, w$ and $2 + 2w$ respectively. So the tropical roots are the

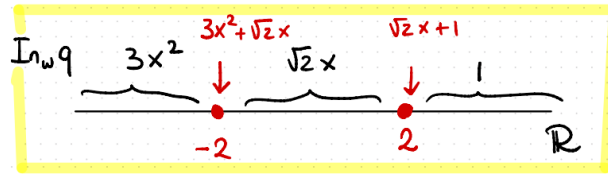


Figure 2.10: Initial form determination and roots

locus where the initial form is not a monomial.

Definition 2.11.3. The complement of the locus where the initial form is a monomial is called the Gröbner complex of $q(x)$.

The Gröbner complex of $q(x)$ is equal to the roots of $\text{Trop}(q)(x)$. This is indeed in correspondence with Gröbner basis, which is very interesting in higher dimension.

⁶Through what? The valuation?

1-dimensional Tropical Geometry

If we have $p(x, y)$ a tropical polynomial in two variables, then we can define its tropical variety to be $V(p)$:

- ◇ The locus in the domain where the piecewise linear function where p is not linear.
- ◇ The locus of points (x, y) where the max associated to each monomial is obtained more than once.

We will have a correspondence theorem which says that if $q(x, y)$ is a polynomial with coefficients over a valued field and the tropicalization of q is p , then

$$V(p) = \overline{\{(\text{val}(x), \text{val}(y)) : (x, y) \in V(q)\}}.$$

Exercise 2.11.4. Show that pairs of rational numbers are dense here. [It has to do with the valuation only taking rational numbers.]

In two dimensions we have way more features, the study of tropical curves will enclose the correspondence statement with subdivisions of Newton Polygon and balancing edge weights. Our objective now is to see the tropical versions of tropical curve theorems. For example, the tropical Bézout and tropical degree/genus formula.

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Tropical Lines

The max convention

If we have a tropical polynomial of degree 1,

$$p(x, y) = a \odot x \oplus b \odot y \oplus c$$

and assume for the sake of drawing pictures, that $-\infty \neq a, b, c$. This corresponds to the piecewise linear function

$$\max(a + x, b + y, c)$$

and if we set any of these two equations equal to each other, we can see that there are three lines that play a role:

$$a + x = b + y \Rightarrow y = x + (a - b)$$

$$a + x = c \Rightarrow x = c - a$$

$$b + y = c \Rightarrow y = c - b$$

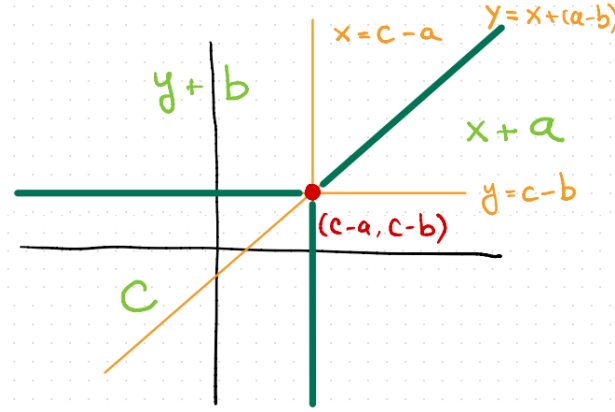


Figure 2.11: Graph of $p(x, y) = 0$ in \mathbb{R}^2

So this is the locus where two functions are equal to each other. In each of the regions the maximum is attained by a particular linear function, the boundary between them is the locus of non-linearity. The point in the middle is $(c - a, c - b)$.

So in general, tropical lines look like this “tripod” and changing the a, b, c shifts the graph.

Exercise 2.12.1 (2). Figure out what happens when a coefficient is $-\infty$.

This is analogous to what we have done with tropical univariate polynomials. The tropical line $V(p)$ is the locus of non-linearity of our function:

$$V(p) = \{ (x, y) \in \mathbb{R}^2 : df_p|_{(x,y)} \text{ isn't defined} \}.$$

The Case of Puiseux Series

In this case, lines will be the zero loci of polynomials of the form

$$p(X, Y) = A(t)X + B(t)Y + C(t)$$

with

$$a = \text{val}(A), \quad b = \text{val}(B) \quad \text{and} \quad c = \text{val}(C).$$

We let $L = \{ (X, Y) \in \mathbb{K}^2 : p(X, Y) = 0 \}$ be the zero locus and then define

$$\text{Trop}(L) = \overline{\{ (\text{val}(X), \text{val}(Y)) : (X, Y) \in L \}} \subseteq \mathbb{T}^2.$$

We may parametrize p in the following way, we let $X = \gamma(t)$ with an arbitrary valuation and then we solve for Y :

$$Y = \underbrace{\frac{-A(t)}{B(t)}\gamma(t)}_{a-b+\text{val } \gamma} - \underbrace{\frac{C(t)}{B(t)}}_{c-b}$$

ASK RENZO Depending on the value of $\text{val } \gamma(t)$, then we may get different values for $\text{val } Y(\gamma(t))$.

- ◊ If $\text{val } \gamma(t) > c - a$, then $\text{val } Y(\gamma(t)) = c - b$.
- ◊ If $\text{val } \gamma(t) < c - a$, then $\text{val } Y(\gamma(t)) = x + a - b$.
- ◊ And if $\text{val } \gamma(t) = c - a$ then $\text{val } Y(\gamma(t))$ can be anything above $c - b$. We thus set

$$\gamma(t) = \left(-\frac{C(t)}{A(t)} \right) (1 + t^{\odot m}), \quad \text{where } m > 0$$

The first two items represent a graph of $(a - b)x + (c - a)$ ⁷

Remark 2.12.2. If we send a, b, c, X and Y to their negatives then $\text{Trop}(L)$ agrees with the previous perspective. Alternatively, we can check that $\text{Trop}(L)$ agrees with the previous perspective but using the min convention.

Example 2.12.3. Let us explicitly choose A, B and C :

$$q(X, Y) = t^a X + t^b Y + t^c.$$

Note that q 's tropicalization is $p(X, Y)$. Points in $V(q)$ can be parametrized as

$$X = \alpha, \quad Y = \frac{-t^a}{t^b} \alpha - \frac{t^c}{t^b} = -t^{a-b} \alpha - t^{c-b}, \quad \text{where } \alpha \in \mathbb{K}^*.$$

Taking valuations of X and Y we get $\text{Trop}(L)$ (but without closing it). Specifically we are looking at the set

$$\text{Trop}(L) = \{ (\text{val}(\alpha), \text{val}(-t^{a-b} \alpha - t^{c-b})) \in \mathbb{T}^2 : \alpha \in \mathbb{K}^* \}$$

We can let α have any valuation we want and depending on that, we determine the valuation of the binomial Y . The possible valuations are

$$\text{val } Y = a - b + \text{val}(\alpha) \quad \text{or} \quad \text{val } Y = c - b$$

which are equal when $\text{val}(\alpha) = c - a$. **ASK RENZO ABOUT THIS HOW TO DETERMINE THE CRITERION ABOUT VAL AND STUFF**

- ◊ What happens if $\text{val}(\alpha) < c - a$ then $-t^{a-b}$
- ◊ Something I fell asleep

Claim: We can obtain any value for y but is to be greater than $c - b$. Let $r \geq 0$ and $\alpha = -t^{c-a}(1 + t^r)$

⁷ask Renzo because I don't understand, page 4 of TG7.

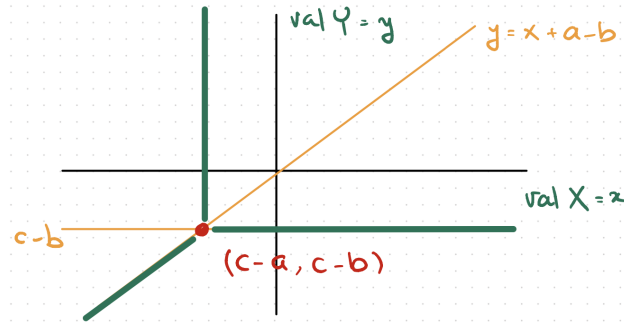


Figure 2.12: The tropical line from the Puiseux perspective

Glimpse into Amoebas

Recall that what matters the most is the logarithm base t of our function. So let us continue with

$$q(x, y) = t^a x + t^b y + t^c$$

and play the same as before. Look for solutions to the equation $q_t = 0$ in \mathbb{C}^2 which is a line intersecting the x axis at $-t^{c-a}$ and the y at $-t^{c-b}$. Every pair of points (x, y) , gives us a pair $(\log_t |x|, \log_t |y|)$. The real trace of this, when $x, y \in \mathbb{R}$ can be parametrized with $x = t^a$ and $y = -t^{a-b+\alpha}$. We analyze the trace in three intervals,

To recap $p(x, y)$ a tropical polynomial, we can define the variety of p $V(p)$, which is either the locus of non-linearity of p , or the locus where the maximum is attained more than once. Then for $q(X, Y)$ a polynomial, we can tropicalize it.

Lines

Lines are $V(p)$ such that $\deg(p) = 1$. Now, to see what happens with tropical lines, consider $p(x, y) = (a \odot x) \oplus (b \odot y) \oplus c$. Assume $-\infty < a < b < c$. Then $p(x, y) = \max\{a + x, b + y, c\}$.

Setting any two equal to each other, we get $a + x = b + y$, $a + x = c$, and $b + y = c$. We get three lines $y = x + (a - b)$, $x = c - a$, and $y = c - b$.

Every tropical line is of the form of a tripod. Even if we only keep $-\infty < a, b, c$, we still keep the tripod, and the corresponding regions of maxima are maintained. This is because the locus are found by setting (constant plus variable) = constant which gives a vertical or horizontal line, or (constant plus variable) = (constant plus variable), which gives a line of slope one

Example 2.12.1. What happens when some of the coefficients are $-\infty$.

As a second perspective, let's let $q(X, Y)$ be a degree 1 polynomial with coefficients from the Puiseux series. We thus have $q(X, Y) = t^a X + t^b Y + t^c$. If we tropicalize q , we get $\text{trop}(q) = (a \odot x) \oplus (b \odot y) \oplus c$. If we take $V(\text{trop}(q))$ we get the thing we had

before modulo adjusting for switch between min and max conventions. In this context, $V(q) = \{(X, Y) \mid q(X, Y) = 0\} \subset (\mathbb{K}^*)^2$

The great thing about lines is that they can be parameterized. We can write $V(q) = \{(\alpha, -t^{a-b}\alpha - t^{c-b}) \mid \alpha \in \mathbb{K}\}$. Now, for any point in $V(q)$, we want to take the valuation $\{(val(\alpha), val(-t^{a-b}\alpha - t^{c-b})) \mid \alpha \in \mathbb{K}^*\} \subset \mathbb{R}^2$. We now study the valuation of $-t^{a-b}\alpha - t^{c-b}$, which is done by studying the valuation of the individual terms. The first term has valuation $a - b + val(\alpha)$, and the valuation on the right equals $c - b$. Interesting things happen when the valuations are equal, when $val(\alpha) = c - a$.

Finally, we consider $val(\alpha) = c - a$ our claim is that we can obtain any value for Y , but it has to be $\geq c - b$

Proof. Let $\gamma \geq 0$, and let $\alpha = -t^{c-a}(1 + t^\gamma)$. We need $\gamma \geq 0$, so that the valuation of $1 + t^\gamma$ equals zero, and so the valuation of γ remains $c - a$. Then $val(Y(\alpha)) = val(-t^{a-b}(-t^{c-a}(1 + t^\gamma)) - t^{c-b})$. This equals $val(t^{c-b(1+t^\gamma)} - t^{c-b}) = val(t^{\gamma+c-b}) = \gamma + c - b$, and so we can make y take valuation any value greater than or equal to $c - b$. \square

If we were to send $a \mapsto -a, b \mapsto -b, c \mapsto -c, X \mapsto -X$, and $Y \mapsto -Y$.

This process can be repeated for the Amoeba perspective. Take a family of polynomials $q_t(X, Y)$. The coefficients are functions of t , but we specify $q_t(X, Y) = t^a X + t^b Y + t^c$. We now want to consider $q_t = 0 \subset \mathbb{C}^2$. For every $(X, Y) \in L$, we consider $(\log_t |X|, \log_t |Y|)$. We first study the real trace of this object, i.e. when $X, Y \in \mathbb{R}$. We now have three cases to consider.

We can consider real image, where $X, Y \in \mathbb{R}$, and further when $0 < X, Y$. Then we can take $(\log_t X, \log_t Y)$. We can once again parameterize to get $X = t^\alpha, Y = -t^{a-b+\alpha} - t^{c-b}$. For each of the three cases, we pick up asymptotes.

2.13 Day 12 | 20230918

The Amoeba Perspective

Our goal is to understand the image of the line

$$L_t = \{t^a x + t^b y - t^c = 0\} \subseteq \mathbb{C}^2$$

via the map $(x, y) \mapsto (\log_t |x|, \log_t |y|)$. The line L_t has three sections, where both x, y are positive and one section corresponding to each x and y being negative. For ease of calculation we may solve the equation as $y = -t^{a-b}x + t^{c-b}$

Let us consider the case where x, y are both positive. We can see that $0 < x < t^{c-a}$, this traces an x in the parameter space such that $-\infty < x < c - a$. Via the solution for y we may write

$$\log_t |y| = \log_t(t^{c-b} - t^{a-b+x})$$

where we have solved the equation for x which is why we have an x exponent and y is positive as we have assumed. We can simplify this as

$$\log_t [t^{c-b} (1 - t^{a-c+x})] = (c - b) + \log_t(1 - t^{a-c+x}).$$

This can be traced as a function of t and in particular

$$\lim_{x \rightarrow -\infty} (c - b) + \log_t(1 - t^{a-c+x}) = c - b \quad \text{and} \quad \lim_{x \rightarrow (c-a)^-} (c - b) + \log_t(1 - t^{a-c+x}) = -\infty.$$

With this information we see two asymptotes for our function, $y = c - b$ and $x = c - a$.

Arbitrary Degree d

Recall that for a polynomial $q \in \mathbb{K}[x, y]$, we may describe its algebraic variety in \mathbb{K}^2 . We may think of that field as Puiseux series. Along it, we may tropicalize it to p and we get its tropical hypersurface, the set of non-linearity.

Kapranov's theorem allows us to see a correspondence as follows:

$$\overline{\text{Trop}(V(q))} = V(\text{Trop}(q)).$$

Left-to-right is still the same idea as the correspondence theorem. If $(x_0, y_0) \in \text{Trop}(V(q))$ then, there exists $(X_0, Y_0) \in \mathbb{K}^2$ such that $\text{val}(X_0) = x_0$, $\text{val}(Y_0) = y_0$ and $q(X_0, Y_0) = 0$. Let

$$q = \sum a_{ij} X^i Y^j$$

If we call m_{ij} each monomial, then $\{m_{ij}(X_0, Y_0)\}_{i,j}$ is a set of elements of \mathbb{K}^* with the property that their sum is zero. Now call

$$\mu = \min\{m_{ij}(X_0, Y_0)\}_{i,j}$$

we claim that there are at least two monomials whose valuation is μ . If there was only one monomial with valuation μ , then that power of μ *cannot* be cancelled. This means that (x_0, y_0) is in $V(p)$. Now we use minimality of closure and we are done.

Now the harder direction will use the fact that we have proven this in dimension zero and proceed by induction. First we want to show that $V(\text{Trop}(q)) \cap \mathbb{Q}^2$ is dense in $V(\text{Trop}(q))$. This is true because all monomials m_{ij} correspond to all linear functions with integer slopes of rational coefficients.

$$a_{ij} X^i Y^j = \text{Trop}(m_{ij}) = \text{val}(a_{ij} \odot x^i \odot y^j) = \text{val}(a_{ij}) + ix + jy$$

It suffices to check that **ERASED TOO QUICK**

Now we wish to proceed by induction. For example a polynomial $q(X, Y)$ can be seen as

$$q(X, Y) = r_0(X) + r_1(X)Y + \cdots + r_d(X)Y^d \quad \text{with} \quad r_i(X) \in \mathbb{K}[X].$$

We do not lose generality when assuming that all r_i 's are monomials⁸. So we have $(x_0, y_0) \in V(\text{Trop}(q))$ and we want to find $(X_0, Y_0) \in (\mathbb{K}^*)^2$ such that

$$q(X_0, Y_0) = 0 \quad \text{and} \quad (\text{val}(X_0), \text{val}(Y_0)) = (x_0, y_0).$$

Choose X_0 , however we want as long as we have the valuation condition. Given our assumption, this implies that $r_i(X_0)$ is non-zero for all i . Now consider the polynomial

$$q(X_0, Y) = \sum r_i(X_0)Y^i \in \mathbb{K}[Y]$$

and its tropicalization $\tilde{p}(y) = \text{Trop}(q(X_0, Y)) = \sum \text{val}(r_i(X_0))y^i = \min(\text{val}(R_i(X_0)) + iy) = \min$ They are hidden in terms of unknown,

To recap, we start with a famil of lines indexed by $t \in \mathbb{R}_{>1}$, dentoed $L_t = \{t^a X + t^b Y - t^c = 0\} \subset \mathbb{C}^2$. We denote We have the function T_t , which makes $x = \log_t |X|$, and $y = \log_t |Y|$. We can solve for thi line, and get $Y = -t^{a-b}X + t^{c-b}$.

We focus momentarily at when $X, Y \in \mathbb{R}$, so we focus on \mathbb{R}^2 . In the case where X, Y are both positive is our first case. In particular, $0 < X < t^{c-a}$. In particular, $-\infty < x < c - a$.

We define a path $X_s := e^{i\pi s} X_0$, where $s \in [0, 1]$. Then for each x_s , we have a corresponding Y_s whcih is the correspodning equation of L for X_s . Now, we can ask about what happens to $T_t(|X_s|, |Y_s|)$. In this case, phase changes are irrelevant to the X corredenate., so $T_t(|X_s|, |Y_s|) = (\log_t(X_0), f(s))$, where $f(s)$ is continuous. This traces a full interval. which allows us to take advantage of the other cases where X and Y can be complex numbers.

We now take $q(X, Y) \in \mathbb{K}[X, Y]$ (Consider Puiseaux series for \mathbb{K}). We then consider the variety $V(q) = \{(X, Y) \mid q(X, Y) = 0\} \subset (\mathbb{K}^*)^2$. We can also take the tropicalization $p(x, y) = \text{trop}(q(X, Y))$. From here we can define the variety of p to be $V(p)$ which is the locus where p fails to be lienar. $V(p) \subset \mathbb{R}^2$. We also have the function $(\mathbb{K}^*)^2 \rightarrow \mathbb{R}^2$ defiend by (val, val) , whcih takes the valuation fo the coordinates. We hope to call (val, val) *trop*.

Theorem 2.13.1 (Kapranou's). $\overline{\text{trop}(V(q))} = V(\text{trop}(q))$, where the closure is with respect to the euclidean topology of \mathbb{R}^2 .

Proof. \subset still the same idea. If $(x_0, y_0) \in \text{trop}(V(q))$, that means that there exists $(X_0, Y_0) \in (\mathbb{K}^*)^2$ such that $\text{val}(X_0) = x_0$, $\text{val}(Y_0) = y_0$, and $q(X_0, Y_0) = 0$. Let $q = \sum a_{ij} X^i Y^j$, let $m_{ij} = a_{ij} X^i Y^j$ be the monomial. We can then consider the collection $\{m_{ij}(X_0, Y_0)\}_{ij}$ whcih is a collection of elements of \mathbb{K}^* with the property that their sum = 0. We let $\mu = \min \text{val}\{m_{ij}(X_0, Y_0)\}_{ij}$. The claim is that there are at least two monomials whose valuaton is μ . This implies that $(x_0, y_0) \in V(p)$. We have shown $\text{trop}(V(q)) \subset V(\text{trop}(q))$. However, $V(\text{trop}(q))$ is closed in the Euclidean topology (since the variety comes from equalities and inequalities).

\supset This direction is a bit tougher. We will prove the claim in dimension 0 and proceed by induction. We first want to show that $V(\text{trop}(q)) \cap \mathbb{Q}^2$ is dense in $V(\text{trop}(q))$. This is

⁸to see next time

true because all monomials m_{ij} correspond to affine linear functions with integer slopes and rational coefficients. We had $\text{trop}(m_{ij}) = \text{val}(a_{ij}) \odot x^i \odot y^j = \text{val}(a_{ij}) + ix + jy$, where $\text{val}(a_{ij}) \in \mathbb{Q}$, and $ix + jy \in \mathbb{N}$.

We can thus focus on rational points, checking that $V(\text{trop}(q)) \cap \mathbb{Q}^2$ lives in $\text{trop}(V(q))$, and then it will follow that the closures gives $V(\text{trop}(q)) = \overline{V(\text{trop}(q))} \cap \mathbb{Q}^2 \subset \overline{\text{trop}(V(q))}$.

We proceed by the following assumption. If we have a polynomial $q(X, Y)$, we can consider it a polynomial in Y with coefficients in X , i.e. $q(X, Y) = r_0(X) + r_1(X)Y + \dots + r_d(X)Y^d$, with $r_i(X) \in \mathbb{K}[X]$. We assume that $r_i(X)$ is a monomial for every i . We have $(x_0, y_0) \in V(\text{trop}(q))$, and we want to find corresponding $(X_0, Y_0) \in (\mathbb{K}^*)^2$ such that

- (a) $q(X_0, Y_0) = 0$, and
- (b) $\text{val}(X_0) = x_0$, and $\text{val}(Y_0) = y_0$

We choose X_0 arbitrarily, so long as $\text{val}(X_0) = x_0$. Because we have made our assumption of $r_i(X)$ being a monomial, no matter how we choose X_0 , X_0 is not a root of these monomials, and so this implies that $r_i(X_0) \neq 0$ for all i . Let us now consider the polynomial $q(X_0, Y) = \sum r_i(X_0)Y^i \in \mathbb{K}[Y]$. Let $\tilde{p}(y) = \text{trop}(q(X, Y)) = \bigoplus_{i=1}^d \text{val}(r_i(X_0)) \odot y^i$. Furthermore, we claim y_0 is a root of $\tilde{p}(y) = \min(\text{val}(r_i(X_0) + iy) = \min(\text{val}(a_{ij} + jx_0 + iy)$ recall that a_{ij} is a Puiseux series. Then

$$\tilde{p}(y) = \min(\text{val}(a_{ij} + jx_0 + iy) = \text{trop}(q(x, y)|_{x=x_0})$$

As we started with (x_0, y_0) is in $V(\text{trop}(q))$, then clearly y_0 is a root of our $\tilde{p}(y)$.

By this univariate case, there exist $Y_0 \in \mathbb{K}^*$ such that Y_0 is a root of $q(X_0, Y)$, and $\text{val}(Y_0) = y_0$.

