MATH601 — Advanced Combinatorics

Based on the lectures by Maria Gillespie

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Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This course will focus on the combinatorics of Young tableaux, crystal bases, root systems, Dynkin diagrams, and symmetric functions arising in representation theory of matrix groups and Lie algebras.

Requirements

Familiarity with the basics of group theory and symmetric functions is helpful.

Contents

Chapter 1

1.1 Day 1 | 20240819

We will start by reviewing the representation theory of finite groups and the Lie group and Lie algebra representations. The objective is to classify semi-simple Lie algebras and groups. This classification is quite combinatorial.

Review of representation theory of finite groups

Recall groups are sets G endowed with a binary operation $\circ$ such that

(a) There is an identity element e: $g \circ e = e \circ g = g$.

(b) Every element possesses an inverse. For each g, there is an h such that $g \circ h = e = h \circ g$.

(c) The operation $\circ$ is associative.

Example 1.1.1. The symmetric group is the set of permutations of $[n]$. We denote it $(S_n, \circ)$ where our operation is composition. We will use this group quite a lot.

Example 1.1.2. We will be working with $\mathrm{GL}_n(\mathbb{C})$ where $\mathbb{C}$ will come in as more useful than $\mathbb{R}$. The general linear group is characterized by the property that $\det(A) \neq 0$ for $A \in \mathrm{GL}_n(\mathbb{C})$.

Example 1.1.3. Given two groups we can construct $G \times H$ by doing operations pointwise. We can also take subgroups and quotient groups.

Example 1.1.4. Take the special linear group $\mathrm{SL}_n(\mathbb{C})$ which is the set of matrices A with $\det(A) = 1$. This is a subgroup of $\mathrm{GL}_n(\mathbb{C})$.

There's a lot more of matrix groups such as $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$ and unitary groups $\mathrm{SU}_n(\mathbb{C})$.

Groups which are representations of themselves

Symmetry groups are groups of linear transformations of $\mathbb{C}^n$ (some Euclidean space) that fix some shape. Any such group is a subgroup of $\mathrm{GL}_n(\mathbb{C})$. Matrices here don't collapse points nor anything.

Example 1.1.5. The symmetry group of a diamond in the plane can be found by analyzing the symmetries of the figure.HMMM The group in question is the Klein-4 group which can be seen as

$$\{\, \mathrm{id}, r_x, r_y, r_x r_y \,\}.$$

Similarly we can see it as

$$\{\, \mathrm{id}, (24), (13), (13)(24) \,\}$$

Fell asleep

1.2 Day 2 | 20240821

We were looking at direct sums of representations. Recall representations are maps which take group elements to matrices.

$$\rho \oplus \sigma : G \to \mathrm{GL}_{n+m}(\mathbb{C})$$

and this map will send g to a block matrix. A central question in representation theory is to classify the irreducible representations of some object. This is a central question because for finite groups, irreducible is the same as indecomposable.

Definition 1.2.1. A representation is indecomposable when it can't be written as a direct sum of smaller representations.

Irreducible means that it has no non-trivial proper representations. This is analogous to the idea of prime and irreducible numbers. In the most general case where groups may be infinite, irreducible implies indecomposable.

Alternative definitions for representations

We may define it as a vector space V with an action $G \times V \to V$ so that

$$g(hv) = (gh)v$$

and it should be a linear action in the sense that $v \mapsto gv$ is a linear transformation.

This is equivalent to the previous definition because V can be seen as $\mathbb{C}^n$. So the definition gives rise to a map

$$G \to \mathrm{Aut}(V),\ g \mapsto g\cdot$$

Even more *objecty* is the next definition. We can see a representation as a module over a group ring $\mathbb{C}G$. This set is made up of formal linear combinations of elements of G.

We endow it with a module structure, for any element $g \in G$ in particular in $\mathbb{C}G$ we can make it a coefficient $gv \in V$ as a $\mathbb{C}G$-module.

Subrepresentations

Now that we have all the algebraic structure we can use it to define subrepresentations. Because a subrepresentation will be a subspace which inherits the action for example.

Definition 1.2.2. $W \subseteq V$ is a subrepresentation of G (when V represents G) if

- W is a subspace of V, and
- W is G-invariant in the sense that the image of $G \times W \to V$ is contained in W.

We will also say that V is irreducible if there's no proper nonzero subrepresentation $W \subseteq V$.

Sometimes it is possible to decompose a representation into a direct sum of subrepresentations.

fell asleep

Definition 1.2.3. A character of a representation is the trace map $g \mapsto \mathrm{tr}(\rho(g))$.

Properties

(a) $\chi_{V \oplus W} = \chi_V + \chi_W$.

(b) $\chi_{V \otimes W} = \chi_V \chi_W$.

(c) χ_V uniquely determines the representation.

1.3 Day 3 | 20240823

Lie groups

Definition 1.3.1. A Lie group is a real smooth manifold G with a group structure such that

$$(g, h) \mapsto gh^{-1}$$

is differentiable.

A manifold is a set such that around each point there's a local neighborhood that's topologically equivalent to $\mathbb{R}^n$. Elliptic curves are examples of manifolds.

Definition 1.3.2. An algebraic group is an algebraic variety with a group structure. In this case the multiplication map should be algebraic.

In certain specializations these two are the same object. In the case of complex Lie groups, we talk about smooth complex manifolds.

Example 1.3.3.

- $(\mathbb{C}^n, +)$ is a Lie group. But it's not compact. sleepy sleepy
- GL_n

Lemma 1.3.4. *(Zariski-)Closed subgroups of a Lie group are also Lie groups.*

Example 1.3.5. In particular B_n, the set of upper triangular matrices in GL_n, forms a Lie group. The torus T_n, the group of diagonal matrices, is also a Lie group.

It is called the torus because it's isomorphic to $(\mathbb{C}\backslash 0)^n$ and $\mathbb{C}\backslash 0$ looks like a circle while $(\mathbb{C}\backslash 0)^2$ is the product of two circles which is the torus.

The Classical Groups

The special linear group SL_n consists of matrices whose determinant is 1. The classical groups are called clasiccal because they have very nice properties. In particular type A is what we call SL_n.

To talk about the special orthogonal group SO_n we should first fix a symmetric bilinear form $(\cdot, \cdot)$ which is positive-definite. The orthogonal group O_n consists of matrices which preserve this form. The special orthogonal group in particular is the subgroup of matrices with determinant 1.

Remark 1.3.6. Over $\mathbb{R}$, O_n is actually the group of rigid transformations which is generated by reflections and rotations. For SO_n, it's only the rotations group.

We can also alternatively define O_n as

$$\{ A : A^\mathsf{T} A = I \}$$

because

$$\langle Av | Aw \rangle = \langle v | w \rangle$$

and from this

$$v^\mathsf{T} A^\mathsf{T} A w = v^\mathsf{T} w.$$

Comparing entry by entry we get the desired property.

It's also a fact that O_n is disconnected, one component is SO_n and the other is the set of matrices with determinant -1. Finally type B means SO_{odd} while D means $\mathrm{SO}_{\text{even}}$. The type C groups are the symplectic groups.

1.4 Day 4 | 20240826

Continuing on with the classical groups, we will be talking about the Symplectic group of even dimension. We will be fixing a symplectic form which is a non-degenerate, skew-symmetric, bilinear form.

Example 1.4.1. The dot product is not symplectic because it's symmetric.

Example 1.4.2. Consider the form

$$v_1 w_{2n} + v_2 w_{2n-1} + \cdots + v_n w_{n+1} - v_{n-1} w_n - v_{n+1} w_n - v_{n+2} w_{n-1} - \cdots - v_{2n} w_1.$$

If Ω is such a matrix of a form, for example when $2n = 6$ we have

$$\Omega := \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & -1 & & & \\ & -1 & & & & \\ -1 & & & & & \end{pmatrix} \Rightarrow (v, w) = v^\mathsf{T} \Omega w$$

From this our first definition of the symplectic group is matrices which preserve this product.

Definition 1.4.3. The symplectic group Sp_{2n} is

$$\{ M : (Mv, Mw) = (v, w) \}$$

or equivalently

$$\{ M : M^\mathsf{T} \Omega M = \Omega \}.$$

We will simplify the notation to type C.

