Exercise 1. A smooth manifold M is called *orientable* if there exists a collection of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ so that, for every α, β such that $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \varnothing$, the differential of the change of coordinates $\phi_\beta^{-1} \circ \phi_\alpha$ has positive determinant.

(a) Show that for any n, the sphere S^n is orientable.

(b) Prove that, if M and N are smooth manifolds and $f : M \to N$ is a local diffeomorphism at all points of M, then N being orientable implies that M is orientable. Is the converse true?

Answer

(a) Consider the sphere without its north and south pole:

$$U = S^n \backslash \{ \mathbf{e}_{n+1} \}, \quad \text{and} \quad V = S^n \backslash \{ -\mathbf{e}_{n+1} \}.$$

These two sets form an atlas of S^n along with the stereographic projections

$$\phi : U \to \mathbb{R}^n, \ \mathbf{u} \mapsto \frac{1}{1 - u_{n+1}}(u_1, \ldots, u_n),$$
$$\psi : V \to \mathbb{R}^n, \ \mathbf{u} \mapsto \frac{1}{1 + u_{n+1}}(u_1, \ldots, u_n).$$

For $\mathbf{u} \in S^n$, call $\mathbf{x} = \phi(\mathbf{u})$ and $\mathbf{y} = \psi(\mathbf{u})$. In order to find the transition function $\psi\phi^{-1}$, we first make the observation that

$$\|\mathbf{x}\|^2 = \|\phi(\mathbf{u})\|^2 = \frac{1}{(1 - u_{n+1})^2}(u_1^2 + \cdots + u_n^2) = \frac{1 - u_{n+1}^2}{(1 - u_{n+1})^2} = \frac{1 + u_{n+1}}{1 - u_{n+1}},$$

and from this we can see that

$$\phi^{-1}(\mathbf{x}) = \frac{1}{1 + \|\mathbf{x}\|^2}(2x_1, \ldots, 2x_n, \|\mathbf{x}\|^2 - 1).$$

Applying ψ we get the transition function to be

$$\begin{aligned}
\mathbf{y} = \psi\phi^{-1}(\mathbf{x}) &= \frac{1}{1 + \left(\frac{\|\mathbf{x}\|^2 - 1}{1 + \|\mathbf{x}^2\|}\right)} \left(\frac{2x_1}{1 + \|\mathbf{x}\|^2}, \ldots, \frac{2x_n}{1 + \|\mathbf{x}\|^2} \right) \\
&= \frac{1}{\frac{1 + \|\mathbf{x}\|^2 + \|\mathbf{x}\|^2 - 1}{\|\mathbf{x}\|^2 + 1}} \frac{2}{1 + \|\mathbf{x}\|^2}\mathbf{x} \\
&= \frac{1 + \|\mathbf{x}\|^2}{2\|\mathbf{x}\|^2} \frac{2}{1 + \|\mathbf{x}\|^2}\mathbf{x} = \frac{\mathbf{x}}{\|\mathbf{x}\|^2}
\end{aligned}$$

The differential of this map can be calculated using the product rule. Call $f = \frac{1}{\|\mathbf{x}\|^2}, G = \mathrm{id}$, then

$$J(fG) = \nabla f \otimes G + fJG = \left(\frac{-1}{\left(\|\mathbf{x}\|^2\right)^2} 2\mathbf{x} \right) \otimes \mathbf{x} + \frac{1}{\|\mathbf{x}\|^2}\mathrm{Id}.$$

Using the matrix determinant lemma we may see that

$$\begin{aligned}
\det(JfG) &= \left(1 + \frac{-2}{\|\mathbf{x}\|^4} \mathbf{x}^\mathsf{T} \left(\|\mathbf{x}\|^2 \mathrm{Id} \right) \mathbf{x} \right) \det \left(\frac{1}{\|\mathbf{x}\|^2} \mathrm{Id} \right) \\
&= (1 - \frac{2}{\|\mathbf{x}\|^2} \mathbf{x}^\mathsf{T}\mathbf{x}) \frac{1}{\|\mathbf{x}\|^{2n}} \\
&= (1-2)\frac{1}{\|\mathbf{x}\|^{2n}} = \frac{-1}{\|\mathbf{x}\|^{2n}}
\end{aligned}$$

This doesn't mean that the sphere is non-orientable, but that my choice of atlas was a poor choice. For our effect then, it suffices to make a small change in our chart. Pick

$$\widetilde{\psi}: V \to \mathbb{R}^n, \ \mathbf{u} \mapsto \frac{1}{1 + u_{n+1}}(u_1, \ldots, -u_n)$$

and observe that this small change in *orientation* will help us recover our desired result. In this case the transition function becomes

$$\mathbf{y} = \widetilde{\psi}\phi^{-1}(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|^2}(x_1, \ldots, x_{n-1}, -x_n).$$

In this case, following a product rule calculation our G function changes to a diagonal matrix $\mathrm{diag}(1, 1, \ldots, 1, -1)$ which means that when taking its determinant we get -1. In the end the whole determinant of the transition function's differential becomes positive, leaving us with the desired result.

