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Cubic surfaces with a double line

By

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Introduction. In [Z2], Zariski has noted that the branch locus of the projection of a cubic surface with a double line is a three cuspidal plane quartic (as will be shown in the present paper, this statement should be interpreted as dealing with a "generic" projection of a "generic" cubic surface with a double line) and there he made use of this fact for deriving the fundamental group of a projective plane minus a three cuspidal quartic in the classical case. In [A5] we have made a similar use of a specific projection of a "nonspecial" cubic surface with a double line for computing the fundamental group of a projective plane minus a three cuspidal quartic in the abstract case (i.e., over a ground field of arbitrary characteristic). Furthermore, in [A6] we have employed a specific projection of a "special" cubic surface with a double line for studying the fundamental group of a projective plane minus a cuspidal cubic together with its flex tangent in the abstract case. Properties of the said projections required in [A5] and [A6] are derived respectively in Sections 3 and 6 and in Section 7 of this paper. We thought it would be of interest to give in this paper a study of cubic surfaces with a double line, together with their projections, branch loci and the corresponding "resolvent surfaces," in general.

In Section 1 (Proposition 1) we give a projective description of cubic surfaces with a double line, showing among other things that there are exactly four projectively distinct such surfaces of which two are cones while the others are non-cones, one of which is "nonspecial" while the other is "special".

In Section 2 we find the discriminants for all the various projections of the various non-conical cubic surfaces with a double line, and from this, in Section 8, we find the corresponding branch loci for the normalizations of the surfaces showing that the branch locus is a three cuspidal quartic for "generic" projections while it consists of a cuspidal cubic together with its flex tangent for "special" projections. For a cone, the branch locus will always consist of a certain number of lines.

In Sections 3 and 6 we study a specific projection of a "non-special" cubic surface V with a double line. Projecting V from a double point of V we obtain a birational representation of V as a projective plane and then via a Cremona quadratic transformation of the second kind we obtain a representation of the normalization of V as an immediate quadratic transform of a projective plane. Furthermore, for the corresponding resolvent surface we obtain a birational representation as a projective plane. The results needed in [A5] are summarized in Proposition 3 of Section 6. Sections 4 and 5 contain the needed auxiliary material. The results of Sections 3 and 6 also suggest a method of studying the local fundamental groups at a cusp and at an ordinary contact in the abstract case and we shall exploit this in a later paper.

In Section 7 we study a specific projection of a "special" cubic surface V with a double line, again obtaining birational representations of V and of the corresponding resolvent surface as projective planes. The results needed in [A6] are summarized in Proposition 4 of Section 7.

As an application of the fundamental group of a projective plane minus a three cuspidal quartic obtained in [A5], we show that the normalization of the two projectively distinct non-conical cubic surfaces with double lines are biregularly equivalent and consequently that the normalization of any non-conical cubic surface with a double line is an immediate quadratic transform of a projective plane (Proposition 7 of Section 8).

Unless otherwise stated, all points, subvarieties, varieties and correspondences will be taken to be defined over an algebraically closed ground field under consideration. We shall use the notations and conventions introduced in [A2, 3, 4]. Also for a polynomial $F(X_1, \cdots, X_n)$ we shall denote the discriminant of F with respect to X_1 by $\Delta_{X_1}F(X_1, \cdots, X_n)$; the subscript X_1 may be dropped when it is clear from the context.

1. Projective description

Let k be an algebraically closed ground field of characteristic p, let P^2 and P^3 be projective spaces over k of dimensions two and three respectively. Note that if V is an irreducible cubic surface in P^3 having a singular line L, then L must be a double line of V and V has no singularities outside L, for if L were a line of multiplicity higher than two for S then V would have to consist of three (distinct or coincidental) planes through L, and if V were to have a singular point P outside L, then the plane joining P and L would have to be a component of V.

Let X, Y, Z, T be projective coordinates in P^3 and let $F_1 = T^3+XYT+X^3$, $F_2 = T^3+X^2Y$, $F_3 = T^3+YXT+X^2Z$, $F_4 = T^3+YT^2+YXT+X^2Z$. Each F_j as a polynomial in Z is linear and in it the coefficients of Z^0 and Z^1 are coprime polynomials in X, Y, T and hence F_j is irreducible in $k[X, Y, Z, T]$; also F_j belongs to the square of the ideal in $k[X, Y, Z, T]$ generated by X and T. Let

$$V_j : F_j = 0\,; \quad L : X = T = 0\,; \quad P : X = Y = T = 0\,.$$

Then V_j is an irreducible cubic surface in P^3 with double line L.

In this section, by a projective transformation of P^3 (respectively : P^2) we shall always mean a nonsingular projective transformation of P^3 (respectively : P^2) onto P^3 (respectively : P^2) defined over k. If $\tau_1, \tau_2, \cdots, \tau_n$ are projective transformations, then by τ_n^* we shall denote the projective transformation $\tau_n \tau_{n-1} \cdots \tau_1$. For a point Q of a surface V in P^3 we shall denote the tangent cone to V at Q by C_Q^V; we may drop the superscript V when it is clear from the context.

Proposition 1. (I : $j=1, 2$). *P is a triple point of V_j and at it the tangent cone of V_1 is V_1 and the tangent cone of V_2 is V_2. Every other point Q of L is a double point of V_1 and V_2; at it the tangent cone to V_1 consists of the two planes ($X=0$ and $T=0$) and the tangent cone to V_2 consists of the plane $X=0$ counted twice. For $j=1, 2$; V_j contains no line which is skew to L. V_1 and V_2 are projectively nonequivalent and if $p \neq 3$ then any irreducible cubic cone in P^3 having a double line is projectively equivalent to V_1 or V_2.*

(II : $j=3$). *Every point of L is a double point of V_3. C_P consists of the plane (α_P : $X=0$) counted twice, and $\alpha_P \cap V_3 = L$, let us denote L by A_P. At any point Q of L other than P, C_Q consists of α_P together with another plane α_Q passing through L and the* *residual intersection of* α_Q *with* V_3 *outside* L *is a line* A_Q *passing through* Q. *If* Q *and* Q' *are two distinct points of* L *other than* P, *then* α_Q *and* $\alpha_{Q'}$ *are distinct and* A_Q *and* $A_{Q'}$ *are skew. As* Q *varies over* L, α_Q *ranges over the set of all planes through* L *and* A_Q *ranges over the set of all lines on* V_3. V_3 *does not contain any line skew to* L. *Any point of* V_3 *not on* L *lies on* A_Q *for exactly one point* Q *of* L. *Thus* V_3 *is a ruled surface with the lines* A_Q *as the ruling and the line* L *as directrix.*

(III: $j=4$). *Each point of* L *is a double point of* V_4; C_P *consists of a plane counted twice and if* $p \neq 2$ *then* $C_{P'}$ *also consists of a plane counted twice where* $P'=(0, 1, \frac{1}{4}, 0)$; *if* Q *is a point of* L *different from* P *and also different from* P' *in case* $p \neq 2$ *then* C_Q *consists of two distinct planes. For any point* Q *of* L *each component plane of* C_Q *passes through* L *and its residual intersection with* V_4 *outside* L *is a line passing through* Q. *If* Q *and* Q' *are any two distinct points of* L *then* C_Q *and* $C_{Q'}$ *have no common component and the residual intersection of any component of* C_Q *with* V_4 *outside* L *is skew to the residual intersection of any component of* $C_{Q'}$ *with* V_4 *outside* L. *The set of all the components of all the various* C_Q *coincides with the set of all planes through* L *and the set* Γ *of the residual intersections with* V_4 *outside* L *of all these various components coincides with the set of all the lines on* V_4 *other than* L *which are not skew to* L. *Every point of* V_4 *not on* L *lies on exactly one member of* Γ. V_4 *contains exactly one line* M *which is skew to* L; M *is given by* $X-T-Y=Z+T=0$, M *intersects each member of* Γ *and if for* A *in* Γ *we denote* $M \cap A$ *and* $L \cap A$ *respectively by* $\phi(A)$ *and* $\psi(A)$ *then* ϕ *maps* Γ *one-to-one onto* M *and hence* $\phi(A) \rightarrow \psi(A)$ *is a two-to-one rational map of* M *onto* L *and it induces a* k*-isomorphism* σ *of* $k(L)$ *into* $k(M)$ *such that* $k(M)/\sigma(k(L))$ *is a cyclic galois extension of degree two and* P *or* (P, P') *is the branch locus on* L *according as* $p=2$ *or* $p \neq 2$; *thus* V_4 *is a ruled surface with ruling* Γ *and directrix* M; *if* $p \neq 2$ *then the four points* P, $\phi\psi^{-1}(P)$, $P', \phi\psi^{-1}(P')$ *do not lie on a plane.*

(IV) *Any two of the surfaces* V_j *are projectively nonequivalent.*

(V) *If* $p \neq 2, 3$ *then any irreducible cubic surface in* P^3 *having a double line is projectively equivalent to one of the* V_j, *and* (VI) *in the* (9 *dimensional*) *projective space* P^9 *of all the cubic surfaces in* P^3 *having* L *as singular line, all points outside a certain proper subvariety of* P^9 *represent cubic surfaces in* P^3 *which are projectively equivalent to* V_4.

First we shall prove the following lemma.

Lemma 1. *Assume that X, Y, T are projective coordinates in P^2 and let W_j be the irreducible cubic curve in P^2 given by $F_j=0$, for $j=1,2$; and let P_1 be the point $X=T=0$ and let P_2 be the point $T=Y=0$. Then* (i) *P_1 is a 2-fold normal crossing and the only singularity of W_1;* (ii) *P_1 is a 2-fold cusp (which is ordinary for P^2) and the only singularity of W_2;* (iii) *if $p\neq 3$ then P_2 is the only flex of W_2 (i.e., the only simple point of W_2 at which the tangent to W_2 has an intersection multiplicity with W_2 which is bigger than two) and the tangent to W_2 at P_2 is the line $N: Y=0$ and $i(N\cdot W_2; P_2)=3$.* (iv) *W_1 and W_2 are projectively nonequivalent. Now let W be any irreducible cubic curve in P^2 having a singular point Q. Then* (v) *at Q, W has either a 2-fold normal crossing or a 2-fold cusp.* (vi) *If W has a normal crossing at Q then W is projectively equivalent to W_1 and* (vii) *if W has a cusp at Q and $p\neq 3$ then W is projectively equivalent to W_2.*

Proof. (i) and (ii) follow from the fact that an irreducible cubic curve can have at most one singularity, and (iv) follows from (i) and (ii). It is clear that N is the tangent to W_2 at P_2 and $i(N\cdot W_2; P_2)=3$; now assume that $p\neq 3$, let P^* be a flex of W_2 and let N^* be the tangent to W_2 at P^*, then N^* cannot pass through P_1 and hence N^* has an equation: $Y=aX+bT$ with a and b in k and substituting this in F_2 we conclude that the polynomial $T^3+bX^2T+aX^3$ must be a cube, since the coefficient of XT^2 in this polynomial is zero and since $p\neq 3$ we must have $b=a=0$, i.e., $N^*=N$ and hence $P^*=N^*\cap W_2=N\cap W_2=P_2$; this proves (iii).

Now let W be an irreducible cubic curve in P^2 having a singular point Q, then it is clear that W has either a 2-fold normal crossing at Q or a 2-fold cusp at Q and this is so according as W has two distinct tangents at Q or only one tangent at Q. First assume that W has two distinct tangents at Q. Make a projective transformation τ_1 such that $\tau_1(Q)=(X=T=0)$ and the two tangent lines to $\tau_1(W)$ at $(X=T=0)$ are $X=0$ and $T=0$. Let H be a cubic form in $k[Y, T, Z]$ such that $\tau_1(W)$ is given by $H=0$. Then $H=aX^3+bT^3+cX^2T+dXT^2+eXTY$ with a, b, c, d, e in k and $e\neq 0$. Dividing out H by e we may assume that $e=1$. Make the projective transformation τ_2: $X\to X$, $T\to T$, $Y\to Y-cX-dT$. Then $\tau_2(H)=aX^3+bT^3+XTY$. Since W is irreducible, we must have

$a \neq 0 \neq b$. Making the projective transformation τ_3: $a^{1/3}X \rightarrow X$, $b^{1/3}Y \rightarrow Y$, $Y/(a^{1/3}b^{1/3}) \rightarrow Y$ where $a^{1/3}$ and $b^{1/3}$ are cube roots respectively of a and b in k, we get $\tau_3\tau_2\tau_1(W) = W_1$. This proves (vi).

Now assume that W has only one tangent at Q and $p \neq 3$. Make a projective transformation τ_1 such that $\tau_1(Q) = (X = T = 0)$ and $X = 0$ is the tangent to $\tau_1(W)$ at $(X = T = 0)$. Then $\tau_1(W)$ has an equation $H = 0$ where $H = aX^3 + bT^3 + cX^2T + dXT^2 + eX^2Y$ with a, b, c, d, e in k and $e \neq 0$. Since H is irreducible, we must have $b \neq 0$, and hence dividing H by b we may assume that $b = 1$. Make the projective transformation τ_2: $Y \rightarrow (1/e)Y - (a/e)X - (c/e)T$, $X \rightarrow X$, $T \rightarrow T$. Then $\tau_2(H) = T^3 + dXT^2 + X^2Y$. If $d = 0$ then $\tau_2\tau_1(W) = W_2$, so now assume that $d \neq 0$. Make the projective transformation τ_3: $dX \rightarrow X$, $d^{-2}Y \rightarrow Y$, $T \rightarrow T$. Then $\tau_3\tau_2(H) = T^3 + XT^2 + X^2Y$. Now make the projective transformation τ_4: $X \rightarrow 27X$, $T \rightarrow -9X + 9T$, $Y \rightarrow -2X + 3T + Y$ $(p \neq 3)$. Then $\tau_4\tau_3\tau_2(H) = 9^3(T^3 + X^2Y)$ and hence $\tau_4\tau_3\tau_2\tau_1(W) = W_2$. This proves (vii) and completes the proof of the lemma.

Now we proceed to the proof of Proposition 1.

Proof of (I). The first two sentences follow by direct computation and from the second sentence it follows that V_1 and V_2 are projectively nonequivalent. If V_j were to contain a line M skew to L then since V_j is a cone with P as vertex, the plane joining M with P would be a component of V_j and that would contradict the irreducibility of V_j. Now assume that $p \neq 3$ and let V be an irreducible cubic cone in P^3 having a double line. Make a projective transformation τ such that the vertex of V is mapped onto P. Then $\tau(V)$ has an equation $H = 0$ where H is a cubic form in $k[X, T, Z]$. If $W: H = 0$ were a nonsingular cubic curve in the projective plane P^2 over k with projective coordinates X, T, Z then P would be the only singularity of $\tau(V)$; this being a contradiction, we are reduced to Lemma 1.

Proof of (II). It is obvious that C_P: $X^2 = 0$. Now let Q be any point of L other than P so that $Q = (0, 1, q, 0)$ where q is a uniquely determined element of k; let $X_1 = X$, $Y_1 = Y$, $Z_1 = Z - qY$, $T_1 = T$, then $F_3 = T_1^3 + Y_1X_1T_1 + X_1^2Z_1 + qX_1^2Y_1$ and hence the tangent cone to V_3 at $X_1 = T_1 = Z_1 = 0$ is given by $X_1T_1 + qX_1^2 = 0$, i.e., the tangent cone C_Q to V_3 at Q is given by $XT + qX^2 = 0$ and clearly $C_Q = \alpha_P \cup (\alpha_Q: T + qX = 0)$ and $\alpha_Q \cap V_3 = A_Q \bigcup L$ where A_Q is a line meeting L in Q. Also $\alpha_P \cap V_3 = \alpha_P \cap (T^3=0) = L$. Now let, if possible, N be a line on V_3 different from all the lines A_Q with Q in L. Then N must be skew to L and hence if we take linear forms π and π' in $k[X, Y, Z, T]$ such that $\pi=0$ and $\pi'=0$ are distinct planes through N, then π and π' must be linearly independent mod (X, T) and hence we may, after replacing π and π' by their suitable linear combinations, assume that $\pi = aX+bT-Y$ and $\pi' = \alpha X + \beta T - Z$. Now N is on V_3 implies that $0 = T^3 + aX^2T + bXT^2 + \alpha X^3 + \beta X^2 T$ and hence equating the coefficients of T^3 we get the contradiction $0=1$.

Proof of (III). Let Q be any point of L other than P, then $Q=(0, 1, q, 0)$ where q is a uniquely determined element of k. The tangent cone C_P consists of the plane α_P: $X=0$ counted twice; let $X_1=X$, $Y_1=Y$, $Z_1=Z-qY$, $T_1=T$, then $F_4 = T_1{}^3 + Y_1 T_1{}^2 + Y_1 X_1 T_1 + X_1^2 Z_1 + qX_1^2 Y_1$ and hence the tangent cone to V_4 at $X_1 = T_1 = Z_1 = 0$ is given by $T_1^2 + X_1 T_1 + qX_1^2 = 0$, i.e., C_Q is given by $G_Q = 0$ where $G_Q = T^2 + XT + qX^2$, and hence C_Q consists either of two distinct planes α_Q, α_Q^*, or of one plane counted twice according as $\Delta_T(G_Q) = X^2 - 4qX^2 \neq 0$ or $=0$, i.e., never in case $p=2$ and for $q=\dfrac{1}{4}$ in case $p \neq 2$. Also it is obvious that $\alpha_P : X=0$ is not a component of $G_Q=0$. Now let q and q' be distinct elements in k, then by a direct computation one sees that the resultant of G_Q and $G_{Q'}$ with respect to T is $(q-q')^2 X^4$ and hence $G_Q=0$ and $G_{Q'}=0$ have no common component. Now $\alpha_P \cap V_4 = \alpha_P \cap (T^3 + YT^2 = 0) = (\alpha_P \cap T = 0) \cup (\alpha_P \cap T + Y = 0) = L \cup (A_P : X = T + Y = 0)$ and hence the line A_P is the residual intersection of α_P with V_4 outside L, and A_P intersects L in P. Next, let β be any plane through L other than α_P, then $\beta : (T = bX)$ with b in k, and it is clear that this plane is a component of $G_Q = 0$ where $q = -b^2 - b$, i.e., $\beta = \alpha_Q$ or $\beta = \alpha_Q^*$, say $\beta = \alpha_Q$, then $\alpha_Q \cap V_4 = \beta \cap V_4 = \beta \cap (b^3X^3 + b^2YX^2 + bYX^2 + X^2Z = 0)$ $= [L : \beta \cap (X^2 = 0)] \cup [A_Q : \beta \cap (b^3X + (b^2+b)Y + Z = 0)]$, and hence the line $A_Q : \beta \cap (b^3X + (b^2+b)Y + Z = 0)$ is the residual intersection of α_Q with V_4 outside L, and A_Q meets L in Q since $q = -b^2 - b$.

Next, it is clear that M and L have no common point and by direct substitution one can verify that M is on V_4. Since Γ is the set of the residual intersections with V_4 outside L of the various planes through L and since M is skew to L, it follows that M intersects each member of Γ and that ϕ maps Γ one-to-one onto

M. Since $M \not\subset (Y=0)$ we can take a k-generic point $(x, 1, z, t)$ of M and then we must have $x+t-1=0$ and $z+t=0$ and hence $k(M)=k(u)$ where $u=t/x$ and consequently the plane joining this point with L must be: $T=uX$ and hence as above the corresponding point on L is $(0, 1, -u-u^2, 0)$; this is a k-generic point of L and calling it $(0, 1, v, 0)$ we get $\sigma(k(L))=k(v)$ and $u^2+u+v=0$, this is a separable equation of degree two in u over $k(v)$ and hence $k(M)/\sigma(k(L))$ is a cyclic galois extension of degree two. Now let N be any other line on V_4 which is skew to L. Let π and π' be linear forms in $k[X, Y, Z, T]$ such that $\pi=0$ and $\pi'=0$ are distinct planes through N. Then π and π' must be linearly independent mod (X, T) and hence we may, after replacing π and π' by their suitable linear combinations, assume that $\pi=aX+bT-Y$ and $\pi'=\alpha X+\beta T-Z$ with a, b, α, β in k. Since N is on V_4, substituting in F_4 we get $0=T^3+(aX+bT)T^2+(aX+bT)XT+X^2(\alpha X+\beta T)=(1+b)T^3+(a+b)XT^2+(a+\beta)X^2T+\alpha X^3$, i.e., $0=1+b=a+b=a+\beta=\alpha$, i.e., $\alpha=0$, $a=1$, $b=\beta=-1$, and hence $N=M$. From these considerations (III) now follows.

Proof of (IV). This now follows from (I), (II), and (III).

Proof of (V). Let V be an irreducible cubic surface in P^3 having a double line; in view of (I) we may assume that V is not a cone. Make a projective transformation τ such that the double line of V is mapped onto L and $(0, 0, 0, 1) \notin \tau(V)$. Then the equation of V is of the form $F=0$ where

$$F = T^3+\phi_1 T+\phi_2 XT+\phi_3 X^2,$$

where ϕ_j is either zero or is a linear form in $k[X, Y, Z]$. *If* $\phi_1 \equiv 0$ *(mod* X*), i.e., if* $\phi_1=hX$ *with* h *in* k*, then* we make the projective transformation $\tau_1: X\to X,\ Y\to Y,\ Z\to Z,\ T\to T-(h/3)X$, so that $\tau_1(F)=T^2+\psi_2 XT+\psi_3 X^2$, where ψ_i is either zero or is a linear form in $k[X,Y,Z]$; since V is not a cone, ψ_2 and ψ_3 must be linearly independent mod X and hence making the projective transformation $\tau_2: X\to X,\ \psi_2\to Y,\ \psi_3\to Z,\ T\to T$, *we get* $\tau_2\tau_1(F)=F_3$. *From now on assume that* $\phi_1 \not\equiv 0$ (*mod* X), we may then make the projective transformation $\tau_1: X\to X,\ \phi_1\to Y,\ Z\to Z,\ T\to T$ and get

$$\tau_1(F)=T^3+YT^2+(aX+bY+cZ)XT+(uX+vY+wZ)X^2 \qquad (1)$$

with a, b, c, u, v, w in k.

