

# MATH619 — Complex Geometry

Based on the lectures by Renzo Cavalieri

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Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This is a topics course on this stuff

## Requirements

Knowledge on stuff

## TO DO:

- ◇ Write 10.1
- ◇ Write 11.3
- ◇ Write 12.4

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# Chapter 1

## Renzo's Complex Projective Exercises

### 1.1 Set of points of the projective line

**Exercise 1.1.1.** Show that there is a bijection between the set  $\text{Set}\mathbb{CP}^1$  and a quotient set of a disjoint union of two copies of  $\mathbb{C}$ .

#### Answer

Indeed consider our copies of  $\mathbb{C}$  embedded into  $\mathbb{C}^2$  as the lines  $\{x = 1\}$  and  $\{y = 1\}$ . Then for a line  $\ell \in \mathbb{CP}^1$  our map is

$$\ell \mapsto \ell \cap (\text{corresponding line}) \mapsto (\text{corresponding coordinate}).$$

Explicitly, if our line is  $[X : Y]$ , then the map is  $[X : Y] \mapsto X/Y$  on one chart while  $Y/X$  on the other.

Observe that this map is surjective as every point in each copy of  $\mathbb{C}$  is hit by a line of a different slope. The only points which are not hit twice are the origins of both lines. From this, we define the quotient by identifying the coordinates as  $x \sim y$  whenever  $y = 1/x$ . Thus our map becomes a bijection at the level of the quotient as we can now properly trace back each point to a particular line.

### 1.2 The projective line as a topological space

**Exercise 1.2.1** (Hopf Fibration). Show there is a fibration of topological spaces:

$$S^1 \rightarrow S^3 \rightarrow S^2$$

meaning that there is a surjective continuous function from the three-dimensional sphere to the two-dimensional sphere, and the inverse image of any point is homeo-

morphic to a circle. This is called the Hopf fibration; notice that while the construction of these maps is rather mysterious in terms of spheres, it becomes transparent when thinking of the two-dimensional sphere as the complex projective line.

### Answer

We have shown that the map

$$\pi_2 : S^3_{\mathbb{R}} \rightarrow \mathbb{CP}^1, (\text{pt. in } S^3) \mapsto (\text{corresponding line through origin in } \mathbb{C}^2)$$

is surjective. Also, we have that

$$\mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\} \simeq S^2$$

where the first homeomorphism comes from previous discussion and the second one from stereographic projection. This means that we have a map  $S^3 \rightarrow S^2$  which is our candidate for the Hopf map. It remains to be seen that this map is continuous and that the fibers are homeomorphic to  $S^1$ .

It suffices to show  $\pi_2$  is continuous as the rest of the maps are homeomorphisms. To that effect, take an open set  $U \subseteq \mathbb{CP}^1$ . This means that in the quotient topology induced by the  $\pi_2$  map,  $U$  is open whenever  $\pi_2^{-1}(U)$  is open. But this proves immediately that  $\pi_2$  is continuous as it takes open sets back to open sets.

Now let  $\ell \in \mathbb{CP}^1$ , we'll analyze what the fibers are:

$$\pi_2^{-1}(\ell) = \{ \lambda z_0 : z_0 \in \ell \cap S^3, \lambda \in \mathbb{C} \}$$

but when restricting to  $S^3$ , we get the condition that  $\lambda$  can only vary an  $S^1$ 's worth of values:

$$\pi_2^{-1}(\ell) = \{ \lambda z_0 : z_0 \in \ell \cap S^3, |\lambda| = 1 \} \simeq S^1.$$

This means that fibers of our map are homeomorphic to  $S^1$  and thus we have the desired fibration structure.

### 1.3 The projective line as a complex manifold

**Exercise 1.3.1.** Compute

$$\phi_{21} := \varphi_2 \circ \varphi_1^{-1} \mid_{\varphi_1(U_1 \cap U_2)} : (\varphi_1(U_1 \cap U_2), x) \rightarrow (\mathbb{C}, y)$$

and show that it is a holomorphic function on its domain of definition. Show that its inverse is also holomorphic on its domain of definition.

These exercises show that  $\mathbb{CP}^1$  has the structure of a complex analytic manifold. The pairs  $(U_i, \varphi_i)$  are called complex charts, the biholomorphic map  $\phi_{21}$  a transition function, and the coordinates  $x$  and  $y$  are called local (or affine) coordinates.

#### Answer

Recall that the open set  $U_1 \cap U_2$  is the collection of lines in  $\mathbb{CP}^1$  which are not  $x$  or  $y$  axes. The image then is all the non-zero  $x$  coordinates of the intersection of those lines with  $x = 1$ . Taking those lines through  $\varphi_2 \circ \varphi_1^{-1}$  gives us the  $y$ -coordinates of the intersections of those lines with the  $y = 1$  line. We get all except the  $y$ -axis. Computing this for a particular line, if  $x_0$  is the intersection with  $x = 1$ , then  $\frac{1}{x_0}$  will be the intersection with  $y = 1$ . Therefore, the map  $\varphi_2 \circ \varphi_1^{-1}$  is  $x_0 \mapsto \frac{1}{x_0}$  of non-zero  $x_0$ . This function and its inverse are holomorphic as the vertical and horizontal lines are excluded from this.

