Exercise 1. Define a *line* in $\mathbb{P}^2$ to be a closed subset of the form $L = \{ [x : y : z] : ax + by + cz = 0 \}$ for some constants $a, b, c \in \mathbb{C}$, not all zero.

i) If $(a, b, c) = (1, 0, 0)$, we saw in class that $\mathbb{P}^2 \backslash L = \{ [x : y : z] : x \neq 0 \} = U_x$ could be identified with $\mathbb{C}^2$.

   Similarly, show that for any line L there is a bijection $\mathbb{P}^2 \backslash L \simeq \mathbb{C}^2$.

ii) Prove that any two distinct lines L_1 and L_2 intersect in a single point.

iii) Prove that there is a unique line L through any two distinct points in $\mathbb{P}^2$.

Answer

i) constants

ii) Let $L_1, L_2 \subseteq \mathbb{P}^2$ be two distinct lines with direction (a, b, c) and (d, e, f). Since they are distinct this means that $\nexists \lambda((d, e, f) = \lambda(a, b, c))$. A point $[x : y : z]$ in the intersection of L_1 and L_2 must satisfy the system of equations

$$\begin{cases} ax + by + cz = 0, \\ dx + ey + fz = 0. \end{cases}$$

   Solutions to this system of equations are parametrized in terms of z in the following manner

$$[x : y : z] = \left[\frac{bf - ce}{ae - bd} z : \frac{cd - af}{ae - bd} z : z \right],$$

   and ordinarily this would give us an infinite number of solutions. However in $\mathbb{P}^2$ this corresponds to the point $[bf - ce : cd - af : ae - bd]$.

iii) Let us now consider two points $[x : y : z], [u : v : w] \in \mathbb{P}^2$ which are distinct. Once again, consider a system of equations

$$\begin{cases} ax + by + cz = 0, \\ au + bv + cw = 0. \end{cases}$$

   There is an infinite number of solutions to this system for $(a, b, c) \in \mathbb{C}^3$.aaaaaaa

Exercise 2. Consider the sequence $(p_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}^3$ with $p_n = (n^3, 2n^2, 3n^3)$. Identifying $\mathbb{C}^3$ with $\{ x_0 \neq 0 \} \subseteq \mathbb{P}^3$, what is the limit of p_n as $n \to \infty$?

Answer

We can identify p_n with the sequence $\widetilde{p}_n = [n^3 : 2n^2 : 3n^3 : 1]$. Now for $n \neq 0$ it holds that

$$\widetilde{p}_n = \left[1 : \frac{2}{n} : 3 : \frac{1}{n^3}\right] \xrightarrow[n\to\infty]{} [1 : 0 : 3 : 0].$$

This coincides with the limit of p_n in the usual sense which is ∞ and $[1 : 0 : 3 : 0]$ is a point at infinity.

Exercise 3. In $\mathbb{A}^2$, let $V = \mathbb{V}(x)$, $W = \mathbb{V}(x-1)$ and $Z = \mathbb{V}(y - x^2)$. Let $\overline{V}, \overline{W}$ and $\overline{Z}$ denote their respective *projective closures* in $\mathbb{P}^2$. Find the points in the intersections $\overline{V} \cap \overline{W}$, $\overline{V} \cap \overline{Z}$ and $\overline{W} \cap \overline{Z}$.

Answer

First, let us parametrize the varieties in question as points of $\mathbb{A}^2$:

$$\begin{cases} \mathbb{V}(x) = \{\, x = 0 \,\} = \{\, (0,t) : \ t \in \mathbb{C} \,\}, \\ \mathbb{V}(x-1) = \{\, x = 1 \,\} = \{\, (1,t) : \ t \in \mathbb{C} \,\}, \\ \mathbb{V}(y - x^2) = \{\, y = x^2 \,\} = \{\, (t,t^2) : \ t \in \mathbb{C} \,\}. \end{cases}$$

For each one of those sets, their projective closure corresponds to the embedding of the points inside $\mathbb{P}^2$ along with their limit points. In the case of V we have

$$\overline{V} = \{\, [0 : t : 1] : \ t \in \mathbb{C} \,\} \cup \{\, \text{limit points} \,\} = \{\, [0 : t : 1] : \ t \in \mathbb{C} \,\} \cup \{\, [0 : 1 : 0] \,\}.$$