We next show that there exist projective transformations τ_2, τ_3, τ_4 such that

$$\tau_4^*(F) = T^3 + YT^2 + ZXT + (\alpha X + \beta Y + \gamma Z)X^2\,;\ \alpha, \beta, \gamma \in k \tag{2}$$

Case 1, $c \neq 0$: We may take $\tau_2 : X \to X,\ Y \to Y,\ aX+bY+cZ \to Z,\ T \to T$ and τ_3, τ_4 to be the identity transformation.
Case 2, $c = 0$: Since V is not a cone, we must have $w \neq 0$ and we may take $\tau_2 : X \to X,\ Y \to Y,\ uX+vY+wZ \to Z,\ T \to T$, thus getting

$$\tau_2^*(F) = T^3 + YT^2 + (dX + eY)XT + ZX^2\,;\ d, e \in k \tag{3}$$

Next take $\tau_3 : X \to X + jT,\ Y \to Y,\ Z \to Z,\ T \to T$, where j is an element in k such that $ej^2 + 2j \neq 0 \neq 1 + dj^2$ (note that $p \neq 2$) and then

$$\tau_3^*(F) = g^3T^3 + \mu_1 T^2 + \mu_2 XT + \mu_3 X^2 \tag{4}$$

where $0 \neq g \in k$, μ_i is either zero or is a linear form in $k[X, Y, Z]$ and

$$\mu_1 \equiv (1 + ej)Y + j^2 Z (\mathrm{mod}\ X)\,;\ \mu_2 \equiv eY + 2jZ (\mathrm{mod}\ X) \tag{5}$$

Now

$$\det \begin{vmatrix} (1+ej) & j^2 \\ e & 2j \end{vmatrix} = ej^2 + 2j \neq 0\,,$$

i.e., μ_1 and μ_2 are linearly independent mod X and hence we may take $\tau_4 :\ X \to X,\ \mu_1/g^2 \to Y,\ \mu_2/g \to Z,\ gT \to T$ and thereby get equation (2).

Now make projective transformation

$$\tau_5 :\ X \to X,\ Y \to -3qX + Y,\ Z \to 3q^2 X - 2qY + Z,\ T \to qX + T\,,$$

where q is an element of k such that

$$q^3 + 3\gamma q^2 - 3\beta q + \alpha = 0\,.$$

Then by direct computation we have

$$\tau_5^*(F) = T^3 + YT^2 + ZXT + (mY + nZ)X^2\,;\ m, n \in k\,. \tag{6}$$

Since $p \neq 2$ and since V is irreducible so that m and n are not simultaneously zero, we can find s in k such that

$$H = -ms^2 + 2ns + 1 = 0. \tag{7}$$

Make the projective transformation

$$\tau_6 :\ X \to X + sT,\ Y \to Y,\ Z \to Z,\ T \to T.$$

Then

$$\tau_6^*(F) = T^3 + (AY + BZ)T^2 + (A^*Y + B^*Z)XT + \nu X^2\,, \tag{8}$$

where ν is a linear form in $k[X, Y, Z]$ and

$$A = 1+ms^2,\ B = s+ns^2,\ A^* = 2ms,\ B^* = 1+2ns\,,$$

and we have

$$AB^* - A^*B = H = 0\,. \tag{9}$$

Note that because of (7),

$$s \neq 0 \text{ and hence } 0 = H = -A+(2/s)B\,;$$

and hence $A=0$ if and only if $B=0$.

Now we divide the argument in three cases.

Case 1, $A=B=0$: Since V is not a cone, A^*Y+B^*Z and ν must be linearly independent mod X and hence we may make the projective transformation $\tau_7: X\to X,\ A^*Y+B^*Z\to Y,\ \nu\to Z,\ T\to T$ and obtain $\tau_7^*(F)=F_3$.

Case 2, $A\neq 0\neq B$, $A^*=0$: Then by equation (7) we have $B^*=0$; since V is not a cone, $AY+BZ$ and ν must be linearly independent mod X and hence making the projective transformation $\tau_7: X\to X$, $AY+BZ\to Y,\ \nu\to Z,\ T\to T$, we get

$$\tau_7^*(F) = T^3+YT^2+X^2Z\,.$$

Consider the transformation

$$\tau_8: X\to X,\ Y\to X-2Y-4T,\ Z\to -2X+2Y-8Z-6T,\ T\to X+2T\,.$$

The determinant of τ_8 is 32 and since $p\neq 2$, τ_8 is a nonsingular projective transformation and by direct computation, we get $\tau_8^*(F) = -8(T^3+YT^2+YXT+ZX^2)$ and since $p\neq 2$ we have $\tau_8^*\tau(V)=V_4$.

Case 3, $A\neq 0\neq B$, $A^*\neq 0$: Let $t=B^*/B$. Then by equation (9) we have $A^*/A=B^*/B=t$ and hence $t\neq 0$. Make the projective transformation $\tau_7: tX\to X,\ AY+BZ\to Y,\ Z\to Z,\ T\to T$. Then

$$\tau_7^*(F) = T^3+YT^2+YXT+\nu^*X^2\,.$$

Since V is not a cone, ν^* and Y must be linearly independent mod X and hence we may make the projective transformation $\tau_8: X\to X$, $Y\to Y,\ \nu^*\to Z,\ T\to T$, and thereby obtain $\tau_8^*(F)=F_4$.

Proof of (VI). The equation of any cubic surface V in P^3 having L as a singular line is of the form $F=0$, where

$$\begin{aligned}F = {} & a_0T^3+(a_1X+a_2Y+a_3Z)T^2+(a_4X+a_5Y+a_6Z)XT \\ & +(a_7X+a_8Y+a_9Z)X^2\,,\end{aligned}$$

with a_i in k. Since there exist irreducible cubic surfaces in P^3, all points in P^9 outside a proper subvariety represent irreducible cubic surfaces in P^3. Hence, in view of (I, II, III, IV, V), it is enough to show that there exists a nonzero form $G(A_0, A_1, \cdots, A_9)$ in $k[A_0, \cdots, A_9]$ such that if $G(a_0, \cdots, a_9) \neq 0$, then V has a line skew to L. Now a line skew to L is given by $\pi = \pi' = 0$ where

$$\pi = bX + cT - Y \text{ and } \pi' = \beta X + \gamma T - Z \text{ with } b, c, \beta, \gamma \text{ in } k .$$

For this line to be on V the condition is

$$\begin{aligned} 0 = a_0 T^3 &+ [a_1 X + a_2(bX + cT) + a_3(\beta X + \gamma T)]T^2 \\ &+ [a_4 X + a_5(bX + cT) + a_6(\beta X + \gamma T)]XT \\ &+ [a_7 X + a_8(bX + cT) + a_9(\beta X + \gamma T)]X^2 , \end{aligned}$$

and equating coefficients, this is equivalent to the following four nonhomogeneous linear equations is b, c, β, γ:

$$\begin{aligned} a_0 + 0b + a_2 c + 0\beta + a_3\gamma &= 0 \\ a_1 + a_2 b + a_5 c + a_3\beta + a_6\gamma &= 0 \\ a_4 + a_5 b + a_8 c + a_6\beta + a_9\gamma &= 0 \\ a_7 + a_8 b + 0c + a_9\beta + 0\gamma &= 0 \end{aligned}$$

Let $G(A_0, \cdots, A_9)$ be the determinant of these equations with A_i replacing a_i. Then as a polynomial in A_6 the coefficient of A_6^2 in G is $\pm A_2 A_8$ and hence G is a nonzero form of degree four. If $G(a_0, \cdots, a_9) \neq 0$ then the above equations can be solved for b, c, β, γ thus giving us a line on V which is skew to L.

Definition 1. Let the notation be as in Proposition 1 and let V be an irreducible cubic surface in P^3 having a double line. *If V is projectively equivalent respectively to* (1) V_1, (2) V_2, (3) V_3, (4) V_4, *then we shall call V respectively*: (1) *a cubic cone with an ordinary double line,* (2) *a cubic cone with a cuspidal double line,* (3) *a special cubic surface with a double line or a cubic surface with one variable tangent plane along a double line,* (4) *a nonspecial cubic surface with a double line or a cubic surface with two variable tangent planes along a double line.*

Proposition 2. *Let X, Y, Z be projective coordinates in P^2, let $F = X^2Y^2 + X^2Z^2 + Y^2Z^2 - 2YZX^2 - 2XZY^2 - 2XYZ^2$ and let W be the curve $F = 0$.* (i) *If $p \neq 2$ then W is an irreducible quartic, its only singularities are $P_1 : (X = Y = 0)$, $P_2 : (X = Z = 0)$, $P_3 : (Y = Z = 0)$, and* *at each of these points W has a 2-fold cusp which is ordinary for P^2.* (ii) *Any irreducible quartic W^* in P^2 having three double points with unique tangents is projectively equivalent to W.* (iii) *If $p=2$ then there does not exist any irreducible quartic in P^2 having three double points with unique tangents and hence there does not exist any irreducible quartic in P^2 having three 2-fold cusps.*

Proof. (i): Take $x=(X-Y)/Z$, $y=Y/Z$ as affine coordinates. Then P_1 is the point $x=y=0$ and W has the affine equation $f=0$ where $f=(x+y)^2y^2+x^2-2(x+y)y^2-2y(x+y)^2$. The leading form of f is x^2, hence P_1 is a double point of W with the unique tangent $N: x=0$. Since $f\equiv y^4-4y^3 \pmod{x}$ and since $p\neq 2$, we conclude that $i(N\cdot W;\, P_1)=3$, i.e., P_1 is a 2-fold cusp of W which is ordinary for P^2. Similarly, this holds for P_2 and P_3. Now $F=(Y-Z)^2X^2-2YZ(Y+Z)X+Y^2Z^2$. Hence in F the coefficients of X^2, X^1, X^0 are coprime polynomials in Y, Z and F is an irreducible element of $k(Y, Z)\,[X]$ because $\Delta_X F=4Y^2Z^2(Y+Z)^2-4(Y-Z)^2Y^2Z^2=16Y^3Z^3$ which is not a square since $p\neq 2$. Therefore F is irreducible in $k[X, Y, Z]$. We shall now show that without any restriction on p, any irreducible quartic E in P^2 can have at most three singularities. For let Q_1, Q_2, Q_3, Q_4 be four singular points of E and let Q_5 be a point of E other than Q_1, Q_2, Q_3, Q_4; then there exists a (unique) curve H of degree 2 in P^2 passing through $Q_1, \cdots, Q_5$; if H is reducible (in which case H consists either of two lines or one line repeated twice) then one component line N of H must pass through three of the points Q_j and since at least two of these points are singular points of E we would have $i(N\cdot E;\, P^2)\geqq 5$ which is a contradiction; if H is irreducible, then $i(H\cdot E;\, P^2)\geqq \sum_{j=1}^4 i(H\cdot E;\, Q_j, P^2)+i(H\cdot E;\, Q_5, P^2)\geqq 2\times 4+1=9>4\times 2$ which is again a contradiction.

(ii) and (iii): Since W^* is irreducible, the three given double points cannot be on a line N, for otherwise we would have $i(N\cdot W^*;\, P^2)\geqq 6$. Make a projective transformation τ such that these three singularities of W^* are mapped onto P_1, P_2, P_3. Let G be a fourth degree form in $k[X, Y, Z]$ such that $\tau(W^*)$ is given by $G=0$. Since P_1 is a double point of $\tau(W^*)$, G cannot have any terms in Z^4, X^3Z, Y^3Z; since P_2 is a double point of $\tau(W^*)$, G cannot have any terms in $\cdots$. Hence $G=a^2X^2Y^2+b^2X^2Z^2+c^2Y^2Z^2+dXYZ^2+eXY^2Z+fX^2YZ$ with a, b, c, d, e, f in k. The points P_1, P_2, P_3 are double points with unique tangents and hence $b^2X^2+c^2Y^2$

$+dXY, \cdots, \cdots$ must be perfect squares and hence we get

$$d = \pm 2bc,\ e = \pm 2ac,\ f = \pm 2ab \tag{I}$$

If p were 2 then equations (I) would tell us that $d=e=f=0$ and hence we would have $G=(aXY+bXZ+cYZ)^2$. Now assume that $p \neq 2$. Then in view of equations (I) we conclude that if a were zero then G would be divisible by Z^2. Hence $a \neq 0$ and similarly $b \neq 0 \neq c$. Consequently, we may (via a projective transformation) replace aX, bY, cZ by X, Y, Z respectively and get $G=X^2Y^2+X^2Z^2+Y^2Z^2 \pm 2XYZ^2 \pm 2XY^2Z \pm 2X^2YZ$. Now we have the following four cases: Case 1: no negative signs, Case 2: one negative sign, Case 3: two negative signs, Case 4: three negative signs. *Case* 1: then $G=(XY+XZ+YZ)^2$ which contradicts the irreducibility of G. *Case* 2: because of symmetry we may assume that the sign of $2XYZ^2$ is negative; then we may (via a projective transformation) replace X by $-X$ and Y by $-Y$ thereby getting $G=F$. *Case* 3: because of symmetry we may assume that the signs of $2XYZ^2$ and $2XY^2Z$ are negative and the sign of $2X^2YZ$ is positive, then $G=(XY+XZ-YZ)^2$ which contradicts the irreducibility of G. *Case* 4: then $G=F$.

2. Discriminants

Let P^3 be a projective three space over an algebraically closed ground field *k of characteristic* $p \neq 2, 3$, let V be an irreducible cubic surface in P^3 having a double line L, let P be a point of P^3 not on V, let P^2 be a plane in P^3 not passing through P and let ϕ be the projection from P of V onto P^2. We want to determine the branch locus $\Delta(V/P^2)=$ the set of all points of P^2 whose image under ϕ^{-1} consists of less than three points. Note that if $\bar{P}^2$ is any other plane in P^3 not passing through P and if $\bar{\phi}$ denotes the projection from P of V onto $\bar{P}^2$ then $\Delta(V/\bar{P}^2)$ and $\Delta(V/P^2)$ are projectively equivalent. If V is a cone, then we can choose projective coordinates X, Y, Z, T in P^3 such that P is: $(0, 0, 0, 1)$ and the vertex of the cone is $Q:(0, 0, 1, 0)$, and then the equation of V is $F=0$ where F is a cubic form in X, Y, T and hence $\Delta_T F$ is a form of degree six in X, Z; consequently $\Delta(V/P^2)$ consists of six lines (not all distinct) through $\phi(Q)$ including $\phi(L)$ which is counted at least twice. From now on we shall assume that V is not a cone.

Lemma 2. *Assume that V is nonspecial. For $j=1, 2$, let P_j be the point of L at which the tangent cone to V consists of a plane α_j counted twice; let A_j be the residual intersection of α_j with V outside L, let M be the line on V which is skew to L, let Q_j be the point of intersection of M and A_j, and let β_j be the plane determined by M and A_j* (see Proposition 1). *Now assume that $P\notin\alpha_1\cup\alpha_2\cup\beta_1\cup\beta_2$. Then* (i) *there exists a projective coordinate system (X, Y, Z, T) in P^3 such that the coordinates of P are* $(0, 0, 0, 1)$ *and V has an equation $F=0$ where*

$$F=T^3+YT^2+ZXT+(\mu Y+\nu Z)X^2,$$

where μ and ν are elements in k such that $\mu=-\nu^2+\frac{1}{4}$, $\nu\neq 1$, $\nu\neq -1$, and $\mu\neq 0$. Furthermore (ii) *$\Delta_T(F)=X^2H$ where*

$$H=-27(\mu Y+\nu Z)^2X^2+[-4Z^3+18YZ(\mu Y+\nu Z)]X \\ -[4Y^3(\mu Y+\nu Z)+Y^2Z^2].$$

(iii) *The polynomial $G(Q)=Q^2-3\nu Q-3\mu$ has two distinct roots q and q_1 in k; and*

$$\tau: X\to X+Y+Z,\ Y\to 3q_1Y+3qZ,\ Z\to 3q_1^2Y+3q^2Z;$$
$$\tau_1: X\to(q-q_1)^3X,\ Y\to -q^3Y,\ Z\to q_1^3Z;$$

give nonsingular projective transformations τ and τ_1 of the (X, Y, Z)-*plane P^2, and we have*

$$\tau_1\tau(H)=s(X^2Y^2+X^2Z^2+Y^2Z^2-2YZX^2-2XZY^2-2XYZ^2)$$

where $0\neq s=-27q_1^6q^6(q-q_1)^6\in k$. Consequently $H=0$ is an irreducible quartic curve in P^2 having cusps at $(1, 0, 0)$, $(1, 3q, 3q^2)$, $(1, 3q_1, 3q_1^2)$.

Proof. By part (III) of Proposition 1 it follows that no four of the points P_1, P_2, Q_1, Q_2, P lie on a plane and hence we can choose a coordinate system X_1, Y_1, Z_1, T_1 in P^3 such that the coordinates of these points are respectively $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(1, 0, 0, 0)$, $(1, 1, 1, 1)$, $(0, 0, 0, 1)$. Then $L, M, A_1, A_2, \alpha_1, \alpha_2$ are respectively given by $X_1=T_1=0$, $Y_1-T_1=Z_1-T_1=0$, $T_1=Z_1=0$, $X_1-T_1=Y_1-T_1=0$, $T_1=0$, $X_1-T_1=0$. Let

$$F=dT_1^3+(aX_1+bY_1+cZ_1)T_1^2+(uX_1+vY_1+wZ_1)X_1T_1 \\ +(\alpha X_1+\beta Y_1+\gamma Z_1)X_1^2,$$

with $a, b, c, d, u, v, w, \alpha, \beta, \gamma$ in k such that V is given by $F=0$. Since $P\notin V$, $d\neq 0$ and hence we may assume that $d=1$. The tangent cones to $F=0$ at P_1 and P_2 are respectively

$$bT_1^2+vX_1T_1+\beta X_1^2=0 \quad \text{and} \quad cT_1^2+wX_1T_1+\gamma T_1^2=0,$$

and hence $v=\beta=0$, $b\neq 0$ and $c=\gamma=-w/2\neq 0$. Thus

$$F = T_1^3+(aX_1+bY_1-(w/2)Z_1)T_1^2+(uX_1+wZ_1)X_1T_1 \\ +(\alpha X_1-(w/2)Z_1)X_1^2 .$$

Since M, A_1, A_2 are on V, substituting the equations of M, A_1, A_2 in F we respectively get:

$$0 = T_1^3+(aX_1+bT_1-(w/2)T_1)T_1^2+(uX_1+wT_1)X_1T_1 \\ +(\alpha X_1-(w/2)T_1)X_1^2 ;$$

$$0 = \alpha X_1^3 ;$$

$$0 = T_1^3+(aT_1+bT_1-(w/2)Z_1)T_1^2+(uT_1+wZ_1)T_1^2 \\ +(\alpha T_1 - (w/2)Z_1)T_1^2 .$$

Equating coefficients to zero, we respectively have:

$$0 = 1+b-w/2 = a+w = u-w/2 = \alpha ;$$

$$0 = \alpha ;$$

$$0 = 1+a+b+u+\alpha = -w/2+w-w/2 .$$

Solving these linear equations we obtain:

$$\alpha = 0,\ b = -1+w/2,\ a = -w,\ u = w/2 ; \quad \text{i.e.,}$$

$$F = T_1^3+[(-w)X_1+(-1+w/2)Y_1+(-w/2)Z_1]T_1^2 \\ +[(w/2)X_1+wZ_1]X_1T_1+(-w/2)X_1^2Z_1 .$$

Now $w\neq 0$ and $b=-1+w/2\neq 0$ (i.e., $w/2\neq 1$) and hence we can take projective coordinates X_2, Y_2, Z_2, T_2 in P^3 given by $T_2=T_1$, $X_2=X_1$, $Y_2=(-w)X_1+(-1+w/2)Y_1+(-w/2)Z_1$, $Z_2=(w/2)X_1+wZ_1$ and setting $m=w/4$ we get

$$F = T_2^3+Y_2T_2^2+Z_2X_2T_2+\left(mX_2-\left(\frac{1}{2}\right)Z_2\right)X_2^2 \text{ with } m\neq 0,\ \frac{1}{2} .$$

Now choose projective coordinates (X, Y, Z, T) in P^3 via the equations

$$X_2 = X,\ Y_2 = -3\delta X+Y,\ Z_2 = 3\delta^2X-2\delta Y+Z,\ T_2 = T+\delta X ,$$

where δ is an element of k satisfying the equation:

$$\delta^3-(3/2)\delta^2+m = 0 .$$

Then

$$\begin{aligned}F &= (T+\delta X)^3+(-3\delta X+Y)(T+\delta X)^2+(3\delta^2X-2\delta Y+Z)X(T+\delta X)\\ &\qquad +[(m-(3/2)\delta^2)X+\delta Y-(1/2)Z]X^2\\ &= T^3+YT^2+ZXT+[(-\delta^2+\delta)Y+(\delta-1/2)Z]X^2 .\end{aligned}$$

Now $m=(3/2)\delta^2-\delta^3$ and hence the inequalities $m\neq 0$, $1/2$ respectively imply that $\delta\neq 3/2$, $-1/2$. Let $\mu=-\delta^2+\delta$ and $\nu=\delta-1/2$. Then $\mu=-\nu^2+1/4$ and $\nu\neq 1$, -1, and we have

$$F = T^3+YT^2+ZXT+(\mu Y+\nu Z)X^2 .$$

It can easily by verified that if μ were zero, then the line $Z=0=Y+T$ is on V, it is skew to L and hence it must be the line M, the tangent cone to V at P_1 is given by $T^2=0$, i.e., α_1 is given by $T=0$ and hence A_1 is given by $T=Z=0$ and hence the plane β_1 containing M and A_1 is given by $Z=0$ and hence $P\in\beta_1$; this is a contradiction. Therefore $\mu\neq 0$.