Representation of Lie groups

Definition 1.4.4. A representation of a Lie group is a map which is also differentiable and a group homomorphism.

1.5 Day 5 | 20240828

For a partition $\lambda \vdash n$, we call $S^\lambda V$

$$\Lambda^{\mu_1} V \otimes \Lambda^{\mu_2} V \otimes \cdots \otimes \Lambda^{\mu_k} V$$

where μ is the conjugate partition.

Example 1.5.1. For example if $\lambda = (5,4,1)$, then $\mu = (3,2,2,2,1)$ and so

$$S^{(5,4,1)} V = \Lambda^3$$

Elements can be written as a filling to the Young diagram. Such an element could be

$$(v_1 \wedge v_2 \wedge v_3) \otimes (a \wedge b) \otimes (c \wedge d) \otimes (x \wedge y) \otimes z$$

and filling the diagram we have

$$\begin{array}{|c|c|c|c|c|} \cline{1-1} r \\ \cline{1-4} q & b & d & y \\ \hline p & a & c & x & z \\ \hline \end{array}.$$

It's important to familiarize ourselves with this idea so we will itechangebly talk about

$$(e_1 \wedge e_4 \wedge e_3) \otimes (e_1 \wedge e_2) \otimes (e_5 \wedge e_3) \otimes (e_2 \wedge e_1) \otimes e_2$$

and

$$\begin{array}{|c|c|c|c|c|} \cline{1-1} 3 \\ \cline{1-4} 4 & 2 & 3 & 1 \\ \hline 1 & 1 & 5 & 2 & 2 \\ \hline \end{array} = - \begin{array}{|c|c|c|c|c|} \cline{1-1} 4 \\ \cline{1-4} 3 & 2 & 3 & 1 \\ \hline 1 & 1 & 5 & 2 & 2 \\ \hline \end{array}$$

The tableau $\begin{array}{|c|c|} \cline{1-1} 1 \\ \hline 1 & 2 \\ \hline \end{array}$ is zero for example.

For a basis of S^λ, we can talk about it being spanned by elementary tableau where we order each column from least to greatest. These are called <u>column-strict tableau</u>. For example

$$\begin{array}{|c|c|c|c|} \cline{1-1} 6 \\ \cline{1-3} 4 & 5 & 3 \\ \hline 1 & 2 & 1 & 2 \\ \hline \end{array}$$

If V is an n-dimensional vector space, then we have a largest element on our basis. This allows us to formulate the question:

How many column strict tableau are there with largest entry n? And shape λ.

From this

$$\binom{n}{\mu_1}\binom{n}{\mu_2}\cdots\binom{n}{\mu_k} = S^\lambda V.$$

Definition 1.5.2. The Schur module V^λ is

$$V^\lambda = S^\lambda \Big/ \left\langle v_T - \sum_S v_S \right\rangle$$

where the sum is over S's obtained from T by

(a) Choose two columns of C_1, C_2 of T.

(b) Choose k elements from C_2.

(c) Exchange them with k elements from C_1 in all ways that preserve the order of the elements.

Example 1.5.3. Take $(4,3,3)$ with the filling

$$\begin{array}{|c|c|c|c|} \cline{1-3} 5 & 7 & 6 \\ \cline{1-3} 2 & 4 & 4 \\ \hline 1 & 1 & 3 & 4 \\ \hline \end{array}$$

so choose the first and third columns as C_1 and C_2. One relation in V_λ

Theorem 1.5.4. *The collection*

$$\{\, e_T : \ T \textit{semistandard} \operatorname{sh}(T) \vdash n \,\}$$

is a basis for the Schur module.

1.6 Day 6 | 20240830

Last time we defined the Schur modules. These are

$$S^\lambda V = \Lambda^{\mu_1} V \otimes \cdots \otimes \Lambda^{\mu_r}$$

where $\mu = \lambda^*$ is the conjugate or transpose. Now V^λ is S^λ modded out by column exchanges. We will show that

$$\{\, e_T : T \in SSYT(\lambda), \text{largest entry} \leqslant n \,\}$$

is a basis for V^λ.

Example 1.6.1. Consider the tableau

$$\begin{array}{|c|c|c|}\cline{1-1} 6 \\ \cline{1-2} 5 & 3 \\ \hline 2 & 1 & 4 \\ \hline\end{array}$$

the second and third row are wrongfully ordered

Sleepy sleepy

We wil show that they are independent in the quotient.

Example 1.6.2. The idea for why D_T's are independent. We can find lex orderings and make D_T have nice leading term and then an ordering on the leading terms. E.g. $1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3$ are independent because the leading terms are all distinct.

In $V^{\begin{array}{|c|c|}\cline{1-1} a \\ \hline b & c \\ \hline\end{array}}$ we have

$$D_{\begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 \\ \hline\end{array}} = \det\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22}\end{pmatrix} z_{11} = \dots$$

And

$$D_{\begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 2 \\ \hline\end{array}} = z_{12} \det$$

In the monomials $z_{11}^2 z_{22}$ is larger than $z_{11}z_{12}z_{22}$ and that's how we show that they're independent of each other. This shows the elementary symmetric functions are independent.

One exciting conclusion to look at it's characters. For a Lie group the right notion is to consider H a maximal torus in a Lie group G. This is the maximal connecated, abelian Lie sub group.

Example 1.6.3. $T_n \subseteq \mathrm{GL}_n$ in this case $\chi_V : H \to \mathbb{C}$ where $h \mapsto \mathrm{tr}(h \text{ acts on } V)$. This χ_V determines V and has nice properties with direct sum and tensor products.

$$\chi_V \operatorname{diag}(x_1, \ldots, x_n)$$

is the trace of that matrix acting on V^λ. It suffices to look at a basis. For a given e_T where T is a SSYT, X acts on each e_i by doing $x_i e_i$. see

$$\begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} = x_1x_1x_2x_2x_3x_3x_4 \cdot \begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

now the trace is the sum of the eigenvalues and this is x^T. So

$$\sum_{T SSYT} x^T = s_\lambda(\underline{x}).$$

1.7 Day 7 | 20240904

Theorem 1.7.1. *A representation of* GL_n *is irreducible if and only if it has a unique highest weight vector.*

Definition 1.7.2. A <u>weight vector</u> of V is $v \in V$ such that for $x \in T_n$ (the torus),

$$x \cdot v = x_1^{\alpha_1} \ldots x_n^{\alpha_n} v$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ is weight.

Recall that being in the torus meant $x = \begin{pmatrix} x_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & x_n \end{pmatrix}$.

Definition 1.7.3. A <u>highest weight vector</u> is a weight vector such that

$$B_n \cdot v = \mathbb{C}^* \cdot v$$

where B_n is the Borel matrices compromised of upper triangular matrices.

A representation is a sum of its weights: $V = \bigoplus_\alpha V_\alpha$ where $V_\alpha = \{ v : x \cdot dotv = x^\alpha v \}$.

Lemma 1.7.4. *The only highest weight vector in* V^λ *is* e_{T_0} *where* T_E

1.8 Day n | 20240930

Combinatorics of $\mathfrak{sl}_3$ representations

Our goal for today is to see how all irreducible $\mathfrak{sl}_3$ representations live in $(V^{(1,0)})^{\otimes n}$. We would like to describe them. Recall $V^{(1,0)}$ means that we have $1L_1$ and no L_2.

As a shorthand we will say

$$F_1 = F_{12}, F_2 = F_{23}, E_1 = E_{12}, \dots$$

Definition 1.8.1. The word $a_1 \dots a_n \in \{1, 2, 3\}^n$ represents the weight space corresponding to the L-diagram calculation in $(V^{(1,0)})^{\otimes n}$ corresponding to $a_1 \otimes \cdots \otimes a_n$.

