Math 601: Advanced Combinatorics I
Homework 1 - Due Aug 30

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 15. The maximum possible score on this homework is 15 points. See the syllabus for details.

Problems

1. (5 points) Prove that all three definitions of representations of finite groups given in the lecture notes are equivalent. Then, for the examples of the groups G and H from Examples 2.1 and 2.2 in the lecture notes, express these representations as a vector space with an action, and as a module.

2. (2 points) Consider the representation of the group
$$B_2 = \{\text{upper triangular matrices in } \mathrm{GL}_2(\mathbb{C})\}$$
given by its defining action on $\mathbb{C}^2$, that is, every matrix is represented by itself. Show that it has a one-dimensional sub-representation, but that it does not decompose as a direct sum of irreducibles.

3. (3 points) Consider the representation of the group
$$B_n = \{\text{upper triangular matrices in } \mathrm{GL}_n(\mathbb{C})\}$$
given by its defining action on $\mathbb{C}^n$. Describe all of its sub-representations.

4. (3 points) Consider the matrix representation of S_3 as the symmetry group of the triangle with coordinates $(1,0), (-1/2, \sqrt{3}/2), (-1/2, -\sqrt{3}/2)$ in the plane. Show that this is an irreducible 2-dimensional representation, even over $\mathbb{C}$.

5. (4 points) Consider the representation of S_3 in which each permutation $\pi \in S_3$ is sent to its corresponding permutation matrix P, in which $P_{i,\pi(i)} = 1$ for all i, and all other entries are 0.
   (a) Find a common eigenvector of all of the permutation matrices.
   (b) Write the representation as a direct sum of irreducible representations.

6. Compute the dimension of each of the following Lie groups as a real manifold.
   (a) (1 point) $\mathrm{SL}_n(\mathbb{R})$
   (b) (1 point) $\mathrm{SL}_n(\mathbb{C})$
   (c) (1 point) $\mathrm{Sp}_{2n}(\mathbb{C})$
   (d) (1 point) $\mathrm{SO}_{2n+1}(\mathbb{R})$

7. (2 points) Show that the commutator bracket $[X, Y] = XY - YX$ is a Lie bracket, by showing that it is antisymmetric and satisfies the Jacobi identity
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

8. Using the formal infinitesimal ϵ satisfying $\epsilon^2 = 0$ as described in class, describe the elements of the Lie algebra corresponding to each of the following Lie groups.
   (a) (1 point) $\mathrm{SO}_n(\mathbb{C})$
   (b) (1 point) $\mathrm{Sp}_{2n}(\mathbb{C})$
   (c) (1 point) The torus $T_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ of diagonal invertible matrices
   (d) (1 point) The Borel subgroup $B_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ of upper triangular invertible matrices

9. (2 points) Show, using the ϵ method, that the Lie algebra of the orthogonal group $O_n(\mathbb{C})$ is isomorphic to that of the special orthogonal group $\mathrm{SO}_n(\mathbb{C})$. Why does this not contradict the bijective correspondence between connected Lie groups and Lie algebras?