We now need to show that the polynomial case proves the general case. TO see why the assumption that $r_i(x)$ is not monomial in X isn't too restrictive, consider $q(X, Y) = XY + X^2Y = (X + X^2)Y$. This is not of the form $\sum r_i(X)Y^i$. However, we can consider $\tilde{q}(X, Y) = q(XY, Y) = XY^2 + X^2Y^3$. This satisfies the assumption of monomials. If $(\tilde{X}_0, \tilde{Y}_0)$ is a solution for $\tilde{q} = 0$, then $(\frac{\tilde{X}_0}{\tilde{Y}_0}, \tilde{Y}_0)$ is a solution for $q = 0$. The key point is that \tilde{q} is obtained from q by an invertible transformation in $(\mathbb{K}^*)^2$.

Given $q(X, Y)$ of degree d , define $\tilde{q}(X, Y) = q(XY, Y^{d-1})$, this satisfies the assumptions we have made. If we have $q(X, Y) = \sum r_{ij}x^iY^j$, then $\tilde{q}(X, Y) = \sum r_{ij}X^iY^{(d+1)j+1}$. Now, if we ask if its possible to have conflicting i, j , i.e. $(d+1)j_1 + i_1 = (d+1)j_2 + i_2$? This says we need $(d+1)j_1 - j_2 = i_2 - i_1$. However, $0 \leq i_1, i_2 \leq d$, so their difference is $\leq d$. Furthermore, $j_1 - j_2 \geq d+1$ when $j_1 = j_2$. (Easier proof, sifting powers of Y first, then assure powers of X cannot cause overlap).

□

We can ask which polynomial with Puiseux valued coefficients has as its tropical polynomial $p(x, y) = xy \oplus x \oplus y \oplus 0$. valuations of zero only occurs with the constant complex numbers. We set the coefficients associated with X and Y 's to be 1, so we let $q(X, Y) = XY + X + Y + C$, where $C \in \mathbb{C}$. q is a polynomial in $\mathbb{K}[X, Y]$ with the property that $\text{trop}(q) = p$. Now, for a particular q , we can ask for $V(q)$. For $q = (X + 1)(Y + 1) + (C - 1) = 0$, we can ask for the set $\{(X + 1)(Y + 1) = C\}$. This is a translation of the \mathbb{C}^2 hyperbola where the asymptotes are the axis $X = -1$ and $Y = -1$.

We can ask to describe this curve in the projective plane. We homogenize with Z , to then describe the points at infinity. It is odd that the curve hits infinity twice, considering that two of the three special points of the projective plane of \mathbb{C}^2 , the other being the origin. Instead, we can compactify via $\mathbb{P}^1 \times \mathbb{P}^1$, the product of $\mathbb{C}^2 \cup \{\infty\}$. We do this by asking the polynomial bi-homogeneous (homogeneous on X , and homogeneous on Y). So we have $\tilde{q} = X_1Y_1 + X_1Y_0 + Y_1X_0 + X_0Y_0C = 0$. This is done by treating Y as a constant, then we get homogeneous in X , and vice versa. This is bihomogeneous of degree 1. Now we do not intersect the four special points at all, and we intersect each of the special lines once. This gives us general behavior (transversal intersection with the boundary).

Somehow, the shape of the tropical curve tells us that it is tropicalization of some plane curve, but it should be compactified in $\mathbb{P}^1 \times \mathbb{P}^1$, not \mathbb{P}^2 .

Theorem 2.13.2. Bummer Let $q(X, Y) = \sum_{i+j \leq d} a_{ij} X^i Y^j$ be a polynomial of degree d in $\mathbb{C}[X, Y] \subset \mathbb{C}\{\{t\}\}[X, Y]$, and all coefficients are $\neq 0$, i.e. $a_{ij} \neq 0$ for all i, j . Then $\overline{\text{trop}(v(q))}$ looks like a tropical line with vertex at $(0, 0)$.

This occurs for trivially valued field, where $0 \mapsto \infty$, and everything else maps to 0.

Proof. The tropicalization of q is $\bigoplus x^i \odot y^j$ (the a_{ij} are all nonzero \mathbb{C} , or their valuation is 0). Then $\text{trop}(q) = \min\{ix + jy\}_{i+j \leq d}$. Then the minimum is always obtained by 0 in the first quadrant when $i = j = 0$, dx in the section containing the second quadrant, and dy in the section containing the fourth quadrant. \square

Definition 2.13.1. Let $p(x, y) = \bigoplus a_{ij} \odot x^i \odot y^j$ be a tropical polynomial. The *Newton polygon* of p is the convex hull of (i, j) such that $a_{ij} \neq -\infty$.

Definition 2.13.2. Let Σ be the convex hull of the points $(i, j, a_{ij}) \subset \text{vert}(NP) \times \mathbb{R}$, Σ ($\text{vert}(NP)$ is the vertex set of the Newton polygon) is a convex polytope in $\mathbb{R}^2 \times \mathbb{R}$. We consider $\pi_z : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $(x, y), z \mapsto (x, y)$, and let \tilde{N} be the subdivision of the Newton polytop by projecting the corners of Σ you can see from above (positive z coordinate). Then the tropical curve $V(p)$ is *dual* to such a subdivision, i.e.

- (a) There is a bijection between vertices/edge of a tropical curve \leftrightarrow faces/edges of \tilde{N} ,
- (b) reversing poset structure given by inclusion into the closure,

- (c) every edge of $V(p)$ is \perp to the edge of \tilde{N} it corresponds to, and
- (d) The coordinates of a vertex are found by solving the linear system of equations obtained by setting equal the linear functions corresponding to monomials corresponding to vertices of the face of the \tilde{N} dual to V .

Example 2.13.1. The Newton polynomial of $p(x, y) = 0 \oplus x^2 \oplus y^2 \oplus 1x \oplus 1y \oplus (1 + xy)$ has six points in a triangle, the three vertices and the three midpoints of the line segments. Each point gets a height corresponding to the value of the coefficient of the monomial (y^2 has coefficient 0, and thus weight 0, while $1xy$ has coefficient 1, and thus weight 1). We then get the three dimensional polytope $(a, b, wt(a, b))$, drape a curtain over the top, and we get a distinguished top corresponding to the triangle with vertices (the midpoints of the line segments which had weight 1).

Theorem 2.13.3. If $q(X, Y) \in \mathbb{K}[X, Y]$ such that $trop(q) = p$ (modulo adapting for min/max). Then the subdivision of the Newton polytope keep track of the initial forms of q , in the sense that for any cell in the Newton polygon subdivision the initial form is given by the monomials corresponding to the lattice points in the cell.

Example 2.13.2. Consider $q(X, Y) = 7 + 3X^2 + Y^2 + t^{-1}X + 2t^{-1}Y + t^{-1}XY$.

The silly but crucial observation to prove this theorem is

Lemma 2.13.4. Evaluating a tropical monomial at a point (x_0, y_0) can be done as a dot product

Proof. A tropical monomial is of the form $m = a \odot x^i \odot y^j$. Then $m(x_0, y_0) = a + ix_0 + jy_0 = (i, j, a) \cdot (x_0, y_0, 1)$. \square

Proof. When we construct the subdivision of the Newton polygon, we consider all points with coordinates (i, j, a_{ij}) as i, j range where $a_{ij} \neq -\infty$. So evaluating at (x_0, y_0) amounts to searching for the maximum of the dot product of the vector $(x_0, y_0, 1)$ with all points (i, j, a_{ij}) .

Evaluating the tropical polynomial at the normal vector to the plane at the top of the polytope for vectors on the edges of that face, we get zero, so the evaluation at the vertices of the vector are equal, so the evaluation of the point at the two monomials is equal. We want the vertex to be on the face of the tropical curve. For any other dot products, the evaluations are negative (the other vertices are below the plane, so the evaluations of the dot product will be negative). We construct the vectors from the plane down, so the vertex on the face is larger than the ones below. So we have $m_{ij}(n_x, y) < \tilde{m}_{ij}(n_x, n_y)$. When m_{ij} correspond to vertex not in face, and \tilde{m}_{ij} correspond to vertex in face.

This is done for every face of the Newton Polygon. then $(n_x, n_y) \in \mathbb{R}^2$ is the vertex of the tropical curve dual to the particular face we were considering.

Precisely, we consider the faces of the convex whose outward pointing normal has positive z coordinate. In higher dimension, we say the last coordinate of the normal vector of the face has positive value.

For the edges, we focus on a particular edge e of the Newton polygon. e bounds two face F_1 and F_2 . F_1 and F_2 have their respective normal vectors n_1 and n_2 . e dots to zero with n_1 and n_2 . The same is true for all linear combinations of n_1 and n_2 . In particular, it is true on the segment connecting n_1 and n_2 . So every point in the segment in \mathbb{R}^2 joining $(n_x, n_y)_1$ and $(n_x, n_y)_2$ has the property that the vector $(n_x, n_y, 1)_i(m_1) = (n_x, n_y, 1)_i(m_2) > (n_x, n_y, 1)_i(m_{other})$. As the maximum is obtained twice, those points belong to the tropical curve.

□

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Theorem 2.14.1. $V(\text{Trop}(q)) = \overline{\text{Trop}(V(q))}$

If we start with a polynomial in valued field, we can tropicalize or look at ... and then take the image of coordinates of points and back in \mathbb{R}^2 then take closure, we end in the same place. The fact that every point of the algebraic curve lies somewhere is an argument of cancellation of Lowest Order Terms. In particular when plugging the value for the Puiseux solution two terms must cancel.

Proof

We have shown that right-to-left is easy, cancellation of L.O.T.

The other direction is trickier, it's a lifting problem. Given $(x_0, y_0) \in V(\text{Trop}(q)) \subseteq \mathbb{R}^2$, then we must find

$$(X_0, Y_0) \in V(q) \subseteq \mathbb{K}^{*2}, \quad \text{val}(X_0) = x_0 \quad \text{val}(Y_0) = y_0$$

If we write q then we will assume that we can write

$$q(X, Y) = \sum r_i(X)Y^i, \quad r_i(X) \text{ monomials}$$

If we first plug in $X = X_0$ (which is any Puiseux series we want with valuation x_0 [We have picked such X_0]),

$$q(X_0, Y) = \sum r_i(X_0)Y^i$$

is a polynomial in Y with Puiseux series coefficients, now tropicalize this q we get

$$\tilde{p}(y) = \sum \text{val } r_i(X_0)y^i \quad (\text{tropical sum and product now}).$$

We claim that y_0 is a root of $\tilde{p}(y)$. Here's where we are using the monomial assumption.

What is the linear function associated to $\tilde{p}(y)$:

$$\tilde{p}(y) = \min(\text{val } r_i(X_0) + iy)$$

and as r_i is monomial, call it $r_i(X_0) = A_{ij}X^j$ where A_{ij} is a Puiseux series. So this \tilde{p} becomes:

$$\tilde{p}(y) = \min(\text{val}(A_{ij}) + jx_0 + iy)$$

which is exactly the tropicalization of $q(x, y)$ and plug in x_0 . This is a univariate polynomial, which allows to apply the univariate case. So there exists a Y_0 , Puiseux series, such that Y_0 is a root of $q(X_0, Y)$ with $\text{val}(Y_0) = y_0$.

It remains to see that our monomial condition is not a restriction.

This allows us not only to lift, but to pick one coordinate freely and then the other one is determined!

Example 2.14.2. Consider the polynomial

$$q(X, Y) = XY + X^2Y = (X + X^2)Y, \quad \text{and} \quad \tilde{q}(X, Y) = q(XY, Y) = XY^2 + X^2Y^3$$

and \tilde{q} does satisfy the previous assumption. If $(\tilde{X}_0, \tilde{Y}_0)$ is a solution to our problem for $\tilde{q} = 0$, then $\left(\frac{\tilde{X}_0}{\tilde{Y}_0}, \tilde{Y}_0\right)$ is a solution for $q = 0$.

The key point is that \tilde{q} is obtained q by an invertible transformation in the torus $(\mathbb{K}^*)^2$.

Given $q(X, Y)$ of degree d , then picking

$$\tilde{q}(X, Y) = q(XY, Y^{d+1})$$

satisfies the monomials assumption. This is because *we are giving enough space*.

$$q(X, Y) = \sum r_{ij}X^iY^j \Rightarrow \tilde{q}(X, Y) = \sum r_{ij}X^iY^{(d+1)j+i}$$

where if we wished to find ... then

$$(d+1)j_1 + i_1 = (d+1)j_2 + i_2 \Rightarrow (d+1)(j_1 - j_2) = i_2 - i_1$$

$$j_1 - j_2 \geq d+1 \text{ when } j_1 = j_2 \text{ and } i_2 - i_1 \leq d.$$

Example 2.14.3. Compute $V(p)$ for the following polynomials

$$\diamond p_1 = 0 + x + y + xy$$

$$\diamond p_2 = 0 + x + y - xy$$

$$\diamond p_3 = 0 - x - y + xy$$

For each of this polynomials there are $\binom{4}{2} = 6$ line possibilities so we check each one.

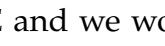
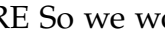
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Consider the polynomial

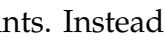
$$q(X, Y) = XY + X + Y + c, \quad c \in \mathbb{C}$$

as a polynomial in $\mathbb{K}[X, Y]$ such that $\text{Trop } q = p$. We can factor q as

$$(X + 1)(Y + 1) + (c - 1) = 0 \Rightarrow (X + 1)(Y + 1) = \tilde{c}$$

which looks hyperbolic. The real locus is  and we would like to compactify. However in \mathbb{P}^2 we get an anomalous curve.  So we would like to compactify instead in $\mathbb{P}^1 \times \mathbb{P}^1$. We bi-homogenize q to obtain

$$\tilde{q} = X_1Y_1 + X_1Y_0 + Y_1X_0 + X_0Y_0c = 0$$

In this case  we won't intersect the special points. Instead just general points. The point we want to emphasise is the fact that the shape of the tropical curve tells us that this could be the tropicalization of a curve but that the plane curve wants to be compactified in $\mathbb{P}^1 \times \mathbb{P}^1$ instead of \mathbb{P}^2 .

Bummer Theorem

Theorem 2.15.1. *Let $q \in \mathbb{C}[X, Y]$ considered as a subset of Puiseux series polynomials where no coefficient is zero. We can write $q(X, Y) = \sum_{i+j \leq d} a_{ij} X^i Y^j$ with $a_{ij} \neq 0$.*

Then $\overline{\text{Trop}(V(q))}$ looks like a tropical line with a vertex at zero.

This is a bummer because any polynomial of this form will look like a tripod. But what happens with *lines off to infinity matches the degree*? We may able to endow lines with information about the degree, but nonetheless, we lose every other piece of information. So choosing coefficients in a trivially valued field reduces all information.

To prove this theorem we will unravel the definitions.

Proof

We have that the tropicalization of q is

$$\text{Trop } q = \bigoplus (x^i \odot y^j) = \min_{i+j \leq d} (ix + jy)$$

and so we can see that the minimum is attained by 

Working with Puiseux series is not only because its fancy, its because we wish to have non-trivial objects.

Structure Theorem for Tropical Curves

Let $p(x, y) = \bigoplus a_{ij} \odot x^i \odot y^j$ be a tropical polynomial. The Newton polygon of p is the convex hull of (i, j) such that $a_{ij} \neq 0$.

Let Σ be the convex hull of the points $(i, j, a_{ij}) \subseteq V(NP) \times \mathbb{R}$, which is for every (i, j) point consider the height a_{ij} so Σ is a convex polytope in $\mathbb{R}^2 \times \mathbb{R}$. Seeing the polytope from the top and projecting down we get a subdivision of the Newton polygon.

Consider $\pi_z((x, y), z) = (x, y)$ and let \tilde{N} be the subdivision of the N.P. obtained by projecting the corners of Σ you can see from above. If we want to say this in fancier words, a polytope is a finite intersection of half-spaces and then we look at intersections of planes with outward normal vector (z positive coordinate) then project down to xy plane.

Then the tropical curve is **DUAL** to such subdivision, meaning that:

- ◊ Vertices of the tropical curve map to faces of \tilde{N} , edges map to edges of \tilde{N} .
- ◊ There is a reversing structure given by inclusion into the closure.
- ◊ Every edge of $V(p)$ is perpendicular to the edges of \tilde{N} , it corresponds to.
- ◊ Coordinates of vertices v are found by solving the linear system obtained by setting equal the linear functions corresponding to monomials corresponding to vertices of the face of \tilde{N} dual to v .

Example 2.15.2. Consider the polynomial

$$p(x, y) = 0 \oplus x^2 \oplus y^2 \oplus 1x \oplus 1y \oplus xy,$$

each monomial corresponds to a vertex in our triangle. Somehow now we know that our tropical curve has 4 vertices, then we can correspond edges.

To get the coordinates of the lower left vertex we look at the vertices surrounding the corresponding triangle. The linear system we ought to solve is

$$0 = 1 + x = 1 + y \Rightarrow x = -1, y = -1$$

and in this fashion we obtained the coordinates.

2.16 Day 15 | 20230925

We were talking about how tropical plane curves are dual to the subdivisions of the Newton polygon of the tropical polynomial. Let's recall this with an example.

Example 2.16.1. Consider the polynomial $0 + x^2 + y^2 + x + y + xy$. All coefficients are non infinity so we get 6 points corresponding to all terms, the convex hull is a triangle. We give each point a height, 0, x^2 and y^2 have 0 coefficient while x, y and xy have coefficient 1.

We imagine throwing a drape over that figure and then divide the faces, edges of tropical curve are perpendicular so the curve should look similar to the following figure.

FIGURE

Now for example, the central vertex is dual to the face F_2 which corresponds to the monomials $1 + x$, $1 + y$ and $1 + x + y$. The solution to the system

$$\begin{cases} 1 + x = 0 \\ 1 + y = 0 \\ 1 + x + y = 0 \end{cases} \Rightarrow (x, y) = (0, 0)$$

so $(0, 0)$ is the coordinate of the central vertex.

Remark 2.16.2. There's not metric duality between the triangle and the curve! It is possible to make a Newton polygon which doesn't fit in the curve as well.

Exercise 2.16.3 (5). Find such an example!

If we did this in the case of Puiseux series, then the subdivision of the Newton polytope keeps track of the initial forms of q in the sense that for any cell in the Newton subdivision, the initial form is given by the monomials corresponding to the lattice points in this set.

Example 2.16.4. Consider the polynomial

$$q(X, Y) = 7 + 3X^2 + 2Y^2 + t^{-1}X + 2t^{-1}Y + t^{-1}XY,$$

this polynomial tropicalizes via $-\text{val}$ to the polynomial p from last example.

FIGURE

The initial form in the face F_1 corresponds to the monomials $XY + 2Y + 2Y^2$, then $XY + 2Y + X$ in F_2 , $X + 2Y + 7$ in F_3 and $3X^2 + X + XY$ in the last F_4 . The Gröbner fan is still the same as in the tropical curve.

Along the edge 01 of F_1 the initial form is $2Y^2 + XY = In_w(q)$ where $w = (w_1, w_2) = -(\text{val } X, \text{val } Y)$ for w in the blue edge of the tropical curve (corresponds to **stuff** which has that initial form).

We will prove this fact by making a crucial observation. Evaluating a tropical monomial at a point (x_0, y_0) can be done as a dot product. Take $m = a \odot x^i \odot y^j$ so the evaluation is at (x_0, y_0) is

$$\langle (i, j, a) | (x_0, y_0, 1) \rangle.$$

When we construct the subdivision of the N.P. we consider all points with coordinates (i, j, a_{ij}) as i, j ranges over $\{a_{ij} \neq -\infty\}$ So evaluating $p(x_0, y_0)$ amounts to looking for the maximum of the dot products of $(x_0, y_0, 1)$ with all (i, j, a_{ij}) .

FIGURE

In other words $m_{ij}(n_x, n_y)$ is equal for all monomials corresponding to vertices of the green face. For vertices below, it occurs that $\langle \mathbf{n} | \mathbf{v} \rangle < 0$ so

$$m_{ij}(n_x, n_y) < m_{\tilde{ij}}(n_x, n_y)$$

when m_{ij} corresponds to vertices not in the face and $m_{\tilde{ij}}(\dots)$ in face.

We have identified why the vertices correspond to tropical subdivisions, but the edges? If we focus on one, it bounds two faces F_1 and F_2 which span two planes with normal vectors $\mathbf{n}_1, \mathbf{n}_2$ with their respective z coordinates equal to 1. So any vector between these two, i.e. any one in with first two coordinates in the segment n_1 to n_2 has the property that

$$\langle n | m_1 \rangle = \langle n | m_2 \rangle > \langle n | m_{\text{other}} \rangle$$

2.17 Day 16 | 20230927

We have discussed that tropical plane curves are dual to a subdivision of the Newton polytope. There's a combinatorial algorithm that will allow us to divide the N.P.

We will study a couple more characteristics to discern between stick figures and tropical curves. We need to introduce the fact that each *stick* gets a weight.

Definition 2.17.1. Any edge of a tropical curve $V(p)$ is given weight w_e equal to the lattice length of the segment of the N.P. subdivision dual to the edge.

Before even defining what things mean, let see an example.

Example 2.17.2. Consider the tropical cubic with subdivision edges $1 \rightarrow yx^2$ and $yx^2 \rightarrow y^2$. We may identify weights of edges with black in the coming figure:

Theorem 2.17.3. Tropical plane curves are balanced, meaning that at every vertex

$$\sum_{v \in e} w_e \mathbf{p}_e = 0.$$

Here w_e is the weight of the edge \mathbf{p}_e outgoing primitive vector in the direction of e .

Continuing the line of the previous example, at the vertex we were looking at we have outgoing primitive vectors

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Observe that when taking the weighted sum at the vertex we get

$$2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We will prove the theorem next.

Proof

Any vertex is dual to a face F_v of N.P. subdivision. For every edge bounding F_v , the vector $w_e \mathbf{p}_e$ is obtained by the vector tracing the dual edge via a 90° rotation. We claim to be done, the fact that the equation is satisfied is equivalent to the fact that the N.P. is a closed polygon.

So imagine a stick figure with weighted edges comes up to you, then we can check the compatibility condition. If it doesn't have weights, does there exist an assignment of weights in order to form a tropical curve?

What do the weights mean?

Example 2.17.4. Suppose we have a subdivision with edges

$$x^2y \rightarrow x^5y^2 \rightarrow x^8y^3 \rightarrow x^{11}y^4$$

and the corresponding edge in the tropical curve. We want to see the initial form. This subdivision *remembers* the monomials that appear in the initial form! It will be a linear combination of the monomials in the edge!

$$\text{In}_\omega(q) = Ax^2y + Bx^5y^2 + Cx^8y^3 + Dx^{11}y^4 = x^2y(A + Bx^3y + Cx^6y^2 + Dx^9y^3) = x^2yP(x^3y)$$

This polynomial factors so nicely because they lie on the same line! If we are just looking for solutions in $(\mathbb{C}^*)^2$ or asymptotically $|x|, |y| \gg 0$ what happens is that the monomial part is irrelevant. Also $\deg(P)$ is equal to the lattice length of the segment. A 1-parameter subgroup orbits $x^3y = r_i$ where the r_i is a root of P counted with multiplicity.

2.18 Sufficient conditions for tropical curves

Definition 2.18.1. Any edge of a tropical plane curve $V(p)$ is given weight ω_e equal to the lattice length of the segment of the Newton Polytope Subdivision dual to the edge.