Lemma 1.8.2. *The weight of $(a_1 \dots a_n)$ is $(\#1's, \#2's, \#3's)$.*

Proof

By induction on n, the base case is a diagram. ASK FOR DIAGRAM,. Then the induction step wishes to show that something is additivw across $\otimes$. So recall, ¿why are weights additive across $\otimes$? Let v_α, v_β be weight vectors. We want to show $\alpha, \beta \in \eta^* = \{H \to \mathbb{C}\}$. Then finish

Corollary 1.8.3. *The highest weight words in $(V^{(1,0)})^{\otimes n}$ are $a_1 \dots a_n$ such that very suffix has $(\#1's) \geqslant (\#2's) \geqslant (\#3's)$.*

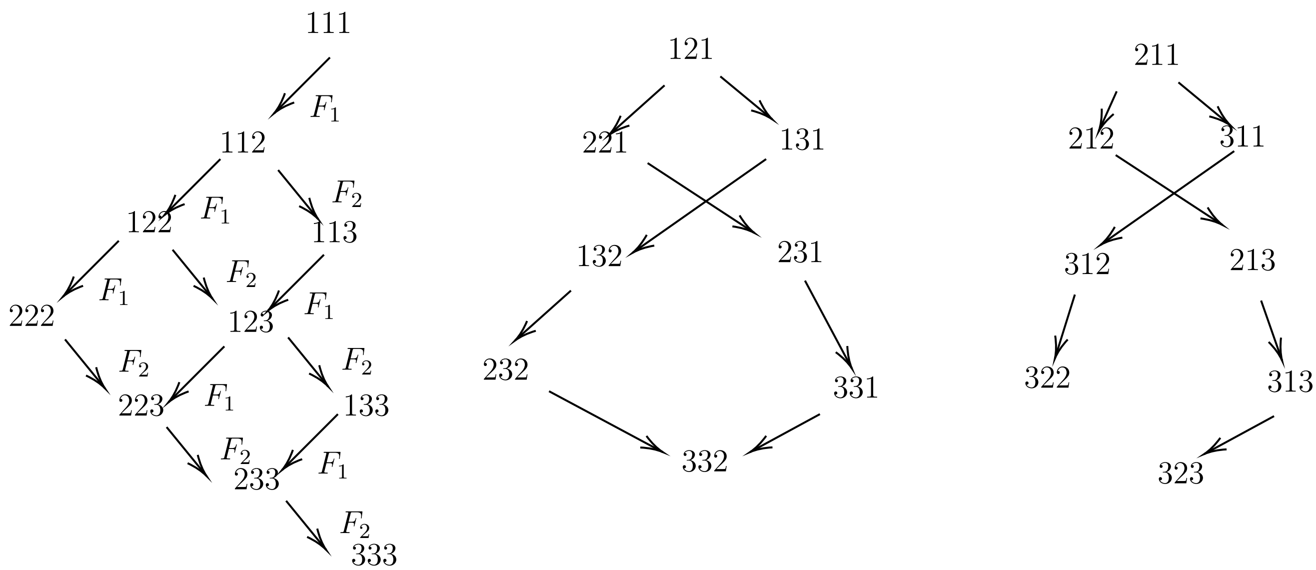
The proof is basically noting that E_1, E_2 map the word to zero when it's of highest weight.

Example 1.8.4. $(V^{(1,0)})^{\otimes 3}$ so first we find the heighest weight words: $111, 121, 211$ and 321. So this first one gives us an adjoint representation, the next one also gives us another adjoint. So this last one for 111 gives us the last 10 elements. The 321 gives us the last one. From this the representation decomposes as

$$V^{(3,0)} \oplus (V^{(2,1)})^(\oplus 2) \oplus V^{(0,0)}.$$

They all have the same weight but are independent one-dimensional weight spaces. Each word on the 111 diagram has a different weight. How many words have weight $(1, 2)$, its $122, 212$ and 221 so the dimesion of that space is $3 = \binom{3}{2} = \binom{3}{1}$.

Question. In $(V^{(1,0)})^{\otimes n}$, ¿what is the dimension of the weight space $\alpha = (a, b, c) = (a - c, b - c)$? It's however many words of length n have a $1'$s, b $2'$s and c $3'$s. This is $\binom{n}{a,b,c} = \frac{n!}{a!b!c!}$, which is counted by taking all the words and then dividing by possible rearrangements. This gives us something with the dots in the diagram.



Recall from when we talked about RSK insertion: It is compatible with $\mathfrak{sl}_2$ crystal operations on tableau reading word. In other words this is, if $\underline{a} \xrightarrow{F_i} \underline{b}$ then

- The RSK insertion tableau of $\underline{a}, \underline{b}$ matches.
- $\mathrm{rw}(\mathrm{ins}(\underline{a})) \xrightarrow{F_i} \mathrm{rw}(\mathrm{ins}(\underline{b}))$.

So in conclusion, each connected component (irreducible representation) in $(V^{(1,0)})^{\otimes n}$ corresponds to a recording tableau. Let's see how this works:

Example 1.8.5. If we take the RSK insertion of the diagram we get diagram. The bumping sequence is all the same! What that means is that we can take the reading word and apply F_1. So all the stuff we did on crystals in 502 is coming back.

If we take the other one for 211, the RSK insetion is $\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}$ but the recording tableau is $\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$ so we're gonna count how many times an irreducible representation shows up by counting tableau.

1.9 Day n+1 | 20241002

Recall $(V^{(1,0)})^{\otimes n}$ is described by words of $1, 2, 3$ of length n with F_1, F_2 bracketing rules. E_1, E_2 also have bracketing rules. Consider the word

$$1223112133212 \rightarrow)((?))()??()($$

so E_1 changes the leftmost unpaired 2 to a 1 which leaves us with

$$1223112133211 \to)((?))()??()).$$

The highest weight is the ballot word which when read right to left has more $\#i$'s than $\#i+1$'s.

Example 1.9.1. We have $(V^{(1,0)})^{\otimes 4}$ with dimension $3^4 = 81$. The highest weight words, or ballot[1] words are

$$1111, 1121, 1211, 1321, 2111, 2121, 2211, 3121, 3211$$

corresponding to

$$V^{(4,0)}, V^{(3,1)}, V^{(3,1)}, V^{(2,1,1)} = V^{(1,0)}, V^{(3,1)}, V^{(2,2)}, V^{(2,2)}, V^{(1,0)}, V^{(1,0)}.$$

So this is

$$V^{(4,0)} \oplus (V^{(3,1)})^{\oplus 3} \oplus (V^{(2,2)})^{\oplus 2} \oplus (V^{(1,0)})^{\oplus 3}.$$

Recall that $V^{(2,1,1)} = V^{(1,0)}$ because $L_1 + L_2 + L_3 = 0$ and we can quotient out by $(1,1,1)$.

Let's recall some crystal/Young tableaux facts:

(a) RSK recording tableau is unchanged via F_1, F_2.

(b) Any highest weight word has RSK insertion tableau that looks like

$$\begin{array}{|c|c|cc}\cline{1-2} 3 & 3 & & \\ \cline{1-3} 2 & 2 & \multicolumn{1}{c|}{2} & \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

where all i's are in row i from the bottom.

This facts imply that two connected components of $(V^{(1,0)})^{\otimes n}$ crystal have different recording tableau as RSK is a bijection.

Example 1.9.2. We have that

$$3121 \xrightarrow{RSK} [[3[]]]$$

[1] Yamanouchi or reverse ballot

1.10 Day n+2 | 20241004

Characters of $\mathfrak{sl}_3$ representations

The last time we talked about

$$\chi(V) = \sum_{\alpha \in \Lambda} \dim(V_\alpha) x^\alpha.$$

There's this fact which is the *highest weight theorem* which states:

Theorem 1.10.1. *V is determined by $\chi(V)$.*

The idea of this for $\mathfrak{sl}_3$ is that we had the hexagonal lattice which was generated by a unique highest weight element.

So given $\mathrm{ch}(V)$, let x^α appear in $\mathrm{ch}(V)$ where α is a highest weight, subtract $\mathrm{ch}(V^\alpha)$ and iterate.

With the notation

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$$

we ask what is the character of $V^{(a,b)}$?

Example 1.10.2. We've written $V^{(1,0)}$ as

FIGURE

Summing up the monomials we can see that this is

$$\chi(V^{(2,1)}) = s_{(2,1)} \bmod x_1 x_2 x_3.$$

Proposition 1.10.3. *The character of $V^{(a,b)}$ is*

$$\chi(V^{(a,b)}) = s_{(a,b)}(x_1, x_2, x_3) \bmod x_1 x_2 x_3.$$

This isn't immediately obvious. We have a bunch of words and insert them, but we need to see that every SSYT occurs.

