Exercise 1 (8.4 Stein& Shakarchi)**.** Does there exist a holomorphic surjection from the unit disc to $\mathbb{C}$? ⟦Hint: Move the upper half-plane "down" and then square it to get $\mathbb{C}$.⟧

Answer

We first consider the map

$$F : B(0,1) \to \mathbb{H}, \quad z \mapsto i\frac{1-z}{1+z}.$$

Now translate by i and finally we square the function. We get the mapping

$$B(0,1) \to \mathbb{C}, \quad z \mapsto \left(i\frac{1-z}{1+z} - i\right)^2$$

which is surjective. Even though $z \mapsto z^2$ is not conformal, the first two mappings are and therefore the composition is surjective.

Exercise 2 (8.11 Stein& Shakarchi)**.** Show that if $f : B(0,R) \to \mathbb{C}$ is holomorphic, with $|f(z)| \leqslant M$ for some $M > 0$, then

$$\left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)}\right| \leqslant \frac{|z|}{MR}.$$

⟦Hint: Use the Schwarz lemma.⟧

Answer

We first rescale our function by considering

$$g(z) = \frac{1}{M} f(Rz).$$

This makes g a function from $B(0,1)$ to $B(0,1)$ unless $|f(z)|$ is exactly M.
In this case f achieved its maximum and so by the maximum modulus principle, f can't be non-constant and so f is constant. In this case the inequality is trivial.
Now if $|f(z)| < M$ for all M, then

$$g : B(0,1) \to B(0,1)$$

is almost the function we wish. We just need $g(0)$ to be 0. Let us consider the Blaschke factor

$$\frac{w - g(0)}{1 - \overline{g(0)}w},$$

if we compose it with g we get

$$\frac{g(z)-g(0)}{1-\overline{g(0)}g(z)} \xrightarrow{z\mapsto 0} 0.$$

We may apply Schwarz' lemma because the Blaschke factor still maps the unit circle to the unit circle, so this means that

$$\left|\frac{g(z)-g(0)}{1-\overline{g(0)}g(z)}\right| \leqslant |z|$$

and substituting for what we know is g we get:

$$\left|\frac{\frac{1}{M}f\left(Rz\right)-\frac{1}{M}f\left(0\right)}{1-\overline{f(0)/M}f(Rz)/M}\right| \leqslant |z|.$$

Taking a substitution $z \mapsto Rz$ and multiplying $1 = M^2/M^2$ all across we obtain

$$\left|\frac{M(f(z)-f(0))}{M^2-\overline{f(0)}f(z)}\right| \leqslant \left|\frac{z}{R}\right| \Rightarrow \left|\frac{f(z)-f(0)}{M^2-\overline{f(0)}f(z)}\right| \leqslant \frac{|z|}{MR}.$$

Exercise 3. Suppose $f, g : U \to V$ are conformal. Explain why there exists some $\delta \in \mathrm{Aut}(V)$ such that $g = \delta \circ f$.

Finally, show that all conformal mappings from the upper half-plane $\mathbb{H}$ to the unit disc $\mathbb{D}$ take the form

$$e^{i\theta}\frac{z-\beta}{z-\overline{\beta}}, \quad \text{where} \quad \theta \in \mathbb{R},\ \beta \in \mathbb{H}.$$

Answer

First observe that as f, g are conformal, they are bijective. So the function $g \circ f^{-1}$ is well defined. Also, this map goes from V to V and it's conformal. Therefore we may take $g \circ f^{-1} \in \mathrm{Aut}(V)$ to be our δ.

Now, we know that $z \mapsto i\frac{1-z}{1+z}$ conformally maps the unit disc to $\mathbb{H}$. On the other direction, consider $f : \mathbb{H} \to \mathbb{D}$. Then $f\left(i\frac{1-z}{1+z}\right)$ is a conformal map from $\mathbb{D}$ to $\mathbb{D}$ which means it's of the form $z \mapsto e^{i\theta}\frac{w-z}{1-\overline{w}z}$ and $w \in \mathbb{D}$. We may now invert our original function:

$$v = i\frac{1-z}{1+z} \Rightarrow z = \frac{1+iv}{1-iv}.$$

Thus replacing into our function we get

$$f(v) = e^{i\theta} \frac{w - \frac{1+iv}{1-iv}}{1 - \overline{w}\left(\frac{1+iv}{1-iv}\right)}.$$

Replacing v by z and rearranging we arrive at the expression:

$$f(z) = e^{i\theta} \frac{w(1-iz) - (1+iz)}{(1-iz) - \overline{w}(1+iz)} = e^{i\theta} \frac{w(1-iz) - (1+iz)}{(1-iz) - \overline{w}(1+iz)}.$$

Observe that if we expand and collect the z terms we arrive at the expression

$$\begin{aligned} \frac{w - 1 - iz(w+1)}{1 - \overline{w} - iz(1+\overline{w})} &= \frac{z + i\frac{w-1}{w+1}}{z + i\frac{1-\overline{w}}{1+\overline{w}}} \\ &= \frac{z - i\frac{1-w}{1+w}}{z - \left(-i\frac{1-\overline{w}}{1+\overline{w}}\right)} \end{aligned}$$

and observe that

$$\beta = i\frac{1-w}{1+w} \Rightarrow \overline{\beta} = -i\frac{1-\overline{w}}{1+\overline{w}}.$$

We know that this quantity is in the upper half plane, so we have our desired function $f(z) = e^{i\theta}\frac{z-\beta}{z-\overline{\beta}}$.