Definition 2.18.2. A *primitive vector* of the direction vector p is the first integral vector (vector with integer coordinates) after the origin which lies on the line defined by p in the direction of p .

Theorem 2.18.1. Tropical plane curves are balanced, i.e. at every vertex, the summation $\sum_{v \in e} \omega_e \vec{p}_e = 0$, where ω_e is the weight of the edge, and \vec{p}_e is the primitive vector in the direction of e .

Proof. Any vertex v is dual to a face F_v of the Newton polygon subdivision. For every edge bounding F_v , the vector $\omega_e \mathbf{p}_e$ is obtained by the vector tracing the dual edge via the linear

transformation $(x, y) \mapsto (y, -x)$. Now, $\sum_{v \in e} \omega_e \vec{p}_e = 0$ is equivalent to the polygon being a closed polygon. \square

Now that we have defined weights, we want to ask what they represent.

If we are just looking for solutions of $x^2yP(x^3y) = 0$ in the torus $(\mathbb{C}^*)^2$ or asymptotically ($|x|, |y| \gg 0$), then (A) the monomial part x^2y is irrelevant (Only vanish at zero or infinity), and (B) We have $\deg(P) = \text{lattice length of the segment}$. On the torus, we have 1-parameter subgroup orbits of the form $x^3y = r_i$, where r_i is a root of p and is counted with multiplicity.

So if we take a point in the tropical curve w . We Think of the Puiseux plane as many planes assembled by valuation. Now that we have fixed valuations, we single out a particular \mathbb{C}^2 . When x, y are very large, the polynomial is approximated by the initial form, so we get (eight of edge) number of torus orbits. The direction of the edge in the subdivision tells us what we get for our 1-parameter subgroup. The primitive vector is $(3, 1)$, so we get x^3y .

The following are things to consider about tropical curves. We can draw all topological types of tropical plane conics and cubics. We can experiment with various tropical plane curves and seek a conjecture to compute their b_1 . Finally, we can study a pencil of tropical conics, i.e. draw a conic, pick four points on it in general position, then find all conics through those four points.

When following four parameter conics through the four points, we find interesting points between leaves and edges.

2.19 Day 17 | 20230929

Last time we talked about edges weighted by lattice length determines the segments of the Newton Polygon it is dual to. Today we will draw topological types of tropical plane curves and cubics. The objective is to experiment with various tropical curves and seek a conjecture to compute their b_1 . Study a pencil of tropical conics, in other words draw a conic and pick 4 points on it in general position. Then find all conics through those 4 points.

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For any interior point of the Newton Polygon, we can make sure that we can find as many cycles can a tropical curve have. There is a correspondence theorem.

The solution to the parameter space of conics is that we get a trivalent tree with 6 edges. This corresponds to \mathbb{P}^5 which has 6 boundary divisors.

Intersections of tropical curves

Definition 2.20.1. Two tropical curves have a transversal intersection if they intersect in finitely many points which are not vertices of either curve.

Example 2.20.2. Two tropical lines seen as tripods which touch just at a point intersect transversally.

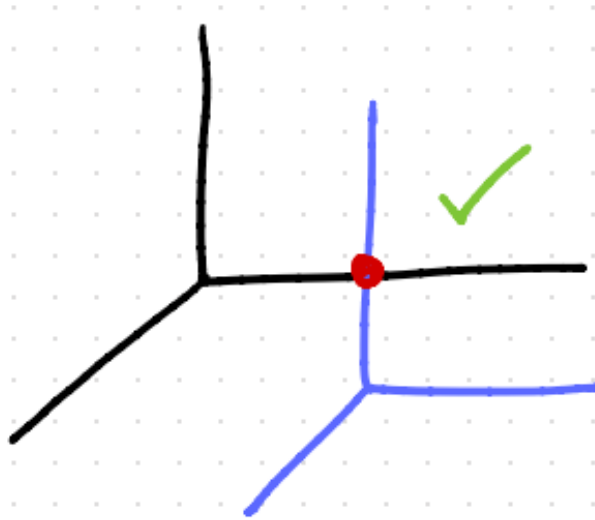


Figure 2.13: Example of a Transversal Intersection

Example 2.20.3. If the line is placed right on top of the other one, they could also intersect non-transversally on the whole edge.

If for example we had a conic, it could intersect the vertex of another tropical line non-transversally. Or two vertices could intersect!

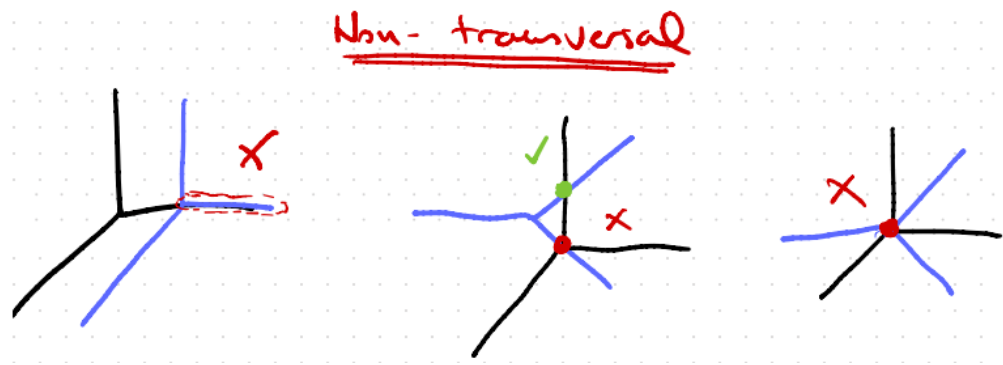


Figure 2.14: Example of Non-Transversal Intersections

It looks like if the intersection is transversal then the set of intersection is just a point. Otherwise let's be very topological and talk about stable intersections.

Definition 2.20.4. For a vector $\mathbf{v} \in \mathbb{R}^2 \setminus \{0\}$, call the vector intersection of Γ_1, Γ_2

$$\Gamma_1 \cap_{\mathbf{v}} \Gamma_2 = \lim_{t \rightarrow 0} (\Gamma_1 \cap (\Gamma_2 + t\mathbf{v})) .$$

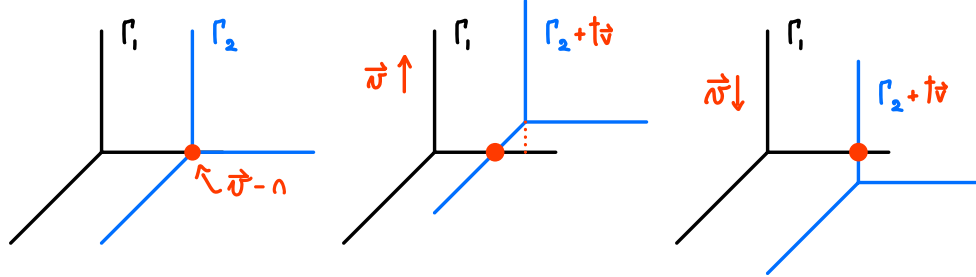


Figure 2.15: Example of Vector Intersections

Observe that this definition may be troublesome because of the choice of the vector \mathbf{v} . What happens if we get another intersection point with another \mathbf{v} ?

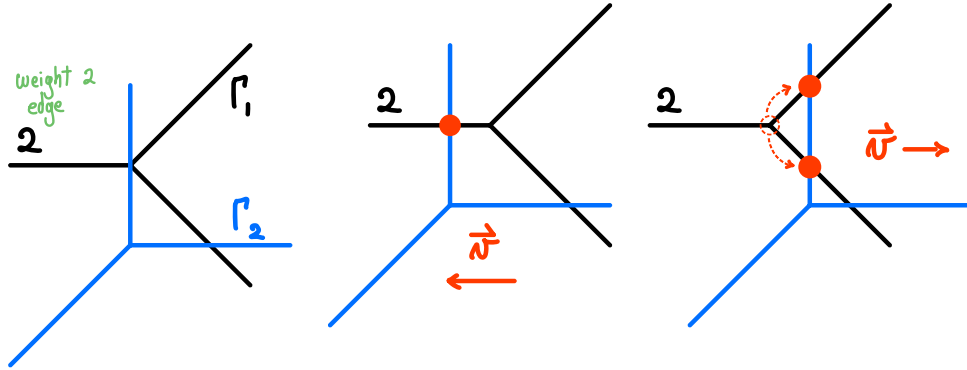


Figure 2.16: Weighted Vector Intersections

From this we deduce that intersection points should be weighted by multiplicity of the edges they belong to. We need more than just the edge weights to define this.

Before we jump into the theory, let's have an aside question, let us ask how the curves

$$C_1 = \{x^a = y^b\}, \quad \text{and} \quad C_2 = \{x^c = y^d\}$$

intersect in \mathbb{C}^2 . Observe that we may parametrize C_1 as $(x, y) = (t^b, t^a)$ and substitute into C_2 's to get

$$t^{bc} = t^{ad} \Rightarrow t^{bc}(t^{ad-bc} - 1) = 0$$

which means that at zero we have a point of high multiplicity and $(ad - bc)$ points away from the origin. The interesting question is how does this relate to our problem. At a glance there doesn't seem to be a connection, but if we look at the *valuation* we might just start to see some hints of it.

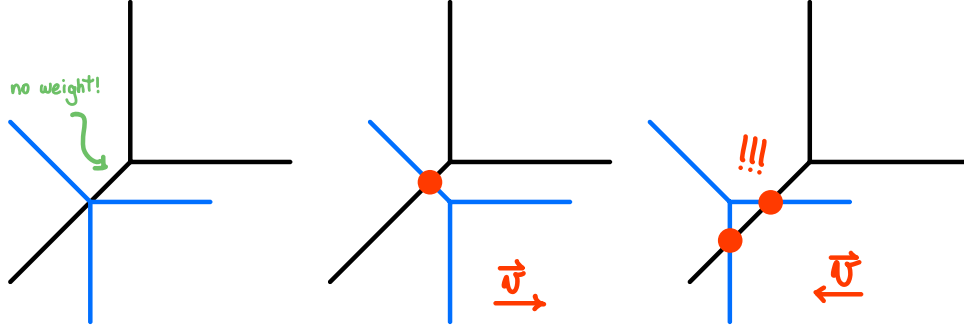


Figure 2.17: Problematic Vector Intersections

Definition 2.20.5. Let q be a point of transversal intersection of two tropical curves Γ_1, Γ_2 . Then the multiplicity of the intersection is determined by the primitive⁹ directions $\mathbf{p}_1, \mathbf{p}_2$ of the intersection as

$$m_q(\Gamma_1, \Gamma_2) = w_1 w_2 |\det(\mathbf{p}_1, \mathbf{p}_2)| = w_1 w_2 [\mathbb{Z}^2 : \mathbf{p}_1 \mathbb{Z} \oplus \mathbf{p}_2 \mathbb{Z}].$$

The last quantity is the subgroup index of $\mathbf{p}_1 \mathbb{Z} \oplus \mathbf{p}_2 \mathbb{Z}$ inside \mathbb{Z}^2 .

2.21 Intersection of Tropical Curves

Definition 2.21.1. Two tropical curves intersect *transversally* if they intersect in finitely many points which are not vertices of either curve.

Our second kind of intersection will be stable intersection. Let $v \in \mathbb{R}^2 - \{(0, 0)\}$, and define $\Gamma_1 \cap_v \Gamma_2 = \lim_{t \rightarrow 0} \Gamma_1 \cap (\Gamma_2 + tv)$, i.e. translate Γ_2 by the vector v . If we were to pick another v , our translations would be different. We are forced to ask ourselves if distinct choices of v may lead to different limits.

Note 2. Points of intersections should be weighted by multiplicities of the edges they belong to.

Example 2.21.1. How do the complex curves $C_1 = \{x^a = y^b\}$ and $C_2 = \{x^c = y^d\}$ intersect in \mathbb{C}^2 ?

We can parameterize $x = t^b, y = t^a$, then we get $t^{bc}(t^{ad-bc} - 1) = 0$

Definition 2.21.2. Let p be a point of transversal intersection of Γ_1, Γ_2 . We let the multiplicity of the point of intersection p to be

$$M_p(\Gamma_1, \Gamma_2) := w_{e_1} w_{e_2} \left| \det \begin{bmatrix} P_{1,x} & P_{2,x} \\ P_{1,y} & P_{2,y} \end{bmatrix} \right| = [\mathbb{Z}^2 : p_1 \mathbb{Z} + p_2 \mathbb{Z}]$$

⁹Primitive lattice vectors are the shortest lattice vectors possible

Where p_i are the direction vectors of the edges for the intersection, $p_{1,x}$ is the x coordinate of the direction vector for edge 1, and w_{ei} is the weight of the corresponding edge

2.22 Day 19 | 20231004

Our objective last time was to motivate the idea of intersections with certain multiplicity. With out new definition of multiplicity, does this resolve the issue we had? **WHAT WAS THE ISSUE?** The issue is that we must define multiplicity in order to talk about weights/multiplicities of non-transversal intersections.

Example 2.22.1. **FIGURE WHICH FIGURE THERES NO PICTURE** In the first case we have intersections at Q_1, Q_2 , where the first intersection has multiplicity

$$m_{Q_1} = \left| \det \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \right|$$

and the other point also has multiplicity 1. But if we move to the non-transversal intersection we get multiplicity 2. This goes according to the definition.

Lemma 2.22.2. *The amount of intersections of two tropical curves is invariant under generic translation.*

The proof follows from two facts:

- ◇ The definition of multiplicity of an intersection:

$$m_q(\Gamma_1, \Gamma_2) = w_1 w_2 |\det(\mathbf{p}_1, \mathbf{p}_2)|$$

- ◇ And the balancing condition at q :

$$\sum_{e \rightarrow v} w_e \mathbf{p}_e = 0$$

where $e \rightarrow v$ means edges incident at v .

It also suffices to analyze cases locally so we may zoom in into an edge of one of our curves.

Proof

Consider an edge of a tropical curve Γ_1 locally intersecting Γ_2 at a vertex. The edge of Γ_1 partitions the plane into two half planes H^+, H^- .

Let us partition the edges into those “in” H^+ and H^- and pick \mathbf{v}_+ in any direction pointing to H^+ and let us take the vector intersection with respect to such \mathbf{v}_+ .

Considering the multiplicities we get a contribution of

$$w_1 \sum_{e \in H^-} w_e \|\det(\mathbf{p}_1, \mathbf{p}_e)\|.^a$$

Similarly if we take a vector \mathbf{v}_- pointing towards H^- and translate Γ_2 in that direction we get a contribution of

$$w_1 \sum_{e \in H^+} w_e \|\det(\mathbf{p}_1, \mathbf{p}_e)\|.$$

Then the contribution to a \mathbf{v} deformed intersection is the same when \mathbf{v} points in the direction of H^+ or H^- .

Suppose \mathbf{v}_1 and \mathbf{v}_2 are two vectors pointing in different directions, if we raise the intersection, we get the lower edges intersecting. Their primitive vectors are all in the clockwise direction from p_e . So the weight of the \mathbf{v}_1 -deformed intersection is

$$\sum_{e' \in E^-} w_e w_{e'} |\det(p_1, p_2)|$$

^aThe edges we have are from H^- because of the relative positions. It does sound counterintuitive however.

Tropical Bézout

From this lemma the result follows immediately!

Theorem 2.22.3. *Let Γ_1, Γ_2 be two tropical curves of degree d_1, d_2 . Then $|\Gamma_1 \cap \Gamma_2| = d_1 d_2$.*

A curve has degree d if its dual to a subdivision of $0, (d, 0)$ and $(0, d)$. We know that if we have a tropical curve of degree d_1 then we can move one of degree d_2 in such a way that just so many edges intersect. Counting down we get the product of the multiplicities.

2.23 Day 20 | 20231009

Last time we started talking about the idea behind Bézout's theorem. Recall, that a curve with degree d_i is dual to the polygon with vertices $0, (0, d_i)$ and $(d_i, 0)$. Intersections, as in the ordinary case, are counted with multiplicities.

The quantity $|\Gamma_1 \cap \Gamma_2|$ is invariant under translation, so we can move two curves so that only unbounded ends intersect. Looking at this picture from far away and squinting our eyes what we see is two tripods with degree d_i . This means that all the information needed to compute the intersection is independent of the subdivisions of the Newton Polygon.

Definition 2.23.1. In $\mathbb{P}^1 \times \mathbb{P}^1$, coordinates are $([x_0 : x_1], [y_0 : y_1])$, a bi-degree (a, b) curve is the zero locus of a bi-homogenous polynomial in the aforementioned variables which is

- ◇ homogenous of degree a is x_i ,
- ◇ homogenous of degree b is y_i .

Tropically, this means that Γ is dual to a subdivision of a rectangle with sides a, b .

Example 2.23.2. The polynomial $x_0y_0 + x_1x_0y_1 = 0$ is a bidegree $(2, 1)$ curve. On the tropical side we have **FIGURE** which is a $(1, 1)$ curve.

The tropical Bézout theorem for $\mathbb{P}^1 \times \mathbb{P}^1$ says that if we have Γ_i with bidegree (a_i, b_i) , then

$$|\Gamma_1 \cap \Gamma_2| = a_1b_2 + a_2b_1.$$

But what if we wanted to intersect a degree d curve with a bidegree (a, b) curve? We just draw the stick figure and notice that degree of the intersection is $d(a + b)$. We can ask how to generalize it and the answer is precisely Bernstein's theorem.

Theorem 2.23.3. Let Γ_i be tropical curves of degree Δ_i . Then

$$|\Gamma_1 \cap \Gamma_2| = \text{MixedArea}(\Delta_1, \Delta_2).$$

In this case the degree of a tropical curve is the Newton polygon of its equation. In other words a lattice polygon.

Minkowski Sum of Polytopes

Once a long time ago we were told that a degree is a number, but a degree is actually a polygon.

Definition 2.23.4. Consider $\Delta_i \subseteq \mathbb{R}^2$ lattice polygon, then the Minkowski sum is

$$\Delta_1 + \Delta_2 = \{ (x_1, y_1) + (x_2, y_2) \in \mathbb{R}^2 : (x_i, y_i) \in \Delta_i \}.$$

This definition is **compatible** with translations! At this moment we can choose how to put two polygons in \mathbb{R}^2 and sum them.

Example 2.23.5. The Minkowski sum of a square and a right triangle is a gem-shaped pentagon. The idea is that we decide one of the vertices to be the origin and then make the polygon travel through the perimeter of the other one.

Exercise 2.23.6. The Minkowski sum appears to be commutative, prove it! Which other properties does the Minkowski sum enjoy?

Example 2.23.7. **FIGURE ABOUT MIXED SUDVISIONS**

Definition 2.23.8. The mixed area of Δ_1, Δ_2 is either of

- i) $A(\Delta_1 + \Delta_2) - A(\Delta_1) - A(\Delta_2)$,
- ii) the area of the mixed cells in any mixed subdivision of $\Delta_1 + \Delta_2$,
- iii) the λ_μ coefficient in $A(\lambda P + \mu Q)$.

Chapter 3

Toric Varieties

3.1 Day 21 | 20231011

IT says that if we have two tropical curves of degree Δ_1, Δ_2 (polytopes), and we wish to compute the intersections, then this is counted via the mixed area. This correspondence is very tight, satisfying. When taking stuff, we get the mixed cells (parallelograms with sides parallel to our original polygons). Having the polygons in positions means that the curves have been moved and the mixed cells correspond to the intersections.

The weight of the edges is the length of sides of the polygon, in particular the Minkowski sum of two lattice polytopes is itself a lattice polytope.

Torus actions

A space X with a torus action $T \cdot X$ and an open set isomorphic to the torus on which T acts by multiplication is a toric variety. The geometry of a torus is fairly simple or trivial, which means that *most* of the geometry of X can be recovered from the complement of this dense open set (the one isomorphic to T). The fact that we have a torus action allows for a *combinatorialization* of the geometric information.

If we don't know what a torus is, it's a group, and the space is just the space. The action moves the points in a certain way. Geometry is connected to the representation theory of the group.

Definition 3.1.1. The algebraic torus of rank k is G_m^k (think of it as $(\mathbb{C}^*)^k$) [[]] If we have a different field than \mathbb{C} , actually G is a multiplicative group *scheme*. This object is a group under pointwise multiplication and in general we can view it as the scheme

$$\text{Spec} \left(\mathbb{C} \left[x_1, \frac{1}{x_1}, \dots, x_k, \frac{1}{x_k} \right] \right).$$

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If we are more familiar with differential geometry, local coordinates are similar but the coordinate is a generalization. We can imagine elements on the scheme as regular functions from the torus to \mathbb{C} . We will usually denote the torus by T .

Recall a group acts on itself via multiplication and in a similar fashion the torus action on a space X will be a map $T \times X \rightarrow X$. We can thus relate the characters of T to invertible regular functions on T and if we are choosing coordinates, then they are also related to monomial functions in x_i 's.

The characters of T are the group homomorphisms from T to \mathbb{C}^* . It is actually the trace of an irreducible representation of T , but they are one dimensional and... Monomial functions on the x_i 's are precisely regular functions vanish. Those places are on the coordinate axis so nowhere near the torus. As a \mathbb{C} -vector space, monomials are a basis of our ring. That is gonna play an important role.

All of these interpretations are groups under multiplication. Given a monomial we can read a vector of exponents:

$$x^3 y^{-5} \mapsto (3, -5)$$

so this is also related to the lattice of exponents of the monomial functions. This gives a multiplicative to additive isomorphism. This lattice is called M , it is also called the character lattice and is isomorphic to $\mathbb{Z}^k \subseteq \mathbb{R}^k$ which we will also call $M_{\mathbb{R}}$. This is actually a *tensor product*. In this sense we can match

$$(3, -5) \mapsto \varphi_{(3, -5)} : T \rightarrow \mathbb{C}^*, (t_1, t_2) \mapsto t_1^3 t_2^{-5}.$$

We also have a relationship between co-characters (linear functions on characters), one parameter subgroups of T (group homomorphisms $\mathbb{C}^* \rightarrow T, t \mapsto (t^a, t^b)$). The co-character lattice N is the dual of the character lattice is isomorphic to $\mathbb{Z}^k \subseteq \mathbb{R}^k = N_{\mathbb{R}}$. There is a natural pairing $M \times N \rightarrow \mathbb{Z}$ which turns out to be the usual dot product. Slightly more satisfying, the natural way to get a pairing is to imagine we have a character χ and a one parameter subgroup γ we get a map

$$\mathbb{C}^* \xrightarrow{\chi} T \xrightarrow{\gamma} \mathbb{C}^*, t \mapsto (t^a, t^b) \mapsto t^{aA+bB}$$

so the exponent is precisely $\langle (a, b) | (A, B) \rangle$.

After talking about so many ingredients, let us talk about one more thing. If we have one parameter subgroups, we can think of them as acting on the torus so we have torus orbits. Let's see some torus orbits we get with different actions.

