MATH 618: Advanced Real Analysis

Homework 1

Problem 1 (Infinite-dimensional spaces). In class, you have seen examples of infinite-dimensional spaces: Notably, (infinite) sequences of numbers and function spaces. But one can come up with many other sets of objects that (i) satisfy the vector space axioms, and (ii) are infinite-dimensional. Come up with your own example of an infinite-dimensional space that doesn't fit the examples you have seen in class. Show that it is a vector space (if you define scalar multiplication and vector addition appropriately) and why you think that the set is infinite-dimensional.

(10 points)

Problem 2 (The dimension of spaces). Defining what the "dimension" of a space is is *intuitively* obvious, but *technically* perhaps not quite as much.

For $\mathbb{R}^n$ and other finite-dimensional spaces, if you have a *basis* of the space with n elements, then we say that the space has dimension n.[1] Importantly, every other basis you can find will then also have exactly n elements. This also means that the operation that converts one basis to another can be written as a square matrix/operator that is invertible. This all will turn out to be more complicated for infinite-dimensional spaces.

- Take $V = \mathbb{R}^3$. Provide a basis $\{\mathbf{a}_i\}_{i=1}^3$ (that is, a set of three vectors) for this space. Then provide another basis $\{\mathbf{b}_i\}_{i=1}^3$.
- There is an operator R (here, a 3×3 matrix) that converts from one basis to another. That is, if I give you a vector $\mathbf{x} \in \mathbb{R}^3$, it can be written as $\mathbf{x} = \sum_{i=1}^3 \alpha_i \mathbf{a}_i$ and as $\mathbf{x} = \sum_{i=1}^3 \beta_i \mathbf{b}_i$. The operator R is then the one that translates between expansion coefficients:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = R \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

  Provide the form of R for your choice of basis, and show that it is invertible.
- Repeat the previous two steps if V is the space of symmetric 2×2 matrices.

(20 points)

[1] Recall: A basis of a space V is a set of vectors $\{\mathbf{a}_i\}$ so that every vector $\mathbf{v} \in V$ can be written as a unique linear combination $\mathbf{v} = \sum_i \alpha_i \mathbf{a}_i$. Note that the basis vectors do not need to be normalized (we are only working with a vector space, no norms so far) and they do not have to be orthogonal (again, we are only working with a vector space, no inner products have been defined so far).

Problem 3 (The dimension of spaces). Let's see how this looks like for the infinite-dimensional case. The upshot of this problem is that infinite-dimensional spaces, obviously, do not have a finite basis but that a space can have both countable and uncountable bases!

As an example, let's consider the vector space of sequences, i.e.,

$$V = \{(q_1, q_2, q_3, \dots) : q_i \in \mathbb{R}\}.$$

Let us think about bases of this space, i.e., sets of vectors $\mathbf{a}_i \in V$ so that every $\mathbf{v} \in V$ can again be written as $\mathbf{v} = \sum_i \alpha_i \mathbf{a}_i$.[2]

- Convince yourself that the set $\{\mathbf{a}_i\}_{i=1}^{\infty}$ with $\mathbf{a}_1 = (1, 0, 0, \dots), \mathbf{a}_2 = (0, 1, 0, \dots), \mathbf{a}_3 = (0, 0, 1, \dots), \dots$ is a basis of V. (To "convince" yourself, look up the formal properties of a basis.) It is obviously countable.
- Create a second countable basis of your choice.
- Can you somehow describe the operator R that translates between these two bases, in the same way as was done in the previous problem?
- Now convince yourself that the set of vectors $\{\mathbf{b}_\lambda\}_{\lambda \in [0,1]}$ where $\mathbf{b}_\lambda = (1, \lambda, \lambda^2, \lambda^3, \dots)$ is also a basis. This is *not* a countable basis because the set is indexed by the real number λ!

  For cases like this, one has to think about what it means to expand a vector in this basis. Before, we had that for every vector $\mathbf{v} \in V$, we can write $\mathbf{v} = \sum_i \alpha_i \mathbf{a}_i$. With the uncountable basis here, this has to be replaced by $\mathbf{v} = \int_0^1 \beta_\lambda \mathbf{b}_\lambda \, d\lambda$.
- Can you come up with a description of the basis transformation operator R for these two bases?

(30 points)

Problem 4 (The dimension of spaces). Let's repeat the previous problem once more for spaces of functions. Concretely, take

$$V = C^0 = \{f : [0,1] \to \mathbb{R}, f \text{ is continuous}\},$$

with norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$.

- Is $\mathbf{a}_n(x) = \sin(n\pi x)$ a countable basis?
- Is

  $$\mathbf{b}_\lambda(x) = \begin{cases} 1 & \text{if } x = \lambda, \\ 0 & \text{otherwise} \end{cases}$$

  for $\lambda \in [0,1]$ an uncountable basis? (If you think about expanding a function as $\mathbf{v} = \int_0^1 \beta_\lambda \mathbf{b}_\lambda \, d\lambda$, you will probably want to replace $\mathbf{b}_\lambda$ by a delta-function, $\mathbf{b}_\lambda(x) = \delta(x - \lambda)$. If you don't know what that means, just ignore the difference.)

(20 points)

[2] This may not be obvious at first: Being able to write $\mathbf{v} = \sum_i \alpha_i \mathbf{a}_i$ with an infinite sum requires that the infinite sum makes sense – which we will interpret as saying that $\lim_{n\to\infty} \sum_{i=1}^n \alpha_i \mathbf{a}_i \to \mathbf{v}$. This in turn requires that we can measure convergence in V, which requires that we have a *norm*. That is, bases in infinite-dimensional spaces inherently only make sense if the vector space V is a *normed* vector space! For the case here, let us assume that the norm on V is $\|\mathbf{v}\| = \sup_i |v_i|$. That is, we take $V = \ell_\infty$.

Problem 5 (Operators on ℓ_2). Recall the set of sequences,

$$V = \{(q_1, q_2, q_3, \ldots) : q_i \in \mathbb{R}\}.$$

When equipped with the norm

$$\|\mathbf{v}\| = \sum_i |q_i|^2,$$

then this defines the normed vector space

$$\ell_2 = (V, \|\cdot\|) = \{\mathbf{v} = (v_1, v_2, v_3, \ldots) : v_i \in \mathbb{R}, \|\mathbf{v}\| < \infty\}.$$

Now consider the set of *linear operators* $\ell_2 \to \ell_2$, i.e., linear operators that map a sequence in ℓ_2 to another sequence in ℓ_2.

Provide three examples of such operators. Argue that the set of linear operators $\ell_2 \to \ell_2$ is (i) a vector space, (ii) infinite dimensional.

(20 points)