MATH502 — Combinatorics 2

Based on the lectures by Maria Gillespie

Notes written by Ignacio Rojas

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This is the second semester of an introductory graduate-level course on combinatorics. We will be covering symmetric function theory, Young tableaux, counting with group actions, designs, matroids, finite geometries, and not-so-finite geometries.

The goal of this class is to give an overview of the wide variety of topics and techniques in both classical and modern combinatorial theory.

Requirements

Knowledge on theory of enumeration, generating functions, combinatorial species, the basics of graph theory, posets, partitions and tableaux, and basic symmetric function theory is required.

Contents

Chapter 1

Symmetric functions

1.1 Day 1 | 20230120

Definition 1.1.1. $f(x_1, x_2, \dots)$ is symmetric if it's fixed under permutations of variables. For a permutation σ this is,

$$f(x_{\sigma(1), x_{\sigma(2)}}, \dots) = f(x_1, x_2, \dots).$$

Example 1.1.2. The function

$$f(x_1, \dots, x_4) = x_1^5 + \cdots + x_4^5$$

is known as p_5 or $m_{(5)}$, where p is the power-sum symmetric function and m, the monomial symmetric function.

We can have the function defined on infinitely many variables. Consider the function g defined as

$$g = x_1^4 x_2 + x_1^4 x_3 + \cdots + x_i^4 x_j + \cdots + 3x_1 + \cdots + 3x_i + \cdots = m_{(4,1)} + 3m_{(1)}.$$

Let us recall some **notation**,

$$\begin{cases} \Lambda_R(x_1, \dots, x_n) \to \text{symmetric functions on } n \text{ variables over } R, \\ \Lambda_R(\underline{x}) \to \text{symmetric functions on \textit{infinitely} many variables over } R. \end{cases}$$

In our case $R = \mathbb{Q}$, so the object of study is $\Lambda_{\mathbb{Q}}$.

Proposition 1.1.3. *The space $\Lambda_{\mathbb{Q}}^n$ is the space of symmetric functions of degree n. Its dimension is $p(n)$, the number of partitions of n.*

This is because, for every such function we can decompose it into monomials and the monomial symmetric functions form a basis.

Bases of Λ_Q

Suppose $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ with $\lambda_1 \geqslant \ldots \geqslant \lambda_k$.

Monomial Symmetric Functions

The function $m_\lambda(\underline{x})$ is the smallest symmetric function which contains the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_k^{\lambda_k}$ as a term. In general

$$m_\lambda = \sum_{i_1 \neq \cdots \neq i_k} x_{i_1}^{\lambda_1} \ldots x_{i_k}^{\lambda_k}.$$

Example 1.1.4. Consider the partition $(5,3) \vdash 8$. The function $m_{(5,3)}$ will be different depending on the number of variables:

- ◇ In one variable we can't have monomials of the form $x_i x_j$, so $m_{(5,3)} = 0$.
- ◇ In two variables we have $m_{(5,3)}(x,y) = x^5 y^3 + y^5 x^3$.
- ◇ In three variables the function is

$$m_{(5,3)}(x,y,z) = x^5 y^3 + y^5 z^3 + z^5 x^3 + y^5 x^3 + z^5 y^3 + x^5 z^3.$$

Considering some special cases, take the partition $(1,1,1,1) \vdash 4$, then

$$\begin{aligned} m_{(1,1,1,1)}(u,v,x,y,z) &= uvxy + vxyz + xyzu + yzuv + zuvx \\ &= uvxy + uxyz + uvyz + uvxz + vxyz. \end{aligned}$$

For cases with less than 4 variables the function is zero and in exactly four, it has 1 term. The partition $(4) \vdash 4$ returns the function

$$m_{(4)}(x) = x^4,\ m_{(4)}(x,y) = x^4 + y^4,\ m_{(4)}(x,y,z) = x^4 + y^4 + z^4,$$

and so on with any number of variables.

Remark 1.1.5. The number of terms in $m_\lambda(x_1, \ldots, x_d)$ is I actually don't know, while the degree of m_λ is $|\lambda| = n$.