Now computing $\Delta_T(F)$ by the textbook formula for the discriminant of a third degree polynomial, we get:

$$\begin{aligned}\Delta_T(F) &= X^2Y^2Z^2-4Z^3X^3-4X^2Y^3(\mu Y+\nu Z)-27X^4(\mu Y+\nu Z)^2\\ &\qquad +18X^3YZ(\mu Y+\nu Z)=X^2H .\end{aligned}$$

Now

$$\Delta(G) = 9\nu^2+12\mu = 9\nu^2-12\nu^2+3 = 3(1-\nu^2) .$$

Since $\nu\neq 1,-1$ and $p\neq 3$, $\Delta(G)\neq 0$ and hence G has two distinct roots q and q_1 in k. Since $\mu\neq 0$, q and q_1 are unequal to zero. The determinant of the transformation τ is $9qq_1(q-q_1)$ and this is unequal to zero since $p\neq 3$. Hence τ is nonsingular. Since $q\neq q_1$, τ_1 is also nonsingular. Now

$$\begin{aligned}\tau(\mu Y+\nu Z) &= \mu(3q_1Y+3qZ)+\nu(3q_1^2Y+3q^2Z)\\ &= (3\mu q_1+3\nu q_1^2)Y+(3\mu q+3\nu q^2)Z\\ &= q_1^3Y+q^3Z .\end{aligned}$$

Hence

$$\begin{aligned}\tau(H) = &-27(q_1^3Y+q^3Z)^2(X+Y+Z)^2\\ &+[-108(q_1^2Y+q^2Z)^3+162(q_1Y+qZ)(q_1^2Y+q^2Z)(q_1^3Y\\ &+q^3Z)](X+Y+Z)-108(q_1Y+qZ)^3(q_1^3Y+q^3Z)+81(q_1Y\\ &+qZ)^2(q_1^2Y+q^2Z)^2 ;\end{aligned}$$

i.e.,

$$(-1/27)\tau(H) = AX^2+2BX+C ,$$

where

$$A = (q_1^3Y+q^3Z)^2\,;$$
$$B = (q_1^3Y+q^3Z)^2(Y+Z)+2(q_1^2Y+q^2Z)^3$$
$$\qquad -3(q_1Y+qZ)(q_1^2Y+q^2Z)(q_1^3Y+q^3Z)\,;$$
$$C = (q_1^3Y+q^3Z)^2(Y+Z)^2+4(q_1^2Y+q^2Z)^3(Y+Z)$$
$$\qquad -6(q_1Y+qZ)(q_1^2Y+q^2Z)(q_1^3Y+q^3Z)(Y+Z)$$
$$\qquad +4(q_1Y+qZ)^3(q_1^3Y+q^3Z)-3(q_1Y+qZ)^2(q_1^2Y+q^2Z)^2\,.$$

Let us write

$$B = B_0Y^3+B_1Y^2Z+B_2YZ^2+B_3Z^3\,,$$
$$C = C_0Y^4+C_1Y^3Z+C_2Y^2Z^2+C_3YZ^3+C_4Z^4\,,$$

with B_i, C_i in k. Then

$$B_0 = q_1^6+2q_1^6-3q_1^6 = 0\,,$$
$$B_1 = (q_1^6+2q_1^3q^3)+2(3q_1^4q^2)-3(q_1^3q^3+q_1^4q^2+q_1^5q)$$
$$= q_1^6-3q_1^5q+3q_1^4q^2-q_1^3q^3 = q_1^3(q_1-q)^3\,.$$

Hence by symmetry, interchanging q and q_1, and Z and Y, we have

$$B_2 = q^3(q-q_1)^3 = -q^3(q_1-q)^3\,;\text{ and } B_4 = 0\,.$$

Further, we have:

$$C_0 = q_1^6+4q_1^6-6q_1^6+4q_1^6-3q_1^6 = 0\,;$$
$$C_1 = (2q_1^6+2q_1^3q^3)+4(q_1^6+3q_1^4q^2)-6(q_1^6+q_1^3q^3+q_1^4q^2+q_1^5q)$$
$$\qquad +4(q_1^3q^3+3q_1^5q)-3(2q_1^4q^2+2q_1^5q) = 0\,;$$
$$C_2 = (q_1^6+4q_1^3q^3+q^6)+4(3q_1^4q^2+3q_1^2q^4)$$
$$\qquad -6(q_1^5q+q_1^4q^2+2q_1^3q^3+q_1^2q^4+q_1q^5)$$
$$\qquad +4(3q_1^2q^4+3q_1^4q^2)-3(q_1^2q^4+4q_1^3q^3+q_1^4q^2)$$
$$= q_1^6-6q_1^5q+15q_1^4q^2-20q_1^3q^3+15q_1^2q^4-6q_1q^5+q^6$$
$$= (q_1-q)^6\,.$$

Again by symmetry, interchanging q and q_1, and Z and Y, we have: $C_3=C_4=0$. Thus

$$(-1/27)\tau(H) = (q_1^3Y+q^3Z)^2X^2+2q_1^3(q_1-q)^3XY^2Z-2q^3(q_1-q)^3XYZ^2$$
$$\qquad +(q_1-q)^6Y^2Z^2$$
$$= q_1^6X^2Y^2+q^6X^2Z^2+(q_1-q)^6Y^2Z^2+2q_1^3q^3YZX^2$$
$$\qquad +2q_1^3(q_1-q)^3XY^2Z-2q^3(q_1-q)^3XYZ^2\,.$$

Consequently we have:

$$\tau_1\tau(H) = s(X^2Y^2+X^2Z^2+Y^2Z^2-2YZX^2-2XY^2Z-2XYZ^2).$$

By Proposition 2, $\tau_1\tau(H)=0$ is an irreducible quartic whose only singularities are 2-fold cusps at (1, 0, 0), (0, 1, 0), (0, 0, 1). Now these points remain fixed under τ_1, and τ^{-1} (1, 0, 0) = (1, 0, 0), $\tau^{-1}(0, 1, 0)=(1, 3q_1, 3q_1^2)$, $\tau^{-1}(0, 0, 1)=(1, 3q, 3q^2)$. This completes the proof.

Lemma 3. *Assume that V is nonspecial and let the notation be as in Lemma 2. Assume that $P\in\alpha_1\cup\alpha_2$ and $P\notin\beta_1\cup\beta_2$. Then* (i) *there exists a projective coordinate system (X, Y, Z, T) in P^3 such that P has coordinates* (0, 0, 0, 1) *and V has an equation $F=0$ where either* (*Case* 1): $F=T^3+YT^2+(\mu X+\nu Y)XT+ZX^2$, *$\mu$ and ν in k such that $9\nu^2+12\mu\neq 0\neq\mu$; or* (*Case* 2): $F=T^3+YT^2+YXT+ZX^2$.

(ii) *In Case* 1, $\Delta_T(F)=X^2H$ *where*

$$H = Y^2(\mu X+\nu Y)^2-4(\mu X+\nu Y)^3X-4Y^3Z-27X^2Z^2 +18XYZ(\mu X+\nu Y);$$

the polynomial $G(Q)=3\mu Q^2+3\nu Q-1$ has two distinct roots q, q_1 in k;

$$\tau: X\to 3q^3X+3q_1^3Y,\ Y\to 3q^2X+3q_1^2Y,\ Z\to 9^{-1}X+9^{-1}Y+9^{-1}Z,$$

and

$$\tau_1: X\to q_1^3X,\ Y\to -q^3Y,\ Z\to (q-q_1)^3Z,$$

are nonsingular projective transformations of the (X, Y, Z)-plane P^2, and we have

$$\tau_1\tau(H) = s[X^2Y^2+X^2Z^2+Y^2Z^2-2YZX^2-2XZY^2-2XYZ^2]$$

where $0\neq s=-3q^6q_1^6(q-q_1)^6\in k$; consequently $H=0$ is an irreducible quartic curve in P^2 having cusps at $(27q^3, 27q^2, 1)$, $(27q_1^3, 27q_1^2, 1)$, $(0, 0, 1)$.

(iii) *In Case* 2, $\Delta_T(F)=X^2H$ *where*

$$H = Y^4-4Y^3X-4Y^3Z-27X^2Z^2+18XY^2Z;$$
$$\tau: X\to X+3^{-1}Y,\ Y\to Y,\ Z\to 3^{-1}Y+Z, \text{ and}$$
$$\tau_1: X\to -X,\ Y\to 3Y,\ Z\to -Z,$$

are nonsingular projective transformations of the (X, Y, Z)-plane P^2 and we have

$$\tau_1\tau(H) = -27(X^2Y^2+X^2Z^2+Y^2Z^2-2YZX^2-2XZY^2-2XYZ^2);$$

consequently $H=0$ *is an irreducible quartic curve in* P^2 *having cusps at* (0,0,1), (1, 0, 0), (1/3, 1, 1/3).

Proof. Because of symmetry, we may assume that $P \in \alpha_1$. We can choose projective coordinates X_1, Y_1, Z_1, T_1 in P^3 such that α_1 is $X_1=0$, L is $X_1=T_1=0$, P_1 is (0, 0, 1, 0), and P is (0, 0, 0, 1); then V has equation $F=0$ where

$$F = T_1^3+(aX_1+bY_1+cZ_1)T_1^2+(uX_1+vY_1+wZ_1)X_1T_1 \\ +(\alpha X_1+\beta Y_1+\gamma Z_1)X_1^2 .$$

Then tangent cone at P_1 is: $cT_1^2+wX_1T_1+\gamma X_1^2=0$ and hence $c=w=0$ and $\gamma\neq 0$. Since V is nonspecial, as in the proof of part (V) of Proposition 1, we must have $b\neq 0$. Consequently we may take $X=X_1$, $Y=aX_1+bY_1+cZ_1$, $Z=\alpha X_1+\beta Y_1+\gamma Z_1$, $T=T_1$, and then

$$F = T^3+YT^2+(\mu X+\nu Y)XT+X^2Z\,;\ \mu,\ \nu\in k .$$

Computing $\Delta_T(F)$ by the textbook formula, we get $\Delta_T(F)=X^2H$. Now α_1, L, P_1 and P are still given by $X=0$, $X=T=0$, (0, 0, 1, 0) and (0, 0, 0, 1), respectively. Consequently the coordinates of P_2 must be (0, 1, h, 0) with h in k; the tangent cone to $F=0$ at this point is: $T^2+\nu XT+hX^2=0$ and the condition that this must be a perfect square yields: $h=\nu^2/4$. Thus P_2 has coordinates (0, 1, $\nu^2/4$, 0). By substituting in F one can verify that the line $Y-\nu X+T=Z+(\nu^2+\mu)T=0$ is on V, and it is clear that this line is skew to L and consequently it must be M. Since β_2 is a plane through M the equation of β_2 is of the form $d(Y-\nu X+T)+e(Z+\nu^2T+\mu T)=0$; since P_2 is on β_2 we get: $d+e(\nu^2/4)=0$ and hence we may take $e=1$ and $d=-\nu^2/4$. Thus β_2 is given by: $-(\nu^3/4)X-(\nu^2/4)Y+Z+((3/4)\nu^2+\mu)T=0$. Hence P is on β_2 if and only if $(3/4)\nu^2+\mu=0$, i.e., if and only if $9\nu^2+12\mu=0$; since by our assumption $P\notin\beta_2$, we conclude that $9\nu^2+12\mu\neq 0$. Now we divide the argument into two cases.

(ii) (Case 1) $\mu\neq 0$. Then $\Delta(G)=9\nu^2+12\mu\neq 0$ and hence $G(Q)$ has two distinct roots q and q_1 in k. The determinant of τ is $q^2q_1^2(q-q_1)$; since $q\neq q_1$, τ as well as τ_1 is nonsingular. Invoking the fact that $G(q)=G(q_1)=0$ we get

$$\tau(\mu X+\nu Y) = (3\mu q^3+3\nu q^2)X+(3\mu q_1^3+3\nu q_1^2)Y = qX+q_1Y .$$

Hence

$$\begin{aligned}\tau(H) = {} & (3q^2X+3q_1^2Y)^2(qX+q_1Y)^2-4(qX+q_1Y)^3(3q^3X+3q_1^3Y)\\ & +[18(3q^3X+3q_1^3Y)(3q^2X+3q_1^2Y)(qX+q_1Y)\\ & -4(3q^2X+3q_1^2Y)^3](9^{-1}X+9^{-1}Y+9^{-1}Z)\\ & -27(3q^3X+3q_1^3Y)^2(9^{-1}X+9^{-1}Y+9^{-1}Z)^2.\end{aligned}$$

Therefore

$$\begin{aligned}(-1/3)\tau(H) = {} & -3(q^2X+q_1^2Y)^2(qX+q_1Y)^2+4(qX+q_1Y)^3(q^3X+q_1^3Y)\\ & +[4(q^2X+q_1^2Y)-6(q^3X+q_1^3Y)(q^2X+q_1^2Y)(qX+q_1Y)](X+Y+Z)\\ & +(q^3X+q_1^3Y)^2(X+Y+Z)^2.\end{aligned}$$

This is the same expression as the one found for $(-1/27)\tau(H)$ in the proof of Lemma 2 except that X and Z have been interchanged. Hence we get:

$$\begin{aligned}(-1/3)\tau(H) = {} & q_1^6Z^2Y^2+q^6Z^2X^2+(q_1-q)^6Y^2X^2+2q_1^3q^3YXZ^2\\ & +2q_1^3(q_1-q)^3ZY^2X-2q^3(q_1-q)^3ZYX^2.\end{aligned}$$

This now immediately yields the asserted expression for $\tau_1\tau(H)$ and the proof of (ii) is now completed by Proposition 2.

(iii) (Case 2) $\mu=0$. Since $9\nu^2+12\mu\neq 0$, we must have $\nu\neq 0$ and hence we may replace νX by X and $\nu^{-2}Z$ by Z, i.e., we may assume that $\nu=1$. By direct computation we get

$$\tau(H) = -3X^2Y^2-27X^2Z^2-3Y^2Z^2-18YZX^2+6XZY^2-18XYZ^2.$$

This yields the asserted expression for $\tau_1\tau(H)$ and the proof of (iii) is now completed by Proposition 2.

Lemma 4. *Assume that V is nonspecial and let the notation be as in Lemma 2. Assume that $P\in\beta_1\cup\beta_2$. Then* (i) *there exists a projective coordinate system (X, Y, Z, T) in P^3 such that P has coordinates* (0, 0, 0, 1) *and V is given by $F=0$ where*

$$F = T^3+YT^2+\mu ZXT+X^2Z.$$

Furthermore (ii) $\Delta_T(F)=X^2ZH$ *where*

$$\begin{aligned}& H = \mu^2Y^2Z-4\mu^3Z^2X-4Y^3-27X^2Z+18\mu XYZ;\\ & \tau: X\to X+(\mu^3/27)Z,\ Y\to Y+(\mu^2/3)Z,\ Z\to Z, \text{ and}\\ & \tau_1: \mu Y-3X\to X,\ (-4)^{1/3}Y\to Y,\ -3Z\to Z,\end{aligned}$$

are nonsingular projective transformations of the (X, Y, Z)-plane P^2 and we have

$$\tau_1\tau(H)=Y^3+X^2Z,\ \tau_1\tau(Z) = -3Z;$$

consequently $H=0$ *is an irreducible cubic in* P^2 *having cusp at* $(\mu^3/27, \mu^2/3, 1)$ *and having* $Z=0$ *as the flex tangent.*

Proof. Because of symmetry we may assume that $P\in\beta_1$. Let $(\bar{X}, \bar{Y}, \bar{Z}, \bar{T})$ be projective coordinates in P^3 and consider $\bar{V}$: $\bar{F}=0$, where $\bar{F}=\bar{T}^3+\bar{Y}\bar{T}^2+\bar{X}^2\bar{Z}$. It is clear that $\bar{V}$ is an irreducible cubic surface with double line $\bar{L}: \bar{X}=\bar{T}=0$. Also $\bar{V}$ contains the line $\bar{Y}+\bar{T}=\bar{Z}=0$ which is skew to $\bar{L}$ and hence by Propositions 1, V and $\bar{V}$ are projectively equivalent and hence we can choose projective coordinates (X_2, Y_2, Z_2, T_2) in P^3 such that V is given by $F=0$ where $F=T_2^3+Y_2T_2^2+X_2^2Z_2$. Then at $(0, 0, 1, 0)$ and $(0, 1, 0, 0)$ the tangent cones to V consist of a plane counted twice and because of symmetry we may assume that $(0, 1, 0, 0)$ is P_1 and $(0, 0, 1, 0)$ is P_2. Now L is $X_2=T_2=0$ and M is $Y_2+T_2=Z_2=0$. Hence β_1 which is the plane joining P_1 and M is given $Z_2=0$. Let (a, b, c, d) be coordinates of P. Since P is on β_1, $c=0$. Since P is not on V we cannot also have $d=0$ and hence we may take $d=1$. Let $X_2=X_1+aT$, $Y_2=Y_1+bT_1$, $Z_2=Z_1$, $T_2=T_1$. Then

$$\begin{aligned} F &= T_1^3+(Y_1+bT_1)T_1^2+(X_1+aT_1)^2Z_1 \\ &= (1+b)T_1^3+(Y_1+a^2Z_1)T_1^2+(2aZ_1)X_1T_1+X_1^2Z_1, \end{aligned}$$

and P is given by $X_1=Y_1=Z_1=0$ and $T_1=1$. Since P is not on V, $1+b\neq 0$. Hence we may replace F by $F/(1+b)$ and setting $X=X_1$, $Y=(Y_1+a^2Z_1)/(1+b)$, $Z=Z_1/(1+b)$, $T=T_1$, $\mu=2a$ we get

$$F = T^3+YT^2+\mu ZXT+X^2Z,$$

and P is given by $X=Y=Z=0$, $T=1$. By the textbook formula we get $\Delta_T(F)=X^2ZH$. It is clear that τ_1 and τ are nonsingular. By direct computation we get

$$\tau(H) = -4Y^3-3Z(\mu Y-3X)^2$$

and hence $\tau_1\tau(H)=Y^3+X^2Z$, $\tau_1\tau(Z)=Z$. The proof is now completed by Lemma 1.

Lemma 5. *Assume that* V *is special and let* P_1 *be the point on* L *at which the tangent cone to* V *consists of a plane* α *counted twice* (*see Proposition* 1) *and assume that* $P\notin\alpha$. *Then there exist projective coordinates* (X, Y, Z, T) *in* P^3 *such that* P *has coordinates* $(0, 0, 0, 1)$ *and* V *is given by* $F=0$ *where*

$$F = T^3+ZT^2+YXT+X^3.$$

Furthermore, $\Delta_T(F)=X^2H$ *where*

$$H = Z^2Y^2-4Y^3X-4Z^3X-27X^4+18X^2YZ;$$

let w *be a primitive cube root of unity in* k. *Then*

$$\tau : X\to(1/3)X+(w^2/3)Y+(w/3)Z,\ Y\to X+wY+w^2Z,\ Z\to X+Y+Z$$

and

$$\tau_1 : X\to X,\ Y\to wY,\ Z\to w^2Z,$$

are nonsingular projective transformations of the (X, Y, Z)*-plane and we have*

$$\tau_1\tau(H) = 9(X^2Y^2+X^2Z^2+Y^2Z^2-2YZX^2-2XZY^2-2XYZ^2.$$

Consequently $H=0$ *is an irreducible quartic curve in* P^2 *having cusps at* $(w/3, w^2, 1)$, $(w^2/3, w, 1)$, $(1/3, 1, 1)$.

Proof. By Proposition 1, we can choose projective coordinates X_1, Y_1, Z_1, T_1 in P^3 such that V is given by $F=0$ where

$$F = T_1^3+Y_1X_1T_1+X_1^2Z_1.$$

α is then given by $X_1=0$ and hence the coordinates of P are $(1, a, b, c)$. Set $X_1=T_2$, $Y_1=Y_2+aT_2$, $Z_1=Z_2+bT_2$, $T_1=X_2+cT_2$. Then P is given by $X_2=Y_2=Z_2=0$, $T_2=1$ and we have

$$F = (c^3+ac+b)T_2^3+(3c^2X_2+aX_2+cY_2+Z_2)T_2^2 \\ +(3cX_2+Y_2)X_2T_2+X_2^3.$$

Let d be a cube root of c^3+ac+b in k. Since P is not on V we have $d\neq 0$. Setting $X=X_2$, $Y=(3cX_2+Y_2)/d$, $Z=(3c^2X_2+aX_2+cY_2+Z_2)/d^2$, $T=dT_2$, we get P: $X=Y=Z=0$ and

$$F = T^3+ZT^2+YXT+X^3.$$

By the textbook formula we get $\Delta_T(F)=X^2H$. The determinant of τ is $w-w^2\neq 0$ and hence τ is nonsingular, and τ_1 is obviously nonsingular. Using the equation $w^2+w+1=0$, we get:

$$\tau(ZY) = (X+Y+Z)(X+wY+w^2Z) = (X^2+wY^2+w^2Z^2-w^2XY \\ -wXZ-YZ),$$

$$\tau(18X^2) = 2(X+w^2Y+wZ)^2 = 2(X^2+wY^2+w^2Z^2+2w^2XY+2wXZ \\ +2YZ),$$

$$\tau(ZY(ZY+18X^2)) = 3(X^2+wY^2+w^2Z^2-w^2XY-wXZ-YZ) \\ (X^2+wY^2+w^2Z^2+w^2XY+wXZ+YZ)$$

$$= 3(X^2+wY^2+w^2Z^2)^2-3(w^2XY+wXZ+YZ)^2$$
$$= 3(X^4+w^2Y^4+wZ^4+wX^2Y^2+w^2X^2Z^2+Y^2Z^2 -2X^2YZ-2w^2Y^2XZ-2wZ^2XY),$$

$$\tau(4Z^3+4Y^3+27X^3) = 4(X+Y+Z)^3+4(X+wY+w^2Z)^3+(X+w^2Y+wZ)^3$$
$$=9(X^3+Y^3+Z^3-w^2X^2Y-wX^2Z-wY^2X -w^2Y^2Z-w^2Z^2X-wZ^2Y+6XYZ).$$

Therefore

$$\tau(-X(4Z^3+4Y^3+27X^3)) = -3(X^4+w^2Y^4+wZ^4-2wX^2Y^2-2w^2X^2Z^2 -2Y^2Z^2+4X^2YZ+4w^2Y^2XZ+4wZ^2XY).$$

Now $H=ZY(ZY+18X^2)-X(4Z^3+4Y^3+27X^3)$, and hence

$$\tau(H) = 9(wX^2Y^2+w^2X^2Z^2+Y^2Z^2-2X^2YZ-2w^2Y^2XZ-2wZ^2XY),$$

and

$$\tau_1\tau(H) = 9(X^2Y^2+X^2Z^2+Y^2Z^2-2YZX^2-2XZY^2-2XYZ^2).$$

The proof is now completed by Proposition 2.

Lemma 6. *Assume that V is special and let α be as in Lemma* 5 *and assume that $P\in\alpha$. Then there exist projective coordinates (X, Y, Z, T) in P^3 such that P has coordinates* $(0, 0, 0, 1)$ *and V is given by $F=0$ where*

$$F = T^3+YXT+X^2Z.$$

Furthermore $\Delta_T(F)=X^3H$ where $H=-4Y^3-27Z^2X$ so that $H=0$ is an irreducible cubic in the (X, Y, Z)-plane P^2 having a cusp at $(1, 0, 0)$ *and having $X=0$ as the flex tangent.*

Proof. By Proposition 1, we can choose projective coordinates (X_1, Y_1, Z_1, T_1) in P^3 such that V is given by $F=0$ where $F=T_1^3+Y_1X_1T_1+X_1^2Z_1$. Then α is given by $X_1=0$ and hence P has coordinates $(0, a, b, c)$. Since P is not on V, we must have $c\neq 0$ and hence we can take $c=1$. Set $X_1=X_2$, $Y_1=Y_2+aT_2$, $Z_1=Z_2+bT_2, T_1=T_2$. Then P is given by $X_2=Y_2=Z_2=0$ and we have

$$F = T_2^3+aX_2T_2^2+(Y_2+bX_2)X_2T_2+X_2^2Z_2.$$

Setting $X_2=X_3$, $Y_2=Y_3$, $Z_2=Z_3$, $T_2=T_3-(1/3)aX_3$, we have P: $X_3=Y_3=Z_3=0$ and

$$F = T_3^3+(Y_3+bX_3-(1/3)a^2X_3)X_3T_3+((2/27)a^3X_3-(1/3)abX_3 \\ -(1/3)aY_3+Z_3)X_3^2.$$

Setting $X=X_3$, $Y=Y_3+bX_3-(1/3)a^2X_3$, $Z=(2/27)a^3X_3-(1/3)abX_3-(1/3)aY_3+Z_3$, everything follows by direct computation and Lemma 1.

