Exercise 1 (Exercise 8, Stanley 1.44.a)**.** Show that the total number of cycles of all even permutations of $[n]$ and the total number of cycles of all odd permutations of $[n]$ differ by $(-1)^n(n-2)!$. Use generating functions.

I must start by making a review of group theory which has helped me throughout the solution of this problem.

Definition 1. Suppose π is a permutation in S_n.

The order of a permutation is the amount of times we need to compose it with itself to obtain the identity permutation.

The parity of a permutation depends on the number of transpositions which compose it. A permutation is even when it is a product of an *even* number of transpositions. Likewise for odd permutations.

The sign of a permutation is 1 when π is even. When π is odd, the sign is -1.

For example, $\operatorname{ord}((123) = 3$ since $(123)(123)(123) = (123)(132) = \mathrm{id}$. Also $(12)(34)$ is an even permutation since it's a product of two transpositions.

Proposition 1. *For any cycle $c = (x_1x_2 \dots x_\ell)$, $\operatorname{ord}(c) = \ell$ the length of the cycle.*
The sign of the cycle c can be computed as $(-1)^{\operatorname{ord}(c)-1}$.
The sign function is multiplicative.

This is because one can decompose a cycle of order ℓ can be decomposed into $\ell - 1$ transpositions.

Theorem 1. *Suppose $\pi \in S_n$ can be decomposed into a product of k cycles, $c_1c_2 \dots c_k$. Then the sign of π is the product of the signs of c_i's. The following formula holds:*[1]

$$\operatorname{sgn}(\pi) = (-1)^{\sum_{j=1}^{k} \operatorname{ord}(c_j)-k}.$$

This is because

$$\operatorname{sgn}(\pi) = \operatorname{sgn}(c_1 \dots c_k) = \operatorname{sgn}(c_1) \dots \operatorname{sgn}(c_k) = (1)^{\operatorname{ord}(c_1)-1} \dots (-1)^{\operatorname{ord}(c_k)-1}$$

and by summing up the exponents we obtain the desired formula.

Remark 2. The formula holds *even when the decomposition includes 1-cycles*. This is because the identity permutation has order 1.

The count of $\sum \operatorname{ord}(c)$ goes up by one, and the k count (amount of cycles) also goes up by one. Therefore parity is preserved.[2]

With this in hand let us proceed.

[1] **Sam** helped me out when verifying that this formula holds.

[2] This observation is key when recognizing the generating function. **Ian** was the one who pointed me out the fact that I could use length 1 cycles to fill out some missing spaces.

Answer

Let us call E_n to be the amount of cycles across all of the even permutations in S_n. Likewise for O_n, the number of cycles across odd permutations.
The quantity we are interested in is $D_n = E_n - O_n$. Suppose $\pi = c_1 \dots c_k$ is an even permutation, this means that π adds k cycles to the count of E_n. Likewise if π were odd, it adds k cycles to O_n.
Since at the end we are subtracting O_n from E_n, then we should take into account the sign when adding. This is our first key point.
In general, π contributes with $\operatorname{sgn}(\pi)k$ cycles to D_n. Counting[a] across all the permutations with k cycles we get

$$D_n = \sum_{k=1}^{n} \operatorname{sgn}(\pi) k c(n,k) = \sum_{k=1}^{n} (-1)^{\sum \operatorname{ord}(c_j) - k} k c(n,k)$$

where the c_j's are the decomposition in disjoint cycles of each permutation and $c(n,k)$ is the unsigned Stirling number of the first kind which counts the amount of permutations of S_n with k cycles in their decomposition.
This formula looks *oddly similar* to the Pochammer symbol's generating function

$$(x)_n = \sum_{k=1}^{n} s(n,k) x^k$$

evaluated at $x = 1$. This is because $s(n,k) = (-1)^{n-k} c(n,k)$.

[a] The idea to count across all permutations given their cycle length using the Stirling numbers comes from stackexchange: `math.se/113202`.

We reach a conundrum at this stage because in general $\sum \operatorname{ord}(c_j) \neq n$. For example consider the transposition (12), but in $S_{10^{10}}$. In this case, the sum of the orders is 2. Because we are only counting the transposition. However $n = 10^{10}$, which most definitely is not equal to 2.

Ian's key observation comes at play here, we can count the 1-cycles which are being multiplied tacitly to (12). We have $(12) = (12)(3)(4)\dots(10^{10})$. All of this transpositions have order 1, save for the first one. Adding up all of the orders, we do indeed get 10^{10}! Now, recall that adding the 1-cycles to our representation does not alter the parity, so the theorem about the parity still holds.

Continuing on with the assumption that we are counting every permutation together with its 1-cycles, our formula for D_n becomes

$$D_n = \sum_{k=1}^{n} (-1)^{n-k} k c(n,k) = \sum_{k=1}^{n} k s(n,k)$$

which we recognize as the derivative of the Pochammer symbol's generating function evaluated at $x = 1$.
The derivative in question is precisely

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=1} (x)_n = \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=1} \left[(x)_{n-1}(x-(n-1))\right] \\
\Rightarrow &\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=1} (x)_n = \left(\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=1} (x)_{n-1}\right)(x-(n-1)) \mid_{x=1} + (x)_{n-1} \mid_{x=1} \\
\Rightarrow &D_n = D_{n-1}(2-n) + \delta_{n1}.
\end{aligned}$$

This recurrence relation allows us to find D_n given the initial condition that $D_1 = 1$, because $E_1 = 1$ (the identity) and $O_1 = 0$. For $n \geqslant 1$ we have $\delta_{n1} = 0$, so

$$D_n = D_{n-1}(2-n) = \left[D_{n-2}(2-(n-1))\right](2-n) = D_{n-2}(3-n)(2-n).$$

Inductively we can see that this quantity is

$$D_1 \ldots (4-n)(3-n)(2-n) = (-1)^{n-2}(n-2)! = (-1)^n (n-2)!$$

and therefore $E_n - O_n = (-1)^n (n-2)!$ as desired.