MATH601 — Advanced Combinatorics

Based on the lectures by Maria Gillespie

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Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This course will focus on the combinatorics of Young tableaux, crystal bases, root systems, Dynkin diagrams, and symmetric functions arising in representation theory of matrix groups and Lie algebras.

Requirements

Familiarity with the basics of group theory and symmetric functions is helpful.

Contents

Chapter 1

1.1 Day 1 | 20240819

We will start by reviewing the representation theory of finite groups and the Lie group and Lie algebra representations. The objective is to classify semi-simple Lie algebras and groups. This classification is quite combinatorial.

Review of representation theory of finite groups

Recall groups are sets G endowed with a binary operation $\circ$ such that

(a) There is an identity element e: $g \circ e = e \circ g = g$.

(b) Every element possesses an inverse. For each g, there is an h such that $g \circ h = e = h \circ g$.

(c) The operation $\circ$ is associative.

Example 1.1.1. The symmetric group is the set of permutations of $[n]$. We denote it $(S_n, \circ)$ where our operation is composition. We will use this group quite a lot.

Example 1.1.2. We will be working with $\mathrm{GL}_n(\mathbb{C})$ where $\mathbb{C}$ will come in as more useful than $\mathbb{R}$. The general linear group is characterized by the property that $\det(A) \neq 0$ for $A \in \mathrm{GL}_n(\mathbb{C})$.

Example 1.1.3. Given two groups we can construct $G \times H$ by doing operations pointwise. We can also take subgroups and quotient groups.

Example 1.1.4. Take the special linear group $\mathrm{SL}_n(\mathbb{C})$ which is the set of matrices A with $\det(A) = 1$. This is a subgroup of $\mathrm{GL}_n(\mathbb{C})$.

There's a lot more of matrix groups such as $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$ and unitary groups $\mathrm{SU}_n(\mathbb{C})$.

Groups which are representations of themselves

Symmetry groups are groups of linear transformations of $\mathbb{C}^n$ (some Euclidean space) that fix some shape. Any such group is a subgroup of $\mathrm{GL}_n(\mathbb{C})$. Matrices here don't collapse points nor anything.

Example 1.1.5. The symmetry group of a diamond in the plane can be found by analyzing the symmetries of the figure.HMMM The group in question is the Klein-4 group which can be seen as

$$\{\, \mathrm{id}, r_x, r_y, r_x r_y \,\}.$$

Similarly we can see it as

$$\{\, \mathrm{id}, (24), (13), (13)(24) \,\}$$

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1.2 Day 2 | 20240821

We were looking at direct sums of representations. Recall representations are maps which take group elements to matrices.

$$\rho \oplus \sigma : G \to \mathrm{GL}_{n+m}(\mathbb{C})$$

and this map will send g to a block matrix. A central question in representation theory is to classify the irreducible representations of some object. This is a central question because for finite groups, irreducible is the same as indecomposable.

Definition 1.2.1. A representation is indecomposable when it can't be written as a direct sum of smaller representations.

Irreducible means that it has no non-trivial proper representations. This is analogous to the idea of prime and irreducible numbers. In the most general case where groups may be infinite, irreducible implies indecomposable.

Alternative definitions for representations

We may define it as a vector space V with an action $G \times V \to V$ so that

$$g(hv) = (gh)v$$

and it should be a linear action in the sense that $v \mapsto gv$ is a linear transformation.

This is equivalent to the previous definition because V can be seen as $\mathbb{C}^n$. So the definition gives rise to a map

$$G \to \mathrm{Aut}(V),\ g \mapsto g\cdot$$

Even more *objecty* is the next definition. We can see a representation as a module over a group ring $\mathbb{C}G$. This set is made up of formal linear combinations of elements of G.

We endow it with a module structure, for any element $g \in G$ in particular in $\mathbb{C}G$ we can make it a coefficient $gv \in V$ as a $\mathbb{C}G$-module.

Subrepresentations

Now that we have all the algebraic structure we can use it to define subrepresentations. Because a subrepresentation will be a subspace which inherits the action for example.

Definition 1.2.2. $W \subseteq V$ is a subrepresentation of G (when V represents G) if

- W is a subspace of V, and
- W is G-invariant in the sense that the image of $G \times W \to V$ is contained in W.

We will also say that V is irreducible if there's no proper nonzero subrepresentation $W \subseteq V$.

Sometimes it is possible to decompose a representation into a direct sum of subrepresentations.

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Definition 1.2.3. A character of a representation is the trace map $g \mapsto \mathrm{tr}(\rho(g))$.

Properties

(a) $\chi_{V\oplus W} = \chi_V + \chi_W$.

(b) $\chi_{V\otimes W} = \chi_V \chi_W$.

(c) χ_V uniquely determines the representation.

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Bibliography

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