### 1.4 Functions on the projective line

**Exercise 1.4.1.** Show that meromorphic functions  $f : \mathbb{CP}^1 \rightarrow \mathbb{C}$  may be described in two equivalent ways:

- (a) As the ratio of two homogeneous polynomials of the same degree in the homogeneous coordinates:

$$f(X : Y) = \frac{P_d(X, Y)}{Q_d(X, Y)}$$

- (b) As a rational function in one of the affine coordinates (with no restrictions on the degrees of the polynomials)

$$f(x) = \frac{p(x)}{q(x)}$$

How do you go from one perspective to the other?

## 1. RENZO'S COMPLEX PROJECTIVE EXERCISES

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### Answer

We begin with the second item by claiming that if  $f$  is meromorphic has a zero of degree  $m$  at  $z = z_0$  then we may write  $f = (z - z_0)^m g$  where  $g$  is meromorphic but has no zeroes at  $z_0$ . Similarly for poles. This means that we may write  $f$  as a product of possibly repeated linear factors over another product of linear factors. These products are the desired polynomials.

We may homogenize to obtain the first characterization.

I don't recall how to construct the function in the homogeneous way :(

## 1.5 Automorphisms of the projective line

**Exercise 1.5.1.** Prove that, given any two ordered triple of points  $P_1, P_2, P_3$  and  $Q_1, Q_2, Q_3$  of the projective line, there exists a unique automorphism  $\Phi$  of the projective line such that  $\Phi(P_i) = Q_i$ . Show that it follows that the only automorphism that fixes three points is the identity. Describe the subgroups of  $\text{Aut}(\mathbb{CP}^1)$  consisting of automorphisms that fix one, or two points in the projective line.

### Answer

In affine coordinates, we can map any triple to  $0, 1, \infty$  by considering the function

$$z \mapsto \frac{z - p_1}{z - p_3} \left( \frac{p_2 - p_3}{p_2 - p_1} \right).$$

This maps  $p_1, p_2, p_3$  to  $0, 1, \infty$  respectively. This is a Möbius transformation, so it is an automorphism of  $\mathbb{CP}^1$ . Call it  $\varphi_P$  and then create  $\varphi_Q$ , the desired function  $\Phi$  is  $\varphi_Q^{-1}\varphi_P$ .

From this we immediately see that if  $P$  is fixed then the function is

$$\Phi = \varphi_P^{-1}\varphi_P = \text{id}.$$

If  $\Phi$  fixed two points we get rotations about the axis passing through those two points. Be it, for example,  $0, \infty$  with scalings  $z \mapsto \alpha z$  or  $1, -1$  with  $z \mapsto 1/z$ .

If only one point is fixed, then it is a translation of the line leaving that point fixed. Say for example maps of the form  $z \mapsto \frac{az+b}{d}$  leave infinity fixed.

## 1.6 Maps to projective spaces

**Exercise 1.6.1.** We define the degree of  $F(\mathbb{CP}^1)$  to be the number of intersections with a general hyperplane in  $\mathbb{CP}^r$ . Prove that if the degree of the polynomials  $P_i(X, Y)$  is equal to  $d$ , then the degree of  $F(\mathbb{CP}^1)$  is less than or equal to  $d$ . When does the strict inequality hold?

### Answer

## 1.7 Line bundles on the projective line

**Exercise 1.7.1.** For  $x_0 \neq 0$ , let  $i_1 : \{x = x_0\} \hookrightarrow \mathcal{O}_{\mathbb{CP}^1}(d)$  and  $i_2 : \{y = \frac{1}{x_0}\} \hookrightarrow \mathcal{O}_{\mathbb{CP}^1}(d)$  be the two inclusions of vertical lines. Show that  $i_2^{-1} \circ i_1 : \mathbb{C} \rightarrow \mathbb{C}$  is a linear isomorphism. Observe that the collection of these linear isomorphisms defines a holomorphic function  $c_{12} : \mathbb{CP}^1 \setminus \{0, \infty\} \rightarrow \mathbb{C} \setminus 0$ .

### Answer

Let's concretely analyze the  $i_2^{-1} \circ i_1$  map. This comes out of the line  $\{x = x_0\}$  and gives us

$$\text{Im } i_1 = \{(x_0, u) : u \in \mathbb{C}\} \subseteq (\mathbb{C}^2, (x, u)).$$

In order to see what  $i_2^{-1}$  does, we translate  $\text{Im } i_1$  into  $(\mathbb{C}^2, (y, v))$  via

$$x \mapsto \frac{1}{y}, \quad u \mapsto \frac{v}{y^d} \Rightarrow v = y^d u.$$

So  $\text{Im } i_1$  on the other chart is

$$\text{Im } i_1 = \left\{ \left( \frac{1}{x_0}, \frac{u}{x_0^d} \right) : u \in \mathbb{C} \right\}$$

and  $i_2^{-1}$  returns us  $\frac{u}{x_0^d}$ . This means that the composition in question is that map  $u \mapsto \frac{u}{x_0^d}$  which means that the map is multiplication by  $x_0^{-d}$ . For a fixed non-zero  $x_0$ , this is a linear isomorphism of  $\mathbb{C}$ .

