Ignacio Rojas (All the proofs are done w/ the help of **Oxley** (not in person), after a thorough reading of Matroid Theory)

But can you imagine?!

3. (2-) [3 points] Let G be a graph with edge set E, and let $\mathcal{C}$ be the set of all cycles of G, considered as subsets of E. Prove that $(E, \mathcal{C})$ satisfies the circuit axioms C1–C3 defining a matroid.

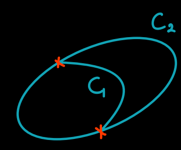
Recall circuit axioms: $C1: \phi \notin \mathcal{C}$, $C2: C_1 \subseteq C_2 \Rightarrow C_1 = C_2$,

$C3: C_1, C_2 \in \mathcal{C} \,(C_1 \neq C_2 \wedge e \in C_1 \cap C_2 \Rightarrow \exists C_3 \in \mathcal{C}\,(C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}))$

Let $\mathcal{C} = \{C \subseteq G : C \text{ is a cycle}\}$

▷ $\mathcal{C}$ satisfies C1: ϕ is not a cycle. If it was ϕ wouldn't be empty.

▷ $\mathcal{C}$ satisfies C2: If $C_1 \subsetneq C_2$ then $\exists v \in C_2 \,(d(v) = 3)$.



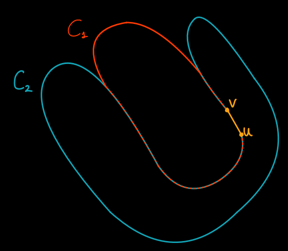
In any cycle, all vertices have degree 2. So containment can't be proper $\Rightarrow C_1 = C_2$

▷ $\mathcal{C}$ satisfies C3: Let $C_1 \neq C_2$ be two different cycles which share an edge.

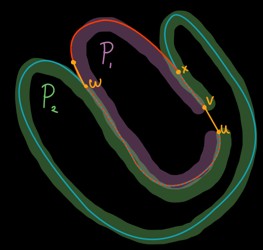
If uv is the common edge, we can build P_1 a path from u to v s.t. $P_1 \subseteq C_1 \setminus \{uv\}$.

Analogously, we build $P_2 \subseteq C_2 \setminus \{uv\}$.

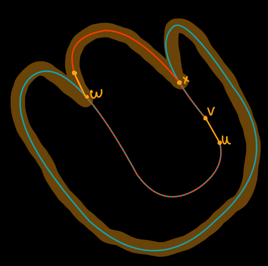
This is possible because C_1, C_2 are cycles.



To build the desired cycle walk P_1 from u to v.



Mark the 1^{st} vtx. $w \in P_1$ s.t. the next edge in P_1 isn't in P_2. Continuing the walk, mark $x \neq w$ as the next intersection of P_1 and P_2.



The walks $w \to x \subseteq P_1$ and $x \to w \subseteq P_2$ form the desired cycle $C \subseteq (C_1 \cup C_2) \setminus \{e\}$.

$\therefore (G, \mathcal{C})$ is a matroid w.r.t. circuit axioms.

5. (2) [3 points] Prove that if $M = (E, \mathcal{I})$ is a matroid with respect to the independence axioms, and $\mathcal{C}$ is the set of circuits of M, then $(E, \mathcal{C})$ satisfies the circuit axioms (C1)-(C3).

Recall circuit axioms:

C1: $\emptyset \notin \mathcal{C}$ C2: $C_1, C_2 \in \mathcal{C} \, (C_1 \subseteq C_2 \Rightarrow C_1 = C_2)$

C3: $C_1, C_2 \in \mathcal{C} \, (C_1 \neq C_2 \wedge e \in C_1 \cap C_2 \Rightarrow \exists C_3 \in \mathcal{C} \, (C_3 \subseteq (C_1 \cup C_2) \setminus \{e\})$

Also: Dep. sets $= \mathcal{P}(E) \setminus \mathcal{I}$, Circuits: minimal dep. sets. wrt inclusion

→ Assume $(E, \mathcal{I})$ is a matroid via independence axioms.

