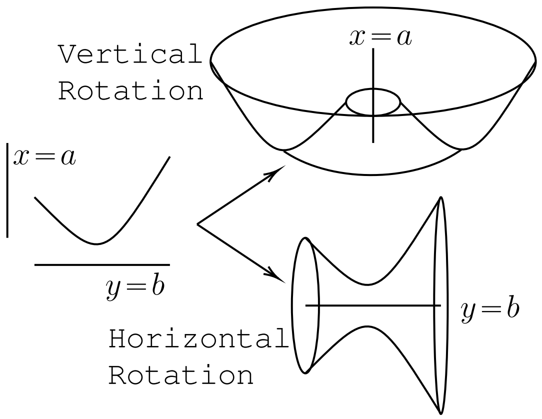
Applications of Integrals

Volume and Solids of Revolution

Definition. A <u>solid of revolution</u> is the figure obtained by rotating a curve about an axis. In most cases the axis will be a *horizontal* or *vertical* line.



Method of the Rings

To obtain the volume of the solid we will cut cross-sectional areas *perpendicular* to the axis of rotation and integrate through the bounds of the curve.

These cross-sectional areas will take the form of **disks**, whose areas can be calculated using the formula

$$\underline{\pi(R^2-r^2)}.$$

Our task is to determine the radii R and r as a function of x or y.

Example 1. We will find the volume of the solid obtained by rotating the region bounded by the curves

$$\begin{cases} f(x)=\sqrt[3]{x}, \\ g(x)=x/4, \end{cases}$$

inside the first quadrant through the x-axis.

I) First we find the bounds of our curves. This is done by equating both expressions:
$$\sqrt[3]{x}=x/4 \iff x=x^3/64 \Rightarrow x^2=64 \text{ or } x=0 \iff x\in\{0,\pm 8\}.$$
Since we are in the first quadrant the intersections must be <u>$x=0$ and $x=8$</u>.

II) After graphing these curves we see that the upper curve is $f(x)$ and the lower is $g(x)$, so we obtain
$$R=f(x),\ r=g(x).$$

III) The area of the larger disk is πR^2 and from that amount we subtract the area of the smaller disk πr^2 to obtain the cross-sectional disk's area:
$$A(x)=\pi(x^{2/3}-x^2/16).$$

IV) Finally we integrate through the bounds we found to obtain the volume:
$$V=\int_0^8 \pi(x^{2/3}-x^2/16)\mathrm{d}x=\frac{128\pi}{15}.$$

Example 2. Let us rotate the region from the last example about the line $x=-2$.

In this case this is now a **vertical rotation**, and we must switch our equations to be in terms of y.

I) We switch the equations:
$$\begin{cases} y=\sqrt[3]{x} \iff x=y^3 \Rightarrow h(y)=y^3, \\ y=x/4 \iff x=4y \Rightarrow k(y)=4y. \end{cases}$$

II) The intersections have not changed, but their y-coordinates are
$$y^3=4y \Rightarrow y^2=4 \text{ or } y=0 \iff y\in\{0,\pm 2\}.$$
Since we are in the first quadrant, the y-bounds are <u>$y=0$ and $y=2$</u>.

III) h and k are the same curves as f and g just that now k is the right one and h is the left one. Thus
$$\begin{cases} R=\text{Right}-\text{axis}=4y-(-2), \\ r=\text{Left}-\text{axis}=y^3-(-2). \end{cases}$$

IV) The area of the cross-sectional disk is
$$A(y)=\pi\big[(4y+2)^2-(y^3+2)^2\big].$$

V) We integrate the area through $y=0$ and $y=2$ to obtain
$$V=\int_0^2 \pi(16y+16y^2-4y^3-y^6)\mathrm{d}y=\frac{848\pi}{21}$$

> *Remark.* When rotation about a line $x=a$ we are doing a **vertical rotation**. While rotations about $y=b$ are **horizontal rotations**. Here a,b are any real number.
> When doing a **vertical rotation** about the axis $x=a$ the radii will be:
> $$\begin{cases} R=\text{Right}-\text{axis}=\text{Right}-a, \\ r=\text{Left}-\text{axis}=\text{Left}-a. \end{cases}$$
> On the other hand, for **horizontal rotations** about $y=b$ we have
> $$\begin{cases} R=\text{Up}-\text{axis}=\text{Up}-b, \\ r=\text{Down}-\text{axis}=\text{Down}-b. \end{cases}$$

Exercise 3. Determine the volume of the solid obtained by rotating the region bounded by the curves $y=6e^{-2x}$, $y=6+4x-x^2$ and $x=1$ about the axis $y=-1$. $\left[V=\frac{937}{15}+\frac{12}{e^2}+\frac{9}{e^4}\right]\pi$.

