Exercise 1 (1.6.D Vakil)**.** Show that a map of complexes induces a map of homology $H^i(A^\bullet) \to H^i(B^8)$ and furthermore, H^i is a covariant functor from $\mathsf{Com}_{\mathsf{C}} \to \mathsf{C}$. ⟦Feel free to deal with the special case Mod_A.⟧

Answer

We will work inside the category of modules in this case. Consider two complexes $A^\bullet, B^\bullet$ with a map of complexes $\varphi : A^\bullet \to B^\bullet$ where $\varphi^i : A^i \to B^i$. To define a map between homology, we will first show that the chain map preserves cycles and boundaries.

- ◇ Suppose $z \in A^i$ is a cycle, then $f^i(z) = 0$. Composing with φ^{i+1} we still get 0. However, by commutativity we have

$$0 = \varphi^{i+1}(f^i(z)) = g(\varphi^i(z)) \Rightarrow g(\varphi^i(z)) = 0$$

which means that $\varphi^i(z)$ is a cycle in B^i. The following diagram represents the previous situation:

$$\begin{tikzcd} z \in A^i \arrow[r, "f^i"] \arrow[d, "\varphi^i"'] & 0 \in A^{i+1} \arrow[d, "\varphi^{i+1}"] \\ \varphi^i(z) \in B^i \arrow[r, "g^i"'] & 0 \in B^{i+1} \end{tikzcd}$$

- ◇ On the other hand suppose $y \in A^i$ is a boundary. Then

$$\exists x (x \in A^{i-1} \wedge f^{i-1}(x) = y).$$

We wish to find an $\tilde{x} \in B^{i-1}$ such that $g^{i-1}(\tilde{x}) = \varphi^i(y)$, so we claim that such $\tilde{x}$ is $\varphi^{i-1}(x)$. By diagram commutativity we have that

$$g^{i-1}(\varphi^{i-1}(x)) =$$