MATH519 — Complex Analysis

Based on the lectures by Jeff Achter

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This course is an introduction to analytic functions of a single complex variable. The subject is beautiful.– it turns out that a function with a complex derivative is highly structured – and enjoys a give and take with many other areas of mathematics.

Requirements

Knowledge of convergence of sequences, series: limits, continuity, differentiation, integration of one-variable functions is required.

Contents

Chapter 1

Symmetric functions

1.1 Recall

Definition 1.1.1. $f(x_1, x_2, \dots)$ is symmetric if it's fixed under permutations of variables.

Example 1.1.2. $f(x_1, \dots, x_4) = x_1^5 + \cdots + x_4^5$. This is known as p_5 or $m_{(5)}$, where p is the power-sum symmetric function and m, the monomial symmetric function.

Example 1.1.3. Consider $g = x_1^4x_2 + x_1^4x_3 + \cdots + x_i^4x_j + \cdots + 3x_1 + \cdots = m_{(4,1)} + 3m_{(1)}$.

Let us recall some **notation**:

i) $\Lambda_R(x_1, \dots, x_n)$ is the ring of symmetric polynomials over R. In *infinitely* many variables we have $\Lambda_R(\underline{x})$.

In the case $R = \mathbb{Q}$, then $\dim \Lambda_Q(\underline{x})_{(d)}$, where every monomial has degree d, is $p(d)$. This is the number of partitions of d. Because for every partition we can form monomials and monomials form a basis.

Bases of Λ_Q

Suppose $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geqslant \dots \geqslant \lambda_k$.

- ⋄ Monomial: $m_\lambda = \sum_{i_1 \neq \dots i_k} x_{i_1}^{\lambda_1} \dots x_{i_k}^{\lambda_k}$.
- ⋄ Elementary: $e_\lambda = \prod e_{\lambda_i}$ where $e_d = m_{(1,1,\dots,1)}$ (d ones).
- ⋄ Homogenous: $h_\lambda = \prod h_{\lambda_i}$ and $h_d = x_1^d + \cdots + x_1^{d-1}x_2 + \cdots + x_1^{d-2}x_2^2 + x_1^{d-2}x_2x_3 + \dots$. In general $h_d = \sum_{\lambda \vdash d} m_\lambda$.
- ⋄ Power sum: $p_\lambda = \prod p_{\lambda_i}$ and $p_d = \sum x_i^d$.

For Schur basis recall SSYT

Example 1.1.4. Consider $\lambda = (5,4,1)$, rows $\leqslant\rightarrow$ and columns $<$, we associate the monomial $x_1^2x_2^3x_3^3x_4^2 := x^T$.

- Schur: $s_\lambda = \sum_{T\in SSYT(\lambda)} x^T$ but also $\sum K_{\lambda\mu}m_\mu$ where the sum is over SSYT of shape λ, content μ.

Schur function motivation (preview)

The first place they showed up is in the representation theory of Lie group. The function $s_\lambda(x_1,\ldots,x_n)$ is a character of irreducible polynomial representations of GL_n. In theoretical physics we have matrix groups acting on particles, representations are smaller matrix groups of things that they are mapping to. We want to take tensor product and direct sums of representations, the tensor product is related to multiplication of Schur function while direct sum into sum of Schur functions.

There's also the Schur-Weyl duality which takes representations into the Weyl group. Under the *Frobenius map*, s_λ corresponds to irreducible representations of S_n.

A more modern application of Schur function goes into geometry, s_λ correspond to Schubert varieties in Grassmannians. Multiplication corresponds to interesections and sum to unions.

There's also context in Probability Theory. But in the end, Schur positivity is important because of this connections.

Definition 1.1.5. $f \in \Lambda$ is Schur-positive if $f = \sum c_\lambda s_\lambda$, $c_\lambda \geqslant 0$.

Example 1.1.6. $3s_{(2,1)} + 2s_{(3)}$ schur pos but change 2 to $-\frac{1}{2}$ then not.

Exercise 1.1.7 (1.1 Stein & Shakarchi)**.** Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$.
(b) $1/z = \overline{z}$.
(c) $\mathrm{Re}(z) = 3$
(d) $\mathrm{Re}(z) > c$, (resp.,$\geqslant c$) where $c \in \mathbb{R}$.
(e) $\mathrm{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$.
(f) $|z| = \mathrm{Re}(z) + 1$.
(g) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$.

