Exercise 1 (Exercise 1)**.** Find the largest possible size of a matching for P_n, and find the smallest possible size of a maximal matching for P_n. Express your answers in terms of n (they may depend on the parity of n or its residue modulo 3).

Answer

Exercise 2 (Exercise 3)**.** Show that the number of spanning trees of $K_{m,n}$ is $m^{n-1}n^{m-1}$.

We will follows the steps described in problem 5.66 in Stanley Vol.2.

Answer

The adjacency matrix of $K_{m,n}$ can be written in block form:

$$A = \begin{pmatrix} 0_{m\times m} & \mathbf{1}_{m\times n} \\ \mathbf{1}_{n\times m} & 0_{n\times n} \end{pmatrix}.$$

Here 0 is the zeroes matrix and $\mathbf{1}$ is the ones matrix. The vertices of $K_{m,n}$ have degree either n or m so the Laplacian matrix of $K_{m,n}$ is

$$L = D - A = \begin{pmatrix} nI_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & mI_{n\times n} \end{pmatrix} - \begin{pmatrix} 0_{m\times m} & \mathbf{1}_{m\times n} \\ \mathbf{1}_{n\times m} & 0_{n\times n} \end{pmatrix} = \begin{pmatrix} nI_{m\times m} & -\mathbf{1}_{m\times n} \\ -\mathbf{1}_{n\times m} & mI_{n\times n} \end{pmatrix}.$$

With this in hand, let us proceed with the computations:

i) The matrix $L - mI$ is precisely

$$L - mI = \begin{pmatrix} (n-m)I_{m\times m} & -\mathbf{1}_{m\times n} \\ -\mathbf{1}_{n\times m} & 0_{n\times n} \end{pmatrix}$$

whose last n rows are all identical. We row reduce this matrix in the following way:

- ◇ Eliminate the last $n-1$ rows subtracting row $n+1$ from them. We initially guess that the rank of this matrix will be $m+1$.
- ◇ Divide the first m rows by $n-m$ and then eliminate the ones in the $(m+1)^{\text{th}}$ row by subtracting the first m rows from that one.
- ◇ Our last row is now $(0, \dots, 0, \frac{m}{n-m}, \dots, \frac{m}{m-n})$ which we will convert to a row of ones after dividing by $r/(n-m)$.
- ◇ We can use the last row to eliminate the $-\mathbf{1}_{m\times n}$ block on top.

The resulting matrix is $\texttt{rref}(L - mI)$, the rank of this matrix is $m + 1$ so the rank of $L - mI$ is also $m + 1$.

By the rank nullity theorem, $\dim\ker(L - mI) + (m + 1) = m + n$ and so the geometric multiplicity of m is $n - 1$. Thus there are *at least* $(n - 1)$ eigenvalues equal to m.

ii) With the same reasoning we can prove that the geometric multiplicity of n is $m - 1$. In which case, there would be at least $m - 1$ eigenvalues of L equal to n.

iii) The matrix L can have at most $m + n$ eigenvalues, assuming that the algebraic and geometric multiplicities coincide for m, n, so summing the multiplicities we get

$$(m - 1) + (n - 1) + \text{remaining} = m + n \Rightarrow \text{remaining} = 2.$$