Exercise 1 (Exercise 1)**.** Find the largest possible size of a matching for P_n, and find the smallest possible size of a maximal matching for P_n. Express your answers in terms of n (they may depend on the parity of n or its residue modulo 3).

Answer

Exercise 2 (Exercise 3)**.** Show that the number of spanning trees of $K_{m,n}$ is $m^{n-1}n^{m-1}$.

We will follows the steps described in problem 5.66 in Stanley Vol.2.

Answer

The adjacency matrix of $K_{m,n}$ can be written in block form:

$$A = \begin{pmatrix} 0_{m\times m} & \mathbf{1}_{m\times n} \\ \mathbf{1}_{n\times m} & 0_{n\times n} \end{pmatrix}.$$

Here 0 is the zeroes matrix and $\mathbf{1}$ is the ones matrix. The vertices of $K_{m,n}$ have degree either n or m so the Laplacian matrix of $K_{m,n}$ is

$$L = D - A = \begin{pmatrix} nI_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & mI_{n\times n} \end{pmatrix} - \begin{pmatrix} 0_{m\times m} & \mathbf{1}_{m\times n} \\ \mathbf{1}_{n\times m} & 0_{n\times n} \end{pmatrix} = \begin{pmatrix} nI_{m\times m} & -\mathbf{1}_{m\times n} \\ -\mathbf{1}_{n\times m} & mI_{n\times n} \end{pmatrix}.$$

As L is a symmetric matrix, it is diagonalizable. This will come in handy when finding the amount of eigenvalues. With this in hand, let us proceed with the computations:

i) The matrix $L - mI$ is precisely

$$L - mI = \begin{pmatrix} (n-m)I_{m\times m} & -\mathbf{1}_{m\times n} \\ -\mathbf{1}_{n\times m} & 0_{n\times n} \end{pmatrix}$$

whose last n rows are all identical. We row reduce this matrix in the following way:

- ◇ Eliminate the last $n-1$ rows subtracting row $n+1$ from them. We initially guess that the rank of this matrix will be $m+1$.
- ◇ Divide the first m rows by $n-m$ and then eliminate the ones in the $(m+1)^{\text{th}}$ row by subtracting the first m rows from that one.
- ◇ Our last row is now $(0,\ldots,0,\frac{m}{n-m},\ldots,\frac{m}{m-n})$ which we will convert to a row of ones after dividing by $m/(n-m)$.

◇ We can use the last row to eliminate the $-\mathbf{1}_{m\times n}$ block on top.

The resulting matrix is $\texttt{rref}(L - mI)$, the rank of this matrix is $m+1$ so the rank of $L - mI$ is also $m+1$.

By the rank nullity theorem, $\dim\ker(L - mI) + (m+1) = m+n$ and so the geometric multiplicity of m is $n-1$. Thus there are *at least* $(n-1)$ eigenvalues equal to m. As L is diagonalizable, the algebraic and geometric multiplicities must coincide, so there are *exactly* $(n-1)$ eigenvalues equal to m.

ii) With the same reasoning we can prove that the geometric multiplicity of n is $m-1$. In which case, there are $m-1$ eigenvalues of L equal to n.

iii) The matrix L can have at most $m+n$ eigenvalues, summing the multiplicities we get

$$(m-1) + (n-1) + \text{remaining} = m+n \Rightarrow \text{remaining} = 2.$$

To find the remaining eigenvalues we will consult the determinant and the trace of L. As L is a block matrix whose diagonal is made of square blocks we have

$$\begin{aligned}\det L &= \begin{pmatrix} nI_{m\times m} & -\mathbf{1}_{m\times n} \\ -\mathbf{1}_{n\times m} & mI_{n\times n}\end{pmatrix} \\ &= \det(nI_{m\times m})\det\big(mI_{n\times n} - (-\mathbf{1}_{n\times m})(nI_{m\times m})^{-1}(-\mathbf{1}_{m\times n})\big) \\ &= n^m \det\big(mI_{n\times n} - (1/n)(\mathbf{1}_{m\times n})^{\mathsf{T}}(\mathbf{1}_{m\times n})\big)\end{aligned}$$

