Homework 7

Due: Friday, March 24

1. [SS] 3.15(c).

2. [SS] 3.15(d). (HINT: *Instead of using the hint in the book, you can also proceed by considering the function* $\exp(f(z))$.)

3. Use Rouché's theorem to give another proof of the fundamental theorem of algebra, as follows.

   Let $p(z) = \sum_{j=0}^{d} a_j z^j$ be a polynomial, where $d \geq 1$ and $a_d \neq 0$.

   In class, we showed that there exist constants $C > 0$ and R_0 such that, if $|z| > R_0$, then $C|z^d| > |p(z)|$.

   Show that, for each $R > R_0$, $p(z)$ has exactly d roots (counted with multiplicity) of size less than R.

4. Let f be nonconstant and holomorphic in an open set containing $\overline{\mathbb{D}}$, the closed unit disk. Further suppose that if $|z| = 1$, then $|f(z)| = 1$.

   (a) Show that $f(z) = 0$ has a root, i.e., that the image of f contains 0. (HINT: *Use the maximum modulus principle.*)

   (b) Show that if $w_0 \in \mathbb{D}$, then there exists some $z_0 \in \mathbb{D}$ such that $f(z_0) = w_0$. (HINT: *Apply the result of (a) to the composition of f with a suitable Blaschke factor, as in [SS] 1.7.*)

   This is the same as [SS]3.17(a).