Exercise 1. Short answers:

1. If $\vec{u}$ and $\vec{v}$ are parallel, what is the algebraic relation between them?
2. Suppose a line ℓ is normal to a plane π. What is the relation between $\vec{v}_\ell$ and $\vec{n}_\pi$?
3. If $f:\mathbb{R}^3 \to \mathbb{R}$ is a function, how many components does its gradient ∇f have?
4. If $\nabla f(\vec{a}) \cdot \vec{u} < 0$, does f increase or decrease in the direction of $\vec{u}$ at $\vec{a}$?
5. If f has $\det Hf > 0$ and $f_{xx} > 0$ at $\vec{x} = \vec{a}$, can you conclude if f has a maximum, minimum, or saddle point at $\vec{a}$?

1. $\vec{u}$ and $\vec{v}$ are multiples. This means $\vec{v} = c\vec{u}$ for some scalar c.
2. If the line is normal to the plane, it's parallel to the normal vector of the plane. So $\vec{v}_\ell \parallel \vec{n}_\pi$.
3. A three-dimensional function has 3 entries on its gradient.
4. This means that the directional derivative is negative in direction of u. So it's aligned opposite to $\nabla f(\vec{a})$, this means that it's decreasing in direction of $\vec{u}$ at $\vec{a}$.
5. Positive determinant of Hessian implies that the function doesn't have a saddle point. As f_{xx} is positive, f looks like *a happy face* in the x direction, therefore it's a minimum.

Exercise 2. Consider the function $f(x,y) = 8x + 8y$ on the region described by $\{4x^2 = y^2 - 1\}$. There are two critical points. Follow these steps to classify them:

1. Identify the function f you're optimizing.
2. Identify the constraint as $g = 0$.
3. Write the Lagrange equation: $\nabla f = \lambda \nabla g$.
4. Solve for $\vec{x}$ in terms of λ, then use $g = 0$ to find λ.
5. Find $\vec{x}$ using λ and classify the points by evaluating f.

1. The function to optimize is $f(x,y) = 8x + 8y$.
2. The constraint is
$$g(x,y) = 4x^2 - y^2 + 1 = 0.$$
3. The Lagrange equation is
$$(8,8) = \lambda(8x, -2y).$$
4. We may solve for x,y via the equations
$$\begin{cases} 8 = 8x \\ 8 = -2y \end{cases} \Rightarrow \begin{cases} \dfrac{1}{\lambda} = x \\ \dfrac{-4}{\lambda} = y \end{cases}$$
So plugging these values into $g = 0$ we get:
$$4\left(\frac{1}{\lambda}\right)^2 - \left(\frac{-4}{\lambda}\right)^2 + 1 = 0 \Rightarrow \frac{-12}{\lambda^2} = -1 \Rightarrow \lambda = \pm 2\sqrt{3}.$$

5. The critical values are thus
$$(x_1,y_1)=\left(\frac{1}{2\sqrt{3}},\frac{-2}{\sqrt{3}}\right) \quad \text{and} \quad (x_2,y_2)=\left(\frac{-1}{2\sqrt{3}},\frac{2}{\sqrt{3}}\right).$$
Plugging them into f we get the values
$$f(x_1,y_1)=\frac{8}{2\sqrt{3}}+\frac{-16}{\sqrt{3}}=\frac{-12}{\sqrt{3}} \quad \text{and} \quad f(x_2,y_2)=\frac{-8}{2\sqrt{3}}+\frac{16}{\sqrt{3}}=\frac{12}{\sqrt{3}}$$
where we see that $f(x_1,y_1)<f(x_2,y_2)$ so (x_2,y_2) is a maximizing point whereas (x_1,y_1) is a minimizing point.