Example 3.1.2. Consider the one parameter subgroup $\gamma_{(1,1)}(t) = (t, t)$. Acting on a point (a, b) we get the point (ta, tb) . So the orbits are lines through the origin. And notice the in fact, $\gamma_{(-1,-1)}(t) = (t^{-1}, t^{-1})$, even if we get the same orbits, on the first case we get flow inwards while on the second one we get *outwards* flow.

We could get $\gamma_{(1,0)}(t) = (t, 1)$, the orbits are horizontal lines. Finally $\gamma_{(1,-1)}(t) = (t, t^{-1})$, then the orbits are hyperbolas.

Now we are gonna say ok, we are gonna allow ourselves to grow the torus a little bit by saying that we want to add limit points of one parameter or allow/decide that some monomial functions are regular and some are not and then add the points. In the case of $\gamma_{(1,1)}$ we are going to need to fill in the point of the origin.

3.2 Day 22 | 20231013

Last time we talked about the setup: the toric variety is a space X . One is that it contains the torus as a dense open set and the other is that the torus acts on X and in particular in the open set it acts by multiplication. We linked objects from the algebraic geometry and representation theory of the torus. The way we think of characters of the torus are monomial functions on the coordinates of the torus. These are regular functions because there are no poles on the torus. On the other hand N is a lattice of one-parameter subgroups, a map from a one dimensional torus to a k -dimensional one. In particular inside the torus, if we take a point and act on it via a one parameter subgroup, we get a one-dimensional orbit. It is always the case that if we let $t = 0$ then the point falls off of the torus. These objects are dual to each other via dot product.

Example 3.2.1. An example of a toric variety is \mathbb{P}^1 , the set of homogenous points $[x : y]$. Take out zero and infinity and we get a one-dimensional torus.

We can also think of points in \mathbb{P}^1 as points of \mathbb{C}^\times as $[x : 1]$ or $[1 : y]$. Our objective is to build toric varieties from simpler varieties.

Definition 3.2.2. A cone in $N_{\mathbb{R}}$ will tell us how to add points to a torus in two ways:

- i) It would tell us what orbits of one-parameter subgroups acquire a limit point when $t \rightarrow 0$.
- ii) Or it would tell us what monomial functions are regular on such limit points.

Let us illustrate with an example taking two dimensional $N_{\mathbb{R}}$:

Example 3.2.3. A cone is what we think is a cone. FIGURE This cone σ_1 contains all lattice points corresponding to one-parameter subgroups of the form

$$t \mapsto (t^a, t^b), \quad a, b > 0$$

Now let's assume that we have a point of the two dimensional torus $P = (r_p, s_p)$, $r_p, s_p \neq 0$. We now look at the orbit of P via the coordinate-wise multiplication action:

$$t \cdot P = (t^a r_p, t^b s_p)$$

and now taking the limit $t \rightarrow 0$ of $t \cdot P$ and we want it to exist, then we introduce three cases:

- ◇ Both a, b are non-zero.
- ◇ $a = 0$ but b not, then we get points of the form $(r_p, 0)$.
- ◇ The other way around for b : $(0, s_p)$.

$$\lim_{t \rightarrow 0} t \cdot P = \begin{cases} (0, 0), & a, b \neq 0 \\ (r_p, 0), & a = 0 \\ (0, s_p), & b = 0 \end{cases}$$

The cone contains all the ways to pick the map. All the dots in the *interior* go to zero! If a, b are both zero the action is trivial. This cone is telling us to add the axes to T to get \mathbb{C}^2 .

Notice that if we had taken the lower cone we would've obtained a similar situation but instead of adding zeroes we would get infinities. We would still get \mathbb{C}^2 but adding different axes.

If we instead were algebraic geometers we would be interested on regular functions at the limit points. Take $f = x^\alpha y^\beta$, when is it regular at the $t \rightarrow 0$ limit point of the orbit of a 1.p.s. $t \mapsto (t^a, t^b)$, then

$$f(t) = t^{a\alpha + b\beta} r_p s_p,$$

the limit exists as long as $a\alpha + b\beta$ is non-negative. This is precisely the dot product, so we are basically saying that we want all vectors in M whose dot products with vectors in N is non-negative. This defines the notion of *dual cone*. This combinatorial object acquires the meaning of telling us what are the regular functions when adding the limit points. In this case σ_1^\vee looks exactly like σ_1 **PICTURE**. This also tells us that x, y are regular functions so every polynomial in x, y is a regular function. So in particular $\mathbb{C}[x, y]$ is the ring of regular functions.

Let us focus our attention now on the projective plane.

Definition 3.2.4. A fan is a collection of cones living *nicely* in N . This means that we are not just wacking at random at some cones. All cones will have the vertex at the origin and the only overlaps occur at boundary edges. **PICTURE**

This particular fan Σ is the fan of \mathbb{P}^2 . To construct it we follow this strategy

- (a) Each top dimensional cone produces an affine open chart in the sense of manifolds.
- (b) Each face, or ray in common, provides transition functions.

If we understand how it works in \mathbb{P}^2 , generalizing is pretty simple. We will look at dual cones, monomial functions there give us regular functions.

Example 3.2.5. The dual to the red cone is itself, the dual to the blue cone is $\{x < 0, 0 < y < -x\}$ and for green its $\{y < 0, 0 < x < -y\}$. Call two monomial functions x_1, y_1 on red^\vee so the regular functions are $\mathbb{C}[x_1, y_1]$.

On the blue we take the points $(-1, 0)$ and $(-1, 1)$, they correspond to the functions $\frac{1}{t_1}$ and $\frac{t_2}{t_1}$. But actually let's just call them x_2, y_2 so the coordinate ring is $\mathbb{C}[x_2, y_2]$ so our affine chart is a copy of \mathbb{C}^2 with coordinates x_2, y_2 . In a similar fashion we will get $\mathbb{C}[x_3, y_3]$ from the green dual.

To make them talk to each other, we take a ray in Σ and ask for its dual. The dual of the positive y -axis is the upper half plane. The regular functions for this space are $\mathbb{C}[\tilde{x}, \tilde{y}, \tilde{x}^{-1}]$ so it is $\mathbb{C}^\times \times \mathbb{C}$ with coordinates \tilde{x}, \tilde{y} . Now notice that this black cone contains the red and blue dual cones. So in particular we can express x_1, y_1 in terms of \tilde{x}, \tilde{y} . The key point is that $\mathbb{C}^\times \times \mathbb{C}$ is contained in the two copies of \mathbb{C}^2 we had before. We get the following relations

$$\tilde{x} = x_1 = x_2^{-1}, \quad \tilde{y} = y_1 = y_2 x_2^{-1}.$$

These are the transition functions which allow us to glue this two patches!

Cones select subsets and produces regular monomials, this produces coordinate ring, take the algebraic variety corresponding to this ring. This variety contains the torus. This can be done in various ways so that's how we build the atlas.

3.3 Day 23 | 20231016

Then the combinatorial data of the fan gives us a way to construct a toric variety as a manifold. Every face gives us the transition functions. The way to construct this chart is the dual cone. Then we take the algebra spanned by the monomials and we ask what variety's coordinate ring is this span.

Consequences of our construction

- (a) There is an inclusion into closure reversing bijection between cones of Σ and the torus orbits of X_Σ , the toric variety. And in particular, also exchanging dimension with codimension.

In action, the fan of \mathbb{P}^2 has 3 two dimensional cones R, G, B and likewise \mathbb{P}^2 has 3 two-codimensional points. Similarly the fan has 3 one dimensional cones. \mathbb{P}^2 has 3 one dimensional torus orbits. From the fan of \mathbb{P}^2 we can immediately read the Euler characteristic of \mathbb{P}^2 . The only things that will contribute are the zero dimensional cones.

In general, to see how the bijection works you can take two different approach:

- ◇ The geometric approach is: for every cone τ of Σ , look at the limits as $t \rightarrow 0$ of the torus orbits of 1-parameter subgroups γ with $\gamma \in \tau^\circ$. (stuff) and instead we pick a 1 parameter subgroup which lies on a ray, take for example $b = 0$, the y coordinate doesn't change and x goes to zero.
- ◇ The algebraic perspective is that for every affine patch dual to a cone, set all the coordinates that you can set to zero to zero.

What we mean by that is that if we look at the spec of the algebra dual to the red cone we get $\mathbb{C}[x_1, y_1]$ that's why it's affine. Setting both to zero we get the origin. When looking for transition functions, the ray has an affine chart, the algebra generated by the dual cone $\mathbb{C}[\hat{x}, \frac{1}{\hat{x}}, \hat{y}]$. In this case \hat{x} can't be set to zero. Doing it, we get any non-zero complex number so we get a one dimensional something. And the origin is itself a cone, the dual cone corresponding to the trivial cone is the whole thing. The corresponding algebra is $\mathbb{C}[x, \frac{1}{x}, y, \frac{1}{y}]$ and that's why we can't set anything equal to zero.

Coordinates here correspond to local coordinates in the affine patch. There's a lot of things with the same name.

Even if in blue we get the same algebra, setting coordinates to zero gives us a different point.

- (b) The second useful thing is very natural. T -equivariant maps of toric varieties correspond to maps of fans. (Maps of fans: Given $\Sigma_k \subseteq N_{k,\mathbb{R}}$, a map of fans is a \mathbb{Z} -linear map $L : N_{1,\mathbb{R}} \rightarrow N_{2,\mathbb{R}}$ such that for τ cone of Σ , $L(\tau) \subseteq \text{cone of } \Sigma_2$.)

Consider the fan of $\mathbb{P}^1 \times \mathbb{P}^1$, a 2 dimensional toric variety, for Σ , $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$. We have a projection map $\pi : X_{\Sigma_1} \rightarrow \mathbb{P}^1$ which is equivalent to the map of fans $\tilde{\pi}$ from Σ_1 to Σ_2 . It is also the case for $\mathbb{P}^2 \setminus \{pt\}$ to \mathbb{P}^1 which is projecting from that point.

The proof of this facts is left to the reader. Even without the proof this is a natural statement which makes sense.

$L(\tau)$ doesn't have to be a full cone. In the previous examples it is, but that's just an accident. We would allow it to not be a full cone because we can allow mor maps.

For example if we wanted to send a line in a plane as a map of.ko

Quotient construction

Given a fan Σ , the toric variety X_Σ can be obtained as a quotient space of the form:

We start with affine space, check the irrelevant locus and then we take something that if we are lucky is a torus:

$$\mathbb{C}^N \setminus \{irrelevant\} / G$$

N is the number of rays in Σ . The irrelevant set is the set determined by rays that don't span cones. And G is given by linear relations among maps.

Example 3.3.1. In \mathbb{P}^2 we have the tripod fan with three coordinates. So we have a \mathbb{C}^3 . We throw away the rays which don't span a cone. In this case the irrelevant. This is $\mathbb{C}^3 \setminus \text{FELL ASLEEP}$

Last week: Fans Σ lead to toric varieties X_Σ . Maximal cones lead to affine charts, and faces lead to transition functions. $\Sigma \subset N_{\mathbb{R}}$.

we get two neat consequences.

- (a) There is an inclusion into closure reversing bijection between cones of σ and the torus orbits of X_Σ (also, exchanging dimension with co dimension).
- (b) T -equivariant maps of toric varieties correspond to maps of fans.

Definition 3.3.1. Given $\Sigma_1 \subset N_{1\mathbb{R}}$ and $\Sigma_2 \subset N_{2\mathbb{R}}$ a *map of fans* is a \mathbb{Z} -linear map $L : N_{1\mathbb{R}} \rightarrow N_{2\mathbb{R}}$ such that for every cone τ of Σ_1 , $L(\tau) \subset$ a cone of Σ_2

with the embedding, we do not necessarily want to subdivide fans, as that introduces non trivial orbits.

If a toric variety is made of toric varieties, torus are homotopic to circles, which have genus 0, as they contribute trivially. To see how this bijection works

- (a) For every cone τ of Σ , look at the limits as $t \rightarrow 0$ of torus orbits of 1-parameter subgroups γ with $\gamma \in \tau^0$ (in the interior of the cone τ).
- (b) For every affine patch dual to a cone, set all the coordinates that you can set to zero to zero

Toric varieties can be viewed as a generalization of projective space (we basically have homogeneous coordinates). We consider the Quotient Construction. Given a fan Σ , the toric variety X_Σ can be obtained as a quotient space of the form $\mathbb{C}^N - \{\text{Irrelevant}\} / G$, where N is the number of rays in the fan Σ (for every ray we get a homogeneous coordinate in the toric variety), the irrelevant stuff is the locus determined by sets of rays that do not span cones of the fan, and G is given by linear relations among rays.

3.4 Day 24 | 20231018

Last time we talked about orbit-cone correspondence for toric varieties. This is a poset-reversing bijection, every biggest cone corresponds to a smallest orbit and vice-versa. In particular, this helps us do really funky things. An orthant corresponds to affine space, the

dual cone is the same orthant and those are the coordinates of affine space. The fan with only the axes is $\mathbb{C}^2 \setminus \{ (0, 0) \}$ and so on.

Maps of fans corresponds to good maps of toric varieties. They both have combinatorial descriptions, maps between fans are the restriction of an integral linear map between vector spaces. Cones of the first fan move to cones of the second fan. Integral in the sense that integer coefficients.

With that now we have a quotient construction: take \mathbb{C}^N remove *stuff* which is obtained from equations coming from rays that do not span a cone and then we mod out by linear relations among rays.

Example 3.4.1. In the case of \mathbb{P}^2 we have the fan which looks like a tripod. As it has 3 rays we begin with \mathbb{C}^3 . Each ray corresponds to a coordinate of \mathbb{C}^3 : x, y and z . The only subset of rays that does not span a cone of the fan is the three rays together $\{ \rho_x, \rho_y, \rho_z \}$. We take the coordinates corresponding to the rays and set them equal to zero. Then we take that locus and throw it away: $\{ x = y = z = 0 \}$. The way to determine the group is that we see the primitive vectors: $\mathbf{p}_x = 1, \dots, \mathbf{p}_z = -e_1 - e_2$ and the fact that we have one relation

$$1 \cdot p_x + 1 \cdot p_y + 1 \cdot p_z = 0$$

means that the torus is one dimensional. t acts on (x, y, z) as $(t^1 x, t^1 y, t^1 z)$. The quotient we obtain is

$$\mathbb{C}^3 \setminus 0 / \mathbb{C}^\times.$$

Observe that the exponents in the action are the coefficients in the relation.

If the coefficients were 2, 3, 4 the action would be $(t^2 x, \dots)$

If we were working with $\mathbb{P}^1 \times \mathbb{P}^1$ the quotient would be by $(\mathbb{C}^\times)^2$ and the action would be $(s, t) \cdot (x_1, x_2, y_1, y_2) = (sx_1, sx_2, ty_1, ty_2)$. More relations means more copies of \mathbb{C}^\times . The real question is *why is this true?* What if we take a fan and declare that any ray of the fan becomes a basis of a new vector space? Artificially we create a vector space.

Example 3.4.2. In the case of the \mathbb{P}^2 fan, \mathbf{p}_x becomes $(1, 0, 0)$ and so on. Now we want to lift the other cones as cones generated by the vectors. The space we just made is the first quadrant. All the cones that we get are in the totally positive quadrant. We are actually missing a three dimensional cone. We have *removed* from this fan that 3-cone. The orbits we are removing are *not* lifts of cones back in the tripod. Now what we want is a map of fans that will make the quadrant go into the tripod. This will be a map of vector spaces. In particular in this case the map is projection by $(1, 1, 1)$. This is a one-parameter subgroup of the torus. Basically the orbits of our one-parameter subgroup get identified, so the projection is the quotient map of the action whose exponent vector is the coefficients of the relation.

Tropicalizing Toric

In tropical geometry, the *field* is the tropical numbers $\mathbb{R} \cup \{\infty\}$, then k^\times is \mathbb{R} . So the torus which is $(k^\times)^n$ is \mathbb{R}^n , since multiplication is the action, then tropical multiplication corresponds to addition.

Example 3.4.3. How do we make the tropical projective plane \mathbb{P}^2 ? Remember we had the fan of \mathbb{P}^2 (the tripod inside N) and then from here we got the dual cones inside M . Each of the dual cones gave us a copy of \mathbb{C}^2 and then we had transition functions of the form $x_2 = \frac{1}{x_1}$, and $y_2 = \frac{y_1}{x_1}$. Also between red and green $x_3 = \frac{x_1}{y_1}$ and $y_3 = \frac{1}{y_1}$. This is what we did a couple of days ago.

To do it tropically we do it exactly the same, but instead of \mathbb{C}^2 we get \mathbb{T}^2 . Specifically three copies with coordinates:

$$\mathbb{T}^2, x_1, y_1, \quad \mathbb{T}^2, x_2, y_2, \quad \mathbb{T}^2, x_3, y_3.$$

These sets are basically copies of \mathbb{R}^2 with lines at infinity so we get points like (∞, ∞) , (∞, r) and (s, ∞) . Now we have to glue these things together to obtain one space. Every time we see a times we see a plus actually:

$$x_2 = -x_1, \quad y_2 = y_1 - x_1, \quad x_3 = x_1 - y_1, \quad y_3 = -y_1$$

To see what these objects glue to, take for example the line (s, ∞) , in the second \mathbb{T}^2 we get $x_2 = -s$ and $y_2 = \infty - s = \infty$ (this is ordinary algebra). So the (s, ∞) line maps to line at infinity *in the reverse direction*. For the $1 \rightarrow 3$ transitions we get $x_3 = s - \infty = -\infty$ and $y_3 = -\infty$. **Finish discussing.**

What happens is that we get an infinite triangle.

Toric varieties have an orbit cone correspondence. This is a bijection between the cones of a toric variety, and the toric orbits. We have that good maps of toric varieties correspond to maps of fans. We also have a quotient construction $X_\Sigma = \mathbb{C}^N - \{\text{stuff}\}$, where N is the number of rays, the stuff is the collection of rays not spanning a cone, and G is a linear relation among rays.

Example 3.4.1. \mathbb{P}^2 has three rays, denoted by P_Y , P_X , and P_Z . The only subset of rays NOT spanning a cone is the subset $\{P_X, P_Y, P_Z\}$ (any two span a cone). so we throw away the locus $\{X = Y = Z = 0\}$, i.e.e the origin. The relation we have between the three is that $1P_X + 1P_Y + 1P_Z = 0$ (the vectors are $P_X = (1, 0)$, $P_Y = (0, 1)$, and $P_Z = (-1, -1)$). all of the 1's tell us we have a one dimensional torus, and our action will be $t(X, Y, Z) = (t^1 X, t^1 Y, t^1 Z)$ With these relations, we get \mathbb{C}^*

Now, we can take the fan, and declare that any ray of the fan is generated by a basis vector of a new vector space. so $P_X = (1, 0, 0)$, $P_Y = (0, 1, 0)$, and $P_Z = (0, 0, 1)$. Then our three corresponding rays in \mathbb{R}^2 become the three positive half lines of the X, Y, Z axis. Now we

want to lift the cones in our toric variety, so we get octant planes. Now, everything we make is contained in the first octant of \mathbb{R}^3 , so this is a subset of all the cones of the first octant, but the first octant with removal. But the first octant is \mathbb{C}^n , and we removed something. We remove orbits corresponding to not being centered in the octant, i.e. we remove the three dimensional cone, so the toric variety is \mathbb{C}^3 minus the orbit corresponding to the center cone, which is $(0, 0, 0)$.

In tropical geometry, k is identified with $\mathbb{T} = \mathbb{R} \cup \{\infty\}$, then k^* corresponds to $\mathbb{R}^>$. Then $T = (k^*)^n$ corresponds to \mathbb{R}^n , and the action $*$ corresponds to $+$.

How to make tropical \mathbb{P}^2

3.5 Day 25 | 20231020

Patching construction for toric varieties: Dual cone gives affine patches, edges give transition functions. This can be switched to tropical numbers. The charts are now affine \mathbb{T} spaces, sometimes it could be without infinity. Transition functions become linear.

With the example of \mathbb{TP}^2 we had an infinite triangle. By removing the line and two points all we get is a copy of \mathbb{T}^2 . The y -axis of the first chart sticks to the y -axis of the second chart, while the x axis becomes $y = x$. **LOOK AT PICTURE**

Remark 3.5.1. When removing the lines at infinity we get a copy of \mathbb{R}^2 , which is just the tropical torus! Even if the lines are arranged differently, the charts differ by an element of $GL(\mathbb{Z})$.

The difficulty of transforming this into a general process is just painful notation. This process can always be done.

- ◇ The tropical toric variety $\mathbb{T}X_\Sigma$ has a *stratification* into \mathbb{R}^k strata which has a natural poset isomorphism with the stratification of the toric variety of X_Σ .

AUDIO AROUND 10min Stratification means a disjoint union into locally closed spaces. Think of it as chopping into pieces of different dimensions. All pieces are tropical tori \mathbb{R}^k for some k .

- ◇ The tropical toric variety has the structure of an “infinite polytope” (not precise, but it’s what I imagine) which is the normal polytope or dual to the fan of the toric variety. In the case of \mathbb{P}^2 , the fan is the tripod fan. For every 0-cell, we have a 2-stratum, for every ray we have a codimension 1 stratum which is incident at exactly one point. And for each 2-cone we have a 0-stratum. This is similar to the orbit-cone correspondence, now it’s a dimension reversing map. Composing both we get a dimension-preserving map. **AUDIO BEFORE 17 min**

The role taken by limits as $t \rightarrow 0$ of $t \cdot p$ in complex toric land, is replaced by $\lim_{T \rightarrow \infty} T + P$. In particular, given a 1-parameter subgroup $(1, 1)$, what are the orbits?

They will act on a point (x_1, y_1) as

$$T \cdot (x_1, y_1) = (x_1 + T, y_1 + T).$$

With this, no matter where we start, we go to (∞, ∞) . The cones of the toric variety tells us how to add the point to infinity, it works the same way here but the action is now additive instead of multiplicative.

The Quotient Construction

Let's go back to \mathbb{P}^1 . In our quotient construction this is

$$\mathbb{P}^1 = \mathbb{C}^2 \setminus \{(0, 0)\} / \mathbb{C}^\times, \quad t \cdot (x, y) = (tx, ty).$$

We now read the same information from the fan, our affine space is over tropical numbers now:

$$\mathbb{TP}^1 = \mathbb{T}^2 \setminus \{(\infty, \infty)\} / \mathbb{R}, \quad T \cdot (x, y) = (T + x, T + y).$$

Without considering anything as of now, just by intuition, the tropical projective line should be a line segment! Formally when constructing like this, we first remove the point at infinity and then look at the orbits. They are all parallel lines and each correspond to one point in the quotient. **PICTURE 32min**

Subvarieties

We aim to answer the question about how subvarieties behave under tropicalization. The main point is that we begin with a curve in a torus and then we tropicalize.

Suppose now $Y \subseteq T \subseteq X_\Sigma$, where T is a torus and the other is a toric variety. What is

$$\text{Trop}(Y) \subseteq \mathbb{R}^n \subseteq \mathbb{TX}_\Sigma?$$

Definition 3.5.2. The extended tropicalization of Y inside X_Σ is the closure of $\text{Trop } Y$ inside \mathbb{TX}_Σ .

