Exercise 1. Define a *line* in $\mathbb{P}^2$ to be a closed subset of the form $L = \{\, [x : y : z] : ax + by + cz = 0 \,\}$ for some constants $a, b, c \in \mathbb{C}$, not all zero.

i) If $(a, b, c) = (1, 0, 0)$, we saw in class that $\mathbb{P}^2 \backslash L = \{\, [x : y : z] : \; x \neq 0 \,\} = U_x$ could be identified with $\mathbb{C}^2$.

   Similarly, show that for any line L there is a bijection $\mathbb{P}^2 \backslash L \simeq \mathbb{C}^2$.

ii) Prove that any two distinct lines L_1 and L_2 intersect in a single point.

iii) Prove that there is a unique line L through any two distinct points in $\mathbb{P}^2$.

Answer

i) constants

ii) Let $L_1, L_2 \subseteq \mathbb{P}^2$ be two distinct lines with direction (a, b, c) and (d, e, f). Since they are distinct this means that $\nexists \lambda((d, e, f) = \lambda(a, b, c))$. A point $[x : y : z]$ in the intersection of L_1 and L_2 must satisfy the system of equations

$$\begin{cases} ax + by + cz = 0, \\ dx + ey + fz = 0. \end{cases}$$

Solutions to this system of equations are parametrized in terms of z in the following manner

$$[x : y : z] = \left[\frac{bf - ce}{ae - bd} z : \frac{cd - af}{ae - bd} z : z \right],$$

and ordinarily this would give us an infinite number of solutions. However in $\mathbb{P}^2$ this corresponds to the point $[bf - ce : cd - af : ae - bd]$.

iii) Let us now consider two points $[x : y : z], [u : v : w] \in \mathbb{P}^2$ which are distinct. Once again, consider a system of equations

$$\begin{cases} ax + by + cz = 0, \\ au + bv + cw = 0. \end{cases}$$

There is an infinite number of solutions to this system for $(a, b, c) \in \mathbb{C}^3$.aaaaaaa

Exercise 2. Consider the sequence $(p_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}^3$ with $p_n = (n^3, 2n^2, 3n^3)$. Identifying $\mathbb{C}^3$ with $\{\, x_0 \neq 0 \,\} \subseteq \mathbb{P}^3$, what is the limit of p_n as $n \to \infty$?

Answer

We can identify p_n with the sequence $\widetilde{p}_n = [n^3 : 2n^2 : 3n^3 : 1]$. Now for $n \neq 0$ it holds that

$$\widetilde{p}_n = \left[1 : \frac{2}{n} : 3 : \frac{1}{n^3}\right] \xrightarrow[n\to\infty]{} [1 : 0 : 3 : 0].$$

This coincides with the limit of p_n in the usual sense which is ∞ and $[1 : 0 : 3 : 0]$ is a point at infinity.

Exercise 3. In $\mathbb{A}^2$, let $V = \mathbb{V}(x)$, $W = \mathbb{V}(x-1)$ and $Z = \mathbb{V}(y - x^2)$. Let $\overline{V}, \overline{W}$ and $\overline{Z}$ denote their respective *projective closures* in $\mathbb{P}^2$. Find the points in the intersections $\overline{V} \cap \overline{W}$, $\overline{V} \cap \overline{Z}$ and $\overline{W} \cap \overline{Z}$.

Answer

First, let us parametrize the varieties in question as points of $\mathbb{A}^2$:

$$\begin{cases} \mathbb{V}(x) = \{\, x = 0 \,\} = \{\, (0, t) :\ t \in \mathbb{C} \,\}, \\ \mathbb{V}(x-1) = \{\, x = 1 \,\} = \{\, (1, t) :\ t \in \mathbb{C} \,\}, \\ \mathbb{V}(y - x^2) = \{\, y = x^2 \,\} = \{\, (t, t^2) :\ t \in \mathbb{C} \,\}. \end{cases}$$

For each one of those sets, their projective closure corresponds to the embedding of the points inside $\mathbb{P}^2$ along with their limit points. In the case of V we have

$$\overline{V} = \{\, [0 : t : 1] :\ t \in \mathbb{C} \,\} \cup \{\, \text{limit points} \,\} = \{\, [0 : t : 1] :\ t \in \mathbb{C} \,\} \cup \{\, [0 : 1 : 0] \,\}.$$