

**Exercise 1.** Consider the dihedral group  $D_4$  (the symmetry group of a square, including reflections and rotations) acting on the plane by the corresponding reflection and rotation matrices. Show that this representation of  $D_4$  is an irreducible 2-dimensional representation over both  $\mathbb{R}$  and  $\mathbb{C}$ .

### Answer

Observe that we may present the dihedral group as

$$\text{Free}(r, s) / \text{gen}(r^4, s^2, (sr)^2).$$

The desired representation is given by the action on the generators:

$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Both matrices have  $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$  as an eigenvector. This can be seen either by checking both eigenspaces or by looking at the row sums. Correspondingly,  $r$  has eigenvalue  $-1$  and  $s$ , a  $1$ .

For both  $\mathbb{R}$  and  $\mathbb{C}$ , the vector  $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  lies in  $\mathbf{v}^\perp$ . To decompose our representation, we consider the basis  $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$  along with the change of basis matrix

$$M = [\text{id}]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow M^{-1} = [\text{id}]_{\mathcal{C}}^{\mathcal{B}} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} M.$$

Conjugating our generators we obtain

$$\begin{aligned} \text{wtr} = M^{-1} r M &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

and for the case of  $s$ :

$$\begin{aligned}
 {}^s M &= M^{-1} s M = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
 \end{aligned}$$

So after conjugating, we can see that we cannot decompose the representation further. As indecomposability is equivalent to irreducibility in this case, we have that this representation is an irreducible 2-dimensional representation.

**Exercise 2.** Prove that if  $G$  is a finite abelian group, then all of its irreducible representations are 1- dimensional. [Hint: you may want to look up some resources showing that commuting matrices are simultaneously diagonalizable.]

### Answer

If  $|G| = n$ , then  $g^n = \text{id}_G$ . If  $\rho : G \rightarrow \text{GL}(V)$  is our representation, then

$$g^n = \text{id}_G \Rightarrow \rho(g)^n = I \Rightarrow \rho(g)^n - I = 0.$$

This means  $\rho(g)$  satisfies the polynomial equation  $z^n - 1 = 0$ . At first, this doesn't directly imply that  $\rho(g)$  is diagonalizable, but with a careful eye we can remember about the minimal polynomial. The minimal polynomial of  $\rho(g)$  divides any polynomial which annihilates  $\rho(g)$  as it is the generator of the ideal of such polynomials:

$$\min_{\rho(g)} \mid z^n - 1$$

and as  $z^n - 1$  splits into distinct linear factors, the minimal polynomial must be a product of such factors. Being the roots of  $\min_{\rho(g)}$  all distinct, we have that  $\rho(g)$  is diagonalizable.

Now, as  $G$  is abelian, we also have that the matrices  $\{\rho(g) : g \in G\}$  commute. Therefore they are all simultaneously diagonalizable.

**Exercise 3.** Suppose  $A$  is an  $m \times m$  matrix and  $B$  is an  $n \times n$  matrix. Recall that the tensor product of the matrices  $A$  and  $B$  is the matrix having block form

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{pmatrix}.$$

Show that, if  $\rho : G \rightarrow \text{GL}(\mathbb{C}^m)$  and  $\sigma : G \rightarrow \text{GL}(\mathbb{C}^n)$  are two representations of a group  $G$  (thought of as collections of matrices  $\rho(g)$  and  $\sigma(g)$ ), then the tensor product of these two representations, defined as the action of  $G$  on  $\mathbb{C}^m \otimes \mathbb{C}^n$  by  $g(v \otimes w) = (gv) \otimes (gw)$ , is the map

$$\rho \otimes \sigma : G \rightarrow \text{GL}(\mathbb{C}^{mn})$$

given by

$$(\rho \otimes \sigma)(g) = \rho(g) \otimes \sigma(g).$$

### Answer

We must check that the new map does indeed define a representation. This means that:

- (a) Each element of  $G$  is mapped to a matrix in  $\text{GL}_{mn}$ .
- (b) The identity is precisely mapped to the identity matrix, and
- (c) The map  $\rho \otimes \sigma$  is a homomorphism.

Firstly, if  $g \in G$ , call  $\rho(g) = A$  and  $\sigma(g) = B$ . Then

$$(\rho \otimes \sigma)(g) = \rho(g) \otimes \sigma(g) = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{pmatrix}$$

which is a block matrix of  $n^2$  blocks, all size  $m \times m$ . The resulting matrix indeed lives in  $\text{GL}_{mn}$ .

