Exercise 1 (Exercise 4)**.** Prove the generating function identity

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \left(\!\binom{n}{k}\!\right) x^k.$$

You may either use induction on n, or a direct combinatorial argument about what the coefficients must be when you expand the product on the left

Answer

Differentiating both sides of the equality $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ n times[a] we get

$$\begin{aligned} D^n\left(\frac{1}{1-x}\right) &= \frac{(n-1)!}{(1-x)^n}, \\ D^n\left(\sum_{k=0}^{\infty} x^k\right) &= \sum_{k=n}^{\infty} (k)(k-1)(k-2)\dots(k-n+1)x^{k-n} \\ \left(\begin{smallmatrix} k=\ell+n \\ k\to n \\ \Rightarrow \ell\to 0 \end{smallmatrix}\right) &= \sum_{\ell=0}^{\infty} (\ell+n)(\ell+n-1)(\ell+n-2)\dots(\ell+1)x^{\ell}. \end{aligned}$$

We get the following equality

$$\frac{1}{(1-x)^n} = \sum_{\ell=0}^{\infty} \frac{(\ell+n)(\ell+n-1)(\ell+n-2)\dots(\ell+1)}{(n-1)!} x^{\ell},$$

and the coefficient in question is precisely

$$\frac{(\ell+n)(\ell+n-1)(\ell+n-2)\dots(\ell+1)}{(n-1)!} = \frac{(\ell+n)!}{(n-1)!\ell!} = \binom{n+\ell-1}{\ell} = \left(\!\binom{n}{\ell}\!\right).$$

[a] Implicitly I'm using induction here

This fact can also be proven using the multiplication principle:

$$\frac{1}{(1-x)^n} = \prod_{k=1}^{n} \left(\frac{1}{1-x}\right).$$

If by induction we assume that the identity holds up to $n-1$, then the product on the right becomes

$$\left[\prod_{k=1}^{n-1} \left(\frac{1}{1-x}\right)\right] \left(\frac{1}{1-x}\right) = \left(\sum_{k=0}^{\infty} \left(\!\binom{n-1}{k}\!\right) x^k\right) \left(\sum_{k=0}^{\infty} x^k\right).$$

After multiplying we obtain the sum

$$\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}\left(\!\!\binom{n-1}{j}\!\!\right)\right]x^k.$$

If we were to prove the identity $\sum_{j=0}^{k}\left(\!\binom{n-1}{j}\!\right) = \left(\!\binom{n}{k}\!\right)$, then we would be done.

Lemma 1. *The following identity holds for* n, k, *positive integers*:

$$\sum_{j=0}^{k}\left(\!\!\binom{n-1}{j}\!\!\right) = \left(\!\!\binom{n}{k}\!\!\right).$$

This is a type of Pascal recurrence for the multichoose coefficient. We can state the first recurrence and the inductively prove this one, or we can prove this one by a counting argument.

Initially consider the recurrence

$$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n}{k-1}\!\!\right).$$

- The quantity on the left counts the number of ways I can distribute k `cookies` among n `grad students`.
- For the quantity on the right, choose the n^{th} `grad student`. There are two ways to give my k `cookies`.
  - Either I exclude the last `grad student` and give out my k `cookies` among the other $n-1$.
  - Or I give *at least* 1 `cookie` to the last one, and I give out the remaining $k-1$ among all the n `grad students`.

With this recurrence it is immediate to prove the identity:

$$\begin{aligned}
\left(\!\!\binom{n}{k}\!\!\right) &= \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n}{k-1}\!\!\right)\\
&= \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n-1}{k-1}\!\!\right) + \left(\!\!\binom{n}{k-2}\!\!\right)\\
&= \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n-1}{k-1}\!\!\right) + \left(\!\!\binom{n-1}{k-2}\!\!\right) + \cdots + \left(\!\!\binom{n}{0}\!\!\right)
\end{aligned}$$

However we can prove the identity in another way:

Consider the same situation where we label the n^{th} `grad student`. Giving out k `cookies` to n `grad students` is the same as giving $k-j$ to the last `grad student` and distribute the remaining j `cookies` among the $n-1$ other `grad students`. Since this events are disjoint, the total number of ways can be obtained by summing for each j, thus obtaining the identity.

