Exercise 1. (Exercise 3.12.11) Show that

$$\mathcal{F}\ell(d_1, \dots, d_k) \cong \mathrm{O}(n)/(\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k)),$$

where $n_1 = d_1$ and $n_i = d_i - d_{i-1}$ for $i = 2, \dots, k$. (In other words, the n_i are the jumps in dimension as we go up the flag.)

Exercise 2. Let M be a manifold with an affine connection ∇. Suppose $\alpha : I \to M$ is a constant curve; that is, $\alpha(t) = p$ for all $t \in I$. Let V be a vector field along α, meaning that $V(t) \in T_{\alpha(t)}M = T_pM$ just gives a curve in the tangent space T_pM. Show that $\frac{DV}{dt} = V'(t)$; that is, the covariant derivative agrees with the usual derivative in this case, regardless of what ∇ is.

Answer

Observe that along a curve α we have

$$\frac{DV}{\mathrm{d}t} = \nabla_{\frac{\mathrm{d}\alpha}{\mathrm{d}t}} V = \sum_{i,j,k} \left(\frac{\mathrm{d}v_k}{\mathrm{d}t} + \frac{\mathrm{d}\alpha_i}{\mathrm{d}t} v_j \Gamma_{ij}^k \right) X_k.$$

As our curve is constant, the terms on the right all cancel out so that we're left with

$$\frac{DV}{\mathrm{d}t} = \sum_k \frac{\mathrm{d}v_k}{\mathrm{d}t} X_k = V'(t).$$

Exercise 3. (Exercise 4.3.4) Show that an affine connection ∇ is compatible with a Riemannian metric g on M if and only if, for any vector fields V and W along a smooth curve $\alpha : I \to M$, we have

$$\left.\frac{d}{dt}\right|_{t=t_0} g_{\alpha(t)}(V(t), W(t)) = g_{\alpha(t_0)}\left(\frac{DV}{dt}, W\right) + g_{\alpha(t_0)}\left(V, \frac{DW}{dt}\right).$$

In other words, for compatible connections we can use the usual product rule to differentiate the inner product.

Answer

Let us suppose first that ∇ is compatible with g. If α is a curve, we may take an orthonormal basis of $T_{\alpha(t_0)}M$:

$$\{ u_1(t_0), \dots, u_n(t_0) \}.$$

As ∇ is compatible with g, we may parallel-transport this basis throughout all the curve α. This means that for any $t \in I$,

$$\langle u_1(t), \ldots, u_n(t) \rangle = T_{\alpha(t)} M.$$

Now, our vector fields V, W may be expressed as linear combinations of these basic elements in the following way:

$$\begin{cases} V(t) = \sum_{k=1}^{n} \alpha_k u_k(t) \\ W(t) = \sum_{k=1}^{n} \beta_k u_k(t) \end{cases} \Rightarrow \begin{cases} \dfrac{DV}{dt} = \sum_{k=1}^{n} \alpha_k' u_k(t) \\ \dfrac{DW}{dt} = \sum_{k=1}^{n} \beta_k' u_k(t) \end{cases}$$

where α_k, β_k are smooth functions. Now if we compute the quantity of the left, we have that

$$\begin{aligned} &\left.\frac{d}{dt}\right|_{t=t_0} g_{\alpha(t)}(V(t), W(t)) \\ &= \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left.\frac{d}{dt}\right|_{t=t_0} \alpha_k \beta_\ell g_{\alpha(t)}(u_k(t), u_\ell(t)) \\ &= \sum_{k=1}^{n} \left.\frac{d}{dt}\right|_{t=t_0} \alpha_k \beta_k \\ &= \sum_{k=1}^{n} \left.\frac{\mathrm{d}\alpha_k}{\mathrm{d}t}\right|_{t=t_0} \beta_k + \sum_{k=1}^{n} \alpha_k \left.\frac{\mathrm{d}\beta_k}{\mathrm{d}t}\right|_{t=t_0} \end{aligned}$$

and then readding indices by multiplying $\delta_{k\ell}$ and a sum through ℓ we recover the final expression:

$$\sum_{\ell=1}^{n} \sum_{k=1}^{n} \left.\frac{\mathrm{d}\alpha_k}{\mathrm{d}t}\right|_{t=t_0} \beta_k g_{\alpha(t_0)}(u_k(t), u_\ell(t)) + \sum_{\ell=1}^{n} \sum_{k=1}^{n} \alpha_k \left.\frac{\mathrm{d}\beta_k}{\mathrm{d}t}\right|_{t=t_0} g_{\alpha(t_0)}(u_k(t), u_\ell(t)).$$

Condensing everything by linearity we recover

$$g_{\alpha(t_0)}\left(\frac{DV}{dt}, W\right) + g_{\alpha(t_0)}\left(V, \frac{DW}{dt}\right).$$

Now on the other hand suppose we have the identity in question. In order to show that our connection is compatible with g, we must show that for V, W

parallel along α, we have that $g_\alpha(t)(V(t), W(t))$ is constant. To that effect, we will show that it has zero derivate.
Let V, W be parallel vector fields along $\alpha(t)$. Then

$$\frac{DV}{\mathrm{d}t} = \frac{DW}{\mathrm{d}t} = 0,$$

and so our identity becomes

$$0 = g_{\alpha(t_0)}\left(0, W\right) + g_{\alpha(t_0)}\left(V, 0\right) = \left.\frac{d}{dt}\right|_{t=t_0} g_{\alpha(t)}(V(t), W(t)).$$

Thus $g_\alpha(t)(V(t), W(t))$ is a constant function because it has zero derivate.