Exercise 1. Define a *line* in $\mathbb{P}^2$ to be a closed subset of the form $L = \{ [x : y : z] : ax + by + cz = 0 \}$ for some constants $a, b, c \in \mathbb{C}$, not all zero.

i) If $(a, b, c) = (1, 0, 0)$, we saw in class that $\mathbb{P}^2 \backslash L = \{ [x : y : z] : x \neq 0 \} = U_x$ could be identified with $\mathbb{C}^2$.

Similarly, show that for any line L there is a bijection $\mathbb{P}^2 \backslash L \simeq \mathbb{C}^2$.

ii) Prove that any two distinct lines L_1 and L_2 intersect in a single point.

iii) Prove that there is a unique line L through any two distinct points in $\mathbb{P}^2$.

Answer

i) I still need to think this one, it's giving me trouble. I thought about it on the real case and I can visualize it. I even tried using the stereographic projection with a computer program to look at it but I can't grasp it.

ii) Let $L_1, L_2 \subseteq \mathbb{P}^2$ be two distinct lines with direction (a, b, c) and (d, e, f). Since they are distinct this means that $\nexists \lambda((d, e, f) = \lambda(a, b, c))$. A point $[x : y : z]$ in the intersection of L_1 and L_2 must satisfy the system of equations

$$\begin{cases} ax + by + cz = 0, \\ dx + ey + fz = 0. \end{cases}$$

Solutions to this system of equations are parametrized in terms of z in the following manner

$$[x : y : z] = \left[\frac{bf - ce}{ae - bd} z : \frac{cd - af}{ae - bd} z : z \right],$$

and ordinarily this would give us an infinite number of solutions. However in $\mathbb{P}^2$ this corresponds to the point $[bf - ce : cd - af : ae - bd]$.

iii) Let us now consider two points $[x : y : z], [u : v : w] \in \mathbb{P}^2$ which are distinct. Once again, consider a system of equations

$$\begin{cases} ax + by + cz = 0, \\ au + bv + cw = 0. \end{cases}$$

There is an infinite number of solutions to this system for $(a, b, c) \in \mathbb{C}^3$. The solution set for this system of equations is precisely the unique line which passes through $[x : y : z], [u : v : w]$

Exercise 2. Consider the sequence $(p_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}^3$ with $p_n = (n^3, 2n^2, 3n^3)$. Identifying $\mathbb{C}^3$ with $\{ x_0 \neq 0 \} \subseteq \mathbb{P}^3$, what is the limit of p_n as $n \to \infty$?

Answer

We can identify p_n with the sequence $\widetilde{p}_n = [n^3 : 2n^2 : 3n^3 : 1]$. Now for $n \neq 0$ it holds that

$$\widetilde{p}_n = \left[1 : \frac{2}{n} : 3 : \frac{1}{n^3}\right] \xrightarrow[n\to\infty]{} [1 : 0 : 3 : 0].$$

This coincides with the limit of p_n in the usual sense which is ∞ and $[1 : 0 : 3 : 0]$ is a point at infinity.

Exercise 3. In $\mathbb{A}^2$, let $V = \mathbb{V}(x)$, $W = \mathbb{V}(x-1)$ and $Z = \mathbb{V}(y - x^2)$. Let $\overline{V}, \overline{W}$ and $\overline{Z}$ denote their respective *projective closures* in $\mathbb{P}^2$. Find the points in the intersections $\overline{V} \cap \overline{W}$, $\overline{V} \cap \overline{Z}$ and $\overline{W} \cap \overline{Z}$.

