MATH519 — Complex Analysis

Based on the lectures by Jeff Achter

Notes written by Ignacio Rojas

Spring 2023

Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This course is an introduction to analytic functions of a single complex variable. The subject is beautiful.– it turns out that a function with a complex derivative is highly structured – and enjoys a give and take with many other areas of mathematics.

Requirements

Knowledge of convergence of sequences, series: limits, continuity, differentiation, integration of one-variable functions is required.

Contents

Chapter 1

First Midterm

1.1 Interim | HW1

Exercise 1.1.1 (1.1 Stein & Shakarchi)**.** Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$.
(b) $1/z = \overline{z}$.
(c) $\mathrm{Re}(z) = 3$
(d) $\mathrm{Re}(z) > c$, (resp.,$\geqslant c$) where $c \in \mathbb{R}$.
(e) $\mathrm{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$.
(f) $|z| = \mathrm{Re}(z) + 1$.
(g) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$.

Answer

i) The first set is the set of points at the same distance from z_1 and z_2. If we consider the line segment z_1z_2, then the set in question is the bisector of that line segment.
ii) Note that

$$1/z = \overline{z} \iff 1 = \overline{z}z \iff 1 = |z|^2 \iff 1 = |z|,$$

thus the set is the unit circle.
iii) The set is a perpendicular line to the real axis at $z = 3$.
iv) This infinite set is an infinite half plane to the right (but not including) of the line $z = c$. In the other case, we do include the line in question.

v) Let us rephrase this inequality in terms of real numbers. Call $a = a_1 + ia_2$, $b = b_1 + ib_2$ and $z = x + iy$. Then

$$\mathrm{Re}(az + b) = \mathrm{Re}[a_1x - a_2y + b_1 + i(a_2x + a_1y + b_2)],$$

thus our desired inequality is true whenever $a_1x - a_2y + b_1 > 0$. Solving for y we get $y > (a_1x + b_1)/a_2$, which is the half plane located above the line $y = (a_1x + b_1)/a_2$.

vi) The equation in question is equivalent to

$$\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 = (\mathrm{Re}(z) + 1)^2.$$

To ease the notation, assume $z = x + iy$. Then the equation reads

$$x^2 + y^2 = x^2 + 2x + 1 \iff y^2 = 2x + 1 \iff x = (y^2 - 1)/2.$$

It holds the the parabola in question contains the points which satisfy the equation.

vii) This set is a line parallel to the real axis at $z = c$

Exercise 1.1.2. Do the following:

i) Show that the complex conjugation map $\kappa : \mathbb{C} \to \mathbb{C},\ z \mapsto \overline{z}$ is an involution, i.e., a ring homomorphism such that $\kappa \circ \kappa = \mathrm{id}$.

ii) Suppose $a \in \mathbb{R},\ z \in \mathbb{C}$. Show that

$$\mathrm{Re}(az) = a\,\mathrm{Re}(z), \quad \text{and} \quad \mathrm{Im}(az) = a\,\mathrm{Im}(z).$$

Answer

Let us take $z = x + iy$ with $x, y \in \mathbb{R}$.

i) We have $\overline{z} = x+i(-y) = x-iy$. Once more we get $\overline{\overline{z}} = x-i(-y) = x+iy = z$. Thus $\overline{\overline{z}} = z$ for any $z \in \mathbb{C}$. In conclusion $\overline{\overline{\cdot}} = \mathrm{id}$.

ii) It holds that

$$\begin{aligned} \mathrm{Re}(az) &= \mathrm{Re}(ax + aiy) = ax = a\,\mathrm{Re}(z), \\ \mathrm{Im}(az) &= \mathrm{Im}(ax + aiy) = ay = a\,\mathrm{Im}(z). \end{aligned}$$

Exercise 1.1.3. Do the following:

i) Prove that $|z+w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\overline{w})$.
ii) Use this to prove the parallelogram rule: $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$.

Answer

i) Note that

$$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + w\overline{z} + z\overline{w} + w\overline{w}.$$

The number $w\overline{z}$ is the conjugate of $z\overline{w}$, and summing a number and its conjugate returns twice its real part. Thus we get the desired identity.

ii) As the past identity holds for all complex numbers, it holds when $w = -w$. This means that $|z-w|^2 = |z|^2 + |-w|^2 + 2\,\mathrm{Re}(z(\overline{-w})) = |z|^2 + |w|^2 - 2\,\mathrm{Re}(z\overline{w})$ and summing this together with the first identity gives us the parallelogram law.

Exercise 1.1.4 (1.5 Stein & Shakarchi)**.** A set Ω is said to be pathwise connected if any two points in Ω can be joined by a (piecewise-smooth) curve entirely contained in Ω. The purpose of this exercise is to prove that an open set Ω is pathwise connected if and only if Ω is connected.

i) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a parametrization $z : [0,1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let

$$t_* = \sup_{0 \leqslant t \leqslant 1} \{\, t : \forall s[(0 \leqslant s < t) \Rightarrow (z(s) \in \Omega_1)] \,\}.$$

Arrive at a contradiction by considering the point $z(t_*)$.

ii) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subseteq \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subseteq \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega 1$ as desired.

Answer

i) We will proceed using a topological argument instead of a metric one. As the function γ is continuous, it pulls back Ω_1 and Ω_2 into $[0,1]$ as open sets. However, as the sets are disjoint, their inverse images are disjoint as well.

In other words, we have found two open disjoint sets which separate $[0,1]$:

$$[0,1] = \gamma^{-1}[\Omega_1] \cup \gamma^{-1}[\Omega_2].$$

But this is impossible because $[0,1]$ is a connected set. Thus, our assumption that Ω was disconnected must be false. We conclude that path-connectedness implies connectedness.

ii) Take Ω_1, Ω_2 as in the statement. Then Ω_1 is non-empty as $w \in \Omega_1$ because it's connected to itself through a trivial path. Suppose now that $z \in \Omega_1$ and that $d(z, \partial\Omega_1) > r > 0$. Take $x \in B(z,r)$, then there exists a line-segment between z and x and there's a smooth curve which connects $z \in \Omega_1$ with w. Thus the piecewise-continuous path from x to z and from z to w is a path which connects x and w. As x is arbitrary, it follows that $B(z,r) \subseteq \Omega_1$, and thus Ω_1 is open.

Formally, if $\gamma : [0,1] \to \Omega_1$ is the map which parametrizes the curve between z and w and $r : [0,1] \to B(z,r)$ is the map $t \mapsto tz + (1-t)x$, then the curve from x to w is parametrized by the function

$$f = \begin{cases} 2tz + (1-2t)x, \ t \in [0, 1/2], \\ \gamma(2t-1), \ t \in [1/2, 1]. \end{cases}$$

On the other hand take a point $z \in \Omega_2$ and let $d(z, \partial\Omega_2) > r > 0$. Consider a point $x \in B(z,r)$ and assume by way of contradiction that such x can be connected to w by a curve which can be parametrized by a smooth function γ. As the ball is convex, we can connect z to x and then to w creating a path between z and w. This is impossible as z cannot be connected to w by a path, thus our assumption must be false. It holds that x cannot be connected to w by a path and thus $x \in \Omega_2$. Therefore Ω_2 is also open. We conclude that $\Omega = \Omega_1 \cup \Omega_2$ is a union of two disjoint open sets, and since Ω is connected, it must hold that Ω_2 is empty. We conclude that Ω is path-connected.

Exercise 1.1.5 (1.7 Stein & Shakarchi)**.** The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

i) Let $z, w \in \mathbb{C}$ such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1$$

if $|z| < 1$ and $|w| < 1$, and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1$$

if $|z| = 1$ or $|w| = 1$. ⟦Hint: Why can one assume that z is real? I then suffices to prove that $(r-w)(r-\overline{w}) \leqslant (1-rw)(1-r\overline{w})$ with equality for appropriate r and $|w|$.⟧⟦Here is an alternate approach, which you may use if you like. Fix $w \in \mathbb{C}$ with $w < 1$, and consider the function $z \mapsto \frac{w-z}{1-\overline{w}z}$. What is $\overline{f(z)}$? By computing $f(z)\overline{f(z)}$, show that $|z| = 1$ implies $|f(z)| = 1$. Find a point z with $|z| < 1$ such that $|f(z)| < 1$. Since f is continuous, this shows that f takes the unit disc to itself. (Why?)⟧

ii) Prove that for a fixed $w \in \mathbb{D}$, the mapping $F : z \mapsto \frac{w-z}{1-\overline{w}z}$ satisfies the following:

a) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.
b) F interchanges 0 and w.
c) $|F(z)| = 1$ if $|z| = 1$.
d) F is bijective. ⟦Hint: Calculate $F \circ F$.⟧

Answer

i) The inequality in question is equivalent to

$$0 \leqslant |w-z| < |1-\overline{w}z|.$$

Since the quantities are positive, we can square them and preserve the order. It holds that

$$0 \leqslant |w-z|^2 < |1-\overline{w}z|^2 \iff 0 \leqslant (w-z)\overline{(w-z)} < (1-\overline{w}z)\overline{(1-\overline{w}z)},$$

Simplifying this expression we get

$$\begin{aligned}
&(w-z)(\overline{w}-\overline{z}) < (1-\overline{w}z)(1-w\overline{z}) \\
\iff\ & w\overline{w} - w\overline{z} - z\overline{w} + z\overline{z} < 1 - w\overline{z} - \overline{w}z + \overline{w}zw\overline{z} \\
\iff\ & |w|^2 + |z|^2 < 1 + |w|^2|z|^2 \\
\iff\ & 0 < (1-|w|^2)(1-|z|^2).
\end{aligned}$$

The inequality is true whenever both moduli are less than one, and whenever either is one equality is achieved.

ii) Now we suppose $w \in \mathbb{D}$ which means that $|w| < 1$. Taking $z \in \mathbb{D}$ and applying F gives us the quantity $\frac{w-z}{1-\overline{w}z}$ which by the previous argument, has modulus less than 1 whenever w, z do.

