Exercise 1. Let $A : V \to W$ be a linear map between vector spaces.

(a) Show that the induced map $\bigwedge^k(V) \to \bigwedge^k(W)$ is well-defined by

$$v_1 \wedge \ldots \wedge v_k \mapsto Av_1 \wedge \ldots \wedge Av_k$$

(extending linearly to sums).

(b) Show that the map $A^* : W^* \to V^*$ defined by $(A^*(\eta))(v) := \eta(A(v))$ determines a map $\bigwedge^k(W^*) \to \bigwedge^k(V^*)$.

(c) Show that, if V is an n-dimensional vector space, then the map $\bigwedge^n(V) \to \bigwedge^n(V)$ is multiplication by $\det A$.

Answer

To prove well-definedness of a map, it suffices to take two representatives of the same class and see that they map to the same place.

(a) Consider then, without loss of generality,

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = -(v_2 \wedge v_1 \wedge \cdots \wedge v_k).$$

This second element we can reinterpret as

$$(-v_2) \wedge v_1 \wedge \cdots \wedge v_k.$$

Applying $\bigwedge^k(A)$ to this we get

$$\begin{cases} Av_1 \wedge Av_2 \wedge \cdots \wedge Av_k \\ A(-v_2) \wedge Av_1 \wedge \cdots \wedge Av_k \end{cases}$$

and using the fact that A is linear we get

$$\begin{aligned} A(-v_2) \wedge Av_1 \wedge \cdots \wedge Av_k &= -(Av_2 \wedge Av_1 \wedge \cdots \wedge Av_k) \\ &= Av_1 \wedge Av_2 \wedge \cdots \wedge Av_k \end{aligned}$$

and this is the desired representation of the image. Via linearity of A, we have that the induced map is multilinear. This allows to see that $\bigwedge^k(A)$ is well-defined.

(b) The map A^* does indeed define a map from the exterior powers, namely $\bigwedge^k(A^*)$. However I must admit I didn't do this one.

(c) Observe that the dimension of $\bigwedge^n(V)$ is 1 so that the induced linear map becomes
$$\bigwedge^n(A)(v) = \lambda v, \quad v \in \bigwedge^n(V).$$
Now the map $\bigwedge^n(A)$ is alternanting, multilinear and we may see that I induces the identity map. By uniqueness of the determinant, this map must be the $\det(A)\cdot$.

Exercise 2. Show that the vectors $v_1, \ldots, v_k \in V$ are linearly independent if and only if $v_1 \wedge \ldots \wedge v_k \neq 0$ as an element of $\bigwedge^k(V)$.

Answer

Assume that $\{ v_1, \ldots, v_k \}$ is linearly dependent, then if $\{ v_1, \ldots, v_\ell \}$ is a maximally independent set, we may write any v_i with $\ell < i \leqslant k$ as a linear combination of $\{ v_1, \ldots, v_l \}$.
This means that

$$\begin{aligned} v_1 \wedge \cdots \wedge v_k &= v_1 \wedge \cdots \wedge v_{\ell+1} \wedge \cdots \wedge v_k \\ &= v_1 \wedge \cdots \wedge \sum_{i=1}^{k} c_i v_i \wedge \cdots \wedge v_k \\ &= \sum_{i=1}^{k} c_i (v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k) \end{aligned}$$

and all the summands will be zero as we will find repeated $v_i's$ in each term.
I couldn't quite piece it together with the other direction.

Exercise 3. We say that an element of $\bigwedge^k(V)$ is *decomposable* if it can be written as $v_1 \wedge \ldots \wedge v_k$.

(a) Suppose $v, w, x, y \in V$. Find necessary and sufficient conditions for $v \wedge w + x \wedge y \in \bigwedge^2(V)$ to be decomposable.

(b) Show that $\omega \in \bigwedge^2(\mathbb{R}^4)$ is decomposable if and only if $\omega \wedge \omega = 0$.

