

Exercise 1. Let $H \subseteq \mathbb{P}^n$ be a hyperplane. H is defined as the zero locus of a linear equation

$$H = \mathbb{V}(a_0x_0 + \cdots + a_nx_n), \quad a_0, \dots, a_n \in \mathbb{C}.$$

Prove that $H \simeq \mathbb{P}^{n-1}$. (Also, be comfortable with the fact that $\mathbb{P}^n \setminus L \simeq \mathbb{A}^n$.)

Answer

Exercise 2. Let $\nu_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^5$ be the Veronese embedding of degree 2. Write out the equations defining the image of ν_2 .

Answer

Recall that this particular Veronese map takes a point in \mathbb{P}^2 to all possible monomials of degree 2 in \mathbb{P}^5 . This means that

$$\nu_2([u : v : w]) = [u^2 : v^2 : w^2 : uv : vw : wu].$$

Recall that the image of the Veronese map is defined using a multi-indexed array, so let us relabel in that sense:

$$[u^2 : v^2 : w^2 : uv : vw : wu] = [z_{2,0,0} : z_{0,2,0} : z_{0,0,2} : z_{1,1,0} : z_{0,1,1} : z_{1,0,1}].$$

The defining equations for the image are given by

$$z_I z_J = z_K z_L, \quad I + J = K + L,$$

where I, \dots, L are our multi-indices. The only ways to sum our multi-indices in a non-trivial manner are:

$$\begin{cases} (2, 0, 0) + (0, 1, 1) = (1, 1, 0) + (1, 0, 1) \\ (0, 2, 0) + (1, 0, 1) = (1, 1, 0) + (0, 1, 1) \\ (0, 0, 2) + (1, 1, 0) = (1, 0, 1) + (0, 1, 1) \end{cases} \quad \begin{cases} (2, 0, 0) + (0, 2, 0) = (1, 1, 0) + (1, 1, 0) \\ (2, 0, 0) + (0, 0, 2) = (1, 0, 1) + (1, 0, 1) \\ (0, 2, 0) + (0, 0, 2) = (0, 1, 1) + (0, 1, 1) \end{cases}$$

If we name $[a : b : c : d : e : f]$ the coordinates of \mathbb{P}^5 , then we have the following system of equations

$$\begin{cases} ae = df \\ bf = de \\ cd = ef \end{cases} \quad \begin{cases} ab = d^2 \\ bc = e^2 \\ ca = f^2 \end{cases}$$

These are the equations which define $\text{Im}(\nu_2)$.

Exercise 3. Recall that \mathbb{P}^5 parametrizes the space of all conics in \mathbb{P}^2 . Show that the image of the Veronese embedding $\nu_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^5$ is exactly those conics which are “double lines”. [Hint: If $L = \mathbb{V}(ax + by + cz)$, what is the (degree 2) equation of the associated double line]

Exercise 4. Given four points and a line in \mathbb{P}^2 , show that typically two conics pass through the four points and are tangent to the line. Under what special conditions on the positions of the points and the line do we fail to get exactly two? [Hint: As we will discuss in detail in Section 6.1, a line is tangent to a conic if the defining quadratic function has a double root when restricted to the line; on the other hand, a quadratic polynomial has a double root if and only if its discriminant, a degree two polynomial in its coefficients, is zero.]