Exercise 1 (Exercise 1)**.** Show that every polynomial in two variables x, y can be written uniquely as a sum of a (two variable) symmetric polynomial and a (two variable) antisymmetric polynomial.

Answer

Suppose $p \in \mathbb{Q}[x, y]$, then

$$p(x, y) = \frac{1}{2}\left(p(x, y) + p(y, x)\right) + \frac{1}{2}\left(p(x, y) - p(y, x)\right).$$

Call the summands $s(x, y)$ and $a(x, y)$ respectively. We have that s is a symmetric polynomial while a is antisymmetric:

$$\begin{cases} s(y, x) = \frac{1}{2}\left(p(y, x) + p(x, y)\right) = \frac{1}{2}\left(p(x, y) + p(y, x)\right) = s(x, y), \\ a(y, x) = \frac{1}{2}\left(p(y, x) - p(x, y)\right) = -\frac{1}{2}\left(p(x, y) - p(y, x)\right) = -a(x, y). \end{cases}$$

Now suppose that there exist $s_1, s_2, a_1, a_2 \in \mathbb{Q}[x, y]$, symmetric and antisymmetric polynomials respectively such that

$$\begin{aligned} p(x, y) = s_1(x, y) + a_1(x, y) &= s_2(x, y) + a_2(x, y), \\ \Rightarrow s_1(x, y) - s_2(x, y) &= a_2(x, y) - a_1(x, y). \end{aligned}$$

From this last equation, after exchanging the variables we get

$$s_1(x, y) - s_2(x, y) = s_1(y, x) - s_2(y, x) = a_2(y, x) - a_1(y, x) = -a_2(x, y) + a_1(x, y)$$

which gives us the equation

$$a_2(x, y) - a_1(x, y) = -a_2(x, y) + a_1(x, y) \Rightarrow a_2(x, y) = a_1(x, y).$$

Now that we have that the antisymmetric parts are equal, we see from the original hypothesis that

$$s_1(x, y) + a_1(x, y) = s_2(x, y) + a_2(x, y) \Rightarrow s_1(x, y) = s_2(x, y).$$

We conclude that the representation is unique.

Exercise 2 (Exercise 2)**.** (Review.) Write the power sum symmetric function p_3 in terms of elementary symmetric functions.

Answer

This result is valid in any number of variables, so let verify it in three variables and then extrapolate the general formula. Recall that in three variables the elementary symmetric functions are

$$e_1 = x + y + z, \; e_2 = xy + yz + zx, \; e_3 = xyz$$

so naively we can cube e_1 first and see what we get:

$$e_1^3 = (x + y + z)^3 = p_3 + 3(x^2y + y^2z + z^2x + y^2x + z^2y + x^2z) + 6e_3.$$

The way to obtain the middle term is to multiply e_1 with e_2:

$$e_1e_2 = x^2y + y^2z + z^2x + y^2x + z^2y + x^2z + 3e_3,$$

solving for the expression we want we obtain

$$3e_1e_2 - 9e_3 = 3(x^2y + y^2z + z^2x + y^2x + z^2y + x^2z)$$

and finally

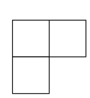
$$e_1^3 = p_3 + 3e_1e_2 - 9e_3 + 6e_3 \Rightarrow \underline{p_3 = e_1^3 - 3e_1e_2 + 3e_3}.$$

We conclude that this is indeed the representation of p_3 in terms of the $e_i's$.

Exercise 3 (Exercise 5)**.** Compute the Schur polynomial $s_{(2,1)}(x, y, z)$ as a ratio of determinants and using semi-standard Young Tableaux. Show that both computations agree.

Answer

[a]The partition $(2, 1)$ is associated to the following Ferrers Diagram[b]:



and since we have 3 variables to work with, we must fill out the diagram with numbers from 1 to 3 with the condition that rows are weakly increasing and columns increase. Out of the possible 27 ways to fill out the diagram, only the following are possible given the condition:

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}.$$

The associated monomials are

$$x^2y,\ xy^2,\ xyz,\ x^2z,\ xyz,\ y^2z,\ xz^2,\ yz^2.$$

So it follows that $s_{(2,1)}(x,y,z) = x^2y + xy^2 + 2xyz + x^2z + y^2z + xz^2 + yz^2$. In this case it's important to notice that $(2,1) = (2,1,0)$ so $s_{(2,1)} = s_{(2,1,0)}$, now using the bi-alternant we have

$$a_{(2,1,0)}(x,y,z) = \det\begin{pmatrix} x^2 & x & 1\\ y^2 & y & 1\\ z^2 & z & 1\end{pmatrix} = x^2y + xz^2 + y^2z - z^2y - zx^2 - y^2x,$$

$$a_{(4,2,0)}(x,y,z) = \det\begin{pmatrix} x^4 & x^2 & 1\\ y^4 & y^2 & 1\\ z^4 & z^2 & 1\end{pmatrix} = x^4y^2 + x^2z^4 + y^4z^2 - z^4y^2 - z^2x^4 - y^4x^2.$$