Example 3.5.3. Lines in \mathbb{P}^2 give rise to tropical lines (tripods) in \mathbb{R}^2 . Now we know that we can close this as tropical \mathbb{P}^2 . Then we take the closure in \mathbb{TP}^2 so we only get 3 additional points. **PIC 41 min**

Example 3.5.4. Same line as before but we close in $\mathbb{P}^1 \times \mathbb{P}^1$. So now we compactify inside an infinite square. Before the 3 points were part of 3 one dimensional orbits, but now one of the points corresponds to a zero-dimensional stratum.

Example 3.5.5. If we let a line go through the origin and close in \mathbb{P}^2 . The corner locus is $\min(x, y)$ which is non linear where $y = x$. Closing in \mathbb{P}^2 gives us a point in a line and a point in the point. But when closing in $\mathbb{P}^1 \times \mathbb{P}^1$ we get only points.

We wish to discuss how the position is related to the original position of the complex variety.

3.6 Tropical Toric Varieties

From the fan Σ we use tropical numbers and tropical operations for transition functions, i.e. \mathbb{TP}^2 provides us with \mathbb{T}^2, x_i, y_i , for $i = 1, 2, 3$. We get relations from $i = 1$ and $i = 2$ via $x_i = -x_2$ & $y_1 = y_2 - x_2$, from 2 to 3 via $x_2 = y_3 - x_3$ & $y_2 = -x_3$, and finally from 1 and 3 via $x_1 = x_3 - y_3$ & $y_1 = -y_3$.

We get invertible in GLZ, which is our notion of automorphism of the torus. We can always do this translation. We observe that the tropical toric variety \mathbb{TX}_Σ has a stratification into " \mathbb{R}^k " strata, which has a natural poset isomorphism with the stratification of the complex toric variety X_Σ . Furthermore, \mathbb{TX}_Σ has the structure of an " ∞ " polytope, i.e. scale the polytope so that lengths go to infinity, which is the normal/dual polytope to the fan of X_Σ .

A stratification is a disjoint union into locally closed spaces. We get 0-dim, 1-dim, and 2-dim spaces, each is a tropical tori. This is similar as how \mathbb{CP}^2 has a stratification as points, lines, and the surface of the triangle.

Now, the shortcut to get the tropical toric variety via the fan of \mathbb{P}^2 , we make rays orthogonal to the rays of \mathbb{P}^2 , introduce fans corresponding to each of the 2-dim pieces of the fan, and set the lines to infinity.

The role taken by $\lim -t \rightarrow 0 t * p$ in \mathbb{C} -toric land is replaced by $\lim -T \rightarrow \infty T + p$ in tropical toric land. So if we take a one parameter subgroup, say $(1, 1)$, and we ask for the orbits of this subgroup, the orbits are found via $T(x_1, y_1) = (T + x_1, T + y_1)$. So regardless of the starting point, the orbit gets to the point at infinity. For the one parameter subgroup $(2, 1)$, we have $T(x_1, y_1) = (2T + x_1, T + y_1)$.

We can also show how the quotient construction works tropically. We begin at \mathbb{P}^1 . In our quotient construction, this is $\mathbb{C}^2 - (0, 0)/\mathbb{C}^*(t(x, y) = (tx, ty))$, where $t * (x, y) = (tx, ty)$. The one parameter subgroup comes from the linear relation on the fans of \mathbb{P}^1 , the relation being $p_1 + p_2 = 0$. Now we do the same from the fan to get us into tropical space. $\mathbb{TP}^1 = \mathbb{T}^2 - (\infty, \infty)/\mathbb{R}(T(X, Y) = (T + X, T + Y))$, where \mathbb{R} is the torus on \mathbb{T} . We expect \mathbb{TP}^1 to be a line segment, as we have \mathbb{P}^1 is two rays connected. We get a point for the first ray, a point for the second, and a line connecting the two. This is related to initial forms (See other texts).

We have parallel lines connecting to the singular point at infinity which we remove. This means that each parallel ray hits a different point at infinity. We can create an orthogonal line to these rays, and we say this ray hits two other points at infinity (at the limits of identifying the x and y axis).

We get some kind of orthogonal line to these parallel rays, and this line intersects each orbit exactly once. This line connects at points which represent $(\infty, 0)$ and $(0, \infty)$.

We have our original construction of tropical curves. Now, assume we have $Y \subset \text{Torus} \subset X_\Sigma$, so a curve in a torus in a toric variety. Now we have $\text{Trop}(Y) \subset \mathbb{R}^n \subset \mathbb{T}X_\Sigma$.

Definition 3.6.1. The *extended Tropicalization* of Y inside of X_Σ is the closure of $\text{Trop}(Y)$ inside of $\mathbb{T}X_\Sigma$.

So we start with a line $L \subset (\mathbb{C}^*)^2 \subset \mathbb{P}^2$. This gives rise to a tropical line $\text{Trop}(L) \subset \mathbb{R}^2$. The closure of $\text{Trop}(L)$ gives three additional points, one for each section of the tropical L which hits lines at infinities.

Next time, we will compare the combinatorics of tropical lines in varieties to toric varieties.

3.7 Day 26 | 20231023

Recall we defined the tropicalization of a toric variety and if we had a $Y \subseteq T \subseteq X_\Sigma$ then the tropicalization in X_Σ is

$$\overline{\text{Trop}(Y)} \subseteq \mathbb{T}X_\Sigma.$$

Example 3.7.1. By taking the equations $x + y + 1 = 0$ and $y + 1 = 0$ then the associated tropical lines are a tripod and a horizontal line. We may either embed them into \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$.

We know that tropical \mathbb{P}^2 looks like an infinite triangle while the other looks like an infinite square. **FIGURE around 5 min**

What will discuss is the relation between how the closure of Y in X_Σ intersects the orbits with respect to the intersections of the tropical toric variety. We summarize this into a question

$$\text{Given } Y \subseteq T \subseteq X_\Sigma$$

In the first example, $x + y + 1 = 0$ is a line which closed into \mathbb{P}^2 intersects 3 lines in \mathbb{P}^2 . In tropical land, it intersects the 3 copies of \mathbb{R} and misses the vertices. While in $\mathbb{P}^1 \times \mathbb{P}^1$, the correspondence persists. The fan of $\mathbb{P}^1 \times \mathbb{P}^1$ consists of 4 rays and 4 2 dimensional fans. In this case one of the edges of the tripods is inside the 2 dimensional fan which is dual to the vertex that **Didn't understand**

Theorem 3.7.2. Assume K has a trivial valuation and $\sigma \in \Sigma$ a cone. Then

$$\overline{Y} \cap O_\sigma \neq \emptyset \iff \text{Trop}(Y) \cap \sigma^\circ \neq \emptyset.$$

Remark 3.7.3. Trivial valuation implies that when *throwing the blanket*, everything stays at the same level.

Proof

To show the result, we will show that both facts are equivalent to the statement. **SLEEP** Put yourself in a local chart that contains this orbit. This is a copy of \mathbb{C} . **sleep** Now we want to show that a face intersection then the non-extended intersects the interior of the cone. This is because the tropical toric variety is dual to the fan. The trivial valuation gives us a tropicalization which is always a cone over the origin.

In conclusion, we know that if we draw a tropical curve which corresponds to an algebraic curve in the torus, after closing it into the toric variety we will know which orbits we intersect just by looking at the variety.

If we are an algebraic variety Y inside a torus, which toric variety do we want to be compactified into? The answer lies in the tropicalization.

But what does this question even mean?

- ◇ The variety Y certainly wants to be compact.
- ◇ It also doesn't want to acquire a whole bunch of singularities. (Think about the Alexandroff compactification, one-point compactification. Good luck doing geometry there.)
- ◇ To compactify we must add some points, and since we are in a *stratified* toric variety, the stratification of \bar{Y} inside X_Σ matches, including dimension grading, with the stratification of Σ .
- ◇ Lastly the compactification must not be wasteful (because varieties care about the environment). The torus itself shouldn't have so many points not needed to compactify.

$x + y + 1 = 0$ likes to be compactified in \mathbb{P}^2 as **picture**. Except none of the infinite points are touched, so it likes to be compactification in \mathbb{P}^2 without the three infinite points. **picture**.

3.8 Day 27 | 20231025

Last time we had a subvariety of a torus which is the open dense set of a toric variety. The question we addressed was

What are the orbits of the toric variety that \bar{Y} intersects?

mn, An orbit intersects the closure of Y iff the tropicalization of Y intersects the relative interior of the cone corresponding to the orbit. This went through an intermediate step, extended tropicalization intersects the face corresponding to σ . Now we know when an orbit intersects \bar{Y} .

Stepping up sophistication, if we have a $Y \subseteq T$, when is the toric variety a good choice to close Y in?

- ◊ When \bar{Y} is compact/complete. (Not synonyms, compact in Zariski is trivial but Euclid not. Complete means all limit points.)
- ◊ When \bar{Y} doesn't become too singular
- ◊ When codimension i strata of \bar{Y} corresponds to codimension i stuff of X_Σ

We can always take a toric variety corresponding to a complete fan which corresponds to a compact toric variety so that's where we wanna close Y into.

Recall we saw the example $x + y + 1 = 0$ inside the torus $(\mathbb{C}^\times)^2$ then we can close it into $\bar{L} \subseteq \mathbb{P}^2 \setminus \{e_1, e_2, e_3\}$. And looking at the tropicalization of L , it happens to be the fan of $\mathbb{P}^2 \setminus \{e_1, \dots, e_3\}$. We ask now, why is this not a coincidence?

In what follows, K is trivially valued. Any Newton Polygon will not be subdivided.

Theorem 3.8.1. $Y \subseteq T_n$ is d -dimensional and irreducible and $\bar{Y} \subseteq X_\Sigma$ the closure of Y in X_Σ then

$$\bar{Y} \text{ is complete} \iff \text{Trop } Y \subseteq |\Sigma|.$$

The support $|\Sigma|$ is the locus of points in N that belong to some cone of Σ .

Proof

Assume by contradiction that $\text{Trop } Y$ is not inside of $|\Sigma|$. Let us draw a picture: **figure 20 min** So let us add fans to get a complete toric variety. Consider a complete fan $\tilde{\Sigma}$ that contains Σ as a sub fan. In the picture $\tilde{\Sigma}$ is both red and green. Complete means that the support is the whole vector space. Then by a known theorem $X_{\tilde{\Sigma}}$ is compact and also has X_Σ .

Their difference is $\bigcup_{\sigma \text{ green}} O_\sigma$ by the orbit cone correspondence something.

Because $\text{Trop } Y$ intersects the green cone, then the closure has a point in the orbit corresponding to the green cone. That means Y has a limit point in the cone corresponding to the difference. So when closing in X_Σ that point...

As $\text{Trop } Y \cap \sigma^\circ \neq \emptyset$, for some green cone σ , then $\bar{Y} \cap O_\sigma \neq \emptyset$. Say y is a point in such intersection, y is a limit point of Y in $X_{\tilde{\Sigma}} \setminus X_\Sigma$. So this means that $\bar{Y} \subseteq X_\Sigma$ is not complete.

Let's go the other direction now, assume $\text{Trop } Y \subseteq |\Sigma|$. We add cones to Σ to get a complete fan $\tilde{\Sigma}$:

$$Y \subseteq T \subseteq X_\Sigma \subseteq X_{\tilde{\Sigma}}(\text{compact}).$$

Lets draw another picture (**fig 30 min**), when closing Y inside $X_{\tilde{\Sigma}}$ we get a compact **sleeping** But $\bar{Y} \cap O_\sigma$ is empty for all orbits in $X_{\tilde{\Sigma}}$.

SLEEP SLEEP

Theorem 3.8.2.

Proof

For every σ , $\bar{Y} \cap O_\sigma \neq \emptyset$ and $\dim(\bar{Y} \cap O_\sigma) = d - \dim \sigma$ iff $\text{Trop } Y = |\Sigma|$.

We must show $|\Sigma| \subseteq \text{Trop } Y$

- (a) Every cone in Σ must intersect $\text{Trop } Y$, in other words $\sigma^\circ \cap \text{Trop } Y \neq \emptyset$.
- (b) The cones of Σ are at most dimension d . Which comes from the hypothesis. If Σ had a $d + 1$ dimensional cone, this corresponds to an orbit of codimension $d + 1$. By the first observation it should intersect \bar{Y} , but that means

$$\dim(\bar{Y} \cap O_\sigma) = d - (d + 1) < 0 \Rightarrow \bar{Y} \cap O_\sigma = \emptyset$$

which contradicts our first observation.

- (c) If we have a d -dim cone of Σ its not possible to be intersected by $\text{Trop } \Sigma$ in a smaller dimension. This means Σ has cones of higher dimension.
- (d) Is it possible to be covered by higher dimensional? Not possible because balancing condition.

3.9 Day 28 | 20231027

The big picture is that our world has trivially valued field and we have a subvariety of a toric variety. We looked at the combinatorial properties of tropicalization, what we should hold is that tropical toric varieties are like adding stuff to a torus which corresponds to cones in a space. We concluded that if a tropicalization is contained in the support of a fan then the variety is compact. The last property we wish to prove is dimension transversality. Those properties are equivalent to the fact the the tropicalization *is* the support of the fan.

Proof

We already know that $\text{Trop}(Y) \subseteq |\Sigma|$, we want to show the other inclusion.

- (a) Every cone σ in Σ must intersect $\text{Trop } Y$. If we have a cone that doesn't intersect $\text{Trop } Y$, we should throw it away.
- (b) The dimension of Σ as a fan is *at most* the dimension of Y . This is because we assumed

$$\text{codim}_{\bar{Y}}(\bar{Y} \cap O_\sigma) = \text{codim}_{X_\Sigma} O_\sigma.$$

- (c) If we have a top-dimensional cone of Σ , then it can't intersect $\text{Trop } Y$ in a positive codimensional locus. Otherwise $\text{Trop } Y$ would intersect $d+1$ -dimensional cones

of Σ (which don't exist). **pic 10 min**

- (d) A top dimensional cone of Σ cannot be partially covered by $\text{Trop } Y$. This would violate the balancing condition for $\text{Trop } Y$. What does it mean to be balanced along τ ? We go to the quotient by projecting down. We quotient away the linear span of τ . So the cones now become 1-dimensional and so we can apply the old definition of balancing.

If that were to be the case, now we take τ a face of $\text{Trop } Y$, then the point is unbalanced.

Altogether, every d -dimensional cone of Σ , must be completely covered by $\text{Trop } Y$ then $|\Sigma| \subseteq \text{Trop } Y$. **Renzo is thinking very hard about the drawing around 25min** Apparently the last comment is not true. When checking incidence conditions, we only care about cones incident to a face.

For the other direction we want to show the properties. We need the following facts:

- (a) There are no positive dimensional compact subvarieties of tori. Tori are full of holes. Prove it for \mathbb{C}^\times and then induction.
- (b) An orbit of codimension k of a toric variety is locally a complete intersection.
- (c) For any $Z \subseteq X_\Sigma$ and a hypersurface, the intersection with Z either remains the same dimension or the dimension goes down by 1 or they have empty intersection.

With this we tackle the other side of the theorem. Take a top dimensional cone σ_d in Σ . Then O_{σ_d} is a codimension d orbit and O_{σ_d} intersects \bar{Y} non-trivially. We know O_{σ_d} is isomorphic to a torus and because \bar{Y} is complete, then $O_{\sigma_d} \cap \bar{Y}$ is complete and so it has a finite number of points. **LAST PICTURE** At every step we have to drop codimension by one.

When all of the conditions happen, we may have a new concept!

Definition 3.9.1. $\bar{Y} \subseteq T \subseteq X_\Sigma$ is a tropical compactification when $\text{Trop } Y = |\Sigma|$. In particular the properties of the previous theorem hold.

Right now we haven't talked about how singular Y become when compactified. Let us just talk a bit. A tropical compactification.

Theorem 3.9.1. $\bar{Y} \cap O_\sigma \neq \emptyset$ and dimensional transversality $\text{codim}_{\bar{Y}}(\bar{Y} \cap O_\sigma) = \text{codim}_{X_\Sigma}(O_\sigma)$ iff $\text{Trop } Y = |\Sigma|$

Proof. \implies we know that $\text{trop}(Y) \subset |\Sigma|$. We want to prove \supset . First, we have that any $\sigma \in \Sigma$ must intersect $\text{trop } Y$, as $\bar{Y} \cap O_\sigma \neq \emptyset$. Second, we have that $\dim(\Sigma)$ is at most $\dim(Y)$, by

the dimensional transversality. Third, we have that a top dimensional cone of Σ cannot intersect $\text{trop}(Y)$ in a positive codimension locus, otherwise $\text{Trop}(Y)$ would intersect $(d+1)$ dimensional cones of Σ (which don't exist). Finally, a top dimensional cone of σ cannot be partially covered by $\text{trop}(Y)$, as either this would violate the balancing condition for $\text{Trop}(Y)$ (balancing condition in higher dimensions is done by quotienting), or $\text{trop } Y$ has to intersect at a face of Σ , not the of a face.

With these four conditions, we get that every $\sigma \in \Sigma$ must intersect $\text{trop}(Y)$, and the intersections are all showing $\text{trop}()$ is covering σ , so we must have that $|\Sigma| \subset \text{trop}(Y)$.

\Leftarrow Now we are assuming that $\text{trop}(Y) = |\Sigma|$. We need some algebraic geometry black boxes.

- (a) There are no positive dimensional compact subvarieties of tori.
- (b) For toric varieties, an orbit O_σ of codimension k is (locally) cut out by exactly k equations
- (c) For any subvariety $Z \subset X_\Sigma$ and any hypersurface $HS \subset X_\Sigma$, $Z \cap HS$ either remains the same dimension of Z , the dimension goes down by exactly one, or the intersection is empty.

With these three facts we can begin to show the other implication. We begin with σ_d being a top dimensional cone in Σ . Then O_{σ_d} is a codimension d orbit. Furthermore, we have $O_{\sigma_d} \cap \bar{Y} \neq \emptyset$, and that O_{σ_d} is isomorphic to some torus. Because \bar{Y} is complete, the intersection $O_{\sigma_d} \cap \bar{Y}$ is complete, and so it has a finite number of points.

All together, this says we can obtain the orbit O_σ by cutting down by k hypersurfaces. In particular, if we have our cone σ , we can take a chain of subsequent surfaces $r_1, \text{span}(r_1, r_2), \text{span}(r_1, r_2, r_3)$, and corresponding to r_1 we have a hypersurface H_1 (orbit of r_1 , $H_1 \cap H_2$ is orbit of σ_2 , and so on), we can take $\bar{Y} \cap H_1$, then $\bar{Y} \cap H_1 \cap H_2$, and then $\bar{Y} \cap H_1 \cap H_2 \cap H_3$. We know \bar{Y} has dimension 3, at the end we have just points at the end, and we drop dimensions by exactly one at every step.

3.10 Day 29 | 20231030

Recall that the theorem we are proving is that the conditions

- $\diamond \bar{Y} \cap O_\sigma \neq \emptyset$ for all σ , and
- $\diamond \text{codim}_{\bar{Y}}(\bar{Y} \cap O_\sigma) = \text{codim}_{X_\Sigma} O_\sigma$

are equivalent to $\text{Trop } Y = |\Sigma|$.

In the course of proving left-to-right, we do know that

$$\text{Trop } Y \subseteq |\Sigma|,$$

otherwise we'd be missing a limit point. We only have to prove the other direction. With a bunch of reductions:

- ◇ Σ should be d -dimensional, where d is $\dim Y$.
- ◇ We also saw that Σ must intersect the top-dimensional cones of Y in top dimension. This means it can't be cutting the space in locus with strictly less dimension.
- ◇ If we fix a cone $\sigma \in \Sigma$, then $\text{Trop } Y$ must contain all of it.

With this the argument is that we imagine a cone $\sigma \in \Sigma$ and a piece of $\text{Trop } Y$ and we wanted to derive a contradiction by balancing. We must analyze two cases:

- (a) Call τ the border between σ and $\text{Trop } Y$. We can call s the part from $\text{Trop } Y$, it is the only top dimensional part of $\text{Trop } Y$ containing τ . And if that's the case the balancing argument holds because $\text{Trop } Y$ is not balanced. This is because all the action occurs *on the lower side of the plane* so we quotient out by τ . This is projecting away the linear subspace generated by τ and we have all of the projection of s on one side *not balancing* the vertex.
- (b) In the other case $\text{Trop } Y$ has other parts incident to τ and not inside of σ . Then it is not possible for τ to be *in the middle* of a cone. picture 14 min

For the other directions we use some facts of algebraic geometry:

- ◇ The only complete/compact subvarieties of a torus are 0-dimensional.
- ◇ In a toric variety, every orbit of σ of codimension k is cut out locally by k equations.
- ◇ If $Y \subseteq X_\Sigma$ and we cut it down with a hypersurface, then dimension of $Y \cap H$ goes down at most by 1.

$$Y \cap H \neq \emptyset \Rightarrow \dim(Y \cap H) \geq \dim(Y) - 1.$$

Grabbing water

Every orbit will correspond to some cone. picture begins at 21 min. Maybe the face is a face of a higher dimensional cone $\tilde{\sigma}$. We choose it as a top dimensional cone of Σ containing σ as a face. Now we choose an ordering of the rays of $\tilde{\sigma}$ in such fashion that the first k rays belong to σ . Every ray corresponds to a codimension 1 orbit. If we keep cutting something.

Let H_i be the closure of the orbit corresponding to the ray ρ_i :

$$H_i = \overline{O}_{\rho_i} \quad \text{a hypersurface in } X_\Sigma.$$

Now $\overline{Y} \cap H_1$ has dimension either $\dim \overline{Y}$ or $\dim \overline{Y} - 1$. Then

$$\dim(\overline{Y} \cap H_1) \cap H_2 = \dim \overline{Y} \quad \text{or} \quad \dim \overline{Y} - 1 \quad \text{or} \quad \dim \overline{Y} - 2,$$

and in general $\dim(\bar{Y} \cap H_1 \cap \cdots \cap H_k)$ can range from $\dim Y$ to $\dim \bar{Y} - k$. We want to show it's the lowest, to rule the other options we go down all the way: In conclusion

- ◊ One can always refine Σ so that \bar{Y} is Cohen-Macaulay. (Understand this as not too badly singular).
- ◊ In characteristic zero, and X_Σ , we can find open subsets such that $\bar{Y} \cap O_\sigma$ is smooth for some σ .

We now reiterate the previously stated algebraic geometry black boxes.

- (a) The only complete/compact subvariety of a torus are 0-dimensional
- (b) In a smooth toric variety, every orbit of sigma O_σ of codimension k is locally cut out by k -equations.
- (c) If $Y \subset X_\Sigma$, and $Y \cap \text{Hypersurface} \neq \emptyset$, then the dimension of the intersection goes down at most by 1.

