Exercise 1 (Mandatory+2). Use the `ytableau` package to typeset the following skew tableau in LaTeX. Rectify the tableau in question:

Answer

Behold:

$$\begin{ytableau} 8 & 10 \\ \none & 5 & 5 \\ \none & \none & 4 & 6 & 6 \\ \none & \none & 1 & 2 & 2 & 3 \\ \none & \none & \none & \none & 1 & 1 & 2 & 7 & 9 \end{ytableau}$$

We apply the topmost inner-slide we can find in each step as follows:

(a)
$$\begin{ytableau} 8 & 10 \\ 5 & 5 \\ \none & \none & 4 & 6 & 6 \\ \none & \none & 1 & 2 & 2 & 3 \\ \none & \none & \none & \none & 1 & 1 & 2 & 7 & 9 \end{ytableau}$$

(b)
$$\begin{ytableau} 8 & 10 \\ 5 & 5 \\ \none & 4 & 6 & 6 \\ \none & \none & 1 & 2 & 2 & 3 \\ \none & \none & \none & \none & 1 & 1 & 2 & 7 & 9 \end{ytableau}$$

(c)
$$\begin{ytableau} 8 \\ 5 & 10 \\ 4 & 5 & 6 & 6 \\ \none & \none & 1 & 2 & 2 & 3 \\ \none & \none & \none & \none & 1 & 1 & 2 & 7 & 9 \end{ytableau}$$

(d)
$$\begin{ytableau} 8 \\ 5 & 10 \\ 4 & 5 & 6 & 6 \\ \none & 1 & 2 & 2 & 3 \\ \none & \none & \none & \none & 1 & 1 & 2 & 7 & 9 \end{ytableau}$$

(e)
$$\begin{ytableau} 8 \\ 5 & 10 \\ 4 & 5 & 6 & 6 \\ 1 & 2 & 2 & 3 \\ \none & \none & \none & \none & 1 & 1 & 2 & 7 & 9 \end{ytableau}$$

(f)
$$\begin{ytableau} 8 \\ 5 & 10 \\ 4 & 5 & 6 \\ 2 & 2 & 3 & 6 \\ 1 & 1 & 1 & 2 & 7 & 9 \end{ytableau}$$

Exercise 2 (Exercise 4). Suppose λ/μ is a horizontal strip skew shape of size $n = |\lambda| - |\mu|$, consisting of rows of lengths $\alpha_1, \ldots, \alpha_k$. Show that

$$s_{\lambda/\mu} = h_{\alpha_1} \ldots h_{\alpha_k} = \sum_{\nu} K_{\nu\alpha} s_\nu.$$

Answer

The word α can be arranged in order $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_k$ to make it a partition, from this, the second equality can be proven as we did in the lecture notes. Now for the first equality we can see that each h_{α_i} accounts (as a generating function) for the ways to fill out the row i. As the tableau is a horizontal strip, the rows are independent of each other so the ways to fill them out are accounted on the product

$$h_{\alpha_1} \ldots h_{\alpha_k}.$$

The function $s_{\lambda/\mu}$'s combinatorial definition accounts for the same thing, so it must hold that both expressions are equal.

Exercise 3 (Exercise 6)**.** Given an (undirected, labeled) graph G, a proper coloring of G is an assignment of a positive integer "color" to each vertex such that no two adjacent vertices have the same color.

If the colors assigned to the vertices are $c_1, c_2, \ldots, c_n$ (with some c_i's possibly being equal to each other), define the monomial of the coloring C to be

$$x^c = x_{c_1} x_{c_2} \cdots x_{c_n}.$$

Finally, define the chromatic symmetric function of G to be

$$X_G(\underline{x}) = \sum_c x^c$$

where the sum ranges over all proper colorings c of G.

i) Show that X_G is indeed a symmetric function for any graph G.
ii) Prove that if K_n is the complete graph on n vertices, then $X_{K_n} = e_n$.
iii) Compute X_{P_3}, X_{P_4} and X_{P_5} and express them in terms of elementary and Schur bases.

