Trigonometric Substitution

We have seen integrals of the type
$$\int x\sqrt{1-x^2}\mathrm{d}x,\ \int\frac{x}{\sqrt{1-x^2}}\mathrm{d}x$$
which we are able to solve by direct u-substitution. In both, by taking $u=1-x^2$, we obtain the following integrals
$$\int\sqrt{u}\left(\frac{-\mathrm{d}u}{2}\right),\ \int\frac{-\mathrm{d}u}{2\sqrt{u}}.$$
However now we are going to face integrals like
$$\int\frac{\sqrt{1-x^2}}{x}\mathrm{d}x.$$
The typical u-sub, $u=1-x^2$ doesn't work here, this integral becomes
$$\int\frac{\sqrt{u}}{\sqrt{1-u}}\left(\frac{\mathrm{d}u}{-2\sqrt{1-u}}\right)=-\frac{1}{2}\int\frac{\sqrt{u}}{1-u}\mathrm{d}u.$$
We will need a new technique to work this type of integrals out.

Sine substitution

For now, let's take $x=\sin(\theta)$ without worrying too much from where that came from.

Example 1. The integral $\int\frac{\sqrt{1-x^2}}{x}\mathrm{d}x$ can be found using the substitution $x=\sin(\theta)$.

In this case $\mathrm{d}x=\cos(\theta)\mathrm{d}\theta$ after differentiating and so the integral becomes
$$\int\frac{\sqrt{1-\sin^2(\theta)}}{\sin(\theta)}(\cos(\theta)\mathrm{d}\theta).$$
By applying the Pythagorean identity we can simplify the square root:
$$\sin^2(\theta)+\cos^2(\theta)=1\Rightarrow\cos^2(\theta)=1-\sin^2(\theta).$$
After taking this back into the integral we obtain
$$\int\frac{\sqrt{\cos^2(\theta)}}{\sin(\theta)}(\cos(\theta)\mathrm{d}\theta)=\int\frac{\cos^2(\theta)}{\sin(\theta)}\mathrm{d}\theta$$
and by using the Pythagorean identity once more and changing the cosine to a sine we get
$$\frac{\cos^2(\theta)}{\sin(\theta)}=\frac{1-\sin^2(\theta)}{\sin(\theta)}=\frac{1}{\sin(\theta)}-\sin(\theta)=\csc(\theta)-\sin(\theta).$$
By linearity the integral separates into
$$\int\csc(\theta)\mathrm{d}\theta-\int\sin(\theta)\mathrm{d}\theta=\underline{-\log[\csc(\theta)+\cot(\theta)]+\cos(\theta)}.$$
The integral is not completely done at this point because we need to substitute back in our $x=\sin(\theta)$ which would become a $\theta=\arcsin(x)$. But *for now* we will be satisfied.

Example 2. Consider the following integral
$$\int\frac{\mathrm{d}x}{x^4\sqrt{4-x^2}}.$$
Since we are thinking in terms of a sine substitution we can take $x=\sin(\theta)$ once more, however that would turn our root into a $4-\sin^2(\theta)$ and **there's no identity for that expression**.

We want that to become a $\cos^2(\theta)$ in some way so if we multiply our sine like this...
$$x=2\sin(\theta)\Rightarrow x^2=4\sin^2(\theta)\Rightarrow 4-x^2=4-4\sin^2(\theta)$$
then we can factor out the 4 and work the integral as before!