Answer

(a) If it was the case that the element is decomposable, then there exist $a, b \in V$

such that

$$v \wedge w + x \wedge y = a \wedge b.$$

Observe now that

$$(a \wedge b)^{\wedge 2} = a \wedge b \wedge a \wedge b = 0$$

so that

$$(v \wedge w + x \wedge y)^{\wedge 2} = 0.$$

Expanding out this quantity we obtain

$$\begin{aligned} 0 =& (v \wedge w)^{\wedge 2} + (v \wedge u) \wedge (x \wedge y) + (x \wedge y) \wedge (v \wedge w) + (x \wedge y)^{\wedge 2} \\ =& v \wedge u \wedge x \wedge y + (-1)^2 (v \wedge w \wedge x \wedge y) \\ =& 2(v \wedge w \wedge x \wedge y) \end{aligned}$$

so it must occur that

$$v \wedge w \wedge x \wedge y = 0$$

or in other words, the vectors are linearly dependent. I couldn't give myself time to finish this one.

(b) Suppose ω is a 2-wedge in $\mathbb{R}^4$ and assume first that ω was decomposable. Then there are $x, y \in \mathbb{R}^4$ such that

$$\omega^{\wedge 2} = (x \wedge y)^{\wedge 2} = (x \wedge y \wedge x \wedge y) = 0.$$

On the other hand, we may expand ω in terms of the basis of $\bigwedge^2(\mathbb{R}^4)$

$$\omega = \sum_{I \in \binom{[n]}{2}} c_I e_I.$$

Observe that if ω is *not* decomposable, then there must exist I, J partitioning $[4]$ and $c_I, c_J \neq 0$. When expanding $\omega^{\wedge 2}$, this gives us a term in the sum

$$c_I c_J e_{I \cup J} = c_I c_J e_1 \wedge \cdots \wedge e_4.$$

which is non-zero.

Exercise 4. Let V be an n-dimensional inner product space. We can extend the inner product from V to all of $\bigwedge(V)$ by setting the inner product of homogeneous elements of different degrees equal to zero and by letting

$$\langle w_1 \wedge \ldots \wedge w_k, v_1 \wedge \ldots \wedge v_k \rangle = \det\left(\langle w_i, v_j \rangle\right)_{i,j}$$

and extending bilinearly.

Since $\bigwedge^n(V)$ is a one-dimensional real vector space, $\bigwedge^n(V) - \{0\}$ has two components. An *orientation* on V is a choice of component of $\bigwedge^n(V) - \{0\}$. If V is an oriented inner product space, then there is a linear map $\star : \bigwedge(V) \to \bigwedge(V)$ called the star map, which is defined by requiring that for any orthonormal basis $e_1, \ldots, e_n$ for V,

$$\star(1) = \pm e_1 \wedge \ldots \wedge e_n, \qquad \star\left(e_1 \wedge \ldots \wedge e_n\right) = \pm 1,$$
$$\star(e_1 \wedge \ldots \wedge e_k) = \pm e_{k+1} \wedge \ldots \wedge e_n,$$

where in each case we take "$+$" if $e_1 \wedge \ldots \wedge e_n$ is in the preferred component of $\bigwedge^n(V)$ and we take "$-$" otherwise. Notice that $\star : \bigwedge^k(V) \to \bigwedge^{n-k}(V)$.

(a) Prove that if $e_1, \ldots, e_n$ is an orthonormal basis for V, then the $e_{i_1} \wedge \ldots \wedge e_{i_k}$ with $1 \leqslant i_1 < \ldots < i_k \leqslant n$ and $1 \leqslant k \leqslant n$ give an orthonormal basis for $\bigwedge(V)$.

(b) Prove that, as a map $\bigwedge^k(V) \to \bigwedge^k(V)$, $\star\star = (-1)^{k(n-k)}$.

(c) Prove that, for $\omega, \eta \in \bigwedge^k(V)$, their inner product is given by

$$\langle \omega, \eta \rangle = \star(\omega \wedge \star\eta) = \star(\eta \wedge \star\omega).$$

Answer

(a) Indeed, if $\{ e_i \}_{i\in[n]}$ forms an orthonormal basis of V, then each collection

$$\{ e_I : \ |I| = k, \ I \subseteq [n] \}, \quad e_I = \textstyle\bigwedge_{i\in I} e_i,$$

spans the k^{th} exterior power of V. In consequence the whole exterior algebra is spanned by the same collection letting $|I|$ range up to n.

