Homework 5
Due: Friday, February 24

1. [SS]2.15. (HINT: *You will need to use the fact that, away from* 0, $z \mapsto \frac{1}{\bar{z}}$ *is continuous; so* z *and* z_0 *are close if and only if* $\frac{1}{\bar{z}}$ *and* $\frac{1}{\bar{z}_0}$ *are close.*)

2. Consider the function
$$f(z) = \frac{2z-3}{z^2-4}.$$
   (a) Find the residue $\operatorname{res}(f;2)$. (HINT: *Try and expand* $f(z)$ *in terms of powers of* $(z-2)$.)
   (b) Find the residue $\operatorname{res}(f;-2)$.
   (c) Let $C = C_5(0)$, the circle of radius 5 centered at the origin. What is
$$\int_C f(z)\,dz?$$
   (d) Find a partial fraction decomposition for $f(z)$.
   (e) Use your partial fraction decomposition to recompute $\int_C f(z)\,dz$.

   Parts (c) and (e) require the Cauchy residue formula, which will be covered in class on Monday, 2/20.

3. Suppose f and g are analytic at z_0, and that $f(z_0) \neq 0$; $g(z_0) = 0$; $g'(z_0) \neq 0$.
   (a) What is the order of the pole of $\frac{f(z)}{g(z)}$ at z_0?
   (b) Show that
$$\operatorname{res}\left(\frac{f(z)}{g(z)};z_0\right) = \frac{f(z_0)}{g'(z_0)}.$$
   (HINT: *Write* $g(z) = (z-z_0)h(z)$...)

4. **Read both of the following problems, and do one of them.**

$\boxed{\text{A}}$ In class, we said that $f(z)$ has a zero of order $n \geq 0$ at z_0 if there is some function $g(z)$, holomorphic and nonvanishing on a neighborhood of z_0, such that

$$f(z) = (z - z_0)^n g(z)$$

(on that neighborhood). More generally, if $f(z)$ is holomorphic on some deleted neighborhood of z_0, we will say that $f(z)$ has a zero of order $n = \mathrm{ord}_{z_0} f(z)$ at z_0 (with n positive *or* negative!) if there is a function $g(z)$, holomorphic and nonvanishing on a neighborhood of z_0, such that

$$f(z) = (z - z_0)^n g(z)$$

on a deleted neighborhood of z_0. (Thus, $\mathrm{ord}_{z_0} \frac{1}{(z-z_0)^3} = -3$.)

Suppose f and h have orders $m = \mathrm{ord}_{z_0}(f)$ and $n = \mathrm{ord}_{z_0}(h)$ at z_0.

(a) What is $\mathrm{ord}_{z_0}(fh)$?

(b) How is $\mathrm{ord}_{z_0}(f + h)$ related to m and n?

$\boxed{\text{B}}$ (a) [SS]3.1.

(b) Check your work using (3).