Exercise 1. Let Ω be a domain containing the unit circle $C_1(0)$. Show that there is no function $F(z)$ holomorphic on Ω such that $e^{F(z)} = z$ on Ω. ⟦ Hint: What would $F'(z)$ be? Can you prove that $F'(z)$ admits no primitive on Ω? ⟧

Answer

Suppose F is a function which satisfies the equation. Differentiating we get

$$e^{F(z)}F'(z) = 1 \to F'(z) = \frac{1}{e^{F(z)}} = \frac{1}{z}.$$

In $\mathbb{C}\setminus\,]-\infty, 0]$ we have that $\frac{1}{z}$ admits $\log(z)$ as a primitive. So let us define $G(z) = F(z) - \log(z)$, this function has derivative 0 so G is constant.
This means that $F(z) = \log(z) + C$

Exercise 2. Let Ω be a domain with $0 \notin \Omega$.

(a) Suppose f, g are continuous branches of the logarithm on Ω. Show that there is some integer n such that $g(z) = f(z) + 2\pi i n$. ⟦ Hint: Ω is connected. ⟧
(b) Suppose $f(z)$ is a continuous branch of the logarithm. Show that $f(z)$ is holomorphic. ⟦ Hint: Ω can be covered by simply connected domains. ⟧