The collection of such isomorphisms is obtained when we let  $x_0$  vary and the function  $c_{12}$  given by  $x_0 \in \mathbb{CP}^1 \setminus \{0, \infty\} \mapsto x_0^{-d} \in \mathbb{C} \setminus \{0\}$  is indeed holomorphic.

## 1.8 Sections of line bundles

**Exercise 1.8.1.** Show that if  $s_0, s_1$  are two sections of the same line bundle, then their ratio is a (meromorphic) function on  $\mathbb{CP}^1$ . Show that if  $s_0, s_1, \dots, s_r$  are  $(r+1)$  sections of the same line bundle, then they define a map  $\mathbb{CP}^1 \rightarrow \mathbb{CP}^r$ .

### Answer

Observe that if  $z \in \mathbb{CP}^1$ , then  $s_0(z), s_1(z)$  lie on the fiber  $\pi^{-1}(z)$  which is isomorphic to  $\mathbb{C}$ . So taking their ratio on this fiber does produce a complex number. However, we must verify that the ratio is well-defined. Assume we picked another element of the base,  $w \in \mathbb{CP}^1$  and asked about the ratio of  $s_0(w)$  with  $s_1(w)$ .



In this case, observe that there's a linear isomorphism between  $\pi^{-1}(z)$  and  $\pi^{-1}(w)$  which scales all vectors by the same length. This means that  $s_i(w) = \alpha s_i(z)$  for some  $\alpha \in \mathbb{C}$  and therefore their ratios are the same.

So  $s_0/s_1$  does define a meromorphic function on  $\mathbb{CP}^1$  thanks to the linear isomorphisms.

In the same fashion, if we instead have  $r + 1$  sections, via a same argument we can see that when changing fibers, the sections only change by a scaling which is the same on all entries. So this means that we may write  $[s_0 : \cdots : s_r]$  as a function to  $\mathbb{CP}^r$ .

## 1.9 Divisors

**Exercise 1.9.1.** When you multiply two meromorphic functions, what happens to their divisors? If two meromorphic functions produce the same divisor, what can you say about them?

### Answer

If we consider our functions as sections of  $\mathcal{O}(d)$  then,

$$s_0(x) = \frac{\prod_{i=1}^m (x - \alpha_i)}{\prod_{j=1}^n (x - \beta_j)} \quad \text{and} \quad s_1(x) = \frac{\prod_{i=1}^r (x - \gamma_i)}{\prod_{j=1}^s (x - \delta_j)}$$

then their divisors are

$$\text{div}(s_0) = \sum_{i=1}^m [\alpha_i] - \sum_{j=1}^n [\beta_j] + (n - m + d)[\infty], \quad \text{div}(s_1) = \sum_{i=1}^r [\gamma_i] - \sum_{j=1}^s [\delta_j] + (s - r + d)[\infty].$$

Where we see the degree at infinity by transitioning via  $s(x) \mapsto y^d s(1/y)$ . Their product is precisely

$$s_0(x)s_1(x) = \frac{\prod_{i=1}^{m+r} (x - \tilde{\alpha}_i)}{\prod_{j=1}^{n+s} (x - \tilde{\beta}_j)},$$

where  $\tilde{\alpha}_i$  corresponds to  $\alpha_i$  when  $i$  is between 1 and  $m$  and  $\gamma_i$  from  $m + 1$  onwards. The same happens for  $\tilde{\beta}_j$ . In this case the divisor is

$$\text{div}(s_0s_1) = \sum_{i=1}^{m+r} [\tilde{\alpha}_i] - \sum_{j=1}^{n+s} [\tilde{\beta}_j] + ((n + s) - (m + r) + d)[\infty]$$

and the  $x = \infty$  still gets the correct coefficient after using the transition function.  
 ¿But why don't we get  $2[\infty]$ ? ¿Is it because this only works in  $\mathcal{O}(0)$ , the trivial bundle?

So all non-infinite coefficients do get added and the divisors are summed.  
 When two meromorphic functions have the same divisor, they should be scalar multiples of each other at most.

**Exercise 1.9.2.** Let  $s$  be a meromorphic section of a line bundle  $\pi : L \rightarrow \mathbb{CP}^1$ , we call the support of  $\text{div}(s)$  the set of points that appear with non-zero coefficients in  $\text{div}(s)$ . Show that there is a natural bijection

$$T_s : \pi^{-1} [\mathbb{CP}^1 \setminus \text{supp}(\text{div}(s))] \rightarrow \mathbb{CP}^1 \setminus \text{supp}(\text{div}(s)) \times \mathbb{C}.$$

Meditate on the following fact: the function  $T_s$  is an isomorphism of complex manifolds, and in fact an isomorphism of line bundles on the punctured  $\mathbb{CP}^1$  (we of course did not precisely define these notions, so try and make a guess of what these things should mean). It is called a *trivialization* of the line bundle  $\pi : L \rightarrow \mathbb{CP}^1$  on the complement of the support of  $\text{div}(s)$ .

