**Exercise 1** (1.3.T)**.** Show that coproduct for Set is disjoint union.

**Answer**

Recall that the disjoint union of $A_1$ and $A_2$ is defined as a set as

$$A_1 \uplus A_2\{\,(a_i, i): \ a_i \in A_i, \ I = 1, 2\,\}.$$

This allows us to define maps $\iota_i : A_i \to A_1 \uplus A_2, \ x \mapsto (x, 1)$ which are morphisms in Set because they're defined everywhere. We are to show that this set satisfies the universal property of coproducts.
Suppose $B$ is a set such that $f_i : \ A_i \to B$ are well defined. We must define a unique function $g : \ A_1 \uplus A_2 \to B$ such that $f_i = g\iota_i$, this is done as follows:

$$g(a, i) = \begin{cases} f_1(a), \ i = 1, \\ f_2(a), \ i = 2. \end{cases}$$

We verify the factoring property:

$$g \circ \iota_1(a_1) = g(a_1, 1) = f_1(a_1), \quad g \circ \iota_2(a_2) = g(a_2, 2) = f_2(a_2).$$

By construction, we have defined $g$ uniquely.

**Exercise 2** (1.3.U)**.** Suppose $A \to B$ and $A \to C$ are two ring morphisms, so in particular $B$ and $C$ are $A$-modules. Recall that $B \otimes_A C$ has a ring structure.

i) Show that there is a natural morphism $\iota_B : B \to B \otimes_A C, \ b \mapsto b \otimes 1$. Similarly for $C$.

ii) Show that this gives a pushout on rings. In other words, the following diagram satisfies the universal property of the pushout.

$$\begin{tikzcd}
B \otimes_A C & C \arrow[l, "\iota_C"'] \\
B \arrow[u, "\iota_B"] & A \arrow[l, "\beta"] \arrow[u, "\alpha"']
\end{tikzcd}$$

**Answer**

i) The map $\iota_B(b) = b \otimes 1$ is a homomorphism in virtue that $B \otimes_A C$ is a tensor product. By construction, all bilinear maps factor through the tensor product as linear maps. This map is one of the factors which should be linear. The same holds for $C$.

ii) Let us now take $M$ an $A$-module with morphisms $f_B : B \to M$ and $f_C : C \to M$. This can described by the following diagram:

$$\begin{tikzcd}
M & & \\
& B \otimes_A C & C \arrow[l, "\iota_C"'] \arrow[ull, bend right, "f_C"'] \\
& B \arrow[u, "\iota_B"] \arrow[uul, bend left, "f_B"] & A \arrow[l, "\beta"] \arrow[u, "\alpha"']
\end{tikzcd}$$

However, let us take advantage of the tensor product, *gatekeeper of bilinear maps*. This morphisms can be combined into a bilinear map from $B \times C \to M$. We define

$$f : B \times C \to M,\ (b, c) \mapsto f_B(b) f_C(c)$$

and by universal property of the tensor product, there exists a unique map $\tilde{f} : B \otimes_A C \to M$ through which $f$ factors. Finally $f_B$ and $f_C$ factor through $\tilde{f}$ by diagram chasing and thus by universality of the tensor product we have that it satisfies the pushout universal property in this case.

**Exercise 3.** Describe the colimit of the diagram $F : J \to \mathsf{Set}$ given by $* \leftarrow * \to *$.

**Answer**

Recall that the colimit of a diagram $F : J \to \mathsf{C}$ is an object $\operatorname{colim} A_i \in \operatorname{Obj}\mathsf{C}$ with morphisms $f_j : A_j \to \operatorname{colim} A_i$ such that if $m : k \to j$ is a morphism in $J$, then the following diagram commutes

$$\begin{tikzcd}
\operatorname{colim} A_i & \\
A_j \arrow[u, "f_j"] & A_k \arrow[l, "F(m)"] \arrow[ul, "f_k"']
\end{tikzcd}$$

In our case, since we only have three objects the diagram looks like this

$$\begin{tikzcd} CL & C \arrow[l] \\ B \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

where $CL$ is the colimit object. In this particular case the colimit coincides with the pushout by universality.