Answer

i) The first set is the set of points at the same distance from z_1 and z_2. If we consider the line segment z_1z_2, then the set in question is the bisector of that line segment.

ii) Note that

$$1/z = \overline{z} \iff 1 = \overline{z}z \iff 1 = |z|^2 \iff 1 = |z|,$$

thus the set is the unit circle.

iii) The set is a perpendicular line to the real axis at $z = 3$.

iv) This infinite set is an infinite half plane to the right (but not including) of the line $z = c$. In the other case, we do include the line in question.

v) DO

vi) The equation in question is equivalent to

$$\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 = (\mathrm{Re}(z) + 1)^2.$$

To ease the notation, assume $z = x + iy$. Then the equation reads

$$x^2 + y^2 = x^2 + 2x + 1 \iff y^2 = 2x + 1 \iff x = (y^2 - 1)/2.$$

It holds the the parabola in question contains the points which satisfy the equation.

vii) This set is a line parallel to the real axis at $z = c$

Exercise 1.1.8. Do the following:

i) Show that the complex conjugation map $\kappa : \mathbb{C} \to \mathbb{C},\ z \mapsto \overline{z}$ is an involution, i.e., a ring homomorphism such that $\kappa \circ \kappa = \mathrm{id}$.

ii) Suppose $a \in \mathbb{R},\ z \in \mathbb{C}$. Show that

$$\mathrm{Re}(az) = a\,\mathrm{Re}(z), \quad \text{and} \quad \mathrm{Im}(az) = a\,\mathrm{Im}(z).$$

Answer

Let us take $z = x + iy$ with $x, y \in \mathbb{R}$.

i) We have $\overline{z} = x + i(-y) = x - iy$. Once more we get $\overline{\overline{z}} = x - i(-y) = x + iy = z$. Thus $\overline{\overline{z}} = z$ for any $z \in \mathbb{C}$. In conclusion $\overline{\overline{\cdot}} = \mathrm{id}$.

ii) It holds that

$$\operatorname{Re}(az) = \operatorname{Re}(ax + aiy) = ax = a\operatorname{Re}(z),$$
$$\operatorname{Im}(az) = \operatorname{Im}(ax + aiy) = ay = a\operatorname{Im}(z).$$

Exercise 1.1.9. Do the following:

i) Prove that $|z + w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w})$.
ii) Use this to prove the parallelogram rule: $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$.

Answer

i) Note that

$$|z + w|^2 = (z + w)\overline{(z + w)} = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + w\overline{z} + z\overline{w} + w\overline{w}.$$

The number $w\overline{z}$ is the conjugate of $z\overline{w}$, and summing a number and its conjugate returns twice its real part. Thus we get the desired identity.
ii) As the past identity holds for all complex numbers, it holds when $w = -w$. This means that $|z-w|^2 = |z|^2+|-w|^2+2\operatorname{Re}(z(\overline{-w})) = |z|^2+|w|^2-2\operatorname{Re}(z\overline{w})$ and summing this together with the first identity gives us the parallelogram law.

Exercise 1.1.10 (1.5 Stein & Shakarchi)**.** A set Ω is said to be pathwise connected if any two points in Ω can be joined by a (piecewise-smooth) curve entirely contained in Ω. The purpose of this exercise is to prove that an open set Ω is pathwise connected if and only if Ω is connected.

i) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a parametrization $z : [0, 1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let

$$t_* = \sup_{0 \leqslant t \leqslant 1} \{\, t : \forall s[(0 \leqslant s < t) \Rightarrow (z(s) \in \Omega_1)] \,\}.$$

Arrive at a contradiction by considering the point $z(t_*)$.
ii) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subseteq \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subseteq \Omega$ denote the set of all points that cannot be joined to w by

a curve in Ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega 1$ as desired.

Answer

i) Recall first, that by definition of supremum we have that if S is our set, then

$$\exists s \in S(s > t_* - \varepsilon)$$

for $\varepsilon > 0$. Following the idea, we consider the point $z(t_*)$. We have two options to place $z(t_*)$, either in Ω_1 or Ω_2.

Let's start by definition of supremum

Exercise 1.1.11 (1.7 Stein & Shakarchi)**.** The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

i) Let $z, w \in \mathbb{C}$ such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1$$

if $|z| < 1$ and $|w| < 1$, and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1$$

if $|z| = 1$ or $|w| = 1$. ⟦Hint: Why can one assume that z is real? I then suffices to prove that $(r-w)(r-\overline{w}) \leqslant (1-rw)(1-r\overline{w})$ with equality for appropriate r and $|w|$.⟧⟦Here is an alternate approach, which you may use if you like. Fix $w \in \mathbb{C}$ with $w < 1$, and consider the function $z \mapsto \frac{w-z}{1-\overline{w}z}$. What is $\overline{f(z)}$? By computing $f(z)\overline{f(z)}$, show that $|z| = 1$ implies $|f(z)| = 1$. Find a point z with $|z| < 1$ such that $|f(z)| < 1$. Since f is continuous, this shows that f takes the unit disc to itself. (Why?)⟧

ii) Prove that for a fixed $w \in \mathbb{D}$, the mapping $F : z \mapsto \frac{w-z}{1-\overline{w}z}$ satisfies the following:

 a) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.
 b) F interchanges 0 and w.
 c) $|F(z)| = 1$ if $|z| = 1$.
 d) F is bijective. ⟦Hint: Calculate $F \circ F$.⟧

