Exercise 1. Describe a "ballot-type" condition for a word of 1's and 2's to be lowest weight for $\mathfrak{sl}_2$, that is, that F sends the word to 0. Prove that your condition is correct. Do the same for $\mathfrak{sl}_3$ and the two lowering operators.

Answer

The condition we are looking for is precisely being a ballot word, this occurs when every prefix left-to-right has more 2's than 1's. Observe that any word like this will be sent by F_1 to zero. F_1 will look for the last unmatched 1, but having a greater number of 2's before it means that there will be no way for a 1 to be unmatched. Thus, F_1 will send such a word to zero.
Similarly for lowest weight words of $\mathfrak{sl}_3$, the condition is that in every prefix when reading left-to-right, we find more 3's than 2's and more 2's than 1's. Once again applying F_1 or F_2, we get nothing, because they will be looking for unmatched 1's or 2's respectively. But as the word is ballot, there are no unmatched 1's nor 2's. Observe also that it is not necessary for there to be double the amount of 2's because of the 1's and 3's. The F_i operators do not interact with each other so there's no qualms about that.

Exercise 2. Write the element E_{12} in $\mathfrak{sl}_3$ as an 8×8 matrix in the adjoint representation, by computing how E_{12} acts on the basis $\{ E_{ij} : i \neq j \} \cup \{ H_{12}, H_{23} \}$ defined in class, via the adjoint operator $[E_{12}, -]$.

For this exercise we shall employ `bra-ket` notation. Here, $\langle u|v\rangle$ is the inner product of vectors u, v and $|u\rangle\langle v|$ is the rank one matrix whose entries are $u_i v_j$. The following property holds:

$$|u\rangle\langle v|\,|x\rangle\langle y| = \langle v|x\rangle\,|u\rangle\langle y|\,.$$

Observe that our E_{ij} matrices are in fact $|e_i\rangle\langle e_j|$ where e_i is the standard basis vector. We will also take the liberty of invoking E_{ii} as $H_{ij} = E_{ii} - E_{jj}$. This will be handy as

$$[E_{ij}, H_{rs}] = [E_{ij}, E_{rr} - E_{ss}] = [E_{ij}, E_{rr}] - [E_{ij}, E_{ss}]\,.$$

With this, we may proceed:

Answer

If we take the Lie bracket of E_{ij} with E_{rs} we get

$$\begin{aligned}[E_{ij}, E_{rs}] &= E_{ij}E_{rs} - E_{rs}E_{ij} \\ &= |e_i\rangle\langle e_j|\,|e_r\rangle\langle e_s| - |e_r\rangle\langle e_s|\,|e_i\rangle\langle e_j| \\ &= \delta_{jr}\,|e_i\rangle\langle e_s| - \delta_{is}\,|e_r\rangle\langle e_j|\end{aligned}$$

In our case this formula specializes to

$$\delta_{2r}\,|e_1\rangle\langle e_s| - \delta_{1s}\,|e_r\rangle\langle e_2|$$

and for this matrix to not be zero we need either $r = 2$ or $s = 1$. This allows us to see

- $[E_{12}, E_{11}] = -\delta_{11}\,|e_1\rangle\langle e_2| = -E_{12}$.
- $[E_{12}, E_{12}] = 0$.
- $[E_{12}, E_{13}] = 0$.
- $[E_{12}, E_{21}] = \delta_{22}\,|e_1\rangle\langle e_1| - \delta_{11}\,|e_2\rangle\langle e_2| = E_{11} - E_{22} = H_{12}$.
- $[E_{12}, E_{22}] = \delta_{22}\,|e_1\rangle\langle e_2| = E_{12}$.
- $[E_{12}, E_{23}] = \delta_{22}\,|e_1\rangle\langle e_3| = E_{13}$.
- $[E_{12}, E_{31}] = -\delta_{13}\,|e_3\rangle\langle e_2| = -E_{32}$.
- $[E_{12}, E_{32}] = 0$.
- $[E_{12}, E_{33}] = 0$.

From this we have

$$[E_{12}, H_{12}] = -E_{12} - E_{12} = -2E_{12}, \quad [E_{12}, H_{23}] = E_{12} - 0 = E_{12},$$

and ordering our basis as

$$\{\, E_{12}, E_{13}, E_{21}, E_{23}, E_{31}, E_{32}, H_{12}, H_{23} \,\}$$

our matrix will be

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 3. Show that the embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3$ that sends

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is indeed an injective homomorphism of Lie algebras.

Answer

To show that this is Lie algebra homomorphism we must show that it respects the Lie bracket. Suppose $A, B \in \mathfrak{sl}_2$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

so that

$$[A, B] = AB - BA = \begin{pmatrix} bg - cf & af + bh - be - df \\ ce + dg - ag - ch & cf - bg \end{pmatrix}.$$

When embedding these matrices into $\mathfrak{sl}_3$ we get

$$\iota A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \iota B = \begin{pmatrix} e & f & 0 \\ g & h & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking their Lie bracket gives us precisely

$$[\iota A, \iota B] = \begin{pmatrix} bg - cf & af + bh - be - df & 0 \\ ce + dg - ag - ch & cf - bg & 0 \\ 0 & 0 & 0 \end{pmatrix} = \iota\,[A, B]$$

showing us that the mapping is indeed a morphism of Lie algebras.
This map is certainly injective for if

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow a = b = c = d = 0 \Rightarrow A = 0.$$

Exercise 4. Recall that the complete homogeneous symmetric function $h_\mu(x_1, \ldots, x_n)$, for a partition $\mu = (\mu_1, \ldots, \mu_k)$, can be defined as the product $h_{\mu_1} \ldots h_{\mu_k}$ where h_d is the sum of all monomials in $x_1, \ldots, x_n$ of degree d.

(a) Which Schur function in n variables is $h_d(x_1, \ldots, x_n)$ equal to?

(b) Show that $h_\mu(x_1, x_2, x_3)$ is the character of the tensor product of the k irreducible $\mathfrak{sl}_3$ representations

$$V^{(\mu_1,0)} \otimes \cdots \otimes V^{(\mu_k,0)}$$

where $V^{(a,b)}$ denotes the irreducible representation with highest weight (a, b).

Answer

(a) We have $h_d = s_d$! This is because s_d's monomials correspond to rows of length d which can be filled as a SSYT. This is the same as all the possible monomials in n variables in h_d.

(b) Recall that $\chi(V^{d,0}) = s_d$ so in this case we have

$$\chi\left(V^{(\mu_1,0)} \otimes \cdots \otimes V^{(\mu_k,0)}\right) = \prod_{j=1}^{k} \chi\left(V^{(\mu_j,0)}\right) = \prod_{j=1}^{k} s_{\mu_j} = \prod_{j=1}^{k} h_{\mu_j} = h_\mu$$

where we use the fact that $s_d = h_d$ for row tableaux.

(c) Kostka numbers count SSYT of shape λ with content μ. When restricting to 3 variables we are only considering fillings of tableaux with $1, 2$ and 3. In this fashion, each term in the decomposition into irreducibles of the tensor product is counted $K_{\lambda\mu}$ times.