There are two more ways in which I'm certain that this problem can be proven:

i) Using the n-fold multiplication principle. The sequence $\mathbf{1} = (1)_{n\in\mathbb{N}}$'s generating function is precisely $1/(1-x)$ so

$$\left(\frac{1}{1-x}\right)^n = \left(\sum_{k=0}^{\infty} x^k\right)^n = \sum_{k=0}^{\infty} \underbrace{(\mathbf{1} * \mathbf{1} * \cdots * \mathbf{1})}_{n \text{ times}} x^k.$$

Using induction and algebraic manipulation, it is possible to prove that the convolution in question is the multichoose coefficient.

ii) The coefficient $\left(\!\binom{n}{k}\!\right)$ also counts *weak compositions* of k into n parts. This is in correspondence with the amount of ways one can form an x^k monomial from the product

$$\left(\sum_{k=0}^{\infty} x^k\right)^n = (1+x+x^2+\dots)(1+x+x^2+\dots)\cdots(1+x+x^2+\dots)$$

since the exponents in the n factors are the *parts* of k.

Exercise 2 (Exercise 6)**.** Find a closed form for the generating function of the sequence b_n defined by $b_0 = 1$ and for all $n \geqslant 0$, $b_{n+1} = \sum_{k=0}^{n} k b_{n-k}$. Use it to find an explicit formula for b_n in terms of n.

Answer

Let us call $B(x) = \sum_{n=0}^{\infty} b_n x^n$. Then

$$B(x) = b_0 + b_1 x + b_2 x^2 + \dots \Rightarrow \frac{B(x) - b_0}{x} = b_1 + b_2 x + b_3 x^2 + \cdots = \sum_{n=0}^{\infty} b_{n+1} x^n.$$

However, from the recurrence we have that

$$\sum_{n=0}^{\infty} b_{n+1}x^n = \sum_{n=0}^{\infty}(n * b_n)x^n = \left(\sum_{n=0}^{\infty} nx^n\right) B(x) = xD\left(\frac{1}{1-x}\right) B(x).$$

Equating this quantities, and using the initial condition, we get

$$\begin{aligned}
\frac{B(x)-1}{x} = \frac{xB(x)}{(1-x)^2} &\Rightarrow B(x)\left(\frac{1}{x} - \frac{x}{(1-x)^2}\right) = \frac{1}{x} \\
&\Rightarrow B(x) = \frac{(1-x)^2}{1-2x} \\
&\Rightarrow B(x) = \frac{1}{1-2x} - \frac{2x}{1-2x} + \frac{x^2}{1-2x}.
\end{aligned}$$

This is the generating function for the sequence (b_n). After converting the functions into sums and rearranging the terms, the closed form of b_n can be obtained. This is done as follows:

$$\begin{aligned}
B(x) &= \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} 2^{n+1}x^{n+1} + \sum_{n=0}^{\infty} 2^n x^{n+2} \\
&= \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=1}^{\infty} 2^n x^n + \sum_{n=2}^{\infty} 2^{n-2} x^n \\
&= 1 + (2-2)x + \sum_{n=2}^{\infty} 2^{n-2} x^n
\end{aligned}$$

From the last equality we extract that

$$b_n = \begin{cases} 1, \text{ when } n = 0 \\ 0, \text{ when } n = 1 \\ 2^{n-2}, \text{ when } n \geqslant 2 \end{cases}$$

Exercise 3 (Exercise 8)**.** Let $p(n,k)$ be the number of partitions of n into exactly k nonzero parts. Show that

$$\sum_{n,k} p(n,k) y^k x^n = \prod_{k=1}^{\infty} \frac{1}{1-yx^k}.$$

Answer

Exercise 4 (Exercise 9)**.** Use generating functions to prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Do NOT give a combinatorial proof. Instead, give a proof by comparing coefficients of two equal generating functions or polynomials.

Answer

Let us begin by considering the binomial formula:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

The square of this function can be computed in two ways:

i) Directly applying the formula

$$(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k.$$

ii) Or by using the multiplication principle

$$((1+x)^n)^2 = \sum_{k=0}^{2n} \left[\sum_{j=0}^{k} \binom{n}{j} \binom{n}{n-j} \right] x^k.$$

It follows that

$$\binom{2n}{k} = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{n-j}$$

since two equal polynomials must share the same coefficients. By setting $k = n$ and relabeling the counter from j to k we arrive at the desired identity.