The function F is holomorphic because it is a quotient of holomorphic functions. The denominator is never zero inside the domain because that would mean that $1 = \overline{w}z$. And taking moduli in both sides of the equation gives us

$$1 = |1| = |w||z| < 1$$

which is impossible.

Now $F(0) = \frac{w-0}{1-0} = w$ and $F(w) = \frac{w-w}{1-|w|^2} = 0$. The denominator in the last expression is never zero because $|w| < 1$.

By the second part of the previous argument it holds that $|z| = 1$ immediately gives us $|F(z)| = 1$. And finally we will see that F is an involution:

$$F(F(z)) = F\left(\frac{w-z}{1-\overline{w}z}\right) = \frac{w - \left(\frac{w-z}{1-\overline{w}z}\right)}{1-\overline{w}\left(\frac{w-z}{1-\overline{w}z}\right)}.$$

Homogenizing and clearing denominators we get

$$\frac{w(1-\overline{w}z) - w + z}{1-\overline{w}z - \overline{w}(w-z)} = \frac{-w\overline{w}z + z}{1-\overline{w}w} = \frac{(-w\overline{w}+1)z}{1-\overline{w}w} = z.$$

This means that F is it's own inverse and therefore, F is bijective.

1.2 Day 1 | 20230120

The Complex Numbers

To construct the complex numbers we take the real numbers, adjoin a variable and mod out by $\langle x^2+1 \rangle$. We can also define $\mathbb{C}$ as $\{ a+bi : a, b \in \mathbb{R} \}$ with the property $i^2 = -1$. This means that we can multiply complex numbers in the following way:

$$(a+bi)(c+di) = ac + (bc+ad)i + bdi^2 = (ac-bd) + (ad+bc)i.$$

Also as x^2+1 is irreducible in $\mathbb{R}[x]$, $\mathbb{C}$ is a finite field extension of $\mathbb{R}$ of degree 2. As a 2-dimensional vector space $\{ 1, i \}$ is a basis for $\mathbb{C}$.

The map $a + bi \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$ is not a ring homomorphism, it's a bijection with a bit of structure. The map $z \mapsto \alpha z$, when $\alpha = a + bi$, is a linear map with the following action over the basis

$$\alpha \cdot 1 = \alpha \Rightarrow [\alpha] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\alpha \cdot i = -b + ai \Rightarrow [\alpha] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

which means that $[\alpha] = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. The converse, if we have a $\mathbb{R}$-linear transformation, then it's $\mathbb{C}$-linear if and only if it looks like $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Definition 1.2.1. The complex conjugation map is $a + bi \mapsto a - bi$, or $z \mapsto \overline{z}$.

This map is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

Example 1.2.2. For $\alpha = a + bi$, we have

$$\overline{2\alpha} = \overline{2(a + bi)} = \overline{2a + 2bi} = 2a - 2bi = 2\overline{al}.$$

Whereas if instead

$$\overline{i\alpha} = \overline{ai - b} = -b - ai \neq i\overline{\alpha} = b + ai.$$

As a $\mathbb{R}$-linear map, we can identify with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By looking at the shape of this matrix we can see that it is not $\mathbb{C}$-linear.

Lemma 1.2.3. *The map $z \mapsto \overline{z}$ is a ring homomorphism*

Proof

$\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\overline{w}$.

With the complex conjugation we can pick out the real and imaginary parts of $\alpha = a + bi$.

$$\alpha + \overline{\alpha} = 2\,\mathrm{Re}(\alpha), \quad \alpha - \overline{\alpha} = 2i\,\mathrm{Im}(\alpha)$$

A Notion of Size

Can't do geometry without one. Notice that for $z = a + bi$

$$z\overline{z} = a^2 + b^2 > 0.$$

From a complex number we have extracted a positive quantity.

Definition 1.2.4. The complex modulus of z is $|z| = \sqrt{z\overline{z}}$.

The fact that every number has n roots is very important in complex analysis.
As a vector in the plane, the norm of z is $|z|$

INC FIG

This means that $a + bi \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$ is an isometry. In this sense the distance between two complex numbers is $d(z, w) = |z - w|$.

Polar Coordinates (*ad hoc*)

For $\theta \in \mathbb{R}$, define

$$\exp(i\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta) \Rightarrow |\exp(i\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

Every point in the unit circle is of the form $e^{i\theta}$ and vice-versa.

INC FIG

For non-zero complex numbers, $z = |z|e^{i\theta}$ for some θ.

Definition 1.2.5. For a complex number $z = re^{i\theta}$, an argument of z is θ.

To have a well defined function, we mod out by multiples of 2π:

$$\arg : \mathbb{C}\backslash\{0\} \to \mathbb{R}\big/2\pi\mathbb{Z},$$

and we obtain a group isomorphism. In general, "lengths multiply, angles add."
For inverses if $z = re^{i\theta}$, then $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$.

Definition 1.2.6. The upper-half plane is $\mathbb{H} = \{\operatorname{Im}(z) > 0\}$.

Lemma 1.2.7. *If H is a half plane* $\operatorname{Im}(z - \beta/\gamma) > 0$

1.3 Day 2 | 20230123

Recall the complex conjugation map and the modulus of a complex number. This gives us an isometry between $\mathbb{R}^2$ and $\mathbb{C}$. Let us prove the lemma from last time.

Lemma 1.3.1. *If $H \subseteq \mathbb{C}$ is a half plane, then there exist $\beta, \gamma \in \mathbb{C}$ such that*

$$H = \left\{ z : \operatorname{Im}\left(\frac{z-\beta}{\gamma}\right) > 0 \right\}.$$

INC FIG

Pick a point $\beta \in H$, then translate H to the origin by $z \mapsto z - \beta$. The plane is now rotated by θ at the origin so we should rotate every point. Then $z \in H - \beta$ whenever $ze^{-i\theta} \in \mathbb{H}$. REDO

Let us see an application, for a polynomial, the coefficients determine the roots. The following lemma is a technical lemma.

Lemma 1.3.2. *Suppose $p \in \mathbb{C}[z]$ and H is a half plane which contains all the roots of p. Then H contains all the roots of p'.*

Proof

We can assume p is monic, so suppose $\alpha_1, \ldots, \alpha_d$ are the roots of $\mathbb{C}$. This means that

$$p(z) = \prod_{k=1}^{d}(z-\alpha_k) \Rightarrow p'(z) = \sum_{k=1}^{d} \frac{p(z)}{z-\alpha_k} \Rightarrow \frac{p'(z)}{p(z)} = \sum_{k=1}^{d} \frac{1}{z-\alpha_k}.$$

Now suppose that H contains all α_k and suppose $z_0 \notin H$, if we show $p'(z_0) \neq 0$ we are done because all the points which make p' vanish won't be outside H. Describe H by the previous lemma, there exist β, γ such that points in H satisfy the inequality $\operatorname{Im}\left(\frac{z-\beta}{\gamma}\right) > 0$. As z_0 is not in H, then $\operatorname{Im}\left(\frac{z_0-\beta}{\gamma}\right) < 0$. For each $k \in [d]$, we have that

$$z_0 - \alpha_k = z_0 - \beta + \beta - \alpha_k = (z_0 - \beta) - (\alpha_k - \beta)$$

so by taking imaginary parts

$$\operatorname{Im}\left(\frac{z-\alpha_k}{\gamma}\right) = \operatorname{Im}\left(\frac{z-\alpha_k}{\gamma}\right) - \operatorname{Im}\left(\frac{z-\alpha_k}{\gamma}\right)$$

The quantity on the right is negative because it's a negative number minus a positive. So it holds that $\mathrm{Im}\left(\frac{\gamma}{z-\alpha_k}\right) > 0$. With this we can calculate the following:

$$\mathrm{Im}\left(\gamma\frac{p'(z_0)}{p(z_0)}\right) = \mathrm{Im}\left(\sum_{k=1}^{d}\frac{\gamma}{z_0-\alpha_k}\right) > 0$$

so in particular this number is non-zero. Thus $p'(z_0) \neq 0$

Definition 1.3.3. A set $S \subseteq \mathbb{R}^n$ is convex if for any two points $x, y \in S$, the line segment between x and y is also contained in S. This is

$$\{\, ty + (1-t)x : \ x, y \in S \,\} \subseteq S.$$

The convex hull of S is the intersection of all convex sets containing S.

In the case of a finite set of complex numbers, the convex hull can be found by intersecting half-planes which contain them.

Corollary 1.3.4 (Gauss-Lucas). *The roots of $p'(z)$ are contained in the convex hull of the roots of $p(z)$.*

Metric Spaces

Definition 1.3.5. A metric space is a set with a distance function.

Example 1.3.6. $\mathbb{R}^n$ is a metric space with $d(x, y) = \|x - y\|$. Subsets of metric spaces with an induced distance are metric spaces.

- nbhd
- open and closed
- Cauchy

Definition 1.3.7. Cauchy sequence

1.4 Day 3 | 20230125

The defining property of $\mathbb{R}$ is that it is complete. In that sense it is possible to prove that $\mathbb{R}^n$ is also complete.

Derivatives

Recall a real function g is differentiable at x_0 if there exists a real number a such that

$$g(x) = g(x_0) + a(x - x_0) + \psi(x), \ \frac{\psi(x)}{x - x_0} \xrightarrow[x \to x_0]{0} .$$

In the same sense a multivariable function is differentiable when there exists a linear transformation such that a similar condition holds.

Definition 1.4.1. f has complex derivative iff real derivative and Cauchy-Riemann equations

Example 1.4.2. The map $z \mapsto \overline{z}$ is not complex-differentiable. First by matrix definition and second with limit.

1.5 Day 4 | 20230127

Lemma 1.5.1. *If $\sum_{n \geqslant 0} z_n$ is absolutely convergent, then it's convergent.*

Proof

If s_n is a partial sum, then

$$|s_n - s_m| = \left| \sum_{i=m+1}^{n} z_i \right| \leqslant \sum_{i=m+1}^{n} |z_i| < \varepsilon$$

because $\sum |z_n|$ is Cauchy.

Power Series

Definition 1.5.2. A <u>power series</u> (centered at 0) is an expression of the form $\sum_{n \geqslant 0} a_n z^n$.