Answer

i) The inequality in question is equivalent to

$$0 \leqslant |w - z| < |1 - \overline{w}z|.$$

Since the quantities are positive, we can square them and preserve the order. It holds that

$$0 \leqslant |w - z|^2 < |1 - \overline{w}z|^2 \iff 0 \leqslant (w - z)\overline{(w - z)} < (1 - \overline{w}z)\overline{(1 - \overline{w}z)},$$

Simplifying this expression we get

$$(w - z)(\overline{w} - \overline{z}) < (1 - \overline{w}z)(1 - w\overline{z})$$

$$\iff$$

1.2 Day 1 | 20230120

The Complex Numbers

To construct the complex numbers we take the real numbers, adjoin a variable and mod out by $\langle x^2 + 1 \rangle$. We can also define $\mathbb{C}$ as $\{ a + bi : a, b \in \mathbb{R} \}$ with the property $i^2 = -1$. This means that we can multiply complex numbers in the following way:

$$(a + bi)(c + di) = ac + (bc + ad)i + bdi^2 = (ac - bd) + (ad + bc)i.$$

Also as $x^2 + 1$ is irreducible in $\mathbb{R}[x]$, $\mathbb{C}$ is a finite field extension of $\mathbb{R}$ of degree 2. As a 2-dimensional vector space $\{ 1, i \}$ is a basis for $\mathbb{C}$.

The map $a + bi \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$ is not a ring homomorphism, it's a bijection with a bit of structure. The map $z \mapsto \alpha z$, when $\alpha = a + bi$, is a linear map with the following action over the basis

$$\alpha \cdot 1 = \alpha \Rightarrow [\alpha] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\alpha \cdot i = -b + ai \Rightarrow [\alpha] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

which means that $[\alpha] = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. The converse, if we have a $\mathbb{R}$-linear transformation, then it's $\mathbb{C}$-linear if and only if it looks like $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Definition 1.2.1. The complex conjugation map is $a + bi \mapsto a - bi$, or $z \mapsto \overline{z}$.

This map is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

Example 1.2.2. For $\alpha = a + bi$, we have

$$\overline{2\alpha} = \overline{2(a+bi)} = \overline{2a + 2bi} = 2a - 2bi = 2\overline{al}.$$

Whereas if instead

$$\overline{i\alpha} = \overline{ai - b} = -b - ai \neq i\overline{\alpha} = b + ai.$$

As a $\mathbb{R}$-linear map, we can identify with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By looking at the shape of this matrix we can see that it is not $\mathbb{C}$-linear.

Lemma 1.2.3. *The map $z \mapsto \overline{z}$ is a ring homomorphism*

Proof

$\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{zw}$.

With the complex conjugation we can pick out the real and imaginary parts of $\alpha = a + bi$.

$$\alpha + \overline{\alpha} = 2\operatorname{Re}(\alpha), \quad \alpha - \overline{\alpha} = 2i\operatorname{Im}(\alpha)$$

A Notion of Size

Can't do geometry without one. Notice that for $z = a + bi$

$$z\overline{z} = a^2 + b^2 > 0.$$

From a complex number we have extracted a positive quantity.

Definition 1.2.4. The complex modulus of z is $|z| = \sqrt{z\overline{z}}$.

The fact that every number has n roots is very important in complex analysis.
As a vector in the plane, the norm of z is $|z|$

INC FIG

This means that $a + bi \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$ is an isometry. In this sense the distance between two complex numbers is $d(z, w) = |z - w|$.

Polar Coordinates (*ad hoc*)

For $\theta \in \mathbb{R}$, define

$$\exp(i\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta) \Rightarrow |\exp(i\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

Every point in the unit circle is of the form $e^{i\theta}$ and vice-versa.

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For non-zero complex numbers, $z = |z|e^{i\theta}$ for some θ.

Definition 1.2.5. For a complex number $z = re^{i\theta}$, an argument of z is θ.

To have a well defined function, we mod out by multiples of 2π:

$$\arg : \mathbb{C}\backslash\{0\} \to \mathbb{R}\big/_{2\pi\mathbb{Z}},$$

and we obtain a group isomorphism. In general, "lengths multiply, angles add."

For inverses if $z = re^{i\theta}$, then $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$.

Definition 1.2.6. The upper-half plane is $\mathbb{H} = \{\operatorname{Im}(z) > 0\}$.

Lemma 1.2.7. *If H is a half plane* $\operatorname{Im}(z - \beta/\gamma) > 0$