Elementary Symmetric Functions

Definition 1.1.6. For any $r \in \mathbb{N}$, the elementary symmetric function e_r is $m_{(1,1,\ldots,1)}$ (r ones). For λ, a partition, $e_\lambda = \prod e_{\lambda_i}$. As an alternative for $m_{(1,1,\ldots,1)}$ we can also write

$$e_r(x_1, \ldots, x_d) = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} x_{i_1} \ldots x_{i_r}.$$

Example 1.1.7. Let us calculate $e_{(2,1)}$ for 1 through 3 variables. When we have $e_{(2,1)}(x) = e_2(x)e_1(x)$, we can't compute $e_2(x)$ because there are no two-term monomials with only one variable. On two variables we have the following

$$e_{(2,1)}(x,y) = e_2(x,y)e_1(x,y) = (xy)(x+y) = x^2y + y^2x$$

and when talking about 3 variables the following happens:

$$\begin{aligned} e_{(2,1)}(x,y,z) &= e_2(x,y,z)e_1(x,y,z) \\ &= (xy+yz+zx)(x+y+z) \\ &= x^2y + y^2z + z^2x + y^2x + z^2y + x^2z + 2xyz. \end{aligned}$$

Consider now the partitions $(2,2,2,2)$ and (5). Then

$$e_{(2,2,2,2)} = e_2^4 \Rightarrow e_{(r,r,\ldots,r)} = e_r^{m_r(\lambda)}$$

where $m_i(\lambda)$ is number of parts of λ equal to i. For the partition (5) we have that $e_{(5)} = e_5$ and in general $e_{(n)} = e_n$.

Remark 1.1.8. As before we don't know how many terms per function, but knowing m implies knowing e. As for the degree, it holds that $\deg(e_\lambda) = |\lambda|$.

Homogenous Symmetric Functions

- Homogenous: $h_\lambda = \prod h_{\lambda_i}$ and $h_d = x_1^d + \cdots + x_1^{d-1}x_2 + \cdots + x_1^{d-2}x_2^2 + x_1^{d-2}x_2x_3 + \ldots$. In general $h_d = \sum_{\lambda \vdash d} m_\lambda$.

- Power sum: $p_\lambda = \prod p_{\lambda_i}$ and $p_d = \sum x_i^d$.

For Schur basis recall SSYT

Example 1.1.9. Consider $\lambda = (5,4,1)$, rows $\leqslant\rightarrow$ and columns $<$, we associate the monomial $x_1^2x_2^3x_3^3x_4^2 := x^T$.

- Schur: $s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$ but also $\sum K_{\lambda\mu} m_\mu$ where the sum is over SSYT of shape λ, content μ.

Schur function motivation (preview)

The first place they showed up is in the representation theory of Lie group. The function $s_\lambda(x_1, \ldots, x_n)$ is a character of irreducible polynomial representations of GL_n. In theoretical physics we have matrix groups acting on particles, representations are smaller matrix groups of things that they are mapping to. We want to take tensor product and direct sums of representations, the tensor product is related to multiplication of Schur function while direct sum into sum of Schur functions.

There's also the Schur-Weyl duality which takes representations into the Weyl group. Under the *Frobenius map*, s_λ corresponds to irreducible representations of S_n.

A more modern application of Schur function goes into geometry, s_λ correspond to Schubert varieties in Grassmannians. Multiplication corresponds to interesections and sum to unions.

There's also context in Probability Theory. But in the end, Schur positivity is important because of this connections.

Definition 1.1.10. $f \in \Lambda$ is Schur-positive if $f = \sum c_\lambda s_\lambda$, $c_\lambda \geqslant 0$.

Example 1.1.11. $3s_{(2,1)} + 2s_{(3)}$ schur pos but change 2 to $-\frac{1}{2}$ then not.

1.2 day 2

Alg defn Schur fncs

Definition 1.2.1. A function is antisymmetric if for $\pi \in S_n$,

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \operatorname{sgn}(\pi) f(x_1, \ldots, x_n).$$

Example 1.2.2. The following functions are antisymmetric:

(a) $f(x, y) = x - y$ then $f(y, x) = -f(x, y)$.
(b) $g(x, y) = (x - y)(x + y)$.
(c) $h(x, y) = x^2 y - y^2 x$.

Notice that the last function can factor as $h = -xy(x - y)$. We claim that this is always the case.

