Exercise 1. Do the following:

i) Let $q = (a_1, \dots, a_n)$ be a point in $\mathbb{A}^n$. Using the fact that $I(q)$ is a maximal ideal in $\mathbb{C}[x_1, \dots, x_n]$, prove that the coordinate ring of q is isomorphic to $\mathbb{C}$.

ii) If $i : \{ q \} \to \mathbb{A}^n$ is the inclusion map, show that the pullback homomorphism

$$i^\sharp : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[q] = \mathbb{C}$$

sends a function $f(x_1, \dots, x_n)$ to the complex number $f(a_1, \dots, a_n)$ obtained by evaluating at that point.

Answer

i) The ideal $I(q)$ is in fact $\operatorname{gen}(x_1 - a_1, \dots, x_n - a_n)$, a maximal ideal in $\mathbb{C}[x_1, \dots, x_n]$. Then the coordinate ring of $\{ q \}$ is precisely

$$\mathbb{C}[q] = \mathbb{C}[x_1, \dots, x_n] \Big/ I(\{ q \}) = \mathbb{C}[x_1, \dots, x_n] \Big/ \operatorname{gen}(x_1 - a_1, \dots, x_n - a_n).$$

The evaluation homomorphism ε_q with help of the 1^{st} isomorphism theorem gives us the desired isomorphism. This is clearly a surjective map since we can get to any complex number by solving a linear equation and its kernel is the aforementioned ideal.

ii) Since the inclusion mapping is a morphism of algebraic varieties, then it induces a pullback homomorphism between the coordinate rings. By definition its action is as follows:

$$i^\sharp : \mathbb{C}[\mathbb{A}^n] \to \mathbb{C}[q], \ g \mapsto g \circ i.$$

Let us unpack the terminology. First, the inclusion homomorphism is the identity mapping restricted to $\{ q \}$. Then the pullback can be expressed as

$$i^\sharp : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}, \ g(\mathbf{z}) \mapsto g(\operatorname{id}|_{\{ q \}}(\mathbf{z})).$$

In this sense the action of $g \circ i$ is

$$\mathbb{A}^n \xrightarrow[\mathbf{z} \mapsto q]{\operatorname{id}|_{\{ q \}}} \mathbb{A}^n \xrightarrow[g \mapsto g(-)]{g} \mathbb{C} \Rightarrow \mathbb{A}^n \xrightarrow[g \mapsto g(q)]{g \circ i} \mathbb{C},$$

and thus, since the action of this map is the same as ε_q, we conclude that $i^\sharp = \varepsilon_q$.

Exercise 2. Prove that if $F : V \to W$ is an isomorphism of affine algebraic varieties, then the pullback homomorphism is a ring isomorphism.

Answer

The pullback homomorphism is precisely $F^\sharp : \mathbb{C}[W] \to \mathbb{C}[V]$ such that $g \mapsto g \circ F$. Since the pullback is already a ring homomorphism, it suffices to show that it is invertible by explicitly constructing an inverse.
The map $F^{-1} : W \to V$ defines a pullback from $\mathbb{C}[V]$ to $\mathbb{C}[W]$ such that $(F^{-1})^\sharp(h) = h \circ F^{-1}$. Now this last map is an inverse to $F^\sharp$ since

$$F^\sharp((F^{-1})^\sharp(h)) = F^\sharp(h \circ F^{-1}) = (h \circ F^{-1}) \circ F = h,$$

and likewise on the other side. It follows that $(F^{-1})^\sharp = (F^\sharp)^{-1}$.

Exercise 3. Let $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ be affine algebraic varieties. Let $\widetilde{F} : \mathbb{A}^n \to \mathbb{A}^m$ be a morphism. Show that

$$\widetilde{F}(V) \subseteq W \iff \widetilde{F}^\sharp : \mathbb{C}[y_1, \ldots, y_m] \to \mathbb{C}[x_1, \ldots, x_n] \text{ sends } I(W) \text{ to } I(V).$$

⟦ Hint: W=V(I(W)). ⟧

Answer

i) Let us consider $q \in \widetilde{F}^\sharp(I(W))$, we want to show that $q(\mathbf{a}) = 0$ for $\mathbf{a} \in V$.

   By definition of direct image, there exists a polynomial $p \in I(W)$ such that $\widetilde{F}^\sharp(p) = q$, thus if $\mathbf{a} \in V$

   $$q(\mathbf{a}) = \widetilde{F}^\sharp(p)(\mathbf{a}) = p(F(\mathbf{a})) = 0$$

   where the last equality follows from the fact that $F(\mathbf{a}) \in W$ and $p(\mathbf{x}) = 0$ for any $\mathbf{x} \in W$.

ii) On the other hand, let us consider a $\mathbf{b} \in \widetilde{F}(V)$. We want to show that $\mathbf{b} \in W$. By the Nullstellensatz, this is equivalent to showing that $\mathbf{b} \in V(I(W))$. Or equivalently, that $q(\mathbf{b}) = 0$ for $q \in I(W)$.