Proof

Recall

$$s_{(a,b)}(x_1, x_2, x_3) = \sum_{\substack{T \in SSYT(a,b) \\ 1,2,3}} x^T.$$

We claim that every SSYT T is obtained from a sequence of F_1, F_2's applied to

$$\begin{array}{|c|c|c|c|c|}\cline{1-3} 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

We ought to show we can get back to it with $E's$. Let's do this more generally for $\mathfrak{sl}_n$. If T is not highest weight, we will show that we can apply a raising operator. Let r be the lowest row such that T doesn't have all r's in that row. Say $r = 3$ then in

$$\begin{array}{|c|c|c|c|c|}\cline{1-2} 4 & 5 \\ \cline{1-4} 3 & 3 & 4 & 5 \\ \cline{1-4} 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

we take the rightmost element i of row r. This means that $i > r$. This is, in this case $i = 5$. We may now apply E_{i-1} which is well defined because in row r, there's no $i-1$ below it. Here $i - 1 \geqslant r$ for all rows below a $r-1$. So bracketing $i, i-1$ there is an upaired i to which we may apply E_i.insertion sleep

This first Littlewood Richardson rule is concatenate reading words like in $\mathfrak{sl}_2$. The coefficient is

$$c^{\nu}_{\lambda\mu} = \#\text{pairs}(T,S)\text{of shapes } \lambda, mu(unfinished)$$

The third LR rule is in terms of skew tableau. The LR coefficient is then # of skew SSYT of shape ν/λ and content μ with a ballot reading word. Proving this is harder but it's not so bad when we think of a skew crystal. It's not obvious what's going on. Recall the Hall inner product which gives us

$$\left\langle s_\mu s_\lambda \middle| s_\nu \right\rangle = c^{\nu}_{\lambda\mu} = \left\langle s_\mu \middle| s_{\nu/\lambda} \right\rangle.$$

1.11 Day n+3 | 20241007

Representations of $\mathfrak{sl}_n$

There's no steps of thinking to this. It's the same as $\mathfrak{sl}_3$. Recall $\mathfrak{sl}_n$ corresponds to $n \times n$ matrices which are traceless. The Cartan subalgebra of $\mathfrak{sl}_n$ is

$$\mathfrak{h} = \{\, \text{diagonal matrices } M,\ \operatorname{tr} M = 0 \,\}.$$

Then elements here are $H = \operatorname{diag}(x_1, \ldots, x_n)$ such that $\sum x_i = 0$. To define weights, these live in $\mathfrak{h}^*$ and they are joint eigenvalues of H in a representation V of $\mathfrak{sl}_n$. In other words, $\alpha \in \mathfrak{h}^*$ with $v_\alpha \in V$ such that

$$Hv_\alpha = \alpha(H)v_\alpha, \quad \text{for} \quad H \in \mathfrak{h}.$$

Example 1.11.1. If $\alpha \operatorname{diag}(x_1, \ldots, x_n) = x_1$ then this is L_1. L_i is such that $L_i \operatorname{diag}(\underline{x}) = x_i$.

The weight space is $L_1 + \cdots + L_n = 0$. What are all the weights which are valid for representations? The weight lattice is $\langle L_1 \ldots, L_n \rangle$ and all weights lie on the lattice. The proof proceeds the same way, taking block copies of $\mathfrak{sl}_2$.

Adjoint representation of $\mathfrak{sl}_n$

In $\mathfrak{sl}_3$, the adjoint representation had dimension 8.

Crystals for $\mathfrak{sl}_n$

$(1, 0, \ldots, 0) = L_1$

Theorem 1.11.2. $(V^{L_1})^{\otimes m} = \bigoplus c_\lambda V^\lambda$ *where* $c_\lambda = \#SYT(\lambda)$

The proof is done using crystals and RSK, every connected component corresponds to a unique recording tableau.

1.12 Day n+4 | 20241009

$\mathfrak{sl}_n$ crystals and Stembridge axioms

Consider the word

$$21132342132$$

we claim that the operations F_1 and F_3 commute in $\mathfrak{sl}_n$. Observe that F_1 brackets $(1, 2)$ and F_3 brackets $(3, 4)$. We thus get:

$$F_1(21132342132) = 21232342132$$
$$F_3(21132342132) = 21132442132$$

and then applying F_3, F_1 respectively we get the same word

$$21232442132.$$

Sometimes F_1F_3 is not defined so we actually mean that

$$F_1F_3(x) = y \Rightarrow F_3F_1(x) = y$$

even when $y = 0$. We have a lot of commuting squares then! We may generalize to

Theorem 1.12.1. *F_i, F_j commute when $|i-j| > 1$.*

This characterizes the crystal graphs but not completely, that's where the Stembridge axioms come into play.

Theorem 1.12.2. *In an $\mathfrak{sl}_n$ crystal, for $F_1(w), F_2(w)$ either:*

(a) $\exists z(z = F_1F_2(w) = F_2F_1(w))$*, or*

(b) $\exists u, v, s, t, z$ *such that the crystal is just like the adjoint representation. This means that*

$$F_2F_1F_1F_2 = F_1F_2F_2F_1.$$

¿Why can't we have more complicated words than that? The proof for this will be very combinatorial and we will analyze a lot of words.

Proof

Let a be the rightmost unpaired 1 in w for $(1, 2)$. Then call b the rightmost unpaired 2 in w for $(2, 3)$.

- In the first case, we assume that removing b does not unbracket a 1. Take for examplesleep
- In the next case $F_1F_1F_2$
- The third case is basically removing $b = 2$ unpairs $c = 1$, where b is left of a and there exists and unpaired 3 to the left of a. This is

$$-2-1-(3)-1-$$

so applying F_1 gets us $-2132-$ and F_2 leaves us with $-3131-$. This both meet up after applying F_2 and F_1 respectively.

- Finally it could be like before but there's no unpaired 3 left of a. Here we get another figure 8 diagram. For this case we get

This all comes from Stembridge's 2004 paper.

1.13 Day n+5 | 20241011

We were talking about how the figure 8 and sqaure characterizes stuff. We will now characterize crystals as graphs.

Crystals of type A

Definition 1.13.1. A type A Kashiwara crystal of finite type is

- ◇ B a crystal base (weight spaces' generators are bases), nonempty set.
- ◇ Two arrow maps $e_i, f_i : B \to B \cup \{ emptyset \}$ for $i \in [n-1]$.
- ◇ Two length maps $\varepsilon_i, \varphi_i : B \to \mathbb{Z}$ which correspond to the number of times you can apply e_i and f_i.
- ◇ And a weight function to thw lattice $\mathrm{wt} : B \to \Lambda$ where $\Lambda\mathbb{Z}\langle L_1, \dots, L_n \rangle$ and $\sum L_i = 0$. Elements here are $\underline{a}/\underline{1}$, i.e. modded by translations.

They must satisfy the properties

(a) If $x, y \in B$, then

$$e_i(X) = y \iff x = f_i(y)$$

and in this case

$$\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i$$

where $\alpha_i = (0, 0, \dots, 1, -1, 0, \dots, 0)$ with 1 in position i and -1 in the next one.

And also

$$\varphi_i(y) = \varphi_i(x) + 1.$$

(b) For all i, x_i

$$\varphi_i(x) - \varepsilon_i(x) = \mathrm{wt}(x)_i - \mathrm{wt}(x)_{i+1}.$$

To state this for general crystals we need an inner product.

Exercise 1.13.2. Analyze how tableaux crystals fit into this picture.

Example 1.13.3. In $n = 2$, we had $\mathfrak{sl}_2$ but now we can make an infinite chain. Let us take our basis

$$B = \{ v_0, v_{-2}, v_{-4}, \dots \}$$

where the f arrow

1.14 Day n+6 | 20241014

Recall the Stembridge axioms, $(B, e_i, f_i, \varepsilon_i, \varphi_i, \mathrm{wt})$.