3. The surface $T^3+YT^2+YXT+X^2Z=0$.

Let X, Y, Z, T be projective coordinates in projective three space P^3 over an algebraically closed ground field k of characteristic $p\neq 2, 3$. Let V be the irreducible cubic surface $F=0$ where

$$F(X, Y, Z, T) = T^3+YT^2+YXT+X^2Z.$$

Let A_2 and A_1 be the lines on V given by $X=T=0$ and $X=T+Y=0$ respectively, and let B be the point on V given by $X=Y=T=0$. Then A_2 is the double line of V and B is the point of intersection of A_1 and A_2.

Let P^2 be the (X, Y, Z)-plane, i.e., the plane $T=0$. Let ϕ be the projection of V onto P^2 from the point P: $X=Y=Z=0$, i.e., the projection along the T-axis. Let $x=X/Y$, $z=Z/Y$, $K=k(x, z)$. Let t be a root of

$$f(T) = F(x, 1, z, T) = T^3+T^2+xT+x^2z$$

in some extension of K, let $K^*=K(t)$, and let K' be a least galois extension of K containing K^*. Then (x, z) and (x, z, t) are affine k-general points of P^2 and V respectively, so that $k(P^2)=K$ and $k(V)=K^*$.

Let A be the line $X=0$ on P^2 and let H be the curve on P^2 given by $\mathfrak{H}=0$ where

$$\mathfrak{H}(X, Y, Z) = Y^4-4XY^3-4Y^3Z-27X^2Z^2+18XY^2Z.$$

Then by Lemma 3 we have:

$$(1) \qquad \Delta(V/P^2) = A\cup H,$$

and

(2) *H is an irreducible quartic curve having for its only singularities 2-fold cusps at Q_1: $(0, 0, 1)$, Q_2: $(1, 0, 0)$, Q_3: $(1/3, 1, 1/3)$.*

Let $\mathfrak{a}$ and $\mathfrak{h}$ be the real discrete valuations of K/k respectively having centers A and H on P^2. Let $\bar{K}$ be the completion of K

with respect to $\mathfrak{a}$ and let us denote the natural extension of $\mathfrak{a}$ to $\bar{K}$ also by $\mathfrak{a}$. It is clear that

(3) *mod* $\mathfrak{a}$, z *is transcendental over* k *and generates the residue field of* $\mathfrak{a}$ *over* k; $\mathfrak{a}(x)=1$ *and* $\bar{K}=k(z)((x))$.

Now $f(T)$ is the minimal monic polynomial of t over K and we have

$$f(T)\equiv(T+1)T^2 \pmod{x}.$$

By Hensel's Lemma, $f(T)=f_1(T)f_2(T)$, where

$$f_1(T) = T+1+\alpha_1x+\alpha_2x^2+\alpha_3x^3+u, \ \alpha_j\in k(z), \ u\in\bar{K}, \ \mathfrak{a}(u)>3;$$

and

$$f_2(T) = T^2+(\beta_1x+\beta_2x^2+\beta_3x^3+v_1)T+\gamma_1x+\gamma_2x^2+\gamma_3x^3+v,$$
$$\beta_j\in k(z), \ \gamma_j\in k(z), \ v_1\in\bar{K}, \ v\in\bar{K}, \ \mathfrak{a}(v_1)>3, \ \mathfrak{a}(v)>3.$$

Multiplying out f_1 and f_2 we get:

$$\begin{aligned}f_1(T)f_2(T)=T^3&+[1+(\alpha_1+\beta_1)x+(\alpha_2+\beta_2)x^2+(\alpha_3+\beta_3)x^3+(u+v_1)]T^2\\&+[(\gamma_1+\beta_1)x+(\gamma_2+\beta_2+\alpha_1\beta_1)x^2+(\gamma_3+\beta_3+\alpha_1\beta_2+\alpha_2\beta_1)x^3+w_1]T\\&+[\gamma_1x+(\gamma_2+\alpha_1\gamma_1)x^2+(\gamma_3+\alpha_1\gamma_2+\alpha_2\gamma_1)x^3+w_2], \ \mathfrak{a}(w_j)>3.\end{aligned}$$

Comparing coefficients with $f(T)$ we get:

$$\begin{aligned}&\alpha_1+\beta_1=0, \ \gamma_1+\beta_1=1, \ \gamma_1=0;\\&\alpha_2+\beta_2=0, \ \gamma_2+\beta_2+\alpha_1\beta_1=0, \ \gamma_2+\alpha_1\gamma_1=z;\\&\alpha_3+\beta_3=0, \ \gamma_3+\beta_3+\alpha_1\beta_2+\alpha_2\beta_1=0, \ \gamma_3+\alpha_1\gamma_2+\alpha_2\gamma_1=0;\\&u+v_1=0.\end{aligned}$$

Solving these four sets of equations successively, we get:

(4) $$f_1(T) = T+1-x+(z-1)x^2+(-2+3z)x^3+u, \ u\in\bar{K}, \ \mathfrak{a}(u)>3$$

(5) $$f_2(T) = T^2+(x+(1-z)x^2+(2-3z)x^3-u)T+(zx^2+zx^3+v),$$
$$v\in\bar{K}, \ \mathfrak{a}(v)>3.$$

Let $T^*=x^{-1}T$ and let $f_2^*(T^*)=x^{-2}f_2(T)$. Then

(5′) $$f_2^*(T^*)\equiv T^{*2}+T^*+z \pmod{x};$$

and $T^{*2}+T^*+z$ is a separable irreducible polynomial over $k(z)$. Now invoking the results of Section 2 of [A1] we deduce the following:

(6) $\mathfrak{a}$ *is unramified in* K^* *and splits into two valuations* $\mathfrak{a}_1^*$ *and* $\mathfrak{a}_2^*$ *such that* $g(\mathfrak{a}_j^*:\mathfrak{a})=j$. *Let* $\bar{K}_j^*$ *be the completion of* K^* *with*

respect to $\mathfrak{a}_j^*$ *and continue to denote the natural extension of* $\mathfrak{a}_j^*$ *to* $\bar{K}_j^*$ *also by* $\mathfrak{a}_j^*$. *Considering* K^* *as a subfield of* $\bar{K}_j^*$ *we have that* $\bar{K}_j^*=\bar{K}(t)$ *and* $f_j(T)$ *is the minimal monic polynomial of* t *over* $\bar{K}$.

Via the polynomials f_1 and f_2 we get $\mathfrak{a}_1^*(t+1)>0$ and $\mathfrak{a}_2^*(t)>0$ and hence

(7) $\mathfrak{a}_j^*$ *has center* A_j *on* V.

In view of (6) we have

(8) K^*/K *is not galois and hence* $[K':K]=6$ *and* $G(K'/K)$ *is isomorphic to the symmetric group* S_3 *on* 3 *letters.*

Since $p\neq 2, 3$, (8) implies that

(9) K^*/P^2 *and* K'/P^2 *are tamely ramified.*

Let V^* be a normalization of V, let ψ_1 be the map of V^* onto V and let $\phi^*=\phi\psi_1$. Then V^* is a K^*-normalization of V. Invoking "purity" for tame extensions [A2, Lemma 17 of Section 2] and noting that P^2 is simply connected, (1) and (6) now yield:

(10) $$\Delta(V^*/P^2)=H.$$

Suppose if possible that $\mathfrak{h}$ has only one extension $\mathfrak{h}^*$ to K^*. Since $[K^*:K]$ equals the prime number 3, we must have $r(\mathfrak{h}^*:\mathfrak{h})=3$ and hence for any K'-extension $\mathfrak{h}'$ of $\mathfrak{h}^*$ we have that $r(\mathfrak{h}':\mathfrak{h})$ equals either 3 or 6; if $r(\mathfrak{h}':\mathfrak{h})$ were 6 then in view of (9), $G_i(\mathfrak{h}':\mathfrak{h})$ would be a cyclic group of order 6 and we would have $G(K'/K)$ $=G_i(\mathfrak{h}':\mathfrak{h})$ which would contradict (8) because S_3 is not cyclic; therefore $r(\mathfrak{h}':\mathfrak{h})=3$. In view of (8), there exists a field K_1 between K and K' such that $[K_1:K]=2$; for any K_1-extension $\mathfrak{h}_1$ of $\mathfrak{h}$ we must have $2\leqq r(\mathfrak{h}_1:\mathfrak{h})$ and $r(\mathfrak{h}':\mathfrak{h})\equiv 0 \pmod{r(\mathfrak{h}_1:\mathfrak{h})}$ and hence we must have $r(\mathfrak{h}_1:\mathfrak{h})=1$. In view of Lemma 17 of [A 2], (10) implies that K_1/P^2 is unramified which contradicts the fact that P^2 is simply connected. Therefore:

(11) $\mathfrak{h}$ *splits in* K^* *into two valuations* $\mathfrak{h}_1^*$ *and* $\mathfrak{h}_2^*$ *such that* $d(\mathfrak{h}_j^*:\mathfrak{h})$ $=r(\mathfrak{h}_j^*:\mathfrak{h})=\bar{r}(\mathfrak{h}_j^*:\mathfrak{h})=j$.

Let H_j^* and H_j be the centers of $\mathfrak{h}_j^*$ on V^* and V respectively. Since V^* is normal, H_1^* and H_2^* are exactly the distinct irreducible components of $\phi^{-1}(H)$. Also V has no singularities outside A_2 and hence the quotient ring of H_j on V is normal and hence H_1, H_2

are exactly the distinct irreducible components of $\phi^{-1}(H)$ and $\psi_1^{-1}(H_j)=H_j^*$.

Consider the point Q_1. Substituting $X=Y=0$ in F we find that B is the only point of V lying above Q_1. Let $x_1=x/z$, $y_1=1/z$, $t_1=t/z$. Then (x_1, y_1) and (x_1, y_1, t_1) are corresponding affine k-general points of P^2 and V respectively, and we have

$$(12) \qquad t_1^3+y_1t_1^2+y_1x_1t_1+x_1^2=0.$$

Let $\xi=x_1/t_1$. Then

$$(13) \qquad t_1+y_1+y_1\xi+\xi^2=0.$$

Let $S=k[x_1, y_1]$, $N=(x_1, y_1)S$, $S^*=k[\xi, y_1, t_1]$, $N^*=(\xi, y_1, t_1)S^*$, and let (R_1, M_1) be the quotient ring of Q_1 on P^2. Then $Q_1=S_N$. By (12), t_1 is integral over S and hence by (13), ξ is integral over S. Hence S^* is integral over S. Again by (12) and (13), t_1 and ξ belong to the radical of NS^* and hence N^* is contained in the radical of NS^*. It is clear that any element of S^* equals some element of k mod N^* and hence N^* must be a maximal ideal in S^* and $N^*\neq S^*$. Therefore $S^*_{N^*}$ is a two-dimensional local ring. Let $R_1^*=S^*_{N^*}$ and $M_1^*=N^*R_1^*$. Then by (13), $M_1^*=(\xi, y_1)R_1^*$ and hence R_1^* is regular (geometrically speaking, we could see this by noting that "$\xi=y_1=0$ is a simple point of the surface (13)"). From this we conclude that

(14) *$\phi^{-1}(Q_1)=B$ and $\phi^{*-1}(Q_1)=\psi_1^{-1}(B)$ consists of a single point B^* of V^*, B^* is a simple point of V^*, (R_1^*, M_1^*) is the quotient ring of B^* on V^*, (ξ, y_1) is a minimal basis of M_1^*, and R_1^* is the integral closure in K^* of the quotient ring R_1 of Q_1 on P^2.*

From (11) and (14) it follows that H_1^* and H_2^* both pass through B^* and $\mathfrak{H}(x_1, y_1, 1)=\delta_1 q_1\bar{q}_1^2$ where δ_1 is a unit in R_1^*, $q_1R_1^*$ is the ideal of H_1^* at B^* and $\bar{q}_1R_1^*$ is the ideal of H_2^* at B^*. Now by (13) we have $x_1=-\xi(y_1+y_1\xi+\xi^2)$. Hence

$$\begin{aligned}\mathfrak{H}(x_1, y_1, 1)&=y_1^4-4x_1y_1^3-4y_1^3-27x_1^2+18x_1y_1^2\\ &=y_1^4+4\xi(y_1+y_1\xi+\xi^2)y_1^3-4y_1^3\\ &\quad -27\xi^2(y_1+y_1\xi+\xi^2)^2-18\xi(y_1+y_1\xi+\xi^2)y_1^2\\ &\equiv -4y_1^3 \pmod{M_1^{*4}}.\end{aligned}$$

Therefore (the R_1^*-leading form of q_1) (the R_1^*-leading form of $\bar{q}_1)^2 = c_1 y_1^3$ with $0 \neq c_1 \in k$. Hence H_j^* has a simple point at B^* and H_1^* and H_2^* are tangential at B^*. Consequently there is a unique immediate quadratic transform $(\bar{R}_1^*, \bar{M}_1^*)$ of R_1^* through which pass the transforms of H_1^* and H_2^*, and then (ξ, η) is a minimal basis of $\bar{M}_1^*$, where $\eta = y_1/\xi$. Now

$$\begin{aligned}\xi^{-3}\mathfrak{H}(x_1, y_1, 1) &= \xi\eta^4 + (\xi\eta + \xi^2\eta + \xi^2)\eta^3 - 4\eta^3 - 27\xi(\eta + \xi\eta + \xi)^2 \\ &\quad - 18\eta^2(\xi\eta + \xi^2\eta + \xi^2) \equiv -4\eta^3 - 27\xi\eta^2 \pmod{\bar{M}_1^{*4}}.\end{aligned}$$

Therefore the $\bar{R}_1^*$-leading form of $\xi^{-1}q_1$ equals η times a non-zero element of k and the $\bar{R}_1^*$-leading form of $\xi^{-1}\bar{q}_1$ equals $(-4\eta - 27\xi)$ times a nonzero element of k. Hence the transforms of H_1^* and H_2^* are nontangential at $\bar{R}_1^*$. Thus

(15) *H_1^* and H_2^* have a simple point and a 2-fold contact at B^*.*

Since $\Delta(V^*/P^2) = H$, [A 4, Lemma 5] tells us that any point of V^* lying above a point of P^2 other than Q_1, Q_2, Q_3 is a simple point of V^*. B^* is simple by (14). Now Q_2 and Q_3 are not on A and V has no singular point outside A_2. Consequently

(16) *V^* is nonsingular.*

Now $x_1 = t_1\xi = -\xi(y_1 + y_1\xi + \xi^2)$ and hence at B^*, A_1^* and A_2^* have a simple point and they are nontangential. Hence by (14), every point of A_1^* is nonsingular. Also outside Q_1, the only other point of intersection of A and H is Q_4: (0, 4, 1), which is a simple point of A and H and at it A and H are nontangential. Hence by [A 4, Lemma 5] A_2^* is also nonsingular. Let Q be any point of A other than Q_1 and Q_4. Then Q is a nonbranch point for ϕ^* and hence $\phi^{*-1}(Q)$ consists of three distinct points. By (6), A_j^* is a j-fold covering of A and hence A_1^* can pass through only one point of $\phi^{*-1}(Q)$ and hence A_2^* must pass through the other two. Noting that ϕ maps A_2 onto A in a one-to-one manner and that a 2-fold tamely ramified covering has at least two branch points, we conclude the following:

(17) *A_1^* and A_2^* are nonsingular and have B^* as the only common point, and at B^* they are nontangential. ψ_1 induces a biregular map of A_1^* onto A_1 and ϕ induces a biregular map of A_2 onto A; under ϕ^*, A_2^* is a 2-fold tamely ramified covering*

of A with branch points Q_1 and Q_4. Under ψ_1, A_2^* is a 2-fold tamely ramified covering of A with branch points B and $\bar{B}$ where $\bar{B}$ is the unique point of $\phi^{-1}(Q_4)$ contained in A_2.

Now consider Q_2. Substituting $Y=Z=0$ in F we see that the point B_2: $(Y=Z=T=0)$ is the only point of V lying above Q_2. Let $y_2=1/x$, $z_2=z/x$, $t_2=t/x$ and let (B_2, M_2) be the quotient ring of B_2 on V. Then $0=F(1, y_2, z_2, t_2)=t_2^3+y_2t_2^2+y_2t_2+z_2$ and hence (y_2, t_2) is a minimal basis of M_2, $z_2=-(y_2t_2+y_2t_2^2+t_2^3)$, and $\mathfrak{H}(1, y_2, z_2)=\delta_2q_2\bar{q}_2^2$ where δ_2 is a unit in R_2, q_2R_2 is the ideal of H_1 at B_2 and $\bar{q}_2$ is the ideal of H_2 at B_2. Substituting the above expression for z_2, we get

$$\begin{aligned}\mathfrak{H}(1, y_2, z_2)&=y_2^4-4y_2^3+4y_2^3(y_2t_2+y_2t_2^2+t_2^3)\\&\quad-27(y_2t_2+y_2t_2^2+t_2^3)^2-18y_2^2(y_2t_2+y_2t_2^2+t_2^3)\\&\equiv-4y_2^3 \pmod{M_2^4}.\end{aligned}$$

Therefore at B_2, H_1 and H_2 have a simple point and are tangential. Let $(\bar{R}_2, \bar{M}_2)$ be the unique immediate quadratic transform through which pass the transforms of H_1 and H_2 and let $\zeta=y_2/t_2$. Then (ζ, t_2) is a minimal basis of $\bar{M}_2$ and we have

$$\begin{aligned}t_2^{-3}\mathfrak{H}(1, y_2, z_2)&=t_2\zeta^4-4\zeta^3+4\zeta^3(\zeta t_2^2+\zeta t_2^3+t_2^3)\\&\quad-27t_2(\zeta+\zeta t_2+t_2)^2-18t_2\zeta^2(\zeta+\zeta t_2+t_2)\\&\equiv-4\mathfrak{s}(\zeta, t_2) \pmod{\bar{M}_2^4};\end{aligned}$$

where

$$\mathfrak{s}(\zeta, t_2)=\zeta^3+(27/4)t_2\zeta^2+(27/2)t_2^2\zeta+(27/4)t_2^3. \tag{18}$$

Now $\mathfrak{s}(\zeta, t_2)=$(a nonzero element of k) (the $\bar{R}_2$-leading form of $t_2^{-1}q_2$) (the $\bar{R}_2$-leading form of $t_2^{-1}\bar{q}_2)^2$, and hence the transforms of H_1 and H_2 are tangential at $\bar{R}_2$ if and only if $\mathfrak{s}(\zeta, t_2)$ is a perfect cube. If $\mathfrak{s}(\zeta, t_2)$ were a perfect cube, we would have $(27/4)^2=3(27/2)$, i.e., $9=8$, i.e., $1=0$, which is a contradiction. Since B_2 is a simple point of V we conclude that

(19) *$\phi^{*-1}(Q_2)$ consists of a single point, this point is simple for V^*, and at it H_1^* and H_2^* have a simple point and a 2-fold contact.*

Now consider the point Q_3. Substituting $X=Z=1/3$ and $Y=1$

in F we find that B_3: $(X=Z=1/3,\ Y=1,\ T=-1/3)$ is the only point of V lying above Q_3. Choose projective coordinates (X_1, Y_1, Z_1, T_1) in P^3 via the equations $X=X_1+(1/3)Y_1$, $Y=Y_1$, $Z=Z_1+(1/3)Y_1$, $T=T_1-(1/3)Y_1$. Then by direct computation we get

$$F_1(X_1, Y_1, Z_1, T_1)=F(X, Y, Z, T)=T_1^3+X_1Y_1T_1$$
$$+(1/9)(-X_1Y_1^2+Z_1Y_1^2+3X_1^2Y_1+6X_1Y_1Z_1+9X_1^2Z_1);$$

and

$$\mathfrak{H}_1(X_1, Y_1, Z_1)=(-1/3)\mathfrak{H}(X, Y, Z)$$
$$=Y_1^2(X_1^2-2X_1Z_1+Z_1^2)+Y_1(6X_1^2Z_1+6X_1Z_1^2)+9X_1^2Z_1^2.$$

Now choose projective coordinates (X_3, Y_3, Z_3, T_3) in P^3 via the equations $X_1=X_3$, $Y_1=Y_3$, $Z_1=X_3-Z_3$, $T_1=T_3$. Then

$$F_3(X_3, Y_3, Z_3, T_3)=F_1(X_1, Y_1, Z_1, T_1)$$
$$=T_3^3+X_3Y_3T_3+X_3^3-X_3^2Z_3+X_3^2Y_3-(2/3)X_3Z_3Y_3-(1/9)Z_3Y_3^2;$$

and

$$\mathfrak{H}_3(X_3, Y_3, Z_3)=\mathfrak{H}_1(X_1, Y_1, Z_1)$$
$$=12X_3^3Y_3+9X_3^4+Z_3(-18X_3^2Y_3-18X_3^3)+Z_3^2(Y_3^2+6X_3Y_3+9X_3^2).$$

Now Q_3 is given by $X_3=Z_3=0$ and B_3 is given by $X_3=Z_3=T_3=0$. Let $x_3=X_3/Y_3$, $z_3=Z_3/Y_3$, $t_3=T_3/Y_3$. Let (R_3, M_3) be the quotient ring of B_3 on V. Then (x_3, t_3) is a minimal basis of M_3, $z_3\in M_3$, and

$$z_3=9x_3^2+9x_3t_3-6x_3z_3+9x_3^3+9t_3^3-9x_3^2z_3. \tag{20}$$

Substituting in $\mathfrak{H}_3$ we get $\mathfrak{H}_3(x_3, 1, z_3)\equiv 12x_3^3 (\operatorname{mod} M_3^4)$. Hence H_1 and H_2 have a simple point at B_3 and are tangential at B_3. Let $(\bar{R}_3, \bar{M}_3)$ be the unique immediate quadratic transform of R_3 through which pass the transforms of H_1 and H_2 and let $\alpha=x_3/t_3$. Then (α, t_3) is a minimal basis of $\bar{M}_3$. Substituting the right-hand expression in (20) in place of z_3 in that expression, we get

$$\begin{aligned} z_3=&(9x_3^2+9x_3t_3+36z_3x_3^2)+(-45x_3^3-54x_3^2t_3+9t_3^3+108z_3x_3^3)\\ &+(-135x_3^4-81x_3^3t_3-54x_3t_3^3+81x_3^4z_3)-81x_3^5-81x_3^2t_3^3. \end{aligned} \tag{21}$$

Consequently we have

$$t_3^{-3}\mathfrak{H}_3(x_3, 1, z_3)\equiv 12\mathfrak{s}(\alpha, t_3) \ (\operatorname{mod} \bar{M}_3^4),$$

where $\mathfrak{s}(\alpha, t_3)=\alpha^3+(27/4)t_3\alpha^2+(27/2)t_3^2\alpha+(27/4)t_3^3$. Now this is the same polynomial $\mathfrak{S}$ as obtained in the proof of (19) (with α and t_3 replacing ζ and t_2) and hence we conclude that

(22) $\phi^{*-1}(Q_3)$ *consists of a single point, this point is simple for* V^*, *and at it* H_1^* *and* H_2^* *have a simple point and 2-fold contact.*

Now (15), (19), (22) and [A 4, Lemma 5] yield the following:

(23) *Above the three singularities of* H *lie unique points of* V^*, *these are the only common points of* H_1^* *and* H_2^* *and at each of them* H_1^* *and* H_2^* *have a 2-fold contact. Also* H_1^* *and* H_2^* *are nonsingular. Consequently we have that*:

$$\nu(H_j^*, H_1^* \cup H_2^*; V^*) = 6 \text{ for } j = 1, 2.$$

Next we would like to represent the surface V as a projective plane by projecting it from the double point B and then we would like to simplify this representation by applying a Cremona quadratic transformation of the second kind. Hence we now interpose the following two sections.