Answer

First, let us parametrize the varieties in question as points of $\mathbb{A}^2$:

$$\begin{cases} \mathbb{V}(x) = \{ x = 0 \} = \{ (0,t) : t \in \mathbb{C} \}, \\ \mathbb{V}(x-1) = \{ x = 1 \} = \{ (1,t) : t \in \mathbb{C} \}, \\ \mathbb{V}(y-x^2) = \{ y = x^2 \} = \{ (t,t^2) : t \in \mathbb{C} \}. \end{cases}$$

For each one of those sets, their projective closure corresponds to the embedding of the points inside $\mathbb{P}^2$ along with their limit points. In the case of V we have

$$\overline{V} = \{ [0 : t : 1] : t \in \mathbb{C} \} \cup \{ \text{limit points} \} = \{ [0 : t : 1] : t \in \mathbb{C} \} \cup \{ [0 : 1 : 0] \}.$$

Likewise we have

$$\begin{cases} \overline{W} = \{ [1 : t : 1] : t \in \mathbb{C} \} \cup \{ [0 : 1 : 0] \}, \\ \overline{Z} = \{ [t : t^2 : 1] : t \in \mathbb{C} \} \cup \{ [0 : 1 : 0] \}. \end{cases}$$

Now their intersections are

$$\begin{cases} \overline{V} \cap \overline{W} = \{ [0 : 1 : 0] \}, \\ \overline{V} \cap \overline{Z} = \{ [0 : 0 : 1], [0 : 1 : 0] \}, \\ \overline{V} \cap \overline{Z} = \{ [1 : 1 : 1], [0 : 1 : 0] \}. \end{cases}$$

This coincides with our intuition. The lines only intersect at infinity, while the parabola and the lines intersect at the finite point and at infinity.

Exercise 4. Do the following:

i) Find a bijection between the set of all homogeneous polynomials in three variables of degree d and the set of all polynomials in two variables of degree at most d. ⟦Hint: Set one the variables to the constant 1.⟧
ii) Use this to show that the subspace topology induced by the affine patches $V \cap \mathbb{A}^2$ from the Zariski topology on a variety $V \subseteq \mathbb{P}^2$ is the same as the Zariski topology on the affine variety $V \cap \mathbb{A}^2$.
iii) Generalize to arbitrary dimension.

Answer

i) Let us call $\mathbb{C}_h[X, Y, Z]$ the set of homogeneous polynomials and $\mathbb{C}[x, y]$ the set of regular polynomials. Consider the mapping

$$\varepsilon_{(x,y,1)} : \mathbb{C}_h[X, Y, Z] \to \mathbb{C}[x, y],\ F([X : Y : Z]) \mapsto F(x, y, 1) = f(x, y)$$

where we evaluate $z = 1$ and consider the resulting polynomial as a non-homogeneous polynomial. The inverse mapping is

$$\varepsilon^{-1} : \mathbb{C}[x, y] \to \mathbb{C}_h[X, Y, Z],\ f(x, y) \mapsto z^d f\left(\frac{x}{z} : \frac{y}{z} : 1\right)$$

where we view the last polynomial as a homogeneous polynomial.

These functions are inverses to one another:

$$\varepsilon(\varepsilon^{-1}(f)) = \varepsilon\left(z^d f\left(\frac{x}{z} : \frac{y}{z} : 1\right)\right) = f(x : y : 1)$$

and the action of this function is the same as $f(x, y)$. The other direction is analogous.

ii) Let us recall that $\mathbb{A}^2$ can be viewed as a topological subspace of $\mathbb{P}^2$ because $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$. So to show that the subspace topology is the same as the ordinary Zariski topology it is enough to show that

$$V \subseteq \mathbb{P}^2 \text{ is closed } \iff V \cap \mathbb{A}^2 \text{ is the zero locus of a set of polynomials.}$$

($\Rightarrow$) Suppose $V = \mathbb{V}(F_1, \dots, F_n)$, where $F_j \in \mathbb{C}_h[X, Y, Z]$, we want to show that $V \cap \mathbb{A}^2 = \mathbb{V}[\varepsilon F_1, \dots, \varepsilon F_n]$. For that effect, take $[a : b : c] \in V \cap \mathbb{A}^2$, this means that $c \neq 0$, so we can take $c = 1$ due to rescaling.

$$\begin{aligned} [a : b : 1] \in V &\Rightarrow \forall i(F_i([a : b : 1]) = 0) \\ &\Rightarrow \forall i[(\varepsilon F_i)(a, b) = 0] \\ &\Rightarrow (a, b) \in \mathbb{V}[\varepsilon F_1, \dots, \varepsilon F_n]. \end{aligned}$$

On the other hand, if $[a : b : c] \notin V \cap \mathbb{A}^2$, we can assume that $[a : b : c] \in \mathbb{A}^2 \backslash V$. Then there exists a homogenous polynomial F_i such that $F_i([a : b : c]) \neq 0$. Then $\varepsilon F_i(a, b) \neq 0$ and so we have the contrapositive. Thus, a zero locus of homogeneous polynomials induces a zero locus of regular polynomials inside the affine patch.

($\Leftarrow$) Let us now assume that we have a zero locus of polynomials $W = \mathbb{V}(f_1, \ldots, f_n)$. We want to find $V \subseteq \mathbb{P}^2$ such that $V \cap \mathbb{A}^2 = W$.

Take $V = \mathbb{V}(\varepsilon^{-1} f_1, \ldots, \varepsilon^{-1} f_n)$, we will prove both inclusions.

$$\begin{aligned}
[a : b : c] \in V \cap \mathbb{A}^2 &\Rightarrow \forall i(\varepsilon^{-1} f_i([a : b : 1]) = 0), \\
&\Rightarrow \forall i(f_i(a, b) = 0), \\
&\Rightarrow (a, b) \in W.
\end{aligned}$$

Following the identification inside projective space, this lets us conclude that $[a : b : 1]$ is in W.

On the other hand if $[a : b : c] \in \mathbb{A}^2 \backslash V$, then

$$\exists i(\varepsilon^{-1} f_i([a : b : c]) \neq 0) \Rightarrow \varepsilon(\varepsilon^{-1} f_i([a : b : c])) = f_i(a, b) \neq 0.$$

It follows that $[a : b : c] \notin W$.

We have thus proven that any projective Zariski closed set induces an ordinary Zariski closed set inside the affine patch and vice-versa. Thus both topologies coincide.[a]

iii) The generalization to higher dimension takes a point $[x_0 : x_1 : \cdots : x_n]$ in projective space to $[1 : x_1 : \cdots : x_n]$ which is in the affine patch. All of my previous arguments are analogous in this case.

For the case of the bijection, we set one coordinate to one for the direction $\mathbb{C}_h[\underline{X}] \to \mathbb{C}[\underline{x}]$. The inverse of this function is analogous with the coordinate we set to one.

[a] I feel like this proof is SO messy and long, I want to polish it to a better version.

Exercise 5. For the following coordinate rings, find affine varieties whose coordinate rings are isomorphic to the ones in questions.

i) $\mathbb{C}[x, 1/x, y]$ (this is, rational functions whose denominator is a polynomial in x.) ⟦ Hint: See exercise 4.1.1 and the preceding paragraphs. ⟧
ii) $\mathbb{C}[x, y, 1/(x^2+y^2)]$, i.e. rational functions whose denominator is a polynomial is x^2+y^2.

Answer

i) If we consider $z = \frac{1}{x}$, then this is precisely the zero locus of the polynomial $xz-1$. In this sense,

$$\mathbb{C}[x, 1/x, y] \simeq \mathbb{C}[x, z, y] \Big/ \mathrm{gen}(xz-1).$$

Then this is the coordinate ring of $V = \mathbb{V}(xz-1)$.
ii) Once again let us consider the equation $z = \frac{1}{x^2+y^2}$. We get that $zx^2 + zy^2 - 1 = 0$. It follows that

$$\mathbb{C}[x, y, 1/(x^2+y^2)] \simeq \mathbb{C}[x, y, z] \Big/ \mathrm{gen}(z(x^2+y^2)-1).$$

Then the variety $V = \mathbb{V}(z(x^2+y^2)-1)$ has this ring as its coordinate ring.