Exercise 1. Do the following exercises from Artin's Algebra:

I) Consider

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \ \mathbf{b} = (b_1, \dots, b_n)$$

be a column and row vector respectively. Compute the products $\mathbf{ab}$ and $\mathbf{ba}$.

II) Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Find a formula for A^n and prove it by induction.

III) Verify the rule

$$\det AB = (\det A)(\det B)$$

for the matrices $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 5 & -2 \end{pmatrix}$.

Answer

I) $\mathbf{ab}$ turns into a $[1 \times 1]$ matrix. This is the inner product of this vectors. The result is $a_1 b_1 + \cdots + a_n b_n$.

On the other hand $\mathbf{ba}$ is a $[n \times n]$ matrix, the outer product of $\mathbf{a}$ and $\mathbf{b}$. This is a rank 1 matrix whose entries are given by $(a_i b_j)_{i,j \in [n]}$.

II) Computing some powers of A we see that

$$A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \ A^3 = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \ A^4 = \begin{pmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

The numbers in the first row of A are easy to characterize, one, n itself, and the n^{th} triangular number. The formula for A^n must be

$$A^n = \begin{pmatrix} 1 & n & \binom{n+1}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

With the matrices above we have the base case, so if we suppose that the formula is valid up to n, we would like to check it for $n+1$. In this case

$$A^nA = \begin{pmatrix} 1 & n+1 & \binom{n+1}{2} + (n+1) \\ 0 & 1 & n+1 \\ 0 & 0 & 1 \end{pmatrix}$$

and since $\binom{n+1}{2} + (n+1) = \binom{n+2}{2}$, we have proven our result.

III) We can see that

$$AB = \begin{pmatrix} 17 & -4 \\ 21 & -7 \end{pmatrix} \Rightarrow \det(AB) = -35.$$

And by themselves, $\det A = 5$ and $\det B = -7$ so the formula holds.