Exercise 1 (Exercise 2)**.** Prove the q-binomial theorem

$$\prod_{j=1}^{n}(1+xq^j) = \sum_{k=0}^{n} q^{\frac{k(k+1)}{2}} \binom{\mathbf{n}}{\mathbf{k}} x^k.$$

You may either use an induction or combinatorial argument.

Answer

Consider the function $p(x) = \prod_{j=1}^{n}(1+xq^j)$, and let us evaluate at $x = qx$:

$$\begin{aligned}
p(qx) &= (1+(qx)q)(1+(qx)q^2)\dots(1+(qx)q^n) \\
&= (1+xq^2)(1+xq^3)\dots(1+xq^{n+1}) \\
\Rightarrow (1+xq)p(qx) &= [(1+xq)\dots(1+xq^n)]\,(1+xq^{n+1}) \\
\Rightarrow (1+xq)p(qx) &= p(x)(1+xq^{n+1}).
\end{aligned}$$

Now, if we expand p as a sum of monomials and call $a_k = a_k(q)$ the coefficient of x^k, we can compare coefficients on both sides: