Exercise 1 (7(a))**.** Show that if two words w and v are Knuth equivalent, then removing the smallest k letters from each of w and v (with ties broken in order from left to right) results in two Knuth equivalent words w' and v'. (Hint: It suffices to show it for just one letter removed, and it also suffices to assume that w and v differ by an elementary Knuth move.)

Let us recall the Knuth move definition from a year ago:

Definition 1. A Knuth move on a permutation swaps two letters a, c if $a < b < c$ (reading order) and one of consecutive subsequences acb, cab, bac, bca appears in the word.

Two words are Knuth-equivalent if they differ by a sequence of Knuth-moves.

Answer

We[a] follow the hint as indicated. This is possible because after doing a sequence of Knuth moves for a particular letter and then doing it for all the letters, we get to the complete result.

So for our case let us take a word w whose smallest letter is x_0. We have two cases:

- If x_0 is not in the affix acb we are Knuth moving then

$$w = \ldots x_0 \ldots acb \ldots.$$

  It is equivalent if x_0 is after the affix. Let us Knuth move and remove x_0 and vice versa.

$$\begin{tikzcd} \ldots x_0 \ldots acb \ldots \arrow[r, "\text{Knuth}"] \arrow[d, "\text{Rem}_{x_0}"'] & \ldots x_0 \ldots cab \ldots \arrow[d, "\text{Rem}_{x_0}"] \\ \ldots acb \ldots \arrow[r, "\text{Knuth}"'] & \ldots cab \ldots \end{tikzcd}$$

  The case where our affix is bac is analogous.

- However if x_0 was in the affix we have that x_0 takes the place of a as the smallest letter. Then we have $w = \ldots x_0cb \ldots$ and so

$$\begin{tikzcd} \ldots x_0cb \ldots \arrow[rr, "\text{Knuth}"] \arrow[dr, "\text{Rem}_{x_0}"'] & & \ldots cx_0b \ldots \arrow[dl, "\text{Rem}_{x_0}"] \\ & \ldots cb \ldots & \end{tikzcd}$$

which leaves us with the same word in both cases. In a similar fashion to the previous part, working with the affix bx_0c is analogous.

In both cases we can see that removing the smallest letter and Knuth moving commutes on words. So this tells us that originally Knuth equivalent words will still be Knuth equivalent after removing the smallest letter.

[a] **Ross, Joel and myself discussed this exercise together.**

Exercise 2 (7(b))**.** Given a pair (S, T) in $P^{\nu}_{\lambda\mu}$, show that T must be the unique highest weight tableau of shape μ. Conclude that the pair (T, T) corresponds under RSK to a two line array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{|\mu|} \\ b_1 & b_2 & \dots & b_{|\mu|} \end{pmatrix}$$

where $a_1, a_2, \dots, a_{|\mu|}$ is the unique weakly increasing word of content μ and $b_1, b_2, \dots, b_{|\mu|}$ is a ballot word.

Answer

As (S, T) is in $P^{\nu}_{\lambda\mu}$, then $\mathrm{rw}(S)\,\mathrm{rw}(T)$ is a ballot word. We have that every suffix of a ballot word is itself ballot. So in particular $\mathrm{rw}(T)$ is ballot and therefore T is of highest weight.