Answer

Let's begin by talking again about the chromatic symmetric function. Suppose $c : G \to [r]$ is an r-coloring of G, so for that particular coloring the monomial will be

$$x_1^{|c^{-1}(1)|} x_2^{|c^{-1}(2)|} \cdots x_r^{|c^{-1}(r)|},$$

where $c^{-1}(i)$ is the inverse image of i, the vertices which are colored i.

i) To show that X_G is symmetric, we must show that

$$X_G(\underline{x}) = X_G(\sigma(\underline{x})) = X_G(x_{\sigma(1)}, x_{\sigma(2)}, \dots).$$

The permutation acts on the indices which represent the colors, so a permutation of the variables is a permutation on the colors used to paint the graph. Let us see that after permuting the colors, we still get a proper coloring.

With our coloring $c : G \to [r]$, let $\sigma \in S_r$. Pick $u, v \in G$ with $uv \in E$ such that $c(u) \neq c(v)$. So, as σ is a permutation we

$$c(u) \neq c(v) \Rightarrow \sigma(c(u)) \neq \sigma(c(v)) \Rightarrow \widetilde{c}(u) \neq \widetilde{c}(v)$$

where we have defined $\widetilde{c} = \sigma \circ c$. The function $\widetilde{c}$ is also a proper coloring of G and so, as σ was arbitrary, we see that a permutation of the colors gives us another proper coloring.

In other words, a particular vertex colored i is colored j after applying σ. As the vertex had no neighbors colored i, it won't have j neighbors so the coloring is proper.

Finally, as X_G runs through all possible colorings, after permuting we get the same sum but in a different order by the previous argument. We conclude that $X_G(\underline{x}) = X_G(\sigma(\underline{x}))$ and therefore X_G is symmetric.

ii) In the complete graph, all the vertices are connected which means proper colorings of K_n use n colors. So expanding X_{K_n} by monomials, we see that each monomial contains n different variables where each one is related to each vertex on K_n. Such expansion can be written as

$$X_{K_n} = \sum_{(*)} x_{i_1} x_{i_2} \dots x_{i_n}$$

where $(*) : i_1 \neq i_2, i_1, \neq i_3, \dots, i_2 \neq i_3 \dots$ and so on. By ordering our vertices we get that $(*)$ becomes $i_1 < i_2 < \cdots < i_n$ which brings us to e_n. However[a] in this count, we are considering unlabeled vertices. By labeling vertices we get that X_{K_n} is actually $n! e_n$ because of the $n!$ ways we can paint n labeled objects with n distinct colors.

iii) The chromatic number of a path graph is 2, however the sum runs over all proper colorings so we may use more than 2.

- Beginning with P_3 we may color by alternating the colors corresponding to monomials of the form $x_i^2 x_j$ or by painting all of the vertices differently ($x_i x_j x_k$). However we have $3! = 6$ ways of painting the 3 vertices with 3 colors. This means that

$$X_{P_3} = m_{(2,1)} + 6m_{(1,1,1)}.$$

Using CoCalc to convert to the elementary and Schur basis we get

$$X_{P_3} = e_{(2,1)} + 3e_3 = s_{(2,1)} + 4s_{(1,1,1)}.$$

- Let us now remember that 4 can be partitioned into

$$(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad \text{and} \quad (1,1,1,1)$$

The partition corresponds to a coloring in the sense that $\lambda_i = |c^{-1}(i)|$ is the number of vertices colored i. While (4) and $(3,1)$ don't work, we may[b] paint with $(2,2)$ because we may have either

$$RBRB \quad \text{or} \quad BRBR.$$

So this means that we have 2 possible colorings with this partition. The partition $(2,1,1)$ is the partition $(2,1)$ after adding the color G. Originally in P_3 we only had RBR but we can insert G at the ends or between any two colors which gives us four options.