4. Projection of a hypersurface of order m from an $(m-1)$-fold point

Definition 2. Let (R, M) be a regular local domain with quotient field K. For $0 \neq a \in R$, set $v(a)$ = the greatest integer m such that $a \in M^m$. For $0 \neq b \in R$, set $v(b/a) = v(b) - v(a)$. Then v is a real discrete valuation of K. *We shall say that* v *is the* R-*adic valuation of* K. If R is the quotient ring of a point P on an algebraic variety V, then we shall also say that v *is the* P-*adic divisor of* V.

Now assume that R is two-dimensional and contains a coefficient field k and that k (i.e., R/M) is algebraically closed. Let (x, y) be a basis of M and let s be an element of k. Then [A3, Sections 3 and 4] there exists a unique immediate quadratic transform (S, N) of R such that $y/x \in S$ and $(y/x) - s \in N$. The S-adic valuation of K will be called *the second* R-*adic valuation of* K *in the direction* $y/x = s$. If R is the quotient ring of a point P on an algebraic surface V over k and if W is a curve on V having a simple point at P such that the ideal of W on V is tangential to the ideal $(y - sx)R$ at R then the S-adic valuation of K will be called *the second* P-*adic divisor of* V *along* W.

Lemma 7. *Let (R, M) be a two-dimensional regular local domain with quotient field K, let (x, y) be a minimal basis of M and let v be a valuation of K having center M in R. Then v is the R-adic valuation of K if and only if $v(x)=v(y)$ and y/x is transcendental over k mod v.*

Proof. The "only if" part follows from well-known properties of regular local domains. So now assume that $v(x)=v(y)$ and that y/x is transcendental over k mod v. Let z be a nonzero element of R and let m be the largest integer such that $z \in M^m$. Then $z=f(x, y)$ where $f(X, Y)$ is a form of degree m with coefficients in R but not all in M. Then $z/x^m=g(y/x)$, where $g(Y)=f(1, Y)$. Then all the coefficients of g have nonnegative v-value and at least one of them has zero v-value. Since y/x is transcendental mod v, we have $g(y/x) \not\equiv 0 \pmod v$ and hence $v(g(y/x))=0$, i.e., $v(z)=mv(x)$. From this we conclude that v is the R-adic valuation of K (or equivalent to it).

Definition 3. Let P^n be a projective n-space over an algebraically closed ground field k, let V be an irreducible hypersurface in P^n of order m $(m>1)$ having an $(m-1)$-fold point A, let P^{n-1} be a hyperplane in P^n not passing through A, and for any point B of P^n other than A let the line joining A and B be denoted by L_B. Let P be a k-general point of V and let $P'=L_P \cap P^{n-1}$. Then P' is a k-general point of P^{n-1} and hence we get a unique rational map τ of V onto P^{n-1} by setting $\tau(P)=P'$. It is clear that if $\bar{P}$ is any other k-general point of V then $\tau(\bar{P})=L_{\bar{P}} \cap P^{n-1}$ and hence τ does not depend on the particular k-general point P of V. Since A is an $(m-1)$-fold point of V, the line L_P does not meet V outside A and P. *Consequently τ is a birational map of V onto P^{n-1}. We shall say that τ is the projection of V from A onto P^{n-1}.*

Lemma 8. *Let the notation be as in Definition* 3. *Let C be the tangent cone of V at A and let $C'=C \cap P^{n-1}$. Then* (I) : *τ is biregular on $V-C\cap V$ and $\tau(V-C\cap V)=P^{n-1}-C'$. A is the only fundamental point of τ and $\tau(A)=C'$. For any point B of V other than A we have $\tau(B)=L_B \cap P^{n-1}$. For any point A' of C' we have $\tau^{-1}(A')=L_{A'} \cap V$ and hence if A' is not on $C\cap V\cap P^{n-1}$ then $\tau^{-1}(A')=A$ and if A' is on $C\cap V \cap P^{n-1}$ then $\tau^{-1}(A')=L_{A'}$. A point of P^{n-1} is fundamental for τ^{-1} if and only if it is on $C\cap V\cap P^{n-1}$.*

(*II*): *Now assume that* $n=3$. *Then* $C\cap V$ *consists of distinct lines* $\gamma_1, \cdots, \gamma_t$ $(t>0)$ *through* A *which intersect* P^2 *in distinct points* $\gamma'_1, \cdots, \gamma'_t$ *lying on* C'. $\gamma'_1, \cdots, \gamma'_t$ *are the only fundamental points of* τ^{-1} *and* $\tau^{-1}(\gamma'_j)=\gamma_j$ *for* $j=1, \cdots, t$. *Now let* V^* *be a normalization of* V, *let* ψ *be the map of* V^* *onto* V, *let* $\tau^*=\tau\psi$, *and assume that* V *is analytically irreducible at* A, *i.e., that* $\psi^{-1}(A)$ *consists of a single point* A^*. *Then* $\psi^{-1}(\gamma_j)$ *is a* (*possibly reducible*) *curve on* V^*. τ^* *is biregular on* $V^*-\psi^{-1}(\gamma_1)-\cdots-\psi^{-1}(\gamma_t)$ *and* $\tau^*(V^*-\psi^{-1}(\gamma_1)-\cdots-\psi^{-1}(\gamma_t))=P^2-C'$.

A^* *is the only fundamental point of* τ^* *and* $\tau^*(A^*)=C'$. $\gamma'_1, \cdots, \gamma'_t$ *are the only fundamental points of* τ^{*-1} *and* $\tau^{*-1}(\gamma'_j)=\psi^{-1}(\gamma_j)$.

(III) *Now assume that* V *is a nonsingular quadric in* P^3 $(m=2)$. *Then the tangent cone* C *is a plane,* $C\cap V$ *consists of two distinct lines* γ_1 *and* γ_2 *through* A, $C'=C\cap P^2$ *is a line,* $\gamma'_1=\gamma_1\cap P^2$ *and* $\gamma'_2=\gamma_2\cap P^2$ *are two distinct points on* C'. *Furthermore the unique valuation of* $k(V)/k$ *having center* γ_j *on* V *is the* γ'_j*-adic divisor of* P^2.

Proof. Choose projective coordinates $X_1, \cdots, X_{n+1}$ in P^n such that A is given by $X_1=\cdots=X_n=0$ and P^{n-1} is given by $X_{n+1}=0$. Then V is given by $F=0$ where

$$F(X_1, \cdots, X_{n+1})=X_{n+1}G(X_1, \cdots, X_n)-H(X_1, \cdots, X_n)$$

where G and H are nonzero forms of degrees $m-1$ and m respectively. Choose $X_1=0$ as the hyperplane at infinity. The k-general point P of V cannot lie on $X_1=0$ and hence P has affine coordinates $(x_2, \cdots, x_{n+1})$ in P^n and $P'=L_P\cap P^{n-1}$ has affine coordinates $(x_2, \cdots, x_n)$ in P^{n-1}. Then $x_2, \cdots, x_n$ are algebraically independent over k, $G(1, x_2, \cdots, x_n)\neq 0$, $x_{n+1}=H(1, x_2, \cdots, x_n)/G(1, x_2, \cdots, x_n)$, $K=k(P^{n-1})=k(x_2, \cdots, x_n)=k(x_2, \cdots, x_{n+1})=k(V)$. Let B be a point of V other than A. Then the X_j-coordinate of B is nonzero for some $j\leq n$. Via a change of coordinates we may assume that B is not at the hyperplane at the infinity $(X_1=0)$. Let $(a_2, \cdots, a_{n+1})$ be the affine coordinates of B in P^n. Let B' be a point in $\tau(B)$. Then B' cannot be on $X_1=0$, i.e., B' is not at infinity in P^{n-1}. Since $x_j-a_j\in M(B, V)$ for $j=1, \cdots, n$; we conclude that B' must be the point having affine coordinates $(a_2, \cdots, a_n)$ in P^{n-1}, i.e., $B'=L_B\cap P^{n-1}$. If $\phi(X_2, \cdots, X_n)$ is a polynomial with coefficients in k such that $\phi(a_2, \cdots, a_n)\neq 0$ then $\phi(a_2, \cdots, a_n)$ is unit in $Q(B, V)$.

Consequently $Q(B', P^{n-1}) \subset Q(B, V)$. Now assume that $B \notin C$. Then $G(1, a_2, \cdots, a_n) \neq 0$ and hence $x_{n+1} \in Q(B', P^{n-1})$. Also $a_{n+1} = H(1, a_2, \cdots, a_n)/G(1, a_2, \cdots, a_n)$ and hence $x_{n+1} \equiv a_{n+1} (\mathrm{mod}\ M(B', P^{n-1}))$. Consequently, if $\phi(X_2, \cdots, X_{n+1})$ is a polynomial with coefficients in k such that $\phi(a_2, \cdots, a_{n+1}) \neq 0$ then $\phi(x_2, \cdots, x_{n+1})$ is a unit in $Q(B', P^{n-1})$. Therefore $Q(B, V) \subset Q(B', P^{n-1})$, i.e., $Q(B, V) = Q(B', P^{n-1})$. Now let A' be a point of C'. Via a change of coordinates, we may assume that A' is not at infinity (i.e., not on $X_1 = 0$). Then $G(1, x_2, \cdots, x_n)$ is a nonzero nonunit in $Q(A', P^{n-1})$. In case $H(X_1, \cdots, X_n)$ equals X_1^m times a nonzero element of k we let v to be any valuation of K/k having center A' on P^{n-1}; in the contrary case $H(1, x_2, \cdots, x_n)$ and $G(1, x_2, \cdots, x_n)$ are coprime nonzero nonunits in $Q(A', P^{n-1})$ (because V is irreducible) so that neither x_{n+1}, nor x_{n+1}^{-1} belongs to $Q(A', P^{n-1})$ and we let v to be a valuation of K/k having center A' on P^{n-1} such that $v(x_{n+1}^{-1}) > 0$ [see Lemma 2 of AZ]; thus in either case v is a valuation of K/k having center A' on P^{n-1} such that $v(x_{n+1}) < 0$. Then $v(1/x_{n+1}) > 0$, $v(x_2/x_{n+1}) > 0$, $v(x_3/x_{n+1}) > 0, \cdots, v(x_n/x_{n+1}) > 0$; and hence v has center A on V. Consequently $A \in \tau^{-1}(A')$. From these considerations, (I) follows immediately and from it (II) follows in view of the form of F.

Now assume that $n = 3$ and that V is a nonsingular quadric. The first statement in (III) is obvious, and because of symmetry, it is enough to prove the second statement in (III) for γ_1. In P^2, $H(X_1, X_2, X_3) = 0$ is either a conic or a pair of distinct lines, and it meets the line $G(X_1, X_2, X_3) = 0$ in P^2 in the distinct points γ_1' and γ_2'. Via a change of coordinates in P^2 we may assume that γ_1' and γ_2' are given in P^2 by $X_2 = X_3 = 0$ and $X_2 = X_1 = 0$ respectively and that also the point $X_1 = X_3 = 0$ in P^2 is on the curve $H = 0$. Then we must have $G(X_1, X_2, X_3) = d_1 X_2$ and $H(X_1, X_2, X_3) = e_1 X_1 X_2 + f_1 X_2 X_3 + g_1 X_3 X_1$ with $d_1, e_1, f_1, g_1 \in k$ and $d_1 \neq 0 \neq g_1$. Let $e = e_1/d_1$, $f = f_1/d_1$, $g = g_1/d_1$. Then $e, f, g_1 \in k$, $g \neq 0$, and

$$x_4 = e + f x_3 + g(x_3/x_2).$$

Now $S = k[x_2, x_3, x_4]$ is an affine coordinate ring of V and $T = (x_2, x_3)S$ is the prime ideal of γ_1 in S. Since T is a one-dimensional ideal, x_4 must be transcendental over k mod T. The above expression for x_4 now implies that x_3/x_2 is in S and it is transcendental over k mod T. Let v be the unique valuation of K/k having center γ_1 on V. Then $R_v = S_T$ and hence $v(x_3) = v(x_2)$ and x_3/x_2 is transcendental over k mod v. Now (x_2, x_3) is a minimal basis of $M(\gamma_1', P^2)$ and hence by Lemma 7, we conclude that v is the γ_1'-adic divisor of P^2.

5. Cremona quadratic transformations of the second kind

Let P^2 and $\bar{P}^2$ be projective planes over an algebraically closed ground field k. Let (X, Y, Z) and $(\bar{X}, \bar{Y}, \bar{Z})$ be homogeneous coordinates in P^2 and $\bar{P}^2$ respectively. Let $x = X/Z$, $y = Y/Z$; $x_1 = X/Y$, $z_1 = Z/Y$; $y_2 = Y/X$, $z_2 = Z/X$. Let $\bar{x} = \bar{X}/\bar{Z}$, $\bar{y} = \bar{Y}/\bar{Z}$; $\bar{x}_1 = \bar{X}/\bar{Y}$, $\bar{z}_1 = \bar{Z}/\bar{Y}$; $\bar{y}_2 = \bar{Y}/\bar{X}$, $\bar{z}_2 = \bar{Z}/\bar{X}$. Then (x, y), (x_1, z_1), (y_2, z_2) are affine k-general points of P^2 and $(\bar{x}, \bar{y})$, $(\bar{x}_1, \bar{z}_1)$, $(\bar{y}_2, \bar{z}_2)$ are affine k-general points of $\bar{P}^2$. Since x and x^2/y are algebraically independent over k we may identify $k(\bar{x}, \bar{y}) = k(\bar{P}^2)$ with a subfield of $K = k(x, y) = k(P^2)$ by setting

$$(1) \qquad \bar{x} = x,\ \bar{y} = x^2/y. \quad \text{Then} \quad x = \bar{x},\ y = \bar{x}^2/\bar{y};$$

and hence $K = k(P^2) = k(x, y) = k(\bar{x}, \bar{y}) = k(\bar{P}^2)$. Let σ be the resulting birational map of P^2 onto $\bar{P}^2$. In homogeneous coordinates, σ and σ^{-1} are given respectively by the following equations:

$$(2) \qquad \bar{X} : \bar{Y} : \bar{Z} = XY : X^2 : YZ;\ X : Y : Z = \bar{X}\bar{Y} : \bar{X}^2 : \bar{Y}\bar{Z}.$$

Lemma 9. (I) *σ is biregular outside the lines $X=0$ and $Y=0$ and $\sigma(P^2-(X=0)-(Y=0)) = \bar{P}^2-(\bar{X}=0)-(\bar{Y}=0)$.* (II) *The points $(X=Z=0)$ and $(X=Y=0)$ are the only fundamental points σ.* (III) *$\sigma((X=Z=0)) = (\bar{Y}=0)$ and the valuation of K/k having center $(\bar{Y}=0)$ on $\bar{P}^2$ is the $(X=Z=0)$-adic divisor of P^2.* (IV) *$\sigma((X=Y=0)) = (\bar{X}=0)$ and the valuation of K/k having center $(\bar{X}=0)$ on $\bar{P}^2$ is the second $(X=Y=0)$-adic divisor of P^2 along $(Y=0)$.* (V) *If $f(X, Y, Z)$ is a form in $k[X, Y, Z]$ not divisible by X or Y and $\bar{f}(\bar{X}, \bar{Y}, \bar{Z})$ is the form obtained by dividing out $f(\bar{X}\bar{Y}, \bar{X}^2, \bar{Y}\bar{Z})$ by the highest possible powers of $\bar{X}$ and $\bar{Y}$ then the curve $\bar{f}=0$ is the σ-transform of the curve $f=0$, and $\bar{f}$ is irreducible if and only if f is irreducible.* (VI) *Statements* (I) *to* (V) *hold for σ^{-1} after the corresponding barred and unbarred letters are interchanged.*

Proof. (I) follows from the equations (1). Now (x_1, z_1) is a basis of the ideal of $(X=Z=0)$ on P^2 and with respect to the affine k-general point $(\bar{y}_2, \bar{z}_2)$ of $\bar{P}^2$ the line $\bar{Y}=0$ is given by $\bar{y}_2=0$. Let $\bar{v}$ be the valuation of K/k having center $\bar{Y}=0$ on $\bar{P}^2$ and let v be any valuation of K/k having center $(X=Z=0)$ on P^2. Then $v(x_1)>0$ and $x_1=x/y=\bar{y}/\bar{x}=\bar{y}_2$ and hence the center of v on $\bar{P}^2$ lies on $(\bar{Y}=0)$. Also $\bar{v}(\bar{y}_2)=1$, $\bar{v}(\bar{z}_2)=0$ and $\bar{z}_2$ is transcendental over k mod $\bar{v}$; and $x_1=\bar{y}_2$ and $x_1/z_1=x=\bar{x}=1/\bar{z}_2$. Hence by Lemma 7 of Section 4, $\bar{v}$ is the $(X=Z=0)$-adic divisor of P^2.

Next, (x, y) is a basis of the ideal of $(X=Y=0)$ on P^2 and with respect to the affine k-general point $(\bar{x}, \bar{y})$ of $\bar{P}^2$ the line $(\bar{X}=0)$ is given by $\bar{x}=0$. Let $\bar{w}$ be the valuation of K/k having center $(\bar{X}=0)$ on $\bar{P}^2$, let w be any valuation K/k having center $(X=Y=0)$ on P^2, and let (S, N) be the immediate quadratic transform of the quotient ring of $(X=Y=0)$ on P^2 such that (x, y^*) is a basis N where $y^*=y/x$. Then $w(x)=w(\bar{x})>0$ and hence the center of w on $\bar{P}^2$ lies on the line $(\bar{X}=0)$. Also $\bar{w}(\bar{x})=1$, $\bar{w}(\bar{y})=0$ and $\bar{y}$ is transcendental over k mod $\bar{w}$; and $x=\bar{x}$ and $x/y^*=x^2/y=\bar{y}$. Consequently, by Lemma 7 of Section 4, $\bar{w}$ is the second $(X=Y=0)$-adic divisor of P^2 along $(Y=0)$. This completes the proof of (II, III, IV); and in view of equations (2), (V) follows from (I, II, III, IV); and finally (VI) follows by symmetry.

Definition 4. *σ will be called a Cremona quadratic transformation of the second kind given by the net of conics passing through $(X=Z=0)$ and having the line $(Y=0)$ as tangent at the point $(X=Y=0)$.*

Lemma 10. *Let V be a projective nonsingular algebraic surface over an algebraically closed ground field k, let $K=k(V)$, and let ψ be a birational map of V onto a projective plane P^2. Let B be a point on V, let A_1 and A_2 be two distinct irreducible curves on V passing through B, let b be a line on P^2, let a_1 and a_2 be two distinct points on b, let w be a line on P^2 different from b and passing through a_1. Assume the following*: (1) *ψ is biregular on $V-A_1-A_2$ and $\psi(V-A_1-A_2)=P^2-b$*; (2) *B is the only fundamental point of ψ and $\psi(B)=b$*; (3) *a_1 and a_2 are the only fundamental points of ψ^{-1}*; (4) *$\psi^{-1}(a_1)=A_1$ and the valuation of K/k having center A_1 on V is the second a_1-adic divisor of P^2 along w*; (5) *$\psi^{-1}(a_2)=A_2$ and the valuation of K/k having center A_2 on V is the a_2-adic divisor of P^2.* Then (I) *$W=\psi^{-1}[w]$ is an irreducible curve on V different from A_1 and A_2*

and V is the immediate quadratic transform of a projective plane with center at a point: more specifically there exists a birational regular map $\bar{\psi}$ of V onto a projective plane $\bar{P}^2$ such that $\bar{\psi}(W)$ is a point $\bar{w}$ in $\bar{P}^2$ which is the only fundamental point of $\bar{\psi}^{-1}$ and the valuation of K/k having center A_1 on V is the $\bar{w}$-adic divisor of $\bar{P}^2$. Furthermore (II) *dim* $|W|=0$ *on V and for any irreducible curve N on V other than W we have dim* $|N|>0$ *on V. Now let h_1 and h_2 be nonsingular conics in P^2 such that $a_1\in h_1$, $a_2\notin h_1$, w is the tangent to h_1 at a_1, $a_2\in h_2$, and $a_1\notin h_2$. Then on V we have dim* $|\psi^{-1}[h_1]|=5$ *and dim* $|\psi^{-1}[h_2]|=6$.

