

**Exercise 1.** A smooth manifold  $M$  is called *orientable* if there exists a collection of coordinate charts  $\{(U_\alpha, \phi_\alpha)\}$  so that, for every  $\alpha, \beta$  such that  $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \emptyset$ , the differential of the change of coordinates  $\phi_\beta^{-1} \circ \phi_\alpha$  has positive determinant.

- (a) Show that for any  $n$ , the sphere  $S^n$  is orientable.
- (b) Prove that, if  $M$  and  $N$  are smooth manifolds and  $f : M \rightarrow N$  is a local diffeomorphism at all points of  $M$ , then  $N$  being orientable implies that  $M$  is orientable. Is the converse true?

### Answer

- (a) Consider the sphere without its north and south pole:

$$U = S^n \setminus \{\mathbf{e}_{n+1}\}, \quad \text{and} \quad V = S^n \setminus \{-\mathbf{e}_{n+1}\}.$$

These two sets form an atlas of  $S^n$  along with the stereographic projections

$$\phi: U \rightarrow \mathbb{R}^n, \quad \mathbf{u} \mapsto \frac{1}{1 - u_{n+1}}(u_1, \dots, u_n),$$

$$\psi: V \rightarrow \mathbb{R}^n, \quad \mathbf{u} \mapsto \frac{1}{1 + u_{n+1}}(u_1, \dots, u_n).$$

For  $\mathbf{u} \in S^n$ , call  $\mathbf{x} = \phi(\mathbf{u})$  and  $\mathbf{y} = \psi(\mathbf{u})$ . In order to find the transition function  $\psi\phi^{-1}$ , we first make the observation that

$$\|\mathbf{x}\|^2 = \|\phi(\mathbf{u})\|^2 = \frac{1}{(1 - u_{n+1})^2}(u_1^2 + \dots + u_n^2) = \frac{1 - u_{n+1}^2}{(1 - u_{n+1})^2} = \frac{1 + u_{n+1}}{1 - u_{n+1}},$$

and from this we can see that

$$\phi^{-1}(\mathbf{x}) = \frac{1}{1 + \|\mathbf{x}\|^2}(2x_1, \dots, 2x_n, \|\mathbf{x}\|^2 - 1).$$

Applying  $\psi$  we get the transition function to be

$$\begin{aligned} \mathbf{y} = \psi\phi^{-1}(\mathbf{x}) &= \frac{1}{1 + \left(\frac{\|\mathbf{x}\|^2 - 1}{1 + \|\mathbf{x}\|^2}\right)} \left( \frac{2x_1}{1 + \|\mathbf{x}\|^2}, \dots, \frac{2x_n}{1 + \|\mathbf{x}\|^2} \right) \\ &= \frac{1}{\frac{1 + \|\mathbf{x}\|^2 + \|\mathbf{x}\|^2 - 1}{\|\mathbf{x}\|^2 + 1}} \frac{2}{1 + \|\mathbf{x}\|^2} \mathbf{x} \\ &= \frac{1 + \|\mathbf{x}\|^2}{2\|\mathbf{x}\|^2} \frac{2}{1 + \|\mathbf{x}\|^2} \mathbf{x} = \frac{\mathbf{x}}{\|\mathbf{x}\|^2} \end{aligned}$$

The differential of this map can be calculated using the product rule. Call  $f = \frac{1}{\|\mathbf{x}\|^2}$ ,  $G = \text{id}$ , then

$$J(fG) = \nabla f \otimes G + fJG = \left( \frac{-1}{(\|\mathbf{x}\|^2)^2} 2\mathbf{x} \right) \otimes \mathbf{x} + \frac{1}{\|\mathbf{x}\|^2} \text{Id}.$$

Using the matrix determinant lemma we may see that

$$\begin{aligned} \det(JfG) &= \left( 1 + \frac{-2}{\|\mathbf{x}\|^4} \mathbf{x}^\top (\|\mathbf{x}\|^2 \text{Id}) \mathbf{x} \right) \det \left( \frac{1}{\|\mathbf{x}\|^2} \text{Id} \right) \\ &= \left( 1 - \frac{2}{\|\mathbf{x}\|^2} \mathbf{x}^\top \mathbf{x} \right) \frac{1}{\|\mathbf{x}\|^{2n}} \\ &= (1 - 2) \frac{1}{\|\mathbf{x}\|^{2n}} = \frac{-1}{\|\mathbf{x}\|^{2n}} \end{aligned}$$

This doesn't mean that the sphere is non-orientable, but that my choice of atlas was a poor choice. For our effect then, it suffices to make a small change in our chart. Pick

$$\tilde{\psi}: V \rightarrow \mathbb{R}^n, \mathbf{u} \mapsto \frac{1}{1 + u_{n+1}}(u_1, \dots, -u_n)$$

and observe that this small change in *orientation* will help us recover our desired result. In this case the transition function becomes

$$\mathbf{y} = \tilde{\psi} \phi^{-1}(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|^2}(x_1, \dots, x_{n-1}, -x_n).$$

In this case, following a product rule calculation our  $G$  function changes to a diagonal matrix  $\text{diag}(1, 1, \dots, 1, -1)$  which means that when taking its determinant we get  $-1$ . In the end the whole determinant of the transition function's differential becomes positive, leaving us with the desired result.