### Answer

The first intuitive way to define the function is take an isomorphism  $\varphi$  of the fiber  $\pi^{-1}(x)$  with  $\mathbb{C}$ . This gives us the map

$$T_s(x) = (\pi(x), \varphi(x)),$$

but this map is very non-canonical and ALSO, doesn't depend on  $s$ . So let us take advantage of a couple of points that we know in the fiber:  $x$  and wherever  $s$  maps  $\pi(x)$  to,  $s(\pi(x))$ . Outside the support of  $s$ ,  $s$  is never zero, so it makes sense to define the quotient  $\varphi(x)/\varphi(s(\pi(x)))$ . The desired function is then

$$T_s(x) = \left( \pi(x), \frac{\varphi(x)}{\varphi(s(\pi(x)))} \right)$$

which is independent of  $\varphi$  But, ¿why was this?, making it natural in that sense. This function is bijective as

$$\left( \pi(x), \frac{\varphi(x)}{\varphi(s(\pi(x)))} \right) = \left( \pi(y), \frac{\varphi(y)}{\varphi(s(\pi(y)))} \right)$$

implies  $x, y$  lie in the same fiber via the first component. This immediately gives us  $s(x) = s(y)$  and from this

$$\frac{\varphi(x)}{\varphi(s(x))} = \frac{\varphi(y)}{\varphi(s(x))} \Rightarrow \varphi(x) = \varphi(y) \Rightarrow x = y.$$

## 1.10 Distinguished line bundles on the projective line

**Exercise 1.10.1.** Recall that the very first definition of the set of points of the projective line is that each point corresponds to a line in  $\mathbb{C}^2$ . We now want to construct a space  $\text{Taut}(\mathbb{CP}^1)$  whose points correspond to the choice of line in  $\mathbb{C}^2$  together with a point on it:

$$\text{Set}(\text{Taut}(\mathbb{CP}^1)) = \{(\ell, P) : \ell \text{ is a line through the origin in } \mathbb{C}^2, P \in \ell\}.$$

- (a) Realize  $\text{Taut}(\mathbb{CP}^1)$  as a subspace of  $\mathbb{CP}^1 \times \mathbb{C}^2$ .
- (b) Show that the first projection restricts to  $\text{Taut}(\mathbb{CP}^1)$  to make it into a line bundle on  $\mathbb{CP}^1$ .
- (c) Describe the fiber over a point  $[\ell] \in \mathbb{CP}^1$ .
- (d) Show that  $\text{Taut}(\mathbb{CP}^1) \simeq \mathcal{O}(-1)$ .
- (e) Show that  $\text{Taut}(\mathbb{CP}^1) \simeq \mathcal{B}_{(0,0)} \mathbb{C}^2$ .

### Answer

- (a) As mentioned, elements of  $\text{Set Taut}(\mathbb{CP}^1)$  are  $A = (\ell, P)$ . This immediately defines a map

$$\text{Taut}(\mathbb{CP}^1) \hookrightarrow \mathbb{CP}^1 \times \mathbb{C}^2, A \mapsto (\ell, P).$$

I had this question for a bit, but, how can I show that this map is a map of projective varieties?

- (b) The map

$$\text{Taut}(\mathbb{CP}^1) \rightarrow \mathbb{CP}^1, (\ell, P) \mapsto \ell$$

describes a structure of a line bundle. To show this we must see that for an open neighborhood  $U$  of  $\ell \in \mathbb{CP}^1$ , we have that  $\pi^{-1}(U) \simeq U \times \mathbb{C}^1$ . We

define the map as

$$U \times \mathbb{C}^1 \rightarrow \pi^{-1}(U), (\ell, z) \mapsto (\ell, zP), P \in \ell.$$

### 1.11 Tautological Bundle

The graph of the function

$$F : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{P}^1, (X, Y) \mapsto [X : Y]$$

is exactly  $(X, Y), [S : T]$  such that

$$F(X, Y) = [S : T] \Rightarrow \exists \lambda \neq 0 (X = \lambda S, Y = \lambda T).$$

If we assume that  $T \neq 0$ , we can divide by  $T$  and then get  $Y/T = X/S$ . This also assumes  $S \neq 0$ , but ok, the points in the closure of this set is obtained by clearing denominators. This leads us to  $SY = TX$ . Spending some time carefully doing this, if one coordinate is zero and the other one isn't, the equation is still satisfied. Even points  $(0, 0)$ , whatever  $S$  and whatever  $T$ , are also in the closure. This means that  $(X, Y)$  belongs to the line  $[S : T]$ , even if  $(X, Y) = (0, 0)$ , it belongs to every line.

