Exercise 1 (Bessel Function)**.** Consider the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{2^{2n}(n!)^2}$, this is $J_0(1)$ where J is called the Bessel J function. Do the following:

I) From the convergence tests we've seen in class name a test that proves that the series converges.
II) State the conditions which guarantee convergence based on the test you named.
III) Apply the test to prove that $J_0(1)$ converges.

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1. It is possible to use either Dirichlet's Alternating Series test or the D'Alembert's Ratio test.
2. Dirichlet's test says that a series $\sum (-1)^n a_n$ converges when a_n is decreasing and $a_n \to 0$ as $n \to \infty$.

   On the other hand D'Alembert's test says that $\sum a_n$ converges when $\left| \dfrac{a_{n+1}}{a_n} \right|$ goes to any value between 0 and 1 but not 1.
3. By Dirichlet's test, we can see that $\frac{1}{2^{2n}(n!)^2}$ clearly goes to zero. While on the other hand by D'Alembert's test

$$\left| \frac{\frac{(-1)^{n+1}}{2^{2(n+1)}((n+1)!)^2}}{\frac{(-1)^n}{2^{2n}(n!)^2}} \right| = \frac{1}{2^2(n+1)^2} \to 0 \text{ as } n \to \infty.$$

   As the consecutive ratio goes to 0, then, by D'Alembert's test, our series converges.

Exercise 2. Suppose a function $f(x)$ can be represented by a polynomial $p(x)$ about $a = 3$. We have
$$p(x) = 5 + 3(x-3)^2 - (x-3)^3 + \frac{2(x-3)^4}{5!} + 2(x-3)^5 + \frac{1}{2}(x-3)^6.$$
Identify the following:

I) $f(3)$ II) $f'(3)$ III) $f''(3)$ IV) $f^{(3)}(3)$ V) $f^{(5)}(3)$

We have

I) $f(3) = 5$ II) $f'(3) = 0$ III) $f''(3) = 6$ IV) $f^{(3)}(3) = -6$ V) $f^{(5)}(3) = 240 = 2 \cdot 5!$

Exercise 3. We will approximate $e^{1/10}$ using Taylor series.

I) Write out the first four terms of e^x's Taylor series.
II) Evaluate $x = \frac{1}{10}$ to obtain an approximation of $e^{1/10}$.

The first four terms are
$$1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$
so the approximation of $e^{1/10}$ is
$$1 + \frac{1}{10} + \frac{1}{200} + \frac{1}{600}.$$