(b) We will build an atlas on M whose transitions functions' differential have positive determinant. To that effect, let $\{(V_\alpha, \psi_\alpha)\}$ be an atlas of N which makes N orientable, this means that we have $\det J(\psi_\beta\psi_\alpha^{-1}) > 0$ for α, β.

Now for $x \in M$, there are neighborhoods $U_x, \widetilde{V}_{f(x)}$ in M, N respectively such that f is a diffeomorphism between these sets. Pick a chart $(V_{f(x)}, \psi_{f(x)})$

from our original atlas such that $f(x) \in V_{f(x)}$. Consider then the new open sets
$$W_{f(x)} = V_{f(x)} \cap \widetilde{V}_{f(x)}$$
and restrict $\psi_{f(x)}$ into $W_{f(x)}$ by calling it $\varphi_{f(x)}$.

This defines a new atlas
$$\{ (W_{f(x)}, \varphi_{f(x)}) \}$$
for N, by virtue of f being bijective, which still preserves the property that its transition functions' differential has positive determinant.

We may pullback this atlas via f into an atlas
$$\{ (f^{-1}(W_{f(x)}), f^{*}\varphi_{f(x)}) \}$$
of M. For $x, y \in M$, we have the expression for the transition function
$$(f^{*}\varphi_{f(x)})(f^{*}\varphi_{f(y)})^{-1} = (\varphi_{f(x)} \circ f)(\varphi_{f(y)} \circ f)^{-1} = \varphi_{f(x)}\varphi_{f(y)}^{-1}.$$
As these are restrictions of our ψ functions, then their differential still has positive determinant. Thus, we have found an atlas of M which makes it orientable.

(c) Consider the quotient map $S^2 \to \mathbb{P}^2$ given by $(x, y, z) \mapsto [x : y : z]$. Locally if we look at the upper and lower hemisphere, this map is the identity which means that its differential is bijective. So the quotient map is a local diffeomorphism from an orientable surface to a non-orientable one.

Quick question for the second item, the fact that M, N are locally diffeomorphic implies that pairs of neighborhoods have the same homology. In particular the relative homology groups, which are used to define local orientation, are isomorphic.

Would this be sufficient to conclude that if N is orientable then M is orientable? Why would this argument fail in the other direction? Say, why if M is orientable, could N be non-orientable?

Exercise 2. Supply the details for the proof that, if $F : \mathrm{Mat}_{d\times d}(\mathbb{C}) \to \mathcal{H}(d)$ is given by $F(U) = UU^{*}$ (where U^{*} is the conjugate transpose [a.k.a., Hermitian adjoint] of U), then the unitary group
$$\mathcal{U}(d) = F^{-1}(I_{d\times d})$$

is a submanifold of $\mathrm{Mat}_{d\times d}(\mathbb{C})$ of dimension d^2. (Hint: it may be helpful to remember that a Hermitian matrix M can always be written as $M = \frac{1}{2}(M + M^*)$.)

Answer

First, let us remind ourselves that a Hermitian matrix is a matrix A such that $A^* = A$. In consequence, for a general U we have

$$F(U)^* = (UU^*)^* = U^{**}U^* = UU^*,$$

meaning that indeed, F maps matrices into the Hermitian matrix space. To verify the unitary group is a submanifold, we must check that I is a regular value of our map. The differential of our map is

$$\frac{\partial}{\partial U}(UU^*) = \frac{\partial U}{\partial U}U^* + \frac{\partial U^*}{\partial U}U = IU^* + \left(\frac{\partial U}{\partial U}\right)^* U = U^* + U.$$

This map is always surjective for any Hermitian matrix M can be written as $\frac{1}{2}(M + M^*)$ which means that if we map $\frac{1}{2}M$ through F we will recover our matrix M. In particular, this means that I is a regular value of F and therefore, $\mathcal{U}$ is a submanifold of the matrix space.