Example 1.5.3. The power series for the exponential function is $e^z = \sum_{n \geqslant 0} \frac{z^n}{n!}$.

Theorem 1.5.4 (Cauchy-Hadamard)**.** *Suppose $\sum_{n \geqslant 0} a_n z^n$ has radius of convergence $\frac{1}{r} = \limsup |a_n|^{\frac{1}{n}}$. Then the series converges for $|z| < r$ and diverges for $|z| > r$.*

Proof

1.6 Day 5 | 20230130

Last time with Hadamard's criterion we learned something that we *already know*. Recall that for radii less than the radius of convergence, power series converge.

As a corollary we can prove the following:

Corollary 1.6.1. *Suppose $f(z) = \sum_{n\geqslant 0} a_n z^n$ has radius of convergence R. Then the following holds*:

i) *The formal derivative of f,*
$$g(z) = \sum_{n\geqslant 1} n a_n z^{n-1}$$
converges absolutely and uniformly on $B(0, R)$.

ii) *$f'(z) = g(z)$.*

Proof

Notice that
$$\lim_{n\to\infty} \sqrt[n]{n} = 1 \Rightarrow g \text{ converges.}$$

This is because $\limsup |na_n|^{1/n} = \limsup |a_n|^{1/n}$.

Call S_N the N^{th} partial sum of f. For $r < R$, suppose $|z - z_0| < r$. Then
$$\left|\frac{f(z) - f(z_0)}{z - z_0} - g(z)\right| = \left|\frac{S_n(z) - E_n(z) - S_n(z_0) + E_n(z_0) - g(z_0)(z - z_0)}{z - z_0}\right|.$$

Let us now add zero carefully and apply the triangle inequality. The previous term is less than
$$\left|\frac{S_N(z) - S_N(z_0)}{z - z_0} - S_N'(z_0)\right| + |S_N'(z_0) - g(z_0)| + \left|\frac{E_N(z) - E_N(z_0)}{z - z_0}\right|.$$

The last term which contains the errors can be written as
$$\left|\sum_{n\geqslant N} \frac{a_n(z^n - z_0^n)}{z - z_0}\right| \leqslant \sum_{n\geqslant N} n|a_n|r^{n-1}$$

and for large N, this quantity is small. With a similar reasoning we get that
$$|S_N'(z_0) - g(z_0)| \leqslant \sum_{n\geqslant N} n|a_n z^{n-1}|.$$

For z close to z_0, the first term is small as well.

Corollary 1.6.2. *A complex power series is infinitely differentiable.*

Lemma 1.6.3. *The power series of the exponential function satisfies the equality $e^{z+w} = e^z e^w$.*

Proof

$$\begin{aligned} e^z e^w &= \left(\sum_{n\geqslant 0} \frac{z^n}{n!}\right)\left(\sum_{n\geqslant 0} \frac{w^n}{n!}\right) \\ &= \sum_{n\geqslant 0} \sum_{k+\ell=n} \frac{z^k}{k!}\frac{w^\ell}{\ell!} \\ &= \sum_{n\geqslant 0} \frac{1}{n!} \sum_{k+\ell=n} \frac{n!}{k!\ell!} z^{k+\ell} \\ \sum_{n\geqslant 0} \frac{1}{n!}(z+w)^n &= e^{z+w}. \end{aligned}$$

Theorem 1.6.4. $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Lemma 1.6.5. *$\theta \in \mathbb{R}$, then $e^{i\theta} = 1$ iff $\theta \in 2\pi\mathbb{Z}$.*

Proposition 1.6.6. *If $z = re^{i\alpha}$, then β is an argument of z iff $\alpha - \beta \in 2\pi\mathbb{Z}$.*

Corollary 1.6.7. *There is a grp isom $\mathbb{C}^\times \to \mathbb{R}_{\geqslant 0} \times \mathbb{R}/2\pi\mathbb{Z}$.*

1.7 Interim 2 | HW2

Exercise 1.7.1. Suppose $S \subseteq \mathbb{C}$ is a domain and $f : S \to \mathbb{C}$ is differentiable at $z_0 \in S$.

i) Compute $f'(z_0)$ along a a trajectory $z_0 + \Delta x$ where $\Delta x \to 0$. Show that

$$f'(z_0) = u_x(z_0) + iv_y(z_0).$$

ii) Compute $f'(z_0)$ along a a trajectory $z_0 + i\Delta y$ where $\Delta y \to 0$. Show that

$$f'(z_0) = (1/i)(u_y(z_0) + iv_x(z_0)).$$

iii) Conclude that f satisfies the Cauchy-Riemann equations.

Answer

By definition, for $h \in \mathbb{C}$, we have

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)$$

whenever f is differentiable at z_0.

i) Take $h = \Delta x$, a number with no imaginary part. Then separating f into its real and imaginary parts we have

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta x\to 0} \frac{f(z_0+\Delta x)-f(z_0)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0)+iv(x_0+\Delta x, y_0)-u(x_0,y_0)-iv(x_0,y_0)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0)-u(x_0,y_0)}{\Delta x} + i\lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0)-v(x_0,y_0)}{\Delta x} \\
&= \frac{\partial}{\partial x}u(x_0,y_0) + i\frac{\partial}{\partial x}v(x_0,y_0)
\end{aligned}$$

ii) On the flipside, take $h = i\Delta y$ with $\Delta y \to 0$. We once again separate f as follows:

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta y\to 0} \frac{f(z_0+i\Delta y)-f(z_0)}{i\Delta y} \\
&= \lim_{\Delta y\to 0} \frac{u(x_0, y_0+\Delta y)+iv(x_0, y_0+\Delta y)-u(x_0,y_0)-iv(x_0,y_0)}{i\Delta y} \\
&= \lim_{\Delta y\to 0} \frac{u(x_0, y_0+\Delta y)-u(x_0,y_0)}{i\Delta y} + i\lim_{\Delta y\to 0} \frac{v(x_0, y_0+\Delta y)-v(x_0,y_0)}{i\Delta y} \\
&= \frac{1}{i}\frac{\partial}{\partial y}u(x_0,y_0) + \frac{\partial}{\partial y}v(x_0,y_0) \\
&= \frac{\partial}{\partial y}v(x_0,y_0) - i\frac{\partial}{\partial y}u(x_0,y_0)
\end{aligned}$$

iii) As the derivatives along both trajectories should match, we have that

$$\frac{\partial}{\partial x}u(x_0,y_0) + i\frac{\partial}{\partial x}v(x_0,y_0) = \frac{\partial}{\partial y}v(x_0,y_0) - i\frac{\partial}{\partial y}u(x_0,y_0).$$

Two complex numbers are equal whenever both the real and imaginary parts coincide, so it must hold that

$$\frac{\partial}{\partial x}u(x_0,y_0) = \frac{\partial}{\partial y}v(x_0,y_0), \qquad \frac{\partial}{\partial x}v(x_0,y_0) = -\frac{\partial}{\partial y}u(x_0,y_0).$$

If a function is holomorphic for every z, then this translates to $u_x = v_y$ and $v_x = -u_y$.

Exercise 1.7.2 ([1] 1.13). Suppose f is holomorphic in an open set Ω. Prove that in any one of the following cases:

$$\mathrm{Re}(f) \text{ is constant;} \quad \mathrm{Im}(f) \text{ is constant;} \quad |f| \text{ is constant;}$$

one can conclude that f is constant.

Answer

As f is holomorphic, f satisfies the Cauchy-Riemann equations. This means that if $f = u + iv$, then

$$u_x = v_y, \quad v_x = -u_y.$$

- If either u or v are constant, then u_x, u_y or v_x, v_y are both zero. In either of those case, by the Cauchy-Riemann equations we can conclude that the other pair of derivatives is zero respectively.
- If the complex modulus is constant, then $|f|^2 = u^2 + v^2$ is constant as well. Differentiating the expression with respect to both variables gives us

$$\begin{cases} 2uu_x + 2vv_x = 0 \\ 2uu_y + 2vv_y = 0 \end{cases} \Rightarrow \begin{cases} uu_x + vv_x = 0 \\ uu_y + vv_y = 0 \end{cases}$$

  Now, for sake of argument suppose u isn't zero. Applying the Cauchy-Riemann equations we can restate the first equation as follows:

$$\begin{cases} uv_y + v(-u_y) = 0 \\ uu_y + vv_y = 0 \end{cases} \Rightarrow \begin{cases} v_y = \frac{v}{u}u_y \\ uu_y + vv_y = 0 \end{cases}$$

  Substituting the first equation into the second we obtain

$$uu_y + v\left(\frac{v}{u}u_y\right) = \left(\frac{u^2+v^2}{u}\right)u_y = 0$$

  from which follows that either $u^2 + v^2 = 0$ or $u_y = 0$. In the first case, as u is a non-zero real function, it is impossible for the sum to be zero. So it must hold that $u_y = 0$.

  Doing a similar process by solving for u_y on the second equation we reach the condition that $v_y = 0$ as well. From here, using the Cauchy-Riemann equations we see that all partial derivatives of u and v are zero as we wished.

In the case that $u = 0$, we refer to the first case, where u is a constant.

Finally we conclude that f is constant in any case.

Exercise 1.7.3. Prove the following:

i) The power series $\sum_{n\geqslant 0} nz^n$ doesn't converge for any point on the unit circle.
ii) The power series $\sum_{n\geqslant 0} \frac{z^n}{n^2}$ converges for *every* point in the unit circle.
iii) The power series $\sum_{n\geqslant 0} \frac{z^n}{n}$ converges for every point in the unit circle, *except* $z = 1$.

Answer

i) We will prove that the series in question isn't Cauchy. Consider S_m, the m^{th} partial sum, then

$$|S_{m+1} - S_m| = m + 1$$

because z has complex modulus 1. Recall that a sequence of complex numbers (z_n) is a Cauchy sequence whenever

$$\forall \varepsilon \exists N \left[\forall m \forall n (m \geqslant N \wedge n \geqslant N \wedge \varepsilon > 0 \Rightarrow |z_m - z_n| < \varepsilon)\right].$$

In order to prove that (S_m) isn't Cauchy we must contradict this statement. Thus we must find an $\varepsilon_0 > 0$ such that for all N, there are m, n for which $|S_m - S_n| > \varepsilon_0$.