▷ C1 is immediate as $\emptyset$ is independent $\Rightarrow \emptyset \notin \mathcal{C}$

▷ For C2: Take $C_1, C_2 \in \mathcal{C}$, assume $C_1 \subseteq C_2$. (Assume by contradiction that C_1 is properly contained in C_2)

Consider this:

Circuit $\Leftrightarrow$ Minimal Dependent $\Leftrightarrow \forall X \, (X \subsetneq C \Rightarrow X$ is independent$)$

$\Leftrightarrow \neg \exists X \, [\neg (X \subsetneq C \Rightarrow X$ is independent$)]$

$\Leftrightarrow \neg \exists X \, (X \subsetneq C \wedge X$ is NOT indepent$)$

$\Leftrightarrow \neg \exists X \, (X \subsetneq C \wedge X$ is dependent$)$

∴ No subsets of circuits are dependent

If $C_1 \subsetneq C_2$, then C_1 is a proper dependent subset of C_1.

This contradicts the minimality of C_2. Thus it must hold $C_1 \subseteq C_2$ but not $C_1 \subsetneq C_2$. So $C_1 = C_2$.

▷ C3: We must show that given 2 diff. circuits C_1, C_2 w/ $e \in C_1 \cap C_2$, then there is a $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

So let us assume there is no C_3 contained in $(C_1 \cup C_2) \setminus \{e\}$

For simplicity, call $D = (C_1 \cup C_2) \setminus \{e\}$

Obs.: D is independent

→ Else, it would be dependent and so one of two occur:

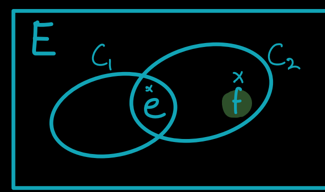
i) D is minimal $\Rightarrow$ D is a circuit

This contradicts the fact that D contains <u>no</u> circuits.

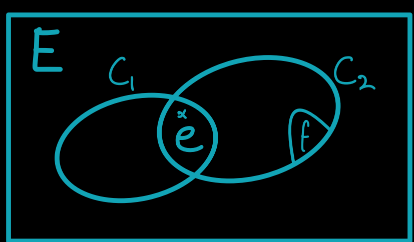
ii) $\exists D'$ dependent $\subseteq D$. Inductively this leads to a minimally dependent set, so D contains a circuit.

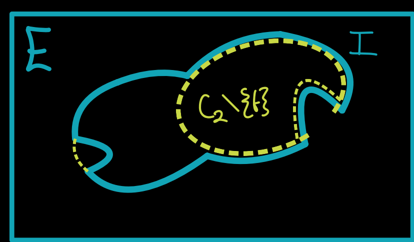
Once again, D contains no circuits

Now $C_1 \neq C_2 \overset{\text{Applying axiom C2}}{\Longrightarrow} C_1 \not\subseteq C_2 \Longrightarrow \exists f \in C_2 \setminus C_1$



$C_2 \setminus \{f\} \subsetneq C_2$ and C_2 is a circuit, thus $C_2 \setminus \{f\} \in \mathcal{I}$





Pick an indep. set which contains $C_2 \setminus \{f\}$.

It could be $C_2 \setminus \{f\}$ if no indep. set contains it. It can't be the whole $C_1 \cup C_2 \setminus \{f\}$. (If it was, $C_1 \subseteq$ such set $\overset{(I2)}{\Longrightarrow} C_1$ is indep.)

Now let $I \supseteq C_2 \setminus \{f\}$ be maximal w.r.t. containing $C_2 \setminus \{f\}$ and I doesn't contain the whole of C_2 (Else C_2 indep.)

So $\exists g \in C_1 \setminus I$. Consider the relation between I and D.

I is missing at least 2 elems. of $C_1 \cup C_2$,

↳ f, g ($g \neq f$ because $f \in C_2 \setminus C_1$, $g \in C_1$)

D is missing <u>only</u> one elem. of $C_1 \cup C_2$

In other words:

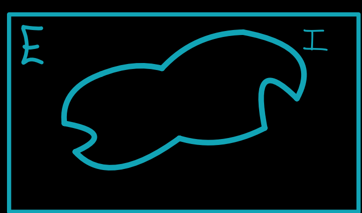
$$|I| \leq |C_1 \cup C_2 \setminus \{f, g\}| = |C_1 \cup C_2| - 2 < |C_1 \cup C_2| - 1 = |D|$$

