

Exercise 1 (5.10(a) Stein& Shakarchi). Find the Hadamard product for $e^z - 1$.

Answer

Recall Hadamard's theorem states that if f is an entire function with order of growth ρ and $k = \lfloor \rho \rfloor$ then

$$f(z) = e^{p(z)} z^m \prod_{n=1}^{\infty} E_k \left(\frac{z}{a_n} \right)$$

where (a_n) is the collection of non-null zeroes of f , p has degree at most k and $m = \text{ord}(f, 0)$.

In our case $e^z - 1$ has order of growth 1 and it has simple zeroes at $z = 2\pi in$ for $n \in \mathbb{Z}$. In particular the order of zero is one. This means that

$$e^z - 1 = e^{a_1 z + a_0} z \prod_{n \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{2\pi in} \right) e^{z/2\pi in}.$$

To simplify this product we multiply opposites across the origin:

$$\begin{aligned} \left[\left(1 - \frac{z}{2\pi in} \right) e^{z/2\pi in} \right] \left[\left(1 - \frac{z}{2\pi i(-n)} \right) e^{z/2\pi i(-n)} \right] &= \left(1 + \left(\frac{z}{2\pi in} \right)^2 \right) e^{z/2\pi in} e^{-z/2\pi in} \\ &= 1 + \frac{z^2}{4\pi^2 n^2} \end{aligned}$$

So we get

$$e^z - 1 = e^{a_1 z + a_0} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2} \right).$$

Dividing both sides by z we get

$$\frac{e^z - 1}{z} = e^{a_1 z + a_0} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2} \right)$$

and as z approaches 0 we get that

$$1 = e^{a_0}(1) \Rightarrow a_0 = 0.$$

Expanding the exponential function as a Taylor series and comparing coefficients we get the following:

$$z + \frac{z^2}{2} + O(z^3) = (1 + a_1 z + \frac{(a_1 z)^2}{2} + O(z^3)) z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2} \right)$$

Thus we obtain

$$z + \frac{z^2}{2} + O(z^3) = z + a_1 z^2 + O(z^3) \Rightarrow a_1 = \frac{1}{2}.$$

In conclusion we have

$$e^z - 1 = e^{z/2} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2} \right).$$

Exercise 2 (5.11 Stein& Shakarchi). Show that if f is an entire function of finite order that omits two values, then f is constant. This result remains true for any entire function and is known as Picard's little theorem. [Hint: If f misses a , then $f(z) - a$ is of the form $e^{p(z)}$ where p is a polynomial.]

Answer

Assume f omits two values a, b which means that

$$f(z) - a = e^{p(z)}, \quad \text{and} \quad f(z) - b = e^{q(z)} \quad \text{for some } p, q \text{ polynomials}.$$

From this, we may subtract one equation from the other to get

$$b - a = e^{p(z)} - e^{q(z)}$$

and now we may differentiate both sides of the equation to obtain

$$0 = p'(z)e^{p(z)} - q'(z)e^{q(z)}.$$

As this equation holds for *all* $z \in \mathbb{C}$ it must happen that $p'(z)$ and $q'(z)$ have the same zeroes with the same multiplicities. Thus $q'(z) = cp'(z)$ for some non-zero $c \in \mathbb{C}$. Returning to our equation we have

$$e^{p(z)} p'(z) = cp'(z) e^{q(z)} \Rightarrow e^{p(z)} = ce^{q(z)} \Rightarrow ce^{q(z)} - e^{q(z)} = b - a.$$

Differentiating this equation we obtain

$$(c - 1)q'(z)e^{q(z)} = 0 \Rightarrow q'(z) = 0 \Rightarrow q \text{ is constant}.$$

This allows us to conclude that f is constant as $f = e^q + b$.

If it occurred that $c = 1$, then $p' = q'$ and so $q(z) = p(z) + d$ for some $d \in \mathbb{C}$.

Replacing this in the equation we have

$$e^{p(z)} - e^{p(z)+d} = b - a \Rightarrow (1 - e^d)e^{p(z)}p'(z) = 0 \Rightarrow p'(z) = 0 \Rightarrow p \text{ is constant}$$

and once again we deduce f is constant. Finally if it was the case that $d = 0$, then $p = q$ but this means that

$$f(z) - e^{p(z)} = a = b$$

and this can't happen as a, b are different values of \mathbb{C} . In conclusion we have that f is constant.