Lemma 1.2.3. *Every antisymmetric polynomial f in two variables x, y can factor as $f(x, y) = (x - y)g(x, y)$ where g is symmetric.*

Proof

Suppose f is antisymmetric, then $f(x,x) = 0$ by taking $y = x$. This means that $(x-y) \mid f$. Thus $f(x,y) = (x-y)g(x,y)$ and we now need to show that g is symmetric.

$$g(y,x) = \frac{f(y,x)}{y-x} = \frac{-f(x,y)}{-(x-y)} = \frac{f(x,y)}{x-y} = g(x,y).$$

Monomial Antisymmetric Functions

Definition 1.2.4. Given a strict partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 > \cdots > \lambda_k$, we define

$$a_\lambda(x_1, \ldots, x_n) = x_1^{\lambda_1} \cdots x_k^{\lambda_k} \pm \text{similar terms} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_k x_{\pi(k)}^{\lambda_k}.$$

This a_λ can be zero.

Example 1.2.5. For two variables we've seen some antisymmetric polynomials. Let us calculate

$$a_{(3,1)}(x,y) = x^3y - y^3x.$$

The smallest possible example in 3 variables is

$$a_{(2,1,0)}(x,y,z) = x^2y + y^2z + z^2x - y^2x - z^2y - x^2z.$$

This can be factored as $(x-y)(y-z)(x-z)$. A similar construction gives us

$$a_{(4,2,0)}(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - y^4x^2 - z^4y^2 - x^4z^2,$$

but how does this factor? We get

$$a_{(4,2,0)}(x,y,z) = (x^2-y^2)(y^2-z^2)(x^2-z^2) = a_{(2,1,0)}(x,y,z)(x+y)(y+z)(x+z).$$

Lemma 1.2.6. *The set $\{ a_\lambda \}_{\lambda \text{ strict}}$ is a basis of the antisymmetric polynomials over $\mathbb{Q}$, $A_{\mathbb{Q}}$. Even more any a_λ is divisible by a_ρ where $\rho = (n-1, n-2, \ldots, 2, 1, 0)$.*

As an algebra generator, a_ρ is a generator.

Proof

WRITE

Proposition 1.2.7. *The a_ρ antisymmetric function is also the* *Vandermonde determinant*:

$$a_\rho = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \dots & x_1^2 & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \dots & x_2^2 & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \dots & x_n^2 & x_n & 1 \end{pmatrix}$$

Schur Polynomials

Definition 1.2.8. The Schur polynomial of $\lambda \in \mathrm{Par}$ is

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})}.$$

Here $\lambda + \rho$ is the pointwise sum as arrays.

Remark 1.2.9. This is the Weyl character proof.

The following proof is due to Proctor(1987) find ref

Lemma 1.2.10. *Any a_λ can be seen as a determinant in the following way*:

$$a_\lambda(\underline{x}) = \det \begin{pmatrix} x_1^{\lambda_1} & x_1^{\lambda_2} & \dots & x_1^{\lambda_n} \\ x_2^{\lambda_1} & x_2^{\lambda_2} & \dots & x_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \dots & x_n \end{pmatrix}$$

Proof

We want to see that

$$\frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})} = \sum x^T$$

where the sum ranges through T's which are SSYT(la) with max entry n.

(a) We will show a recursion for the combinatorial definition that the character formula will also satisfy. It holds that

$$s_\lambda(\underline{x}) = \sum s_\mu(\underline{x}) x_n^{|\lambda|-|\mu|}$$

where μ has $n-1$ parts with $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \dots$.

(b) We also show that the ratio of determinants satisfies the same recursion.

Example 1.2.11. Consider $\lambda = (8, 8, 4, 1, 1)$ and $\mu = (8, 5, 2, 1)$, then $\lambda \backslash \mu$ is a skew-table in which we can fill in n's

Corollary 1.2.12. *The Schur polynomials are a basis of $\Lambda_{\mathbb{Q}}$.*

1.3 Day 3 | 20230125

Recall $\Lambda = \mathbb{Q}[e_1, e_2, \dots]$ where the e_j's are the elementary symmetric functions. So the e_j's are algebraic generators of Λ and they're algebraically independent. Equivalently, as a vector space, $\{ e_\lambda : \lambda \in \text{Par} \}$ is a basis.