Now we assume $\text{Trop}(Y) = |\Sigma|$. We can consider some gace $\sigma \in \Sigma$, and let $\tilde{\sigma}$ to be a top dimensional cone of Σ containign σ as a face. We choose an ordeing o the rays of $\tilde{\sigma}$ such that the first k rays belong to σ . (Each ray corresponds to a codimension 1 orbit, after the k steps we get to an intersection with $O_\sigma \cap Y$). We let H_i be the closure $\overline{O_{\rho_i}}$, i.e. the hupersurface in X_σ corresponding to the ray ρ_i . Now, $\dim(\bar{Y} \cap H_1) \geq \dim(\bar{Y}) - 1$. Then, $\dim((\bar{Y} \cap H_1) \cap H_2) \geq \dim(\bar{Y}) - 2$. Eventually we get to $\dim(\bar{Y} \cap H_1 \cap \cdots \cap H_k) \geq \dim(\bar{Y}) - k$. To check that we get minimal dimensions, we terminate at $\dim(\bar{Y} \cap H_1 \cap \cdots \cap H_d) = \dim(\bar{Y} \cap O_{\tilde{\sigma}})$. Note $\bar{Y} \cap \overline{O_{\tilde{\sigma}}} = \bar{Y} \cap O_{\tilde{\sigma}}$ (inherently the closure of Y misse s any missing limit points of the orbit). So we know that $\bar{Y} \cap \overline{O_{\tilde{\sigma}}}$ is compact, comapct living in the torus $O_{\tilde{\sigma}}$, but by our black box we get that subvariety of a torus is 0 dimension, at every step we needed to reduce our dimension by exactly once, so at no point would our dimenion ever stay the same.

□

Definition 3.10.1. $\bar{Y} \subset T \subset X_\Sigma$ is a *tropical compactification* when $\text{trop}(Y) = |\Sigma|$

This definition says that Y thinks X_Σ is a good place to be comapctified in. This is because X_Σ doesnt waste any orbits, and all orbit of X_Σ are dimensionaly transverse to \bar{Y} .

In particular, the nice properies of the left hand side of the theorem hold. So far we have not talekd about the singularity of \bar{Y} when we compactify it. It is possibly that \bar{Y} could have hotten singular in this process. Via a technical process and analysis with a toolkit from a third course in algebraic geometry, we notice that he statement of tropical compactificatioojon is discussing the support of Σ , not the fan itself.

In torc geoemtry if we take a fan and subdivide into cones, this corresponds to blow ups in strata of troic varieties (whcih are used to resolve singularities). So maybe we had singlarities, but we use blow ups to reslve singularities.

- (a) One can always refine Σ so that \bar{Y} is Cohen-Macaulay (read “not too badly singular”)
- (b) In Char 0, X_Σ projective, then we can find an open subset such that $\bar{Y} \cap O_\sigma$ is smooth for all σ .

3.11 Day 30 | 20231101

Geometric Tropicalization

This is the *philosophical* reverse direction of compactification. If $Y \subseteq X_\Sigma$ and it sits nicely inside it, then X_Σ allows us to compactify Y to figure out $\text{Trop } Y$.

Example 3.11.1. We will consider the line in the projective plane. Take $X + Y - Z = 0$ inside \mathbb{P}^2 . The stratification of \mathbb{P}^2 induces a stratification on our line. Our line has 3 codimension 1 strata and one codimension 0 stratum. We may encode this into a boundary complex which turns out to be 3 points.

Then we may construct the cone over the boundary complex by just taking a point outside in an independent dimension and then joining them. We call this $C\Delta_{\partial L}$.

We have 3 boundary divisors on \mathbb{P}^2 which determine 3 boundary divisors on L . These give rise to divisorial valuations:

$$\text{val}_D : K(L) \rightarrow \mathbb{Z} \cup \{\infty\}, \quad f \mapsto \text{order of vanishing or pole of } f \text{ along } D.$$

something The valuation val_a naturally sits inside N_T the co-character lattice. First let's see the recipe and then see that it works:

- (a) Use val_a , val_b and val_c to get 3 points in N_T . But think of each of these points as the boundary complex.
- (b) Take the cone in N_T .
- (c) This delivers $\text{Trop } Y$.

In our case

$$a = [0 : 1 : 1], \quad b = [1 : 0 : 1], \quad \text{and} \quad [1 : -1 : 0].$$

If we place ourselves at a , then we can give local coordinates for the line as t . This allows to view rational functions as functions of t and observe the order of vanishing and poles. We may parametrize the line in the affine chart $Z = 1$ to get

$$x(t) = t, \quad y(t) = 1 - t.$$

We compute val_a on the generators of \mathbb{Z}^2 to get $(1, 0)$. Another way to see this is to see that a lies in the y axis and not on the x axis. To find the exact value we must compute. At b we are switching the roles of x and y to get

$$x(t) = 1 - t, \quad y(t) = t \Rightarrow \text{val}_b(x, y) = (0, 1).$$

Finally at c , the point at infinity, we may parametrize L centered at c as

$$x(t) = \frac{1}{t}, \quad y(t) = 1 - \frac{1}{t} \Rightarrow \text{val}_c(x) = -1 \quad \text{and} \quad \text{val}_c(y) = -1.$$

Those three points in N_T determine $\text{Trop } L$. [pics 27 min](#)

In general, we start with a space Y whose boundary is composed of multiple divisors. Also $Y \setminus \partial Y$ is isomorphic to a subvariety of a torus. We will say that the boundary has combinatorial normal crossing if every time k divisors intersect, they intersect in codimension k . If the intersection is transversal, then the intersection is a simple normal crossing. This means that the boundary looks like hyperplanes, the local model of this is coordinate $x_i = 0$. Just a normal crossing is something which is locally a simple normal crossing.

What kind of divisorial valuation we have access to? In practice [stuff](#) so when $Y \setminus \partial Y \subseteq T$, then any toric variety containing T as their dense torus gives rise to divisorial valuations.

(a) $\text{Trop } Y$ is equal to the cone

$$\{c \cdot \text{val}_D : c \geq 0, D \text{ divisor coming from any } X_\Sigma \supseteq T\} \subseteq N_T(\otimes_{\mathbb{Z}} \mathbb{R}).$$

(b) If ∂Y is a combinatorial normal crossing, then we may find a map $C\Delta_{\partial Y} \rightarrow N_T$ which maps $(D_i, 1) \mapsto \text{val}_{D_i}$, then extend by linearity. So $\text{Im } \mathbb{T} \supseteq \text{Trop } Y$ and if we want equality we can either take it to be a simple normal crossing or assume only combinatorial normal crossing and characteristic zero.

and so somehow such a variety knows its tropicalization without going through Puiseux because a source of valuations comes from the divisors.

Now, K -trivially valued, tropical compactification tells us that if $Y \subset T$, we get that $\text{trop}(Y)$ determines a toric variety inside which Y compactifies nicely!

3.12 Geometric Tropicalization

If $Y \subset X_\Sigma$ is a subset of a toric variety, and Y sits “nicely” in X_Σ , then the toric variety allows us to know $\text{trop}(Y)$. We illustrate this in an example, and then generalize.

Example 3.12.1. We take the line in \mathbb{P}^2 , $\{X + Y + Z = 0\} \subset \mathbb{P}^2$. The strata of the line is defined via a boundary complex, where smaller stratifications lead to larger structures in the boundary complex. In the case of \mathbb{P}^2 and the boundary complex of L , we get three points $\{a, b, c\}$. We can construct the cone over the boundary complex cx by taking a point in an independent dimension from a, b, c , and join the points with half lines. The toric variety has 3 divisors (codimension 1 subvarieties, in \mathbb{P}^2 it is the lines, not lying within the torus) on \mathbb{P}^2 that induce (via intersection) 3 divisors on L (the points of intersection of L with the toric invariant lines of \mathbb{P}^2). Divisors give rise to “divisorial valuation” $\text{val}_D : K(L) \rightarrow \mathbb{Z} \cup \infty$,

where for every function $f \in K(L)$ it defines the order of vanishing or pole of f along D . so as long as the space is not too singular, locally and divisor has one local equation, take the rational function where on this local set is valid, and see how many times we can factor the equation with our function.

Notice that $val_a : K(L) \rightarrow \mathbb{Z} \cup \infty$, where we get a rational function on \mathbb{P}^2 , restrict to L , then do the evaluation, but we really like monomials as rational functions. So we actually take $val_a : M_T \rightarrow K(L) \rightarrow \mathbb{Z} \cup \infty$, where $M_T \rightarrow K(L)$ is via restriction, and $K(L) \rightarrow \mathbb{Z} \cup \infty$ is the order of vanishing. So we assign an element of M a valuation, which is dual to M , so $image(val_a) \subset N_T$, which is connected to the co-character lattice.

M_T is the character space of the torus i.e. monomials, and N_T is the co-character 1-parameter subgroups.

For our recipe, we use val_a, val_b , and val_c to get three points in N_T . But we think of each of these points as points of height one in the cone of the boundary complex, and we take the cone over these point in N^T . We then get $Trop(Y)$.

Now we do computations, we take the point $a = (0 : 1 : 1)$. We can parameterize the line L around the neighborhood of a via t by defining affine coordinates in \mathbb{P}^2 , so we take $x = \frac{x}{z}$, $y = \frac{y}{z}$, as the line we parameterize as $x(t) = t$, and $y(t) = 1 - t$. We now need to compute val_a . In this case, M_T is \mathbb{Z}_2 , so we can test on the generating set x and y . So $val_a(x) = 1$, as the order of vanishing of t at 0 is 1. $val_a(y) = 0$, as the order of vanishing of $1 - t$ at $t = 0$ is 0. Similarly, $b = (1 : 0 : 1)$, this switches the role of x and y , so $val_b(x) = 0$ and $val_b(y) = 1$. The trickier point is c , which is the point at infinity. We can choose a parameterization of L near this point centered at c , which becomes $x(t) = \frac{1}{t}$ and $y(t) = 1 - \frac{1}{t}$. Now $val_c(x) = -1$ and $val_c(y) = -1$. We thus define $val_a = (1, 0)$, $val_b = (0, 1)$, and $val_c = (-1, -1)$. This gives us $Trop(L)$.

In general, we start with a very affine variety with divisorial boundary. A divisorial boundary, we start with a space Y , with codimension 1 subvarieties, very affine means that $Y - \partial Y$ is isomorphic to a closed subvariety of a torus (closed inside the torus, typically the complement lies in a torus).

Definition 3.12.1. We say ∂Y has combinatorial normal crossings if every time n divisors intersect they intersect in codimension n . If the intersection is transversal, we say this is a simple normal crossing

Simple normal crossing means that the boundary locally looks like hyperplane coordinates in our space.

Definition 3.12.2. A normal crossing is locally SNC

We can ask for divisorial evaluations. We embed the curve minus its boundary into a torus. In practice we can embed the boundary into any toric variety we choose. When

$Y - \partial Y \subset T$, and toric variety containing T as their dense torus gives rise to a divisorial valuation.

Lemma 3.12.1. (a) $Trop(Y) = \{c \cdot val_D \mid c \in \mathbb{R}_{\geq 0}, D \text{ is any divisor coming from } X_\Sigma \supset T\} \subset N_T$

(b) If ∂Y is combinatorial normal crossing, then there exists a map $\pi : C\Delta_{\partial Y} \rightarrow N_T$, Y was a variety with some divisor, so we use those varieties, so that $\pi(D_i, 1) \mapsto val_{D_i}$, and we extend by linearity. With combinatorial normal crossings, we can guarantee that $Trop(Y) \subset Im(\pi)$. If we want equality, we ensure simple normal crossing boundaries, or (if char 0 is fine), combinatorial normal crossing with characteristic 0 is sufficient.

3.13 Day 31 | 20231103

3.14 Day 32 | 20231106

Essentially we started with an equation

$$1 + x + y + txy = 0$$

and at the beginning of the semester we tropicalized to a stick figure. Last time, starting from this equation we can construct a family of surfaces in $\mathbb{P}^3 \times \mathbb{A}^1$ simply by putting a two-dimensional torus (the one the curves live in) in \mathbb{P}^3 and then think of the t parameter as a *time deformation parameter*. **stuff** And then we can cut this family of surfaces via the hyperplane

$$x_0 + \cdots + x_3 = 0$$

because there's **some relation with the torus**. This hyperplane cuts generically on the smooth but on the special fiber it cuts **in a cool way**.

The subdivision of the Newton polygon tells us how the degenerations occur onto the special fiber. This is because each of the triangles in the Newton polygon is a copy of \mathbb{P}^2 . **something** corresponds to the vertices of the curves being $(0, 0)$ and $(1, 1)$. So what the picture tells us **something**. And then there's two planes one with valuation zero and one with valuation -1 that play a specific part in the sense that they contain lines that meet $x_0x_3 = 0$ which holds regardless of the valuation and this gives us a vertical line.

We break the surface into two surfaces and the curve into two simple curves.

Somehow all this information can be described combinatorially with the information of toric geometry. When we see this sort of things we should think of a fan or a sort of cone on $N_T \times \mathbb{R}$. Then we say at $t = 1$ we see the tropical curve. Then we take the cone over the tropical curve. **flaps expand**. Pointing out the $t = 0$ what we see is the recession fan of the tropical curve. This picks up the unbounded directions of the tropical curve. And this is a fan which allows to make a map from one toric variety to another. The inverse image of

the generic fiber, is **something projective**. That means that our old picture can be described with toric geometry. **There's a map**

$$T^2 \Rightarrow \mathbb{P}_{\mathbb{K}}^3, [s : u] \mapsto [1 : s : u : tsu] \Rightarrow X_0 X_3 = t X_1 X_2.$$

GFAN | Joel BN

Gfan is used to compute Gröbner complexes. In terms of graphical interface, there's no interface. It all runs on the terminal. Gröbner basis gives us a Gröbner cone which is a sub-cone of our usual tropical cone. In order to visualize, the app uses an auxiliary called `xfig`. The utility of this is that it exports figures to \LaTeX .

Chapter 4

Projects

Joel, Gfan

We get a map from the torus T^2 to $\mathbb{P}_{\mathbb{K}}^3$ via $(s : u) \mapsto (1 : u : tsu)$. We get the conic $x_0x_3 = tx_1x_2$. For $t \neq 0$ we get the smooth quadratic, for $t = 1$ we get the intersection of planes. The two copies of \mathbb{P}^2 which have the line is also given by the lines of Puiseux valued points corresponding to the fact that the vertices of the curve are $(0, 0)$ and $(1, 1)$. The tropical curve tells us that the central fiber of the family of curves corresponds to $x + y = 0$, when x_0 and x_3 both equal 0.

When the valuation of x and y are both zero, then the valuation of txy is irrelevant.

We get a fan in $N_T \times \mathbb{R}$ and a ray dealing with t depending on expanding on t from $1 + x + y + txy = 0$. We have the inverse image of a fan $\mathbb{P}^1 \times \mathbb{P}^1$, under the special./generic fiber, we have the actual tropical curve.

Gfan is a software package for computing Grobner fans and tropical varieties. Overview of installation. Gfan runs on the kernel. Gfan consists of various programs. Gfan can be run by creating scripts to pass to the Gfan library. Gfan natively supports \mathbb{Q} and finite fields. Integers can be used with some work. For example, we can begin with an ideal, such as $\mathbb{Q}[x, y, z]$ and the ideal $\{xyx - z, yzy - x, zzx - y\}$. The library can compute a Grobner basis for any such ideal.

There is a relationship between Grobner basis and tropical varieties. A Grobner basis can give a Grobner cone. A collection of these Grobner cones construct a Grobner fan (distinct from tropical fan). A tropical variety is a union of Grobner cones, and is thus a subfan of a Grobner fan. Gfan does not have the most advanced visualization techniques. Gfan can visualize figure files (only with xfig). We can imagine the Grobner fan exists in \mathbb{R}^3 in the positive quadrant. The Grobner cone is 2-D cones. The unit vectors in \mathbb{R}^3 form a triangle, and the image is the intersection of these cones with that triangle. Given an ideal and a permutation in a permutation group, it can compute a cone in an orbit of the permutation group.

If we want to consider tropical varieties, we can give Gfan a principle ideal $(x + y + z + w)$

within $\mathbb{Q}[x, y, z, w]$. If we want to ask more of the tropical verities, we would run tropical intersection to compute the rays, the cones generated by the rays. This is a tropical hyperplane in 4-dim.

Generically, Gfan does trivially valued fields. It is possible to add nontrivial valuations, but that takes longer to compute.

4.1 Day 33 | 20231108

Polymake | Andre

Polymake was made to study combinatorial properties of polyhedra. It has been extended to study far more. It is open source.

In programming terms, objects are divided in two categories: big and small. Big objects are the most abstract mathematical objects while small ones are the usual programming objects. Groups can be used to see group actions for example.

Example 4.1.1. The handle `pm` allows code to be run in the context of polymake inside Julia.

We can also use it to visualize fans, there is a context menu to turn option on/off. In this particular example, we create three tropical numbers and it's important to note that we must define the convention we are using.

In principle, polymake can visualize polynomials, hypersurfaces, but there's no documentation on how to do that with Julia. It can be used with `C++` in a more easier way.

Persistence Landscapes and Tropical Rational Functions | Tatum

The focus is to view persistence landscapes as Tropical Rational Functions. This is based on work by Peter Bubenik.

Suppose we have a data set, the topology is boring because it's just points. We could connect points if they are *close enough*. The threshold can be determined by a parameter α . The idea of persistence is that we have the choice made for us.

What people want to do with TDA is

- ◇ Persistence with statistics and machine learning
- ◇ No kernel methods with persistence diagrams
- ◇ Basic stats is hard

The ultimate goal is to have a representation in Hilbert spaces. This way we can represent them as functions. A way to define a persistence landscape is hard, this is a piecewise linear function. We can create it using the points

$$f_{(a,b)} = \max(0, \min(a - t, t - b)) \Rightarrow \lambda(k, t) = k \max_i f_{(a_i, b_i)}.$$

Can we view this as a tropical rational function? Observe that

$$f_{(a,b)} = 0 \oplus ((a \odot t^{-1}) \oplus (t \odot b^{-1}))^{-1}$$

We can see that the roots of these functions are the birth, the death and the midpoint which is the peak. Adding the degrees of this points gives us zero. This makes sense because convex shapes have only positive degree. We may also factor our function as

$$0 \odot (t \oplus a) \odot (t \oplus b) \odot (t \oplus \frac{a+b}{2})^{-2}.$$

The constant coefficient can be found using persistence as $f(\text{midpoint}) = \frac{b-a}{2}$. To talk about the $k \max$ we must use elementary symmetric polynomials. Tropical e_k looks like

$$\max_{S_n} \sum x_{\pi(k)} \Rightarrow \lambda(k, t) = e_k(f_{(a_i, b_i)}) - e_{k-1}(f_{(a_i, b_i)}).$$

The minus can be written tropically.

Recall that the elementary symmetric function recovers the coefficient of x^k when operated with the roots. Does that also happen tropically?

Example 4.1.2. We will see two small cases. Take a diagram with two points $(1, 4)$ and $(2, 3)$. We recover a two mountain tropical landscape. Then we have

$$\lambda_1(t) = f_{(1,4)}(t), \quad \lambda_2(t) = f_{(2,3)}(t).$$

If we have points with the same persistence $(2, 4)$ and $(1, 3)$ we get two mountains of the same height. Here we see that

$$\lambda_1(t), \lambda_2(t) \neq f_{(2,4)}, f_{(1,3)}.$$

Instead we have 5 roots: 1, 2, 5/2, 3, 4. 5/2 is a root for λ_1 but a pole for λ_2 .

With this in mind, there arise questions:

- ◇ λ_k functions seem to have combinatorial properties. Regarding number of points in persistence diagrams and so on.
- ◇ For theoretical reasons we can't use the original definition for locally finite posets.
- ◇ Instead of viewing λ as a function of f_i 's we could view it as a $2n + 1$ variable function a_i, b_i 's and t . Peter Bubenik mentions that it can be done.

4.2 Day 34 | 20231110

Neural Networks | Kristina

Example 4.2.1. What features of two animals let us decide whether a picture of a cat and a dog is either? There are more important feature than others. Also, does the orientation matter? Our brains focus on different features and where those features are.

A neural network is an attempt to realize the way that brain works. Say we pixelate a picture so for each pixel we hive it to a program as an input. Then the input is sent to hidden functions and then give us an output.

For each input we add a weight and have a bias. In this way for example we get

$$a_1^{(1)} = \sigma \left(\sum_{i=1}^n w_{1,i} a_i^{(0)} + b_1^{(0)} \right)$$

where b is a bias and σ is an activation factor.

There's a problem with this, assigning the weights! This can be solved via the *machine learning*. What we give to the machine is a cost function, as we know what is it that we want.

The activation function can be a sigmoid, or a ReLU which is $\max(0, Ax + b)$. We can apply tropical geometry to this! We are assuming that

- ◇ Weight matrices are integer valued.
- ◇ Bias vectors have real values.
- ◇ And activation functions have the form $\max(x, t^{(\lambda)})$.

We can rewrite our activation function as

$$\max(Ax + b, t) = \max(A_+x + b, A_-x + t) - A_-b$$

Under the previous assumptions we can write our neural network as a tropical rational function. The paper explores number of linear regions of the decision boundary. This can be used to find the complexity of the neural network.

There is an equivalence between feedforward neural networks with ReLU activation and tropical rational functions.

(Disucssion of the cat vs dog pictures for neural networks) Orientation, identifiable features, replicate human brain processing. Neural networks have hidden layers. Each layer is a matrix product (weight assignment) and vector addition (bias). We then introduce cost functions to compare results, back propogation (gradient descent) to adjust weights.

We have activation functions. THE first is the sigmoid $\sigma(Ax + b) = \frac{1}{1 + e^{-(Ax + b)}}$ we also have the rectified linear unit ReLU $\sigma(Ax + b) = \max(0, Ax + b)$. ReLU makes data more sparse,

and looks like tropical geometry. First we make assumptions about our L -layer network. We assume that the weight matrices $A^{(1)}, \dots, A^{(l)}$ are integer values, the bias vectors $b^{(1)}, \dots, b^{(l)}$ are real valued, and the activation functions take the form $\sigma(Ax + b) = \max(0, Ax + b)$.

To build our equivalence, we first consider the output from the first layer in the neural network $\nu(x) = \max\{Ax + b, t\}$, where $t \in (\mathbb{R} \cup \infty)^l$. So we can rewrite $\max\{Ax + b, t\} = \max\{A_+x + b, A_-x + t\} - A_-x$. So every coordinate of a one layer network is the difference of two tropical polynomials. For networks with multiple layers, apply this decomposition recursively.

Theorem 4.2.1. A feedforward neural network under the assumptions is a function $\nu : \mathbb{R}^d \rightarrow \mathbb{R}^p$ whose coordinates are tropical rational functions of the input, i.e. $\nu(x) = F(x) \oslash G(x) = F(x) - G(x)$, where F and G are tropical rational functions.

We can use this equivalence to consider decision boundaries of a neural network. The input space of a neural network is partitioned into disjoint subsets, where each subset determines a final decision (what is a dog, what is a cat). So our input space might be a tropical curve, and the 2-cells give decision boundaries. We can bound the number of linear regions of a NN by bounding vertices in the dual subdivision of the Newton polytope. This number of linear regions measures complexity of a neural network. These don't make better bounds, but it shows that tropical geometries can do the same work.

