Exercise 1. Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$, and $a \neq 0$. ⟦Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.⟧

Answer

The function $g(z) = f(1/z)$ has a singularity at $z = 0$. If it were removable, then g is bounded on $B(0, R)$ for some $R > 0$.
This means that f is bounded on $\{ |z| > R \}$, but as f is entire, it's continuous and so it's bounded in $\overline{B}(0, R)$, the *closed* ball. From this, we see that f is bounded in all of $\mathbb{C}$.
By Liouville's theorem f is constant. But that contradicts the fact that f is injective.
Now assume g has an essential singularity at $z = 0$. By the Casorati-Weierstrass theorem, we have a neighborhood of the origin $B(0, R)$ with $R > 0$, such that $g[B(0, R)]$ is dense in $\mathbb{C}$. This means that $f[\{ |z| > R \}]$ is dense in $\mathbb{C}$.
Recall that dense sets intersect every non-trivial open set, so in particular we find an intersection with $f[B(0, R)]$ (which is open by the open mapping theorem). This means that there exists $w \in f[\{ |z| > R \}] \cap f[B(0, R)]$ such that

$$w = f(z_1) = f(z_2), \quad \text{where} \quad |z_1| > R, \quad \text{and} \quad |z_2| < R.$$

In particular $z_1 \neq z_2$. So this contradicts the injectivity of f.
Finally this means that g has a finite-order pole at $z = 0$. When taking the Laurent expansion of g, this corresponds to having finitely many terms of the form $\frac{a_k}{z^k}$.
As for f, the positive degree part of its Laurent expansion is a finite degree polynomial. There are no negative power terms because f is entire.
This lets us conclude that f is a polynomial. The degree of f can't be anything other than 1 because otherwise it won't be injective. Therefore, we conclude that f is a linear function.

Exercise 2. As in class, consider the unit sphere

$$X = \{ (a, b, c) : a^2 + b^2 + c^2 = 1 \} \subseteq \mathbb{R}^3$$

Let $N = (0, 0, 1), S = (0, 0, -1), U_N = X \backslash N, U_S = X \backslash S$. Consider the following three charts on X:

- $\phi_N : U_N \to \mathbb{C},\ (a, b, c) \mapsto \frac{a+ib}{1-c}$.

- $\phi_S : U_S \to \mathbb{C},\ (a,b,c) \mapsto \frac{a+ib}{1+c}$.
- $\psi_S : U_S \to \mathbb{C},\ (a,b,c) \mapsto \frac{a-ib}{1+c}$.

Do the following:

i) The inverse of ϕ_N is

$$\phi_N^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right).$$

Calculate ϕ_S^{-1} and ψ_S^{-1}.

ii) Among the three charts $\{\, (U_N, \phi_N), (U_S, \phi_S), (U_S, \psi_S) \,\}$, one pair is compatible and the other two are not. Which is which? Why?

$[\![$ Hint: Remember a function is holomorphic if and only if $\partial_{\overline{z}} f = 0$. $]\!]$

Answer

We claim that

$$\phi_S^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1} \right).$$

When composing this function with ϕ_S we obtain

$$\phi_S^{-1}(\phi_S(a,b,c)) = \phi_S^{-1}\left(\frac{a+bi}{1+c}\right)$$

To ease our calculations we may calculate the modulus of this complex number beforehand:

$$\left|\frac{a+bi}{1+c}\right|^2 = \frac{a^2+b^2}{(1+c)^2} = \frac{1-c^2}{(1+c)^2} = \frac{1-c}{1+c}.$$

From this we can also see

$$\frac{1-c}{1+c} + 1 = \frac{2}{1+c}, \quad \text{and} \quad 1 - \frac{1-c}{1+c} = \frac{2c}{1+c}.$$

Applying this to our calculation we obtain

$$\phi_S^{-1}\left(\frac{a+bi}{1+c}\right) = \left(\frac{(2a)/(1+c)}{2/(1+c)}, \frac{(2b)/(1+c)}{2/(1+c)}, \frac{(2c)/(1+c)}{2/(1+c)}, \right) = (a,b,c).$$

In a similar fashion we have

$$\begin{aligned}\phi_S(\phi_S^{-1}(a,b,c)) &= \phi_S\left(\frac{2\operatorname{Re}(z)}{|z|^2+1}, \frac{2\operatorname{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right)\\ &= \frac{(2\operatorname{Re}(z))/(|z|^2+1) + i(2\operatorname{Im}(z))/(|z|^2+1)}{1+(1-|z|^2)/(|z|^2+1)}\\ &= \frac{2z}{|z|^2+1+1-|z|^2} = z.\end{aligned}$$

