Exercise 1 (14.2.B Vakil)**.** Suppose $\mathcal{F}, \mathcal{G}$ are locally free sheaves on X of rank m and n respectively. Show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a locally free sheaf of rank mn.

Answer

Observe that if $\mathcal{F}$ is locally free of rank m then, there is a basis (U_i) of X such that $\mathcal{F} \mid_{U_i} \simeq \mathcal{O}_{U_i}^m$. Similarly $\mathcal{G} \mid_{V_j} \simeq \mathcal{O}_{V_i}^n$.
Now the collection $(U_i \cap V_j)_{i,j}$ is also a cover for X. From this we get that

$$\mathcal{F} \mid_{U_i \cap V_j} \simeq \mathcal{O}_{U_i \cap V_j}^m, \quad \text{and} \quad \mathcal{G} \mid_{U_i \cap V_j} \simeq \mathcal{O}_{U_i \cap V_j}^n.$$

Observe now that for open sets $W \subseteq U_i \cap V_j$ we have that

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(W) = \mathrm{Hom}(\mathcal{F} \mid_W, \mathcal{G} \mid_W) \simeq \mathcal{H}om(\mathcal{O}_W^m, \mathcal{O}_W^n) \simeq \mathcal{O}_W^{mn}.$$

This means that the Hom-sheaf locally looks like copies $\mathcal{O}$ which means it's locally free.

Exercise 2 (14.2.C Vakil)**.** If $\mathcal{E}$ is a locally free sheaf on X of rank n, then $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ is also locally free of rank n. This is the <u>dual</u> of $\mathcal{E}$.

i) Given transition functions for $\mathcal{E}$, describe the transition functions for $\mathcal{E}$ for $\mathcal{E}^\vee$. ⟦Note that if $\mathcal{E}$ is rank 1, i.e., invertible, the transition functions of the dual are the inverse of the transition functions of the original.⟧
ii) Show $\mathcal{E} \simeq \mathcal{E}^{\vee\vee}$. ⟦Caution: your argument showing that there is a canonical isomorphism $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$ better not also show that there is an isomorphism $\mathcal{F} \to \mathcal{F}^\vee$! We will see an example in 15.1 of a locally free $\mathcal{F}$ that is not isomorphic to its dual: the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n$.⟧

Answer

Exercise 3. Show that every invertible sheaf on $\mathbb{P}_k^1$ is of the form $O(n)$ for some n. ⟦Hint: Use the classification of finitely generated modules over a principal ideal domain to show that all invertible sheaves on $\mathbb{A}_k^1$ are trivial. Reduce to determining possible transition functions between the two open subsets in the standard cover of $\mathbb{P}_k^1$.⟧