To see linear independence it suffices to see orthogonality. Between different index sizes it is clear via hypothesis. So let $I, J \in \binom{[n]}{k}$ and consider $\langle e_I | e_J \rangle$. The obtained quantity is the determinant of the Gram matrix formed by the bases $\{ e_i \}_{i\in I}$ and $\{ e_j \}_{j\in J}$. The only possibility for the determinant in question to be non-zero is if J is a permutation of I. But up to sign, this is the same basic element. Therefore, we get an orthonormal basis as desired.

(b) Proving this fact for a basic element suffices, so take e_I. $\star e_I$ is the element such that

$$e_I \wedge \star e_I = e_{I \cup I^c} = e_1 \wedge \cdots \wedge e_n$$

so that we must multiply by $(-1)^{\text{inv}(\texttt{ToList}(I \cup I^c))}$. Such inversions can only happen between I and I^c because I and I^c themselves are already ordered. Thus

$$\star e_I = (-1)^{\text{inv}(II^c)} e_{I^c}.$$

Taking the star again we get

$$\star \star e_I = (-1)^{\text{inv}(II^c) + \text{inv}(I^c I)} e_I.$$

I couldn't finish the combinatorial argument which shows that the number of inversions in question is indeed $k(n-k)$.

(c) Observe that if we prove this for basic elements, we are done by multilinearity. Consider the wedge:

$$\begin{aligned} \star(e_I \wedge \star e_J) &= \star(e_I \wedge e_{J^c}) \\ &= \star(e_{I \cup J^c}) \\ &= e_{I^c \cap J} \end{aligned}$$

On the other hand we obtain $e_{I \cap J^C}$. This calculation aligns with what it's supposed to be as the element $e_{I \cup J^c}$ lives in the top exterior power of V as it has k indices from I and $n-k$ from J^c. Now we must interpret the set $I^c \cap J$ to see that the non-zero conditions of the inner product match up.

Observe that if $I = J$ then $I^c \cap J$ and $I \cap J^c$ are the empty set. This leaves us with no wedges and so the result of the operation is 1. In the other case, we get 0.

Exercise 5. Let M^n be a closed manifold (i.e., a compact manifold without boundary) and let $\omega \in \Omega^1(M)$ so that $\omega_p \neq 0$ for all $p \in M$ (i.e., for all p, there exists $v \in T_pM$ so that $\omega_p(v) \neq 0$). Show that ω is not exact.

Answer

It is equivalent to show that if ω is exact, then there exists $p \in M$ so that $\omega_p = 0$. So to our effect, assume ω is exact. Then there is an $f : M \to \mathbb{R}$ such that $\mathrm{d}f = \omega$.

As M is compact, then there exists a point $p \in M$ at which f attains a minimum (otherwise, we'll just have to switch signs). Our claim is that

$$\mathrm{d}f_p = 0,$$

and to show this we take $v \in T_pM$ and a curve $\alpha :]-\varepsilon, \varepsilon[\to M$ such that

$$\alpha(0) = p, \quad \alpha'(0) = v.$$

We have that

$$\begin{aligned} \mathrm{d}f_p v &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\alpha(t)) \\ &= \lim_{h\to 0} \frac{f(\alpha(h)) - f(\alpha(0))}{h} \\ &= \lim_{h\to 0} \frac{f(\alpha(h)) - f(p)}{h} \end{aligned}$$

And as $f(p)$ is a minimum of f at $h = 0$, it occurs that $f(\alpha(h)) \geqslant f(p)$ for all h. For $h > 0$ it happens that the quotient

$$\frac{f(\alpha(h)) - f(p)}{h} \quad \text{is positive}$$

whereas for $h < 0$ the quotient is negative, as the numerator is always positive. Then, as our limit exists, it should happen that it's zero. In conclusion p is the point which makes ω vanish.