Exercise 1 (5.2 Stein& Shakarchi)**.** Find the order of growth of the following entire functions:

i) $p(z)$, p is a polynomial. ii) e^{bz^n}, and iii) e^{e^z}.

Answer

Recall an entire function f has order of growth at most ρ if there exist A, B such that

$$|f(z)| \leqslant Ae^{B|z|^\rho}$$

We will use the fact that if f, g have order of growth ρ_f and ρ_g, then $\operatorname{ord}(fg) \leqslant \max \rho_f, \rho_g$. This can be seen to be true as follows:

$$|fg(z)| \leqslant A_1e^{B_1|z|^{\rho_f}} A_2e^{B_2|z|^{\rho_g}} = A_1A_2e^{B_1|z|^{\rho_f}+B_2|z|^{\rho_g}}.$$

If it happens that $\rho_f = \rho_g$, then $\operatorname{ord}(fg) \leqslant \rho_f$. Otherwise, suppose $\rho_f > \rho_g$ then

$$|z|$$

i) For this case, assume first that p is linear, so $p(z) = az + b$ with $a \neq 0$. Without losing generality we may take $a = 1$ because $|az + b| = |a| \left|z + \frac{b}{a}\right|$. Now for $t \in \mathbb{R}$ we have $e^t \geqslant 1 + t$, so for $t = n|z|^{\frac{1}{n}}$ where $n \in \mathbb{N}$ we have

$$e^{n|z|^{1/n}} \geqslant 1 + n|z|^{\frac{1}{n}}$$

ii) Note that

$$|e^{bz^n}| = \left|\exp\left(b\sum_{k=0}^{n}\binom{n}{k}x^k(iy)^{n-k}\right)\right|.$$

If $z = re^{i\theta}$ then $z^n = r^n\cos(n\theta) + isin(n\theta)$ so

$$|e^{bz^n}| = |\exp\left(br^n\cos(n\theta) + ibr^n\sin(n\theta)\right)| = |\exp\left(br^n\cos(n\theta)\right)|$$