Exercise 1. Describe a "ballot-type" condition for a word of 1's and 2's to be lowest weight for $\mathfrak{sl}_2$, that is, that F sends the word to 0. Prove that your condition is correct. Do the same for $\mathfrak{sl}_3$ and the two lowering operators.

Answer

The condition we are looking for is precisely being a ballot word, this occurs when every prefix left-to-right has more 2's than 1's. Observe that any word like this will be sent by F_1 to zero. F_1 will look for the last unmatched 1, but having a greater number of 2's before it means that there will be no way for a 1 to be unmatched. Thus, F_1 will send such a word to zero.
Similarly for lowest weight words of $\mathfrak{sl}_3$, the condition is that in every prefix when reading left-to-right, we find more 3's than 2's and more 2's than 1's. Once again applying F_1 or F_2, we get nothing, because they will be looking for unmatched 1's or 2's respectively. But as the word is ballot, there are no unmatched 1's nor 2's. Observe also that it is not necessary for there to be double the amount of 2's because of the 1's and 3's. The F_i operators do not interact with each other so there's no qualms about that.

Exercise 2. Recall that the complete homogeneous symmetric function $h_\mu(x_1, \ldots, x_n)$, for a partition $\mu = (\mu_1, \ldots, \mu_k)$, can be defined as the product $h_{\mu_1} \ldots h_{\mu_k}$ where h_d is the sum of all monomials in $x_1, \ldots, x_n$ of degree d.

(a) Which Schur function in n variables is $h_d(x_1, \ldots, x_n)$ equal to?