- (b) We will build an atlas on  $M$  whose transitions functions' differential have positive determinant. To that effect, let  $\{(V_\alpha, \psi_\alpha)\}$  be an atlas of  $N$  which makes  $N$  orientable, this means that we have  $\det J(\psi_\beta \psi_\alpha^{-1}) > 0$  for  $\alpha, \beta$ .

Now for  $x \in M$ , there are neighborhoods  $U_x, \tilde{V}_{f(x)}$  in  $M, N$  respectively such that  $f$  is a diffeomorphism between these sets. Pick a chart  $(V_{f(x)}, \psi_{f(x)})$

from our original atlas such that  $f(x) \in V_{f(x)}$ . Consider then the new open sets

$$W_{f(x)} = V_{f(x)} \cap \tilde{V}_{f(x)}$$

and restrict  $\psi_{f(x)}$  into  $W_{f(x)}$  by calling it  $\varphi_{f(x)}$ .

This defines a new atlas

$$\{ (W_{f(x)}, \varphi_{f(x)}) \}$$

for  $N$ , by virtue of  $f$  being bijective, which still preserves the property that its transition functions' differential has positive determinant.

We may pullback this atlas via  $f$  into an atlas

$$\{ (f^{-1}(W_{f(x)}), f^* \varphi_{f(x)}) \}$$

of  $M$ . For  $x, y \in M$ , we have the expression for the transition function

$$(f^* \varphi_{f(x)})(f^* \varphi_{f(y)})^{-1} = (\varphi_{f(x)} \circ f)(\varphi_{f(y)} \circ f)^{-1} = \varphi_{f(x)} \varphi_{f(y)}^{-1}.$$

As these are restrictions of our  $\psi$  functions, then their differential still has positive determinant. Thus, we have found an atlas of  $M$  which makes it orientable.

- (c) Consider the quotient map  $S^2 \rightarrow \mathbb{P}^2$  given by  $(x, y, z) \mapsto [x : y : z]$ . Locally if we look at the upper and lower hemisphere, this map is the identity which means that its differential is bijective. So the quotient map is a local diffeomorphism from an orientable surface to a non-orientable one.

Quick question for the second item, the fact that  $M, N$  are locally diffeomorphic implies that pairs of neighborhoods have the same homology. In particular the relative homology groups, which are used to define local orientation, are isomorphic.

*Would this be sufficient to conclude that if  $N$  is orientable then  $M$  is orientable?  
Why would this argument fail in the other direction? Say, why if  $M$  is orientable,  
could  $N$  be non-orientable?*

**Exercise 2.** Supply the details for the proof that, if  $F: \text{Mat}_{d \times d}(\mathbb{C}) \rightarrow \mathcal{H}(d)$  is given by  $F(U) = UU^*$  (where  $U^*$  is the conjugate transpose [a.k.a., Hermitian adjoint] of  $U$ ), then the unitary group

$$\mathcal{U}(d) = F^{-1}(I_{d \times d})$$

is a submanifold of  $\text{Mat}_{d \times d}(\mathbb{C})$  of dimension  $d^2$ . (Hint: it may be helpful to remember that a Hermitian matrix  $M$  can always be written as  $M = \frac{1}{2}(M + M^*)$ .)

### Answer

First, let us remind ourselves that a Hermitian matrix is a matrix  $A$  such that  $A^* = A$ . In consequence, for a general  $U$  we have

$$F(U)^* = (UU^*)^* = U^{**}U^* = UU^*,$$

meaning that indeed,  $F$  maps matrices into the Hermitian matrix space. To verify the unitary group is a submanifold, we must check that  $I$  is a regular value of our map. The differential of our map is

$$\frac{\partial}{\partial U}(UU^*) = \frac{\partial U}{\partial U}U^* + \frac{\partial U^*}{\partial U}U = IU^* + \left(\frac{\partial U}{\partial U}\right)^* U = U^* + U.$$

This map is always surjective for any Hermitian matrix  $M$  can be written as  $\frac{1}{2}(M + M^*)$  which means that if we map  $\frac{1}{2}M$  through  $F$  we will recover our matrix  $M$ . In particular, this means that  $I$  is a regular value of  $F$  and therefore,  $\mathcal{U}$  is a submanifold of the matrix space.

To count the dimensions, it's important to recall that as a real-vector space, the dimension of the space of matrices is  $2d^2$  because of complex coefficients. For the Hermitian matrices, we have 1 real degree of freedom across the diagonal and on the upper triangle we have  $\binom{d}{2}$  complex degrees of freedom. This means that the dimension of the unitary group is

$$\dim \mathcal{U} = 2d^2 - (d + 2\binom{d}{2}) = 2d^2 - d - d(d-1) = d^2$$

as desired.