Take $\varepsilon_0 = 1$, m any sufficiently large natural number and $n = m + 1$ as we did before. Thus we have that $m + 1 > 1$ which lets us conclude that (S_m) isn't Cauchy. There are no non-Cauchy convergent sequences in $\mathbb{C}$ so it must hold that our series diverges given the condition that $|z| = 1$.

ii) Recall the Weierstrass M-test which states that if $(f_n(z))$ is a sequence of functions and there are $M_n > 0$ such that $|f_n(z)| \leqslant M_n$ and $\sum M_n$ is a convergent series, then $\sum f_n$ converges uniformly.

In this case, pick $M_n = \frac{1}{n^2}$. The series $\sum \frac{1}{n^2}$ converges as it is a p-series. Then

$$|z| = 1 \Rightarrow \left|\frac{z^n}{n^2}\right| \leqslant \frac{1}{n^2}$$

and thus we can conclude that $\sum \frac{z^n}{n^2}$ converges uniformly for points in the unit circle.

iii) The series in question is the harmonic series when $z = 1$ so it diverges. We will prove that when $|z| = 1$, but $z \neq 1$ this series is Cauchy. So let us fix z with $|z| = 1$ and call $S_m = \sum_{k=0}^{m} \frac{z^k}{k}$, then let $\varepsilon > 0$. Assume $n > m$ for sake of argument and then

$$\begin{aligned}
|S_n - S_{m-1}| &= \left|\sum_{k=m}^{n} \frac{z^k}{k}\right| \\
&= \left|\frac{1}{n}\sum_{k=1}^{n} z^k - \frac{1}{m}\sum_{k=1}^{m-1} z^k - \sum_{k=m}^{n-1}\left(\frac{1}{k+1} - \frac{1}{k}\right)\sum_{j=1}^{k} z^j\right| \\
&\leqslant \frac{1}{n}\left|\frac{z^{n+1}-z}{z-1}\right| + \frac{1}{m}\left|\frac{z^m - z}{z-1}\right| + \sum_{k=m}^{n-1}\left|\frac{z^{n+1}-z}{(z-1)(k^2+k)}\right| \\
&\leqslant \frac{2}{n}\left|\frac{1}{z-1}\right| + \frac{2}{m}\left|\frac{1}{z-1}\right| + \sum_{k=m}^{n-1}\left|\frac{1}{z-1}\right|\frac{2}{k^2+k}
\end{aligned}$$

Now let us state a couple of facts:

- ◇ $\left|\frac{1}{z-1}\right|$ might be arbitrarily large, but z is fixed. This means that $\left|\frac{1}{z-1}\right|$ is finite.
- ◇ Call $\widetilde{S}_r = \sum_{k=1}^{r} \frac{2}{k^2+k}$, it is important to note that this a sequence of *positive numbers*. $\widetilde{S}_\infty$ converges after comparing with $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

We will name $M = \left|\frac{1}{z-1}\right|$ so that the last expression can be written as follows:

$$\frac{2M}{n} + \frac{2M}{m} + M(\widetilde{S}_{n-1} - \widetilde{S}_{m-1}).$$

Now, as $\frac{1}{n}$ converges to zero, there exists $N_1 \in \mathbb{N}$ such that

$$n \geqslant N_1 \Rightarrow \frac{1}{n} < \frac{\varepsilon}{6M}, \ \varepsilon > 0$$

On the other hand as $\widetilde{S}_r$ converges, there exists an $N_2 \in \mathbb{N}$ such that

$$m, n \geqslant N_2 \Rightarrow |\widetilde{S}_m - \widetilde{S}_n| < \frac{\varepsilon}{3M}, \ \varepsilon > 0$$

Pick $N = \max N_1, N_2$ and let $\varepsilon > 0$, then whenever $m, n \geqslant N$, the following holds

$$\frac{2M}{n} + \frac{2M}{m} + M(\widetilde{S}_{n-1} - \widetilde{S}_{m-1}) \leqslant 2M\frac{\varepsilon}{6M} + 2M\frac{\varepsilon}{6M} + M\frac{\varepsilon}{3M} = \varepsilon.$$

Therefore the series in question is Cauchy and we can conclude that it converges.

Exercise 1.7.4. Let $\alpha \in \mathbb{C}$, $r > 0$ and $\gamma_r : [0, 2\pi[\to \mathbb{C}$ given by $t \mapsto re^{it} + \alpha$. Let $n \in \mathbb{N}$, calculate the integral $\int\limits_{\gamma_1(0)} z^n \mathrm{d}z$.

Answer

We can parametrize with $t \mapsto e^{it}$ with $r \in [0, 2\pi[$ so that

$$\int\limits_{\gamma_1(0)} z^n \mathrm{d}z = \int_0^{2\pi} (e^{it})^n (ie^{it}\mathrm{d}t) = \int_0^{2\pi} ie^{i(n+1)t}\mathrm{d}t = \left.\frac{e^{i(n+1)t}}{n+1}\right|_0^{2\pi} = \frac{e^{2\pi i(n+1)}}{n+1} - \frac{1}{n+1} = 0.$$

Exercise 1.7.5. Consider the following three groups:

- $\mathbb{C}^\times$ with multiplication as binary operation.
- $\mathbb{R}_{>0}$ with multiplication as binary operation.
- $\mathbb{R}/_{2\pi\mathbb{Z}}$ with addition as binary operation.

Show that

$$\alpha : \mathbb{C}^\times \to \mathbb{R}_{>0} \oplus \mathbb{R}\big/_{2\pi\mathbb{Z}}, \; z \mapsto (|z|, \arg(z))$$

is a group isomorphism as follows:

i) Show that α is a group homomorphism. ⟦Hint: This comes down to show that $|zw| = |z||w|$ and $\arg(zw) = \arg(z) + \arg(w)$.⟧
ii) Show that α is surjective. ⟦Hint: For $r \in \mathbb{R}_{>0}$ and representative $\theta \in \mathbb{R}$ show that there is some $\mathbb{C}^\times$ such that $|z| = r$ and $\arg(z) = \theta$.⟧
iii) Show that α is injective. ⟦Hint: Suppose $\alpha(z) = (1, 0)$, then show that $z = 1$.⟧

Answer

i) The function α is a homomorphism because

$$\begin{aligned} \alpha(wz) &= (|wz|, \arg(wz)) = (|w||z|, \arg(w) + \arg(z)) \\ &= (|w|, \arg(w)) \circ (|z|, \arg(z)) = \alpha(w) \circ \alpha(z) \end{aligned}$$

where $\circ$ is the group operation in the direct product. To prove the equalities hold, take $wz = r_1 e^{i\theta_1}$, $w = r_2 e^{i\theta_2}$ and $z = r_3 e^{i\theta_3}$. Then

$$wz = (r_2 e^{i\theta_2})(r_3 e^{i\theta_3}) = (r_2 r_3) e^{i(\theta_2 + \theta_3)} = r_1 e^{i\theta_1},$$

and as the polar representation of a complex number is unique we have that $r_1 = r_2 r_3$ and $\theta_1 = \theta_2 + \theta_3$.

ii) Take (r, θ) in the codomain of α. As $r > 0$, we can write it as $x^2 + y^2$ for $x, y \in \mathbb{R}$. Given that condition we may find the angle by the relation $\tan(\theta) = \frac{y}{x}$. Taking r and θ as given lets us construct a complex number $z = x + iy$ such that $\alpha(z) = (r, \theta)$.

iii) If it happened that $r = 1$ and $\theta = 0$, then the complex number in question could be represented as $1 \cdot e^0 = 1$. Thus $z = 1$. This means that $\ker(\alpha) = \{ \mathrm{id} \}$ and thus, as α is a morphism, it's also injective.

1.8 Day 6 | 20230201

Definition 1.8.1. A parametrization of a curve is a function $z : [a, b] \to \mathbb{C}$.

It is smooth if it's differentiable and piecewise smooth if for a partition of $[a, b]$, z is smooth on the parts.

Example 1.8.2. The function $z : [0, 2\pi] \to \mathbb{C},\ t \mapsto e^{it}$ is a parametrization of the unit circle.

Definition 1.8.3. Two parametrizations w, z are equivalent if there exists a bijection $[a, b] \to [c, d]$ such that $w(s) = z(s(t))$.

Example 1.8.4. An equivalent parametrization of e^{it} is $[0, 1] \to \mathbb{C},\ t \mapsto e^{2\pi it}$.

The reverse parametrization of $z : [a, b] \to \mathbb{C}$ is $z^- : [-b, -a] \to \mathbb{C},\ t \mapsto z(-t)$. A curve is closed if it starts where it ends. Simple curves don't cross themselves.

Definition 1.8.5. The integral over a curve γ is

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t$$

where $z : [a, b] \to \gamma$ parametrizes the curve.

Example 1.8.6. Consider the integral of $\overline{z}$ over the unit circle. This is

$$\int_{\{ |z|=1 \}} \overline{z}\mathrm{d}z = \int_0^{2\pi} \overline{e^{it}} i e^{-it}\mathrm{d}t = \int_0^{2\pi} i\mathrm{d}t = 2\pi i.$$

Integrals over the complex numbers obey the same properties as over the real numbers. The arc length of a curve is the same as in multivariable calculus. The integral also obeys the triangle inequality.

Definition 1.8.7. A domain is a non-empty, open and connected subset of $\mathbb{C}$.

Lemma 1.8.8. *If F, f are functions defined on Ω, a domain, with $F' = f$, and $w, z \in \Omega$, then*

$$\int_\gamma f(z)\mathrm{d}z = F(z) - F(w)$$

where $\gamma \subseteq \Omega$ is a curve connecting w to z.

Corollary 1.8.9. *If the curve is closed the integral is zero.*

As a consequence, the function $\overline{z}$ has no antiderivative in any ball around the origin.

