**Exercise 1** (7(a))**.** Show that if two words $w$ and $v$ are Knuth equivalent, then removing the smallest $k$ letters from each of $w$ and $v$ (with ties broken in order from left to right) results in two Knuth equivalent words $w'$ and $v'$. (Hint: It suffices to show it for just one letter removed, and it also suffices to assume that $w$ and $v$ differ by an elementary Knuth move.)

Let us recall the Knuth move definition from a year ago:

**Definition 1.** A <u>Knuth move</u> on a permutation swaps two letters $a, c$ if $a < b < c$ (reading order) and one of consecutive subsequences $acb, cab, bac, bca$ appears in the word.

Two words are <u>Knuth-equivalent</u> if they differ by a sequence of Knuth-moves.

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**Answer**

We[a] follow the hint as indicated. This is possible because after doing a sequence of Knuth moves for a particular letter and then doing it for all the letters, we get to the complete result.

So for our case let us take a word $w$ whose smallest letter is $x_0$. We have two cases:

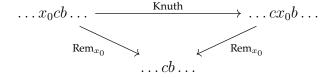⋄ If $x_0$ is not in the affix $acb$ we are Knuth moving then

$$w = \ldots x_0 \ldots acb \ldots.$$

It is equivalent if $x_0$ is after the affix. Let us Knuth move and remove $x_0$ and vice versa.

$$
\begin{array}{ccc}
\ldots x_0 \ldots acb \ldots & \xrightarrow{\text{Knuth}} & \ldots x_0 \ldots cab \ldots \\
\Big\downarrow{\scriptstyle \text{Rem}_{x_0}} & & \Big\downarrow{\scriptstyle \text{Rem}_{x_0}} \\
\ldots acb \ldots & \xrightarrow[\text{Knuth}]{} & \ldots cab \ldots
\end{array}
$$

The case where our affix is $bac$ is analogous.

⋄ However if $x_0$ was in the affix we have that $x_0$ takes the place of $a$ as the smallest letter. Then we have $w = \ldots x_0 cb \ldots$ and so

$$
\ldots x_0 cb \ldots \xrightarrow{\quad\text{Knuth}\quad} \ldots c x_0 b \ldots
$$

with $\text{Rem}_{x_0}$ arrows pointing down to

$$\ldots cb \ldots$$

> which leaves us with the same word in both cases. In a similar fashion to
> the previous part, working with the affix $bx_0c$ is analogous.
>
> In both cases we can see that removing the smallest letter and Knuth moving
> commutes on words. So this tells us that originally Knuth equivalent words
> will still be Knuth equivalent after removing the smallest letter.
>
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>
> [a]**Ross, Joel and myself discussed this exercise together.**

**Exercise 2** (7(b)). Given a pair $(S, T)$ in $P^\nu_{\lambda\mu}$, show that $T$ must be the unique highest weight tableau of shape $\mu$. Conclude that the pair $(T, T)$ corresponds under RSK to a two line array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{|\mu|} \\ b_1 & b_2 & \dots & b_{|\mu|} \end{pmatrix}$$

where $a_1, a_2, \dots, a_{|\mu|}$ is the unique weakly increasing word of content $\mu$ and $b_1, b_2, \dots, b_{|\mu|}$ is a ballot word.

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**Answer**

As $(S, T)$ is in $P^\nu_{\lambda\mu}$, then $\mathrm{rw}(S)\,\mathrm{rw}(T)$ is a ballot word. We[a] have that every suffix of a ballot word is itself ballot. So in particular $\mathrm{rw}(T)$ is ballot.

On every row, $T$ has the same entries and so its structure is determined. This means that it is of highest weight. Having shape $\mu$ uniquely determines it.

Now, instead of applying reverse RSK to $(T, T)$, we consider the two-line array comprised of the reverse column reading word of $T$, $b$ and the reverse reading word of $T$, $a = \mathrm{rev}\,\mathrm{rw}(T)$.

The reverse column reading word is obtained by reading left-to-right, and down-up through $T$, so when verifying it's ballot we must read right-to-left and down-up. This produces a ballot word $b$ because of the structure of $T$, always before adding a higher number, there will come a smaller number. And the reverse reading word of $T$ is weakly increasng due to its structure.

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[a]**Ross, Joel and myself worked on this, but then Parker helped me out with the final details.**

**Exercise 3** (7(c)(i)). Insert $b_1, \dots, b_n$ into $S$ and write $a_1, \dots, a_{|\mu|}$ into the boxes that appear in order. Let $R$ be the resulting skew tableau with letters $a_1, \dots, a_\mu$.

Show that $R$ has shape $\nu/\lambda$ with content $\mu$.

> **Answer**
>
> Observe the $|\lambda| + |\mu| = |\nu|$ by the fact that $(S,T) \in P_{\lambda\mu}^{\nu}$. So $|\nu| - |\lambda| = |\mu|$. Now, our assumption of $\mathrm{rw}(T)\,\mathrm{rw}(S)$ being ballot guarantees that when inserting $a$, we will get shape $\nu$, thus when skewing $S$ we get shape $\nu/\lambda$. The content of $b$ is precisely $\mu$.

**Exercise 4** (7(c)(ii))**.** To show it (the reading word of $R$) is ballot, consider the tableau $\tilde{R}$ formed by filling the shape $\lambda$ underneath $R$ with a highest weight word of negative entries $0, -1, -2, \ldots$, in other words, fill the top row of $\lambda$ with $0$'s, the next with $-1$, the next with $-2$, and so on. Let $V$ be the highest weight tableau of shape $\nu$. Apply RSK to the pair $(\tilde{R}, V)$, and use part $(a)$ of this problem to show that the letters corresponding to the entries of $\tilde{R}$ in the resulting two-line array form a highest weight word. Conclude that the reading word of $R$ is ballot

> **Answer**
>
> As $V$ is of highest weight, its rows are all the same elements and it has content $\nu$. In order to apply RSK we once again do it in reverse, following the same strategy as before.
>
> We look at the reverse column reading word of $\tilde{R}$ which we've seen, inserts into the tableau. Following the previous algorithm we will get that $\tilde{b}$ is a ballot word and using the first part of this problem we have that when removing the negative letters we still obtain a ballot word. This is because Knuth moves commute with crystal operators and thus preserve weight. In particular, our word is of highest weight and therefore is ballot.

**Exercise 5** (7(c)(iii))**.** Conclude that the map $(S,T) \mapsto R$ using the above construction is a well-defined function from $P_{\lambda\mu}^{\nu}$ to $Q_{\lambda\mu}^{\nu}$.

> **Answer**
>
> The previous two problems show that $(S,T)$ indeed land in $Q_{\lambda\mu}^{\nu}$ when applying this map as $\mathrm{sh}(R) = \nu/\lambda$ with content $\mu$ and its reading word is ballot.
>
> Showing that the map is well defined amounts to showing that if we have $(S_1, T_1) = (S_2, T_2) \in P_{\lambda\mu}^{\nu}$ then their image coincides in $Q_{\lambda\mu}^{\nu}$. As we have specified the process above, applying it to the same pair of tableau should give us the same result guaranteeing that their image is the same.