Therefore ϕ_S^{-1} is indeed the inverse map of ϕ_S. Now, observe that $\psi_S = \overline{\phi_S}$ from which we may conclude that $\psi_S^{-1}(z) = \phi_S^{-1}(\overline{z})$, this is

$$\psi_S^{-1}(z) = \left(\frac{2\operatorname{Re}(z)}{|z|^2+1}, \frac{-2\operatorname{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right).$$

Finally considering the transition maps we may see after calculating that

$$\phi_S \circ \phi_N^{-1} = \frac{1}{\overline{z}}, \quad \psi_S \circ \phi_N^{-1} = \frac{1}{z}, \quad \text{and} \quad \psi_S \circ \phi_S^{-1} = \overline{z}.$$

Among these three, the only holomorphic transition map is $\psi_S \circ \phi_N^{-1}$. From this, we see that $\mathbb{CP}$ with the atlas $\{ (U_N, \phi_N), (U_S, \psi_S) \}$ is a complex manifold.

Exercise 3. If f is meromorphic on Ω and $z_0 \in \Omega$, we define the order of f by

$$\operatorname{ord}_{z_0}(f) = \begin{cases} 0 & \text{when } f \text{ is holomorphic at } z_0 \text{ and } f(z_0) \neq 0,\\ m & \text{when } f \text{ has a zero of order } m \text{ at } z_0,\\ -m & \text{when } f \text{ has a pole of order } -m \text{ at } z_0.\end{cases}$$

Do the following:

i) Let $p(z)$ be a polynomial of degree d, thought of as a meromorphic function $\hat{C} \to \hat{C}$. Use the definition of a pole at infinity ([SS, p. 87]) to show that $\operatorname{ord}_\infty p = -d$.
ii) Show that if $p(z)$ is a polynomial, then

$$\sum_{z_0 \in \hat{\mathbb{C}}} \operatorname{ord}_{z_0}(f) = 0.$$

⟦ Hint: Use the fundamental theorem of algebra. ⟧

iii) Show that if $f(z) = \frac{p(z)}{q(z)}$ is a rational function, then

$$\sum_{z_0 \in \hat{\mathbb{C}}} \operatorname{ord}_{z_0}(f) = 0.$$

Answer

i) The behavior of p at infinity is the behavior of $p\left(\frac{1}{z}\right)$ at the origin. Observe that if p had degree d then

$$p(z) = a_0 + a_1 z + \cdots + a_d z^d, \quad \text{where} \quad a_d \neq 0$$
$$\Rightarrow p\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \cdots + \frac{a_d}{z^d} = \frac{1}{z^d}(a_0 z^d + a_1 z^{d-1} + \ldots a_d).$$

Observe that at $z = 0$, the function $a_0 z^d + a_1 z^{d-1} + \ldots a_d$ doesn't vanish because $a_d \neq 0$ and it's holomorphic. Then we see that the order of the pole at the origin is $-d$. Thus for p, $-d = \operatorname{ord}_\infty p$.

ii) We may factor p as

$$p(z) = a \prod_{k=1}^{r} (z - z_k)^{\alpha_k}$$

where $z_1, \ldots, z_k$ are the roots of p. Now

$$\sum_{z_0 \in \hat{\mathbb{C}}} \operatorname{ord}_{z_0}(f) = \operatorname{ord}_\infty(p) + \sum_{k=1}^{r} \operatorname{ord}_{z_k}(p) = -d + \sum_{k=1}^{r} \alpha_k = 0$$

which follows from $\sum_{k=1}^r \alpha_k = d$.

iii) Finally consider a rational function $\frac{p}{q}$. Then

$$\sum_{z_0 \in \hat{\mathbb{C}}} \operatorname{ord}_{z_0}(f) = \sum_{z_0 \in \hat{\mathbb{C}}} \operatorname{ord}_{z_0}(p) + \sum_{z_0 \in \hat{\mathbb{C}}} \operatorname{ord}_{z_0}\left(\frac{1}{q}\right) = \sum_{z_0 \in \hat{\mathbb{C}}} \operatorname{ord}_{z_0}\left(\frac{1}{q}\right).$$

As $\operatorname{ord}_{z_0}\left(\frac{1}{q}\right) = -\operatorname{ord}_{z_0}(q)$, the other sum is also zero, because $\sum \operatorname{ord}_{z_0}(q) = 0$. In conclusion we have $\operatorname{ord}_{z_0}(f) = 0$.