MATH 620: Homework 4

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Problem 1

Problem 2

Norm axioms

$|| \cdot ||_1$

All the axioms follow from the properties of the max

- $\max_{x\in[0.1]} |au(x)| = |a| \max_{x\in[0.1]} |u(x)|$
- $\max_{x\in[0.1]} |u(x)| = 0 \implies \forall x \in [0,1](|u(x)| \leq 0) \implies u \equiv 0$
- $\max_{x\in[0.1]} |u(x) + v(x)| \leq \max_{x\in[0.1]} |u(x)| + \max_{x\in[0.1]} |v(x)|$

The last statement is true because if we find x_M that maximizes the sum $u + v$ then

$$\begin{aligned}\max_{x\in[0.1]} |(u+v)(x)| = |(u+v)(x_M)| &\leq |u(x_M)| + |v(x_M)| \\ &\leq \max_{x\in[0.1]} |u(x)| + \max_{x\in[0.1]} |v(x)|.\end{aligned}$$

$|| \cdot ||_2$

Notice that $||u||_2 = ||u||_1 + ||u'||_1$. So all the norm requirements follow from the fact that $|| \cdot ||_1$ is a norm and that the derivative is linear.

$|| \cdot ||_3$

In these case the only property that is not immediate is the triangular inequality. So let's proof that.

By Cauchy-Schwarz:

$$\int_0^1 |u||v| \leq \left(\int_0^1 |u|^2\right)^{1/2} \left(\int_0^1 |v|^2\right)^{1/2}$$

Then:

$$\begin{aligned}\int_0^1 |u+v|^2 &= \int_0^1 |u|^2 + \int_0^1 |v|^2 + 2\int_0^1 uv \\ &\leq \int_0^1 |u|^2 + \int_0^1 |v|^2 + 2\int_0^1 |u||v| \\ &\leq \int_0^1 |u|^2 + \int_0^1 |v|^2 + 2\left(\int_0^1 |u|^2\right)^{1/2}\left(\int_0^1 |v|^2\right)^{1/2} \\ &= \left(\left(\int_0^1 |u|^2\right)^{1/2} + \left(\int_0^1 |v|^2\right)^{1/2}\right)^2.\end{aligned}$$

Taking the square root we obtain the result.

Equivalence of norms

No two norms are equivalent in this case. Notice that a consequence of the definition of equivalence of norms is the following:

If $||\cdot||_a$ and $||\cdot||_b$ are equivalent then any sequence that is convergent in $||\cdot||_a$ must also be convergent to the same limit point in $||\cdot||_b$. This is because $x_n \to x$ in norm $||\cdot||_a$ means $||x_n - x||_a \to 0$, and due to the equivalence of norms

$$||x_n - x||_b \leq c||x_n - x||_a$$

we have

$$||x_n - x||_b \to 0.$$

$||\cdot||_1$ vs $||\cdot||_2$

Since $||u||_2 = ||u||_1 + ||u'||_1$ it is clear that convergence in $||\cdot||_2$ implies convergence in $||\cdot||_1$ but the converse is not true. Pick for example $x_n = x^n/n$. It is easy to see that $x_n \to 0$ in $||\cdot||_1$ but because $\frac{d}{dx}(x^n/n) = x^{n-1}$ then $||x_n||_2 = 1$. So $x_n \nrightarrow 0$ in $||\cdot||_2$. So these norms are not equivalent.

$||\cdot||_1$ vs $||\cdot||_3$

Notice that:

$$||\cdot||_3^2 = \int_0^1 |u|^2 \leq \int_0^1 ||u||_1^2 = ||u||_1^2.$$

So convergence in $||\cdot||_1$ implies convergence in $||\cdot||_3$. But the converse is not true. Consider $x_n = x^n$. Clearly $||\cdot||_1 = 1$ but $||x_n||_3 = (1/(2n+1))^{1/2}$. So $x_n \to 0$ in $||\cdot||_3$ but $x_n \nrightarrow 0$ in $||\cdot||_1$. So these norms are not equivalent.

$||\cdot||_2$ vs $||\cdot||_3$

Convergence in $||\cdot||_2$ implies convergence in $||\cdot||_1$ which in turn implies convergence in $||\cdot||_3$, but the converse is not true. Take the same example as before. Again these norms are not equivalent.

Problem 3

Norm axioms

In the previous section we proved this for any continuously differentiable function on [0,1] so in particular this is true for all polynomials.

Norm equivalence

In order to prove that these norms are equivalent we can just repeat the proof that all norms are equivalent which only uses the fact that the norm is continuous and the unit ball is compact, but given the conversation we had in class we want to find an explicit constant for each case.

$||\cdot||_1$ vs $||\cdot||_3$

We want to find c and C so that

$$c||p||_3 \leq ||p||_1 \leq C||p||_3.$$

Notice that we can take $c = 1$ because

$$||p||_3^2 = \int_0^1 |p|^2 \leq \int_0^1 ||p||_1^2 = ||p||_1^2.$$

Now let's find C.

Notice that this is equivalent to finding C such that

$$\left|\left|\frac{p}{||p||_3}\right|\right|_1 \leq C,$$

so without loss of generality we can consider p with $||p||_3 = 1$. And show that

$$||p||_1 \leq C.$$

In order to continue with the proof consider the shifted Lengedre polynomials (the usual Legendre polynomials are defined on [-1,1] but the shifted Legendre polynomials are defined on [0,1]). We will denote the $i-th$ shifted Legendre polynomial by L_i. These polynomials have the following properties which I will not proof because it would be too long:

1. $\{L_i\}_{i=0,\ldots,n}$ forms a basis for X
2. $\int_0^1 L_i L_j = \frac{\delta_{ij}}{2i+1}$
3. $||L_i||_1 = \max_{x\in[0,1]} |L_i| \leq 1 \quad \forall i$

Proof:

Take p with $||p||_3 = 1$. By property (1) we can write p as:

$$p = \sum_0^n a_i L_i,$$

then using property (2) we get

$$1 = \int_0^1 |p|^2 = \int_0^1 \left(\sum_0^n a_i L_i \right)^2 = \sum_0^n \frac{a_i^2}{2i+1},$$

which implies that $\frac{|a_i|}{\sqrt{2i+1}} \leq 1$ for $i = 0, \dots, n$.

Now notice that

$$||p||_1 \leq \sum_0^n |a_i| ||L_i||_1 \leq \sum_0^n \sqrt{2i+1}.$$

Where we used the previous inequality and property (3). So our explicit constant is:

$$C = \sum_0^n \sqrt{2i+1}.$$

Problem 4