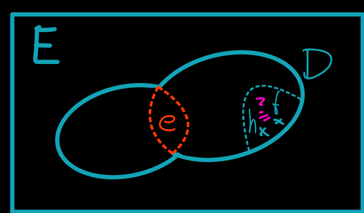
$$\implies |I| < |D|$$

We may now apply the exchange axiom $(I3)$ to I & D

$\exists h \in D \setminus I$ ($I \cup \{h\}$ is independent)

Observe that $I \cup \{h\} \supseteq I$





Such h is not f:

$$h = f \implies I \cup \{f\} \supseteq (C_2 \setminus \{f\}) \cup \{f\} = C_2 \implies C_2 \text{ indep. } (\Rightarrow\Leftarrow)$$

$\therefore$ $I \cup \{h\}$ is a larger indep. set than I which contains $C_2 \setminus \{f\}$

This contradicts I's maximality. In conclusion our assumption must be false.

$\therefore$ $\exists C_3 \subseteq D$ and so (E, I) satisfies $C1, C2, C3$.

6. (2+) [4 points] Prove that if $M = (E, \mathcal{C})$ is a matroid with respect to the circuit axioms, and $\mathcal{I}$ is the set of subsets of E that contain no member of $\mathcal{C}$, then $(E, \mathcal{I}$ satisfies the independence axioms (I1)-(I3).

(Hint: You may want to use proof by contradiction as we did in class.)

→ Let $\mathcal{I} \subseteq P(E)$ be the set $\{I \subseteq E : \nexists C \in \mathcal{C} (I \supseteq C)\}$.

▷ $\mathcal{I}$ satisfies (I1): As ϕ is empty, it only contains itself.

so it doesn't contain any $C \in \mathcal{C}$.

$\therefore \phi \in \mathcal{I}$

▷ (I2): Let $I \in \mathcal{I}$ and $J \subseteq I$. If there was a $C \in \mathcal{C}$ s.t.

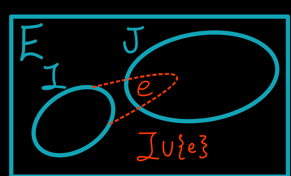
$C \subseteq J$, then $C \subseteq I$. But this is impossible as $I \in \mathcal{I}$.

$\therefore \exists C \in \mathcal{C} (J \supseteq C) \Rightarrow J \in \mathcal{I}$.

Recall I3: If $I_1, I_2 \in \mathcal{I}$ with $|I_1| > |I_2|$, then $\exists e \in I_1 \setminus I_2 (I_2 \cup \{e\} \in \mathcal{I})$

▷ (I3): Suppose I, J are indep. w/ $|I| < |J|$ but there's no

$e \in J \setminus I$ s.t. $I \cup \{e\} \in \mathcal{I}$



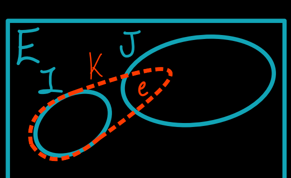
* There's no such e

↳ set of indep. subsets inside $I \cup J$ bigger than I

The set $S_I := \{K \in \mathcal{I} : K \subseteq I \cup J \wedge |K| > |I|\}$ is non-empty as it

contains J.

Obs.: We may pick a K such that $I \setminus K$ is non-empty



$I \setminus K = \phi$ as $K \supseteq I$

→ If $I \setminus K$ were empty, then $K \supseteq I$

As $|K| > |I|$, $K \supsetneq I$ and $K \cap J$ is non empty, so there is $e \in (K \cap J) \setminus I$

$I \cup \{e\} \subseteq K$ is indep. by (I2)

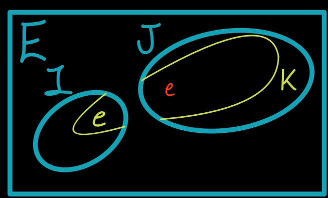
BUT $e \in J$ and e augments I (Impossible! No $e \in J$ can augment I)

Let us pick a $K \in S_I$ which makes $|I \setminus K|$ be as small as possible.

Obs.: (I3) also fails for (K, I)

$\rightarrow$ Any element $e \in K \setminus I$ used to augment I should be in J because $K \subseteq I \cup J$.

But once again we find $e \in J$ which augments I and this is impossible.