Proposition 1.3.1. *A homomorphism $f : \Lambda \to \Lambda$ ($f(a+b) = f(a) + f(b),\ f(ab)f(a)f(b)$ for $a, b \in \Lambda$) is fully determined by where it sends the e_i's.*

Definition 1.3.2. The map $\omega \in \text{End}(\Lambda)$ will send e_j to h_j.

Example 1.3.3. Consider $f = 3e_{(2,1)} + 2e_3$, then applying ω we get

$$\omega(f) = \omega(3e_{(2,1)} + 2e_3) = 3h_{(2,1)} + 2h_3.$$

For p_2, we can decompose to $e_1^2 - 2e_2$. So

$$\omega(p_2) = \omega(e_1^2 - 2e_2) = h_1^2 - 2h_2$$

and we can expand this last expression into

$$(x_1 + x_2 + \dots)^2 - 2(x_1^2 + x_2^2 + \dots + x_1x_2 + x_1x_3 + \dots) = -x_1^2 - x_2^2 - \dots$$

and we recognize this last term as $-p_2$. *This is not a coincidence.*

Theorem 1.3.4. *The map ω is involutive.*

Proof

It suffices to prove that $\omega(h_j) = e_j$. We will use power expansions and generating functions. We have

$$H(t) = \frac{1}{1 - x_1 t} \frac{1}{1 - x_2 t} \cdots = \sum h_n(\underline{x}) t^n,$$

and this comes from expanding the $1/(1-y)$'s as geometric series. When collecting the coefficients of t^n we get exactly $h_n(\underline{x})$. Similarly, for the elementary symmetric functions,

$$E(t) = (1 + x_1 t)(1 + x_2 t) \cdots = \sum e_n t^n.$$

When multiplying to obtain the coefficient of t^n we get a plethora of different x_j's which form the e_j's. Now from this expressions we have $H(t)E(-t) = 1$ which means that

$$\left(\sum h_n(\underline{x})t^n\right)\left(\sum e_n(\underline{x})(-t)^n\right) \Rightarrow \sum_{k=0}^{n} (-1)^k e_k h_{n-k} = 0, \; n \geqslant 1.$$

Now applying the map to the equation we get

$$\omega\left(\sum_{k=0}^{n} (-1)^k e_k h_{n-k}\right) = \sum_{k=0}^{n} (-1)^k h_k \omega(h_{n-k}) = 0.$$

After reindexing, we get that both e_j's and $\omega(h_j)$'s are determined recursively by the h_j's in the same way. Thus we conclude that $\omega(h_j) = e_j$.

Lemma 1.3.5. *The following equation holds for the power-sum symmetric functions*:

$$\exp\left(\sum \frac{1}{n} p_n(\underline{x}) p_n(\underline{y})\right) = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} =: \Omega(\underline{x}, \underline{y}).$$

It also holds that

$$\Omega(\underline{x}, \underline{y}) = \sum_{l} a \frac{1}{z^{\lambda}} p_\lambda(\underline{x}) p_\lambda(\underline{y})$$

where $z_\lambda = \prod k^{m_k} m_k!$ *where* m_k *is the number of parts of* λ *equal to* k.

Proof

We will prove both parts separately. For the first equation we will take the logarithm on both sides:

$$\sum \frac{1}{n} p_n(\underline{x}) p_n(\underline{y}) = \log\left(\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j}\right)$$

and after manipulating the logarithm we get

$$\sum_{i,j=1}^{\infty} (\log(1) - \log(1 - x_i y_j)) = \sum_{i,j=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} x_i^n y_j^n.$$

We can separate[a] into

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_i x_i^n \right) \left(\sum_j y_j^n \right).$$

Now taking exp on both sides we get equality.
By not removing the exponential we get the following expression

$$\exp\left(\sum \frac{1}{n} p_n(\underline{x}) p_n(\underline{y}) \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum \frac{1}{n} p_n(\underline{x}) p_n(\underline{y}) \right)^k.$$