   Once again, using the definition of the direct image, we may find an element $\mathbf{a} \in V$ such that $\widetilde{F}(\mathbf{a}) = \mathbf{b}$. Thus if we take $q \in I(W)$, then

   $$q(\mathbf{b}) = q(\widetilde{F}(\mathbf{a})) = (q \circ \widetilde{F})(\mathbf{a}) = (\widetilde{F}^\sharp(q))(\mathbf{a}) = 0$$

   where the last equality follows from the fact that $\widetilde{F}^\sharp(q)$ is a polynomial in $I(V)$.

I wanted to answer the last exercise directly without separating the directions in the equivalence. However the proof would go along the lines of

$$\widetilde{F}(V) \subseteq W \iff I(F(V)) \supseteq I(W) \iff I(V) \supseteq \widetilde{F}^\sharp(I(W))$$

using I as a contravariant functor. I feel that this diagram comes in handy: My question

$$\begin{tikzcd}
 & & W \arrow[dr, "i"] \arrow[dd, "I"] & \\
V \arrow[urr] \arrow[dr, "i"] \arrow[dd, "I"] & & & \mathbb{A}^m \arrow[dd, "I"] \\
 & \mathbb{A}^n \arrow[urr, "\widetilde{F}"] \arrow[dd, "I"] & I(W) \arrow[dll] \arrow[dr, "i"] & \\
I(V) \arrow[dr, "i"] & & & \mathbb{C}[y_1, \dots, y_m] \arrow[dll, "\widetilde{F}^\sharp"] \\
 & \mathbb{C}[x_1, \dots, x_n] & &
\end{tikzcd}$$

is, can this statement be proven in a direct way? Using the equivalence of categories maybe?

Exercise 4 (2.6.1)**.** Prove that the spectrum $\operatorname{Spec}(R)$ of a commutative ring R can be given the structure of a topological space whose closed sets are of the form $V(I) = \{\, \mathfrak{p} \in \operatorname{Spec}(R) : \ \mathfrak{p} \supseteq I \,\}$ for $I \trianglelefteq R$.

Answer

First, the whole set and the empty set are closed:

- ◇ Since every prime ideal $\mathfrak{p}$ is an ideal, we get $0 \in \mathfrak{p}$ and then $\{\, 0 \,\} \subseteq \mathfrak{p}$. Then $V(0) = \{\, \mathfrak{p} \in \operatorname{Spec}(R) : \ \mathfrak{p} \supseteq \{\, 0 \,\} \,\}$, and this is the whole set. Thus $V(0) = \operatorname{Spec}(R)$.
- ◇ On the other hand, if $V(I) = \varnothing$, then, there's no prime ideal which contains I besides the whole ring. It follows that $V(R) = \varnothing$.

Let us now prove that the finite union of closed sets is closed. This is, we are looking for $K \trianglelefteq R$ such that $V(I) \cup V(J) = V(K)$. Let us take $K = IJ$, then

$$IJ \leqslant I, J \Rightarrow V(I) \cup V(J) \subseteq V(IJ), \ (\mathfrak{p} \supseteq IJ) \Rightarrow [(\mathfrak{p} \supseteq I) \vee (\mathfrak{p} \supseteq J)]$$

and the last assertion gives us the other inclusion. We conclude that $V(IJ) = V(I) \cup V(J)$ and by induction we can prove the equality for a countable number of ideals.
Finally, consider a collection of ideals $(I_\alpha)_{\alpha\in\mathcal{A}}$. We want to find an ideal $J \trianglelefteq R$ such that $\bigcap_{\alpha\in\mathcal{A}} V(I_\alpha) = V(J)$. For that effect we shall take $J = \sum_{\alpha\in\mathcal{A}} I_\alpha$.
Since J is the smallest ideal which contains all of the I_α's, then

$$\forall\alpha\,(V(J) \subseteq V(I_\alpha)) \Rightarrow V(J) \subseteq \bigcap_{\alpha\in\mathcal{A}} V(I_\alpha).$$

On the other hand, by minimality of J,

$$\forall\alpha(\mathfrak{p} \supseteq I_\alpha) \Rightarrow \mathfrak{p} \supseteq \sum I_\alpha$$

and this guarantees the other side of the inclusion.
We conclude that in fact the Zariski topology defined on $\mathrm{Spec}(R)$ is in fact a topology.

Exercise 5 (2.5.(1,2))**.** Do the following:

i) Show that the pullback $\mathbb{C}[W] \xrightarrow{F^\sharp} \mathbb{C}[V]$ is injective if and only if F is *dominant*. This is, $F(V)$ is dense in W.
ii) Show that the pullback $\mathbb{C}[W] \xrightarrow{F^\sharp} \mathbb{C}[V]$ is surjective if and only if F defines an isomorphism between V and some algebraic subvariety of W.

Answer

i) Recall F is dense in E whenever $\overline{F} = E$. The Zariski closure operator is $V(I(\cdot))$, so it is equivalent to show that $F^\sharp$ is injective if and only if $V(I(F(V))) = W$.
ii) (Hmmm?)

I must admit that this time I've got to optimize since there's 161 exam on Thursday and I haven't started my Combinatorics homework due Friday. Today is (202209131754).