Exercise 1. Draw the $\mathfrak{sl}_3$ crystal for weight $(3, 3, 0)$.

Exercise 2. Prove that the elements of the hyperoctahedral group, written in cycle notation as a permutation on $\{ \pm 1, \ldots, \pm n \}$, has all of its cycles coming in either pairs of the form $(a_1 \ldots a_k)(-a_1 \cdots - a_k)$, or of the form $(a_1 \ldots a_k - a_1 \ \ - a_2 \cdots - a_k)$.

Exercise 3. Define the Lie algebra $\mathfrak{so}_{2n+1}$ as $\{ X : X^\intercal S + SX = 0 \}$ where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0_n & I_n \\ 0 & I_n & 0_n \end{pmatrix}$$

and I_n is the $n \times n$ identity matrix and 1 is in the upper left corner. Write down what an arbitrary element X looks like, and using the fact that with respect to this setup the torus is simply the set of diagonal matrices X satisfying these conditions, explain how one obtains the type B root system.

Answer

First observe that the matrix S is a permutation matrix which acts by row permutation when applied as left multiplication. The permutation it applies is a product of disjoint transpositions of the form $\begin{pmatrix} i & n+i \end{pmatrix}$ for $i \in [n+1] \backslash \{ 1 \}$.

Exercise 4. What is the dimension of the adjoint representation of $\mathfrak{so}_7$?

Answer

Via the isomorphism $X \mapsto [X, -]$ we have that the dimension of the adjoint representation is the same as $\dim \mathfrak{so}_7$ which is $\binom{7}{2} = 21$.

Exercise 5. Explain why the set of 5^{th} roots of unity in the plane don't form a root system. Which axioms of root systems does it satisfy?

Answer

The axioms we should check are:

(a) The roots span our vector space.

(b) The reflections across hyperplanes are still roots.

(c) Projections onto the span of a single root are an integer multiple or a half-integer multiple of the root.

(d) If α, β are roots such that $\beta = \lambda\alpha$ then $\lambda = \pm 1$.

The first axiom is satisfied as any non-zero complex number spans $\mathbb{C}$. The last axiom is satisfied vacuously.
The second axiom isn't satisfied as reflections across hyperplanes send the 5^{th} roots to 10^{th} (primitive) roots of unity. Projections are also not integer multiples nor half-integer multiples of other roots.

Exercise 6. Compute the evacuation of the Young tableau below, and then evacuate again, and show you have returned to the starting tableau.

$$\begin{array}{cccc} 5 & & & \\ 2 & 7 & 8 & \\ 1 & 3 & 4 & 6 \end{array}$$

Answer

We switch entries following the rule $k \mapsto n+1-k$ and then rotating $180°$:

$$\begin{array}{cccc} 5 & & & \\ 2 & 7 & 8 & \\ 1 & 3 & 4 & 6 \end{array} \longrightarrow \begin{array}{cccc} 4 & & & \\ 7 & 2 & 1 & \\ 8 & 6 & 5 & 3 \end{array} \longrightarrow \begin{array}{cccc} 3 & 5 & 6 & 8 \\ \times & 1 & 2 & 7 \\ & & & 4 \end{array}.$$

Here we have already marked the first inner corner we will move. This leads us to

$$\begin{array}{cccc} 3 & 5 & {\color{green}8} & \\ {\color{green}1} & {\color{green}2} & {\color{green}6} & 7 \\ & & \times & 4 \end{array} \longrightarrow \begin{array}{cccc} 3 & 5 & 8 & \\ 1 & 2 & 6 & \\ & \times & {\color{green}4} & {\color{green}7} \end{array} \longrightarrow \begin{array}{cccc} 3 & {\color{green}8} & & \\ 1 & {\color{green}5} & 6 & \\ \times & {\color{green}2} & 4 & 7 \end{array} \longrightarrow \begin{array}{cccc} {\color{green}8} & & & \\ {\color{green}3} & 5 & 6 & \\ {\color{green}1} & 2 & 4 & 7 \end{array}$$

where every green character moved when clearing out the inner corner in the previous step. Redoing the process we obtain the skew tableau