Likewise we have

$$\begin{cases} \overline{W} = \{\, [1 : t : 1] : \ t \in \mathbb{C} \,\} \cup \{\, [0 : 1 : 0] \,\}, \\ \overline{Z} = \{\, [t : t^2 : 1] : \ t \in \mathbb{C} \,\} \cup \{\, [0 : 1 : 0] \,\}. \end{cases}$$

Now their intersections are

$$\begin{cases} \overline{V} \cap \overline{W} = \{\, [0 : 1 : 0] \,\}, \\ \overline{V} \cap \overline{Z} = \{\, [0 : 0 : 1], [0 : 1 : 0] \,\}, \\ \overline{V} \cap \overline{Z} = \{\, [1 : 1 : 1], [0 : 1 : 0] \,\}. \end{cases}$$

This coincides with our intuition. The lines only intersect at infinity, while the parabola and the lines intersect at the finite point and at infinity.

Exercise 4. Do the following:

i) Find a bijection between the set of all homogeneous polynomials in three variables of degree d and the set of all polynomials in two variables of degree at most d. ⟦Hint: Set one the variables to the constant 1.⟧
ii) Use this to show that the subspace topology induced by the affine patches $V \cap \mathbb{A}^2$ from the Zariski topology on a variety $V \subseteq \mathbb{P}^2$ is the same as the Zariski topology on the affine variety $V \cap \mathbb{A}^2$.
iii) Generalize to arbitrary dimension.

Answer

i) Let us call $\mathbb{C}_h[X, Y, Z]$ the set of homogeneous polynomials and $\mathbb{C}[x, y]$ the set of regular polynomials. Consider the mapping

$$\varepsilon_{(x,y,1)} : \mathbb{C}_h[X, Y, Z] \to \mathbb{C}[x, y],\ F([X : Y : Z]) \mapsto F(x, y, 1) = f(x, y)$$

where we evaluate $z = 1$ and consider the resulting polynomial as a non-homogeneous polynomial. The inverse mapping is

$$\varepsilon^{-1} : \mathbb{C}[x, y] \to \mathbb{C}_h[X, Y, Z],\ f(x, y) \mapsto z^d f\left(\frac{x}{z} : \frac{y}{z} : 1\right)$$

where we view the last polynomial as a homogeneous polynomial.

These functions are inverses to one another:

$$\varepsilon(\varepsilon^{-1}(f)) = \varepsilon\left(z^d f\left(\frac{x}{z} : \frac{y}{z} : 1\right)\right) = f(x : y : 1)$$

and the action of this function is the same as $f(x, y)$. The other direction is analogous.

ii) Let us recall that $\mathbb{A}^2$ can be viewed as a topological subspace of $\mathbb{P}^2$ because $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$. So to show that the subspace topology is the same as the ordinary Zariski topology it is enough to show that

$V \subseteq \mathbb{P}^2$ is closed $\iff$ $V \cap \mathbb{A}^2$ is the zero locus of a set of polynomials.

$\Rightarrow$ Suppose $V = \mathbb{V}(F_1, \ldots, F_n)$, where $F_j \in \mathbb{C}_h[X, Y, Z]$, we want to show that $V \cap \mathbb{A}^2 = \mathbb{V}[\varepsilon F_1, \ldots, \varepsilon F_n]$. For that effect, take $[a : b : c] \in V \cap \mathbb{A}^2$, this means that $c \neq 0$, so we can take $c = 1$ due to rescaling.

$$\begin{aligned} [a : b : 1] \in V &\Rightarrow \forall i(F_i([a : b : 1]) = 0) \\ &\Rightarrow \forall i[(\varepsilon F_i)(a, b) = 0] \\ &\Rightarrow (a, b) \in \mathbb{V}[\varepsilon F_1, \ldots, \varepsilon F_n]. \end{aligned}$$

Exercise 5. For the following coordinate rings, find affine varieties whose coordinate rings are isomorphic to the ones in questions.

i) $\mathbb{C}[x, 1/x, y]$ (this is, rational functions whose denominator is a polynomial in x.)