Exercise 1 (Exercise 5)**.** A **binary tree** of length n constructed recursively as follows.

- ⋄ The empty set is a binary tree of length 0.
- ⋄ Otherwise a binary tree has a *root vertex* v, a *left subtree* T_1 and a *right subtree* T_2, each of which is also a binary tree having a root vertex.
  We draw the root vertex at the top with an edge going down to the root vertices of T_1, T_2. Then draw each tree recursively in the same manner.

Prove that the number of binary trees on n vertices is the n^{th} Catalan number C_n.⟦ Hint: Show that they satisfy the recursion for the Dyck paths ⟧

Answer

Let us call $f(n)$ the number of binary trees on n vertices. The initial condition is $f(0) = 1$ because the empty set is a binary tree.
To create a binary tree with $n+1$ vertices we choose the root and then we still have n vertices to go.
Fix ℓ to be the number of vertices we assign to the left tree then, the the remaining $n-\ell$ vertices go to the right tree. The number of ways to build right and left subtrees this way is $f(\ell)f(n-\ell)$.
However, running ℓ through all possible options of n gives us a plethora of disjoint events. We can sum those possibilities to get the total number of binary trees on $n+1$ vertices which is

$$f(n+1) = \sum_{\ell=0}^{n} f(\ell)f(n-\ell).$$

It follows that $f(n) = C_n$.

Exercise 2 (Exercise 6)**.** A **triangulation** of a convex $(n+2)$-gon is a collection $(n-1)$ diagonals that do not intersect each other. Show that the number of triangulations of a convex $(n+2)$-gon is the n^{th} Catalan numbers C_n. ⟦ Hint: Show that they satisfy the recursion for the Dyck paths ⟧

Answer

In the same way we chose a *left* and *right* trees, here we will chose L-and-R triangulations.
Once again let us begin by verifying the initial condition, for $n = 0$ we have a 2-gon which is a line. There's no possible triangulation there. The definition of triangulation starts making sense at $n = 1$ because we have a triangle and

$1 - 1 = 0$ diagonals.

Exercise 3 (Exercise 8)**.** A **derangement** of $[n]$ is a permutation $\pi \in S_n$ with no fixed points. That is $\forall i(\pi(i) \neq i)$. Let D_n be the number of derangements of $[n]$. Prove that

$$\sum_{n=0}^{\infty} \frac{D_n}{n!} x^n = \frac{e^{-x}}{1-x}.$$

Answer

Let us begin by establishing a recurrence relation for D_n. We will do this by considering `grad students` and their `preffered place to sit at`. Then the number D_n is the number of ways not `grad student` sits at their `preffered desk`.
Suppose that the first `grad student` enters the room and sits on desk i. When the i^{th} `grad student` enters the room there are two possibilities:

- They sit on desk 1, and then the problem reduces to the case with $n-2$ `grad students`.
- Otherwise we may relabel `grad student` i as the first `grad student` and then say that desk 1 is i's preferred desk. This reduces to the case of $n-1$ `grad students`.

Since this events are disjoint, the possibilities for each must be summed. But our choice for the first one's preference was arbitrary, there are other $n-1$ possible choices. It follows that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

By taking the exponential generating function on both sides we get

$$\sum_{n=0}^{\infty} D_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (n-1) D_{n-1} \frac{x^n}{n!} + \sum_{n=0}^{\infty} (n-1) D_{n-2} \frac{x^n}{n!} +$$