Joswig Algorithm | Jacob

The Joswig algorithm is an algorithm for graphs. We will use the min convention to minimize. For example Dijkstra's algorithm optimizes path cost when traveling graphs. In general fixed weight edges are too restrictive. For a communication network the cost might change. If we assume that some edges have cost $x, y \in \mathbb{R} \cup \{\infty\}$ then the Joswig algorithm solves this problem. So as an idea, the Joswig algorithm generalizes Dijkstra's algorithm.

Example 4.2.2. In this graph, edges have weights $x, y, 4, 5$ (each variable only appears once, for independence) so the optimization between $4 + y$ or $5 + x$ becomes

$$\min(4 + y, 5 + x) = 4 \odot y \oplus 5 \odot x.$$

Observe that the regions where one solution is better than another are bounded by a tropical variety. In this case $4 \odot y = 5 \odot x$.

To be tropically interesting, what happens when there are multiple regions? In terms of the graph we can imagine that there are multiple destinations.

Example 4.2.3. Consider the weighted graph. The Joswig algorithm reduces a big polynomial and saves time by introducing redundancies.

Not every decomposition can be described as a tropical variety of some polynomial. The counterexample is the one just mentioned. In summary the Joswig algorithm produces a subdivision of the Newton Polygon **and things happen**.

We let $\mathbb{T} := (\mathbb{R} \cup \{\infty\}, \min, +)$. We begin by saying that networks can be modeled using graphs. We have Dijkstra's Algorithm which gives us shortest path (including weights). This is relevant to transversing cities with weights on roads. Sometimes, fixed edge weights are too limiting. We let $x, y \in \mathbb{T}$ be parameters. We have separated graphs, only one copy of each variable associated to each edge (no two edges have the same variable). We then have a parametric shortest path. We let \odot be a concatenation of edges/paths, and \oplus a comparison. We interpret ∞ as being an edge that doesn't exist. We can consider the two parametric equations to be $4 + y \leq 5 + x$ to go down the left path, or $4 + y \geq 5 + x$ to go down the right path, so we have $\min\{4 + y, 5 + x\} = (4 \odot y) \oplus (5 \odot x)$.

The key observation is that the regions of optimal solutions are separated by tropical varieties. Now, what if our graph has multiple tropical polynomials. Each tropical polynomial corresponds to a different destination node. We can instead consider $(A \rightarrow B, 2)$, $(A \rightarrow B, y)$, $(B \rightarrow D, 1)$, $(A \rightarrow D, 2)$, $(A \rightarrow C, x)$, $(A \rightarrow C, 2)$, and $(C \rightarrow D, 1)$. We then have polynomials with residues when how we end at D . The Joswig Algorithm is reducing these polynomials. A selection of a solution is selecting a path for each destination node. In each region we select a path for that destination.

We decompose our parameter space into cells where within each cell we have an optimal solution. If we have a path through our x-y plane, we can segment that path into optimal segments.

This decomposition of parameter space came from three tropical polynomials. We can ask if we can describe the decomposition as the tropical varieties of some polynomials. We cannot! We have a proof by pictorial contradiction. We have vertices on the corners, which have coefficients ∞ , but that would cause kinds and additional cells (new vertices).

You are guaranteed convex cells (for any two solutions in a cell, any solution in between is a solution). The computation are also doable and efficient. In summary, the Joswig algorithm produces a decomp of param space into convex cells via tropical varieties. The algorithm works on the order of 10's parameters, as the parametric shortest path problem is (probably) NP-complete (or hard).

4.3 Day 35 | 20231113

Group Theory and Tropical | Natalie

Consider a $\xi \in \mathbb{R}$ non-zero. Consider the group G_ξ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}.$$

Our big question is: is G_ξ finitely presented? For our effects, suppose ξ is the root of an irreducible polynomial $f \in \mathbb{Z}[x]$. The idea is the if ξ or ξ^{-1} are algebraic integers over \mathbb{Q} , then the highest/lowest order term of f is ± 1 .

For sake of notation, call S the set of Laurent polynomials over \mathbb{Z} . Observe that units in this ring are monomials. Recall also that an initial form of a polynomial $f(x)$ with respect to w is another polynomial that records the terms from $f(x)$ that admit the minimum when tropicalized with $w = \text{val}(x)$.

Example 4.3.1. For simplicity in one variable take

$$f(x) = 4x^3 + 3x + 2 \Rightarrow \text{Trop}(f) = \min(3 \text{val}(x), \text{val}(x), 0).$$

If $w = 1 = \text{val}(x)$, then $\text{Trop}(f) = \min(3, 1, 0) = 0$ and so $\text{In}_{w=1}(f) = 2$. So when we evaluate at $\text{val}(x) = 0$ for example, then the initial form is the whole polynomial.

Definition 4.3.2. Take $I \triangleleft S$, then the initial ideal $\text{In}_w(I)$ is the ideal generated by the initial forms of $f \in I$.

Definition 4.3.3. The tropical variety of I is or equivalently, this is the vectors such that $\text{In}_w(I)$ has no unit of S .

Example 4.3.4.

$$\text{Trop}(f) = \min(\text{val}(s) + \text{val}(x), \text{val}(s) + \text{val}(y), \text{val}(s)).$$

We could be in a lot of possibilities for $(a, b) = (\text{val}(x), \text{val}(y))$.

If $a < 0, b$, then the initial form is sx which is a unit so its not in the tropical variety. But if $a, b > 0$ then the initial form would be $3s$. So this idel wouldn't contain a unit and so it's a variety. **TABLE**

If we were to instead work with rational coefficients 3 wouldn't be a unit. So our variety would be the usual tripod.

Example 4.3.5. Take $p = ax^2 + bx + c$, irreducible, then $\text{Trop}(p) = \min(2 \text{val}(x), \text{val}(x), 0)$.

Remember that if ξ algebraic integer, then ξ^{-1} is not so $b = \pm 1 = c$. We have a lot cases, in essence we recover pieces of the real line.

We let ξ be a nonzero real number, and denote G_ξ by the group generated by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & \xi \\ 0 & 1 \end{bmatrix}$. We then ask if G_ξ is finitely presented. If ξ is transcedental, the answer is no. However, let ξ be the root of some irreducible polynomial $f(x) \in \mathbb{Z}[x]$. Then G_ξ is finitely presented iff ξ or $\frac{1}{\xi}$ is an algebraic integer over \mathbb{Q} .

The statement that ξ or $\frac{1}{\xi}$ is an algebraic integer over \mathbb{Q} implies that highest order term or lowest order term of $f(x)$ is ± 1 .

Consider the Laurent Polynomial ring $S = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Units are monomials $\pm x_1^{a_1} \cdots x_n^{a_n}$.

Definition 4.3.1. The *initial form* of $f(\vec{x})$ wrt \vec{w} is a polynomial that records the terms that admit the minimum when tropicalized with $\vec{w} = \text{val}(\vec{x})$.

Example 4.3.1. Consider $f(x) = 4x^3 + 3x + 2$. Then $\text{Trop}(f) = \min(3\text{val}(x), \text{val}(x), 0)$. Now, let $w = 1 = \text{val}(x)$. Then $\text{trop}(f) = \min\{3, 1, 0\}$, the initial form $\epsilon_{w=1}(f) = 2$, as 2 is where the valuation of zero comes from. Now, let $w = 0$. Then the minimum is the same, so $\text{in}_{w=0}(f) = 4x^3 + 3x + 2$, as everything hits the minimum.

Definition 4.3.2. Let I be a proper ideal of S . Then the *initial ideal* $\text{in}_w(I)$ is the ideal generated by all initial forms $\text{in}_w(f)$ where f runs through I .

Definition 4.3.3. The *Tropical variety* of I is $V_{\text{trop}}(I) = \{w \in \mathbb{R}^n \mid \text{in}_w(I) \neq S\}$, or $V_{\text{trop}}(I) = \{w \in \mathbb{R}^n \mid \epsilon_w(I) \text{ does not contain a unit of } S\}$.

Example 4.3.2. Let I be the principal ideal $I = \langle x_1 + x_2 + 3 \rangle = \{sx_1 + sx_2 + 3s \mid s \in S\}$. We want to find $V_{\text{trop}}(I)$. First we find $\text{in}_w(f)$ for possible values of $w \in \mathbb{R}^2$. Note that $\text{trop}(f) = \min\{\text{val}(s) + \text{val}(x_1), \text{val}(s) + \text{val}(x_2), \text{val}(s)\}$. Now we let $w = (a, b)$, and so $\text{val}(x_1) = a$ and $\text{val}(x_2) = b$. Now, if $a < 0, b$, then $\epsilon_w(f) = sx_1$. If we let x_1 have valuation 1, we would get a unit, so it is not in the variety. If $0 < a, b$, then $\epsilon_w(f) = 3s$, not a unit, so w is in the variety.

Example 4.3.3. Let ξ be a root of an irreducible polynomial $p(X) = a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x^{\pm 1}]$. Then $\text{trop}(p(x)) = \min\{2\text{val}(x), \text{val}(x), 0\}$. We have $\epsilon_{w<0} = a_2x^2$, $\text{in}_{w=0} = a_2x^2 + a_1x + a_0$, and $\epsilon_{w>0} = a_0$.

Tropical Cryptography | Kylie

Cryptography is sending messages to one another and reduce probabilities that another person might decode it. There are two major schools in cryptography, public and private key. The first one has the idea that there's two keys, a public and a secret one. The public one is used to encrypt information and the secret one decrypts information. For the second one, the private key is used for both encryption and decryption. An advantage of private key is that it's symmetric, this means that the protocols are computationally faster than the public key protocols.

An idea is to do the key exchange for private keys via public methods. The key will be based on the algorithm used.

The Tropicalness of it all

We will be working with the min convention.

Example 4.3.6. Quick tropical sum of matrices:

$$\begin{pmatrix} 1 & 2 \\ \infty & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 4 & -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 \\ \infty & -1 \end{pmatrix} \odot \begin{pmatrix} 0 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix}.$$

Tropical multiplication is not commutative. And also the tropical identity is composed of infinities all across except in the diagonal. In that place looks like zeroes. Diagonal matrices have non-zero entries on the diagonal.

We want two public matrices A, B such that they are not commutative. This doesn't change a lot of the algorithm, but it makes sure that the space of matrices is large enough. Alice picks two tropical polynomials p_1, p_2 with integer coefficients. Bob also picks two polynomials secretly. Then they compute $p_1(A) \odot p_2(B)$ and $q_1(A) \odot q_2(B)$. Call these matrices P_A, Q_B . Then Alice computes $p_1(A) \odot Q_B \odot p_2(B)$ while Bob does $q_1(A) \odot P_A \odot q_2(B)$.

Lemma 4.3.7. *We have*

$$p_1(A) \odot Q_B \odot p_2(B) = q_1(A) \odot P_A \odot q_2(B).$$

This is the private key.

The proof goes along the lines that any matrix commutes with itself. Now, the key is also a matrix, so it suffices to multiply by the inverse. However, tropical invertible matrices are rare.

Proposition 4.3.8. *The only tropical invertible matrices are permutations of a diagonal matrices.*

It takes a little bit of thought as to why a tropical invertible matrix is a diagonal matrix but we know that $A \odot A^{-1} = I$. **finish** An idea is to only get the matrix key and then use usual matrix multiplication.

Cryptography is a format for sending secret methods that can't be read by anyone besides the recipient. The two major schools are public and private key cryptography. Within Public key cryptography, there are two keys: one public and one private. The public key is used to encrypt information, while the private key is used to decrypt information. In Private Key cryptography, there is a single private key used for both encryption and decryption.

Public key is more secure, while private key is faster. In the discussed paper, a key exchange protocol is used. This is a secure way to exchange a key using public methods between parties.

We will take the min convention. So $\begin{bmatrix} 1 & 2 \\ \infty & -1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ \infty & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$ \otimes is matrix multiplication (and then using plus and min) is not commutative, and the tropical identity matrix is the matrix with zero on the diagonal, *infinity* elsewhere. Diagonal matrices have something on the diagonal.

We want our public key to be $n \times n$ matrices A and B with tropical entries, with the requirement that $A \otimes B \neq B \otimes A$.

Alice has a secret: two tropical polynomials p_1 and p_2 with integer coefficients. Bob does the same with q_1 and q_2 . Alice computes $p_1(A) \otimes p_2(B) = P_{Alice}$, while Bob computes $q_1(A) \otimes q_2(B) = Q_{Bob}$. Then Alice sends Bob her P_{Alice} , Bob sends Alice Q_{Bob} . Alice knows $P_1(A)$ and $P_2(B)$. Then Alice computes $p_1(A) \otimes Q_{Bob} \otimes p_2(B) = K_{Alice}$, while Bob computes $q_1(A) \otimes P_{Alice} \otimes q_2(B) = K_{Bob}$. It turns out that

Lemma 4.3.1. $K_{Alice} = K_{Bob}$

Proof. Tropical matrices commute with themselves, so $q_1(A) \otimes p_1(A) = p_1(A) \otimes q_1(A)$. \square

The key will then be the matrix $K_a = K_b = K$. We can then use these matrices to code and decode matrices via vector multiplication. PROBLEM! Tropical invertible matrices are rare.

Lemma 4.3.2. The only tropical invertible matrices are permutations of a diagonal matrix.

Proof. That permutations of a diagonal matrix is easy. To see that invertibles must be permutation of a diagonal, we note that $A \otimes B = I$, we have $\oplus a_{ik} \otimes b_{kj}$ which is 0 if $i = j$ and ∞ if $i \neq j$. Keep going to see issues. \square

4.4 Day 36 | 20231115

Tropical Eigenvalues | Ian

Example 4.4.1. The first example comes from cooking mac and cheese. It might be efficient to do some steps in parallel, this creates a network of flow. To do step 2 and 5 you need to have done some other steps and to do step 6 we need all the previous steps. Call x_j the soonest time you can start step j . Say steps 2 and 5 take 3 and 9 minutes, so step 6 can be started at least after 9 minutes, then

$$x_i = \max_j (A_{ij} + x_j)$$

where A_{ij} is how long does it take to go from $j \rightarrow i$. Observe that this corresponds to the tropical matrix equation $A \odot x = x$ and this is an eigenvalue problem for $\lambda = 0$.

Remark 4.4.2. This is also corresponding to the longest path search in the Joswig algorithm.

Definition 4.4.3. A *tropical eigenvalue* is a solution λ to the equation $A \odot x = \lambda \odot x$.

Example 4.4.4. Consider

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -\infty & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

and think of it as an (transpose) adjacency matrix for a directed graph.

Lemma 4.4.5. *If we have an eigenvalue of A , then there exists a normalized cycle of average weight λ .*

Example 4.4.6. Consider the matrix

$$\begin{pmatrix} -\infty & 2 & -\infty \\ -\infty & 4 & -\infty \\ -\infty & -\infty & 5 \end{pmatrix}$$

with eigenvector $(0, 2, -\infty)$. When applying the matrix product to it we get $(4, 6, -\infty)$ with eigenvalue 4. This corresponds to the cycle about vertex 2. As we also notice a cycle about vertex 3 then we may ask if it corresponds to an eigenvector $(-\infty, -\infty, 5)$ with eigenvalue 5.

In general however this is not true. Not all cycles corresponds to an eigenvalue.

Lemma 4.4.7. *If $C \subseteq G$ is a cycle in our graph then*

$$\lambda = \max_C \frac{w(C)}{|C|}$$

is an eigenvalue of the adjacency matrix of G whenever the quantity is not infinite.

Observe that in the previous case, the eigenvalue 4 was extraneous as it didn't correspond to a max weight cycle.

Theorem 4.4.8. *If the graph is strongly connected, then the adjacency matrix has a unique eigenvalue which is the previous eigenvalue mentioned, the normalized weight.*

We can decompose the eigenvalue equation as

$$\lambda + x_i = \max_j (A_{ij} + x_j) \Rightarrow A_{ij} + x_j \leq \lambda + x_i \Rightarrow A_{ij} + x_j - x_i \leq \lambda$$

and this gives a system of linear inequalities. Using linear programming we can find a solution to that inequality.

- ◇ Are there conditions when the maximum is achieved by a unique cycle? *It is not necessarily unique, even if the graph is strongly connected.*
- ◇ Are the eigenvalues located in the matrix? *Not necessarily.*

- ◇ Is there a characteristic polynomial for tropical matrix? *Squinting it is the same as the usual, as we also have a fundamental theorem of algebra, we have a correspondence.*

Max convention As an example, we have instructions for making mac and cheese s1) boil whatever s2) cook pasta s3) grate cheese s4) make bachelmel s4.5) lightly cook tomatoes s5) melt cheese s6) combine. But there are steps that can happen at the same time. step 2 depends on step 1, but grating cheese and making bachelmel can happen at the same time. But melting cheese requires rating cheese. Step six is a final step that everything else needs to happen first. This provides a flow of operations. We want to ask when is the soonest we can start a step.

We create a vector \vec{x} where x_j says when we can start task j . For example, step 6 can start some time after starting step 2. We get a weighted graph where the weights say when can subsequent steps can occur. The max of all paths leading to step 6 gives us x_6 . Thus $x_i = \max\{A_{ij} + x_j\}$, where A_{ij} is how long it takes to go from j to i . This correpsodns to the matrix equation $A \odot x = x$

Definition 4.4.1. $\lambda \odot x = A \dot{x}$, where λ is an eigenvalue with eigenvector x

We let A be a equare matrix, and we consider it as an adjacency matrix, where $-\infty$ denotes no edge. existing, and zero and negative edges are allowed, A_{ij} is $j \rightarrow i$.

We can think of eigenvalues with respect to this graph.

Lemma 4.4.1. If $A \odot x = x \odot \lambda$, there exists a normalized cycle of averaged wieght λ

A normalized cycle is the sum of weights devided by the number of edges

Example 4.4.1. Let $A = \begin{bmatrix} -\infty & 2 & -\infty \\ -\infty & 4 & -\infty \\ -\infty & -\infty & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -\infty \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -\infty \end{bmatrix}$. Here $lambda = 4$, and we have another igenvector $(-\infty, -\infty, 5)$ with eigenvalue 5.

Lemma 4.4.2. Let σ be any cycle in our graph. Then the maximum normalized cyucle $\max_{\sigma} w(\sigma)/|\sigma| (\neq \pm\infty)$ is an eigenvalue for the associated matrix A .

Theorem 4.4.3. If the graph associated with A is strongly connected then there is exactly one eigenvalue, $\lambda = \max_{\sigma} w(\sigma)/|\sigma|$.

In terms of our computation, we have $A \odot x = \lambda \odot x$, which rewrites as $\max_j (A_{ij} + x_j) = \lambda + c_i$, sow get an inequality $a_{ij}x_j \leq \lambda + x_i$, so we have $A_{ij} + x_j - x_i \leq \lambda$.

Tropical Varieties of Higher Codimension | Seth

Recall most our varieties have had codimension one. This topic is a connection between algebraic geometry and tropical geometry.

Definition 4.4.9. A hypersurface is the vanishing set of a single polynomial in classical algebraic sense. We denote $V(f)$ the set of points x such that $f(x) = 0$ for $f \in K[x_1, x_2, \dots, x_n]$.

Example 4.4.10. The variety $V(x + y + 1)$ is a line with slope -1 .

Most objects are *not* going to be hypersurfaces. Points for example are an intersection of two hypersurfaces in the affine plane. We could say that $\text{pt} = V(f, g)$ but what we are really doing is looking at the vanishing set of an ideal. If $I = \text{gen}(f, g)$ then everything in I is a polynomial combination of f and g and polynomials there vanish whenever f, g vanish. This means that $V(I) = \{x : \forall f \in I (f(x) = 0)\}$.

This is a sensible way to reason because

- ◇ I is always finitely generated by the Hilbert basis theorem.
- ◇ We can think of $V(I)$ as the intersection of all the hypersurfaces, but we can do better, it is the intersection of the hypersurfaces generated by the generators of the ideal. However, this won't work in the tropical sense.

Let us walk the same path through tropical hypersurfaces. Remember that we are now looking at polynomials in the Laurent ring while $\text{Trop}(f)(w_1, \dots, w_n) = \max(\text{valuation of terms})$. We can define a tropical hypersurface with this, instead of a vanishing set it is a tying set.

$$\text{Trop}(V(f)) = \{w : \text{Trop}(f) \text{ attains max at least twice}\}.$$

In this case we don't want K to be trivially valued.

Example 4.4.11. Let us take the same polynomial as before but now it lives somewhere different. Now $\text{Trop}(f)$ takes (w_1, w_2) to $\max(w_1, w_2, 0)$. Now our tropical variety contains the sets

$$w_1 = w_2 \geq 0, \quad w_1 = 0 \geq w_2, \quad \text{and} \quad w_2 = 0 \geq w_1.$$

This is the usual tripod.

We also have Kapranov's theorem which basically tells us that $\text{Trop}(V(f)) = \overline{\text{val}(V(f))}$. However this theorem only works for hypersurfaces.

Definition 4.4.12. For an ideal I we define its tropical variety as

$$\text{Trop}(V(I)) = \bigcap_{f \in I} \text{Trop}(V(f)).$$

When defining it this way then Kapranov's theorem also holds. However we the small caveat that $\text{Trop}(V(I))$ is not necessarily the intersection of the tropical hypersurfaces of the generators. This has a very *Gröbner basis*-like feeling. So we define the basis that works.

Definition 4.4.13. A tropical basis for an ideal I is any basis such that

$$\text{Trop}(V(I)) = \bigcap_{g \in B \subseteq I} \text{Trop}(V(g)).$$

Tropical basis exist and are computable, for example we can use Gfan.

Example 4.4.14. If $f = x + y + 1$ and $g = x + 2y$. The tropical hypersurface of g is the $w_1 = w_2$ line. The intersection of this two varieties is a ray which is most definitely not a tropical variety. This is because f, g don't form a tropical basis of $\text{gen}(f, g)$, but if we take $\text{gen}(f, g, y - 1)$ it still generates the same ideal and now its a tropical basis. When intersecting we now get a point.

- ◇ In general the right hand side is too big? *stuff*
- ◇ Ross
- ◇ Correspondence between grobner and tropical?
- ◇ Bound on Joseph?

We begin by discussing classic algebraic geometry and the varieities there. In this setting

Definition 4.4.2. A *hypersurface* is a vanishing set of a single polynmlial $f \in K[x_1, \dots, x_n]$, denoted $V(f) = \{(x_1, \dots, x_n) \mid f(x_1, \dots, x_n) = 0\}$

Example 4.4.2. For $f(x + y + 1)$, then $V(f)$ is the line $y = -x - 1$.

In higer cosien sion varieties are vanishing sets of ideals $V(I)$, whre $I = (f, g)$. Then we get $V(I) = \bigcap -f \in IV(f)$. THis is nice because I is finitely generated (Hilbert Basis Theorem), and $V(I) = \bigcap_{f \in \text{gen}(I)} V(f)$. In the tropical setting, $V(I) = \bigcap V(f)$ for f a genratr does not work.