To count the dimensions, it's important to recall that as a real-vector space, the dimension of the space of matrices is $2d^2$ because of complex coefficients. For the Hermitian matrices, we have 1 real degree of freedom across the diagonal and on the upper triangle we have $\binom{d}{2}$ complex degrees of freedom. This means that the dimension of the unitary group is

$$\dim \mathcal{U} = 2d^2 - (d + 2\binom{d}{2}) = 2d^2 - d - d(d-1) = d^2$$

as desired.

Exercise 3. Let M be a compact manifold of dimension n and let $f : M \to \mathbb{R}^n$ be a smooth map. Prove that f must have at least one critical point.

Answer

We will first observe that the result is true in one dimension. If $f : M \to \mathbb{R}$ is smooth, then its image is compact and therefore f must attain an extreme value. Let $p \in M$ be the point where f reaches an extreme, for a smooth curve α about

p such that
$$\alpha(0) = p, \quad \alpha'(0) = v.$$

As a real function then, $f \circ \alpha$ has an extreme value at $t = 0$ which means that $t = 0$ is a critical point of $f \circ \alpha$. Thus, we have $(f \circ \alpha)'(0) = 0$.
On the other hand, this is the differential of f at p:

$$df_p v := (f \circ \alpha)'(0)$$

which means that the differential is zero and therefore non-surjective. Thus p is a critical value of f.
In general, consider a coordinate projection map $\pi_i = \langle \mathbf{e}_i |$[a] for some i. The function $\pi_i \circ f$ then becomes a smooth function to $\mathbb{R}$ with a critical point p. This means that
$$d(\pi_i \circ f)_p$$

is not surjective. By the chain rule this is $(d\pi_i)_{f(p)} df_p$ and as this are linear maps, product is a composition. Observe that the differential of our projection is itself as it's a linear map and so the failure of surjectivity must come from df_p. We conclude that p is a critical point of f.

[a] The linear map, take the dot product of input with $\mathbf{e}_i$.

Exercise 4. Prove that, if X, Y, and Z are smooth vector fields on a smooth manifold M and $a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$, then

(a) $[X, Y] = -[Y, X]$ (anticommutivity)

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ (linearity)

(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity)

(d) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.

Answer

We have defined the Lie bracket as the commutator of vector fields

$$[X, Y]f := X(Yf) - Y(Xf)$$

and so we have:

(a) $[Y, X]f = Y(Xf) - X(Yf) = -(X(Yf) - Y(Xf)) = -[X, Y]f$.

(b) To show linearity we have

$$\begin{aligned}&[aX+bY,Z]f\\=&(aX+bY)(Zf)-Z((aX+bY)f)\\=&aX(Zf)+bY(Z(f))-Z(aXf+bYf)\end{aligned}$$

where the first equality comes by definition and the second one is the definition of the sum of linear operators. Now applygin the fact that Z is linear we get:

$$aX(Zf)+bY(Z(f))-aZ(Xf)-bZ(Yf)$$

which we rearrenge as a sum of smooth functions now:

$$a(X(Zf)-Z(Xf))+b(Y(Z(f))-Z(Yf))=(a[X,Z]+b[Y,Z])f.$$

(c) Let us take the first two terms in the sum and see that

$$\begin{aligned}&([[X,Y],Z]+[[Y,Z],X])f\\=&[X,Y](Zf)-Z([X,Y]f)+[Y,Z](Xf)-X([Y,Z]f)\\=&X(YZf)-Y(XZf)-Z(XYf)+Z(YXf)\\&+Y(ZXf)-Z(YXf)-X(YZf)+X(ZYf)\end{aligned}$$

Observe now that the 1st and 7th, and 4th and 6th terms cancel out. We are left with a term which we rearrange into...

$$\begin{aligned}&-Y(XZf)-Z(XYf)+Y(ZXf)+X(ZYf)\\=&Y(ZXf)-Y(XZf)-(ZX(Yf)-XZ(Yf))\\=&Y([Z,X]f)-[Z,X](Yf)\\=&[Y,[Z,X]]f=-[[Z,X],Y]f\end{aligned}$$

as desired.

(d) Finally if h is another smooth function on M:

$$\begin{aligned}&[fX,gY]h\\=&fX(gYh)-gY(fXh)\\=&fXgYh+fgXYh-gYfXh-gfYXh\\=&fXgYh+fg(XYh-YXh)-gYfXh\\=&(f(Xg)Y+fg[X,Y]-g(Yf)X)h\end{aligned}$$

where in the second equality we have applied the product rule for vector fields.