Chapter 1

Introduction and background

Our goal is to understand the calculation of Gromov-Witten invariants of the space $\overline{M}_{g,n}(\mathbb{P}^r, d)$, the moduli space of degree d maps to $\mathbb{P}^r$ using techiniques of Atiyah-Bott localization. To begin this endeavor, we must explore $\overline{\mathcal{M}}_{g,n}$ first and its intersection theory. This is the space of genus g Riemann surfaces with n marked points which was studied by Deligne and Mumford originally. Afterwards, we will introduce the concept of equivariant cohomology and the Atiyah-Bott theorem. The theorem's usefulness will be presented by showing several examples and finally we will apply it to calculate more examples in the setting of the moduli space of maps.

1.1 Moduli of curves

Definition 1.1.1. A Riemann surface is a complex analytic manifold of dimension 1.

For every point, there's a neighborhood which is isomorphic to $\mathbb{C}$ and transition functions are linear isomorphisms of $\mathbb{C}$. We will interchangeably say Riemann surface or *smooth compact complex curve*.

Example 1.1.2. The following classes define Riemann surfaces.

(a) $\mathbb{C}$ itself is a Riemann surface with one chart.

(b) Any open set of $\mathbb{C}$ is a Riemann surface.

(c) A holomorphic function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ defines a Riemann surface by considering $\Gamma_f \subseteq \mathbb{C}^2$. There's only one chart determined by the projection and the inclusion i_{Γ_f} is its inverse.

(d) Take another holomorphic function f, then $\{ f(x, y) = 0 \}$ is a Riemann surface such that

$$\mathrm{Sing}(f) = \{\partial_x f = \partial_y f = f = 0\} = \varnothing.$$

This means that at every point the gradient identifies a normal direction to the level set $f = 0$. In particular, there's a well defined tangent line. The inverse function theorem guarantees that this is a complex manifold.

(e) Our first compact example is $\mathbb{CP}^1$.

Definition 1.1.3. The moduli space $\mathcal{M}_{g,n}$ is the set of isomorphism classes of genus g, n-pointed Riemann surfaces.

Remark 1.1.4. This immediately implies that the parametrized curves are smooth complex algebraic curves.

Example 1.1.5. The space $\mathcal{M}_{0,4}$ parametrizes genus 0, 4-pointed Riemann surfaces. These are $(\mathbb{P}^1, p_1, \ldots, p_4)$. We have that

$$(\mathbb{P}^1, p_1, \ldots, p_4) \sim (\mathbb{P}^1, q_1, \ldots, q_4)$$

whenever there is exists a Möbius transformation, $T \in \mathrm{PGL}_2$, such that

$$(q_1, \ldots, q_4) = (Tp_1, \ldots, Tp_4).$$

Any such Möbius transformation is determined by where it maps the points $0, 1$ and ∞. With this fact in hand we may map the first 3 points of our curve to $0, 1, \infty$, and let the last one map to an arbitrary but fixed t:

$$(Tp_1, \ldots, Tp_4) = (0, 1, \infty, t), \quad t \in \mathbb{P}^1.$$

At the level of equivalence classes this means:

$$[(\mathbb{P}^1, p_1, \ldots, p_4)] = [(\mathbb{P}^1, 0, 1, \infty, t)]$$

and so every equivalence class is determined by a unique $t \in \mathbb{P}^1$. We call this value the <u>cross-ratio</u> of $(p_1, \ldots, p_4)$. The Möbius transformation in question is

$$T(z) = \frac{(z - p_1)(p_2 - p_3)}{(z - p_3)(p_2 - p_1)}$$

and the image of p_4, $t = T(p_4)$ is the aformentioned cross-ratio of $p_1, \ldots, p_4$. This leads us to see that

$$\mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

Example 1.1.6. The space $\mathcal{M}_{1,1}$ parametrizes 1-pointed *elliptic curves*.

