Exercise 1 (Exercise 1)**.** Evaluate the following expressions using the properties of the Hall inner product discussed in class:

i) $\left\langle s_{(2,1)} \middle| h_{(1,1,1)} \right\rangle$.
ii) $\left\langle s_{(3,1,1)} \middle| s_{(3,2)} \right\rangle$.
iii) $\left\langle e_{(2,1)} \middle| h_{(2,1)} \right\rangle$.
iv) $\left\langle p_{(3,2,2,1)} \middle| p_{(3,2,2,1)} \right\rangle$.

Answer

(a) From the previous homework we have calculated

$$s_{(2,1)}(x,y,z) = x^2y + xy^2 + 2xyz + x^2z + y^2z + xz^2 + yz^2$$

which tells us in a general that $s_{(2,1)} = m_{(2,1)} + 2m_{(1,1,1)}$. Applying the inner product we get

$$\left\langle s_{(2,1)} \middle| h_{(1,1,1)} \right\rangle = \left\langle m_{(2,1)} \middle| h_{(1,1,1)} \right\rangle + 2\left\langle m_{(1,1,1)} \middle| h_{(1,1,1)} \right\rangle = 2.$$

(b) As s_λ form an orthonormal basis, it holds that $\left\langle s_{(3,1,1)} \middle| s_{(3,2)} \right\rangle = 0$.

(c) Turning $e_{(2,1)}$ to the monomial basis we obtain

$$\begin{aligned} e_{(2,1)}(x,y,z) &= (xy + yz + zx)(x + y + z) \\ &= x^2y + y^2z + z^2x + y^2x + z^2y + x^2z + 3xyz \\ &= m_{(2,1)}(x,y,z) + 3m_{(1,1,1)}(x,y,z). \end{aligned}$$

Inputting into the inner product we get

$$\left\langle e_{(2,1)} \middle| h_{(2,1)} \right\rangle = \left\langle m_{(2,1)} \middle| h_{(2,1)} \right\rangle + 3\left\langle m_{(1,1,1)} \middle| h_{(2,1)} \right\rangle = 1$$

and this is because m and h form a dual pair.

(d) As (p_λ) forms an orthogonal basis, the following holds $\langle p_\lambda | p_\lambda \rangle = z_\lambda$ where $z_\lambda = \prod k^{m_k} m_k!$ and m_k is the number of parts of λ equal to k. In this case

$$\left\langle p_{(3,2,2,1)} \middle| p_{(3,2,2,1)} \right\rangle = z_{(3,2,2,1)} = (3^1)(1!)(2^2)(2!) = 24.$$