Proof. Choose projective coordinates X, Y, Z in P^2 such that a_1 is $(X=Y=0)$, a_2 is $(X=Z=0)$ and w is $Y=0$. Let $\bar{X}, \bar{Y}, \bar{Z}$ be projective coordinates in a projective plane $\bar{P}^2$ over k. Let σ be the Cremona quadratic transformation of P^2 onto $\bar{P}^2$ given by

$$X:Y:Z=\bar{X}\bar{Y}:\bar{X}^2:\bar{Y}\bar{Z}.$$

Let $\bar{\psi}=\sigma\psi$. Then (I) follows by Lemma 9, where $\bar{w}=\bar{\psi}(W)=\sigma[(Y=0)]=(\bar{X}=\bar{Z}=0)$. From (I) it follows that dim $|W|=0$. Now let N be an irreducible curve on V different from W. Then $\bar{\psi}[N]$ is an irreducible curve on $\bar{P}^2$ of some order s. Let t be the multiplicity of $\bar{\psi}[N]$ at $\bar{w}$. Then $t\leqq s$. Using the technique of the proof of [A3, Proposition 5 of Section 6] we conclude that dim $|N|=$ dim $|\bar{\psi}[N]|-\left(\frac{1}{2}\right)t(t+1)=\left(\frac{1}{2}\right)s(s+3)-\left(\frac{1}{2}\right)t(t+1)>0$. This proves (II).

h_1 is given by $f_1(X, Y, Z)=0$ where

$$f_1(X, Y, Z)=\alpha_1X^2+\alpha_2Y^2+\alpha_3XY+YZ,\ \alpha_i\in k,\ \alpha_2\neq 0.$$

Now

$$f_1(\bar{X}\bar{Y}, \bar{X}^2, \bar{Y}\bar{Z})=\bar{X}^2\bar{f}_1(\bar{X}, \bar{Y}, \bar{Z}),\ \text{where}$$
$$\bar{f}_1(\bar{X}, \bar{Y}, \bar{Z})=\alpha_1\bar{Y}^2+\alpha_2\bar{X}^2+\alpha_3\bar{X}\bar{Y}+\bar{Y}\bar{Z}.$$

Then $\bar{f}_1$ is divisible neither by $\bar{X}$ nor by $\bar{Y}$ (since $\alpha_2\neq 0$) and hence by Lemma 9, $\sigma[h_1]$ is given by $\bar{f}_1=0$. Hence $\sigma[h_1]$ is of order 2 and does not pass through $\bar{w}$. Therefore dim $|\psi^{-1}[h_1]|=$ dim $|\sigma[h_1]|=5$.

h_2 is given by $f_2(X, Y, Z)=0$, where

$$f_2(X, Y, Z)=\beta_1X^2+\beta_2Z^2+\beta_3XY+\beta_4XZ+\beta_5YZ,\ \beta_i\in k,\ \beta_2\neq 0.$$

Now

$$f_2(\bar{X}\bar{Y}, \bar{X}^2, \bar{Y}\bar{Z}) = \bar{Y}\bar{f}_2(\bar{X}, \bar{Y}, \bar{Z}), \quad \text{where}$$
$$\bar{f}_2(\bar{X}, \bar{Y}, \bar{Z}) = \beta_1\bar{X}^2\bar{Y} + \beta_2\bar{Y}\bar{Z}^2 + \beta_3\bar{X}^3 + \beta_4\bar{X}\bar{Y}\bar{Z} + \beta_5\bar{X}^2\bar{Z}.$$

Since $\beta_2 \neq 0$, $\bar{f}_2$ is not divisible by $\bar{X}$. Since f_2 is irreducible, β_3 and β_5 are not simultaneously zero and hence $\bar{f}_2$ is not divisible by $\bar{Y}$. Hence $\sigma[h_2]$ is given by $\bar{f}_2 = 0$. Therefore $\sigma[h_2]$ is of order three and has a double point at $\bar{w}$. Hence $\dim |\psi^{-1}[h_1]| = \dim |\sigma[h_1]| - 3 = 9 - 3 = 6$.

6. The surface $T^3 + YT^2 + YXT + X^2Z = 0$

(Continued)

$$\begin{tikzcd}
P'^2 & \bar{V}' \arrow[l, "\tau'"] \arrow[d, "\tau"] & & V' \arrow[ll, "\tau_1"] \arrow[lll, bend right, "\psi'"'] \arrow[d, "\varphi_1"] \arrow[dd, bend left] & \\
& P^{*2} & V \arrow[l, "\psi"] \arrow[dr, "\varphi"] & V^* \arrow[l, "\psi_1"] \arrow[ll, bend right, "\psi^*"'] \arrow[r, "\bar{\psi}"] \arrow[d, "\varphi^*"] & \bar{P}^2 \\
& & & P^2 &
\end{tikzcd}$$

Figure 1

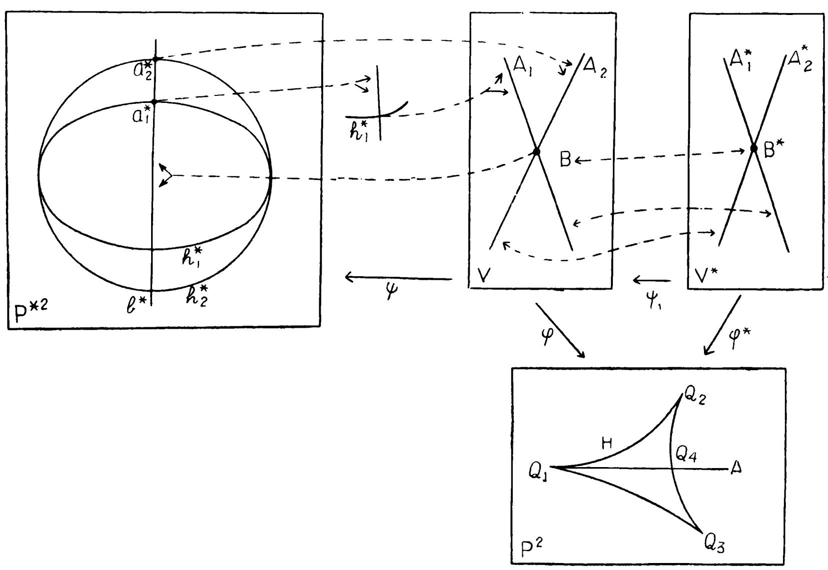

Figure 2

Now let us resume the considerations of Section 3. Let P^{*2} be the (X, Y, T)-plane, i.e., the plane $Z=0$; and let ψ be the projection of V from B onto P^{*2}. The tangent cone C to V at B is $X^2=0$, and $C\cap V=A_1\cup A_2$. Let $b^*=C\cap P^{*2}$, and $a_j^*=A_j\cap P^{*2}$. Then b^* is a line in P^{*2} and a_1^* and a_2^* are points on b^*. In P^{*2}, b^* is given by $X=0$, a_1^* is given by $X=Y+T=0$, and a_2^* is given by $X=T=0$. Let $\psi^*=\psi\psi_1$. By Lemma 8 we have:

(24) *ψ^* is biregular on $V^*-A_1^*-A_2^*$, $\psi^*(V^*-A_1^*-A_2^*)=P^{*2}-b^*$, B is the only fnndamental point of ψ^* and $\psi^*(B^*)=b^*$, a_1^* and a_2^* are the only fundamental points of ψ^{*-1} and $\psi^{*-1}(a_j^*)=A_j^*$.*

Let $h_j^*=\psi^*[H_j^*]$ be the (proper) ψ^*-transform of H_j^*, i.e., h_j^* is the center of $\mathfrak{h}_j^*$ on P^{*2}. Since $H_j^*\not\subset A_1^*\cup A_2^*$, h_j^* is an irreducible curve on P^{*2}. Since $B^*\in H_j^*$, the total ψ^*-transform $\psi^*(H_j^*)$ equals $h_j^*\cup b^*$. Now we have

$$zx^2 = -(t^3+t^2+xt), \quad \text{and}$$
$$\mathfrak{H}(x, 1, z) = 1-4x-4z-27x^2z^2+18xz\,.$$

Multiplying $\mathfrak{H}(x, 1, z)$ by x^2 and substituting the expression for zx^2 we get $x^2\mathfrak{H}(x, 1, z)=S(x, t)$, where

$$\begin{aligned} S(x, t) &= x^2-4x^3+4(t^3+t^2+xt)-27(t^3+t^2+xt)^2-18x(t^3+t^2+xt) \\ &= -4x^3+\alpha x^2+\beta x+\gamma\,, \end{aligned}$$

where

$$\alpha = 1-27t^2-18t\,, \quad \beta = 4t-54(t^3+t^2)t-18(t^3+t^2)\,,$$
$$\gamma = 4(t^3+t^2)-27(t^3+t^2)^2\,.$$

By (11), (i) $\mathfrak{h}_1^*(S(x, t))=1$ and $\mathfrak{h}_2^*(S(x, t))=2$. Now (ii) the K^*-extensions of $\mathfrak{a}$ are $\mathfrak{a}_1^*$ and $\mathfrak{a}_2^*$ and both of these are divisors of the second kind on P^{*2}, (iii) the ϕ^{-1}-transform of the line $Y=0$ of P^2 consists of the irreducible components of the intersection of V with the plane $Y=0$ and these curves make up exactly the ψ^{-1} transform of the line $Y=0$ of P^{*2}, and (iv) the only divisor of the first kind on P^{*2} which is a divisor of the second kind on V^* is the divisor b^* and by directly substituting $x=0$ in $S(x, t)$ we note that $S(0, t)$ is a nonzero polynomial in t. From (i), (ii), (iii), (iv), we conclude that $\mathfrak{h}_1^*(S(x, t))=1$, $\mathfrak{h}_2^*(S(x, t))=2$, and $S(x, t)$ has value zero with respect to any other divisor of K/k which is of first kind for P^{*2} and is at finite distance with respect to the affine k-general point (x, t) of P^{*2} (i.e., other than the divisor of $Y=0$ on P^{*2}). Consequently

(25) $$S(x, t) \doteq \mathfrak{H}_1^*(x, t)\mathfrak{H}_2^*(x, t)^2, \quad \textit{where}$$
$$\mathfrak{H}_j^*(x, t) = 0 \textit{ is an affine equation of } h_j^*.$$

Now $S(x, t)$ is not divisible by any nonconstant polynomial in t because the coefficient of x^3 is -4, and hence $\mathfrak{H}_j^*$ must be of degree at least one in x. Since $S(x, t)$ is of degree 3 in x we can take

$$\mathfrak{H}_1^*(x, t) = -4x+\mu, \; \mathfrak{H}_2^*(x, t) = x-\nu\,; \; \mu,\nu \in k[t].$$

Also $S(x, t) \not\equiv 0 \pmod{x}$ implies that μ and ν are nonzero. Let m and n be respectively the degrees of μ and ν in t.
Now

$$\mathfrak{H}_1^*(x, t)\mathfrak{H}_2^*(x, t)^2 = -4x^3+(\mu+8\nu)x^2+(-2\mu\nu-4\nu^2)x+\mu\nu^2\,.$$

Hence (I) $\mu+8\nu=\alpha$, (II) $-2\mu\nu-4\nu^2=\beta$, (III) $\mu\nu^2=\gamma$. (I) and (III) respectively imply that (I′) degree of $\mu+8\nu$ is 2 and (III′) $m+2n=6$. (III′) implies the following possibilities: (I*) $m=0$, $n=3$; (II*) $m=2$, $n=2$; (III*) $m=4$, $n=1$; (IV*) $m=6$, $n=0$. Because of (I′), (I*, III*, IV*) cannot occur. Therefore we must have (II*) $m=n=2$. Now $\gamma=(t^3+t^2)(-27t^3-27t^2+4)=t^2(t+1)(-27t^3-27t^2+4)$. Now $N(-1)=27-27+4=4\neq 0$ where

$$N(t) = -27t^3-27t^2+4\,,$$

and hence $N(t)$ is not divisible by $t+1$. Consequently in view of the last factorization of γ, (III) and (II*) imply that $N(t)$ must have a double root in k. Now $dN(t)/dt=-81t^2-54t=-27t(3t+2)$. Since $N(0)\neq 0$, $N(t)$ must be divisible by $(3t+2)^2$. We divide $N(t)$ by $(3t+2)^2$ thus:

$$\begin{aligned} N(t) &= -27t^3-18t^2-9t^2-6t+6t+4 = (3t+2)(-9t^2-3t+2) \\ &= (3t+2)(-9t^2-6t+3t+2) = (3t+2)^2(-3t+1)\,. \end{aligned}$$

Consequently

$$\gamma = t^2(3t+2)^2(t+1)(-3t+1)\,.$$

Hence by (III) and (II*) we must have

$$\begin{aligned} \nu &= \delta t(3t+2) = (3\delta t^3+2\delta t)\,, \\ \mu &= \delta^{-2}(t+1)(-3t+1) = (-3\delta^{-2}t^2-2\delta^{-2}t+\delta^{-2})\,, \end{aligned}$$

with $0 \neq \delta \in k$.

By (I) we get

$$(24\delta - 3\delta^{-2})t^2 + (16\delta - 2\delta^{-2})t + \delta^{-2} = \mu + 8\nu = \alpha = -27t^2 - 18t + 1.$$

Hence $\delta^{-2} = 1$ and $24\delta - 3\delta^{-2} = -27$, and hence $\delta^2 = 1$ and $24\delta = -24$, and hence $\delta = -1$. Thus

(26) $\mathfrak{H}_1^*(x, t) = -4x - 3t^2 - 2t + 1$ *and* $\mathfrak{H}_2^*(x, t) = x + 3t^2 + 2t$.

Now

$$\mathfrak{H}_2^*(0, t) = 3t^2 + 2t = t(3t+2);$$
$$\mathfrak{H}_1^*(0, t) = -3t^2 + 2t - 1 = (t+1)(-3t+1);$$

and hence

(27) *h_1^* meets b^* in a_1^* and in another point, h_2^* meets b^* in a_2^* and in another point; all these four points are distinct and hence at each of them $h_1^* \cup h_2^* \cup b^*$ has a 2-fold strong normal crossing.*

From the form of $\mathfrak{H}_1^*$ and $\mathfrak{H}_2^*$ it is clear that $Y = T = 0$ is a common point of h_1^* and h_2^* and that at this point they have the common tangent $Y = 0$. Also $\mathfrak{H}_1^*(1/3, -1/3) = 0 = \mathfrak{H}_2^*(1/3, -1/3)$, and the value of the derivatives of $\mathfrak{H}_1^*$ and $\mathfrak{H}_2^*$ with respect to t at $x = 1/3$, $t = -1/3$, can be seen to be zero and hence $X = 1/3$, $Y = 1$, $T = -1/3$ is also a common point of h_1^* and h_2^* at which they are tangential. Consequently,

(28) *h_1^* and h_2^* meet in two distinct points and at each of them these curves have a 2-fold contact. Hence $\nu(h_j^*, h_1^* \cup h_2^* \cup b^*; P^{*2}) = \nu(h_j^*, h_1^* \cup h_2^*; P^{*2}) = 4$ for $j = 1, 2$; and $\nu(b^*, h_1^* \cup h_2^* \cup b^*; P^{*2}) = \nu(b^*, b^*; P^{*2}) = 0$.*

Referring to (3), (5), (5′), and (6) we deduce that: $\mathfrak{a}_2^*(x) = \mathfrak{a}_2^*(t) = 1$ and t/x is transcendental over k mod $\mathfrak{a}_2^*$. Now (x, t) is a minimal basis of $M(a_2^*, P^{*2})$ and hence by Lemma 7, we conclude that

(29) *$\mathfrak{a}_2^*$ is the a_2^*-adic divisor of P^{*2}.*

Referring to (3), (4), (6) we deduce that: $\mathfrak{a}_1^*(x) = \mathfrak{a}_1^*(t+1) = \mathfrak{a}_1^*((t+1)x^{-1} - 1) = 1$ and $((t+1)x^{-1} - 1)x^{-1}$ is transcendental over k mod $\mathfrak{a}_1^*$. Now $(x, t+1)$ is a minimal basis of $M(a_1^*, P^{*2})$ and the tangent line to h_1^* at a_1^* is $t + 1 = x$ and hence by Lemma 7, we conclude that

(30) *$\mathfrak{a}_1^*$ is the second a_1^*-adic divisor of P^{*2} along h_1^*.*

In view of Lemma 10, from (16), (24), (27), (29), (30) we get the following:

(31) *There exists a regular birational map $\bar{\psi}$ of V^* onto a projective plane $\bar{P}^2$ such that $\bar{\psi}^{-1}$ has only one fundamental point $\bar{w}$, V^* is an immediate quadratic transform of $\bar{P}^2$ with center at $\bar{w}$, $W^*=\bar{\psi}^{-1}(\bar{w})$ is an irreducible curve on V^* different from A_1^*, A_2^*, H_1^*, H_2^*. On V^*, dim $|W^*|=0$, dim $|H_1^*|=5$, dim $|H_2^*|=6$ and dim $|N^*|>0$ for any irreducible curve N^* different from W^*.*

Referring to (4), (6), and (26), we see that

$$\mathfrak{H}_1^*(x,t) = -4x-3t^2-2t+1 = x[-4+(t+1)x^{-1}(-3t+1)];$$

$$(t+1)x^{-1} = 1+(1-z)x+u'; \ (-3t+1) = 4-3x+v',$$

with u' and v' in $\bar{K}$ such that $\mathfrak{a}(u')>1$ and $\mathfrak{a}(v')>1$. Therefore

$$-4+(t+1)x^{-1}(-3t+1) = (1-4z)x+w',$$

with w' in $\bar{K}$ such that $\mathfrak{a}(w')>1$. Hence

(32) *$\mathfrak{H}_1^*(x,t)=x^2(1-4z+w^*)$ where w^* is an element of K^* with $\mathfrak{a}_1^*(w^*)>0$ and consequently $\mathfrak{a}_1^*(\mathfrak{H}_1^*(x,t))=2$.*

Since V^* is nonsingular and birationally equivalent to P^{*2}, by Lemma 30 of [A2] we have:

(33) *V^* is simply connected.*

Since V^* is nonsingular, from (8), (9), (10), (11), and Lemmas 5 and 17 of [A 2] we deduce:

(34) *$[K':K^*]=2$, K'/V^* is tamely ramified, and $\Delta(K'/V^*)=H_1^*$.*

Let X, Y, T, E be homogeneous coordinates in projective three space P^{*3} over k and let $\bar{V}'$ be the surface in P^{*3} given by $F'=0$ where

$$F'(X,Y,T,E) = E^2+4XY+3T^2+2TY-Y^2.$$

By the Jacobian criterion we can verify that $F'=0$ has no singularities and hence

(35) *$\bar{V}'$ is an irreducible nonsingular quadric.*

We may consider P^{*2} as the $E=0$ plane in P^{*3}. Let τ be the

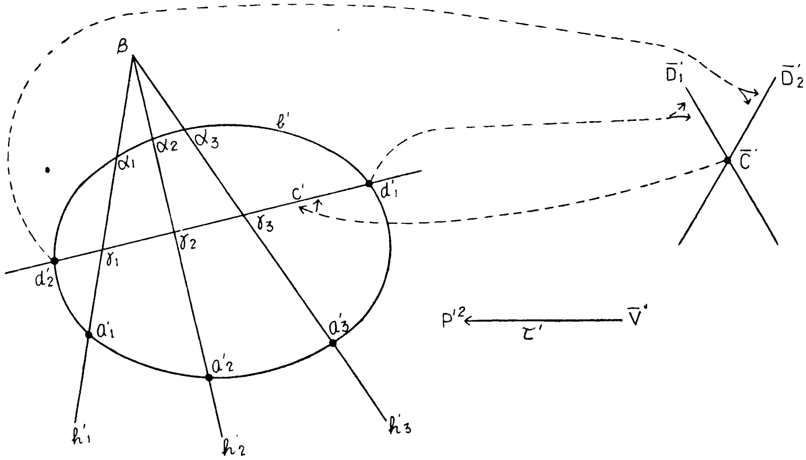

Figure 3

projection of $\bar{V}'$ from $X=Y=T=0$ onto P^{*2}. Then $\bar{V}'$ becomes a covering of P^{*2}. From the form of F' it is clear that

$$(36) \qquad \Delta(\bar{V}'/P^{*2}) = h_1^* .$$

Let e be a root of

$$f'(E) = F'(x, 1, t, E) = E^2+4x+3t^2+2t-1 = E^2-\mathfrak{H}_1^*(x, t)$$

in some algebraic extension of K'. Then (x, t, e) is an affine k-general point of $\bar{V}'$ and $k(\bar{V}')=K^*(e)$.

Now $f'(E)\equiv E^2-1\equiv(E+1)(E-1)$ $(\bmod\ \mathfrak{a}_2^*)$ and hence, in view of Lemma 7, and (29) we conclude the following:

(37) $\mathfrak{a}_2^*$ *is unramified in* $k(\bar{V}')$ *and splits into two valuations* $\mathfrak{a}_2'$ *and* $\mathfrak{a}_3'$; *the center of* $\mathfrak{a}_2'$ *on* $\bar{V}'$ *is the point* $\bar{A}_2'$: $(X=T=0, E=1, Y=1)$ *and* $\mathfrak{a}_2'$ *is the* $\bar{A}_2'$*-adic divisor of* $\bar{V}'$; *the center of* $\mathfrak{a}_3'$ *on* $\bar{V}'$ *is the point* $\bar{A}_3'$: $(X=T=0,\ E=-1,\ Y=1)$ *and* $\mathfrak{a}_3'$ *is the* $\bar{A}_3$*-adic divisor of* $\bar{V}'$.

In view of (32), the form of f' tells us that $\mathfrak{a}_1^*$ is unramified in $k(\bar{V}')/k$. This together with (37), (36), (24), (16) implies that $\Delta(k(\bar{V}')/V^*)=H_1^*$. Consequently in view of (16), (33), (34), and Lemma 36 of [A 2] we conclude that

$$(38) \qquad K^*(e) = k(\bar{V}') = K' .$$

Consequently (6) and the form of f' tell us that:

(39) $\mathfrak{a}_1^*$ *is unramified in* K' *and has a unique extension* $\mathfrak{a}_1'$. *Also* $g(\mathfrak{a}_1':\mathfrak{a}_1^*)=g(\mathfrak{a}_1':\mathfrak{a})=2$. *The center of* $\mathfrak{a}_1'$ *on* $\bar{V}'$ *is the point* $\bar{A}_1'$: $(X=0,\ T=-1,\ E=0,\ Y=1)$.

(11) now implies that:

(40) $\mathfrak{h}_2^*$ *is unramified in* K' *and splits into two valuations* $\mathfrak{h}_2'$ *and* $\mathfrak{h}_3'$, $d(\mathfrak{h}_2':\mathfrak{h}_2^*)=1=d(\mathfrak{h}_3':\mathfrak{h}_2^*)$. $\mathfrak{h}_1^*$ *has a unique extension* $\mathfrak{h}_1'$ *to* K' *and* $d(\mathfrak{h}_1':\mathfrak{h}_1^*)=r(\mathfrak{h}_1':\mathfrak{h}_1^*)=\bar{r}(\mathfrak{h}_1':\mathfrak{h}_1^*)=2$.