Lemma 1.8.10. *Suppose f is holomorphic on Ω and $f' = 0$ on Ω. Then f is constant.*

Proof

If $w, z \in \Omega$, then

$$0 = \int f' = f(z) - f(w) \Rightarrow f(z) = f(w)$$

so f must be constant.

Next time: Goursat's theorem.

1.9 Day 7 | 20230203

1.10 Day 8 | 20230206

Last time we proved Morera's theorem. Recall Goursat's theorem, which tells us that along a contour a holomorphic function has zero integral. From this it is extracted that holomorphic functions have primitives.

Corollary 1.10.1. *If f is holomorphic on an open disk D and $\gamma \subseteq D$ is a closed contour, then $\int_\gamma f(z)\mathrm{d}z = 0$.*

Proof

As f admits a primitive F on D, the integral in question is $F(end) - F(begin) = 0$.

Toy Contours

A toy contour is a closed curve with well defined interior and exterior and it's *easy to describe*^{TM}.

Example 1.10.2. Squares are toy contours. A hollow circle with a rectangle is a toy contour.

Theorem 1.10.3. *If γ is a toy contour, f is holomorphic on an open set containing γ in its interior, then $\int f(z)\mathrm{d}z = 0$.*

Example 1.10.4. Let us calculate $\int\limits_0^\infty \frac{1-\cos(x)}{x^2}\mathrm{d}x$.

For that, consider the function $z \mapsto \frac{1-e^{iz}}{z^2}$. DRAW FIG. The integral can be separated into

$$\int\limits_{-R}^{-\varepsilon} f + \int\limits_{\gamma_\varepsilon} f + \int\limits_{\varepsilon}^{R} f + \int\limits_{\gamma_R} f.$$

We can parametrize γ_R by $t \mapsto R^{it}$ with $t \in [0, \pi]$. Computing the integral we get the function

$$\frac{1-\exp(iRe^{it})}{(Re^{it})^2} \Rightarrow |f(t)| \leqslant \frac{1+|\exp(iRe^{it})|}{|(Re^{it})^2|} \leqslant \frac{2}{R^2}.$$

1.11 Interim 3 | HW3

Exercise 1.11.1 (Exercise 2)**.** Evaluate the integral $\int\limits_{\gamma_1(0)} \mathrm{Re}(z)\mathrm{d}x$ in two ways:

i) Directly using the definition. ⟦Hint: You can model your calculation on the work we did in class to compute the integral of $\overline{z}$.⟧

ii) Using the fact that $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$.

Answer

Both integrals will use the parametrization $t \mapsto e^{it}$ with $t \in [0, 2\pi[$.

i) The first integral is

$$\int_0^{2\pi} \mathrm{Re}(e^{it})(ie^{it})\mathrm{d}t = \int_0^{2\pi} ie^{it}\cos(t)\mathrm{d}t = \int_0^{2\pi} (i\cos^2(t) - \sin(t)\cos(t))\mathrm{d}t$$
$$= i\int_0^{2\pi} cos^2(t)\mathrm{d}t - \int_0^{2\pi} \sin(t)\cos(t)\mathrm{d}t = i\pi + 0 = i\pi.$$

ii) The second integral is

$$\int_{\gamma_1(0)} \frac{z+\overline{z}}{2} = \frac{1}{2}\int_{\gamma_1(0)} z\mathrm{d}z + \frac{1}{2}\int_{\gamma_1(0)} \overline{z}\mathrm{d}z = 0 + \frac{1}{2}\int_0^{2\pi} e^{-it}(ie^{it})\mathrm{d}t = \frac{1}{2}(2\pi i) = i\pi.$$

Both calculations coincide in the value of $i\pi$.

Exercise 1.11.2. Suppose f is defined on a domain Ω with $\gamma \subseteq \Omega$, a closed contour. Additionally, suppose that for $\varepsilon > 0$, there exists a polynomial $P_\varepsilon(z)$ such that $|f(z) - P_\varepsilon(z)| < \varepsilon$ for $z \in \gamma$. Show that $\int_\gamma |f(z)|\mathrm{d}z = 0$ and thus conclude that $\int_\gamma f(z)\mathrm{d}z = 0$.

Consider γ an arbitrary, but fixed contour inside Ω. For $\varepsilon/\ell(\gamma) > 0$, there exists $P_{\varepsilon/\ell(\gamma)}(z)$ such that
$$|f(z) - P_{\varepsilon/\ell(\gamma)}(z)| < \frac{\varepsilon}{\ell(\gamma)}.$$
Then, applying the triangle inequality we have

$$\int_\gamma f(z)\mathrm{d}z \leqslant \int_\gamma |f(z)|\mathrm{d}z \leqslant \int_\gamma |f(z) - P_{\varepsilon/\ell(\gamma)}(z)|\mathrm{d}z + \int_\gamma |P_{\varepsilon/\ell(\gamma)}(z)|\mathrm{d}z$$

The first integral can be bounded in the contour γ by hypothesis as follows:

$$\int_\gamma |f(z) - P_{\varepsilon/\ell(\gamma)}(z)|\mathrm{d}z \leqslant \sup_{z\in\gamma} |f(z) - P_{\varepsilon/\ell(\gamma)}(z)| \int_\gamma \mathrm{d}z \leqslant \frac{\varepsilon}{\ell(\gamma)}\ell(\gamma) = \varepsilon.$$

The other integral can't be shown to be zero using the theorems we have at hand.

Lemma 1.11.3. *Suppose f is holomorphic, then there are two possibilities*:

i) *Either* $|f|$ *is holomorphic and therefore constant* (*from which we conclude that* f *is constant*).
ii) *Or* $|f|$ *is not holomorphic.*

Proof

The function $|f|$ is a real valued complex function. This means that

$$|f(z)| = g(z) + ih(z), \quad \text{with} \quad h = 0.$$

If $|f(z)|$ was holomorphic, it would satisfy the Cauchy-Riemann equations which means that $g_x = g_y = 0$. This can be used to conclude that f is also constant (by a previous homework exercise.)
In the other case, $|f(z)|$ is simply not holomorphic.

By the previous lemma, assuming P_ε is not a constant polynomial, $|P_\varepsilon|$ is not holomorphic. We cannot state results at this moment about the integral $\int_\gamma |P_{\varepsilon/\ell(\gamma)}(z)|\mathrm{d}z$.

As an alternative approach without showing that $\int_\gamma |f(z)|\mathrm{d}z$ we have the following:

Answer

Recall the reverse triangle inequality:

$$|x+y| \leqslant |x| + |y| \Rightarrow |x+y| - |y| \leqslant |x| \xrightarrow{x=\tilde{x}-y} |\tilde{x}| - |y| \leqslant |\tilde{x} - y|.$$

Now take $P_{\varepsilon/\ell(\gamma)}(z)$ as before, then

$$\left|\int_\gamma (f(z) - P_{\varepsilon/\ell(\gamma)}(z))\mathrm{d}z\right| \geqslant \left|\int_\gamma f(z)\mathrm{d}z\right| - \left|\int_\gamma P_{\varepsilon/\ell(\gamma)}(z)\mathrm{d}z\right|$$

and the rightmost integral is zero as $P_{\varepsilon/\ell(\gamma)}$ is a polynomial and therefore, a holomorphic function. Then, applying the integral triangle inequality we have

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leqslant \left|\int_\gamma (f(z) - P_{\varepsilon/\ell(\gamma)}(z))\mathrm{d}z\right| \leqslant \int_\gamma |f(z) - P_{\varepsilon/\ell(\gamma)}(z)|\mathrm{d}z \leqslant \varepsilon.$$

The last integral is smaller than ε by the previous attempted argument. As $\varepsilon > 0$ is arbitrary and $\left|\int_\gamma f(z)\mathrm{d}z\right|$ is a real number, the only possibility is that it's equal to zero. The only complex number with zero modulus is the origin, so we conclude that $\int\limits_\gamma f(z)\mathrm{d}z = 0$.

Exercise 1.11.4. Prove that $\int\limits_0^\infty \sin(x^2)\mathrm{d}x = \int\limits_0^\infty \cos(x^2)\mathrm{d}x = \frac{\sqrt{2\pi}}{4}$. These are the Fresnel integrals. Here, $\int\limits_0^\infty$ is interpreted as $\lim_{R\to\infty}\int\limits_0^R$.⟦Hint: Integrate the function e^{-z^2} over the path in Figure 14. Recall that $\int\limits_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$.⟧

Answer

The path in question is boundary of the circular sector of fixed radius from $\theta = 0$ to $\theta = \frac{\pi}{4}$. Call γ the curve which bounds the sector. Now notice that

$$\cos\left(x^2\right) + i\sin\left(x^2\right) = e^{ix^2},$$

so we will work with the function e^{iz^2} through the circular sector in question. We have that $\int\limits_\gamma e^{iz^2}\mathrm{d}z$ is zero because the exponential function is holomorphic, but also that integral can be broken down into three pieces as follows:

$$\int\limits_0^R e^{ix^2}\mathrm{d}x + \int\limits_0^{\pi/4} \exp\left[i(Re^{it})^2\right](iRe^{it}) + \int\limits_0^1 \exp\left[i((1-t)Re^{i\frac{\pi}{4}})^2\right](-Re^{i\frac{\pi}{4}})\mathrm{d}t.$$

The second integrand can be bounded as follows:

$$\begin{aligned}
\left|\exp\left[i(Re^{it})^2\right](iRe^{it})\right| \leqslant R|\exp\left[i(Re^{it})^2\right]| &= R|\exp\left[iR^2(\cos(2t) + i\sin(2t))\right]| \\
&= R|\exp\left[iR^2\cos(2t) - R^2\sin(2t)\right]| \\
&= Re^{-R^2\sin(2t)}\left|e^{i(R^2\cos(2t))}\right| \\
&= Re^{-R^2\sin(2t)} \xrightarrow[R\to\infty]{} 0
\end{aligned}$$

So, as R grows, we can bound the second integral by small quantity which decreases to zero. The third integral can be manipulated as follows, first consider the exponent in the integrand:

$$i((1-t)Re^{i\frac{\pi}{4}})^2 = i(1-t)^2R^2e^{\frac{i\pi}{2}} = -(1-t)^2R^2.$$

Taking the substitution $u = (1-t)R$ we have $\mathrm{d}u = -R\mathrm{d}t$ and as $t \to 0$, $u \to R$ while $t \to 1 \Rightarrow u \to 0$. The third integral can be written as

$$\int_R^0 e^{-u^2} e^{i\frac{\pi}{4}} \mathrm{d}u = -e^{\frac{i\pi}{4}} \int_0^R e^{-u^2} \mathrm{d}u.$$

We have the following equation at this point

$$\int_0^R e^{ix^2} \mathrm{d}x - e^{\frac{i\pi}{4}} \int_0^R e^{-u^2} \mathrm{d}u = 0$$

where we now take the limit as $R \to \infty$. From here we get

$$\int_0^\infty e^{ix^2} \mathrm{d}x = e^{\frac{i\pi}{4}} \frac{\sqrt{\pi}}{2}.$$

Taking real and imaginary parts of the integral in question gives us the desired result.

