Exercise 1 (Exercise 1, Bonus)**.** What is your favorite bijection, and why? (This problem does not count towards your 10 point total problem limit; it will be an extra fun point added on top to whatever you hand in for real below.)

Answer

I have a whole set of bijections which I may call my favorite. Consider (f_d) where $d \in \texttt{DATES}$, the set `DATES` consists of all the days where I went to class. For each d, we have

$$f_d : \{\, \texttt{students in 502 class on day } d \,\} \to \{\, \text{seats in room E204} \,\},$$
$$\texttt{student} \mapsto \text{preferred seat on day } d.$$

The function f_d is a bijection when restricted to its image because no two `grad students` can sit in the same chair.
In particular, amongst the f_d's, the ones I'm most partial to, are the ones in $\texttt{DATES}'$ where this is the subset of days in which I did not fall asleep in class.
These are my favorite bijections because they are unique and different. I'm almost sure no one in class has a bijection like this as their favorite. Also, the idea of writing this made me chuckle, it's not everyday that you can have fun like this.[a]

[a] **Everyone** in class collaborated with me on this problem. If it weren't for them, I wouldn't have been able to do this problem quite literally.

Exercise 2 (Exercise 2)**.** Apply Franklin's involution to $(7, 6, 4, 3)$ and check that applying it again returns the original partition.

Exercise 3 (Exercise 3, Sagan 3.16(c))**.** Show that the generating function for the number of partitions of n with $\lambda_1 = k$ equals the generating function for the number of partitions of n with exactly k parts and that both are equal to the product $x^k/[(1-x)\ldots(1-x^k)]$.

Answer

Let us call $p(n, k)$ the number of partitions of n with exactly k parts and $\widehat{p}(n, k)$, the number of partitions of n with $\lambda_1 = k$. To show that their generating functions are equal, it suffices to prove that $p = \widehat{p}$ for all n, k.
To that effect, take $\lambda \vdash n$ with k parts. When conjugating λ we see that λ^* now has largest part equal to k. In terms of their Young tableaux, λ has dimensions $k \times \lambda_1$ while λ^* has dimensions $\lambda_1 \times k$.

Conjugation of partitions is a bijective map, which means that every partition with k parts is uniquely associated to a partition with largest part k and back. In conclusion it must hold that $p = \widehat{p}$ and therefore the generating functions are equal.
To see the product formula let us consider instead the number of partitions with largest part *at most* k. The generating product function for that sequence is

$$\big((1+x+x^2+\dots)((x^2)^0+(x^2)^1+(x^2)^2+\dots)\dots((x^k)^0+(x^k)^1+(x^k)^2+\dots)\big) = \frac{1}{(1-x)(1-x^2)}$$

The last infinite sum counts the number parts in λ equal to k. If we wish to guarantee that there is one part equal to k we must multiply that factor by x^k. Therefore we have that the product which counts $\widehat{p}$ is

$$\big((1+x+x^2+\dots)((x^2)^0+(x^2)^1+(x^2)^2+\dots)\dots((x^k)^1+(x^k)^2+(x^k)^3+\dots)\big) = \frac{x^k}{(1-x)(1-x^2)}$$

And since $p = \widehat{p}$ it holds that both functions are equal to the product.

Exercise 4 (Exercise 3, Sagan 3.16(d))**.** The Durfee square of λ is the largest square partition (d^d) such that $(d^d) \subseteq \lambda$. Use this concept to prove

$$\sum_{n=0}^{\infty} p(n)x^n = \sum_{d=0}^{\infty} \frac{x^{d^2}}{(1-x)^2(1-x^2)^2\dots(1-x^d)^2}.$$

Answer

What the problem is telling us is to count the total number of partitions in a different way than the one we know.
For that effect consider a partition λ with its corresponding Young diagram. Such a partition can be decomposed into a Durfee square, a *right* component and an *up* component. Still viewing such components as Young diagrams, we can see a characteristic which defines them:

- The *right* component is a partition of $n-d$ with at most d parts.
- The *up* component is a partition (of some number) with largest part at most d.

By the previous problem we know that these types of partitions are in correspondence and each is counted by a product generating function. So for all sizes of Durfee squares we can count all partitions these way: fix a Durfee square size, then attach the *right* and *up* components to the Young diagram. Since each partition necessarily contains a Durfee square, we add a factor of x^{d^2} while each of the two components are each counted by $1/((1-x)\ldots(1-x^d))$. Multiplying these together gives us the desired identity

$$\sum_{n=0}^{\infty} p(n)x^n = \sum_{d=0}^{\infty} \frac{x^{d^2}}{(1-x)^2(1-x^2)^2\ldots(1-x^d)^2}.$$