Let us take the substitution
$$x=2\sin(\theta)\Rightarrow\mathrm{d}x=2\cos(\theta)\mathrm{d}\theta$$
and replacing inside the integral we obtain
$$\begin{aligned}\int\frac{\mathrm{d}x}{x^4\sqrt{4-x^2}}&=\int\frac{2\cos(\theta)\mathrm{d}\theta}{(2\sin(\theta))^4\sqrt{4-4\sin^2(\theta)}}\\&=\int\frac{2\cos(\theta)\mathrm{d}\theta}{16\sin^4(\theta)\sqrt{4\cos^2(\theta)}}\\&=\frac{1}{16}\int\csc^4(\theta)\mathrm{d}\theta\end{aligned}$$
This type of integral can be solved by separating the cosecant and using a trigonometric identity. Recall from the Pythagorean identity:
$$\sin^2(\theta)+\cos^2(\theta)=1\Rightarrow 1+\cot^2(\theta)=\csc^2(\theta).$$
Then separating the cosecant into two squares we get:
$$\begin{aligned}\frac{1}{16}\int\csc^4(\theta)\mathrm{d}\theta&=\frac{1}{16}\int\csc^2(\theta)\csc^2(\theta)\mathrm{d}\theta\\&=\frac{1}{16}\int(1+\cot^2(\theta))\csc^2(\theta)\mathrm{d}\theta\\\left({\scriptstyle u=\cot(\theta) \atop \scriptstyle \mathrm{d}u=-\csc^2(\theta)\mathrm{d}\theta}\right)&=\frac{-1}{16}\int 1+u^2\mathrm{d}u\\&=\frac{1}{16}\left(u+\frac{1}{3}u^3\right)\\&=\underline{\frac{1}{16}\cot(\theta)+\frac{1}{48}\cot^3(\theta)}\end{aligned}$$

Proposition 3. *Integrals with quadratic expressions inside of radicals such as* $\underline{\sqrt{a^2-b^2x^2}}$ *will be worked using the substitution* $\underline{x=\frac{a}{b}\sin(x)}$.

Practice

Consider the following integral:
$$\int e^x\sqrt{1-9e^{2x}}\mathrm{d}x.$$

I) Transform this integral by using a u-substitution into another one that could be done with a trigonometric substitution.
II) Which of the following substitutions would lead to a correct answer? $u=3\sin(\theta)$, $u=\sin(3\theta)$ or $u=(1/3)\sin(\theta)$. Discuss with your group members!

Tangent Substitution

When we have radicals involving an expression similar to $\sqrt{1+x^2}$ we will instead take $x=\tan(\theta)$.

Remark. Notice the difference in the sign between the sine subs and the tangent subs. With sine it's a minus, and with tangent, a plus!

Example 4. Consider the following integral
$$\int x^3\sqrt{1+x^2}\mathrm{d}x.$$
The expression should remind us of the Pythagorean identity in another way. We have that
$$\sin^2(\theta)+\cos^2(\theta)=1\Rightarrow\tan^2(\theta)+1=\sec^2(\theta).$$
Let us substitute then
$$x=\tan(\theta)\Rightarrow\mathrm{d}x=\mathrm{d}(\tan(\theta))=\sec^2(\theta)\mathrm{d}\theta.$$
Replacing this into the integral we obtain
$$\int(\tan(\theta))^3\sqrt{1+\tan^2(\theta)}(\sec^2(\theta)\mathrm{d}\theta)=\int\tan^3(\theta)\sec^3(\theta)\mathrm{d}\theta.$$
This trigonometric integral can be done by using the Pythagorean identity and separating the cubes into smaller powers:
$$\begin{aligned}\tan^3(\theta)\sec^3(\theta)&=\tan^2(\theta)\sec^2(\theta)(\tan(\theta)\sec(\theta))\\&=(\sec^2(\theta)-1)\sec^2(\theta)(\tan(\theta)\sec(\theta))\\&=(\sec^4(\theta)-\sec^2(\theta))(\sec(\theta)\tan(\theta)).\end{aligned}$$
This is now the argument of our integral. We have an expression involving the secant and it's derivative so we will take:
$$u=\sec(\theta)\Rightarrow\mathrm{d}u=\mathrm{d}(\sec(\theta))=(\sec(\theta)\tan(\theta))\mathrm{d}\theta.$$
Our integral finally becomes
$$\int(u^4-u^2)\mathrm{d}u=\underline{\frac{\sec^5(\theta)}{5}-\frac{\sec^3(\theta)}{3}}$$

Proposition 5. *Integrals with quadratic expressions inside of radicals such as* $\underline{\sqrt{a^2+b^2x^2}}$ *will be worked using the substitution* $\underline{x=\frac{a}{b}\tan(x)}$.