To get a term of the form $p_\lambda(\underline{x})p_\lambda(\underline{y})$ we have to choose which parts of the λ come from each of the factors in $\sum \frac{1}{n} p_n(\underline{x})p_n(\underline{y})$. If $\ell(\lambda) = k$ then it comes from the k^{th} term in the exponential sum. If $\lambda = (\lambda_1, dots, \lambda_1, \dots, 2, \dots, 2, 1, \dots, 1)$ with m_{λ_1} λ_1's, m_1 1's, then out of k elements we have to choose m_1 1's and so on. Thus there are $\binom{k}{m_{\lambda_1}, \dots, m_1}$ choices and each i in λ comes with a $\frac{1}{i}$. Therefore the coefficient of $p_\lambda(\underline{x})p_\lambda(\underline{y})$ is

$$\frac{1}{k!} \frac{k!}{m_1! m_2! \dots} \frac{1}{1^{m_1}} \frac{1}{2^{m_2}} \cdots = \frac{1}{z_\lambda}.$$

[a] Are we using Fubini-Tonelli here?

Lemma 1.3.6. *We have the following identities*

$$\exp\left(\sum \frac{(-1)^{n-1}}{n} p_n(\underline{x}) p_n(\underline{y}) \right) = \prod_{i,j=1}^{\infty} \frac{1}{1 + x_i y_j} = \sum_\lambda \frac{(-1)^{n-\ell(\lambda)}}{z_\lambda} p_\lambda(\underline{x}) p_\lambda(\underline{y}).$$

Lemma 1.3.7. *Another equality for $\Omega(\underline{x}, \underline{y})$ is*

$$\Omega(\underline{x}, \underline{y}) = \sum_\lambda m_\lambda(\underline{x}) h_\lambda(\underline{y})$$

Theorem 1.3.8. *It holds that $\omega(p_\lambda) = (-1)^{n-k} p_\lambda$ where k is the number of parts of λ.*

Proof

Applying ω to Ω, but *only working with $\underline{y}$ variables* we get

$$\omega(\Omega) = \omega\left(\sum_\lambda m_\lambda(\underline{x}) h_\lambda(\underline{y}) \right) = \sum_\lambda m_\lambda(\underline{x}) e_\lambda(\underline{y}) = \prod_{i,j=1}^{\infty} (1 + x_i y_j) = \sum_\lambda \frac{1}{z_\lambda} (-1)^{n-k_\lambda} p_\lambda(\underline{x}) p_\lambda(\underline{y}).$$

Comparing coefficients with

$$\omega\left(\sum_{l} a\frac{1}{z^{\lambda}}p_{\lambda}(\underline{x})p_{\lambda}(\underline{y})\right)$$

we get the result.

1.4 Day 4 | 20230127

To continue exploring the ring of symmetric functions we need a couple of tools. One of them is the involution which we have already seen. But the other one is a scalar product which is compatible with the multiplication.

Hall Inner Product

Recall an inner product is a function

$$\langle -|-\rangle\colon\ V \times V \to \mathbb{Q}$$

which is bilinear $\langle u+v|w\rangle = \langle u|w\rangle + \langle v|w\rangle$ and the same on the other entry. For scalars the following behavior is expected $\langle \lambda u|v\rangle = \langle u|\lambda v\rangle = \lambda\langle u|v\rangle$. Recall that if the base field is the complex numbers, then the inner product is Hermitian.

Definition 1.4.1. We say that two vectors are orthogonal when $\langle u|v\rangle = 0$.

This gives us a possible decomposition of space into several components. Suppose that $\{ u_\lambda \}_{\lambda\in\mathrm{Par}(n)}, \{ v_\lambda \}_{\lambda\in\mathrm{Par}(n)}$ are basis of Λ^n. So we would like a condition such as

$$\langle u_\lambda|v_\mu\rangle = \begin{cases} 0 \ \lambda \neq \mu, \\ 1 \ \lambda = \mu. \end{cases}$$

If we cap the dimension this says that $\langle u|v\rangle$ is the usual dot product. But in infinite dimensions we don't have matrices. We'll call this basis dual to one another. If miraculously we have the same basis, then this basis is orthonormal.

Definition 1.4.2 (Phillip Hall)**.** The Hall inner product is defined so that $\langle m_\lambda|h_\mu\rangle = \delta_{\lambda\mu}$.

By defining the product on two basis, we have defined it for all other elements by bilinearity.

