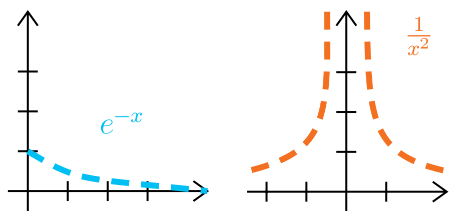
Improper Integration

The integral of a function f represents the area enclosed by f's graph and the x axis.

We have studied this type of integrals and these are called *definite integrals*. But what happens when there's a discontinuity in the function? Or when we want to find an infinite area? These questions are illustrated in the following examples:

$$\int_{-1}^{1} \frac{dx}{x^2} \quad \text{and} \quad \int_0^{\infty} e^{-x} dx.$$

These integrals can be represented graphically as



To address this integrals we will treat their troublesome points as limits.

Example 1. To find the integral $I = \displaystyle\int_{-1}^{1} \frac{dx}{x^2}$, we must recognize that $\frac{1}{x^2}$ has a singularity at $x=0$. We must, separate the integral in two, and then take limits in both of them. We have

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} \quad = \lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{x^2} + \lim_{a\to 0^+} \int_a^1 \frac{dx}{x^2}$$

and the integrals inside the limits are our day-to-day, usual definite integrals. We evaluate them to see that

$$I = \lim_{b\to 0^-} \left(\frac{-1}{b} - \frac{-1}{(-1)} \right) + \lim_{a\to 0^+} \left(\frac{-1}{1} - \frac{-1}{a} \right).$$

We can see that b's limit is positive infinity and a's limit is negative infinity. After solving the rest of the algebra we can see that the limit is infinite.

Definition. We say that an improper integral diverges when its corresponding limit doesn't exist or is infinite.

Let us take a look at the other example:

Example 2. To calculate the integral $\displaystyle\int_0^{\infty} e^{-x} dx$, we first notice that the function e^{-x} is continuous everywhere. So the only troublesome point is at infinity. We setup the integral as follows

$$\int_0^{\infty} e^{-x} dx = \lim_{b\to\infty} \int_0^b e^{-x} dx = \lim_{b\to\infty} (-e^{-b} - (-e^0))$$
$$= -\lim_{b\to\infty} e^{-b} + 1 = 0 + 1 = 1.$$

In this case we will say that $\displaystyle\int_0^{\infty} e^{-x} dx = 1$ since the limit exists and its value is 1.

Definition. We say that an improper integral converges when its corresponding limit exists and it's finite. The value of that limit is the value we will assign to the improper integral.

It may the case that we have both infinite limits! Consider the integral $\displaystyle\int_{-\infty}^{\infty} |x| e^{-x^2} dx$. While it would be tempting to consider that

$$\int_{-\infty}^{\infty} |x| e^{-x^2} dx = \lim_{a\to\infty} \int_{-a}^{a} |x| e^{-x^2} dx,$$

we can't do this. The behavior at each of the limits could be different.

Example 3. To evaluate the integral in question, we must separate by using any point in the middle which isn't a point of discontinuity.

The function $|x| e^{-x^2} dx$ is continuous everywhere so we pick $x = 0$ as our point in the middle. Then $\int_{-\infty}^{\infty} |x| e^{-x^2} dx = \int_{-\infty}^{0} |x| e^{-x^2} dx + \int_0^{\infty} |x| e^{-x^2} dx$ and now we take the limits.

But first, let's get rid of the absolute value by recalling that

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

The integral in question becomes

$$\lim_{r\to-\infty} \int_r^0 (-x) e^{-x^2} dx + \lim_{s\to\infty} \int_0^s x e^{-x^2} dx$$
$$= \lim_{r\to-\infty} -\left(\frac{-1}{2} - \frac{-e^{-r^2}}{2} \right) + \lim_{s\to\infty} \left(\frac{-e^{-s^2}}{2} - \frac{-1}{2} \right)$$
$$= \frac{1}{2} - 0 + 0 + \frac{1}{2} = 1.$$

So in the end, we say that $\displaystyle\int_{-\infty}^{\infty} |x| e^{-x^2} dx = 1.$

Remark. In cases like this one, if at least one of the integrals diverge, we will say that the whole integral diverges.

Practice

In groups, analyze the integral $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x^2}$.

i) Sketch a graph of the function from 0 to ∞.
ii) Evaluate the integral by taking the limit. Determine if it converges.
iii) Does the lower limit come into play?
   - What happens if we switch 1 for an a and let $a > 1$, for example 36?
   - Or if $a < 1$? Maybe even $a < 0$?

iv) How does the power of x contribute? Consider switching the 2 for a p. Examine different values of p:
   - $p < 2$, like $p = 3/2$.
   - Also $p = 1$ or $p = 1/2$.
   - What happens with negative values of p? Like $p = -9$?

Practice

Consider the integral $\displaystyle\int_{-3}^4 \frac{\mathrm{d}x}{x^3}$.

As in the last exercise analyze this integral's behavior.

i) Change both limits to other values, including one where the interval doesn't contain zero. What happens in that case?
ii) What role does the exponent 3 play? Change the values of the exponent and predict a general result.

Practice

Study the integral $\displaystyle\int_0^a \frac{\mathrm{d}x}{\sqrt{a-x}}$, where $a \in \mathbb{R}$.

i) First consider a positive value of a, such as $a = 2$. Does the integral
$$\int_0^2 \frac{\mathrm{d}x}{\sqrt{2-x}}$$
converge or diverge?
ii) Can you come up with a general result considering different values of a? This must *all* numbers, not only the whole negative numbers.

Practice

Consider the function $f(t) = \frac{4t}{\sqrt[3]{t^2-4}}$.

i) Find the points of discontinuity of f. Is this function discontinuous on a larger set?
ii) Determine if the integral
$$\int_1^2 f(t)\mathrm{d}t$$
is improper or not. In case it is, determine if it converges or not. In the case it does, find its value.

Practice

Consider the function $f(z) = \frac{1}{z^2} e^{\frac{1}{z}}$.

i) Where is this function discontinuous?
ii) If possible, find the value of the integral
$$\int_{-\infty}^0 f(z)\mathrm{d}z.$$

Practice

How does the integral $\displaystyle\int_{-\infty}^1 \sqrt{a-y}\,\mathrm{d}y$, where $a \in \mathbb{R}$.

i) Suppose a is positive, such as $a = 9$. Is the function continuous over the specified domain of integration? What if we switched the 1 to a 10?
ii) If we replace the 1 for a real number b, what can you say about the convergence of the integral? *The answer must depend on a and b.*