Exercise 1 (7.8 Stein& Shakarchi)**.** The function ζ has infinitely many zeros in the critical strip. This can be seen as follows.

i) Let
$$F(s) = \xi(1/2 + s), \quad \text{where} \quad \xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$
Show that $F(s)$ is an even function of s and as a result, there exists G such that $G(s^2) = F(s)$.

ii) Show that the function $(s-1)\zeta(s)$ is an entire function of growth order 1, that is
$$|(s-1)\zeta(s)| \leqslant A_\varepsilon e^{a_\varepsilon |s|^{1+\varepsilon}}.$$
As a consequence $G(s)$ is of growth order $1/2$.

iii) Deduce from the above that ζ has infinitely many zeros in the critical strip.

⟦Hint: To prove the first two parts use the functional equation for $\zeta(s)$. For the last one, use a result of Hadamard, which states that an entire function with fractional order has infinitely many zeros (Exercise 14 in Chapter 5).⟧

Answer

i) Observe that
$$\begin{aligned} F(-s) = \xi(1/2 - s) &= \xi\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - s\right) = \xi\left(1 - \frac{1}{2} - s\right) \\ &= \xi\left(1 - \left(\frac{1}{2} + s\right)\right) = \xi\left(\frac{1}{2} + s\right) \end{aligned}$$
where the last equality comes from the identity $\xi(s) = \xi(1-s)$ for all $s \in \mathbb{C}$.

ii) We know that $\zeta(s)$ has a pole of order 1 at $s = 1$ and that's it's only pole. So the function $(s-1)\zeta(s)$ is holomorphic on the whole plain which means it's entire. Show order of growth

iii) Finally our function has non-integral order so it has an infinite number of roots. This follows from a exercise where we use Hadamard's factorization theorem.

Exercise 2 (7.6 Stein& Shakarchi)**.** Read [SS]7.6, assume its result, and proceed as follows. Let δ be the function defined in [SS]7.6:

$$\delta(a) = \begin{cases} 1 & 1 < a \\ \frac{1}{2} & a = 1 \\ 0 & 0 \leqslant a < 1 \end{cases}$$

Fix a positive real number X which is not an integer.

i) Show that $\Psi(X) = \sum_{n \geqslant 1} \Lambda(n) \delta\left(\frac{X}{n}\right)$.

ii) Consider $G(s) = \frac{X^s}{s}\left(\frac{-\zeta'(s)}{\zeta(s)}\right)$, show that

$$\Psi(X) = \frac{1}{2\pi i} \int\limits_{\{\operatorname{Re}(s)=c\}} G(s)\mathrm{d}s.$$

⟦Hint: Assume you can exchange summation and integration; you will need to use our formula from class for $L(\zeta(s))$, which is also in [SS] Chapter 7, section 2.⟧

Answer

i) Observe that the δ function can be expressed as a sum of indicator functions:

$$\delta(a) = \mathbf{1}_{\{a>1\}} + \frac{1}{2}\mathbf{1}_{\{a=1\}} + 0\mathbf{1}\{0 \leqslant a < 1\}.$$

In this sense we have that for a fixed n, the function $\delta\left(\frac{X}{n}\right)$ is

$$\delta\left(\frac{X}{n}\right) = \mathbf{1}_{\{X>n\}} + \frac{1}{2}\mathbf{1}_{\{X=n\}} + 0\mathbf{1}_{\{X<n\}}.$$

So reminding ourselves that X is a positive non-integer we have that $\delta(X/n)$ is never $\frac{1}{2}$. Now fix X so that

$$\sum_{n\geqslant 1} \Lambda(n)\delta\left(\frac{X}{n}\right) = \sum_{n<X} \Lambda(n) + 0\sum_{n>X} \Lambda(n).$$

As the rest of the sum is zero because of the indicator, we have that the whole sum actually is $\sum_{n<X} \Lambda(n)$ which is precisely our Ψ function.

ii) If we now have the integral in question, we may replace G by its definition and see that

$$\frac{1}{2\pi i}\int_{\{\operatorname{Re}(s)=c\}} \frac{X^s}{s}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) \mathrm{d}s = \frac{1}{2\pi i}\int_{\{\operatorname{Re}(s)=c\}} X^s \frac{\mathrm{d}s}{s} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Subtly applying the dominated convergence theorem we may interchange the series with th integral to obtain

$$\sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i}\int_{\{\operatorname{Re}(s)=c\}} \left(\frac{X}{n}\right)^s \frac{\mathrm{d}s}{s}$$

and by exercise 7.6 we know that the integral in question is

$$\frac{1}{2\pi i}\int_{\{\operatorname{Re}(s)=c\}} \left(\frac{X}{n}\right)^s \frac{\mathrm{d}s}{s} = \mathbf{1}_{\{X>n\}}$$

which means that the whole expression is

$$\sum_{n=1}^{\infty} \Lambda(n)\mathbf{1}_{\{X>n\}} = \sum_{n<X} \Lambda(n)$$

as desired. As both expressions equal the same sum, we have that Ψ is the integral in question.

Exercise 3. One uses the results of the previous problems in the following way.

i) Show that $\operatorname{res}(G, 1) = X$. ⟦Hint: Use the fact that $\zeta(s)$ has a pole at $s = 1$ of order 1.⟧

ii) Show that $\operatorname{res}(G, 0) = \lim_{s\to 0} \frac{-\zeta'(s)}{\zeta(s)}$. *It turns out that this is* $-\log(2\pi)$.

iii) Show that $\sum \operatorname{res}(G, \rho) = -\frac{1}{2}\log(1 - X^{-2})$, where the sum is over the trivial zeros of $\zeta(s)$.

From here, moving c "all the way to the left" means that we pick up all the residues of $G(s)$*, and we are left with von Mangoldt's explicit formula:*

$$\psi(X) = X - \sum \frac{X^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - X^{-2}\right)$$

where the sum is over all critical zeroes of $\zeta(s)$.

Answer

i) Observe that

$$\operatorname{res}(G,1) = \lim_{s\to 1}(s-1)\frac{X^s}{s}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) = \lim_{s\to 1}\frac{X^s}{s}\lim_{s\to 1}(s-1)(L(\zeta(s))).$$

The limit on the right is the residue at $s=1$ of the logarithmic derivative of ζ, it is know that this residue is the order of the point in question of the function. This means that

$$\operatorname{res}(G,1) = X\cdot -\operatorname{ord}(\zeta,1) = X\cdot 1 = X.$$

ii) In this case, we have that

$$\operatorname{res}(G,0) = \lim_{s\to 0}(s)\frac{X^s}{s}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) = \left(\lim_{s\to 0}X^s\right)\left(\lim_{s\to 0}\frac{-\zeta'(s)}{\zeta(s)}\right)$$

and the left limit turns to 1 so we obtain the desired result.

iii) It is a subtle observation that

$$-\frac{1}{2}\log\left(1-X^{-2}\right) = \frac{1}{2}\sum_{n\geqslant 1}\frac{\left(\frac{1}{X^2}\right)^n}{n} = \sum_{n\geqslant 1}\frac{X^{-2n}}{2n}.$$

Now, the trivial zeroes of the zeta function are at $s=-2n$, so

$$\operatorname{res}(G,-2n) = \lim_{s\to -2n}(s+2n)\frac{X^s}{s}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) = \left(\lim_{s\to -2n}\frac{X^s}{s}\right)(-\operatorname{ord}(\zeta,-2n))$$

where the limit evaluates to $\frac{X^{-2n}}{-2n}$ and the order is 1 so we obtain $\frac{X^{-2n}}{2n}$ and summing through all trivial zeroes we obtain the desired result.