1.12 Day 9 | 20230208

The Cauchy Integral Formula

Suppose f is holomorphic on a domain Ω which contains a disk D with boundary C. Then for $z \in D$

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z} \mathrm{d}w.$$

Proof

Consider the function $g(w) = \frac{f(w)}{w-z}$ and a keyhole contour around z. We know that inside the contour g is holomorphic, so this means that $\oint_{\Gamma(\delta,\varepsilon)} g(w)\mathrm{d}w = 0$.

Away from z, g is continuous, so this means that

$$\lim_{\delta \to 0} \oint_{\Gamma(\delta,\varepsilon)} g(w)\mathrm{d}w = \oint_{-B(z,\varepsilon)} g(w)\mathrm{d}w + \oint_C g(w)\mathrm{d}w$$

where $B(z,\varepsilon)$ is the ball of radius ε centered at z. We can now write

$$\oint_{-B(z,\varepsilon)} g(w)\mathrm{d}w = \oint_{-B(z,\varepsilon)} \frac{f(w)-f(z)}{w-z}\mathrm{d}w + \oint_{-B(z,\varepsilon)} \frac{f(z)}{w-z}\mathrm{d}w$$

so as f is holomorphic we can bound $\frac{f(w)-f(z)}{w-z}$ with $\sup_B |f'(z)|$ (WATCH OUT). On the other hand

$$\oint_{-B(z,\varepsilon)} \frac{f(z)}{w-z}\mathrm{d}w = f(z)\int_0^{2\pi} \frac{(-i\varepsilon e^{-it})}{z+\varepsilon e^{-it}-z}\mathrm{d}t = f(z)\int_0^{2\pi} -i\mathrm{d}t = f(z)(-2\pi i).$$

We conclude that

$$\oint_C g(w)\mathrm{d}w = 2\pi i f(z)$$

which we can rearrange to the desired equality.

Example 1.12.1. With the formula we can compute

$$\oint_{B(0,1)} \frac{z}{2z+1} = \frac{-i\pi}{2}.$$

Theorem 1.12.2. *Suppose f is holomorphic on Ω, then*

i) f is infinitely differentiable.
ii) If C is a curve inside Ω,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{n+1}}\mathrm{d}w.$$

1.13 Interim 4 | HW4

Exercise 1.13.1. In this problem, we'll use the Cauchy integral formula to show that an analytic function has a power series representation.

Suppose f is analytic on an open set containing $\overline{B}(0,R)$. We will show that, on $B(0,R)$, there is an equality of functions

$$f(z) = \sum_{n\geqslant 0} a_n z^n, \quad \text{where} \quad a_n = \frac{f^{(n)}(0)}{n!}.$$

Let $\mathcal{C} = \partial B(0, R)$ be the circle of radius R centered at the origin oriented in positive direction. Fix z with $|z| < r = R$.

i) Show that

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} g(w) f(w) \mathrm{d}w, \quad \text{where} \quad g(w) = \frac{1}{w} \frac{1}{1 - z/w}.$$

ii) Let $N \in \mathbb{N}$. Show that

$$g(w) = \sum_{n=0}^{N-1} \frac{z^n}{w^{n+1}} + \frac{z^N}{(w-z)w^N}.$$

iii) Show that

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z), \quad \text{where} \quad \rho_N(z) = \frac{z^N}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z)w^N} \mathrm{d}w.$$

iv) Let $M = \sup_{z \in \mathcal{C}} |f(z)|$, show that

$$|\rho_N(z)| \leqslant \frac{rM}{R-r} \left(\frac{r}{R} \right)^{N-1}.$$

⟦Hint: If $w \in \mathcal{C}$, then $|w - z| \geqslant R - r$.⟧

v) Show that $\lim_{N \to \infty} \rho_N(z) = 0$.

Answer

i) By Cauchy's formula for $z \in B(0, R)$ we have

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{w - z} \mathrm{d}z.$$

Taking the $\frac{1}{w-z}$ and factoring a w on the bottom we get

$$\frac{1}{w - z} = \frac{1}{w(1 - z/w)} = \frac{1}{w} \frac{1}{1 - z/w}.$$

Replacing inside the integral we get the desired equality.

ii) Consider the geometric series with common term $\frac{z}{w}$, we have the following:

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n = \frac{1}{1 - z/w} \\ \displaystyle\sum_{n=0}^{N-1} \left(\frac{z}{w}\right)^n = \frac{1 - (z/w)^N}{1 - (z/w)} \end{cases}$$

Now notice that

$$\begin{aligned} \frac{1}{1 - z/w} - \frac{1 - (z/w)^N}{1 - (z/w)} &= \frac{(z/w)^N}{1 - (z/w)} \\ \Rightarrow g(w) - \frac{1}{w} \sum_{n=0}^{N-1} \left(\frac{z}{w}\right)^n &= \frac{(z/w)^N}{w - z} \\ \Rightarrow g(w) &= \sum_{n=0}^{N-1} \frac{z^n}{w^{n+1}} + \frac{z^N}{(w - z)w^N}. \end{aligned}$$

iii) Replacing the last value on the identity of the first item we get

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(\sum_{n=0}^{N-1} \frac{z^n}{w^{n+1}} + \frac{z^N}{(w - z)w^N} \right) f(w) \mathrm{d}w$$

and as the sum is finite, it commutes with the integral without any restrictions. Exchanging the integral we get

$$\sum_{n=0}^{N-1} \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(\frac{z^n}{w^{n+1}} f(w) \mathrm{d}w \right) + \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{z^N}{(w - z)w^N} f(w) \mathrm{d}w.$$

Using Cauchy's integral formula for derivatives we have

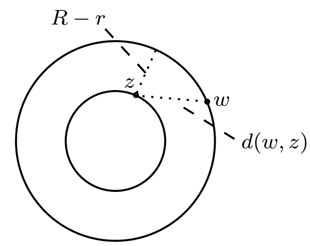
$$\begin{aligned} f^{(n)}(0) &= \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w - 0)^{n+1}} \mathrm{d}w \\ \Rightarrow \frac{f^{(n)}(0)}{n!} &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{w^{n+1}} \mathrm{d}w. \end{aligned}$$

So when factoring out the z's from the previous expressions we obtain

$$\sum_{n=0}^{N-1} z^n \frac{f^{(n)}(0)}{n!} + \frac{z^N}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w - z)w^N} \mathrm{d}w.$$

This is the desired expression.

iv) For this item, it is paramount to remember that $|z| = r < R$ and points $w \in \mathcal{C}$ satisfy $|w| = R$. Now look at the following diagram:



Recall that for a closed set C $d(z, C) = \min_{w \in C}\{\, d(w, z)\,\}$. In the case of the disk of radius R we have that

$$d(z, \mathcal{C}) = R - r \geqslant d(z, w), \quad \text{where} \quad w \in \mathcal{C}.$$

This means that

$$|w - z| \geqslant R - r \Rightarrow \frac{1}{|w-z|} \leqslant \frac{1}{R-r}.$$

Now taking the complex modulus of ρ_N for z with $|z| = r$, we have

$$|\rho_N(z)| \leqslant \frac{r^n}{2\pi} \oint_{\mathcal{C}} \frac{|f(z)|}{|w-z||w|^N} \mathrm{d}w \leqslant \frac{r^N}{2\pi} \frac{M}{(R-r)R^N}(2\pi R) = \frac{Mr}{R-r}\left(\frac{r}{R}\right)^{N-1}.$$

v) It suffices to show that $|\rho_N| \to 0$ as this implies $\rho_N \to 0$. Notice that $\frac{r}{R} < 1$ because $r < R$. So this means that $\left(\frac{r}{R}\right)^{N-1} < \frac{\varepsilon(R-r)}{rM}$ where $\varepsilon > 0$ and N is large enough. Thus for such an N we have

$$|\rho_N(z)| \leqslant \frac{rM}{R-r}\left(\frac{r}{R}\right)^{N-1} < \frac{rM}{R-r}\frac{\varepsilon(R-r)}{rM} = \varepsilon.$$

We thus conclude that $|\rho_N(z)| \to 0$ as $N \to \infty$ and therefore $\rho_N \to 0$ as well.

Exercise 1.13.2. For $0 \leqslant r \leqslant n$, the binomial coefficient $\binom{n}{r}$ is defined by $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

i) Let $\gamma = \partial B(0,1)$, the positive, circular arc around 0 with radius 1. Show that

$$\binom{n}{r} = \frac{1}{2\pi i} \oint_{\gamma} \frac{(1+z)^n}{z^{r+1}} \mathrm{d}z.$$

⟦In fact, this works for any simple closed curve around the origin.⟧

ii) Use the previous item to show that $\binom{n}{r} \leqslant 2^n$. *This can also be shown directly by computing* $(1+1)^n$.

