Exercise 1 (Exercise 1)**.** Prove that all three definitions of representations of finite groups given in the lecture notes are equivalent. Then, for the examples of the groups G and H from Examples 2.1 and 2.2 in the lecture notes, express these representations as a vector space with an action, and as a module.

The definitions in question are:

Definition 1. A representation of a group G over a field $\mathbb{F}$ is a homomorphism

$$\rho : G \to \mathrm{GL}_n(\mathbb{F})$$

where $\mathrm{GL}_n(\mathbb{F})$ is the group of invertible $n \times n$ matrices over $\mathbb{F}$.

Definition 2. A representation of a group G over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space V along with an action $G \rhd V$ by linear transformations, i.e. a homomorphism $\rho : G \to \mathrm{GL}(V)$.

Definition 3. A representation of a group G over a field $\mathbb{F}$ is an $\mathbb{F}G$-module V. (Here $\mathbb{F}G$ is the group ring consisting of formal linear combinations of elements of G over $\mathbb{F}$. A module is essentially a "vector space over a ring".)

Answer

The first definition introduces a map

$$G \to \mathrm{Aut}(V), \quad g \mapsto \rho(g),$$

where $V = \mathbb{F}^n$, and $\rho(g)$ acts as a linear transformation on $v \in V$. This map defines an action since ρ is a homomorphism, specifically:

- $\rho(e) = I_{n\times n}$.
- $\rho(gh) = \rho(g)\rho(h)$, by virtue that ρ is a homomorphism.

This leads to the definition of a representation of G on the vector space V.
Conversely, any $\mathbb{F}G$-module is, by definition, also a vector space over $\mathbb{F}$, given that $\mathbb{F} \subseteq \mathbb{F}G$. To equip this vector space with the structure of a $\mathbb{F}G$-module, we define the action of elements of $\mathbb{F}G$ using the representation. Specifically, for any $v \in V$ and $\sum_{g\in G} c_g g \in \mathbb{F}G$, we define:

$$\left(\sum_{g\in G} c_g g\right) \cdot v := \sum_{g\in G} c_g[\rho(g)v].$$

This shows how the representation can be viewed as an $\mathbb{F}G$-module.

Now the groups in question are

$$G = \{\, e, a, b, c : ab = c,\ a^2 = b^2 = e \,\} \simeq (\mathbb{Z}/2\mathbb{Z})^2$$
$$H = \langle a :\ a^4 = e \rangle \simeq \mathbb{Z}/4\mathbb{Z}$$

To realize the group representations as vector spaces with particular transformations let us follow the example in the notes. First, both groups can be represented by the space $\mathbb{F}^2$.For the first group we have the transformations

$$e \mapsto \mathrm{id}, \quad a \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

while for the cyclic group we have

$$e \mapsto \mathrm{id}, \quad \text{and} \quad a \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In order to see the group rings, we will use the following facts:

$$\mathbb{F}(\mathbb{Z}/n\mathbb{Z}) \simeq {}^{\mathbb{F}[x]}\big/_{\langle x^n - 1 \rangle}, \quad \text{and} \quad \mathbb{F}(G \times H) \simeq \mathbb{F}G \otimes_{\mathbb{F}} \mathbb{F}H.$$

The group rings are thus

$${}^{\mathbb{F}[x]}\big/_{\langle x^4 - 1 \rangle}, \quad \text{and} \quad \left({}^{\mathbb{F}[x]}\big/_{\langle x^2 - 1 \rangle} \right)^{\otimes 2}$$

and observe that in the case that $\mathbb{F}$ is algebraically closed both group rings are just $\mathbb{F}$.
In both cases, these are $\mathbb{F}$-vector spaces to which we extend the matrix action.

Exercise 2. Consider the representation of S_3 in which each permutation $\pi \in S_3$ is sent to its corresponding permutation matrix P , in which $P_{i,\pi(i)} = 1$ for all i, and all other entries are 0.

(a) Find a common eigenvector of all of the permutation matrices.

(b) Write the representation as a direct sum of irreducible representations.

Exercise 3. Let N be any open neighborhood of the identity element e in a connected Lie group G. Show that N generates G as a group.

Before proceeding with the proof, let's briefly review the multiplication map in a topological group and recall some essential properties of images and inverse images.

Remark 1. In any topological group, the multiplication map is a homeomorphism. Specifically, the maps

$$x \mapsto gx \quad \text{and} \quad x \mapsto g^{-1}x$$

are inverses of each other and are continuous by definition. Additionally, if f is a bijective function, then

$$f^{-1}[f[A]] = f[f^{-1}[A]] = A.$$

With these concepts in mind, we can now state and prove the following lemma.

Lemma 1. *For a topological group G, a subset $U \subseteq G$ is a neighborhood of an element $g \in G$ if and only if $g^{-1}U$ is a neighborhood of the identity element e.*

Proof

Define the map m_g by $x \mapsto gx$. Notice that

$$g^{-1}U = m_{g^{-1}}[U].$$

Since m_g is a homeomorphism, U is open if and only if $g^{-1}U$ is open.

Answer

Now, consider the subgroup $\langle N \rangle$ generated by N. This subgroup is non-empty, as it contains at least the identity element e.

- First, we show that $\langle N \rangle$ is open by proving that it's a neighborhood of each of its elements. Let $x \in \langle N \rangle$. Since N is a neighborhood of e, the set xN is a neighborhood of x. Observe that $xN \subseteq \langle N \rangle$, as products of elements of N with x definitely are products of elements of N to begin with. This means that $\langle N \rangle$ is open.

- Next, we prove that $\langle N \rangle$ is closed by showing that its complement is open. Suppose $x \notin \langle N \rangle$. The set xN is a neighborhood of x, and we aim to demonstrate that it's entirely contained outside of $\langle N \rangle$. Assume, for the sake of contradiction, that

$$\langle N \rangle \cap xN \neq \varnothing.$$

Then there exists an element

$$z \in \langle N \rangle \cap xN.$$

Being in xN means that $z = xy$ for some $y \in N$. Consequently, $x = zy^{-1}$, and since both z and y^{-1} belong to $\langle N \rangle$, it follows that $x \in \langle N \rangle$. This contradicts the assumption that x was outside $\langle N \rangle$. Therefore, xN does not intersect $\langle N \rangle$, meaning the complement of $\langle N \rangle$ is open, so $\langle N \rangle$ is closed.

In conclusion, the subgroup $\langle N \rangle$ is both open and closed, and it's non-empty. In a connected topological space, the only set with these properties is the entire space. Hence, we conclude that $\langle N \rangle = G$, as required.