To factor this polynomials we have to *creatively sum zero*, in this case $0 = xyz - xyz$ and $0 = x^2y^2z^2 - x^2y^2z^2$. For $a_{(4,2,0)}$ we have

$$\begin{aligned}
&x^4y^2 + x^2z^4 + y^4z^2 - z^4y^2 - z^2x^4 - y^4x^2 + x^2y^2z^2 - x^2y^2z^2\\
=&x^2(x^2y^2 - z^2x^2 - y^4 + y^2z^2) - z^2(z^2y^2 - x^2z^2 - y^4 + x^2y^2)\\
=&(x^2 - z^2)[x^2(y^2 - z^2) - y^2(y^2 - z^2)]\\
=&(x^2 - z^2)(x^2 - y^2)(y^2 - z^2)\\
=&(x - z)(x + z)(x - y)(x + y)(y - z)(y + z)
\end{aligned}$$

In the case of $a_{(2,1,0)}$ we get $(x-z)(x-y)(y-z)$. It thus follows that

$$s_{(2,1)} = \frac{a_{(4,2,0)}}{a_{(2,1,0)}} = (x+z)(x+y)(y+z) = x^2y + x^2z + xy^2 + 2xyz + xz^2 + y^2z + yz^2$$

so both computations agree on the value of $s_{(2,1)}$.

[a] I started doing this problem because **Kelsey** wanted to check the calculations with me.

[b] Is there a way to draw Ferrers diagrams in French notation? The package I'm using `youngtab` only admits English notation. I mean, **you** must surely know.

Exercise 4 (Exercise 7, Stanley 7.3)**.** Expand the power series $\prod_{i \geqslant 1}(1 + x_i + x_i^2)$ in terms of elementary symmetric functions.

Answer

We will begin with a lower dimensional cases, call f our infinite product and f_n its partial products. Consider the case when $n = 2$, we have the following decomposition for f_2:

$$\begin{aligned}(1+x+x^2)(1+y+y^2) &= x^2y^2 + x^2y + x^2 + xy^2 + xy + x + y^2 + y + 1\\ &= e_2^2 + e_1^2 + e_2e_1 - e_2 + e_1 + 1.\end{aligned}$$

Likewise for f_3 we have

$$f_3 = e_3^2 + e_2^2 + e_1^2 + e_2e_1 + e_3e_2 - e_3e_1 - 2e_3 - e_2 + e_1 + 1.$$

Expanding[a] f_4 we see a similar pattern with the second powers however:

$$f_4 = p_2(e_1, \dots, e_4) + e_2e_1 + e_3e_2 + e_4e_3 - 2e_4e_1 - e_3e_1 - e_4e_2 - e_4 - 2e_3 - e_2 + e_1 + 1.$$

A pattern that we can observe is the sum of powers, and also, all the previous terms are used in the coming expansion. Notice that

$$\begin{aligned}f_3 - f_2 &= e_3^2 + e_3e_2 - e_3e_1 - 2e_3,\\ f_4 - f_3 &= e_4^2 + e_4e_3 - 2e_4e_1 - e_4e_2 - e_4.\end{aligned}$$

This strange coefficients might come from an unexpected place. Let us recall[b] that

$$(x-1)^3 = (x-1)(x^2+x+1) = (x-1)(x - e^{\frac{i\pi}{3}})(x - e^{i\frac{2\pi}{3}}).$$

Now, notice that the last two factors can be rearranged to be

$$e^{i\frac{\pi}{3}}e^{i\frac{2\pi}{3}}(xe^{-i\frac{\pi}{3}} - 1)(xe^{-i\frac{2\pi}{3}} - 1) = (1 - xe^{-i\frac{\pi}{3}})(1 - xe^{-i\frac{2\pi}{3}}).$$

Thus we can rearrange the product in question as follows:

$$\prod_{i \geqslant 1}(1 + x_i + x_i^2) = \left(\prod_{k \geqslant 1}(1 - x_k e^{-i\frac{\pi}{3}})\right)\left(\prod_{k \geqslant 1}(1 - x_k e^{-i\frac{2\pi}{3}})\right)$$

and the products on the right are the generating functions of the elementary symmetric functions evaluated at $e^{-i\frac{\pi}{3}}$ and $e^{-i\frac{2\pi}{3}}$ respectively. We can rewrite them as

$$\left(\sum_{n \geqslant 1} e_n e^{i\frac{\pi n}{3}}\right)\left(\sum_{n \geqslant 1} e_n e^{i\frac{2\pi n}{3}}\right) = \sum_{n \geqslant 1}\left(\sum_{k=1}^{n}\left(e^{i\frac{\pi k}{3}}\right)\left(e^{i\frac{2\pi(n-k)}{3}}\right)\right)e_n$$

Simplifying the exponentials we get

$$\left(e^{i\frac{\pi k}{3}}\right)\left(e^{i\frac{2\pi(n-k)}{3}}\right) = e^{\frac{i\pi}{3}(k+2n-2k)} = e^{\frac{i\pi(2n-k)}{3}}.$$

We conclude that

$$\prod_{i\geqslant 1}(1+x_i+x_i^2) = \sum_{n\geqslant 1}\left(\sum_{k=1}^{n} e^{\frac{i\pi(2n-k)}{3}}\right) e_n.$$

[a] This work was done using the **Macaulay2** software.

[b] This idea was brought to my attention by **Jae Hwang** while discussing the problem the problem with him.