Answer

i) Notice that the integral in question, by the binomial theorem is:

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z^{r+1}} \left(\sum_{k=0}^{n} \binom{n}{k} z^k \right) \mathrm{d}z = \frac{1}{2\pi i} \sum_{k=0}^{n} \binom{n}{k} \oint_{\gamma} z^{k-r-1} \mathrm{d}z.$$

Notice that most of the terms in the sum are actually zero because

$$\oint_{\gamma} z^{k-r-1} \mathrm{d}z = (2\pi i)\delta_{kr}.$$

With this, we have the desired equality as the expression is equal to

$$\frac{1}{2\pi i} \binom{n}{r} \oint_{\gamma} z^{-1} \mathrm{d}z = \binom{n}{r}.$$

ii) Now bounding terms in the integral we have that

$$\binom{n}{r} \leqslant \frac{1}{2\pi}(2^n)\mathrm{len}(\gamma) = 2^n.$$

Exercise 1.13.3 ([1] 2.7)**.** Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies $2|f'(0)| \leqslant d$.

Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = az + b$. ⟦In connection with this result, see the relationship between the diameter of a curve and Fourier series described in Problem 1, Chapter 4, Book I.⟧ ⟦Hint: $2f'(0) = \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{f(\zeta)-f(-\zeta)}{\zeta^2} \mathrm{d}\zeta$ when $0 < r < 1$.⟧

Answer

To show the desired identity for $f'(0)$ it suffices to use Cauchy's formula:

$$f'(z) = \frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{(w-z)^2} \mathrm{d}w \Rightarrow f'(0) = \frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{w^2} \mathrm{d}w.$$

Now take the change of variable $w = -u \Rightarrow \mathrm{d}w = -\mathrm{d}u$. The curve through which we are integrating remains the same, as $|w| = |-u| = |u|$. Thus

$$f'(0) = \frac{1}{2\pi i} \oint\limits_{|u|=r} \frac{-f(u)}{u^2}\mathrm{d}u.$$

Renaming variables and adding the last two results we get

$$2f'(0) = \frac{1}{2\pi i} \oint\limits_{|\zeta|=r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2}\mathrm{d}\zeta.$$

Taking the complex modulus we see that

$$2|f'(0)| \leqslant \frac{1}{2\pi} \oint\limits_{|\zeta|=r} \frac{|f(\zeta) - f(-\zeta)|}{|\zeta|^2}\mathrm{d}\zeta \leqslant \frac{1}{2\pi r^2} \sup_{|\zeta|=r} |f(\zeta) - f(-\zeta)|(2\pi r)$$

As $\{\, |\zeta| = r \,\} \subseteq \mathbb{D}$ we have that

$$\sup_{|\zeta|=r} |f(\zeta) - f(-\zeta)| \leqslant d \Rightarrow 2|f'(0)| \leqslant \frac{d}{r}.$$

As the last inequality holds for all $0 < r < 1$, in particular we have that

$$|f'(0)| \leqslant \inf_{0<r<1} \frac{d}{r} = d$$

which is the desired inequality.

Exercise 1.13.4. Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then φ is linear. $\llbracket$ Hint: Why can one assume that $z_0 = 0$? Write $z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\varphi_k = \varphi \circ \varphi \circ \cdots \circ \varphi$ (where φ appears k times), then $\varphi_k(z) = k a_n z^n + O(z^{n+1})$. Apply the Cauchy inequalities and let $k \to \infty$ to conclude the proof. $\rrbracket$

Answer

Let us begin by making some observations:

- As φ is holomorphic inside Ω, it is analytic inside Ω and thus we may write it as a power series centered at z_0

$$\varphi(z) = \sum_{n\geqslant 0} a_n(z-z_0)^n.$$

The conditions in the problem imply that $a_0 = z_0$ and $a_1 = 1$ which means that

$$\varphi(z) = z_0 + (z-z_0) + \sum_{n\geqslant 2} a_n(z-z_0)^n.$$

- We can thus consider the function $\varphi(z) - z_0$ and move the whole domain to the origin by translating by z_0. This way we can reinterpret

$$\varphi(z) = z + \sum_{n\geqslant 2} a_n z^n.$$

- Suppose $m \geqslant 2$ is the smallest integer such that $a_m \neq 0$ in the expansion of φ, we this we have

$$\varphi(z) = z + a_m z^m + O(z^{m+1}).$$

- φ is a bounded function because $\mathrm{im}(\varphi) \subseteq \Omega$ and as Ω is bounded, there exists $r > 0$ such that $\Omega \subseteq B(0,r)$. this means that $\|\varphi\|_\infty \leqslant r$.

With this in hand we are ready to proceed. Consider the composition of φ with itself. We have

$$\begin{aligned}
&\varphi\left(z + a_m z^m + O(z^{m+1})\right) \\
=&\left(z + a_m z^m + O(z^{m+1})\right) + a_m\left(z + a_m z^m + O(z^{m+1})\right)^m + O(z^{m+1}) \\
=&z + a_m z^m + a_m z^m + O(z^{m+1}) \\
=&z + 2a_m z^m + O(z^{m+1}),
\end{aligned}$$

where the second-to-last equality comes after expanding the m^{th} power and realizing that all the other terms in the expansion belong in $O(z^{m+1})$. Inductively we have

$$\begin{aligned}
&\varphi\left(z + (k-1)a_m z^m + O(z^{m+1})\right) \\
=&\left(z + (k-1)a_m z^m + O(z^{m+1})\right) + a_m\left(z + (k-1)a_m z^m + O(z^{m+1})\right)^m + O(z^{m+1}) \\
=&z + (k-1)a_m z^m + a_m z^m + O(z^{m+1}) \\
=&z + ka_m z^m + O(z^{m+1}),
\end{aligned}$$

so in general φ_k is what we expect it to be. Notice that vf_k is still a function from Ω to Ω which means it's uniformly bounded independent of k. Now by using the Cauchy inequality we have

$$|D^m\varphi_k(z)| = km!|a_m| \leqslant \frac{m!}{R^m}\|\varphi_k\|_\infty \Rightarrow |a_m| \leqslant \frac{\|\varphi_k\|_\infty}{kR^m} \xrightarrow{k\to\infty} 0.$$

As m is the smallest index such that a_m isn't zero, we conclude that there can be no such smallest index. All the a_m are zero except for the linear coefficient and thus we conclude that φ is a linear function.

1.14 Day 11 | 20230213

We have used the Cauchy integral formula for derivatives. As a corollary we have that

Corollary 1.14.1. *If f is holomorphic about $z_0 \in \Omega$, then f is analytic at z_0.*

Theorem 1.14.2. *Suppose f is holomorphic on a domain Ω with $(z_n) \subseteq \Omega$ such that $z_n \to z_0 \in \Omega$. Suppose that $f(z_n) = 0$ for all n, then f is identically zero.*

Proof

About z_0 we have

$$f(z) = \sum_{n\geqslant m} a_n(z-z_0)^n = (z-z_0)^m \sum_{n\geqslant m} a_n(z-z_0)^{n-m} = (z-z_0)^m g(z),\ a_m \neq 0.$$

We also have that $g(z_0) = a_m \neq 0$. As g is holomorphic, $|z-z_0| < \varepsilon$ implies $g(z) \neq 0$. But for large enough k, if $|z_k - z_0| < \varepsilon$, if $f(z_k) = 0$, then

$$f(z_k) = (z_k - z_0)^m g(z_k).$$

Theorem 1.14.3. *Suppose f, g are holomorphic on a domain Ω. Suppose also that there is a non-constant sequence $(z_n) \subseteq \Omega$ such that $z_n \to z \in \Omega$ and $f(z_n) = g(z_n)$. Then $f = g$.*

Proof

Letting $h = f - g$, we see that h is identically zero with the previous result so $f = g$.

Definition 1.14.4. We say a that a sequence of functions (f_n) converges to f pointwise on S, if

$$\forall z(f_n(z) \xrightarrow{n\to\infty} f(z)).$$

Meanwhile the convergence is uniform if

$$\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0.$$

Example 1.14.5. The sequence of functions (z^n) converges pointwise to δ_{z0} which is not continuous. Uniform convergence preserves continuity so (z^n) is not uniformly convergent.

Power series converge uniformly.

1.15 Day 12 | 20230215

Theorem 1.15.1. *Suppose (f_n) is a sequence of holomorphic functions such that $f_n \to f$ uniformly on compact sets. Then f is holomorphic.*

Lemma 1.15.2 (Uniform Convergence Theorem). *$f_n \to f$ uniformly $\Rightarrow \int f_k \to \int f$.*

Proof

By Morera's theorem, it's enough to show that for triangular contours in Ω, $\int_T f(z)\mathrm{d}z = 0$. The triangle is a compact set and so we are done.

FINISH

1.16 Day 13 | 20230217

Recall the Schwarz reflection principle. We will move into meromorphic functions now, so to begin we will assume f is holomorphic on a domanin Ω and that it's not identically zero.

1.17 Day 14 | 20230220

Cauchy residue formula for one residue and multiple residues.

Example 1.17.1. We will compute the following principal value:

$$\lim_{R\to\infty} \frac{-R}{R} \frac{x^2}{x^6+1}\mathrm{d}x.$$

The function $f(z) = \frac{z^2}{z^6+1}$ has poles at the roots of $z^6 + 1$. This means that

$$z^6 + 1 = 0 \iff e^{6it} = e^{\pi i + 2\pi i k}, \; k \in \mathbb{Z}.$$

So solutions to this equation are

$$t = \frac{\pi}{6} + \frac{2\pi i k}{6}, \; k \in \mathbb{Z}.$$

We will only use $0 \leqslant k \leqslant 5$ in this case.