(a) seminormality

(b) length axioms

(c) diamond shape

(d) figure 8

So let us finish the proof

Theorem 1.14.1. *Connecteed Stembridge crystals are crystals whose graph is connected. They have a unique highest weight element (unique maximal, all e_i's send it to zero).*

Proof

Let x be a highest weight element, assume for contradiction it's not unique. Let us define

$$S = \{\, z_i : \exists f_{i_1}, \dots, f_{i_k},\ f_{i_1} \circ \cdots \circ f_{i_k} x = z \,\} = \{\, \text{pts. below } x \,\}.$$

There could be some other stuff not below x, outside S. Let us take $z \in S$ be maximal (in the crystal) such that z is covered by some y not in S. This is the highest z which has something from the outside pointing to it.
We have that $z \neq x$ as x is maximal (highest weight) by assumption (otherwise $y \to x$, contradiction). Let $z' \in S$, $f_i(z') = z$ for some i.
If $S2'$ applies, then there's a w

$$w \xrightarrow{f_j} z', \quad w \xrightarrow{f_i} y,$$

because z was maximal, $w \in S$.

(a) $S1$ implies that if $f_i(x) = y$

(b) sleep

Theorem 1.14.2. *If two connected Stembridge crystals C, C' have highest weight elements x, x' and weight $\mathrm{wt}(x) = \mathrm{wt}(x')$ then $C \simeq C'$. This means that we get the same graph, weight, φ_i, ε_i values.*

This theorem is proved in Shilling and Bald's book. The idea starts with the weight function, axiom $K2$ relates weight with ε_i, φ_i. Then by induction each successive row starting from the top is determined. (Row means how many f's you have to apply to get there. Can't have non-graded posets) This is uniquely determined via $S1, S2$ and $S3$.

We can now connect this with Lie algebras.

Corollary 1.14.3. *Every Stembridge crystal is isomorphic to a crystal of tableaux.*

Proof

Tableaux crystals satisfy K_1, K_2 and $S0$ through $S3'$. And every highest weight appears as the highest weight of some tableau crystal. So by the previous theorem, we are done.

Remark 1.14.4. It is hard to find local axioms to globally determine the structure of the crystal. Recall that we know that the character of a Stembridge crystal $\chi(V^\lambda) = s_\lambda$ (irreducible representation of $\mathfrak{sl}_n$), then

$$\chi(C) = \sum_{b \in C} x^{\mathrm{wt}(b)}$$

sum is over dots of the crystal. So

$$\chi(\text{tableau crystal } C^\lambda) = s_\lambda.$$

Corollary 1.14.5. *Say we have a symmetric function $f = \sum c_\lambda s_\lambda$. If we want to show $c_\lambda \in \mathbb{Z}_{\geqslant 0}$, then if the symmetric function is of the form $f = \sum x^P$, then the method is to try to define raising and lowering operators such that Stembridge's axioms are satisfied.*

This method has been done a few times now. Stanley symmetric functions and ordered symmetric functions are some of the examples where this can be used.

Beyond type A: General Lie algebras and Root systems

Recall $\mathfrak{g}$ is a $\mathbb{C}$-vector space with a Lie bracket. Any Lie algebra has a system of roots.

Definition 1.14.6. The roots of $\mathfrak{g}$ are weights of the adjoint representation acting on itself via the Lie bracket.

The Cartan subalgebra $\mathfrak{h}$ is the maximal Abelian Lie subalgebra.

Definition 1.14.7. We say $\mathfrak{g}$ is semisimple if its roots span $\mathfrak{h}^*$.

Example 1.14.8. Recall $\mathfrak{gl}_n$ is the whole set of $n \times n$ matrices. The roots of $\mathfrak{gl}_n$ are $L_i \operatorname{diag}(\underline{x}) = x_i$, $L_i - L_j$ for $i \neq j$. This is the same as $\mathfrak{sl}_n$, but the weight space is $\langle L_1, \dots, L_n \rangle$ which is not $\langle L_1, \dots, L_n \rangle / \sum L_i = 0$ which is $\mathfrak{sl}_n$'s weight space.

The goal is to classify semisimple Lie algebras by classifying sets of roots. We will use certain facts about Lie algebras:

Proposition 1.14.9. *For a semisimple Lie algebra $\mathfrak{g}$ the following hold*:

(a) $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ *where $\mathfrak{g}_\alpha$ is 1 dimensional. Say for example $\mathfrak{g}_{L_1 - L_2} = \langle E_{12} \rangle$.*

(b) *Little $\mathfrak{sl}_2$'s for each root. $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \hookrightarrow \mathfrak{g}$. In our case this correspond to E, F and H.*

1.15 Day n+7 | 20241016

General Lie algebra representation theoru and root systems

Recall that the adjoint representation of a Lie algebra decomposes as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where R is a set of roots which are non-zero weights of $\mathrm{Ad}(\mathfrak{g})$. We have the properties

- Each $\mathfrak{g}_\alpha$ is 1 dimensional.
- For $\alpha \in R$ we also have $-\alpha \in R$.
- We will have little $\mathfrak{sl}_2$:

$$(\mathfrak{sl}_2)_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \,.$$

- In $\mathfrak{h}$ choose a hyperplane, positive roots and simple roots (roots which are not a positive sum of other positive roots.)
- Positive simple roots are in correspondence with raising operators $E_\alpha,\ \alpha \in R^+$.

From this highest weight is such that it is killed by all the raising operator.

Definition 1.15.1. A vector $v \in V$ is of highest weight if $E_\alpha v = 0$ for all $\alpha \in R^+$.

Proposition 1.15.2. *Irreducible representations of any semisimple $\mathfrak{g}$ have a unique highest weight vector, one for each highest weight V^β is the irreducible representation with highest weight*

1.16 Day n+8 | 20241018

Root systems

Recall a root system in an inner product space is a finite set R with

(a) $r_\alpha(\beta) \in R$ for all α, β where

$$r_\alpha(\beta) = \beta - \frac{2\langle\beta|al\rangle}{\langle\alpha|\alpha\rangle}\alpha.$$

(b) The coefficient $\frac{2\langle\beta|al\rangle}{\langle\alpha|\alpha\rangle}\alpha$ is an integer.

(c) If $\beta = \lambda\alpha$ then $\lambda = \pm 1$.

Example 1.16.1. All the root systems that fit in two dimensions are

(a) For $\mathfrak{sl}_n$: $R = \{ L_i - L_j \}$ and positive roots $R^+ = \{ L_i - L_j : i < j \}$ and the simple roots $R_0^+ = \{ L_i - L_{i+1} : i \in [n-1] \}$.

(b) For $\mathfrak{sl}_3$, we have $L_1 - L_2$ and $L_2 - L_3$.

Exercise 1.16.2. Draw the root system of $\mathfrak{sl}_4$.

Example 1.16.3. For type sleep

1.17 Day n+9 | 20241021

CLassifying root systems

Last time we connected root systems with Dynkin diagrams. Let's prove some lemmas:

Lemma 1.17.1. *In a root system R, if α, β are two roots, their configuration (if $|\alpha| = 1$) is one of the following: Add figure*

Proof

By definition of root system, we have

$$\frac{2\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} = \frac{2|\alpha||\beta|\cos(\theta)}{|\beta|^2} = \frac{2\cos(\theta)|\alpha|}{\beta} \in \mathbb{Z}$$

but also

$$\frac{2\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle} = \frac{2\cos(\theta)|\beta|}{|\alpha|} \in \mathbb{Z}.$$

If we normalize by $|\alpha|$ then in particular we have

$$2|\beta|\cos(\theta)\cdot\frac{2\cos(\theta)}{|\beta|}\in\mathbb{Z}$$

there's not that many values of θ such that $4\cos^2(\theta)\in\mathbb{Z}$. As $\cos^2(\theta)\leqslant 1$, our expression is smaller than 4. Our possibilities are

$$\cos^2(\theta)=0,\frac{1}{4},\frac{2}{4},\frac{3}{4},1\Rightarrow\cos(\theta)=0,\pm\frac{1}{2},\pm\frac{1}{\sqrt{2}},\pm\frac{\sqrt{3}}{2},\pm 1.$$

This gives us the angles

$$0,\pi,\pi/6,5\pi/6,\pi/4,3\pi/4,\pi/3,2\pi/3,\pi/2$$

and for the lengths, take for example $\theta=\pi/6$, then

$$\frac{\sqrt{3}}{|\beta|},\quad \sqrt{3}|\beta|\in\mathbb{Z}\Rightarrow|\beta|=\sqrt{3}.$$

It really limits the possibilities, so there's only so many finite semisimple Lie algebras. An important corollary of this is:

Corollary 1.17.2. *If α,β are simple roots, then their angle θ is one of*

$$\pi/2,2\pi/3,3\pi/4,5\pi/6,$$

which means that only obtuse angles occur, or in other words, simple roots can't form acute angles.