Lemma 1.4.3. *The Hall inner product is symmetric.*

Theorem 1.4.4. *The Hall inner product is positive definite, this is $\langle f|f\rangle \geqslant 0$ and equality is achieved when $f = 0$.*

It's important to note that this statement is symmetric. However we are talking about an asymmetric definition. Last, before proving the statement we need a criteria for dual bases. But importantly, recall the result from last lecture: 1.3.7

Theorem 1.4.5. *If $u_\lambda, \{v_\mu\}$ are dual, then $\sum_\lambda u_\lambda v_\lambda = \Omega$.*

Proof

Fix a partition of n, then

$$\delta_{\lambda\mu} = \langle m_\lambda | h_\mu\rangle = \left\langle \sum_{\rho\vdash n} \alpha_{\lambda_\rho} u_\rho \middle| \sum_{\tau\vdash n} \beta_{\mu_\tau} v_\tau \right\rangle = \sum_{\rho,\tau} \alpha_{\lambda_\rho}\beta_{\mu_\tau} \langle u_\rho | v_\tau\rangle .$$

We want $\langle u_\rho|v_\tau\rangle = \delta_{\rho\tau}$, to that effect name $A_{\rho\tau}$ the matrix whose entries are $\langle u_\rho|v_\tau\rangle$.
As u and v are dual bases, we have that $A = \mathrm{id}$. Thus $I = \alpha\beta^\mathsf{T}$ and now $\delta_{\rho\tau} = \sum \alpha_{\lambda_\rho}\beta_{\lambda_\tau}$. We are now going to use the hypothesis and the interpretation of m, h in the u, v basis. We have

$$\Omega = \sum \left(\sum \alpha u\right)\left(\sum \beta v\right) = \sum\left(\sum \alpha\beta\right) uv = \sum uv$$

so the inner sum must be one and thus we are done.

Corollary 1.4.6. *For the Hall inner product it holds that $\langle p_\lambda|p_\mu\rangle = z_\lambda\delta_{\lambda\mu}$.*

The key is to recall that p_λ is an eigenfunction of ω. Also 1.3.5. By using a power-sum decomposition it is possible to prove that the Hall inner product is positive definite.

Corollary 1.4.7. *The ω involution is orthogonal with respect to $\langle -|-\rangle$. This is $\langle \omega f|\omega g\rangle = \langle f|g\rangle$.*

Once again, the idea is to transfer to power-sum and use the fact that it's an eigenfunction.

1.5 Interim 1

Theorem 1.5.1 (Fundamental Theorem of Sym. Fnc. Thry.). *Every symmetric function can be written uniquely in the form $\sum_\lambda c_\lambda e_\lambda$ with $c_\lambda \in \mathbb{Q}$.*

There are at least two proofs if not more of this fact. The first comes from Maria Gillespie's blog which Mark Haiman presented to her.

Proof

It suffices to prove the transition matrix between m and e is invertible.

For proof 2 read [10] pg. 290. Proof 3 in another Maria post

1.6 Day 5 | 20230130

Exercise 1.6.1. Compute $\omega(s_{(3,1)})$.

Answer

We have that By Jacobi-Trudi

$$s_{(3,1)} = \det \begin{pmatrix} h_3 & h_4 \\ 1 & h_1 \end{pmatrix} = h_{(3,1)} - h_4.$$

Using the omega involution, we get

Recall that $\omega : h_n \leftrightarrow e_n$, $\omega p_k = (-1)^{k-1} p_k$. We have the following questions, where do m and s map to? Also

$$\langle m|h\rangle = \delta,\ \langle p|p/z\rangle = \delta,$$

but what are e and s dual to?

Definition 1.6.2. We call $\omega m_\lambda = f_\lambda$ the <u>forgotten basis</u>.

There's not much we could say about them, they are not Schur positive and there's no patterns.

Dual to e

Recall ω is an isometry, so $\langle \omega f|\omega g\rangle = \langle f|g\rangle$, so

$$\langle e_\lambda|?\rangle = \langle h_\lambda|\omega ?\rangle = \delta_{\lambda\mu}.$$

Since $\langle h|m\rangle = \delta$, then applying ω again we get that $\langle e_\lambda|f_\mu\rangle = \delta_{\lambda\mu}$.