We now integrate over the half disk contour $\mathcal{C}$ with radius R of the upper half-plane. By the residue theorem we have that

$$\int\limits_{\mathcal{C}} \frac{z^2}{z^6+1} \mathrm{d}z = 2\pi i \sum_{k=0}^{2} \operatorname{res}\left(\frac{z^2}{z^6+1}, e^{\frac{\pi}{6} + \frac{2\pi i k}{6}} \right).$$

1.18 Interim 5 | HW5

Exercise 1.18.1 (Stein & Shakarchi 2.15)**.** Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Suppose that

$$|f(z)| = 1 \quad \text{when} \quad |z| = 1,$$

then f is constant. ⟦Hint: Extend f to $\mathbb{C}$ by $f(z) = \frac{1}{\overline{f(1/\overline{z})}}$ when $|z| > 1$. Argue as in the Schwarz reflection principle.⟧⟦Hint: You will need to use the fact that, away from 0, $z \mapsto \frac{1}{z}$ is continuous; so z and z_0 are close if and only if $\frac{1}{z}$ and $\frac{1}{z_0}$ are close.⟧

Answer

Let us extend f to F defined as

$$F(z) = \begin{cases} f(z) & \text{if} \quad |z| \leqslant 1, \\ \frac{1}{\overline{f(1/\overline{z})}} & \text{when} \quad |z| \geqslant 1. \end{cases}$$

To show that f is constant, we will show that F is constant by Liouville's theorem. Notice first that $\frac{1}{f(1/z)}$ is holomorphic when $|z| > 1$ because $\frac{1}{z} \in \mathbb{D}$ and f is non-vanishing in $\mathbb{D}$. Now, for z_0 with $|z_0| > 1$ we also have $|\overline{z_0}| > 1$, this means that we may write $\frac{1}{f(1/z)}$ as a power series centered at $\overline{z_0}$:

$$\frac{1}{f(1/z)} = \sum_{n=0}^{\infty} a_n (z - \overline{z_0})^n$$

Conjugating twice, z and the whole expression, we obtain

$$\frac{1}{\overline{f(1/\overline{z})}} = \overline{\sum_{n=0}^{\infty} a_n(\overline{z} - \overline{z_0})^n} = \sum_{n=0}^{\infty} \overline{a_n}(z - z_0)^n$$

which means F is analytic for $|z| > 1$.
Now, for the finishing touch, notice that for triangles T contained entirely in $\{\,|z| > 1\,\}$ or $\{\,|z| < 1\,\}$, $\int_T F(z)\mathrm{d}z = 0$. For triangles which overlap between both regions, we may apply the symmetry principle in order to conclude that the integral of F is also zero.
Also F is well defined at the boundary of the unit disk because of the given conditions of the problem.
Finally F is bounded because f is defined on a compact set and the composition $f(1/z)$ is also bounded when $|z| > 1$. Taking complex moduli and a reciprocal away from zero doesn't change this fact.
By Liouville's theorem we conclude that F is constant as it is an entire and bounded function. Therefore f is also constant.

Exercise 1.18.2. Consider the function

$$f(z) = \frac{2z-3}{z^2-4}.$$

i) Find the residue $\operatorname{res}(f, 2)$.⟦Hint: Expand $f(z)$ in terms of powers of $(z-2)$.⟧
ii) Find the residue $\operatorname{res}(f, -2)$.
iii) Let $\mathcal{C} = \partial B(0, 5)$, find $\int_{\mathcal{C}} f(z)\mathrm{d}z$.
iv) Find a partial fraction decomposition for $f(z)$.
v) Use the decomposition to recompute $\int_{\mathcal{C}} f(z)\mathrm{d}z$.

Answer

Notice that we can factor

$$f(z) = \frac{2z-3}{(z+2)(z-2)}$$

and we can see that 2 and -2 are simple poles of f.

i) By definition we have that

$$\operatorname{res}(f,2) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{2z-3}{z+2} = \frac{1}{4}.$$

ii) In the same fashion as before

$$\operatorname{res}(f,-2) = \lim_{z\to -2}(z+2)f(z) = \lim_{z\to 2}\frac{2z-3}{z-2} = \frac{7}{4}.$$

iii) By the residue theorem the integral in question is equal to

$$2\pi i(\operatorname{res}(f,2)+\operatorname{res}(f,-2)) = 4\pi i.$$

iv) We partially decompose f to

$$\begin{aligned}
&\frac{2z-3}{(z+2)(z-2)} = \frac{A}{z+2} + \frac{B}{z-2} \\
\Rightarrow &2z-3 = Az+2A+Bz-2B \\
\Rightarrow &\begin{cases} A+B=2 \\ 2A-2B=3 \end{cases}
\end{aligned}$$

This linear system solves to $A=\frac{7}{4}$ and $B=\frac{1}{4}$.

v) We can now separate the integral into

$$\frac{7}{4}\int_{\mathcal{C}}\frac{\mathrm{d}z}{z+2} + \frac{1}{4}\int_{\mathcal{C}}\frac{\mathrm{d}z}{z-2}$$

and the residues of both functions at their respective poles are $2\pi i$. Summing this values we obtain the desired result $4\pi i$ as before.

Exercise 1.18.3 (Residue Lemma)**.** Suppose f,g are analytic at z_0 with $f(z_0)\neq 0$, $g(z_0)=0$, and $g'(z_0)\neq 0$.

i) What is the order of the pole of $f(z)/g(z)$ at z_0?

ii) Show that $\operatorname{res}(f/g,z_0) = f(z_0)/g'(z_0)$. ⟦Hint: Write $g(z)=(z-z_0)h(z)$.⟧

Answer

i) Notice that f has no zeroes at z_0 while g has a zero of order one at z_0 because $g'(z_0) \neq 0$. So this means that f/g has a *simple pole* at z_0.

ii) Recall we may calculate the residue of a function f at z_0 as

$$\operatorname{res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left((z-z_0)^n f(z)\right)$$

so in the case of f/g we have

$$\operatorname{res}\left(f/g, z_0\right) = \lim_{z \to z_0} \left((z - z_0)\frac{f(z)}{g(z)}\right)$$

and as $g(z_0) = 0$ this expression equals

$$f(z_0) \lim_{z \to z_0} \left(\frac{z - z_0}{g(z) - g(z_0)}\right) = \frac{f(z_0)}{g'(z_0)}.$$

Exercise 1.18.4 (4.B, [1] 3.1)**.** Using Euler's formula:

$$\sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i},$$

show that the complex zeros of $\sin(\pi z)$ are exactly at the integers, and that they are each of order 1.

Calculate the residue of $1/\sin(\pi z)$ at $z = n \in \mathbb{Z}$.

Answer

Notice that

$$\begin{aligned} & \sin(\pi z) = 0 \\ \iff & \frac{e^{i\pi z} - e^{-i\pi z}}{2i} = 0 \\ \iff & e^{i\pi z} - e^{-i\pi z} = 0 \\ \iff & e^{i\pi z} = e^{-i\pi z} \\ \iff & e^{2\pi i z} = 1. \end{aligned}$$

The last equation is satisfied whenever $z \in \mathbb{Z}$ which means that $\sin(\pi z) = 0$ only when $z \in \mathbb{Z}$.

To prove they are simple zeroes, it suffices to see that $\sin(\pi z)$'s derivative is never zero at such zeroes. Notice that

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin(\pi z) = \pi\cos(\pi z) \xrightarrow{z \mapsto n} \pi(-1)^n \neq 0.$$

This holds for all integers so it follows that $n \in \mathbb{Z}$ is a simple zero of $\sin(\pi z)$.
The residue can be calculated as follows:

$$\operatorname{res}(\csc(\pi z), n) = \lim_{z\to n}\frac{z-n}{\sin(\pi z)} = \lim_{z\to n}\frac{1}{\pi\cos(\pi z)} = \frac{(-1)^n}{\pi}$$

where in the last step we have applied L'Hôpital's rule as we have an indeterminate form.
Notice that when the residue lemma we should have a quotient f/g where the numerator doesn't vanish. However in the quotient $(z-n)/\sin(\pi z)$, the numerator also vanishes at $z = n$. In this sense we can't apply the residue lemma.

1.19 Day 15 | 20230222

Continuing with the example from last class we have the following:

By the residues lemma, if z_k are the roots of the polynomial:

$$\operatorname{res}(f, z_k) = \frac{1}{6z_k^3}.$$

We get the following

$$\int_{\text{Line}} f(z)\mathrm{d}z + \int_{\text{half-circle}} f(z)\mathrm{d}z = 2\pi i\left(\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i}\right) = \frac{\pi}{3}.$$

Across the half-circle we can bound the integral

$$\pi R\frac{R^2}{R^6-1} \xrightarrow{R\to\infty} 0$$

where we have used the reverse triangle inequality. Thus, we conclude that the integral we wanted to find is $\frac{\pi}{3}$.

Example 1.19.1. Take $a \in]0,1[$. Consider the function

$$f(z) = \frac{e^{az}}{1+e^z},$$

we will find $\int\limits_{-\infty}^{\infty} f(z)\mathrm{d}z$. To do this we use the rectangular contour with sides $2R$ and 2π. Inside this contour, the only pole is $z = i\pi$ so

$$\int\limits_{\mathcal{C}} f(z) = 2\pi i \operatorname{res}(f, i\pi).$$

As $i\pi$ is a simple zero of the denominator, we can use the residue lemma to find

$$\operatorname{res}(f, i\pi) = \frac{e^{ia\pi}}{e^{i\pi}} = -e^{ia\pi}.$$

Be quick to notice that $\frac{f(z+2\pi i)}{f(z)} = e^{2\pi i a}$. This means that the integral on the top side is $-e^{2\pi i a}$ times our desired integral. On the short sides we have to calculate the following:

$$\int\limits_0^{2\pi} f(R+it)i\mathrm{d}t$$

after using the parametrization $R+it$ with $t \in]0, 2\pi]$, thus

$$\int\limits_0^{2\pi} \left|\frac{e^{a(R+it)}}{1+e^{R=it}}\right| \leqslant \int\limits_0^{2\pi} \frac{e^{(a-1)R}}{|e^{-R}+e^{it}|}\mathrm{d}t \leqslant \frac{2\pi e^{(a-1)R}}{1-e^{-R}} \xrightarrow[subscript]{R\to\infty} 0.$$

From this we can find that

$$I - e^{2\pi i a}I = -2\pi i e^{i\pi a} \Rightarrow I = \frac{\pi}{\sin(\pi a)}.$$

Definition 1.19.2. A function f has a removable singularity at z_0 if it can be analitically extended to a domain containing z_0.

Example 1.19.3. The function $\frac{e^z}{z}$ has a removable singularity at 0 because we can define it as 1 at the origin.

1.20 Day 16 | 20230301

Recall we have talked about removable singularities.

Index

Bibliography

[1] Lawrence C. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010.