For the second item, if  $\text{id}_G$  is our identity element, then  $\rho(\text{id}_G) = I_m$  and  $\sigma(\text{id}_G) = I_n$ . This means that

$$\begin{aligned}
 (\rho \otimes \sigma)(\text{id}_G) &= \rho(\text{id}_G) \otimes \sigma(\text{id}_G) \\
 &= I_m \otimes I_n \\
 &= \begin{pmatrix} \delta_{11}I_n & \delta_{12}I_n & \cdots & \delta_{1m}I_n \\ \delta_{21}I_n & \delta_{22}I_n & \cdots & \delta_{2m}I_n \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{m1}I_n & \delta_{m2}I_n & \cdots & \delta_{mm}I_n \end{pmatrix} \\
 &= \begin{pmatrix} I_n & 0 & \cdots & 0 \\ 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_n \end{pmatrix}
 \end{aligned}$$

which is the  $m \times m$  block matrix containing  $I_n$  blocks through its diagonal. This is exactly  $I_{mn}$ .

Finally call  $C = \rho(h)$  and  $D = \sigma(h)$  for another  $h \in G$ . We wish to show that

$$(\rho \otimes \sigma)(gh) = (\rho \otimes \sigma)(g)(\rho \otimes \sigma)(h).$$

Expanding the definition on the left side we have

$$(\rho \otimes \sigma)(gh) = \rho(gh) \otimes \sigma(gh) = (\rho(g)\rho(h)) \otimes (\sigma(g)\sigma(h)) = (AC) \otimes (BD).$$

Meanwhile on the right hand side we have

$$(\rho \otimes \sigma)(g)(\rho \otimes \sigma)(h) = (\rho(g) \otimes \sigma(g))(\rho(h) \otimes \sigma(h)) = (A \otimes B)(C \otimes D).$$

Comparing entries will show us that the result<sup>a</sup> is indeed true: Observe that

$$(AC)_{ij} = \sum_{k=1}^m a_{ik}c_{kj}$$

so that  $(AC) \otimes (BD)$  has block form:

$$\begin{pmatrix} \sum_{k=1}^m a_{1k}c_{k1}BD & \sum_{k=1}^m a_{1k}c_{k2}BD & \cdots & \sum_{k=1}^m a_{1k}c_{km}BD \\ \sum_{k=1}^m a_{2k}c_{k1}BD & \sum_{k=1}^m a_{2k}c_{k2}BD & \cdots & \sum_{k=1}^m a_{2k}c_{km}BD \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^m a_{mk}c_{k1}BD & \sum_{k=1}^m a_{mk}c_{k2}BD & \cdots & \sum_{k=1}^m a_{mk}c_{km}BD \end{pmatrix}.$$

But observe that we may rearrange the terms of an entry of this matrix as follows:

$$\sum_{k=1}^m a_{ik}c_{kj}BD = \sum_{k=1}^m (a_{ik}B)(c_{kj}D).$$

This means that this sum computes the entry of the product of matrices whose entries are the blocks  $a_{ij}B$  and  $c_{ij}D$ . But the matrices with those entries are precisely  $A \otimes B$  and  $C \otimes D$ .

After showing this three properties we have shown that the tensor product of representations is indeed a representation.

<sup>a</sup>This property is the mixed product property of the tensor product and matrix multiplication!

**Exercise 4.** Recall the RSK bijection from Math 502, and use an appropriate version of it to give a combinatorial proof that the dimensions of the spaces on both sides of the Schur-Weyl duality formula

$$\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = \bigoplus_{\lambda \vdash k} V_\lambda \otimes V^\lambda$$

are equal. (Recall that the left hand side is a tensor product of  $k$  copies of  $\mathbb{C}^n$ , and the right hand side is summing over all  $\lambda$  of size  $k$  with at most  $n$  rows.) [Hint: You do not need to know the actual RSK algorithm to do this problem, only what objects it forms bijections between. This can be found in Stanley chapter 7, for instance.]

### Answer

Observe that the dimension of the space on the left is  $n^k$ . For the right,  $V_\lambda$  is the

Specht module with dimension

$$\dim(V_\lambda) = \#SYT(\lambda)$$

and  $V^\lambda$  is the Schur module with basis  $\{e_T\}$  where  $T$  runs through the possible  $SSYT(\lambda)$ . Now recall that the particular Schensted bijection gives us the correspondence

$$\{\text{words of length } k \text{ in the alphabet } [n]\} \leftrightarrow \{(S, T) \in SYT(\lambda) \times SSYT(\lambda)\}.$$

The amounts counted by each side correspond to the dimensions of spaces in question, thus the dimensions of both spaces must be equal.

**Exercise 5.** Use the  $\varepsilon$  method to find the Lie algebra associated to the Lie group  $B_n \subseteq \mathrm{SL}_n(\mathbb{C})$  of upper triangular matrices in  $\mathrm{SL}_n(\mathbb{C})$ .