Exercise 4. Determine the volume of the solid obtained by rotating the region bounded by the curves $x=y^2-4$ and $x=6-3y$ about the axis $x=24$. $[V=(31556/15)\pi]$.

Exercise 5. Determine the volume of the solid obtained by rotating the triangle bounded by $y=2x+1$, $x=4$ and $y=3$ about the axis $x=-4$. $[V=126\pi]$.

Method of the Cylinders

We know cut cross-sectional areas *parallel* to the axis of rotation. These areas look like *cylinders* (or shells). Their area is given by

$$\underline{2\pi r h}$$

and once again we find r and h as functions of x and y.

Example 6. Let us find the solid of revolution formed by rotating the region bounded by the curves

$$\begin{cases} f(x)=(x-1)(x-3)^2, \\ y=0 \ (x \text{ axis}), \end{cases}$$

inside the first quadrant about the axis $x=0$ (y-axis).

I) We first find the bounds of integration by equating the expressions:
$$(x-1)(x-3)^2=0 \iff x=1 \text{ or } x=3.$$
So the intersections are $\underline{x=1 \text{ and } x=3}$.

II) We graph the curves to see that $f(x)$ lies above the x axis inside the interval $[1,3]$ but now we have that the height of the cylinders will be
$$h(x)=\text{Up}-\text{Down}=(x-1)(x-3)^2-0.$$
The radius r is the distance from the axis $x=0$ to the cylinders, this distance is precisely x.

III) Thus the area of the cylinder at any point will be
$$A(x)=2\pi(x)[(x-1)(x-3)^2].$$

IV) We integrate the area through $x=1$ and $x=3$ to obtain the volume:
$$V=\int_1^3 [2\pi(x^4-7x^3+15x^2-9x)]\mathrm{d}x=24\pi/5.$$

Remark. If we had used the method of the rings in the last example, we would have to express $f(x)$ in terms of y, and inverting a cubic polynomial is not an easy task.

Example 7. We will find the volume of the solid of revolution formed by rotating the region bounded by

$$\begin{cases} x=(y-2)^2, \\ y=x, \end{cases}$$

inside the first quadrant, about the axis $y=-1$.

I) We first get the intersections:
$$(y-2)^2=y \iff y^2-5y+4=0 \iff y\in\{1,4\}.$$
Our bounds will be $\underline{y=1 \text{ and } y=4}$.

II) After graphing the curves, we see that the <u>rightmost one</u> is $x=y$ and the <u>left one</u> is $x=(y-2)^2$. Thus
$$h(y)=\text{Right}-\text{Left}=y-(y-2)^2.$$
In this case the radius is the distance from the axis $y=-1$ to the point on the cylinder which is precisely y. So $\underline{r=y-(-1)}$.

III) The area of the cylinder is
$$2\pi r h=2\pi(y+1)[y-(y-2)^2].$$

IV) We integrate to obtain the volume
$$V=\int_1^4 2\pi(-y^3+4y^2+y-4)\mathrm{d}y=63\pi/2.$$

> *Remark.* In general for the method of the cylinders, when doing a **vertical rotation** about the axis $x=a$ we have:
> $$\begin{cases} r=\text{Up}-\text{Down}, \\ h=x-\text{axis}=x-a. \end{cases}$$
> On the other hand, for **horizontal rotations** about $y=b$ we have
> $$\begin{cases} r=\text{Right}-\text{Left}, \\ h=y-\text{axis}=y-b. \end{cases}$$

Exercise 8. Find the volume of the solid obtained by rotating the region bounded by the curves

$$\begin{cases} y=e^{x/2}/(x+2), \\ y=5-x/4, \\ x=-1, \text{ and } x=6, \end{cases}$$

about the axis $x=-2$. $[V=2\pi(392/3+2/\sqrt{e}-2e^3)]$.

Exercise 9. Determine the volume of the solid obtained by rotating the region bounded by the curves

$$\begin{cases} x=y^2-4, \\ x=6-3y, \end{cases}$$

about the axis $y=-8$. $[V=(4459/6)\pi]$.

Exercise 10. Determine the volume of the solid obtained by rotating the region bounded by the curves

$$\begin{cases} y=x^2-6x+9, \\ y=-x^2+6x-1, \end{cases}$$

about the axis $x=8$. $[V=(640/3)\pi]$.