Proof

Recall that simple roots are positive roots which are not sums of other positive roots, where positive roots were positive relative to a hyperplane.

(a) Suppose $\theta=\pi/3$ for example, then if α,β are length 1, we may reflect α across β to get another root. If α' was such a root, then $\alpha+\alpha'=\beta$. If $\alpha'\in R^+$, then this is a contradiction as β is simple.

Otherwise, the reflection of β across α is still in R^+ so that $\beta+\beta'=\alpha$.

Similar cases involve discarding cases like this.

The point is that simple roots are the roots close to the hyperplane instead of the ones pointing up.

Definition 1.17.3. The Dynkin diagram of a root system is a multigraph whose vertices are the roots and the edges are as follows:

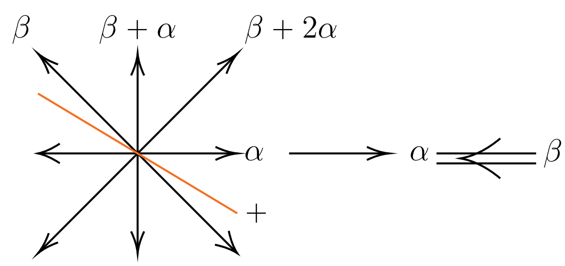
- There's no edge between α, β if $\alpha \perp \beta$.
- There's an edge between them if $\langle \alpha | \beta \rangle = -\frac{1}{2}$.
- There's two edges if $\langle \alpha | \beta \rangle = -1$ as $\theta = \frac{3\pi}{4}$.
- Three edges if the angle is $\frac{5\pi}{6}$, in this case $\langle \alpha | \beta \rangle = -\frac{3}{2}$.

Question. ¿Why can't simple roots be π apart?

This is a subtlety, as both roots are $\alpha, -\alpha$. They can't be both positive as the hyperplane cuts both of them. We wiggle it a bit and get that they can't be both positive.

We will classify Dynkin diagrams.

Example 1.17.4. One of the type B_2 systems was



Example 1.17.5. Another was G_2, add figure

We will get the graph structure by using undirected Dynkin diagrams. These correspond to normalized root systems.

Definition 1.17.6. An admissible diagram is a graph with n vertices representing n independent unit vectors $e_1, \ldots, e_n$ with angles θ_{ij} being in

$$\left\{ \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6} \right\}.$$

In each of these cases we draw no, one, two and three edges respectively but without the arrows.

After finding the graph shapes, we will add the orientation again. Further we will show that admissible diagrams are trees, vertices have degree 3 and much more.

1.18 Day n+10 | 20241023

Admissible diagrams

Maria has been looking up to this moment the whole morning. Recall that the setup is that we have n unit vectors which are independent in $\mathbb{R}^N$:

- ◇ We have $e_1, \ldots, e_n$ independent vectors.
- ◇ The length of the angle classifies the edges.

1.19 Day n+12 | 20241028

Weyl Groups

We went up to Lie groups, representations of Lie algebras and now we are going back to finite groups once again.

Recall that given a root system R, for $\alpha, \beta \in R$, the reflection of β through H_α is

$$r_\alpha(\beta) = \beta - \frac{2\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle}\alpha.$$

Definition 1.19.1. A simple reflection s_α is r_α when α is a simple root. The Weyl group of R is the group of reflections generated by s_α. It is a subgroup of all reflections in n-space.

In particular this is a subgroup of GL_n. For a root system, this is a finite group. We can also think of them as acting on the roots.

Lemma 1.19.2. *Weyl groups are finete and are generated by simple reflections.*

W acts on the root system by permutations, so $W \subseteq \mathrm{Sym}(R)$ because the induced permutation on the roots has to determine the transformation.

In general, we should show that if $\beta = \alpha + \alpha'$ then we can express s_β in terms of $s_\alpha, s_{\alpha'}$.

Proposition 1.19.3. *Let s_i, s_j be simple reflections (generators in a Weyl group W). Then*

- ◇ $s_i^2 = s_j^2 = 1$, *and*
- ◇ $\exists m_{ij}(s_i s_j)^{m_{ij}} = 1$ *where* $m_{ij} \in \{2, 3, 4, 6\}$ *is minimal. In other words, it's the order.*

Observe that it's not necessary that $m_{ij} = 2$ always as s_i, s_j may not necessarilly commute.

Proof

For the first item, observe that s_i, s_j are involutions. So immediately we get the result. Now let's analyze cases. Simple reflections correspond to nodes of the Dynkin diagram. Take roots α_i, α_j:

- If there's no edge between them, then they are orthogonal. Then $s_i s_j$ is a rotation by π. Thus s_i, s_j commute and $m_{ij} = 2$.
- If there's one edge, then the angle between α_i, α_j is $2\pi/3$. Hyperplanes between the roots differ by $\pi/3$. We can simplify by considering an $\mathfrak{sl}_3$ L diagram. We get

$$s_1 = (23), \quad s_2 = (13) \Rightarrow s_1 s_2 s_1 = s_2 s_1 s_2$$

  which is the braid relation. This simplifies to $(s_1 s_2)^3 = 1$.
- The third case corresponds to 2 edges and a $3\pi/4$ angle. This gives us $m_{ij} = 4$.
- Finally with 6 edges and a $5\pi/6$ angle we get $m_{ij} = 6$.

Theorem 1.19.4. *The Weyl group is actually generated by simple reflections modulo the previous relations.*

$$W = \operatorname{gen}(s_i : \; s_i^2 = 1, \; (s_i s_j)^{m_{ij}} = 1).$$

Weyl groups sit inside a more general class of groups.

Definition 1.19.5. A Coxeter group is the free group generated by $(s_i)_{i \in I}$ such that

- $s_i^2 = 1$, and
- $(s_i s_j)^{m_{ij}} = 1$ for $m_{ij} \in \mathbb{Z}_{\geqslant 0}$.

Example 1.19.6. In type A, our Dynkin diagram looks like a line. Assume it's $\alpha_1 \ldots \alpha_5$. Vertices connected by an edge have a braid relation. In this case W is generated by s_i's such that

$$s_i^2 = 1, \; s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \; s_i s_j = s_j s_i, \; |i - j| \geqslant 2.$$

Observe that we have $s_i = (i \; i+1)$ and

$$s_i s_j = (i \; i+1)(j \; j+1) = s_j s_i$$

when $|i-j|>2$. On a simpler note observe that

$$s_1s_2s_1 = s_2s_1s_2$$

because

$$(12)(23)(12) = (13) = (23)(12)(23) = 321.$$

This happens in general for any i so that we have the desired braid relation.

It is also possible to show that there's $(n+1)!$ elements by noticing that how W acts on $L_1, \ldots, L_n$ determines a permutation, so $|W| \leqslant (n+1)!$.

For type B, let's go a little smaller. Type B and C have the same Dynkin diagram.

Example 1.19.7. In this case let's label the edge between s_0, s_1 as the double edge. Between s_1, s_2 we have an ordinary braid relation and s_0 commutes with s_2. But we have the relation

$$s_0s_1s_0s_1 = s_1s_0s_1s_0.$$

The model of this group is the hyperoctahedral group, or signed permutations group. Call $s_1 = (12), s_2 = (23)$, but the element s_0 *flips* the sign of the first letter. In list notation

$$s_0: \quad -21-3 \mapsto 21-3.$$

Definition 1.19.8. A signed permutation of $[n]$ is a permutation in list notation along with a $+$ or $-$ sign on each letter.

Example 1.19.9. $5-143-3$ is a signed permutation, here $w \in S_{[\pm n]}$ such that $w(-i) = -w(i)$ for all i.