**Exercise 3.** Let  $M$  be a compact manifold of dimension  $n$  and let  $f : M \rightarrow \mathbb{R}^n$  be a smooth map. Prove that  $f$  must have at least one critical point.

### Answer

We will first observe that the result is true in one dimension. If  $f : M \rightarrow \mathbb{R}$  is smooth, then its image is compact and therefore  $f$  must attain an extreme value. Let  $p \in M$  be the point where  $f$  reaches an extreme, for a smooth curve  $\alpha$  about

$p$  such that

$$\alpha(0) = p, \quad \alpha'(0) = v.$$

As a real function then,  $f \circ \alpha$  has an extreme value at  $t = 0$  which means that  $t = 0$  is a critical point of  $f \circ \alpha$ . Thus, we have  $(f \circ \alpha)'(0) = 0$ .

On the other hand, this is the differential of  $f$  at  $p$ :

$$df_p v := (f \circ \alpha)'(0)$$

which means that the differential is zero and therefore non-surjective. Thus  $p$  is a critical value of  $f$ .

In general, consider a coordinate projection map  $\pi_i = \langle \mathbf{e}_i |^a$  for some  $i$ . The function  $\pi_i \circ f$  then becomes a smooth function to  $\mathbb{R}$  with a critical point  $p$ . This means that

$$d(\pi_i \circ f)_p$$

is not surjective. By the chain rule this is  $(d\pi_i)_{f(p)} df_p$  and as these are linear maps, product is a composition. Observe that the differential of our projection is itself as it's a linear map and so the failure of surjectivity must come from  $df_p$ . We conclude that  $p$  is a critical point of  $f$ .

<sup>a</sup>The linear map, take the dot product of input with  $\mathbf{e}_i$ .

**Exercise 4.** Prove that, if  $X, Y$ , and  $Z$  are smooth vector fields on a smooth manifold  $M$  and  $a, b \in \mathbb{R}$ ,  $f, g \in C^\infty(M)$ , then

- (a)  $[X, Y] = -[Y, X]$  (anticommutativity)
- (b)  $[aX + bY, Z] = a[X, Z] + b[Y, Z]$  (linearity)
- (c)  $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$  (Jacobi identity)
- (d)  $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ .

### Answer

We have defined the Lie bracket as the commutator of vector fields

$$[X, Y]f := X(Yf) - Y(Xf)$$

and so we have:

$$(a) \quad [Y, X]f = Y(Xf) - X(Yf) = -(X(Yf) - Y(Xf)) = -[X, Y]f.$$

(b) To show linearity we have

$$\begin{aligned} & [aX + bY, Z]f \\ &= (aX + bY)(Zf) - Z((aX + bY)f) \\ &= aX(Zf) + bY(Zf) - Z(aXf + bYf) \end{aligned}$$

where the first equality comes by definition and the second one is the definition of the sum of linear operators. Now applyin the fact that  $Z$  is linear we get:

$$aX(Zf) + bY(Zf) - aZ(Xf) - bZ(Yf)$$

which we rearrange as a sum of smooth functions now:

$$a(X(Zf) - Z(Xf)) + b(Y(Zf) - Z(Yf)) = (a[X, Z] + b[Y, Z])f.$$

(c) Let us take the first two terms in the sum and see that

$$\begin{aligned} & ([X, Y], Z] + [[Y, Z], X])f \\ &= [X, Y](Zf) - Z([X, Y]f) + [Y, Z](Xf) - X([Y, Z]f) \\ &= X(YZf) - Y(XZf) - Z(XYf) + Z(YXf) \\ &\quad + Y(ZXf) - Z(YXf) - X(YZf) + X(ZYf) \end{aligned}$$

Observe now that the 1<sup>st</sup> and 7<sup>th</sup>, and 4<sup>th</sup> and 6<sup>th</sup> terms cancel out. We are left with a term which we rearrange into...

$$\begin{aligned} & -Y(XZf) - Z(XYf) + Y(ZXf) + X(ZYf) \\ &= Y(ZXf) - Y(XZf) - (ZX(Yf) - XZ(Yf)) \\ &= Y([Z, X]f) - [Z, X](Yf) \\ &= [Y, [Z, X]]f = -[[Z, X], Y]f \end{aligned}$$

as desired.

(d) Finally if  $h$  is another smooth function on  $M$ :

$$\begin{aligned} & [fX, gY]h \\ &= fX(gYh) - gY(fXh) \\ &= fXgYh + fgXYh - gYfXh - gfYXh \\ &= fXgYh + fg(XYh - YXh) - gYfXh \\ &= (f(Xg)Y + fg[X, Y] - g(Yf)X)h \end{aligned}$$

where in the second equality we have applied the product rule for vector fields.