Let $\bar{C}'$ be the point $Y=T=E=0$ of P^{*3} and let P'^2 be the plane $X=0$ in P^{*3}. Then $\bar{C}'$ is a (simple) point of $\bar{V}'$ and $\bar{C}'\notin P'^2$. Let τ' be the projection of $\bar{V}'$ from $\bar{C}'$ onto P'^2. Then (x, t, e) and (t, e) are corresponding affine k-general points of $\bar{V}'$ and P'^2 respectively, so that $k(P'^2)=k(t, e)=K'=k(\bar{V}')$. Let c' be line $Y=0$ on P'^2, let $\sqrt{-3}$ be a square root of -3 in k, let d_1' be the point $(Y=0,\ T=1,\ E=\sqrt{-3})$ of P'^2, let d_2' be the point $(Y=0,\ T=1,\ E=-\sqrt{-3})$ of P'^2, and let $\bar{D}_j'$ be the line (in P^{*3}) joining d_j' and $\bar{C}'$. Then the tangent cone to $\bar{V}'$ at $\bar{C}'$ intersects $\bar{V}'$ and P'^2 in $\bar{D}_1'\cup\bar{D}_2'$ and c' respectively. Therefore by Lemma 8, we have the following description for the birational map τ':

(41) τ' *is biregular on* $\bar{V}'-\bar{D}_1'-\bar{D}_2'$ *and* $\tau'(\bar{V}'-\bar{D}_1'-\bar{D}')=P'^2-c'$. *The only fundamental point of* τ' *is* $\bar{C}'$ *and* $\tau'(\bar{C}')=c'$. *The only fundamental points of* τ'^{-1} *are* d_1' *and* d_2'; $\tau'^{-1}(d_j')=\bar{D}_j'$; *and denoting by* $\mathfrak{d}_j'$ *the unique (real discrete) valuation of* K'/k *having center* $\bar{D}_j'$ *on* $\bar{V}'$ *we have that* $\mathfrak{d}_j'$ *is the* d_j'*-adic divisor of* P'^2.

None of the points $\bar{A}_j'$ are on $\bar{D}_1'\cup\bar{D}_2'$ and hence by (41), $\tau'(\bar{A}_j')$ consists of a unique point a_j' of P'^2 and these points are given by

(42) a_1': $(Y=1,\ T=-1,\ E=0)$; a_2': $(Y=1,\ T=0,\ E=1)$;
a_3': $(Y=1,\ T=0,\ E=-1)$.

By (37), (39), (41), we have that $\mathfrak{a}_2'$ is the a_2'-adic divisor of P'^2, $\mathfrak{a}_3'$ is the a_3'-adic divisor of P'^2, and $\mathfrak{a}_1'$ has center a_1' op P'^2. Now let $\sigma=t+1$. Then (σ, e) is a minimal basis of $M(a_1', P'^2)$. Now $e^2=\mathfrak{H}_1^*(x, t)$ and hence from (3), (4), (6), (32), (39) we deduce that $\mathfrak{a}_1'(x)=\mathfrak{a}_1'(e)=\mathfrak{a}_1'(\sigma)=1$ and that $(e/x)^2$ is transcendental over k mod $\mathfrak{a}_1'$ and hence e/x is transcendental over k mod $\mathfrak{a}_1'$. By (4), (6) and (39), $\sigma/x\equiv 1 \pmod{\mathfrak{a}_1'}$ and hence e/σ is transcendental over k mod

$\mathfrak{a}_1'$. In view of Lemma 7 we can now state the following:

(43) *For* $j=1, 2, 3$; $\mathfrak{a}_j'$ *has center* a_j' *on* P'^2 *and* $\mathfrak{a}_j'$ *is the* a_j'*-adic divisor of* P'^2.

Let H_j' be the center of $\mathfrak{h}_j'$ on P'^2. Now $\tau(\bar{D}_1' \cup \bar{D}_2')$ is the line d^* on P^{*2} given by $Y=0$. It is clear that $h_1^* \not\subset d^*$ and $h_2^* \not\subset d^*$ and hence (41) tells us that h_j' is an irreducible curve on P'^2 which is at finite distance with respect to the k-general point (t, e). From the equation $F'=0$ of $\bar{V}'$ it is clear that h_1' is $\mathfrak{H}_1'=0$ where $\mathfrak{H}_1'=E$. Now

$$\begin{aligned} 4\mathfrak{H}_2^*(x, t) &= 4x+12t^2+8t \\ &= 4x+3t^2+2t-1+9t^2+6t+1 \\ &= -e^2+(3t+1)^2 \\ &= (3t+1-e)(3t+1+e). \end{aligned}$$

Consequently, after relabelling $\mathfrak{h}_2'$, $\mathfrak{h}_3'$ suitably we have that h_2' is given by $\mathfrak{H}_2'=0$ and h_3' is given by $\mathfrak{H}_3'=0$ where $\mathfrak{H}_2'(Y, T, E)=Y+3T-E$ and $\mathfrak{H}_3'(Y, T, E)=Y+3T+E$.

Now consider the line b^* which is given in P^{*2} by $X=0$. Let $\bar{B}'=\tau^{-1}(b^*)$ and $b'=\tau'[\bar{B}']$. Again b^* is not contained in d^* and hence b' is a curve in P'^2. Substituting $X=0$ in F' we have that b' is contained in the curve $\mathfrak{B}'=0$ where $\mathfrak{B}'=E^2+3T^2+2TY-Y^2$. By Jacobian criterion, $\mathfrak{B}'=0$ is a nonsingular irreducible conic and hence

(44) $b'=\tau'[\tau^{-1}(b^*)]$ *is the irreducible nonsingular conic* $E^2+3T^2+2TY-Y^2=0$ *in* P'^2.

Let $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma_1, \gamma_2, \gamma_3$ be the points in P'^2 given respectively by α_1: $(Y=1,\ T=1/3,\ E=0)$;

α_2: $(Y=1,\ T=-2/3,\ E=-1)$; α_3: $(Y=1,\ T=-2/3,\ E=1)$;
β: $(Y=1,\ T=-1/3,\ E=0)$; γ_1: $(Y=0,\ T=1,\ E=0)$;
γ_2: $(Y=0,\ T=1,\ E=3)$; γ_3: $(Y=0,\ T=1,\ E=-3)$.

Then it can be verified that:

(45) *The twelve points* $\alpha_1, \alpha_2, \alpha_3, \beta, a_1', a_2', a_3', d_1', d_2', \gamma_1, \gamma_2, \gamma_3$ *are all distinct.* b' *meets* c' *in* d_1' *and* d_2'. *For* $j=1, 2, 3$; b' *meets* h_j' *in* a_j' *and* α_j; *and* c' *meets* h_j' *in* γ_j. *The lines* h_1', h_2', h_3' *have the point* β *in common. Consequently all the singularities of the curve* $h_1' \cup h_2' \cup h_3' \cup b' \cup c'$ *lying on* b' *are strong normal crossings.*

Now let V' be a K'-normalization of V^*. Let ϕ_1 be the map of V' onto V^* and let ϕ' be the map of V' onto V, i.e., $\phi'=\phi^*\phi_1$. Let τ_1 be the map of V' onto $\bar{V}'$, and let ψ' be the map of V' onto P'^2, i.e., $\psi'=\tau'\tau_1$. Now $\tau(\bar{C}')$ is the point on P^{*2} given by $Y=T=0$ and this is not on $b^*: X=0$ and hence by (24), $\tau_1^{-1}(\bar{C}')$ consists of a single point C' of V' and $Q(C', V')=Q(\bar{C}', \bar{V}')$. By (15), H_1^* contains B^*; and by (31), $\Delta(K'/V^*)=H_1^*$ and $[K':K^*]=2$; consequently $\phi_1^{-1}(B^*)$ consists of a single point B' of V' and $B'\neq C'$. From (24), (37), (41), (43), we deduce the following:

(46) *Let H'_j be the center of $\mathfrak{h}'_j$ on V'. Then H'_1, H'_2, H'_3 are distinct irreducible curves on V'. ψ' is a birational map of V' onto the projective plane P'^2. The centers of $\mathfrak{d}'_1, \mathfrak{d}'_2, \mathfrak{a}'_1, \mathfrak{a}'_2, \mathfrak{a}'_3$ are distinct irreducible curves $D'_1, D'_2, A'_1, A'_2, A'_3$ on V'. ψ' is biregular on $V'-D'_1-D'_2-A'_1-A'_2-A'_3$ and $\psi'(V'-D'_1-D'_2-A'_1-A'_2-A'_3)=P'^2-b'-c'$. The only fundamental points of ψ' are two distinct points B' and C' and $\psi'(B')=b'$ and $\psi'(C')=c'$. The only fundamental points of ψ'^{-1} are $d'_1, d'_2, a'_1, a'_2, a'_3$ and $\psi'^{-1}(d'_j)=D'_j$ and $\psi'^{-1}(a'_j)=A'_j$. $\mathfrak{d}'_j$ is the d'_j-adic divisor of P'^2 and $\mathfrak{a}'_j$ is the a'_j-adic divisor of P'^2.*

Let W be the transform of P'^2 obtained by applying to P'^2 immediate successive quadratic transformations with centers at $d'_1, d'_2, a'_1, a'_2, a'_3$ respectively; let α be the regular map of W onto P'^2 and let β be the map of W onto V'. Then from (46) we deduce that β is regular, B' and C' are the only fundamental points of β^{-1} and $\beta^{-1}(B')$ is the irreducible curve $\alpha^{-1}[b']$ and $\beta^{-1}(C')$ is the irreducible curve $\alpha^{-1}[c']$. Therefore by the Zariski factorization theorem [$Z1$, Section 24] W can be obtained from V' by applying immediate successive quadratic transformations with centers at B' and C' respectively. From this we conclude that

(47) *$\mathfrak{b}'$ is the B'-adic divisor of V' and $\mathfrak{c}'$ is the C'-adic divisor of V'.*

Observing that $\tau(\bar{C}')$ is a common point of h_1^* and h_2^*, and that B^* is a common point of H_1^* and H_2^*; from (16), (23), (34), and [$A4$, Lemma 5] we deduce the following:

(48) *H'_1, H'_2, H'_3 and V' are all nonsingular. Above each of the three singularities of H (or of $H_1^*\cup H_2^*$) lies exactly one point of V'; each of these three points is a common point of H'_1,*

H'_2, H'_3 and $H'_1\cup H'_2\cup H'_3$ has a 3-fold ordinary point there. $H'_1\cup H'_2\cup H'_3$ has no singularities outside these three points. B' and C' are among the singularities of $H'_1\cup H'_2\cup H'_3$. Also $\nu(H'_j, H'_1\cup H'_2\cup H'_3; V')=3$ for $j=1, 2, 3$.

Now we wish to determine dim $|H_j|$ on V'. From (46), (47), (48), it follows that for an element γ of K'/k we have $(\gamma)+H'_1\geqq 0$ on V' if and only if $(\gamma)+h'_1+b'+c'\geqq 0$ on P', and $\mathfrak{a}'_j(\gamma)\geqq 0$ for $j=1, 2, 3$, and $\mathfrak{d}'_i(\gamma)\geqq 0$ for $j=1, 2$. Consequently dim $|H'_1|$ on V' equals the dimension of the linear system S of quartics in P'^2 passing through a'_2, a'_3 and having at least double points at a'_1, d'_1, d'_2. Since the points a'_1, a'_2, a'_3, d'_1, d'_2 are on the irreducible conic b'; no three of them can be colinear. Hence we can choose a projective coordinate system $(\bar{X}, \bar{Y}, \bar{Z})$ in P'^2 such that a'_1, a'_2, a'_3, d'_1, d'_2 are respectively given by (1, 0, 0), (1, 1, 1), (1, u, v), (0, 1, 0), (0, 0, 1) with $u\neq 0\neq v$. Let s be a member of S. Since s has at least double points at a'_1, d'_1, d'_2, the equation of s must be of the form $\mathfrak{S}=0$ with $\mathfrak{S}=\alpha_1\bar{Y}^2\bar{Z}^2+\alpha_2\bar{X}^2\bar{Z}^2+\alpha_3\bar{X}^2\bar{Y}^2+\beta_1\bar{X}^2\bar{Y}\bar{Z}+\beta_2\bar{X}\bar{Y}^2\bar{Z}+\beta_3\bar{X}\bar{Y}\bar{Z}^2$. Since s contains a'_1 and a'_2 we must have

$$\alpha_1+\alpha_2+\alpha_3+\beta_1+\beta_2+\beta_3 = 0 \text{ and}$$
$$\alpha_1u^2v^2+\alpha_2v^2+\alpha_3u^2+\beta_1uv+\beta_2u^2v+\beta_3uv^2 = 0\,.$$

It can be easily checked that $u\neq 0\neq v$ and $(1, u, v)\neq(1, 1, 1)$ imply that these two linear conditions on α_1, α_2, α_3, β_1, β_2, β_3 are linearly independent. Consequently S is of dimension 3. Similarly treating the cases of H'_2 and H'_3 we thus have:

(49) $$\dim|H'_j| = 3 \ (\textit{on } V') \textit{ for } j = 1, 2, 3\,.$$

For future reference, we shall summarize some of the results of Section 3 and this section in the following proposition.

Proposition 3. *Let P^2 be a projective two space over an algebraically closed ground field k of characteristic $p\neq 2, 3$. Let (x, z) be an affine k-general point of P^2 so that $k(P^2)=K=k(x, z)$. Let H be the curve in P^2 given by the affine equation $1-4x-4z-27x^2z^2+18xz=0$. Let K' be a splitting field over K of the polynomial $T^3+T^2+xT+x^2z$. Let V' be a K'-normalization of V and let ϕ' be the rational map of V' onto P^2. Then: (I) H is an irreducible three cuspidal quartic. (II) K'/P^2 is tamely ramified and $\Delta(K'/P^2)=H$. (III) $G(K'/K)$ is the symmetric group S_3 on three symbols and hence $[K':K]=6$. (IV) $\phi'^{-1}(H)$ consists of three distinct irreducible* *curves* H'_1, H'_2, H'_3. (*V*) H'_1, H'_2, H'_3 *and* V' *are all nonsingular. There exists a birational map* ψ' *of* V' *onto a projective plane* P'^2 *over* k *with the following properties*: (*VI*) ψ' *has exactly two fundamental points* B' *and* C' *on* V' *and* ψ'^{-1} *has exactly five fundamental points* $a'_1, a'_2, a'_3, d'_1, d'_2$ *on* P'^2. (*VII*) $b'=\psi'(B')$ *is a nonsingular conic in* P'^2 *and* $c'=\psi'(C)$ *is a line in* P'^2. (*VIII*) $A'_1=\psi'^{-1}(a'_1)$, $A'_2=\psi'^{-1}(a'_2)$, $A'_3=\psi'^{-1}(a'_3)$, $D'_1=\psi'^{-1}(d'_1)$, $D'_2=\psi'^{-1}(d'_2)$ *are distinct irreducible curves on* V' *other than* H'_1, H'_2, H'_3. (*IX*) $h'_1=\psi'[H'_1]$, $h'_2=\psi'[H'_2]$, $h'_3=\psi'[H'_3]$ *are distinct lines in* P'^2 *other than* c'. (*X*) *The* (*unique*) *valuation of* K'/k *having center* A'_j *on* V' *is the* a'_j-*adic divisor of* P'^2 *and the* (*unique*) *valuation of* K'/k *having center* D'_j *on* V' *is the* d'_j-*adic divisor of* P'^2. (*XI*) *The three lines* h'_1, h'_2, h'_3 *meet in a point*; b' *and* c' *intersect in* d'_1 *and* d'_2; $a'_j \in b' \cap h'_j$ *for* $j=1, 2, 3$; *all the singularities of the curve* $h'_1 \cup h'_2 \cup h'_3 \cup b' \cup c'$ *lying on* b' *are strong normal crossings.* (*XII*) *dim* $|H'_j|=3$ (*on* V') *for* $j=1, 2, 3$.

7. The surface $T^3+YXT+X^2Z=0$.

Let X, Y, Z, T be projective coordinates in projective three space P^3 over an algebraically closed ground field k of characteristic $p\neq 2, 3$. Let V be the irreducible cubic surface $F=0$ where

$$F = (X, Y, Z, T) = T^3+YXT^2+X^2Z\,.$$

Let P^2 be the (X, Y, Z)-plane and let ϕ be the projection of V onto P^2 from the point $X=Y=Z=0$. Let $x=X/Y$, $z=Z/Y$, and $K=k(x, z)=k(P^2)$. Let t be a root of

$$f(T) = F(x, 1, z, T)= T^3+xT+x^2z$$

in some extension of K, let $K^*=K(t)=k(V)$, and let K' be a least galois extension of K containing K^*. Let A be the line $X=0$ on P^2 and let H be the curve on P^2 given by $\mathfrak{H}=0$ where

$$\mathfrak{H}(X, Y, Z) = 4Y^3+27Z^2X\,.$$

Then by Lemma 1 of Section 1,

(1) *H is an irreducible cubic curve with a 2-fold cusp at* $Q_1: Y=Z=0$ *as the only singularity and* $Q_2: X=Y=0$ *as the only flex.* A *is the flex tangent to* H *and* Q_1 *and* Q_2 *are the only singularities of* $A\cup H$.

Let $\mathfrak{a}$ and $\mathfrak{h}$ be the real discrete valuations of K/k respectively having centers A and H on P^2. Mod $\mathfrak{a}$, z is transcendental over k and generates the residue field of $\mathfrak{a}$ over k and the completion of K with respect to $\mathfrak{a}$ is $k(z)((x))$. Let $x^{\frac{1}{2}}$ be a square root of x in some extension of $k(z)((x))$. Let

$$g(T) = (x^{\frac{1}{2}})^{-3}f(x^{\frac{1}{2}}T).$$

Then

$$g(T) = T^3+T+zx^{\frac{1}{2}} \equiv (T^2+1)T \pmod{x^{\frac{1}{2}}}.$$

Hence by Hensel's Lemma, $g(T)$ and hence $f(T)$ factors completely in $k(z)((x^{\frac{1}{2}}))$. Since $[k(z)((x^{\frac{1}{2}})) : k(z)((x))]=2$ and $f(T)$ is of degree 3, we conclude that $f(T)$ is reducible in $k(z)((x))$. Since the coefficients of T^2, T in $f(T)$ are of positive $\mathfrak{a}$-value and since the constant term in $f(T)$ has $\mathfrak{a}$-value two, we must have

$$f(T) = (T+\sum_{i=1}^{\infty}\alpha_i x^i)(T^2+(\sum_{i=1}^{\infty}\beta_i x^i)T+\sum_{i=1}^{\infty}\gamma_i x^i)$$
$$\text{with} \quad \alpha_i,\ \beta_i,\ \gamma_i \in k(z).$$

Mod x^3, the right-hand side equals

$$T^3+[(\alpha_1+\beta_1)x+(\alpha_2+\beta_2)x^2]T^2+[\gamma_1 x+(\alpha_1\beta_1+\gamma_2)x^2]T+\alpha_1\gamma_1 x^2.$$

Equating coefficients with $f(T)$ we get $\gamma_1=1$ and $\alpha_1\gamma_1=z$, i.e., $\gamma_1=1$ and $\alpha_1=z$. Therefore

$$(2)\quad f(T) = (T+zx+\sum_{i=2}^{\infty}\alpha_i x^i)(T^2+(\sum_{i=1}^{\infty}\beta_i x^i)T+x+\sum_{i=2}^{\infty}\gamma_i x^i),$$
$$\text{with} \quad \alpha_i,\ \beta_i,\ \gamma_i \text{ in } k(z).$$

By Eisenstein's theorem and the results of Section 2 of [$A1$], we conclude that

(3) *$\mathfrak{a}$ has two extensions $\mathfrak{a}_1^*$ and $\mathfrak{a}_2^*$ in K^* and* $r(\mathfrak{a}_1^* : \mathfrak{a})=\bar{r}(\mathfrak{a}_1^* : \mathfrak{a})=1$ *and* $r(\mathfrak{a}_2^* : \mathfrak{a}) =\bar{r}(\mathfrak{a}_2^* : \mathfrak{a})=2$.

By (3), Lemma 6 of Section 2, Theorem 3 of Section 13 of [$A2$], and the technique of the proof of (11) of Section 3, we deduce the following:

(4) *K^*/K is not galois; $G(K'/K)$ is isomorphic to the symmetric group S_3 on three letters; $\mathfrak{a}_1^*$ has a unique extension $\mathfrak{a}_1'$ to K'; $\mathfrak{a}_2^*$ has two extensions $\mathfrak{a}_2'$ and $\mathfrak{a}_3'$ to K'; $\mathfrak{h}$ has two extensions $\mathfrak{h}_1^*$ and $\mathfrak{h}_2^*$ to K^*; $\mathfrak{h}_1^*$ has a unique extension $\mathfrak{h}_1'$ to K'; $\mathfrak{h}_2^*$ has two extensions $\mathfrak{h}_2'$ and $\mathfrak{h}_3'$ to K';* $r(\mathfrak{a}_1' : \mathfrak{a}_1^*)=r(\mathfrak{a}_1' : \mathfrak{a})=2$; $r(\mathfrak{a}_2' : \mathfrak{a}_2^*)=r(\mathfrak{a}_3' : \mathfrak{a}_2^*)=1$; $r(\mathfrak{h}_1^* : \mathfrak{h})=1$; $r(\mathfrak{h}_2^* : \mathfrak{h})=2$; $r(\mathfrak{h}_1' : \mathfrak{h}_1^*)=$

$r(\mathfrak{h}_1' : \mathfrak{h})=2$; $r(\mathfrak{h}_2' : \mathfrak{h}_2^*)=r(\mathfrak{h}_3' : \mathfrak{h}_2^*)=1$; $g(\mathfrak{a}_j' : \mathfrak{a})=g(\mathfrak{h}_j' : \mathfrak{h})=1$ *for all* j. *Also* K^*/P^2 *and* K'/P^2 *are tamely ramified and* $\Delta(K^*/P^2)=\Delta(K'/P^2)=A\cup H$.

By (4), there exists a field K_1 between K' and K such that K_1/K is cyclic of degree 2, K_1/P^2 is tamely ramified and $\Delta(K_1/P^2) \subset A\cup H$. By Section 13 of [A2], it follows that $\Delta(K_1/P^2)=A\cup H$ and that there exists a primitive element $\bar{e}$ of K_1/K with the minimal monic polynomial

$$E^2-x^3\mathfrak{H}(x, 1, z)\,.$$

Since $[K_1 : K]=2$ and $[K^* : K]=3$, it follows that $K'=K^*(\bar{e})$ and that the above polynomial is also the minimal monic polynomial of $\bar{e}$ over K^*. From $f(t)=0$ we get

$$zx^2 = -(t^3+xt)$$

and hence

$$\begin{aligned} x^3\mathfrak{H}(x, 1, z) &= 4x^3+27z^2x^4 = 4x^3+27(t^3+xt)^2 \\ &= 4x^3+27t^2x^2+54t^4x+27t^6\,, \end{aligned}$$

so that

(5) $$x^3\mathfrak{H}(x, 1, z) = \mathfrak{H}_1^*(x, t)\mathfrak{H}_2^*(x, t)^2\,,$$
$$\textit{where}\quad \mathfrak{H}_1^*(x, t) = 4x+3t^2$$
$$\textit{and}\quad \mathfrak{H}_2^*(x, t) = x+3t^2\,.$$

Let $e=\bar{e}(x+3t^2)^{-1}$. Then

(6) $K'=K^*(e)$ *and the minimal monic polynomial of* e *over* K^* *is* $E^2-(4x+3t^2)$.

