Exercise 1. Consider the dihedral group D_4 (the symmetry group of a square, including reflections and rotations) acting on the plane by the corresponding reflection and rotation matrices. Show that this representation of D_4 is an irreducible 2-dimensional representation over both $\mathbb{R}$ and $\mathbb{C}$.

Answer

Observe that we may present the dihedral group as

$$\text{Free}(r,s)\Big/_{\text{gen}(r^4, s^2, (sr)^2)}.$$

The desired representation is given by the action on the generators:

$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Both matrices have $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as an eigenvector. This can be seen either by checking both eigenspaces or by looking at the row sums. Correspondingly, r has eigenvalue -1 and s, a 1.

For both $\mathbb{R}$ and $\mathbb{C}$, the vector $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ lies in $\mathbf{v}^\perp$. To decompose our representation, we consider the basis $\mathcal{B} = \{\,\mathbf{u}, \mathbf{v}\,\}$ along with the change of basis matrix

$$M = [\text{id}]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow M^{-1} = [\text{id}]_{\mathcal{C}}^{\mathcal{B}} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}M.$$

Conjugating our generators we obtain

$$\begin{aligned}
\text{wt}r = M^{-1}rM &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{aligned}$$

and for the case of s:

$$\begin{aligned}
\mathrm{wt}s = M^{-1}sM &= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.
\end{aligned}$$

So after conjugating, we can see that we cannot decompose the representation further. As indecomposability is equivalent to irreducibility in this case, we have that this representation is an irreducible 2-dimensional representation.

Exercise 2. Prove that if G is a finite abelian group, then all of its irreducible representations are 1- dimensional. [Hint: you may want to look up some resources showing that commuting matrices are simultaneously diagonalizable.]

Answer

If $|G| = n$, then $g^n = \mathrm{id}_G$. If $\rho : G \to \mathrm{GL}(V)$ is our representation, then

$$g^n = \mathrm{id}_G \Rightarrow \rho(g)^n = I \Rightarrow \rho(g)^n - I = 0.$$

This means $\rho(g)$ satisfies the polynomial equation $z^n - 1 = 0$. At first, this doesn't directly imply that $\rho(g)$ is diagonalizable, but with a careful eye we can remember about the minimal polynomial. The minimal polynomial of $\rho(g)$ divides any polynomial which annihilates $\rho(g)$ as it is the generator of the ideal of such polynomials:

$$\min_{\rho(g)} \mid z^n - 1$$

and as $z^n - 1$ splits into distinct linear factors, the minimal polynomial must be a product of such factors. Being the roots of $\min_{\rho(g)}$ all distinct, we have that $\rho(g)$ is diagonalizable.
Now, as G is abelian, we also have that the matrices $\{\, \rho(g) : \ g \in G \,\}$ commute. Therefore they are all simultaneously diagonalizable.

Exercise 3. Suppose A is an $m \times m$ matrix and B is an $n \times n$ matrix. Recall that the tensor product of the matrices A and B is the matrix having block form

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{pmatrix}.$$

Show that, if $\rho : G \to \mathrm{GL}(\mathbb{C}^m)$ and $\sigma : G \to \mathrm{GL}(\mathbb{C}^n)$ are two representations of a group G (thought of as collections of matrices $\rho(g)$ and $\sigma(g)$), then the tensor product of these two representations, defined as the action of G on $\mathbb{C}^m \otimes \mathbb{C}^n$ by $g(v \otimes w) = (gv) \otimes (gw)$, is the map

$$\rho \otimes \sigma : G \to \mathrm{GL}(\mathbb{C}^{mn})$$

given by

$$(\rho \otimes \sigma)(g) = \rho(g) \otimes \sigma(g).$$

Exercise 4. Recall the RSK bijection from Math 502, and use an appropriate version of it to give a combinatorial proof that the dimensions of the spaces on both sides of the Schur-Weyl duality formula

$$\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = \bigoplus_{\lambda} V_\lambda \otimes V^\lambda$$

are equal. (Recall that the left hand side is a tensor product of k copies of $\mathbb{C}^n$, and the right hand side is summing over all λ of size k with at most n rows.) [Hint: You do not need to know the actual RSK algorithm to do this problem, only what objects it forms bijections between. This can be found in Stanley chapter 7, for instance.]

Exercise 5. Use the ε method to find the Lie algebra associated to the Lie group $B_n \subseteq \mathrm{SL}_n(\mathbb{C})$ of upper triangular matrices in $\mathrm{SL}_n(\mathbb{C})$.

Exercise 6. Let V_i be the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with highest weight i. Compute the decomposition of $V_3 \otimes V_5$ into irreducibles.

Exercise 7. Do the following:

(a) Show that the total number of ballot sequences of 1's and 2's of length $2n$ is $\binom{2n}{n}$, and that the number of ballot sequences of 1's and 2's of length $2n+1$ is $\binom{2n+1}{n+1}$.

(b) If V_1 is the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ having highest weight 1, what does this imply about the decompositions of $V_1^{\otimes 2n}$ and $V_1^{\otimes 2n+1}$ into irreducibles?

Exercise 8. Starting with the ballot word 12211122121111, draw the corresponding $\mathfrak{sl}_2$ chain by applying the lowering operator F until it is no longer possible.