Problem 18

Repeat problem 17 but traverse the triangle clockwise.
(Problem 17 had the vector field $\mathbf{F}(x, y) = (2xy^2,\ 2x^2y)$ and C was the triangle with vertices $(0, 0), (0, 1), (1, 0)$.)

We parametrize the paths between the vertices following the orientation

$$\begin{cases} r_1(t) = (0, t) \Rightarrow r_1'(t) = (0, 1) \\ r_2(t) = (t, -t+1) \Rightarrow r_2'(t) = (1, -1) \\ r_3(t) = (-t+1, 0) \Rightarrow r_3'(t) = (-1, 0) \end{cases}$$

where all paths have $0 \leqslant t \leqslant 1$. Evaluating F at each path we obtain

$$\begin{cases} F(r_1(t)) = (0, 0) \\ F(r_2(t)) = (2(t)(-t+1)^2, 2(t)^2(-t+1)) = (2t^3 - 4t^2 + 2t, -2t^3 + 2t^2) \\ F(r_3(t)) = (0, 0) \end{cases}$$

From this

$$\begin{aligned} \oint_T F \cdot \mathrm{d}\mathbf{x} &= \int_{L_1} F \cdot \mathrm{d}\mathbf{x} + \int_{L_2} F \cdot \mathrm{d}\mathbf{x} + \int_{L_3} F \cdot \mathrm{d}\mathbf{x} \\ &= 0 + \int_0^1 F(r_2(t)) \cdot r_2'(t)\mathrm{d}t + 0 \\ &= \int_0^1 (2t^3 - 4t^2 + 2t, -2t^3 + 2t^2) \cdot (1, -1)\mathrm{d}t \\ &= \int_0^1 (2t^3 - 4t^2 + 2t) - (-2t^3 + 2t^2)\mathrm{d}t \\ &= \int_0^1 (4t^3 - 6t^2 + 2t)\mathrm{d}t \\ &= \frac{4}{4} - \frac{6}{3} + \frac{2}{2} = 0. \end{aligned}$$

Thus our desired integral is

$$\boxed{\oint_T F \cdot \mathrm{d}\mathbf{x} = 0}$$

Problem 19

Determine whether the vector field $\mathbf{F}(x, y) = 3x^2\mathbf{i} + y^3\mathbf{j}$ *is conservative. If it is, compute a potential function.*

Let $\mathbf{F} = (P, Q)$ with $P = 3x^2$ and $Q = y^3$. Via the conservative criterion, $\mathbf{F}$ is conservative iff $\partial P/\partial y = \partial Q/\partial x$.

Computing:

$$\frac{\partial P}{\partial y} = 0, \qquad \frac{\partial Q}{\partial x} = 0,$$

so they are equal. Hence $\mathbf{F}$ is conservative.

Find a potential $\varphi(x, y)$ with $\varphi_x = 3x^2$ and $\varphi_y = y^3$. Integrate φ_x w.r.t. x:

$$\varphi(x, y) = \int 3x^2 \, \mathrm{d}x = x^3 + g(y).$$

Differentiate with respect to y and match φ_y:

$$\varphi_y(x, y) = g'(y) = y^3 \implies g(y) = \frac{y^4}{4} + C.$$

Thus a potential function is

$$\boxed{\varphi(x, y) = x^3 + \frac{y^4}{4} + C.}$$

Problem 21

Determine whether the vector field $\mathbf{F}(x, y) = ye^x\mathbf{i} + xe^y\mathbf{j}$ *is conservative. If it is, compute a potential function.*

Let $\mathbf{F} = (P, Q)$ with $P = ye^x$ and $Q = xe^y$. Computing the mixed partials:

$$\frac{\partial P}{\partial y} = e^x, \qquad \frac{\partial Q}{\partial x} = e^y.$$

These are not equal for general (x, y) (they would be equal only on the line $x = y$), so $\mathbf{F}$ is *not* conservative on any domain containing points with $x \neq y$. Therefore no global potential function exists.

$$\boxed{F \text{ is not conservative (since } \partial P/\partial y = e^x \neq e^y = \partial Q/\partial x\text{).}}$$