MATH 620: Variational Methods and Optimization I

Homework 6

Problem 1 (A small variation for the Dirichlet problem). In class, we have gone through the details of a proof for guaranteeing that a minimizer exists for the functional

$$I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2$$

over the (affine) space

$$X_g = \left\{u \in W^{1,2}(\Omega) : u|_{\partial\Omega} = g\right\}.$$

Among the other consequences of the theorem were that the (unique) minimizer $\bar{u}$ had to satisfy the weak Euler-Lagrange equation

$$\int_\Omega \nabla \bar{u} \cdot \nabla \varphi = 0 \qquad \forall \varphi \in X_0,$$

where X_0 is the tangent space to X_g (i.e., consists of functions with zero boundary values), and that if $\bar{u}$ happens to be smooth enough, that it then has to satisfy the partial differential equation

$$\begin{aligned} -\Delta \bar{u} &= 0 && \text{in } \Omega, \\ \bar{u} &= g && \text{on } \partial\Omega, \end{aligned}$$

i.e., it has to solve the Laplace equation.

Repeat some of the steps of the proof for the following variation:

$$I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - hu,$$

where $h \in L^2(\Omega)$ is a given function. For simplicity take $X_0 = W_0^{1,2}$ as the set to find a minimum over, i.e., $g = 0$.

In particular, do the following:

- Repeat the first step of showing that a minimizer exists. Namely, we needed to show that for a minimizing sequence $\{u_n\} \subset X_g$ so that $I(u_n) \to m = \inf_{u \in X_g} I(u)$, there exists an N and $\gamma < \infty$ so that for all $n \geq N$, we have that $\|u_n\|_{W^{1,2}} \leq \gamma$.

  $$\|u\|_{W^{1,2}} \leq \gamma.$$

  The key to this was to show that

  $$\|u\|^2_{W^{1,2}} \leq c_1 I(u) + c_2.$$

  If this is true, then we know – because u_n is a *minimizing sequence* – that there are $N < \infty, |a| < \infty, b < \infty$ so that

  $$I(u_n) \leq am + b$$

for all sufficiently large $n \geq N$. As a consequence, we know that after that point in the sequence, $\|u\|_{W^{1,2}} \leq \sqrt{c_1(am+b)+c_2} = \gamma$ and the weak compactness of the ball of radius γ in $W^{1,2}$ then guarantees that there is a weakly convergent subsequence.

Show a similar proof with the variation of the functional $I(u)$ above.

- Show the weak Euler-Lagrange equation a minimizer has to satisfy.
- Show the strong Euler-Lagrange equation a minimizer has to satisfy if it is regular (smooth) enough.

(40 points)

The remainder of the homework is concerned with finding counter-examples for extensions of the general theorem we have mentioned in class. It reads as follows:

Theorem: *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary. Let $f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, $f = f(x,u,\xi)$ be a function that satisfies the following conditions:*

(i) $\xi \mapsto f(x,u,\xi)$ *is convex for all* $x \in \Omega, u \in \mathbb{R}$*;*

(ii) there exist $p > q \geq 1$ *and* $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$ *(i.e., they must be finite) so that*

$$f(x,u,\xi) \geq \alpha_1|\xi|^p + \alpha_2|u|^q + \alpha_3$$

for all $x \in \Omega, u \in \mathbb{R}, \xi \in \mathbb{R}^n$*.*

Then the functional

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx$$

has a minimizer $\bar{u}$ *in*

$$X_g = \left\{u \in W^{1,p}(\Omega) : u|_{\partial\Omega} = g\right\},$$

where g is the restriction of some $\tilde{g} \in W^{1,p}(\Omega)$ to $\partial\Omega$. (Or viewed differently, g are prescribed boundary values that are nice enough so that we can find an extension of g called $\tilde{g}$ so that $\tilde{g} \in W^{1,p}(\Omega)$ and so that $\tilde{g}|_{\partial\Omega} = g$.)

If, furthermore,

(iii) $f \in C^1$ *and if there is a* $\beta \geq 0$ *so that*

$$|f_u(x,u,\xi)| \leq \beta(1 + |u|^{p-1} + |\xi|^{p-1}),$$
$$|f_\xi(x,u,\xi)| \leq \beta(1 + |u|^{p-1} + |\xi|^{p-1}),$$

for all $x \in \Omega, u \in \mathbb{R}, \xi \in \mathbb{R}^n$*,*

then $\bar{u}$ *satisfies the weak Euler-Lagrange equations*

$$\int_\Omega (f_u(x, \bar{u}(x), \nabla\bar{u}(x))\varphi + f_\xi(x, \bar{u}(x), \nabla\bar{u}(x)) \cdot \nabla\varphi)\, dx = 0$$

for all $\varphi \in X_0$*.*

The theorem as stated seems to have a lot of restrictions, but it turns out that they all seem necessary since one can find counter-examples without too much trouble. The following exercises are therefore meant to probe the applicability of the theorem.

Problem 2 (Application 1 of the general theorem). Consider the function $f(x, u, \xi) = \frac{1}{4}|\xi|^4 + g(x, u)$ where $g \in C^{0,1}(\Omega \times \mathbb{R})$. Show that the theorem applies.

(20 points)

Problem 3 (Application 2 of the general theorem). Consider the function $f(x, u, \xi) = \frac{1}{2}|\xi|^2 - \frac{1}{2}\lambda^2 u^2$ where λ is large – say, larger than the constant in the Poincaré inequality for functions in $W_0^{1,2}(\Omega)$. Show that the theorem does not apply by checking each condition individually. Then try to construct a sequence u_n so that $I(u_n) \to -\infty$, i.e., show that $I(u)$ is not bounded from below on $X_0 = W_0^{1,2}$. For this part of the example, choose $\Omega = (0, 1)$ and $\lambda > \pi$.

(20 points)

Problem 4 (Application 3 of the general theorem). Consider the function $f(x, u, \xi) = (|\xi|^2 - 1)^2$ on $\Omega = (0, 1) \subset \mathbb{R}$ and with $X_g = W_0^{1,4}(0, 1)$. Show that the theorem does not apply by checking each condition individually.

Derive the weak and strong Euler-Lagrange equations for this case. Show that $u = 0$ satisfies both of these equations; then show that it is not a minimizer of $I(u)$, for example by finding another function $v \in X_g$ so that $I(v) < I(u)$.

(20 points)