$$RBRG, \quad RBGR, \quad RGBR, \quad \text{and} \quad GRBR.$$

However we can now consider $RGRB$ and $BRGR$ which were previously not allowed because the coloring wouldn't have been proper without the green in the middle. In total we have a $6m_{(2,1,1)}$ term. The partition $(1,1,1,1)$ accounts for all the possible 4-colorings of P_4 which are 4. Summarizing we have

$$\begin{cases} 2x_i^2 x_j^2 \leftarrow \text{2 by alternating 2 colors.} \\ 6x_i^2 x_j x_k \leftarrow \text{inserting in alternating } P_3 \text{ plus not allowed.} \\ 4! x_i x_j x_k x_\ell \leftarrow \text{4 colors.} \end{cases}$$

This means that

$$\begin{aligned} X_{P_4} &= 2m_{(2,2)} + 6m_{(2,1,1)} + 24m_{(1,1,1,1)} \\ &= 2e_{(3,1)} + 2e_{(2,2)} + 4e_4 \\ &= 2s_{(2,2)} + 4s_{(2,1,1)} + 8s_{(1,1,1,1)}. \end{aligned}$$

◇ Following the idea we consider the valid partitions of 5 which can be used to color. These are:

$$(3,2), \quad (3,1,1), \quad (2,2,1), \quad (2,1,1,1), \quad \text{and} \quad (1,1,1,1,1).$$

There's only one possible coloring with $(3,2)$ but two with $(3,1,1)$ because we can switch the second and third colors. So we begin with a $m_{(3,2)} + 2m_{(3,1,1)}$.
The partition $(2,2,1)$ allows us to insert G in the middle of any pair of our previous two colorings. This gives us 10 possibilities for coloring but additionally we had previously unavailable colorings which are now allowed by inserting the green vertex. These are

$$RBGBR, \quad \text{and} \quad BRGRB$$

which brings us to a total of 12 possible colorings and a factor of $12m_{(2,2,1)}$.
Finally the partition $(2,1,1)$ previously gave us 6 colorings, inserting Y^c in between any of the 5 possible slot in these colorings gives us 30 colorings. Now the disallowed block of colors RR becomes available when inserting yellow in the middle. The remaining BG can receive this whole block in the ends or in the middle. We also have GB so in total we have 6 more possibilities. In total this account for 36 counts of $m_{(2,1,1,1)}$. The final coloring can be done in $5!$ ways so in summary:

$$\begin{cases} x_i^3 x_j^2 \leftarrow \text{alternating 2 colors.} \\ 2x_i^3 x_j x_k \leftarrow \text{truncating with color } j \text{ and } k. \\ 12x_i^2 x_j^2 x_k \leftarrow \text{like } (2,2) \text{ but inserting and 2 disallowed.} \\ 36x_i^2 x_j x_k x_\ell \leftarrow \text{same as } P_4 \text{ and inserting plus 6 disallowed.} \\ 120x_i x_j x_k x_\ell x_m \leftarrow \text{all colors.} \end{cases}$$

We obtain the following:

$$\begin{aligned} X_{P_5} &= m_{(3,2)} + 2m_{(3,1,1)} + 12m_{(2,2,1)} + 36m_{(2,1,1,1)} + 120m_{(1,1,1,1,1)} \\ &= 5e_5 + e_{(4,1)} + 7e_{(3,2)} + e_{(2,2,1)} \\ &= s_{(3,2)} + s_{(3,1,1)} + 9s_{(2,2,1)} + 12s_{(2,1,1,1)} + 16s_{(1,1,1,1,1)}. \end{aligned}$$

[a] I owe this observation **Kelsey** and **Trent**. Originally they said that $RBRB$ and $BRBR$ were different whence I said the contrary. I argued that in that case X_{K_n} should be $n!e_n$, but it *wasn't*. **You** were there for the climax of the story.

[b] I will color according to the following order (red,blue,green,yellow,…).

[c] There's a yellow Y here.