The matrix $(\mathbf{1}_{m\times n})^{\mathsf{T}}(\mathbf{1}_{m\times n})$ is an $[n\times n]$ with entries $\langle \mathbf{1}_{1\times m} | \mathbf{1}_{m\times 1}\rangle = m$. Thus the matrix inside the determinant is

$$mI_{n\times n} - (m/n)\mathbf{1}_{n\times n} \Rightarrow (mI_{n\times n} - (m/n)\mathbf{1}_{n\times n})\mathbf{1}_{n\times 1} = 0.$$

As the rows of both matrices sum to the same value m, then the corresponding ones vector $\mathbf{1}_{n\times 1}$ is in their kernel. Our matrix in question has non-trivial kernel and thus is singular. Then it has determinant zero. It follows that L has determinant zero[a] and so it has zero as an eigenvalue.

The trace of our matrix is $\operatorname{tr}(L) = nm + mn = 2mn$. And as the sum of the eigenvalues is the trace, we have that

$$(m-1)n + (n-1)m + 0 + \text{last} = 2mn \Rightarrow \text{last} = m+n.$$

iv) Finally, using the Matrix-Tree Theorem we conclude that the number of spanning trees is

$$\frac{1}{m+n}(m+n)m^{n-1}n^{m-1} = m^{n-1}n^{m-1}.$$

[a] I should've realized earlier without invoking the Schur Decomposition, that the rows of L sum to the same value. This means that $\mathbf{1}_{(m+n)\times 1}$ is in the kernel and thus L is singular. Still, it was a fun exercise to compute that determinant.

Exercise 3 (Exercise 4)**.** Let m and n be positive integers with $m < n$. How many saturated matchings does the complete bipartite graph $K_{m,n}$ have?

Answer

Call $M \uplus N$ our partition of the vertices. Then any saturated matching must saturate M. Pick any vertex $v \in M$, then we have n possibilities from where to pick our first edge for the matching. Now pick another vertex $u \in M \backslash \{ v \}$, we have $n-1$ possibilities to pick another edge for our matching because we have already picked one edge and u can't be connected to that same vertex on our matching.
Iterating this process we see that the total number of ways to construct our matching is

$$n(n-1)(n-2)\dots(n-m+1) = n^{\underline{m}},$$

the Pochammer Symbol.

Exercise 4 (Exercise 9)**.** An undirected graph is k-regular if every vertex has degree k.

i) Show that a bipartite k-regular graph must have the same number of vertices of each color in a two-coloring.

ii) Show that such a graph has a perfect matching (that saturates both vertex colors).

Answer

Call $G = U \cup V$ our graph. We will begin by proving that U and V must be of the same size given that G is regular. The number of edges in G can be counted in two ways:

- Every vertex sends k edges from U to V so there are in total $k|U|$ edges.
- By the same reasoning, counting on the other side, there are $k|V|$ edges in

total.

This means that $k|U| = k|V|$ and thus $|U| = |V|$.

i) Color the vertices according to which set they are in. Paint a vertex red if it's in U and blue if it's in V. No other two-coloring is proper because if we paint any vertex in V red, it will be connected to all the other red vertices in U.

ii) Now let us verify the condition of Hall's theorem. Pick any subset of $S \subseteq U$ and look at edges coming out of it, there are $k|S|$ edges which land in $N(S)$. Now

$$E(N(S)) = \{\text{ edges to } S\,\} \cup \{\text{ edges to } U \backslash S\,\}.$$

There are two possibilities:

a) If the second set is empty then $E(N(S)) = E(S)$ but we can count $E(N(S)) = k|N(S)|$, so $|N(S)| = |S|$ and thus we have Hall's condition.

b) On the other hand $|E(N(S))| > |E(S)|$ and so $k|N(S)| > k|S|$, so once again we have Hall's condition.

By Hall's theorem, there must exist a perfect matching, as such matching saturates U and therefore saturates V.