

Exercise 1 (Exercise 1). Among a family of 20 people, 9 of them like chocolate ice cream, 7 of them like vanilla ice cream, and 2 like both chocolate and vanilla. How many don't like either flavor of ice cream? Explain how the Inclusion-Exclusion principle applies here.

Answer

Call F the set of family members, C the set of people who like chocolate ice cream and V the ones who like vanilla. The set F can be partitioned as

$$F = (C \cup V) \cup (C \cup V)^c$$

where $(C \cup V)^c$ is the set of family members who dislike both flavors. This is the set we are looking for.

By Inclusion-Exclusion applied to the two sets C, V we have

$$|C \cup V| - |C| - |V| + |C \cap V| = 0 \Rightarrow |C \cup V| = 9 + 7 - 2 = 14.$$

As the whole family has 20 members and 14 of them like either flavor of ice cream, then 6 of them dislike both flavors.

We must use the Inclusion-Exclusion formula to find the size of the union, by summing only $|C|$ and $|V|$ we would be overcounting the the ones who like both flavors.

Exercise 2. Let P be a poset in which every interval $[x, y]$ is finite. Show that, in the incidence algebra $\mathcal{I}(P)$:

- (a) f is invertible if and only if $\forall x(f(x, x) \neq 0)$.
- (b) $fg = \delta \iff gf = \delta$, this is, inverses are two sided.
- (c) If f is invertible then f^{-1} is unique.

Really quickly, recall that the incidence algebra is the set of *interval functions* from P to \mathbb{C} . In other words, we can describe $\mathcal{I}(P)$ as

$$\mathcal{I}(P) = \{ f : P^2 \rightarrow \mathbb{C} : x > y \Rightarrow f(x, y) = 0 \}.$$

Answer

- (a) Suppose f is invertible with $fg = \delta$. If $x \in P$:

$$(f \cdot g)(x, x) = f(x, x)g(x, x) = \delta(x, x) = 1.$$

This means that, as complex numbers, $f(x, x)g(x, x) = 1$ thus none can be zero and $f(x, x) = \frac{1}{g(x, x)}$.

On the other hand, suppose $f(x, x) \neq 0$. We will construct an inverse for f inductively using the fact the every interval is finite. So, by induction on the length of the interval, if $|[x, y]| = 1$, then $x = y$ and $g(x, x) = \frac{1}{f(x, x)}$.