### 1.12 A quick recap

Originally, we saw  $\mathbb{CP}^1$  as a complex manifold of dimension 1, we studied things by restricting them to charts and reducing problems to complex analysis. Locally around every point we had complex numbers.

We like to study functions on the space because this translate geometric data to algebraic data. For example we understand  $\mathbb{R}^2$  by considering the *coordinate functions*! Take a point  $P$  where the  $x$  function takes value 33 and the  $y$  function  $-37$ .

Bummer,  $\mathbb{CP}^1$  has few functions. Only the constants in fact. This gives us directions of study, meromorphic functions for example give us the whole collection  $\mathbb{C}(x)$ . This field of rational functions is a birational invariant of  $\mathbb{CP}^1$ , in a sense it's a very rich invariant but it's insensitive to small changes in the space.

Otherwise we can consider the local functions. For every open set, consider holomorphic functions on that open set. We have a lot of them, and they are not completely unrelated. There's actually a whole series of connections between these open sets. This is the notion of a sheaf.

$$\text{regular} \leftrightarrow \text{holomorphic} \quad \text{and} \quad \text{rational} \leftrightarrow \text{meromorphic}.$$

So now,  $(\mathbb{CP}^1, \mathcal{O})$  is a locally ringed space which is the datum of a scheme.

The third perspective is to consider sections of a line bundle instead of functions! This comes from replacing a function with its graph.

$$F : X \rightarrow Y \leftrightarrow i_{\Gamma_F} : X \rightarrow \Gamma_F \subseteq X \times Y,$$

we have the same information of a function as  $x \mapsto (x, F(x))$ . The idea is that we will allow the graph to not live in the product  $\mathbb{CP}^1 \times \mathbb{C}$  but instead on a space which *locally* looks like that. We allow the graph of our *kinda* function to live in spaces that are not necessarily products  $\mathbb{CP}^1 \times \mathbb{C}$  but locally are products with  $\mathbb{C}$ .

Sections of a line bundle give us maps to projective space.  $r + 1$  sections give us a map to  $\mathbb{P}^r$ . This gets us to another invariant, the Picard group  $\text{Pic}(\mathbb{CP}^1)$ . This is the group of isomorphism classes of line bundles. For a line bundle, we can compute the Čech cohomology which give us another powerful invariant. Most of our work is to construct such invariants. We don't have a complete set of invariants for algebraic varieties.

### Why do divisors determine the line bundle?

Given a divisor  $D$  we can construct a line bundle  $L$  with a section  $s$  such that  $\text{div}(s) = D$ .

**Example 1.12.1.** Consider the base  $\mathbb{CP}^1$  and the divisor  $D = d[\infty]$ . Over  $\mathbb{CP}^1$  we have  $\text{supp}(D) = \{\infty\}$  so we build two open sets

$$U_\infty = \text{neighborhood of } \text{supp}(D), \quad \text{and} \quad U_0 = \mathbb{CP}^1 \setminus U_\infty.$$

In the chart  $U_0$  we have the zero section and if there was a non-zero section  $s$ , then we might as well rescale all the fibers so that the section becomes  $s(x) = 1$ . This is done by rescaling the fiber by  $\frac{1}{s(x)}$ . About  $U_\infty$ , we have that our section is  $\{v = y^d\}$ . So a good transformation would be

$$y = \frac{1}{x}, \quad v = uf(x) = u/x^d.$$

In general for an arbitrary base of dimension 1, we have a divisor

$$D = \sum_{i=1}^n a_i P_i.$$

To construct  $L$ , we need an open cover of  $X$  plus transition functions for any pair of intersecting sets. Let us take then

$$U_0 = \{p_i\}_{i \in [n]}^c, \quad U_i = \text{open disk about } p_i.$$

[Ask Simeon for video](#)

### 1.13 Serre Duality

We have observed that

$$\dim(H^0(\mathbb{P}^1, \mathcal{O}(d))) = \dim(H^1(\mathbb{P}^1, \mathcal{O}(-d-2))).$$

This follows from the fact that  $H^0(\mathbb{P}^1, \mathcal{O}(d))$  is always isomorphic to the dual of  $H^1(\mathbb{P}^1, \mathcal{O}(-d) \otimes T_{\mathbb{P}^1}^*)$ . This is also sometimes denoted the canonical divisor of  $\mathbb{P}^1$ : And a modification if we have  $X$  of dimension greater than 1, then we want to write this as

$$H^i(X, \mathcal{L}) \simeq H^{n-i}(X, \mathcal{L}' \otimes K_X)$$

where  $K_X \simeq \bigwedge^n T^*X$  is the canonical sheaf. This amounts to having a perfect pairing (non-degenerate bilinear form), a bilinear map

$$H^0(\mathbb{P}^1, \mathcal{O}(d)) \times H^1(\mathbb{P}^1, \mathcal{O}(-d) \otimes T_{\mathbb{P}^1}^*) \rightarrow \mathbb{C}$$

which means that if we represent this map by a basis and a matrix, the matrix is invertible. There is no vector on the left which maps to zero with a vector on the right unless it's zero. It's precisely like a dot product.