Any such curve is isomorphic to

$$\mathbb{C} \big/ L, \quad L = \mathbb{Z}u + \mathbb{Z}v, \quad \text{where} \quad u, v \in \mathbb{Z},$$

and the image of the origin under the quotient map is the natural choice for the marked point. We have that two lattices L_1, L_2 determine the same elliptic curve whenever

$$\exists \alpha \in \mathbb{C}^\times (L_2 = \alpha L_1).$$

So that

$$\mathcal{M}_{1,1} = \{\text{lattices}\} \big/ \mathbb{C}^\times$$

but we can be more precise!

Explicitly, if

$$L = \mathrm{gen}_{\mathbb{Z}}(u, v) \Rightarrow \tilde{L} = \frac{1}{u} L = \mathrm{gen}_{\mathbb{Z}}(1, \tau).$$

This quantity τ always lies in the upper half plane when

$$\arg(v) > \arg(u) \mod [-\pi, \pi]$$

which means that $\tau \in \mathbb{H}$ determines $\left[{}^{\mathbb{C}}/_{L}\right]$. Let us apply two $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{gen}(S,T)$ actions on τ which will leave the quotient unchanged:

$$\begin{cases} T : \tau \mapsto \tau + 1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \tau = \dfrac{\tau+1}{0+1}, \\ S : \tau \mapsto -\dfrac{1}{\tau} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ \tau = \dfrac{0-1}{\tau+0}. \end{cases}$$

Then observe that the lattices

$$\mathrm{gen}_{\mathbb{Z}}(1, T\cdot\tau) \quad \text{and} \quad \mathrm{gen}_{\mathbb{Z}}(1, S\cdot\tau)$$

give us the same quotient. From this we can be more specific and say

$$\mathcal{M}_{1,1} = {}^{\mathbb{H}}\Big/_{\mathrm{SL}_2(\mathbb{Z})}.$$

Stable curves

Definition 1.1.7 ([12], pg. 16). A genus g, n-pointed <u>stable curve</u> $(C, p_1, \ldots, p_n)$ is a compact complex algebraic curve[1] satisfying:

(a) The only singularities of C are simple nodes.
(b) Marked points and nodes are all distinct. Marked points and nodes do not coincide.
(c) $(C, p_1, \ldots, p_n)$ has a finite number of automorphisms, or equivalently:

$$2 - 2g - n < 0.$$

We assume these curves are connected. The *genus* of C is the arithmetic genus, or equivalently, the genus of the curve obtained when *smoothing the nodes*.

Question. How to explain normalization? smoothing and separating the nodes?

Look at Mumford's red book and math.of/109395 and math.of/60097

Theorem 1.1.8. *A stable curve admits a finite number of automorphisms (as in condition c) if and only if every connected component C_i of its normalization with genus g_i and n_i special points satisfies*

$$2 - 2g_i - n_i < 0.$$

TODO

(a) Write proof of sufficient condition of top theorem
(b) Write about compactifications of previous moduli spaces. Can use families or not, our choice.
(c) Write about tautological ring of Mgn and start describing graphs.

[1] It's almost a manifold, but it's not because it can lack smoothness

1.2 The tautological ring

(a) ψ, λ classes
(b) Intersection product Examples
(c) Projection formula
(d) String and Dilaton relations
(e) Integral examples

1.3 Moduli space of maps

Chapter 2

Equivariant Cohomology and Localization

2.1 Basics of equivariant cohomology

(a) Borel Construction of Equivariant Cohomology
(b) Examples of point equivariant Cohomology
(c) Equivariant Cohomology of projective space

2.2 Atiyah-Bott Localization

(a) Example of $H_T^*(\mathbb{P}^r)$ through Localization
(b) Toric varieties Euler characteristic via Atiyah-Bott
(c) Hodge integral $\int_{\overline{M}_{0,2(\mathbb{P}^2,1)}} \mathrm{ev}_1^*([1:0:0])\mathrm{ev}_2^*([0:1:0])$ via localization.

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