RSK algorithm

We want to show two things:

$$\omega s_\lambda = s_{\lambda^\intercal},\ \langle s_\lambda | s_\mu \rangle = \delta_{\lambda\mu}.$$

Proposition 1.6.3. *It holds that*

$$\sum_\lambda s_\lambda(\underline{x}) s_\lambda(\underline{y}) = \Omega = \sum_\lambda m_\lambda(\underline{x}) h_\lambda(\underline{y})$$

Proof

The sum on the left is

$$\sum_{(S,T) SSYT} x^S y^T$$

so we will study pairs (S, T) of SSYT of the same shape to show that they're equal to the sum on the right.

algorithm: process of doing the bijection.

The RSK bijection takes a pair (S, T) of SSYT of the same shape and it maps it to "two-line arrays" of length n.

Definition 1.6.4. A <u>two-line array</u> is a matrix in $\mathcal{M}_{2\times n}(\mathbb{Z}_{\geqslant 0})$ such that

i) The bottom row is weakly increasing.
ii) If $b_i = b_{i+1}$, then $a_i \leqslant a_{i+1}$, where a's are the top row and b's the bottom row.

Example 1.6.5. Consider the matrix

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{pmatrix}$$

Within "blocks", there is a weak increment. From right-to-left we will find a pair of SSYT. We will "insert" top row letters from left-to-right.

(a) Place 1st letter $\boxed{1}$

(b) For each letter, if it can go at the end of last row, put it there

$$\boxed{1}\boxed{1} \leftarrow 2,\ \boxed{1}\boxed{1}\boxed{2} \leftarrow 1$$

but one can't go after 2.

(c) Otherwise if inserting b_1, let c be the leftmost $> b$, "bump c", then insert c into the next row.

$$\begin{ytableau} 1 & 1 & 1 \\ 2 \end{ytableau}$$

For the bottom row, place in a new square at each step to form a "recording tableau". The recording tableau always matches the shape of the insertion one. The first three steps lead to $\begin{ytableau} 1 & 1 & 1 \end{ytableau}$ in the recording one. But in the fourth step we get $\begin{ytableau} 1 & 1 & 1 \\ 2 \end{ytableau}$. The next step leads us to

$$\begin{ytableau} 1 & 1 & 1 & 4 \\ 2 \end{ytableau}\,, \quad \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 \end{ytableau}$$

then in insertion, 2 bumps 4 and 4 doesn't bump 2 on next row, so we get

$$\begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 4 \end{ytableau}\,, \quad \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 3 \end{ytableau}$$

The three is no problem so

$$\begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ 2 & 4 \end{ytableau}\,, \quad \begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 \end{ytableau}$$

then the next one bumps out the 2, the 2 bumps the 4 on the second row to get

$$\begin{ytableau} 1 & 1 & 1 & 1 & 3 \\ 2 & 2 \\ 4 \end{ytableau}\,, \quad \begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 \\ 4 \end{ytableau}$$

Finally

$$\begin{ytableau} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 3 \\ 4 \end{ytableau}\,, \quad \begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 \end{ytableau}\,.$$

Why do we get SSYT. The insertion tableau gives us the question, can we make a column non-increasing? No, we are always bumping something bigger. Imagine we bump $c > b$ with b, then c replaces something that goes to the left.

$$\begin{ytableau} \leqslant & b & c \\ {} & {} \\ {} \end{ytableau} \Rightarrow \begin{ytableau} {} & b & {} \\ {} & d \\ {} \end{ytableau}$$

and $d > c$ so it bumps something else. The recording tableau is also a SSYT. Let us prove it.

Lemma 1.6.6 (Key Lemma 1)**.** *The insertion path (sequence of squares that are bumped) moves up and weakly left.*

Lemma 1.6.7 (Key Lemma 2)**.** *If $a \leqslant b$ and T is a SSYT, computing*

$$T \leftarrow \boxed{a} \leftarrow \boxed{b},$$

the intersection path of a in T lies strictly left of the intersection path of b in $T \leftarrow \boxed{a}$.

Proof

We will do induction on the rows with an example.

Example 1.6.8. Consider

$$\begin{ytableau} 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 3 & 4 \\ 3 & 3 & 5 & 5 \\ 4 & 4 \end{ytableau}$$

Inserting 1 we bump the 2, then the 3 and finally the 5. We get

$$\begin{ytableau} 1 & 1 & 1 & o & 2 & 3 \\ 2 & 2 & t & 3 & 4 \\ 3 & 3 & t & 5 & 5 \\ 4 & 4 & f \end{ytableau}$$

so inserting the 2 we bump 3,4,5. And they will be to the side of the last sequence.

