Exercise 1. Let $n \geqslant 2$ and consider fractions of the form $\frac{1}{ab}$ where a and b are relatively prime integers such that

$$a < b \leqslant n, \quad \text{and} \quad a + b > n$$

Prove that for all n, the sum of these fractions is equal to $\frac{1}{2}$.

Exercise 2. Consider a set S of $2n - 1$ distinct irrational numbers for $n \in \mathbb{N}$. Prove that there exist n distinct elements $x_1, \ldots, x_n \in S$ such that for any $a_1, \ldots, a_n$ non-negative rational numbers with $a_1 + \cdots + a_n > 0$, we have that $a_1x_1 + \cdots + a_nx_n$ is irrational.

Exercise 3. Let (a_n) be a sequence of real numbers such that $a_{i+j} \leqslant a_i + a_j$ for all $i, j = 1, 2, \ldots$. Prove that for all n we have

$$a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n} \geqslant a_n.$$

Exercise 4. Suppose $f : \mathbb{N} \to \mathbb{N}$ is an strictly increasing function such that

$$\begin{cases} f(2) = 2, \\ f(mn) = f(m)f(n) \quad \text{when} \quad m, n \quad \text{are relatively prime.} \end{cases}$$

Prove that $f(n) = n$ for all n.

Exercise 5. Consider the sequence (a_n) defined as follows:

$$\begin{cases} a_1 = 1, \\ a_{2n} = 1 + a_n \\ a_{2n+1} = \dfrac{1}{a_{2n}} \end{cases}$$

Prove that every positive rational number appears on this sequence exactly once.

Exercise 6. Consider $a_0, \ldots, a_n$ positive real numbers such that $a_{k+1} - a_k \geqslant 1$ for all $k = 0, \ldots, n-1$. Prove that

$$1 - \frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right) \ldots \left(1 + \frac{1}{a_n - a_0}\right) \leqslant \left(1 + \frac{1}{a_0}\right)\left(1 + \frac{1}{a_1}\right) \ldots \left(1 + \frac{1}{a_n}\right)$$

Exercise 7 (IMO77)**.** Suppose $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(n+1) \geqslant f(f(n))$ for all natural numbers. Prove that $f(n) = n$ for all n.