

Exercise 1 (5.1.A Vakil). Show that \mathbb{P}_k^n is irreducible.

Answer

Exercise 2 (5.1.G Vakil). Show that affine schemes are quasi-separated. [Hint: 5.1.F Vakil]

To solve this exercise we will use the equivalent condition in the following lemma:

Lemma 1 (5.1.F Vakil). *Show that a scheme is quasi-separated if and only if the intersection of any two affine open subsets is a finite union of affine open subsets.*

Answer

To show that an affine scheme X is quasi-separated it suffices to show that the intersection of $U, V \subseteq X$, affine and open, is a finite union of affine and open subsets. As U, V are affine and open we have that they are isomorphic to Spec of something. This means that

$$U \simeq \text{Spec } A, \quad \text{and} \quad V \simeq \text{Spec } B$$

and as the distinguished open sets form a basis we have

$$U \subseteq \bigcup_{f \in A} D(f) \Rightarrow U \subseteq \bigcup_{i=1}^n D(f_i)$$

because of quasi-compactness. Similarly $V = \bigcup_{j=1}^m D(g_j)$ and this way we have

$$U \cap V = \bigcup_{i=1}^n \bigcup_{j=1}^m D(f_i) \cap D(g_j) = \bigcup_{i=1}^n \bigcup_{j=1}^m D(f_i g_j).$$

The double union in question is a finite union of affine open sets, we conclude that X is quasi-separated.

Exercise 3 (5.2.D Vakil). Show that $(k[x, y] / \langle y^2, xy \rangle)_x$ has no nonzero nilpotent elements. [Hint: Show that it is isomorphic to another ring, by considering the geometric picture. Exercise 3.2.L may give another hint.]

Show that the only point of $\text{Spec } k[x, y] / \langle y^2, xy \rangle$ with a non-reduced stalk is the origin.

Answer

Exercise 4 (5.2.I Vakil). Suppose X is an integral scheme. Then X (being irreducible) has a generic point η . Suppose $\text{Spec } A$ is any nonempty affine open subset of X . Show that $\mathcal{O}_{X,\eta}$ (the stalk of \mathcal{O}_X at η) is naturally identified with $K(A)$, the fraction field of A .

This is called the function field $K(X)$ of X . It can be computed on any nonempty open set of X , as any such open set contains the generic point. The elements of $K(X)$ are called rational functions on X (to be generalized further in Definition 6.5.35).

Answer

As η is generic, we may identify it with a prime ideal in A and so $\eta \in \text{Spec } A$. Now, as $\text{Spec } A$ is an open set, we have

$$\mathcal{O}_X(\text{Spec } A) = A$$

and as X is integral, A is an integral domain. This implies that A is reduced and therefore

$$\langle 0 \rangle = \text{Nil}(A) = \bigcap_{\mathfrak{p} \in \text{Spec } A} \mathfrak{p}.$$

We have that η is generic in X so it's generic in A . Taking closures inside A we get

$$\bar{\eta} = \{ \mathfrak{p} \in \text{Spec } A : \mathfrak{p} \supseteq \eta \} = \text{Spec } A.$$

This means that any prime ideal is a prime ideal containing η , thus

$$\eta \subseteq \bigcap_{\mathfrak{p} \in \text{Spec } A} \mathfrak{p} = \langle 0 \rangle$$

and so we conclude that

$$\mathcal{O}_{X,\eta} \simeq \mathcal{O}_{\text{Spec } A, \langle 0 \rangle} = (A \setminus \langle 0 \rangle)^{-1} A = K(A)$$

as we desired.