To translate into tropical geometry, we let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, then $\text{trop}(f) : \mathbb{R}^n \rightarrow \mathbb{R}$, where $w = (w_1, \dots, w_n) \mapsto \max$ of hte valuation of terms. A tropical hypersurface is the set of ties, $\text{trop}(v(f)) = \{w \in \mathbb{R}^n \mid \text{trop}(f) \text{ attains max at least twice}\}$.

Example 4.4.3. Let $f = x + +1 \in K[X^{\pm 1}]$. Then $\text{trop}(f) : \mathbb{R}^2 \rightarrow \mathbb{R}$, where $(w_1, w_2) \mapsto \max(w_1, w_2, 0)$. Then $\text{trop}(V(f)) =_1 = w_2 \geq 0, w_1 = 0 \geq w_2$, or $w_2 = 0 \geq w_1$, so we get a tropical curve.

By Kapranov's Theorem, $\text{trop}(V(f)) = \overline{\text{val}(V(f))}$.

Definition 4.4.3. A *tropical variety* is $\text{trop}(V(I)) = \bigcap_{f \in I} \text{trop}(V(f))$

Theorem 4.4.4. Fundamental Theorem of tropical algebraic geometry $\text{trop}(V(I)) = \overline{\text{val}(V(f))}$.

However, $\bigcap_{f \in I} \text{trop}(V(f)) \neq \bigcap_{g \in \text{gens}(I) \in I} \text{trop}(V(g))$. Similarly to Grobner basis, we define a basis which works to work In

Definition 4.4.4. A *Tropical basis* T for an ideal I is any basis for which $\bigcap_{f \in I} \text{trop}(V(f)) = \bigcap_{g \in T \in I} \text{trop}(V(g))$

So when we consider $f = x + y + 1$ and $g = x + 2y$ (the line is $w_1 = w_2$), we consider $\max(w_1, w_2)$. When we intersect the two we get a ray. This is not balanced. So $I = (x + y + 1, x + 2y)$ generates an ideal, but it is not a tropical basis. If we add in $y - 1$ (coming from $x + 2y - (x + y + 1)$), we then get a tropical basis. This adds $h = y - 1$ which is a horizontal line, which the additional intersection gives us a point.

4.5 Day 37 | 20231127

Line bundles over \mathbb{P}^1 and tropical \mathbb{P}^1 | Sam

Recall that the complex projective plane can be defined via an equivalence relation and it has an open cover which is the affine charts. These charts have the usual transition functions.

Definition 4.5.1. A vector bundle of rank n over a space X (can be replaced with manifold/-variety) is a space L together with a projection $\pi : L \rightarrow X$ which satisfies two conditions:

- i) There is an open cover of X (U_i) satisfying

$$\pi^{-1}[U_i] \simeq U_i \times \mathbb{C}^n,$$

that's where the rank n comes in. And if φ_i is such isomorphism, then $\pi = p_1 \phi_i$ where p_1 is the projection $U_i \times \mathbb{C}^n \rightarrow U_i$.

- ii) And the other condition is that the map

$$\varphi_j \varphi_i^{-1} : (U_i \cap U_j) \times \mathbb{C}^n \rightarrow (U_i \cap U_j) \times \mathbb{C}^n$$

is a linear map on \mathbb{C}^n and the identity on the intersection.

Particularly, a line bundle is a vector bundle of rank 1.

Example 4.5.2. Consider the real projective line. It is covered by two circles with one point removed. Then $\pi^{-1}(U_i)$ "looks like" a squiggly cylinder and the map φ_1 "straightens it out". The map $\varphi_2 \varphi_1^{-1}$ basically straightens and shrinks. But remember *not all scalars are the same*, take for example multiplication by -1 . This is an example of a line bundle.

Definition 4.5.3. Suppose $L \xrightarrow{\pi} X$ is a line bundle, $U \subseteq X$ is open. Then a section of L over U is a morphism $s : U \rightarrow L$ such that πs is the identity on U .

What this is doing is choosing a vector space element for each point of X .

Example 4.5.4. The trivial line bundle over complex projective space is $\mathbb{P}^1 \times \mathbb{C}$. A section has the form

$$s : x \mapsto (x, f(x))$$

and what this is doing is defining a function on \mathbb{P}^1 .

Example 4.5.5. The tautological line bundle is

$$L = \{ ([X : Y], (x, y)) : (x, y) = (\lambda X, \lambda Y) \}$$

this gathers all the points in \mathbb{P}^1 and lines in \mathbb{C}^2 . This has a lot of information.

Let's go now to the tropical setting. Similar to our line bundle definition,

Definition 4.5.6. a line bundle over X a tropical space, is a space L and a projection π such that

- i) There is an open cover of X , (U_i) , and
- ii) there are maps $\varphi_i : \pi^{-1}[U_i] \rightarrow U_i \times \mathbb{T}$ where they are tropical isomorphisms and the same commuting relation holds.

In this case, the local trivializations induce automorphisms of \mathbb{T} as $\tilde{\varphi}_{ij} = \varphi_j \varphi_i^{-1}$. Actually $\text{Aut}(\mathbb{T}) \simeq \mathbb{R}$. This lets us define φ_{ij} such that $\tilde{\varphi}_{ij}(x, y) \mapsto (x, \varphi_{ij}(x) \odot y)$.

Example 4.5.7. If we define tropical projective space as $\mathbb{T} \cup \mathbb{T}$ via $x = -y$ then the trivializations are both 0 so

$$\varphi_{12} :]-\infty, \infty[\rightarrow \mathbb{R}$$

is the constant zero function.

Manifold Structure on \mathbb{CP}^1

. We define \mathbb{CP}^1 as $(\mathbb{C}^2 - \{(0, 0)\}) / ((X, Y) (\lambda X, \lambda Y))$ for all $\lambda \in \mathbb{C}$. We take open covers

$U_1 = \{[X : Y] \in \mathbb{CP}^1 \mid Y \neq 0\}$ and $U_2 = \{[X : Y] \in \mathbb{CP}^1 \mid X \neq 0\}$, an maps

$\varphi_1, \varphi_2 : \mathbb{CP}^1 \rightarrow \mathbb{C}$ defined by $\varphi_1 : [X : Y] \mapsto \frac{X}{Y} =: x$ and $\varphi_2 : [X : Y] \mapsto \frac{Y}{X} =: y$.

We have an equivalent construction $\mathbb{CP}^1 := ((\mathbb{C}, x) \sqcup (\mathbb{C}, y)) / (x = \frac{1}{y})$.

line bundles

Definition 4.5.1. A *vector bundle* of rank n over a space X is itself a space L together with a projection $\pi : L \rightarrow X$ such that

- (a) There exists an open cover of X $U := \bigcup_i U_i$ satisfying $\pi^{-1}(U_i) \simeq U_i \times \mathbb{C}^n$, and if ϕ_i is such a morphism, $\pi : (\pi^{-1}(U_i)) \rightarrow U_i$ is equal to $p_1 \circ \phi_i$, where p_1 is the projection onto the first coordinate.
- (b) The map $\phi_j \circ \phi_i^{-1} : (U_i \cap U_j) \times \mathbb{C}^n \rightarrow (U_i \cap U_j) \times \mathbb{C}^n$ is a linear map on \mathbb{C} and the identity on the intersection.

Definition 4.5.2. the ϕ_i are called *trivializations*.

Definition 4.5.3. A *line bundle* is a rank 1 vector bundle

Definition 4.5.4. Let $\pi : L \rightarrow X$ be a vector bundle (line bundle). Let $U \subset X$ be open. A *section of the bundle* over U is a morphism $s : U \rightarrow L$ such that $\pi \circ s = id_U$.

Sections choose a vector space element for each point of X .

Example 4.5.1. We can take $L = \mathbb{CP}^1 \times \mathbb{C}$ which is the trivial line bundle over complex projective space. Sections have the form $s : x \mapsto (x, f(x))$, where $f(x)$ is a complex number associated to x . So sections of $\pi : L \rightarrow \mathbb{P}^1$ define functions on complex projective space.

Example 4.5.2. The Tautological Line bundle is

$$L = \{([X : Y], (x, y)) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid (x, y) = (\lambda X, \lambda Y)\}$$

So for each line in \mathbb{CP}^1 we pick a representative point on that line.

Tropical Line Bundles

We have a similar definition when defining line bundles over a tropical space X .

Definition 4.5.5. A *line bundle* over a tropical space X is a space L and a projection $\pi : L \rightarrow X$ so that

- (a) There exists an open cover $U = \bigcup_i U_i$, and
- (b) there exist $\phi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{T}$, which are tropical isomorphisms, and such that $p_1 \circ \phi_i = \pi$ on $\pi^{-1}(U_i)$.

These maps induce automorphisms on the tropical semiring \mathbb{T} . We define $\tilde{\phi}_{ij} := \phi_j \circ \phi_i^{-1} : (U_i \cap U_j) \times \mathbb{T} \rightarrow (U_i \cap U_j) \times \mathbb{T}$. In particular, $\tilde{\phi}_{ij}$ is an automorphism on each $\{x\} \times \mathbb{T}$.

Note 3. $Aut(\mathbb{T}) \simeq \mathbb{R}$.

This lets us define the map $\phi_{ij} : U_i \cap U_j \rightarrow \mathbb{R}$ via $\tilde{\phi}_{ij}(x, t) = (x, \phi_{ij}(x) \odot t)$.

Example 4.5.3. Take $\mathbb{TP}^1 := (\mathbb{T}, x) \sqcup (\mathbb{T}, y) / (x = -y) = [-\infty, \infty]$. We take an open cover $U_1 := [-\infty, \infty)$ and $U_2 := (-\infty, \infty]$. We can take the trivial line bundle $L := \mathbb{TP}^1 \times \mathbb{T}$. Then the morphisms are $\phi_1 = \phi_2 = 0$, and so the induced map we get is $\phi_{12} : (-\infty, \infty) \rightarrow \mathbb{R}$, which is the constant 0 function.

Ross, Counting Curves Like I Count Stacks

Example 4.5.4. How many circles are tangent to three circles in a plane? The answer is three

Example 4.5.5. How many lines go through two points? It's one.

Example 4.5.6. How many conics go through five points? It's one.

We want to tropicalize these questions so we can more efficiently answer these questions.

We typically moduli and consider $f : \mathbb{P}^1 \rightarrow \mathbb{P}^2$. Each condition brings down the dimension of the solution set, until we get to zero dimension, which is a finite number of points.

The general dimensionality of our solution is $3d+g-1$

We have a related notion in tropical geometry. Given two points, we can find a single tropical line passing through the two, and given five points we have a single tropical conic going through the five points (the conic is degree two).

To be careful, we say that a tropical curve has degree d if there are exactly d rays going in each direction (for which rays go).

We have lengths of a graph, and we define $\gamma : \Gamma \rightarrow \mathbb{R}^2$ via $\{w_l, P_l\}$, where we have weights of each edge l , and some P .

We can tropicalize a moduli space to get the tropical moduli space.

There are many determinants, and multiplicities. $N(d, g)$ is the number of curves satisfying n point conditions, g is genus

Take some complex polygon, project down some vector, to make sure the points in the polygon are separated nicely. We call the vector we project along λ we call a path λ increasing if a path along vertices goes along an increasing set of points when considering the projections.

Now only do we have $N(d, g) = N^{trop}(d, g)$, we have $N(d, g) = N^{path}(d, g)$, the number of λ increasing paths counting multiplicity.

We now give a recursive definition. $\mu(\gamma) = \mu_+(\gamma)\mu_-(\gamma)$, where $\mu_\pm(\gamma) = 2 * \text{Area}(T)\mu_\pm(\gamma'_\pm) \pm \mu(\gamma''_\pm)$. There are two kinds of paths. γ_+ looks for when the path turns left, the γ' is forming triangle, γ'' is making a parallelogram, γ_- is when you turn right. This makes subdivisions, base case definitions are the side lengths, where $\mu(\delta_\pm) = 1$.

Sandra, Intro to Berkovich spaces

What if instead of algebraically closed characteristic 0 field, we had a non archimedean field. We want to build an analog of complex geometry over a non archimedean field

Definition 4.5.6. A *valued field* is a pair $(K, |\cdot| : K \rightarrow \mathbb{R}_{\geq 0})$, where

(a) $|a| = 0 \iff a = 0,$

(b) $|a - b| \leq |a| + |b|$, and

(c) $|ab| = |a| * |b|$

Example 4.5.7. \mathbb{C} with the infinity norm.

A valued field might be complete. For notation $(\hat{k}, |\cdot|)$ denotes the completion of K with respect to the norm $|\cdot|$. If we take $K = \mathbb{Q}$, and the normal Euclidean norm, $\hat{\mathbb{Q}} = \mathbb{R}$. If instead we had the p-adic norm, $\hat{\mathbb{Q}} = \mathbb{Q}_p$.

Definition 4.5.7. K is complete if $k = \hat{k}$

We now consider Laurent series. Take $K = \tilde{k}((T)) = \{f = \sum_{i=-\infty}^{\infty} a_i T^i \mid a_i \in k, (i \ll 0, a_i = 0)\}$. Then the order of the zero of f is defined to be $\min\{i \mid a_i \neq 0\}$. Then $|f|_* := e^{ord_0(f)}$. We then claim that $(K, |\cdot|_*)$. The Archimedean principle is $|a + b| > \max\{|a|, |b|\}$. $(k, |\cdot|)$ is non-Archimedean if $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in k$.

Example 4.5.8. $(\mathbb{Q}, |\cdot|_p)$ for all primes p .

Note 4. If $(K, |\cdot|)$ is NA, then the following are true:

(a) If $|a| > |b|$, then $|a + b| = |a|$.

(b) All open balls over k are clopen.

(c) k is totally disconnected (each connected component is a singleton)

These properties make building a manifold theory complicated.

When we build a manifold, we have certain desired properties of a space.

Our first step is to define a space. Some essential ingredients:

(a) We need a set X

(b) We need a topology on X

(c) We need a structure sheaf \mathcal{O}_X such that if we pick a section $f \in \mathcal{O}_X(U)$ (U open) is analytic.

4.6 Day 39 | 20231201

Tropical Intersection Theory | Jake and Conager

For the Grassmanian, the Chow ring coincides with cohomology. Elements of this ring are subvarieties of X , it is graded by codimension and operations are the usual set operations.

Example 4.6.1. Consider \mathbb{P}^2 , A^0 is the codimension 0 part and so on. In this case A^0 is only \mathbb{P}^2 so we have that it is \mathbb{Z} . For A^2 its the same story. And A^1 is generated by a line and all lines are rationally equivalent.

If we had two lines

$$L_1 : ax + by = 0, \quad L_2 : cx + dy = 0$$

then we get a rational equivalence by setting $\alpha L_1 + \beta L_2 = 0$.

Example 4.6.2. If $X = \mathbb{A}^2$, the Chow ring is trivial for if we have $F : V(f(x, y))$ then $\alpha F + \beta(1) = 0$ is a rational equivalence. So the Chow ring of affine space is \mathbb{Z} generated by the space itself. This equivalence works here because 1 is not the same degree as F (most likely).

Example 4.6.3. If we have a subvariety X of Y then the Chow ring of X includes in the Chow ring of Y and then we can restrict into $Y \setminus X$. This means that the sequence

$$A^*(X) \rightarrow A^*(Y) \rightarrow A^*(Y \setminus X) \rightarrow 0$$

is exact and $\text{Im}(i_*) = \ker(r_*)$ and $\text{Im}(r_*) = A^*(Y \setminus X)$. So by the first isomorphism theorem we have

$$A^*(Y) / \text{Im}(i_*) \simeq \text{Im}(r_*) = A^*(Y \setminus X).$$

Applying this to the coordinate axes of \mathbb{C}^2 we get

$$A^*(Z) \rightarrow A^*(\mathbb{C}^2) \rightarrow A^*((\mathbb{C}^\times)^2) \rightarrow 0.$$

Example 4.6.4. For a toric variety X_Σ with torus T , then

$$A^*(X_\Sigma \setminus T) \rightarrow A^*(X_\Sigma) \rightarrow A^*(T) \rightarrow 0$$

so as long as $\alpha \in A^*(X_\Sigma)$ doesn't go zero then its no the whole space. This means that $\alpha \neq X_\Sigma$ and $\alpha \in \ker r_*$. By exactness $\alpha \in \text{Im } i_*$.

Minkowski weights

Giving intersection numbers of points, lines and conics, we can describe classes of them.

Example 4.6.5. For \mathbb{P}^2 we can take a generic line which will intersect **SLEEP**

Example 4.6.6. For the blowup of \mathbb{P}^2 the fan looks like a tripod with an extra leg.

Jake, Tropical Intersection Theory

Consider a space X . We have the Chow ring $A^*(X)$, which is a graded ring (via the codimension). Elements are subvarieties of X . We have the translation that $+$ is the union, and \times is the intersection. If we consider $X = \mathbb{P}^2$, then $A^0(\mathbb{P}^2) = \mathbb{Z}$, and so on. The collection of them is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \dots$.

If we consider two polynomials $F : ax + by = 0$ and $G : cx + dy = 0$, we can consider $\lambda F + \mu G = 0$. We get that the equations come from $[0 : 1]$ and $[1 : 0]$. This gives polynomials $x^2 + y^2 = 0$, which comes from $(ax + by)(cx + dy) = 0$.

Now, if $X = \mathbb{A}^2$, then $F : V(f(x, y))$. Then $\lambda F + \mu(1) = 0$ gives $A^*(\mathbb{A}^2) = \mathbb{Z}$.

Conegor, Minkowski Weights

This describes classes (pts, lines, conics, etc.) by giving their intersection number with all toric invariant subvarieties (boundaries). We define $X = B|_p \mathbb{P}^2$. Looking at the fans of X , we can ask what happens when L , line, intersects, a vertex of X .

Daniel, Tropical Covers & Tropical Riemann Hurwitz formula

We are trying to emulate Riemann surfaces.

Definition 4.6.1. A Riemann surface is a complex analytic manifold of dimension 1.

Note 5. A space is a compact Riemann surface if and only if it is a smooth projective curve over \mathbb{C} .

Compact Riemann surfaces are classified by their genus. We say X has genus n if and only if $x \hookrightarrow \mathbb{R}^3$ is an n -holed torus. Furthermore, holomorphic maps between compact Riemann surfaces $f : X \rightarrow Y$ are branch coverings, i.e. almost everywhere d -to-one, where cutting out finitely many problematic points and their pre-images, we get d -to-one. Where the map is not d -to-one we call that ramification. i.e. we say z^2 is ramified over 0 and ∞ . We have global information, which is the genus of X and Y , and local information, which is the ramification profile. This information is more combinatorial. The Riemann-Hurwitz formula is how we relate the two. If we have $f : X \rightarrow Y$ is nonconstant, holomorphic, and degree d , we get the equality $2g(X) - 2 = d(2g(Y) - 2) + \sum_{x \in X} (e_x - 1)$, where d is the degree of the map, $2g(X) - 2$ is the Euler characteristic of the space, and e_x is the ramification index of the point i.e. looking at the pre-images and how many branches intersect. This is a realizability condition.

Definition 4.6.2. An abstract tropical curve is a connected metric graph Γ with unbounded rays called ends and a genus function $g : V \rightarrow \mathbb{N}$ which assigns a genus to every vertex of our graph.

We have the genus function so that we can think of the curves as being dual to deformations of Riemann surfaces. We can think of the single vertex of degree two as being dual to the genus 2 torus. The deformation splits the genus two surface into two surfaces connected at a single point, so we get two vertices of degree 1 with an edge connecting them. Vertices of degree 1 correspond to the “irreducible” surfaces of genus one (the torus), and the vertices connecting them correspond to the surfaces intersecting at a point.

Definition 4.6.3. The local degree at a vertex is the sum of the surrounding weights

Definition 4.6.4. A tropical cover $\Pi : \Gamma_1 \rightarrow \Gamma_2$ is a surjective map satisfying

- (a) locally integer affine linear (Π scales length by an integer factor). This factor is called the weight of the edge $w(e)$.
- (b) Harmonic/Balancing condition

The first condition allows us to label edges by weights. For the second condition, returning to earlier in these notes. If we have a tropical curve, we label with weights so that we get a “balancing” of the weights on either sides of vertices.

Definition 4.6.5. The local Riemann-Hurwitz condition states that for $v \in \Gamma_1$, $v \mapsto v'$ with local degree d_v , we have $2g(v) - 2 \geq d_v(2g(v') - 2) + \sum_{e \in E, v \in e} (w(e) - 1)$.

When we consider a cover of a tropical curve Π , we get a duality too a cover of Riemann surfaces $\hat{\Pi}$. So satisfying local condition at vertices allows us to satisfy global condition of covers on surfaces.

Even, (Tropical) Hurwitz numbers

If we have two maps $f : X \rightarrow Y$ and $g : \tilde{X} \rightarrow Y$ between Riemann surfaces we say f and g are isomorphic if there exists $\varphi : X \rightarrow \tilde{X}$ such that $f = \varphi \circ g$. We can allow $X = \tilde{X}$. Let Y be a connected compact Riemann surface with genus g . Fix $b_1, \dots, b_h \in Y$, and $\lambda_1, \dots, \lambda_h$ be a partition of $d \geq 1$. Then $f : X \rightarrow Y$ is a Hurwitz cover if

- (a) f is holomorphic,
- (b) X is a connected compact genus h ,
- (c) Branch locus of f is $\{b_1, \dots, b_h\}$,
- (d) λ_i is the ramification product of b_i .

From here, we can say the Hurwitz Number is $H_{d:h \rightarrow g}(\lambda_1, \dots, \lambda_n) := \sum -[f] \frac{1}{|Aut(f)|}$. Now, a tropical cover of $\mathbb{R} \cup \{\pm\infty\}$. We need to send one valent vertices (ends) to $\pm\infty$. We can count these tropical covers via more discrete data. We fix μ and ν as partitions of $d \in \mathbb{Z}_{\geq 1}$. If our degree of a tropical curve is 4, we can get the left end has partition 4, our right end has partition (2, 2), the genus of our graph is 1. We let $r := 2g - 2 + l(\mu) + l(\nu) > 0$ (length of partition μ and ν), and we then fix $p_1, \dots, p_r \in \mathbb{R}$. Then the tropical Hurwitz number is $H_{d:g \rightarrow 0}^{trop}(\mu, \nu) : \sum_f m(f)$, where $M(f)$ is the multiplicity of f , defined as

$$\frac{1}{|Aut(f)|} \prod_{e \text{ a bounded edge}} w(e)$$

The automorphisms of our graph are either swap two vertices or swap to edges, so the automorphism group has size 4.

Example 4.6.1.

$$H_{4:1 \rightarrow 0}^{trop}((4), (2, 2)) = \left(\frac{1}{4} 2 * 2 * 4 \right) + 6 + 3 + 1 = 14$$

We get four total graph coverings.

We have the following correspondence: $H_{d:g \rightarrow 0}^{trop}(\mu, \nu) = H_{d:g \rightarrow 0}(\mu, \nu)$

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