Exercise 1. Prove that all entire functions that are also injective take the form $f(z) = az + b$with $a, b \in \mathbb{C}$, and $a \neq 0$. ⟦Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.⟧

Answer

The function $g(z) = f(1/z)$ has a singularity at $z = 0$. If it were removable, then g is bounded on $B(0, R)$ for some $R > 0$.
This means that f is bounded outside $B(0, R)$, but as f is entire, it's continuous and so it's bounded *inside* $B(0, R)$.
By Liouville's theorem f is constant. But that contradicts the fact that f is injective.
Now assume g has an essential singularity at $z = 0$. By the Casorati-Weierstrass theorem, we have a neighborhood of the origin $B(0, R)$ with $R > 0$, such that $g\left[B(0, R)\right]$ is dense in $\mathbb{C}$. This means that $f\left[\{\, |z| > R \,\}\right]$ is dense in $\mathbb{C}$ and we have that $f[B(0, R)]$ is an open set.
Recall now that dense subsets of $\mathbb{C}$ intersect every non trivial open set, in particular this means that

$$f[B(0, R)] \cap f\left[\{\, |z| > R \,\}\right] \neq \varnothing$$

and so for any $w \in f[B(0, R)] \cap f\left[\{\, |z| > R \,\}\right]$ we can find z_1 with $|z_1| < R$ and z_2 with $|z_2| > R$ such that

$$f(z_1) = f(z_2) = w, \quad \text{and} \quad z_1 \neq z_2.$$

This contradicts the fact that f is injective. Thus, the only type of singularity that may occur at $z = 0$ is a pole.
From here we see that in the Taylor expansion of f, the analytic part coincides with g's principal part. As g's principal part must be finite, f must be a polynomial. The degree of f can't be larger than 1 because f is injective, it can't also be 0 because f is injective and so we conclude that f is linear as desired.

Exercise 2. As in class, consider the unit sphere

$$X = \{\, (a, b, c) : \ a^2 + b^2 + c^2 = 1 \,\} \subseteq \mathbb{R}^3$$

Let $N = (0, 0, 1), S = (0, 0, -1), U_N = X \backslash N, U_S = X \backslash S$. Consider the following three charts on X:

- ◇ $\phi_N : U_N \to \mathbb{C}, \ (a, b, c) \mapsto \frac{a+ib}{1-c}$.

- $\diamond$ $\phi_S : U_S \to \mathbb{C},\ (a, b, c) \mapsto \frac{a+ib}{1+c}$.
- $\diamond$ $\psi_S : U_S \to \mathbb{C},\ (a, b, c) \mapsto \frac{a-ib}{1+c}$.

Do the following:

i) The inverse of ϕ_N is

$$\phi_N^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2 + 1}, \frac{2\,\mathrm{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

Calculate ϕ_S^{-1} and ψ_S^{-1}.

ii) Among the three charts $\{\,(U_N, \phi_N), (U_S, \phi_S), (U_S, \psi_S)\,\}$, one pair is compatible and the other two are not. Which is which? Why?

$[\![$ Hint: Remember a function is holomorphic if and only if $\partial_{\overline{z}} f = 0.$ $]\!]$

Answer

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