Exercise 1 (Exercise 1)**.** Prove that addition and multiplication of cardinalities satisfies the distributive law, that is, that

$$|A|(|B| + |C|) = |A|\cdot|B| + |A|\cdot|C|,$$

using the definition of addition and multiplication of cardinalities that we defined in class.

Answer

To prove the equality using the definition of cardinalities, we would like to exhibit a bijection from a set whose size is the amount on the left to a set whose size is the amount on the right.
Consider any sets A, B and C. The following table summarizes the information we are working with

Set	Cardinality	Elements
$A \times (B \uplus C)$	$\lvert A\rvert(\lvert B\rvert + \lvert C\rvert)$	$(a, (x, n))$
$(A \times B) \uplus (A \times C)$	$\lvert A\rvert\cdot\lvert B\rvert + \lvert A\rvert\cdot\lvert C\rvert$	$((a, x), n)$

with $n \in \{0, 1\}$ and $n = 0 \Rightarrow x \in B$, while $n = 1 \Rightarrow x \in C$. This is because the disjoint union can be constructed in the following way

$$B \uplus C = (B \times \{0\}) \cup (C \times \{1\}).$$

Because of this construction it doesn't matter if B and C share elements since they will be labeled.
Now the function

$$f : A \times (B \uplus C) \to (A \times B) \uplus (A \times C),\ (a, (x, n)) \mapsto ((a, x), n)$$

can be proven to be well-defined, injective and surjective. It is therefore bijective and its inverse is the function that follows the rule $((a, x), n) \mapsto (a, (x, n))$.
Since we have found the bijection in question, it follows that both sets have the same cardinalities and thus the quantities in question are equal.

Exercise 2. Prove the binomial theorem using a combinatorial argument as follows.

1) Show that, for all positive integers s, t, and n, we have

$$\sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k} = (s + t)^n.$$

In particular, do not treat s and t as variables; rather, interpret $(s+t)^n$ as counting something parameterized by the integers s, t, n and show that the right hand side counts the same thing.

II) Defining the polynomials $p(x) = (x+1)^n$ and $q(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$, we have that $p(s) = q(s)$ for all positive integers s. Use the fact that polynomials in one variable that agree on infinitely many values must be the same to conclude that $p(x) = q(x)$ as polynomials.

III) Finally, plug in x/y and clear the denominators on both sides of the equation $p(x/y) = q(x/y)$ to show that the binomial theorem holds.

Answer

(a) Consider finite sets S, T with cardinalities s, t respectively. The quantity $s+t$ is the cardinality of the set $S \uplus T$ which can be realized as $(S \times \{0\}) \cup (T \times \{1\})$. Now the n^{th} power of that quantity counts the number of elements in the n-fold cartesian product of $S \uplus T$.

We can expand the sum of the equality to begin analyzing the sets

$$\binom{n}{0} t^n + \binom{n}{1} s t^{n-1} + \cdots + \binom{n}{n-1} s^{n-1} t + \binom{n}{n} s^n.$$