We have two ingredients to show this:

- (a) We can identify  $\mathbb{C}$  with  $H^1(\mathbb{P}^1, T_{\mathbb{P}^1}^*)$ . The particular cohomology group is one-dimensional.
- (b) We have a natural map from the cartesian product to that  $H^1$ .

Spelling this out carefully will give us our natural map. Even if we have  $T_{\mathbb{P}^1}^* \simeq \mathcal{O}(-2)$  here, we will see it with differential forms in order to apply it to other spaces.

In the more general case what we have is a map

$$H^i(X, \mathcal{L}_1) \times H^j(X, \mathcal{L}_2) \rightarrow H^{i+j}(X, \mathcal{L}_1 \otimes \mathcal{L}_2).$$

Let's rewrite the Čech complex for the cotangent bundle:

$$T_{\mathbb{P}^1}^*(U_0) \oplus T_{\mathbb{P}^1}^*(U_\infty) \rightarrow T_{\mathbb{P}^1}^*(U_0 \cap U_\infty)$$

This is

$$\mathbb{C}[x]dx \oplus \mathbb{C}[y]dy \rightarrow \mathbb{C}[x, 1/x]dx$$

and the map acts on monomials as

$$\begin{cases} (x^k dx, 0) \mapsto x^k dx \\ (0, y^\ell dy) \mapsto -(-x^{-\ell} dx/x^2) = x^{-2-\ell} dx. \end{cases}$$

At the end of the day  $H^1(\mathbb{P}^1, T_{\mathbb{P}^1}^*)$  is represented by the cocycle  $dx/x$ . So  $H^1(\mathbb{P}^1, T_{\mathbb{P}^1}^*) = \langle dx/x \rangle_{U_0 \cap U_\infty}$ . The only monomial we are not catching is  $1/x$ , every other we've caught.

If we did this whole process tensoring with  $\mathcal{O}(-d) \otimes -$ , then on left we still have functions times  $dx$  and functions times  $dy$ . The  $y$  now transitions as

$$(0, y^\ell dy) \mapsto -(-x^{-\ell-d} dx/x^2) = x^{-2-\ell-d} dx.$$

So in this case, for example in  $d = 5$ , we will not catch things between  $-1$  and  $-5 - 1$ :

$$H^1(\mathbb{P}^1, \mathcal{O}(-5) \otimes T_{\mathbb{P}^1}^*) \simeq \left\langle \frac{dx}{x}, \frac{dx}{x^2}, \dots, \frac{dx}{x^6} \right\rangle.$$

We don't get  $dx/x^7$  because it's in the image of the map!

Now we would like to show that we have a map

$$H^0(\mathbb{P}^1, \mathcal{L}_1) \times H^1(\mathbb{P}^1, \mathcal{L}_2) \rightarrow H^1(\mathbb{P}^1, \mathcal{L}_1 \otimes \mathcal{L}_2)$$

at the level of the Čech complex:

$$C^0(\mathbb{P}^1, \mathcal{L}_1) \times C^1(\mathbb{P}^1, \mathcal{L}_2) \rightarrow C^1(\mathbb{P}^1, \mathcal{L}_1 \otimes \mathcal{L}_2).$$

Elements of the first group are  $\{s_0, s_\infty\}$  where  $s_0 \in L_1(U_0)$  and  $s_\infty \in L_1(U_\infty)$  and on the other we have  $u_{0\infty} \in L_2(U_0 \cap U_\infty)$ . Given this we want to produce a section  $v_{0\infty}$  of  $\mathcal{L}_1 \otimes \mathcal{L}_2(U_0 \cap U_\infty)$ .

We can take  $s_0$  and restrict it:  $s_0|_{U_0 \cap U_\infty} \cdot u_{0\infty}$ . This gives us a  $v_{0\infty}$ .

Our choice was biased! Why didn't we choose  $s_\infty$ ? The fact that

$$\{s_0, s_\infty\} \in \ker d$$

means that  $s_0|_{U_0 \cap U_\infty} - s_\infty|_{U_0 \cap U_\infty} = 0$  which means that on the intersection they're the same.

So basically we have

$$\begin{aligned} H^0(\mathcal{O}(d)) \times H^1(\mathcal{O}(-d) \otimes T^*) &\rightarrow H^1(T^*) \\ \Rightarrow \langle 1, x, \dots, x^d \rangle_{U_0} \otimes \left\langle \frac{dx}{x}, \frac{dx}{x^2}, \dots, \frac{dx}{x^{d+1}} \right\rangle &\rightarrow \left\langle \frac{dx}{x} \right\rangle. \end{aligned}$$

The matrix of this pairing is the identity matrix and then it's obviously the *dot product*. All the other  $x^k dx = 0$ .





# Chapter 2

## Higher Genus

### 2.1 Riemann surfaces

**Definition 2.1.1.** A Riemann surface is a complex analytic manifold of dimension 1.

For every point, there's a neighborhood which is isomorphic to  $\mathbb{C}$  and transition functions are linear isomorphisms of  $\mathbb{C}$ .

