Exercise 1. Do the following:

i) Let $q = (a_1, \ldots, a_n)$ be a point in $\mathbb{A}^n$. Using the fact that $I(q)$ is a maximal ideal in $\mathbb{C}[x_1, \ldots, x_n]$, prove that the coordinate ring of q is isomorphic to $\mathbb{C}$.

ii) If $i : \{ q \} \to \mathbb{A}^n$ is the inclusion map, show that the pullback homomorphism

$$i^\sharp : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[q] = \mathbb{C}$$

sends a function $f(x_1, \ldots, x_n)$ to the complex number $f(a_1, \ldots, a_n)$ obtained by evaluating at that point.

Answer

i) The ideal $I(q)$ is in fact $\operatorname{gen}(x_1 - a_1, \ldots, x_n - a_n)$, a maximal ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Then the coordinate ring of $\{ q \}$ is precisely

$$\mathbb{C}[q] = \mathbb{C}[x_1, \ldots, x_n] \Big/ I(\{ q \}) = \mathbb{C}[x_1, \ldots, x_n] \Big/ \operatorname{gen}(x_1 - a_1, \ldots, x_n - a_n).$$

The evaluation homomorphism ε_q with help of the 1^{st} isomorphism theorem gives us the desired isomorphism. This is clearly a surjective map since we can get to any complex number by solving a linear equation and its kernel is the aforementioned ideal.

ii) Since the inclusion mapping is a morphism of algebraic varieties, then it induces a pullback homomorphism between the coordinate rings. By definition its action is as follows:

$$i^\sharp : \mathbb{C}[\mathbb{A}^n] \to \mathbb{C}[q], \; g \mapsto g \circ i.$$

Let us unpack the terminology. First, the inclusion homomorphism is the identity mapping restricted to $\{ q \}$. Then the pullback can be expressed as

$$i^\sharp : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}, \; g(\mathbf{z}) \mapsto g(\operatorname{id}|_{\{ q \}}(\mathbf{z})).$$

In this sense the action of $g \circ i$ is

$$\mathbb{A}^n \xrightarrow[\mathbf{z} \mapsto q]{\operatorname{id}|_{\{ q \}}} \mathbb{A}^n \xrightarrow[g \mapsto g(-)]{g} \mathbb{C} \Rightarrow \mathbb{A}^n \xrightarrow[g \mapsto g(q)]{g \circ i} \mathbb{C},$$

and thus, since the action of this map is the same as ε_q, we conclude that $i^\sharp = \varepsilon_q$.

Exercise 2. Prove that if $F : V \to W$ is an isomorphism of affine algebraic varieties, then the pullback homomorphism is a ring isomorphism.

Answer

The pullback homomorphism is precisely $F^\sharp : \mathbb{C}[W] \to \mathbb{C}[V]$ such that $g \mapsto g \circ F$. Since the pullback is already a ring homomorphism, it suffices to show that it is invertible by explicitly constructing an inverse.
The map $F^{-1} : W \to V$ defines a pullback from $\mathbb{C}[V]$ to $\mathbb{C}[W]$ such that $(F^{-1})^\sharp(h) = h \circ F^{-1}$. Now this last map is an inverse to $F^\sharp$ since

$$F^\sharp((F^{-1})^\sharp(h)) = F^\sharp(h \circ F^{-1}) = (h \circ F^{-1}) \circ F = h,$$

and likewise on the other side. It follows that $(F^{-1})^\sharp = (F^\sharp)^{-1}$.

Exercise 3. Let $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ be affine algebraic varieties. Let $\tilde{F} : \mathbb{A}^n \to \mathbb{A}^m$ be a morphism. Show that

$$\tilde{F}(V) \subseteq W \iff \tilde{F}^\sharp : \mathbb{C}[y_1, \ldots, y_m] \to \mathbb{C}[x_1, \ldots, x_n] \text{ sends } I(W) \text{ to } I(V).$$

⟦ Hint: W=V(I(W)). ⟧

Exercise 4 (2.6.1)**.** Prove that the spectrum $\operatorname{Spec}(R)$ of a commutative ring R can be given the structure of a topological space whose closed sets are of the form $V(I) = \{\, \mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq I \,\}$ for $I \trianglelefteq R$.

Answer

First, the whole set and the empty set are closed:

- Since every prime ideal $\mathfrak{p}$ is an ideal, we get $0 \in \mathfrak{p}$ and then $\{0\} \subseteq \mathfrak{p}$. Then $V(0) = \{\, \mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \{0\} \,\}$, and this is the whole set. Thus $V(0) = \operatorname{Spec}(R)$.
- On the other hand, if $V(I) = \varnothing$, then, there's no prime ideal which contains I besides the whole ring. It follows that $V(R) = \varnothing$.

Let us now prove that the finite union of closed sets is closed. This is, we are looking for $K \trianglelefteq R$ such that $V(I) \cup V(J) = V(K)$. Let us take $K = IJ$, then

$$IJ \leqslant I, J \Rightarrow V(I) \cup V(J) \subseteq V(IJ), \; (\mathfrak{p} \supseteq IJ) \Rightarrow [(\mathfrak{p} \supseteq I) \vee (\mathfrak{p} \supseteq J)]$$

and the last assertion gives us the other inclusion. We conclude that $V(IJ) = V(I) \cup V(J)$ and by induction we can prove the equality for a countable number of ideals.
Finally, consider a collection of ideals $(I_\alpha)_{\alpha\in\mathcal{A}}$. We want to find an ideal $J \trianglelefteq R$ such that $\bigcap_{\alpha\in\mathcal{A}} V(I_\alpha) = V(J)$. For that effect we shall take $J = \sum_{\alpha\in\mathcal{A}} I_\alpha$.
Since J is the smallest ideal which contains all of the I_α's, then

$$\forall\alpha\,(V(J) \subseteq V(I_\alpha)) \Rightarrow V(J) \subseteq \bigcap_{\alpha\in\mathcal{A}} V(I_\alpha).$$

On the other hand, by minimality of J,

$$\forall\alpha(\mathfrak{p} \supseteq I_\alpha) \Rightarrow \mathfrak{p} \supseteq \sum I_\alpha$$

and this guarantees the other side of the inclusion.
We conclude that in fact the Zariski topology defined on $\operatorname{Spec}(R)$ is in fact a topology.

Exercise 5 (2.5.(1,2))**.** Do the following:

i) Show that the pullback $\mathbb{C}[W] \xrightarrow{F^\sharp} \mathbb{C}[V]$ is injective if and only if F is *dominant*. This is, $F(V)$ is dense in W.
ii) Show that the pullback $\mathbb{C}[W] \xrightarrow{F^\sharp} \mathbb{C}[V]$ is surjective if and only if F defines an isomorphism between V and some algebraic subvariety of W.