Practice

Consider the integral
$$\int e^{4x}\sqrt{1+e^{2x}}\mathrm{d}x.$$

1. Would the substitution $u=e^{2x}$ transform this into a trigonometric integral?
2. What if we took $e^x=\tan(\theta)$ at once? Can the integral be solved?
3. Compare with your group members either approach, first a u-sub and then a trig. one and the direct substitution $e^x=\tan(\theta)$.

Secant Substitution

The secant substitution comes into play with radicals of the form
$$\sqrt{x^2-1}\Rightarrow x=\sec(\theta)$$
because of the identity $\tan^2(\theta)=\sec^2(\theta)-1$. When taking the secant inside, the root becomes a root of $\tan^2$.

Example 6. Consider the integral
$$\int\frac{\mathrm{d}x}{\sqrt{x^2-2x-3}}.$$
This polynomial inside the radical can be simplified by completing the square:
$$\begin{aligned}x^2-2x-3=x^2-2x(+1-1)-3&=(x^2-2x+1)-1-3\\&=(x-1)^2-4.\end{aligned}$$
Even if we have that -1 we will take the whole of it to become our secant:
$$\begin{aligned}x-1=2\sec(\theta)&\Rightarrow\mathrm{d}(x-1)=\mathrm{d}(2\sec(\theta))\\&\Rightarrow\mathrm{d}x=2\sec(\theta)\tan(\theta)\mathrm{d}\theta.\end{aligned}$$
Thus replacing in the integral we get
$$\begin{aligned}\int\frac{\mathrm{d}x}{\sqrt{(x-1)^2-4}}&=\int\frac{2\sec(\theta)\tan(\theta)}{\sqrt{(2\sec(\theta))^2-4}}\mathrm{d}\theta\\&=\int\frac{2\sec(\theta)\tan(\theta)}{\sqrt{4\tan^2(\theta)}}\mathrm{d}\theta\\&=\int\frac{2\sec(\theta)\tan(\theta)}{2\tan(\theta)}\mathrm{d}\theta\\&=\int\sec(\theta)\mathrm{d}\theta=\underline{\log(\sec(\theta)+\tan(\theta))}\end{aligned}$$

Proposition 7. *We have the following summary*

- $\sqrt{a^2-b^2x^2}$ *reminds us of* $1-\sin^2(\theta)=\cos^2(\theta)$ *so we take* $x=\sin(\theta)\Rightarrow\mathrm{d}x=\cos(\theta)\mathrm{d}\theta$.
- $\sqrt{a^2+b^2x^2}$ *reminds us of* $1+\tan^2(\theta)=\sec^2(\theta)$ *so we take* $x=\tan(\theta)\Rightarrow\mathrm{d}x=\sec^2(\theta)\mathrm{d}\theta$.
- $\sqrt{b^2x^2-a^2}$ *reminds us of* $\sec^2(\theta)-1=\tan^2(\theta)$ *so we take* $x=\sec(\theta)\Rightarrow\mathrm{d}x=\sec(\theta)\tan(\theta)\mathrm{d}\theta$.

Exercise 8. Compute the following integrals, in some of them, it will be necessary to complete the square:

I) $\displaystyle\int\frac{x}{\sqrt{x^2-4x}}\mathrm{d}x$

II) $\displaystyle\int\sqrt{4x^2+4x+2}\mathrm{d}x$

III) $\displaystyle\int\frac{x^3}{\sqrt{16-9x^2}}\mathrm{d}x$

Is there an easier way to compute the last integral? Discuss with your group members to know each other's approach.