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Exercise 1.7.1. Apply RSK to $\begin{pmatrix} 3 & 2 & 4 & 1 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

Answer

We get $\begin{ytableau} 1 & 4 & 5 \\ 2 \\ 3 \end{ytableau}$, $\begin{ytableau} 1 & 3 & 5 \\ 2 \\ 4 \end{ytableau}$.

Notice that we got STANDARD Young tableau. So to prove it's a bijection we will begin with all different numbers.

Lemma 1.7.2. *The RSK bijection is a bijection between pairs of standard Young tableaux of the same shape and "permutations"* ($2 \times n$ *matrices whose rows are permutations.*)

To prove it's a bijection we will find an inverse by reversing the process. Look at the recording tableau, we will bump out the largest number. We will take S as the recording tableau. Then we start with the spot on S, T which corresponds to largest label in S.

- ⋄ If b is the item in such a square we "un-bump" it.
  - If in bottom row, just remove.
  - Else, let c be the rightmost entry in row below b that is less than b. Then replace b with c and repeat the process with c until the letter that is removed is done by the just removing it.

  Then we add the two letters to the matrix from right-t-left.

With the original tableau we remove the 5 and the 5 to get

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}\ , \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

then the 4 indicates that in T we must "un-bump" the 3. The three un-bumps the 2, the 2 to the 1 so that we get

$$\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\ , \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ .$$

Now we get the matrix $\begin{pmatrix} x & x & x & 1 & 5 \\ x & x & x & 4 & 5 \end{pmatrix}$ and removing the 3 from S just removes the 4 from T as it is in the bottom row.

Now as this two sets are in bijection, this means that they have the same size.

Corollary 1.7.3. *Let f^λ be the number of standard Young tableau of shape λ. Then*

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!.$$

We will generalize one step at a time. Let us now assume that T is semi-standard. On the matrix, we will have that the top row is now random, but the bottom row is still from 1 to n.

Lemma 1.7.4 (Schensted)**.** *There is a bijection between (S, T), S is standard, T is SSYT, and words of length n.*

Example 1.7.5. Consider the matrix $\begin{pmatrix} 2 & 1 & 3 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ which returns the two Young tableau

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}\ , \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}\ .$$

The proof of the inverse is similar but when un-bumping, we must bump the rightmost entry *strictly* smaller than b. But we don't need this, we will do it more creatively.

Definition 1.7.6. Suppose T is a Young tableau. Then

i) The reading word of T $\mathrm{rw}(T)$ is the concatenation of rows from top to bottom.
ii) The standarization of an SSYT T, $\mathrm{std}(T)$, is the unique SYT with same relative order of entries, ties broken with "reading order".
iii) The standarization of a word is similar

In the previous example, the reading word is

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \to 23113.$$

The standarization are as follows:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}\ , \quad 23113 \to 34125.$$

We can standarize the matrix

$$\begin{pmatrix} 2 & 1 & 3 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 3 & 4 & 1 & 2 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

and this matrix corresponds to the pair (S, T) where T is $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$. In essence, the following diagram commutes

$$\begin{tikzcd}
(S, T) \arrow[d, "std"'] & 21313 \arrow[l, "RSK"'] \arrow[d] \\
(S, T') & 31425 \arrow[l, "RSK"]
\end{tikzcd}$$

Definition 1.7.7. Given a content $\mu = (\mu_1, \dots, \mu_k)$ with $\sum \mu_k = n$ (not nec. partition). Then the de-standarization with respect to μ of a SYT T is a $SSYT$ T' such that $\text{std}(T') = T$.

In this case

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \xrightarrow[\text{std}^{-1}(2,1,2)]{} \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}.$$

Recall now lemma 1.6.7 about consecutive insertions.

The Full RSK

We are now going to prove that there is an inverse to the original RSK function. Consider the following example

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} \to \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 7 \\ \hline 4 & 6 & 9 \\ \cline{1-3} 8 \\ \cline{1-1}\end{array}$$

The matrix $\begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{pmatrix}$ can be standarized to our word matrix. Then the table $\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$ also standarizes to the word table.

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Index

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