

Exercise 1 (Exercise 1). Find the largest possible size of a matching for P_n , and find the smallest possible size of a maximal matching for P_n . Express your answers in terms of n (they may depend on the parity of n or its residue modulo 3).

Answer

Exercise 2 (Exercise 3). Show that the number of spanning trees of $K_{m,n}$ is $m^{n-1}n^{m-1}$.

We will follow the steps described in problem 5.66 in Stanley Vol.2.

Answer

The adjacency matrix of $K_{m,n}$ can be written in block form:

$$A = \begin{pmatrix} 0_{m \times m} & \mathbf{1}_{m \times n} \\ \mathbf{1}_{n \times m} & 0_{n \times n} \end{pmatrix}.$$

Here 0 is the zeroes matrix and 1 is the ones matrix. The vertices of $K_{m,n}$ have degree either n or m so the Laplacian matrix of $K_{m,n}$ is

$$L = D - A = \begin{pmatrix} nI_{m \times m} & 0_{m \times n} \\ 0_{n \times m} & mI_{n \times n} \end{pmatrix} - \begin{pmatrix} 0_{m \times m} & \mathbf{1}_{m \times n} \\ \mathbf{1}_{n \times m} & 0_{n \times n} \end{pmatrix} = \begin{pmatrix} nI_{m \times m} & -\mathbf{1}_{m \times n} \\ -\mathbf{1}_{n \times m} & mI_{n \times n} \end{pmatrix}.$$

As L is a symmetric matrix, it is diagonalizable. This will come in handy when finding the amount of eigenvalues. With this in hand, let us proceed with the computations:

i) The matrix $L - mI$ is precisely

$$L - mI = \begin{pmatrix} (n-m)I_{m \times m} & -\mathbf{1}_{m \times n} \\ -\mathbf{1}_{n \times m} & 0_{n \times n} \end{pmatrix}$$

whose last n rows are all identical. We row reduce this matrix in the following way:

- ◇ Eliminate the last $n - 1$ rows subtracting row $n + 1$ from them. We initially guess that the rank of this matrix will be $m + 1$.
- ◇ Divide the first m rows by $n - m$ and then eliminate the ones in the $(m + 1)^{\text{th}}$ row by subtracting the first m rows from that one.
- ◇ Our last row is now $(0, \dots, 0, \frac{m}{n-m}, \dots, \frac{m}{m-n})$ which we will convert to a row of ones after dividing by $m/(n - m)$.

◇ We can use the last row to eliminate the $-\mathbf{1}_{m \times n}$ block on top.

The resulting matrix is $\text{rref}(L - mI)$, the rank of this matrix is $m + 1$ so the rank of $L - mI$ is also $m + 1$.

By the rank nullity theorem, $\dim \ker(L - mI) + (m + 1) = m + n$ and so the geometric multiplicity of m is $n - 1$. Thus there are *at least* $(n - 1)$ eigenvalues equal to m . As L is diagonalizable, the algebraic and geometric multiplicities must coincide, so there are *exactly* $(n - 1)$ eigenvalues equal to m .

- ii) With the same reasoning we can prove that the geometric multiplicity of n is $m - 1$. In which case, there are $m - 1$ eigenvalues of L equal to n .
- iii) The matrix L can have at most $m + n$ eigenvalues, summing the multiplicities we get

$$(m - 1) + (n - 1) + \text{remaining} = m + n \Rightarrow \text{remaining} = 2.$$

To find the remaining eigenvalues we will consult the determinant and the trace of L . As L is a block matrix whose diagonal is made of square blocks we have

$$\begin{aligned} \det L &= \det \begin{pmatrix} nI_{m \times m} & -\mathbf{1}_{m \times n} \\ -\mathbf{1}_{n \times m} & mI_{n \times n} \end{pmatrix} \\ &= \det(nI_{m \times m}) \det(mI_{n \times n} - (-\mathbf{1}_{n \times m})(nI_{m \times m})^{-1}(-\mathbf{1}_{m \times n})) \\ &= n^m \det(mI_{n \times n} - (1/n)(\mathbf{1}_{m \times n})^T(\mathbf{1}_{m \times n})) \end{aligned}$$

The matrix $(\mathbf{1}_{m \times n})^T(\mathbf{1}_{m \times n})$ is an $[n \times n]$ with entries $\langle \mathbf{1}_{1 \times m} | \mathbf{1}_{m \times 1} \rangle = m$. Thus the matrix inside the determinant is

$$mI_{n \times n} - (m/n)\mathbf{1}_{n \times n} \Rightarrow (mI_{n \times n} - (m/n)\mathbf{1}_{n \times n})\mathbf{1}_{n \times 1} = 0.$$

As the rows of both matrices sum to the same value m , then the corresponding ones vector $\mathbf{1}_{n \times 1}$ is in their kernel. Our matrix in question has non-trivial kernel and thus is singular. Then it has determinant zero. It follows that L has determinant zero^a and so it has zero as an eigenvalue.

The trace of our matrix is $\text{tr}(L) = nm + mn = 2mn$. And as the sum of the eigenvalues is the trace, we have that

$$(m - 1)n + (n - 1)m + 0 + \text{last} = 2mn \Rightarrow \text{last} = m + n.$$

iv) Finally, using the Matrix-Tree Theorem we conclude that the number of spanning trees is

$$\frac{1}{m+n}(m+n)m^{n-1}n^{m-1} = m^{n-1}n^{m-1}.$$

^aI should've realized earlier without invoking the Schur Decomposition, that the rows of L sum to the same value. This means that $\mathbf{1}_{(m+n) \times 1}$ is in the kernel and thus L is singular. Still, it was a fun exercise to compute that determinant.

Exercise 3 (Exercise 4). Let m and n be positive integers with $m < n$. How many saturated matchings does the complete bipartite graph $K_{m,n}$ have?

Answer

Call $M \cup N$ our partition of the vertices. Then any saturated matching must saturate M . Pick any vertex $v \in M$, then we have n possibilities from where to pick our first edge for the matching. Now pick another vertex $u \in M \setminus \{v\}$, we have $n - 1$ possibilities to pick another edge for our matching because we have already picked one edge and u can't be connected to that same vertex on our matching.

Iterating this process we see that the total number of ways to construct our matching is

$$n(n-1)(n-2)\dots(n-m+1) = n^{\underline{m}},$$

the Pochhammer Symbol.

Exercise 4 (Exercise 9). An undirected graph is k -regular if every vertex has degree k .

- i) Show that a bipartite k -regular graph must have the same number of vertices of each color in a two-coloring.
- ii) Show that such a graph has a perfect matching (that saturates both vertex colors).

Answer

Call $G = U \cup V$ our graph. We will begin by proving that U and V must be of the same size given that G is regular. The number of edges in G can be counted in two ways:

- ◇ Every vertex sends k edges from U to V so there are in total $k|U|$ edges.
- ◇ By the same reasoning, counting on the other side, there are $k|V|$ edges in

total.

This means that $k|U| = k|V|$ and thus $|U| = |V|$.

- i) Color the vertices according to which set they are in. Paint a vertex red if it's in U and blue if it's in V . No other two-coloring is proper because if we paint any vertex in V red, it will be connected to all the other red vertices in U .
- ii) Now let us verify the condition of Hall's theorem. Pick any subset of $S \subseteq U$ and look at edges coming out of it, there are $k|S|$ edges FINISH