Ejercicio 1. Sea $\varnothing \neq E \subsetneq \mathcal{C}(\mathbb{R})$, muestre que

$$I_E = \{ f \in \mathcal{C}(\mathbb{R}) \mid \forall x \in E(f(x) = 0) \}$$

no es un ideal principal.

Prueba

Suponga a manera de contradicción que $I_E = \operatorname{gen}(g)$ para $g \in \mathcal{C}(\mathbb{R})$. Luego toda función en I_E se puede escribir como $f = gh$ y $g(x) = 0$ para $x \in E$. Suponga adicionalmente que E no es denso. De esta manera hay un abierto U tal que $U \cap E = \varnothing$. Dadas las condiciones g se anula en la frontera de U.
Considere así la función

$$h = \sqrt{|g|}\mathbf{1}(U) + g\mathbf{1}(U^c)$$

que es positiva en U y además $h \in I_E$. Así $h = fg$ para alguna f continua y de inmediato vemos que

$$f \restriction_U = \frac{\sqrt{|g|}}{g} \sim \frac{1}{\sqrt{g}} \xrightarrow[x \to \partial U]{} \infty.$$

Esto contradice la continuidad de f por lo que nuestra suposición está errada. Así I_E no es principal.

To determine whether the series $\sum_{n=2}^{\infty} \log(1 + (-1)^n/\sqrt{n})$ is convergent or divergent, we can use the alternating series test.
First, note that the terms of the series alternate in sign, since each term is multiplied by $(-1)^n$.
Next, consider the absolute value of the terms:
$|\log(1 + (-1)^n/\sqrt{n})|$
$= \log(1 + (-1)^n/\sqrt{n})$
Since $\log(x)$ is an increasing function for $x > 1$, we can say that:
$\log(1 + (-1)^n/\sqrt{n}) \leqslant \log(1 + 1/\sqrt{n})$
Now we can apply the limit comparison test by comparing our series with the series $1/\sqrt{n}$, which is a p-series with $p = 1/2$.
By taking the limit of the ratio of the two series, we get:
$\lim_{n\to\infty} \frac{\log(1+(-1)^n/\sqrt{n})}{1/\sqrt{n}}$
$= \lim_{n\to\infty} \sqrt{n}\log(1 + (-1)^n/\sqrt{n})$

Using L'Hopital's rule on this limit, we can get:
$\text{lím}_{n\to\infty} \frac{\log(1+(-1)^n/\sqrt{n})}{1/(2\sqrt{n})}$
$= \text{lím}_{n\to\infty} 2\sqrt{n}\log(1+(-1)^n/\sqrt{n})$
Since the limit of $2\sqrt{n}$ as n approaches infinity is infinity, the limit of our ratio is 0, meaning that our series converges by the limit comparison test.
Therefore, the series $\sum_{n=2}^{\infty} \log(1+(-1)^n/\sqrt{n})$ is convergent.
And for the second part of your question:
To compute the limit:
$\text{lím}_{n\to\infty} \left|\frac{\log(1+(-1)^n/\sqrt{n})}{(-1)^n/\sqrt{n}}\right|$
we can simplify the expression first:
$\left|\frac{\log(1+(-1)^n/\sqrt{n})}{(-1)^n/\sqrt{n}}\right|$
$= |\log(1+(-1)^n/\sqrt{n}) \cdot (-1)^n\sqrt{n}|$
$= \log(1+(-1)^n/\sqrt{n}) \cdot \sqrt{n}$
Now, since the expression inside the absolute value is positive for all $n \geqslant 2$, we can drop the absolute value sign.
Taking the limit as n approaches infinity, we can apply L'Hopital's rule:
$\text{lím}_{n\to\infty} \log(1+(-1)^n/\sqrt{n}) \cdot \sqrt{n}$
$= \text{lím}_{n\to\infty} \frac{1}{2n(1+(-1)^n/\sqrt{n})}$
Since the denominator goes to infinity as n approaches infinity, and the numerator is bounded, the limit is 0.
Therefore, the limit of the expression is 0 as n approaches infinity.
$\left|\frac{\log(1+(-1)^n/\sqrt{n})}{(-1)^n/\sqrt{n}}\right|$
$= |\log(1+(-1)^n/\sqrt{n}) \cdot (-1)^n\sqrt{n}|$
$= \log(1+(-1)^n/\sqrt{n}) \cdot |\sqrt{n}|$.
Then we can proceed with taking the limit as n approaches infinity using L'Hopital's rule, and the final limit of the expression is still 0.