#### Answer

First let us do a sanity check. It is not that easy for my to outright believe that  $B_n^{\mathrm{SL}}$  is totally inside  $B_n^{\mathrm{GL}}$ . Let's see why this is the case:

In  $\mathrm{GL}_n$  we have the subgroups  $\mathrm{SL}_n$  and  $B_n^{\mathrm{GL}}$ . It is true that  $\mathrm{SL}_n \cap B_n$  is nonempty, but we have determinant 1 matrices which are not triangular, and triangular matrices without determinant 1. Now let us assume that there are matrices in  $B_n^{\mathrm{SL}}$  not in  $B_n^{\mathrm{GL}}$ . Such a matrix fixes the complete flag of  $\mathbb{C}^n$ , but by virtue of being invertible, that matrix should be in  $B_n^{\mathrm{GL}}$ . This means that

$$B_n^{\mathrm{SL}} = B_n^{\mathrm{GL}} \cap \mathrm{SL}_n$$

and so taking their Lie algebras we get

$$\mathfrak{b}_n^{\mathrm{sl}} = \mathfrak{b}_n^{\mathrm{gl}} \cap \mathfrak{sl}_n$$

which means that the Borel Lie subalgebra of  $\mathfrak{sl}_n$  is the triangular matrices with zero trace.

Knowing that the Borel subgroup inside  $\mathrm{SL}_n$  is the triangular matrices with determinant one, we may apply the  $\varepsilon$  method to arrive to

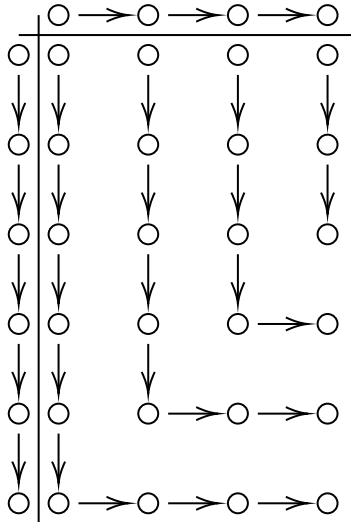
$$\det(I + \varepsilon X) = 1 + \varepsilon \operatorname{tr}(X)$$

with the additional condition that  $X$  is triangular. In the same fashion, we conclude that  $\mathfrak{b}_n^{\mathrm{sl}}$  is the collection of triangular, traceless matrices.

**Exercise 6.** Let  $V_i$  be the irreducible representation of  $\mathfrak{sl}_2(\mathbb{C})$  with highest weight  $i$ . Compute the decomposition of  $V_3 \otimes V_5$  into irreducibles.

### Answer

We may use the L-diagram process to find the tensor product decomposition:



Observe that this means that the desired decomposition is

$$V_3 \otimes V_5 = V_8 \oplus V_6 \oplus V_4 \oplus V_2.$$

**Exercise 7.** Do the following:

- Show that the total number of ballot sequences of 1's and 2's of length  $2n$  is  $\binom{2n}{n}$ , and that the number of ballot sequences of 1's and 2's of length  $2n + 1$  is  $\binom{2n+1}{n+1}$ .
- If  $V_1$  is the irreducible representation of  $\mathfrak{sl}_2(\mathbb{C})$  having highest weight 1, what does this imply about the decompositions of  $V_1^{\otimes 2n}$  and  $V_1^{\otimes 2n+1}$  into irreducibles?

### Answer

- I found this one on your website but didn't give myself enough time to read it through. :(
- This means that when decomposing the tensor product we will have exactly  $\binom{2n}{n}$  and  $\binom{2n+1}{n+1}$  direct summands respectively. This is because each direct summand's highest weight vector corresponds to a ballot word in 1's and 2's.

**Exercise 8.** Starting with the ballot word  $12211122121111$ , draw the corresponding  $\mathfrak{sl}_2$  chain by applying the lowering operator  $F$  until it is no longer possible.

### Answer

Recall that the lowering operator does the following:

- ◊ Replaces all the 1's with  $)$  and all the 2's with  $($ .
- ◊ Then matching parenthesis get matched.
- ◊  $F$  turns the last unmatched 1 to a 2.

Turning our word into parenthesis form we get

$$12211122121111 \rightarrow )(( ))(( ))(( ))$$

and from this we can actually see it can't actually be raised more as there's no unmatched 2. The chain goes as follows:

$$\begin{aligned} 12211122121111 &\rightarrow )(( ))(( ))(( )) \\ \xrightarrow{F} 12211122121122 &\rightarrow )(( ))(( ))(( ( \\ \xrightarrow{F} 12211222121122 &\rightarrow )(( ))(( ))(( ( ( \\ \xrightarrow{F} 22211222121122 &\rightarrow (( ))(( ))(( ( ( \end{aligned}$$

Had fun doing this homework. I believe that the fact that you asked for all the problems motivated me a lot. In a sense, when there's a chance to *do so many points*, I feel like I can take the easy way out. But here it's like, tough luck, gotta do them all even if they're hard. The course's been a bit rough for me as I took 502 a year ago, but I feel like I'm getting back into action. Thanks once again.