**Example 2.1.2.** The following classes define Riemann surfaces.

- (a)  $\mathbb{C}$  itself is a Riemann surface with one chart.
- (b) Any open set of  $\mathbb{C}$  is a Riemann surface.
- (c) A holomorphic function  $f: U \subseteq \mathbb{C} \rightarrow \mathbb{C}$  defines a Riemann surface by considering  $\Gamma_f \subseteq \mathbb{C}^2$ . There's only one chart determined by the projection and the inclusion  $i_{\Gamma_f}$  is its inverse.
- (d) Take another holomorphic function  $f$ , then  $\{f(x, y) = 0\}$  is a Riemann surface such that

$$\text{Sing}(f) = \{ \partial_x f = \partial_y f = f = 0 \} = \emptyset.$$

This means that at every point the gradient identifies a normal direction to the level set  $f = 0$ . In particular, there's a well defined tangent line. To show that this is a complex manifold, we will use the inverse function theorem.

- (e) The first compact example is  $\mathbb{CP}^1$ .

### 2.2 20241009

Our first examples of non-compact Riemann surfaces are images of holomorphic functions  $\mathbb{C}^2 \rightarrow \mathbb{C}$  such that  $\text{Sing}(f) = \emptyset$ . This implies that

$$V(f) = \{ (x, y) \in \mathbb{C}^2 : f(x, y) = 0 \}$$

is a Riemann surface via the implicit function theorem.

#### Compact Riemann surfaces

Our first example is  $\mathbb{CP}^1$ . But the next one is complex tori  $\mathbb{C}/\Lambda$ . Here  $\Lambda$  is a non-degenerate lattice:

$$\Lambda = z_1\mathbb{Z} \oplus z_2\mathbb{Z}, \quad z_1, z_2 \in \mathbb{C}$$

where these numbers are linearly independent over  $\mathbb{R}$ . This quotient is given by the equivalence relation

$$x \sim y \iff x - y \in \Lambda.$$

Another way to see this is to choose a fundamental domain which is the parallelogram  $0, z_1, z_2, z_1 + z_2$ . Any point in  $\mathbb{C}$  is equivalent to a point inside, the pairs of parallel edges are equivalent and all the vertices are equivalent as well. Essentially what we are doing is building a torus (a real 2-torus) with this parallelogram.

This is a Riemann surface because  $\mathbb{C}$  induces a natural atlas via the quotient. For a point  $x \in \mathbb{C}/\Lambda$  we get a chart by:

- (a) Picking a point  $z_x \in \mathbb{C}$  a representative of the equivalence class of  $x$ .
- (b) Then take a neighborhood about  $z_x$ , and call  $U_x$  its image in  $\mathbb{C}/\Lambda$
- (c) Consider the inverse of the projection function  $U_x \rightarrow \mathbb{C}$  as our chart.

Now pick a  $y$  and a representative  $z_y$ . For elements in the intersection of the neighborhood, transition functions are given by translations.

**Example 2.2.1.** Projective plane curves will be our next example. If we let  $F \in {}^h\mathbb{C}_d[X, Y, Z]$  be a homogeneous, degree  $d$  polynomial such that  $\text{Crit}(F) = \emptyset$  in  $\mathbb{P}^2$ , then

$$V(F) = \{ [X : Y : Z] : F(X, Y, Z) = 0 \}$$

is a Riemann surface. To prove this, we rely on the implicit function theorem. We dehomogenize  $F$  and then chart it via

$$V(F) \cap U_z = \{ (x, y) : F(x, y, 1) = 0 \}.$$

**Remark 2.2.2.** Recall that homogeneous  $\text{Crit}(F)$  contains  $V(F)$  because  $F$  is in the image of the Euler operator  $x_i \partial_i = d \text{ id}$ .

**Example 2.2.3.** Complete intersections of  $(n-1)$  hypersurfaces in  $\mathbb{P}^n$  are the generalize the previous example. Take  $F_i \in {}^h\mathbb{C}_{d_i}[x]$  and

$$V(F_1, \dots, F_{n-1}) = \{ [\underline{X}] \in \mathbb{P}^n : F_1(\underline{X}) = \dots = F_{n-1}(\underline{X}) = 0 \}$$

then this is a Riemann surface. There is however a condition on the  $F_i$ 's, the gradient of  $F_i$  at  $P$  must give us different directions. This means that the matrix whose columns are  $\nabla_p F_i, i \in [n-1]$ , has full rank.

**Exercise 2.2.4.** Find an expression in terms of determinants for that condition.

**Example 2.2.5.** As a last example, we have local complete intersections. Consider

$$C = V(X_0 X_3 - X_1 X_2, X_0 X_2 - X_1^2, X_1 X_3 - X_2^2) \subseteq \mathbb{P}^3,$$

this is the image of the rational normal curve

$$\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^3, [s : t] \mapsto [s^3 : s^2 t : s t^2 : t^3].$$

Choosing only two of the equations, for example the first two, if we set  $X_0 = X_1 = 0$  we get a whole line's worth of points because  $X_2, X_3$  are free to vary. So this gives us the curve plus a line. As it is the image of a map, then it's a Riemann surface.

