Chapter 1

Introduction and background

Our goal is to understand the calculation of Gromov-Witten invariants of the space $\overline{M}_{g,n}(\mathbb{P}^r, d)$, the moduli space of degree d maps to $\mathbb{P}^r$ using techiniques of Atiyah-Bott localization. To begin this endeavor, we must explore $\overline{\mathcal{M}}_{g,n}$ first and its intersection theory. This is the space of genus g curves with n marked points which was studied by Deligne and Mumford originally. Afterwards, we will introduce the concept of equivariant cohomology and the Atiyah-Bott theorem. The theorem's usefulness will be presented by showing several examples and finally we will apply it to calculate more examples in the setting of the moduli space of maps.

1.1 Moduli of curves

Definition 1.1.1. A genus g, n-pointed stable curve is a projective curve $(C, p_1, \ldots, p_n)$ satisfying the following properties:

(a) Marked points p_i are all distinct and lie in the smooth locus of the curve.

(b) Singularities of C are, at worst, nodal.

(c) Each connected component of the curve with genus g_i and n_i nodes or marks satisfies

$$2g_i - 2 + n_i > 0.$$

The moduli space of curves, $\overline{\mathcal{M}}_{g,n}$, parametrizes stable curves with arithmetic genus g and n marked points. This means that any point of the moduli space is an equivalence class of curves up to homeomorphism type.

Example 1.1.2. The moduli space $\overline{\mathcal{M}}_{0,4}$ parametrizes stable rational curves with 4 marks.

Inside this moduli space lies the dense open subset $\mathcal{M}_{0,4}$ of smooth rational curves with 4 marks. We have that

$$(\mathbb{P}^1, p_1, \ldots, p_4) \sim (\mathbb{P}^1, q_1, \ldots, q_4)$$

whenever there is exists a Möbius transformation $T \in \mathrm{PGL}_2$ such that
$$(q_1,\dots,q_4) = (Tp_1,\dots,Tp_4).$$
Any such Möbius transformation is determined by where it maps the points $0,1$ and ∞. With this fact in hand we may map the first 3 points of our curve to $0,1,\infty$, and let the last one map to an arbitrary but fixed t:
$$(Tp_1,\dots,Tp_4) = (0,1,\infty,t), \quad t \in \mathbb{P}^1.$$
At the level of equivalence classes this means:
$$[(\mathbb{P}^1,(p_1,\dots,p_4))] = [(\mathbb{P}^1,(0,1,\infty,t))]$$
and so every equivalence class is determined by a unique $t \in \mathbb{P}^1$. We call this value the <u>cross-ratio</u> of $(p_1,\dots,p_4)$. The Möbius transformation in question is
$$T(z) = \frac{(z-p_1)(p_2-p_3)}{(z-p_3)(p_2-p_1)}$$
and the image of p_4, $t = T(p_4)$ is the aformentioned cross-ratio. This leads us to see that
$$\mathcal{M}_{0,4} \simeq \mathbb{P}^1 \backslash \{0,1,\infty\}.$$
When we let $t \to 0,1$ or ∞, this appears to break the condition that marks must be distinct. But what happens is that we blow up the curve at that point of collision and attach our marks the exceptional divisor at different places. The point of contact then becomes a node and we do, indeed, get a stable curve. need to explain better.

Chapter 2

Moduli Spaces

2.1 Toy example: quadruplets of points

2.2 Quadruplets along $\mathbb{P}^1$

2.3 The moduli space of curves with n marked points

(a) Stable curves
(b) Examples $\overline{M}_{0,4}$,$\overline{M}_{0,5}$

2.4 The tautological ring inside $A^*(\overline{M})$

(a) ψ, λ classes
(b) Intersection product Examples
(c) Projection formula
(d) String and Dilaton relations
(e) Integral examples

2.5 Moduli space of maps

Chapter 3

Equivariant Cohomology and Localization

3.1 Basics of equivariant cohomology

(a) Borel Construction of Equivariant Cohomology
(b) Examples of point equivariant Cohomology
(c) Equivariant Cohomology of projective space

3.2 Atiyah-Bott Localization

(a) Example of $H_T^*(\mathbb{P}^r)$ through Localization
(b) Toric varieties Euler characteristic via Atiyah-Bott
(c) Hodge integral $\int_{\overline{M}_{0,2}(\mathbb{P}^2,1)} \mathrm{ev}_1^*([1:0:0])\mathrm{ev}_2^*([0:1:0])$ via localization.