

Exercise 1. Let $A : V \rightarrow W$ be a linear map between vector spaces.

- (a) Show that the induced map $\bigwedge^k(V) \rightarrow \bigwedge^k(W)$ is well-defined by

$$v_1 \wedge \dots \wedge v_k \mapsto Av_1 \wedge \dots \wedge Av_k$$

(extending linearly to sums).

- (b) Show that the map $A^* : W^* \rightarrow V^*$ defined by $(A^*(\eta))(v) := \eta(A(v))$ determines a map $\bigwedge^k(W^*) \rightarrow \bigwedge^k(V^*)$.
- (c) Show that, if V is an n -dimensional vector space, then the map $\bigwedge^n(V) \rightarrow \bigwedge^n(V)$ is multiplication by $\det A$.

Answer

To prove well-definedness of a map, it suffices to take two representatives of the same class and see that they map to the same place.

- (a) Consider then, without loss of generality,

$$v_1 \wedge v_2 \wedge \dots \wedge v_k = -(v_2 \wedge v_1 \wedge \dots \wedge v_k).$$

This second element we can reinterpret as

$$(-v_2) \wedge v_1 \wedge \dots \wedge v_k.$$

Applying $\bigwedge^k(A)$ to this we get

$$\begin{cases} Av_1 \wedge Av_2 \wedge \dots \wedge Av_k \\ A(-v_2) \wedge Av_1 \wedge \dots \wedge Av_k \end{cases}$$

and using the fact that A is linear we get

$$\begin{aligned} A(-v_2) \wedge Av_1 \wedge \dots \wedge Av_k &= -(Av_2 \wedge Av_1 \wedge \dots \wedge Av_k) \\ &= Av_1 \wedge Av_2 \wedge \dots \wedge Av_k \end{aligned}$$

and this is the desired representation of the image. **FINISH MULTILINEAR**

This allows to see that $\bigwedge^k(A)$ is well-defined.

- (b) The map A^* does indeed define a map from the exterior powers, namely $\bigwedge^k(A^*)$. **FINISH**

- (c) **NO IDEA**

Exercise 2. Show that the vectors $v_1, \dots, v_k \in V$ are linearly independent if and only if $v_1 \wedge \dots \wedge v_k \neq 0$ as an element of $\bigwedge^k(V)$.

Answer

Assume that $\{v_1, \dots, v_k\}$ is linearly dependent, then if $\{v_1, \dots, v_\ell\}$ is a maximally independent set, we may write any v_i with $\ell < i \leq k$ as a linear combination of $\{v_1, \dots, v_\ell\}$.

This means that

$$\begin{aligned} v_1 \wedge \dots \wedge v_k &= v_1 \wedge \dots \wedge v_{\ell+1} \wedge \dots \wedge v_k \\ &= v_1 \wedge \dots \wedge \sum_{i=1}^k c_i v_i \wedge \dots \wedge v_k \\ &= \sum_{i=1}^k c_i (v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_k) \end{aligned}$$

and all the summands will be zero as we will find repeated v_i 's in each term.

Exercise 3. We say that an element of $\bigwedge^k(V)$ is *decomposable* if it can be written as $v_1 \wedge \dots \wedge v_k$.

- (a) Suppose $v, w, x, y \in V$. Find necessary and sufficient conditions for $v \wedge w + x \wedge y \in \bigwedge^2(V)$ to be decomposable.
- (b) Show that $\omega \in \bigwedge^2(\mathbb{R}^4)$ is decomposable if and only if $\omega \wedge \omega = 0$.

Exercise 4. Let V be an n -dimensional inner product space. We can extend the inner product from V to all of $\bigwedge(V)$ by setting the inner product of homogeneous elements of different degrees equal to zero and by letting

$$\langle w_1 \wedge \dots \wedge w_k, v_1 \wedge \dots \wedge v_k \rangle = \det(\langle w_i, v_j \rangle)_{i,j}$$

and extending bilinearly.

Since $\bigwedge^n(V)$ is a one-dimensional real vector space, $\bigwedge^n(V) - \{0\}$ has two components. An *orientation* on V is a choice of component of $\bigwedge^n(V) - \{0\}$. If V is an oriented inner product space, then there is a linear map $\star : \bigwedge(V) \rightarrow \bigwedge(V)$ called the star map, which is defined by requiring that for any orthonormal basis e_1, \dots, e_n for V ,

$$\begin{aligned} \star(1) &= \pm e_1 \wedge \dots \wedge e_n, & \star(e_1 \wedge \dots \wedge e_n) &= \pm 1, \\ \star(e_1 \wedge \dots \wedge e_k) &= \pm e_{k+1} \wedge \dots \wedge e_n, \end{aligned}$$

where in each case we take “+” if $e_1 \wedge \dots \wedge e_n$ is in the preferred component of $\bigwedge^n(V)$ and we take “−” otherwise. Notice that $\star : \bigwedge^k(V) \rightarrow \bigwedge^{n-k}(V)$.

- (a) Prove that if e_1, \dots, e_n is an orthonormal basis for V , then the $e_{i_1} \wedge \dots \wedge e_{i_k}$ with $1 \leq i_1 < \dots < i_k \leq n$ and $1 \leq k \leq n$ give an orthonormal basis for $\bigwedge(V)$.
- (b) Prove that, as a map $\bigwedge^k(V) \rightarrow \bigwedge^k(V)$, $\star\star = (-1)^{k(n-k)}$.
- (c) Prove that, for $\omega, \eta \in \bigwedge^k(V)$, their inner product is given by

$$\langle \omega, \eta \rangle = \star(\omega \wedge \star\eta) = \star(\eta \wedge \star\omega).$$

Exercise 5. Let M^n be a closed manifold (i.e., a compact manifold without boundary) and let $\omega \in \Omega^1(M)$ so that $\omega_p \neq 0$ for all $p \in M$ (i.e., for all p , there exists $v \in T_p M$ so that $\omega_p(v) \neq 0$). Show that ω is not exact.