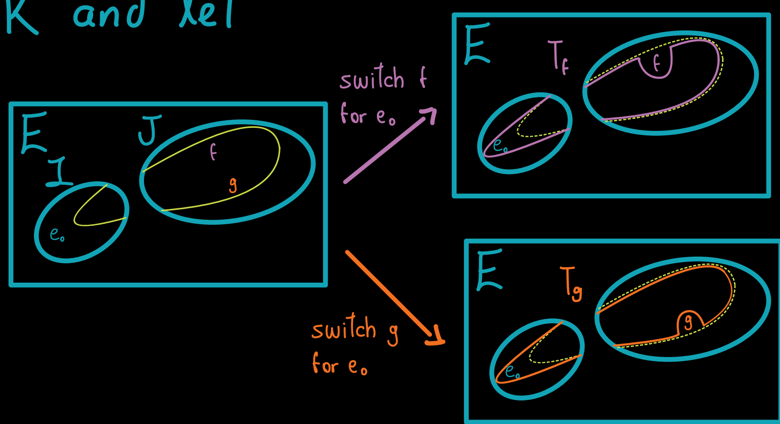


* If there was an $e \in K$ used to augment I, it should be in J. $(\Rightarrow\!\Leftarrow)$

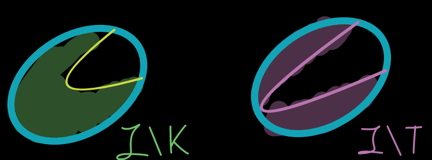
* If e was in I it wouldn't augment I.

Now let's pick an $e_0 \in I \setminus K$ and let

$$T_f := (K \cup \{e_0\}) \setminus \{f\}, \quad f \in K \setminus I$$



We have that $T_f \subseteq I \cup J$, $|T_f| = |K| > |I|$, and $|I \setminus T_f| \overset{(*)}{<} |I \setminus K|$



$\overset{(*)}{\rightarrow}$ This is because $T_f \cap I = (K \cap I) \cup \{e_0\}$

So $|I \setminus T_f| = |I \setminus (T_f \cap I)| = |I \setminus [(K \cap I) \cup \{e_0\}]|$

$= |[I \setminus (K \cap I)] \setminus \{e_0\}| = |(I \setminus K) \setminus \{e_0\}|$

$= |I \setminus K| - 1 < |I \setminus K|$

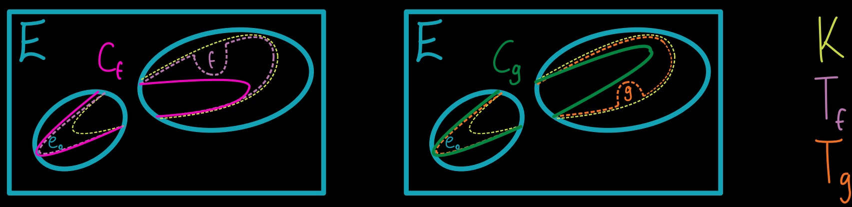
Obs.: T_f is not independent.

$\rightarrow$ If it was then $T_f \in S_I$ and $|I \setminus T_f| < |I \setminus K|$, but this contradicts K's minimality w.r.t. $|I \setminus K|$

Not being independent means T_f contains a circuit C_f.

$\implies C_f \subseteq (K \cup \{e_0\}) \setminus \{f\} \implies f \notin C_f \wedge e_0 \in C_f$

If C_f didn't contain e_0 then $C_f \subseteq K$, but $K \in \mathcal{I}$ means K doesn't contain circuits.



Also C_f must intersect $K \setminus I$, else $\underline{C_f} \subseteq (K \cup \{e_0\}) \cap I \underline{\subseteq I}$

which is $\underline{\text{impossible}}$ as I is indepent.

$\implies \exists g \in C_f \cap (K \setminus I)$ (I know I've drawn g before, but now it really comes into play.)

We construct a circuit C_g in an analogous way. We have:

$\longrightarrow e_0 \in C_f \cap C_g$

$\longrightarrow C_f \neq C_g$ (They differ by $g \in C_f \setminus C_g$)

Using circuit elimination, we find $C \subseteq (C_f \cup C_g) \setminus \{e\}$

$\implies \underline{C \subseteq K}$ ($\Rightarrow\Leftarrow$) Impossible as K was independent.

In conclusion, our assumption was wrong, there do not exist two sets which don't satisfy I3.

$\therefore (E, \mathcal{I})$ satisfies I1, I2 and I3.