If we take any point on the curve besides the intersection point, we do get a complete intersection.

**Exercise 2.2.6.** Verify that the image satisfied the polynomial equations.

## 2.3 20241011

Every space is locally approximated by tangent spaces. So giving orientation of manifolds is giving an orientation to all tangent in way that is coherent. And of course, in a chart, the tangent bundle trivializes so then we spread the orientation via fibers. On each individual chart we choose an orientation. But between charts we need to look at transition functions.

If for every transition function, the determinant of the Jacobian of the transition is positive implies that all are orientation preserving.

**Lemma 2.3.1.** Any Riemann surface is orientable.

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### Proof

A surface is orientable if and only if for any transition function  $\phi_{ij}$  between  $U_i, U_j$  we have

$$\det d\phi_{ij} > 0.$$

For a Riemann surface,  $\phi_{ij}$  are holomorphic and in particular satisfy Cauchy-Riemann equations. Recall we identify  $1 = (1, 0)$  and  $i = (0, 1)$  so that  $\phi = u + iv$  where  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ . **Finish writing derivation for CR eqns via drawing**

$$u_x = v_y \quad \text{and} \quad -u_y = v_x.$$

So for our transition function the Jacobian is

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} = u_x^2 + u_y^2 > 0.$$

So holomorphicity implies this and therefore any Riemann surface is orientable.

Restricting our attention to compact Riemann surfaces we obtain a classification theorem. Non orientable are connected sums of projective planes while orientable ones are connected sums of tori.

**Corollary 2.3.2.** *Any compact Riemann surface is homeomorphic to a genus  $g$  surface.*

We must interpret connected sum of zero tori as a sphere. Connected sum is surgery from topology. We remove a small open disk from both surfaces and then identifying both boundaries gives us the result.

The number of holes we obtain is an important invariant called the genus. This is a topological invariant.

*Remark 2.3.3.* If two R.S. walk about to you with different genera, then they are most certainly non-isomorphic. In genus 1 there's infinitely many non-isomorphic R.S.

Some questions arise which will be able to answer:

*Question.* ¿Are there Riemann surfaces of any genus?

*Question.* ¿Are there general genus Riemann surfaces which are plane projective planes? This is, as the zero locus of polynomials in  $\mathbb{P}^2$ .

The answer to the first question is yes, and the simplest way to construct one is to construct a hyperelliptic curve of genus  $g$ . These are built out of equations of the form

$$y^2 = f(x), \quad (x, y) \in \mathbb{C}^2.$$

Generically this curves admit a degree 2 map to  $\mathbb{C}$ .

The second question has a negative answer. This should weird us out as we are defining hyperelliptic curves via a polynomial (this only works if the degree of  $f$  is 3 or 4). In particular if  $C = V(f)$  where  $f$  is homogeneous of degree  $d$  then

$$g_C = \binom{d-1}{2}$$

so as not all integers are of the form  $\binom{d-1}{2}$ , then there's no genus 5 hyperelliptic curve in  $\mathbb{P}^2$  for example.

### Families, singular curves and genus

**Definition 2.3.4.** A family of Riemann surfaces is a space  $\mathfrak{X} \xrightarrow{\pi} B$  such that

- (a) For every  $b \in B$ ,  $\pi^{-1}(b)$  is a Riemann surface.

**Example 2.3.5.** Consider  $F, G \in {}^h\mathbb{C}[X, Y, Z]_d$  then  $\lambda F + \mu G$  is also homogeneous of degree  $d$ .  $V(\lambda F + \mu G)$  defines a subvariety or a bihomogenous equation (degree 1 in  $\lambda, \mu$  and degree  $d$  in  $X, Y, Z$ ). This is our space  $\mathfrak{X}$  in  $\mathbb{P}^1 \times \mathbb{P}^2$  and projecting  $\pi_1$  gives us the family.

**Example 2.3.6.** In a more specific example we have

$$\lambda[(Y^2 - X^2)Z + X^3] + \mu XZ^2 = 0.$$

It is well defined to say whether a point  $(\lambda_0 : \mu_0), (X_0 : Y_0 : Z_0)$  is a solution to the equation above. So it makes sense to ask for points  $((\lambda_0 : \mu_0), (X_0 : Y_0 : Z_0)) \in \mathbb{P}^1 \times \mathbb{P}^2$  solutions of this equation. As the parameters vary we get Riemann surfaces.

However dehomogenizing  $(Y^2 - X^2)Z + X^3$  gives us  $y^2 - x^2 + x^3$ , the nodal curve. The singular locus of this curve contains  $(0, 0)$ , so it's singular at  $(0, 0)$ . About the origin this looks like  $dy^2 = dx^2 - dx^3 = dx^2$  which is the union of the diagonals. This is a nodal singularity. It turns out that if we want, we can care only about nodal singularity. If we get a family where curves vary and at some point it degenerates into worse than nodal singularities, then we can replace it with a nodal curve.



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# Bibliography

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