Observe that W_{B_n} is a subgroup as

$$w\sigma(-i) = w(-\sigma(i)) = -w\sigma(i).$$

So in more generality

$$s_1 = (12)(3)(-3)(-1-2), s_2 = (23)(1)(-1)(-2-3).$$

The longer relation holds in this following example:

$$1234 \mapsto -1234 \mapsto 2-134 \mapsto -2-134 \mapsto -1-234$$

where this is $s_0s_1s_0s_1$ and the relation also holds. It's also possible to see in cycle notation that we get a 4-cycle.

1.20 Day n+13 | 20241030

Recall that the Weyl group is the symmetry group of a root system.

The type D_n looks like a Y, recall that if s_i, s_j:

- ◇ share no edge, we have the relation $s_i s_j = s_j s_i$, and
- ◇ if they share an edge we have $s_i s_j s_i = s_j s_i s_j$.

These are the only relations that we need for D_n's Weyl group. We have that $W_{D_n} \subseteq W_{B_n}$ which was the octahedral group, the group of signed permutations.

- ◇ s_i will correspond to switching i with $i+1$, and
- ◇ s_0 negates and swaps *the first two entries*.

Observe that

$$1234 \xrightarrow{s_0} 6534 \xrightarrow{s_2} 5634 \xrightarrow{s_0} 2134$$

where as on the other direction

$$1234 \xrightarrow{s_2} 1324 \xrightarrow{s_0} 7524 \xrightarrow{s_2} 7254$$

Exercise 1.20.1. Check

Lemma 1.20.2. *The size of W_{D_n} is $n!2^{n-1}$.*

This amount is calculated via $n! \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{2i}$.

Lemma 1.20.3. $|W_{B_n}| = n!2^n$.

Now instead we choose a sign without restrictions, there's 2 ways per entry and there's n entries. This means that $W_{D_n} \lhd W_{B_n}$ so it's a normal subgroup and there's possibilities of taking semidirect products and other stuff.

Definition 1.20.4. The Weyl group can be defined abstractly as

$$N_G(T) \Big/ T$$

where G is a Lie group and T is the torus, the maximal abelian subgroup.

Example 1.20.5. For SL_n, T is the set of diagonal matrices of determinant 1. Then the normalizer, $N_G(T)$ is the permutation matrices times a diagonal matrix. Finall $N_G(T)/T$ is the set of permutation matrices.

Lemma 1.20.6. *The action of W fixes the set of weights of any representation of $\mathfrak{g}$.*

Proof

s_i reflects about the hyperplane orthogonal to α_i. The $\mathfrak{sl}_2$ string in α_i direction is symmetric. That means $s_i(\beta)$ is a weight of V if β is a weight.

Therefore the action of S_n on A_{n-1} is:

- ◇ Pair $i+1$'s and i's.
- ◇ The remaining unpaired $i^a(i+1)^b$ are replaced by $i^b(i+1)^a$.

The word

$$iii+1ii+1iii+1 \to iii()()ii+1$$

On Friday we will talk about evacuation and on Monday we will talk about Springer theory.

There's a vertical symmetry exhibited by type A crystals. Is there a natural involution that gives vertical symmetry of the type A crystals where we swap F_i with E_{n-i}? It turns out that taking the action almost does what we want.

If w_0 is the word $n(n-1)(n-2)\dots 321$ so one minimal way of writing it is

$$\dots(s_1s_2s_3)(s_1s_2)(s_1)$$

This is not the answer, the actual answer is the Schützenberger involution.

1.21 Day n+14 | 20241101

Evacuation

To evacuate a tableau T we:

- ◇ Rotate it 180 degrees.
- ◇ Change all i to $n+1-i$.
- ◇ Finally JDT rectify.

Example 1.21.1. Evacuate

$$\begin{array}{|c|c|c|c|}\hline 3 \\ \cline{1-1} 2 & 3 & 4 \\ \cline{1-3} 1 & 1 & 2 & 3 \\ \hline\end{array} \to \begin{array}{|c|c|c|c|}\hline 4 \\ \cline{1-1} 2 & 3 & 3 \\ \cline{1-3} 1 & 2 & 2 & 4 \\ \hline\end{array}$$

Alternatively

- ⋄ Remove bottom left corner i,
- ⋄ Rectify,
- ⋄ Fill evacuated outer square with $n+1-i$ and curly decorate it.
- ⋄ Repeated on non-curly until all are curly.

Call η our first evacuation. Then

Proposition 1.21.2.

η gives a vertical symmetry of type A_{n-1} crystals under $F_i \leftrightarrow E_{n+1-i}$.

Chapter 2

Springer Theory

2.1 Day n+15 | 20241104

Even less details

There's going to be EVEN LESS DETAIL, look at Maria.com at the end of the webpage for more references to this.

The Springer correspondence is between

$$\{\,\text{irr. reps. of } W_R\,\} \leftrightarrow \{\,\text{Nilp. conj. classes in } \gg\,\}.$$

Where nilpotent conjugacy classes come with a representation of the associateed group A_G. In type A isn't needed. Depending on opinions, the Springer correspondence is incomplete. However in type A is a bijection. So to obtain information on the Weyl group, we look at the Lie algebra.

Not only it's a bijection, it's an *explicit* bijection. This is constructed via the Springer resolution, coming from the geometry of the flag variety.

Definition 2.1.1. A flag is a chain

$$V_\bullet : 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n$$

where $\dim V_i = i$.

Example 2.1.2. If e_i is the standard basis vector of $\mathbb{C}^n$, then

$$0 \subseteq \operatorname{gen} e_1 \subseteq \operatorname{gen}(e_1, e_2) \subseteq \cdots \subseteq \mathbb{C}^n$$

is the standard flag.

Out of flags we would like to construct manifolds or algebraic varieties. In particular consider GL_n acting on flags, $\mathcal{F}\ell_n$. The action is

$$\mathfrak{g}\cdot V_\bullet : 0 \subseteq gV_1 \subseteq gV_2 \subseteq \cdots \subseteq gV_n.$$

Example 2.1.3. For $n = 2$, consider the standard flag and $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The the standard flag becomes

$$0 \subseteq \mathrm{gen}(e_2) \subseteq \mathbb{C}^2.$$

Lemma 2.1.4. *The action of* GL_n *on* $\mathcal{F}\ell_n$ *is transitive.*

Proof

It suffices to show this with the standard flag $E_\bullet$. Choose the right g, given $W_\bullet$, we can take g as the matrix whose columns are the (w_i)'s.

Example 2.1.5. The stabilizer of $E_\bullet$ counts the amount of overcounting. This is

$$\mathrm{Stab}(E_\bullet) = \{\, g : \ g\cdot E_\bullet = E_\bullet \,\}.$$

Inductively we can show that this stabilizer is the set of upper triangular matrices which are invertible. In conclusion

$$\mathrm{Stab}\, E_\bullet = B_n.$$

Theorem 2.1.6. $\mathcal{F}\ell_n = \mathrm{GL}_n/B_n$.

Proof

Flags represent cosets

$$\{\, gB_n : \ gE_\bullet = V_\bullet \,\}.$$

From this correspondence we inherit the manifold structure of GL_n. Recall that GL_n is a Lie group so the quotient inherits the manifold structure.

From this we will now call the flag as the flag variety or manifold. We can embed the flag into projective space via Plücker relations.

Example 2.1.7. In different types we have:

(a) In type A, the flag is equivalent to $\mathrm{SL}_n/B_n^{\mathrm{SL}}$.

(b) For type B, the flag is SO_{2n+1} modded by its corresponding Borel subgroup.

Recall that the Borel subgroup is the maximal, closed, connected, solvable subgroup.

Remark 2.1.8. Recall that solvable groups are those whose successive commutator subgroups eventually trivialize: $B, B_1 = [B, B]$, $B_2 = [B_1, B_1]$ and so on.

Let's check it in dimensoin 2:

Example 2.1.9. Observe that

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} a^{-1} & -b/ac \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} x^{-1} & -y/xz \\ 0 & z^{-1} \end{pmatrix}.$$

Multiplying this out we get a matrix of the form

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

In third dimension we get

$$\left[\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 2.1.10. For types C and D the flags are Sp_{2n} and SO_{2n} modded by their corresponding Borel subgroups.

In general it's possible to do this even for the exceptional types.