$$\begin{array}{cccc} 8 & & & \\ 3 & 5 & 6 & \\ 1 & 2 & 4 & 7 \end{array} \longrightarrow \begin{array}{cccc} {\color{green}1} & & & \\ {\color{green}6} & {\color{green}4} & {\color{green}3} & \\ {\color{green}8} & {\color{green}7} & {\color{green}5} & {\color{green}2} \end{array} \longrightarrow \begin{array}{cccc} {\color{green}2} & {\color{green}5} & {\color{green}7} & {\color{green}8} \\ \times & {\color{green}3} & {\color{green}4} & {\color{green}6} \\ & & & {\color{green}1} \end{array}.$$

With the first inner corner marked, we move it out and continue the process:

$$\begin{array}{cccc} {\color{green}5} & {\color{green}7} & {\color{green}8} & \\ {\color{green}2} & 3 & 4 & 6 \\ & & \times & 1 \end{array} \longrightarrow \begin{array}{cccc} 5 & 7 & 8 & \\ 2 & 3 & 4 & \\ & \times & {\color{green}1} & {\color{green}6} \end{array} \longrightarrow \begin{array}{cccc} 5 & 7 & & \\ 2 & 3 & {\color{green}8} & \\ \times & 1 & {\color{green}4} & 6 \end{array} \longrightarrow \begin{array}{cccc} 5 & & & \\ 2 & {\color{green}7} & 8 & \\ {\color{green}1} & {\color{green}3} & 4 & 6 \end{array}.$$

As we have returned to our original tableau we conclude that the process is correct.

Exercise 7. Compute the Hall-Littlewood polynomial $\tilde{H}_{(2,1,1)}(x;q)$.

Answer

We first find all SSYT with content $(2,1,1)$. These are:

$$\begin{ytableau} 3 \\ 2 \\ 1 & 1 \end{ytableau}, \quad \begin{ytableau} 2 & 3 \\ 1 & 1 \end{ytableau}, \quad \begin{ytableau} 3 \\ 1 & 1 & 2 \end{ytableau}, \quad \begin{ytableau} 2 \\ 1 & 1 & 3 \end{ytableau}, \quad \begin{ytableau} 1 & 1 & 2 & 3 \end{ytableau}.$$

The cocharge labeling of their reading words is respectively

$$2100, \quad 1100, \quad 1000, \quad 1001, \quad 0000$$

giving us cocharges: $3, 2, 1, 2, 0$. This means that the Hall-Littlewood polynomial is

$$\begin{aligned} & q^3 s_{(2,1,1)} + q^2 s_{(2,2)} + q s_{(3,1)} + q^2 s_{(3,1)} + q^0 s_4 \\ = & q^3 s_{(2,1,1)} + q^2 s_{(2,2)} + (q + q^2) s_{(3,1)} + s_4. \end{aligned}$$

Exercise 8. Let $w = w_1 \dots w_n$ be a word of partition content, and suppose $w_1 \neq 1$. Let $w' = w_2 \dots w_n w_1$ be formed by cycling w_1 around to the end of the word. Show that $\boldsymbol{c}(w') = \boldsymbol{c}(w) - 1$ where $\boldsymbol{c}$ is cocharge. This operation is called *cyclage*.

Answer

Observe that it suffices to view this on standard words. This is because we may separate a word into standard subwords and calculate cocharge[a]. Consider the subword $\tilde{w}$ of w which contains w_1 in the previous decomposition sense, as w has partition content so does $\tilde{w}$.

When cycling w_1 to the end of $\tilde{w}$, cocharge is reduced by 1 as there is a element in $\tilde{w}$ smaller than w_1 which was to the right of w_1. After cycling, it's to the *left* and so the cocharge labeling drops by one.

[a] Ah! Inadvertently **you** helped me with this problem as the decomposition idea was written on your thesis!

Exercise 9. Give a counterexample showing that the formula in the above problem does not hold in general when $w_1 = 1$.

Answer

The word 121 has cocharge labeling 000 giving it a cocharge of 0 whereas 211 has cocharge labeling 100 with cocharge 1.