Exercise 3. Assume $\operatorname{Re}(s) = \sigma > 0$. For $n, N \in \mathbb{N}$ define

$$\delta_n(s) = \frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s} = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \quad \text{and} \quad F_N(s) = \sum_{n=1}^N \delta_n(s).$$

- i) Show that $|\delta_n(s)| \leq \frac{|s|}{n^{\operatorname{Re}(s)+1}}$. [Hint: Represent the integrand in the definition of δ_n using the observation that $\int_n^x \frac{du}{u^{s+1}} = \frac{-1}{s}(x^{-s} - n^{-s})$.]
- ii) Show that $(F_N(s))$ converges uniformly on any half-plane of the form $\operatorname{Re}(s) \geq \alpha > 0$.
- iii) Show that $\zeta(s) - \frac{1}{s-1}$ is bounded and holomorphic near $s = 1$. [Hint: Use the fact that $\frac{1}{s-1} = \int_1^\infty x^{-s} dx$.]

Answer

- i) Observe that from the hint we have

$$s \int_n^x \frac{du}{u^{s+1}} = \left(\frac{1}{n^s} - \frac{1}{x^s} \right).$$

Then replacing this expression inside δ_n we get

$$|\delta_n(s)| = \left| \int_n^{n+1} s \int_n^x \frac{du}{u^{s+1}} dx \right| \leq \int_n^{n+1} |s| \left| \int_n^x \frac{du}{u^{s+1}} \right| dx$$

and we can estimate the inner integral by taking the integrand's sup-norm. We have

$$\left| \int_n^x \frac{du}{u^{s+1}} \right| \leq \sup_{n \leq u \leq x} \left| \frac{1}{u^{s+1}} \right| (x - n)$$

and for positive real numbers u we can estimate $|u^z|$ as

$$|u^x| |u^{iy}| = u^x |e^{iy \log(u)}| = u^x$$

so this means that

$$\sup_{n \leq u \leq x} \left| \frac{1}{u^{s+1}} \right| (x - n) = \sup_{n \leq u \leq x} \frac{1}{u^{s+1}} (x - n) \leq \frac{x - n}{n^{s+1}}$$

where the last inequality holds because $\frac{1}{u^{s+1}}$ is a decreasing function. Returning to our δ_n estimate we have

$$\int_n^{n+1} |s| \left| \int_n^x \frac{du}{u^{s+1}} \right| dx \leq \int_n^{n+1} |s| \frac{x - n}{n^{s+1}} dx = \frac{|s|}{n^{s+1}} \frac{1}{2} \leq \frac{|s|}{n^{s+1}}.$$

- ii) We now consider the partial sums $F_N = \sum_{n=1}^N \delta_n$. Observe that we may bound F_N with the countable triangle inequality:

$$|F_N(s)| \leq \sum_{n=1}^N |\delta_n(s)| \leq \sum_{n=1}^N \frac{|s|}{n^{s+1}}$$

and this is a series of real numbers which converges when $\sigma > 0$. This implies that F_N converges uniformly for $\sigma \geq \alpha > 0$.

- iii) Observe that from our initial identity

$$\delta_n(s) = \frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s}$$

we can sum up to N to obtain

$$F_N(s) = \sum_{n=1}^N \frac{1}{n^s} - \sum_{n=1}^N \left(\int_n^{n+1} \frac{dx}{x^s} \right) = \sum_{n=1}^N \frac{1}{n^s} - \int_1^{N+1} \frac{dx}{x^s}.$$

Letting N grow without bound we arrive at

$$\lim_{N \rightarrow \infty} F_N(s) = \zeta(s) - \int_1^{\infty} \frac{dx}{x^s} = \zeta(s) - \frac{1}{s-1}.$$

As the properties of the sequence $(F_N(s))$ are preserved through uniform limit, we have that $F_{\infty}(s)$ is bounded and holomorphic for $\sigma > 0$. So our expression for the equality is only valid wherever the improper integral converges, and this is where $\sigma > 1$. So we obtain the desired result as F_{∞} is $\zeta(s) - \frac{1}{s-1}$.