Let V^* be a normalization of V, let ψ_1 be the map of V^* onto V and let $\phi^*=\phi\psi_1$. Let B be the double point $X=Y=T=0$ of V. Substituting $X=Y=0$ in F we find that $B=\phi^{-1}(Q_2)$. Let $x_1=x/z$, $\eta=1/z$, $t_1=t/z$. Then $t_1^3+\eta x_1t_1+x_1^2=0$. Let $\xi=x_1/t_1$. Then $t_1+\eta\xi+\xi^2=0$, i.e., $t_1=-(\eta\xi+\xi^2)$. Now $x=\xi t_1/\eta$ and $t=t_1/\eta$. As in the proof of (12), (13), (14) of Section 3, we can deduce that $\psi_1^{-1}(B)$ consists of a single point B^* of V^*, B^* is a simple point of V^*, and (ξ, η) is a basis of $M(B^*, V^*)$. Consequently we get the following:

(7) $\phi^{-1}(Q_2)$ *consists of a single point* B. $\phi^{*-1}(Q_2)=\psi_1^{-1}(B)$ *consists of a single point* B^*. B^* *is a simple point of* V^*. $M(B^*, V^*)$

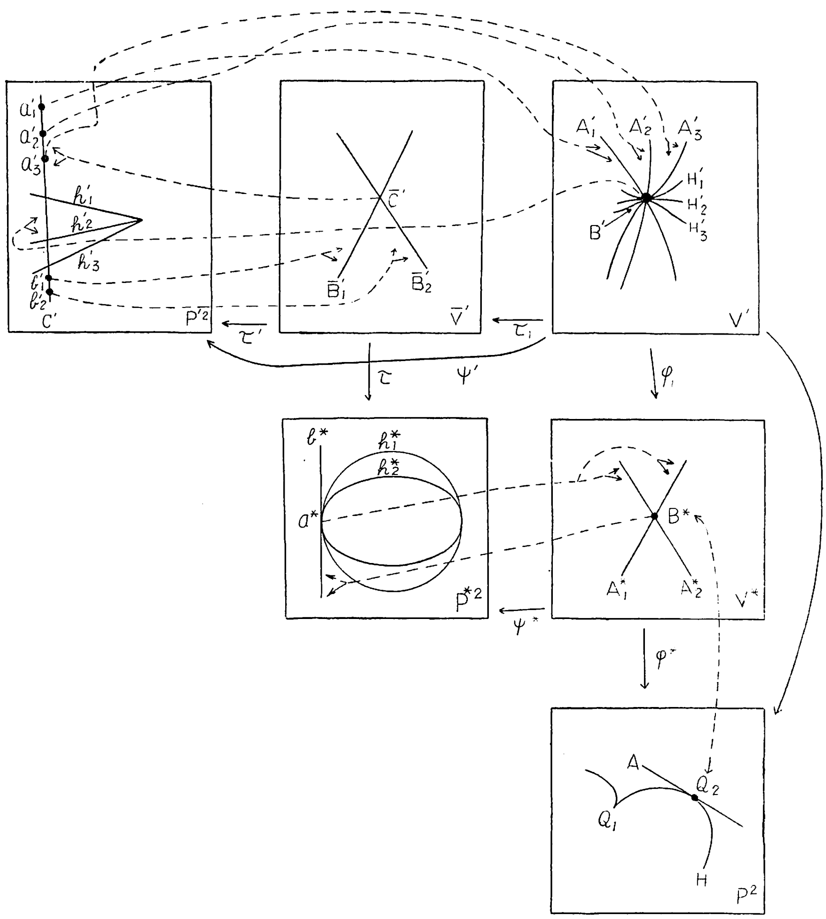

Figure 4

has a certain basis (ξ, η) *such that* $x=-\xi^2(\xi+\eta)/\eta$ *and* $t=-\xi(\xi+\eta)/\eta$.

Let A_1 be the line $X=T=0$ of V. Then $\phi^{-1}(A)=A_1$ and hence by (3) it follows that

(8) *For almost all* (*i.e., all except a finite number*) *points* D *of* A_1, $\psi_1^{-1}(D)$ *consists of two distinct points of* V^*, *i.e.,* V *is analytically reducible at almost all points of* A_1.

Let ψ be the projection of V from B onto the (X, Y, T)-plane P^{*2} and let $\psi^*=\psi\psi_1$. The tangent cone to V at B is $X=0$, it intersects P^{*2} and V respectively in the lines $b^*: X=0$ and A_1, which meets P^{*2} in the point $a^*: X=T=0$. Let H_j^*, A_j^* be the centers respectively of $\mathfrak{h}_j^*$ and $\mathfrak{a}_j^*$ on V^* and let h_j^* and a_j^* be the centers respectively of $\mathfrak{h}_j^*$ and $\mathfrak{a}_j^*$ on P^{*2}. By Lemma 8 of Section 4 we conclude that:

(9) *ψ^* is a birational map of V^* onto P^{*2}. ψ^* is biregular on $V^*-A_1^*-A_2^*$. $\psi^*(V^*-A_1^*-A_2^*)=P^{*2}-b^*$. B^* is the only fundamental point of ψ^* and $\psi^*(B^*)=b^*$. a^* is the only fundamental point of ψ^{*-1} and $\psi^{*-1}(a^*)=A_1^*\cup A_2^*$.*

As in the proof of (25) of Section 6, by (5) we deduce that

(10) *$h_j^*=\psi^*[H_j^*]$ is the irreducible conic $\mathfrak{H}_j^*=0$.*

By direct verification we have that:

(11) *At a^* each of the pairs h_1^* and b^*, h_2^* and b^*, h_1^* and h_2^* has a 2-fold contact. b^* and h_j^* have no other common point. Outside a^*, the intersection of h_1^* and h_2^* consists of the point $Y=T=0$ at which they have a 2-fold contact.*

Let (X, Y, T, E) be homogeneous coordinates in projective three space P^{*3} over k, let $\bar{V}'$ be the nonsingular quadric in P^{*3} given by $F'=0$ where

$$F'(X, Y, T, E) = E^2-4XY-3T^2,$$

and considering P^{*2} to be the plane $E=0$ in P^{*3} let τ be the projection of $\bar{V}'$ from the point $X=Y=T=0$ onto P^{*2}. Then $\bar{V}'$ is a K'-normalization of P^{*2}. Let τ' be the projection of $\bar{V}'$ from the point $\bar{C}': X=T=E=0$ of $\bar{V}'$ onto the (X, T, E)-plane P'^2. Then $k(P'^2)=k(t', e')$ where $t'=t/x$ and $e'=e/x$. The tangent cone to $\bar{V}'$ at $\bar{C}'$ is $X=0$; it intersects P'^2 and $\bar{V}'$ respectively in the line $c': X=0$ and in the lines $\bar{B}_1': (X=E+\sqrt{3}T=0)$ and $\bar{B}_2': (X=E-\sqrt{3}T=0)$ where $\sqrt{3}$ is a square root of 3 in k. The lines $\bar{B}_1'$ and $\bar{B}_2'$ meet P'^2 in the points $b_1': (X=E+\sqrt{3}T=0)$ and $b_2': (X=E-\sqrt{3}T=0)$ respectively. By Lemma 8 of Section 4 we have that:

(12) *τ' is biregular on $\bar{V}'-\bar{B}_1'-\bar{B}_2'$ and maps it onto P'^2-c'. The only fundamental point of τ' is $\bar{C}'$ and we have $\tau'(\bar{C}')=c'$. The only fundamental points of τ'^{-1} are b_1' and b_2' and we have $\tau'^{-1}(b_j')=\bar{B}_j'$.*

Let $\bar{H}_j'$ and h_j' be the centers of $\mathfrak{h}_j'$ on $\bar{V}'$ and P'^2 respectively.

Eliminating Y between (5) and $F'(T)$ and labelling $\mathfrak{h}'_2$, $\mathfrak{h}'_3$ suitably, we get that

(13) *h'_1 is the line $E=0$, h'_2 is the line $E+\sqrt{-9}T=0$, h'_3 is the line $E-\sqrt{-9}T=0$, where $\sqrt{-9}$ is a square root of -9 in k. The lines h'_1, h'_2, h'_3 meet in the point $E=T=0$ which is not on c'. Consequently, h'_1, h'_2, h'_3 meet c' in three distinct points. Also $\tau'[\bar{H}'_j]=h'_j$.*

From (2), (3), (4), we deduce that $\mathfrak{a}'_1(x)=\mathfrak{a}'_1(t)=2$ and hence from $f'(e)=0$ we get $\mathfrak{a}'_1(t')=0$ and $\mathfrak{a}'_1(e')=-1$. Therefore the center of $\mathfrak{a}'_1$ on P'^2 is the point $a'_1: X=T=0$. Again from (2) and (3) we get that $\mathfrak{a}^*_2(t)=1$, $\mathfrak{a}^*_2(x)=2$, and $t^2/x\equiv-1 \pmod{\mathfrak{a}^*_2}$. Therefore $4x+3t^2=x(4+3t^2/x)$ and $4+3t^2/x=1 \pmod{\mathfrak{a}^*_2}$. Hence from $f'(e)=0$ we get that $\mathfrak{a}^*_2(e^2)=2$ and $(e/t)^2\equiv-1 \pmod{\mathfrak{a}^*_2}$. Hence by (4) we get that $\mathfrak{a}'_2(t')=\mathfrak{a}'_3(t')=\mathfrak{a}'_2(e')=\mathfrak{a}'_3(t')=-1$ and that after a suitable labelling of $\mathfrak{a}'_2$, $\mathfrak{a}'_3$ we have $e'/t'=e/t\equiv\sqrt{-1} \bmod \mathfrak{a}'_2$ and $e'/t'=e/t\equiv-\sqrt{-1} \pmod{\mathfrak{a}'_3}$. Therefore the centers of $\mathfrak{a}'_2$ and $\mathfrak{a}'_3$ on P'^2 are respectively the points $a'_2: (X=E-\sqrt{-1}T=0)$ and $a'_3: (X=E+\sqrt{-1}T=0)$ on the line c'. Note that a'_1, a'_2, a'_3, $h'_1\cap c'$, $h'_2\cap c'$, $h'_3\cap c'$, b'_1, b'_2 are all distinct points.

Let V' be a K'-normalization of V^*. Let A'_j and H'_j be the centers on V' of $\mathfrak{a}'_j$ and $\mathfrak{h}'_j$ respectively. Let τ_1, ψ', ϕ_1, ϕ' be the maps of V' onto $\bar{V}'$, P'^2, V^*, P^2 respectively. Then $\psi'=\tau'\tau_1$ and $\phi'=\phi^*\phi_1$. Since $Q_2=\phi^*(B^*)$ is on H and since $r(\mathfrak{h}'_1:\mathfrak{h}^*_1)=2=[K':K^*]$, we conclude that $\phi_1^{-1}(B^*)$ consists of a single point B' and hence by (7), $B'=\phi'^{-1}(Q_2)$. Since Q_2 is the only common point of H and A; given i and j, A'_i meets H'_j only in B'. Also B' is a common point of all the six curves H'_1, H'_2, H'_3, A'_1, A'_2, A'_3. From (9) and (12) and from all the considerations made after (13) up to this point, we now deduce the following:

(14) *The centers H'_1, H'_2, H'_3, A'_1, A'_2, A'_3 respectively of $\mathfrak{h}'_1$, $\mathfrak{h}'_2$, $\mathfrak{h}'_3$, $\mathfrak{a}'_1$, $\mathfrak{a}'_2$, $\mathfrak{a}'_3$ on V' are distinct curves and they have only one point B' in common. For any i, j; B' is the only common point of A'_i and H'_j. ψ' is a birational map of V' onto P'^2. ψ' is biregular on $V'-A'_1-A'_2-A'_3$ and maps it onto P'^2-c' where c' is a line in P'^2. B' is the only fundamental point of ψ' and we have $\psi'(B)=c'$. ψ'^{-1} has exactly three fundamental points a'_1, a'_2, a'_3. The point a'_j is on the line c' and we have $\psi'^{-1}(a'_j)=A'_j$. $h'_1=\psi'[H'_1]$, $\psi'[H'_2]$, $h'_3=\psi'[H'_3]$ are distinct lines in P'^2 meeting in a point d' not on c'. The points a'_1, a'_2, a'_3,*

$h_1'\cap c'$, $h_2'\cap c'$, $h_3'\cap c'$, d' are all distinct.

By (7) we get

$$-(4x+3t^2) = [4\xi^2(\xi+\eta)/\eta]-[3\xi^2(\xi+\eta)^2/\eta^2]$$
$$= (\xi^2/\eta^2)(\eta+\xi)(\eta-3\xi)\,.$$

Let $\zeta=e\eta/\xi$, $\xi_1=\xi+\eta$, $\eta_1=3\xi-\eta$. Then ξ_1, η_1 is a basis of $M(B^*, V^*)$, ζ is a primitive element of K'/K^* and the minimal monic polynomial of ζ over K^* is $E^2-\xi_1\eta_1$. From this it can be verified that the integral closure of $Q(B^*, V^*)$ is a nonregular local ring and hence B' is a singular point of V'. Now, outside of Q_2, the only other singular point of $\Delta(K'/P^2)=A\cup H$ is Q_1. Hence any point of V' lying above a point of P^2 other than Q_1 and Q_2 is a simple point of V'. Also $Q_1\notin A$ implies that $\phi'^{-1}(Q_1)$ has no point in common with $A_1'\cup A_2'\cup A_3'$ and hence by (14), every point of V' lying above Q_1 is also simple. Thus

(15) *B' is the only singular point of V'.*

For future reference, we shall summarize some of the results of this section in the following proposition.

Proposition 4. *Let P^2 be the projective plane over an algebraically closed ground field k of characteristic $p\neq 2, 3$. Let x, z be an affine k-general point of P^2 so that $k(P^2)=K=k(x, z)$. Let H and A be the curves in P^2 given by the affine equations $4+27z^2x=0$ and $x=0$ respectively. Then H is an irreducible cuspidal cubic and A is the flex tangent of H. Let K' be a splitting field over K of the polynomial T^3+xT+x^2z, let V' be a K'-normalization of P^2 and let ϕ' be the map of V' onto P^2. Then $G(K'/K)$ is isomorphic to the symmetric group S_3 on three letters; K'/P^2 is tamely ramified; $\Delta(K'/P^2)=A\cup H$, $\phi'^{-1}(A)$ has three irreducible components A_1, A_2, A_3; $\phi'^{-1}(H)$ has three irreducible components H_1, H_2, H_3; A_1, A_2, A_3, H_1, H_2, H_3 have a point B in common; B is the only singularity of V'; and any A_i meets any H_j only in B. Furthermore there exists a birational map ψ' of V onto a projective plane P'^2 with the following properties: (I) ψ' is biregular on $V'-A_1-A_2-A_3$ and maps it onto P'^2-b where b is a line in P'^2. (II) B is the only fundamental point of ψ' and $\psi'(B)=b$. (III) ψ'^{-1} has exactly three fundamental points a_1, a_2, a_3 and all these lie on b. (IV) After a suitable labelling of a_1, a_2, a_3 we have that $\psi'^{-1}(a_j)=A_j$. (V) $h_1=\psi'[H_1]$, $h_2=\psi'[H_2]$, $h_3=\psi'[H_3]$ are distinct lines in P'^2 other than b. (VI)*

h_1, h_2, h_3 have a point d in common. (VII) The points a_1, a_2, a_3, d, $h_1 \cap b$, $h_2 \cap b$, $h_3 \cap b$ are all distinct.

8. Normalizations

Let P^3 be a projective three space over an algebraically closed ground field k of characteristic $p \neq 2, 3$. Let V be an irreducible cubic surface in P^3 having a double line L such that V is not a cone. If V is nonspecial, then let M denote the line on V skew to L, let P_1 and P_2 denote the points on L at which the cones to V consist of single planes counted twice, and let β_j denote the plane joining M and P_j; if V is special then let α denote the fixed tangent plane to V along L; (Proposition 1 and Definition 1 of Section 1). Let P be a point of P^3 not on V, let P^2 be a plane in P^3 not containing P, and let ϕ be the projection of V onto P^2 from P. Let $K=k(P^2)$, $K^*=k(V)$, and let K' be a least galois extension of K containing K. Let V^* and V' be normalizations of V in K^* and K' respectively, let ψ be the map of V^* onto V, and let ϕ^* and ϕ' be the maps respectively of V^* and V' onto P^2. Let S_3 denote the symmetric group on 3 letters.

Proposition 5. *Assume that either V is nonspecial and $P \in \beta_1 \cup \beta_2$ or V is special and $P \in \alpha$. Then: (I) $\Delta(V^*/P^2) = \Delta(V'/P^2)$ consists of an irreducible cuspidal cubic H together with its flex tangent A. (II) $\phi^{*-1}(H)$ has two irreducible components H_1^* and H_2^* and $r(H_j^* : H) = \bar{r}(H_j^* : H) = j$. (III) $\phi'^{-1}(H)$ has three irreducible components H_1', H_2', H_3', and $r(H_j' : H) = \bar{r}(H_j' : H) = 2$. (IV) $\phi^{*-1}(A)$ has two irreducible components A_1^* and A_2^* and $r(A_j^* : A) = \bar{r}(A_j^* : A) = j$. (V) $\phi'^{-1}(A)$ has three irreducible components A_1', A_2', A_3' and $r(A_j' : A) = \bar{r}(A_j' : A) = 2$; (VI) $[K^* : K] = 3$, $[K' : K] = 6$ and $G(K' : K)$ is isomorphic to S_3.*

Proof. First assume that V is nonspecial. Then in the notation of Lemma 4, $\Delta(V^*/P^2) \subset H \cup A \cup (X=0)$ where A is the line $Z=0$. Now $\phi^{-1}((X=0)) = L \cup L_1$ where L_1 is the line $(X=T+Y=0)$. By Proposition 1 of Section 1, V is projectively equivalent to the surface of Section 3 and hence by (17) of Section 3, $\psi^{-1}[L]$ is irreducible and above all except two points of L lie two distinct points of V^*. Hence above almost all points of the line $X=0$ of P^2 there lie three distinct points of V^*, the line $X=0$ is unramified in K^* and $\phi^{*-1}((X=0))$ consists of two irreducible components.

Consequently $\Delta(V^*/P^2)=\Delta(V'/P^2)\subset H\cup A$ and we also have (VI). Let K_1 be the field between K^* and K' such that K_1/K is cyclic of order 2, let V_1 be a K_1-normalization of P^2 and let ϕ_1 be the map of V_1 onto P^2. Then $\Delta(K'/P^2)\subset H\cup A$ and hence [A3, Section 13], $\Delta(K'/P^2)=H\cup A$, $\phi_1^{-1}(H)$ and $\phi_1^{-1}(A)$ are irreducible and $r(\phi_1^{-1}(H):H)=r(\phi_1^{-1}(A):A)=2$. Suppose if possible that $\phi^{*-1}(A)$ is irreducible. Then $r(\phi^{*-1}(A):A)=3$. Hence the ramification index over A of any irreducible component of $\phi'^{-1}(A)$ is divisible by 2 as well as 3 and hence it must equal 6, thus implying that $G(K'/K)$ is cyclic. This contradicts (VI). Similarly $\phi^{-1}(H)$ is not irreducible. Now everything follows immediately.

Now assume that V is special. Then in view of Lemma 6 of Section 2, everything follows from Section 7.

Proposition 6. *Assume that either V is nonspecial and $P\notin\beta_1\cup\beta_2$ or V is special and $P\notin\alpha$. Then $\Delta(V^*/P^2)=\Delta(V'/P^2)$ is an irreducible three cuspidal quartic.*

Proof. By Lemmas 2, 3, and 5 of Section 2, $\Delta(V^*/P^2)\subset H\cup A$ where H is an irreducible three cuspidal quartic. $A=\phi(L)$, and $\phi^{-1}(A)=L\cup L_1$ where L_1 is a line on V different from L. In view of Proposition 1 of Section 1, (17) of Section 3, and (8) of Section 7, it follows that A is not ramified in K^*.

Now invoking Theorem 2 of [A5], the results of Sections 3 and 6 yield the following two propositions.

Proposition 7. *The normalizations of any two non-conical irreducible cubic surfaces in P^3 having double lines are biregularly equivalent. More specifically the normalization of any non-conical irreducible cubic surface in P^3 having a double line is an immediate quadratic transform of a projective plane with center at a point.*

Proposition 8. *Assume that either V is nonspecial and $P\notin\beta_1\cup\beta_2$, or V is special and $P\notin\alpha$. Let Q_1, Q_2, Q_3 be the singularities of $H=\Delta(V^*/P^2)$. Then we have the following: (I) $\phi^{*-1}(H)$ has two irreducible components H_1^* and H_2^* where $r(H_j^*:H)=\bar{r}(H_j^*:H)=j$. (II) V^*, H_1^* and H_2^* are all nonsingular. (III) $\phi^{*-1}(Q_j)$ consists of a single point Q_j^*. (IV) Q_1^*, Q_2^*, Q_3^* are exactly the common points of H_1^* and H_2^*, and H_1^* and H_2^* have a 2-fold contact at each Q_j^*. (V) $\nu(H_j^*, H_1^*\cup H_2^*; V^*)=6$, $dim|H_1^*|=5$ and $dim|H_2^*|=6$. (VI) $\phi'^{-1}(Q_j)$ consists of a single point Q_j'. (VII) $\phi^{-1}(H)$ consists of three irreducible components H_1', H_2', H_3' and $r(H_j':H)=\bar{r}(H_j':H)$*

$=2$. *(VIII)* V', H'_1, H'_2, H'_3 *are all nonsingular.* *(IX) For any of the two components* H'_j; Q'_1, Q'_2, Q'_3 *are exactly the common points.* *(X)* $H'_1 \cup H'_2 \cup H'_3$ *has an ordinary* 3-*fold point at each* Q'_j. *(XI)* $\dim |H'_j| = 3$. *(XII)* $G(K'/K)$ *is isomorphic to* S_3. *(XIII) There exists a birational map* ψ' *of* V' *onto a projective plane* P'^2 *satisfying the description given in (VI) to (XII) of Proposition* 3 *of Section* 6.

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