Only Type A

We would like to investigate the cohomology of $\mathcal{F}\ell_n$. Recall the cohomology ring is graded via codimension. In this case

$$H^*(\mathcal{F}\ell_n) = H^0(\mathcal{F}\ell_n) \oplus H^2(\mathcal{F}\ell_n) \oplus \cdots \oplus H^{2d}(\mathcal{F}\ell_n)$$

and the product respects codimension. Elements of this cohomology are closed subvarieties modded out by cohomology.

Theorem 2.1.11. *$H^*(\mathcal{F}\ell_n) \simeq \mathbb{C}[\underline{x}]/\langle \underline{e} \rangle$ where e_d is the sum of squarefree monomials of degree d.*

Example 2.1.12. For $n = 1$ we get $\mathbb{C}[x]/\langle x \rangle = \mathbb{C}$. So the flag variety of one dimension is a point. In $n = 2$ we get

$$\mathbb{C}[x, y]/\langle x + y, xy \rangle = \mathbb{C}[x]/\left\langle -x^2 \right\rangle = \{ a + bx : \ a, b \in \mathbb{C} \}.$$

2.2 Day n+16 | 20241106

Invariance under variable permutation (i.e. the action of S_n on $\mathbb{C}[\underline{x}]$) comes from symmetric polynomials. So that's why when modding by $\underline{e}$ we get the coinvariance.

Example 2.2.1. For $n = 2$ we had the coinvariant ring to be $\{ a + bx : a, b \in \mathbb{C} \} = \mathbb{C}[x] / \langle x^2 \rangle$.

So the cohomology ring of the flag varierty gives us $\mathbb{C} \oplus \mathbb{C}x$. This can be seen as a $\mathbb{C}$-vector space with dimenion 2.

Example 2.2.2. The coinvariant ring in $n = 3$ is

$$\mathbb{C}[x,y,z] \Big/ \langle x+y+z.xy+yz+zx, xyz \rangle.$$

If we say $x = -y - z$ then substitute we get

$$\mathbb{C}[y,z] \Big/ \langle y^2 + yz + y^2 \rangle$$

2.3 Day n+18 | 20241111

Recall

$$\mathcal{B}_\mu = \{ V_\bullet \in \mathcal{F}\ell_{|\mu|} : X_{\mu_i} V_i \subseteq V_i \}$$

where X_μ is a nilpotent matrix of Jordan type μ, in the sense that its block sizes are μ_i.

Example 2.3.1. For example $\mathcal{B}_{0_3}$ is the full flag variety as it is $\mathcal{B}_{(1,1,1)}$.

Example 2.3.2.

$$\mathrm{Frob}(R_{(2,1)}) = \sum_{T \in SSYT \text{ content } (2,1)} q^{\mathbf{c}(T)} s_{\mathrm{sh}(T)} = q s_{(2,1)} + s_{(3)}.$$

We will use this as motivation to do chromatic symmetric functions.

Definition 2.3.3. The <u>chromatic symmetric function</u> is

$$X_G(\underline{x}) = \sum_{c \text{ coloring}} \underline{x}_{c(1,\dots,n)}.$$

Recall proper coloring is a coloring where adjacent vertices are differently colored.

Example 2.3.4. For the graph P_1, we can color each vertex i and j and backwards. So we get the monomial $2x_i x_j$ for each coloring and its reverse. This means that $X_{P_1} = 2e_2$.

Example 2.3.5. Instead if we get P_2, we have the following

- ◇ 3 different colors on each vertex which amount to $3!e_3$.
- ◇ Then we can color the center vertex different from the leaves to get $x_i^2 x_j$ or $x_i x_j^2$. This is the monomial function $m_{(2,1)}$ which we may write as

$$m_{(2,1)} = e_1 e_2 - 3e_3$$

so that $X_G(\underline{x}) = 3e_3 + e_{(2,1)}$.

Chapter 3

Stanley and Stembridge's conjecture

3.1 Day n+19 | 20241113

The conjecture basically states that $K_{1,3}$-free graphs have an e positive X_G.

Example 3.1.1. The claw graph $K_{1,3}$'s chromatic symmetric function is

$$X_G = 24e_4 + 6m_{(2,1,1)} + m_{(3,1)} = 4e_4 + e_{(2,1,1)} - 2e_{(2,2)} + 5e_{(3,1)}.$$

Lemma 3.1.2. *If* $G = H \coprod K$ *then* $X_G = X_H X_K$.

Proof

The monomial for a proper coloring of H, K is the product of monomials obtained by restricting the coloring to H, K.

Example 3.1.3. For a complete graph, its chromatic symmetric function is $n!e_n$.

Remark 3.1.4. Recall e-positivity implies s-positivity. This is because e_λ is Schur positive via the Pieri rule:

$$s_\lambda e_d = \sum_{\mu = \lambda + \text{h.s.}(d)} s_\mu.$$

In terms of representation theory, it's stronger.

It was known that X_G was already Schur-positive for $K_{1,3}$-free graphs. Claw graphs are still not Schur-positive so it doesn't improve the conjecture.

Remark 3.1.5. Actually it's still a conjecture that claw-free graphs are Schur-positive.

Definition 3.1.6. A unit interval graph is a graph whose vertices are intervals of length 1 in $\mathbb{R}$ and edges occur when the intervals overlap.

Lemma 3.1.7. *Unit interval graphs are claw free.*

Theorem 3.1.8 (Guay-Paquet)**.** *It suffices to show SSC for unit interval graphs instead of for all graphs.*

Chromatic quasisymmetric functions

This is like a q-analogue of certain cohomology of X where that space is similar to Springer fibers.

Definition 3.1.9. An <u>ascent</u> of a proper coloring is an edge labeled $i \to j$ with $c(i) < c(j)$.

Example 3.1.10. Consider the graph

$$\{ (1,2), (1,3), (2,3), (2,4) \}$$

and the coloring

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

there's two ascents in the edges $(1,3)$ and $(2,3)$. So that $\operatorname{asc}(c) = 2$.

Definition 3.1.11. The <u>chromatic quasisymmetric function</u> of a graph G is the q-analogue of asc:

$$X_G(\underline{x}; q) = \sum_c q^{\operatorname{asc}(c)} \underline{x}_c.$$

Example 3.1.12. For a path graph $\{ (1,2), (2,3) \}$ colored

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

we get the q-analogue

$$2(1 + q + q^2)e_3$$

however on two colors we get

$$q^2 x_1^2 x_2 + x_2^2 x_1, \quad q^2 x_2^2 x_5 + x_5^2 x_2.$$

Definition 3.1.13. A <u>quasisymmetric function</u> in $\underline{x}$ is a bounded degree sum of monomials such that the coefficient of $\underline{x}^{\underline{\alpha}}$ matches the coefficient of

$$x_{i_1}^{\alpha_1} \dots x_{i_n}^{\alpha_n}, \quad i_1 < i_2 < \dots < i_n.$$

Quasisymmetric functions captures tuples of $\underline{\alpha}$ which form compositions.

Definition 3.1.14. For a composition α, define the monomial quasisymmetric function as

$$M_\alpha = \sum_{i_1 < \cdots < i_n} x_{i_1}^{\alpha_1} \dots x_{i_n}^{\alpha_n}.$$

This is the minimal quasisymmetric function which contains a particular monomial.

Proposition 3.1.15. *The set $\{ M_\alpha \}_\alpha$ is a basis of the ring of quasisymmetric functions over any ring.*

As in the case of monomials in symmetric functions this basis is so nice and nothing can go wrong. There's a little theorem hiding in this previous result:

Theorem 3.1.16. *The sum and product of quasisymmetric functions is quasisymmetric.*

In terms of quasisymmetric functions, our previous graph's function can be written as

$$X_G(x;q) = 2([2]_q!)e_3 + M_{1,2} + q^2 M_{2,1} = 2([2]_q!)M_{1,1,1} + M_{1,2} + q^2 M_{2,1}.$$

Example 3.1.17. Imagine that now we color our graph

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

then there's two ascents and we get

$$q(M_{1,2} + M_{2,1}) = qm_{2,1}$$

so that

$$X_G = [2]_q! e_3 + qe_{(2,1)}.$$

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