

Exercise 1. Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$, and $a \neq 0$. [Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.]

Answer

The function $g(z) = f(1/z)$ has a singularity at $z = 0$. If it were removable, then g is bounded on $B(0, R)$ for some $R > 0$.

This means that f is bounded on $\{|z| > R\}$, but as f is entire, it's continuous and so it's bounded in $\overline{B}(0, R)$, the *closed* ball. From this, we see that f is bounded in all of \mathbb{C} .

By Liouville's theorem f is constant. But that contradicts the fact that f is injective.

Now assume g has an essential singularity at $z = 0$. By the Casorati-Weierstrass theorem, we have a neighborhood of the origin $B(0, R)$ with $R > 0$, such that $g[B(0, R)]$ is dense in \mathbb{C} . This means that $f[\{|z| > R\}]$ is dense in \mathbb{C} .

Recall that dense sets intersect every non-trivial open set, so in particular we find an intersection with $f[B(0, R)]$ (which is open by the open mapping theorem). This means that there exists $w \in f[\{|z| > R\}] \cap f[B(0, R)]$ such that

$$w = f(z_1) = f(z_2), \quad \text{where } |z_1| > R, \quad \text{and } |z_2| < R.$$

In particular $z_1 \neq z_2$. So this contradicts the injectivity of f .

Finally this means that g has a finite-order pole at $z = 0$. When taking the Laurent expansion of g , this corresponds to having finitely many terms of the form $\frac{a_k}{z^k}$.

As for f , the positive degree part of its Laurent expansion is a finite degree polynomial. There are no negative power terms because f is entire.

This lets us conclude that f is a polynomial. The degree of f can't be anything other than 1 because otherwise it won't be injective. Therefore, we conclude that f is a linear function.

Exercise 2. As in class, consider the unit sphere

$$X = \{(a, b, c) : a^2 + b^2 + c^2 = 1\} \subseteq \mathbb{R}^3$$

Let $N = (0, 0, 1)$, $S = (0, 0, -1)$, $U_N = X \setminus N$, $U_S = X \setminus S$. Consider the following three charts on X :

$$\diamond \phi_N : U_N \rightarrow \mathbb{C}, (a, b, c) \mapsto \frac{a+ib}{1-c}.$$

$$\diamond \phi_S : U_S \rightarrow \mathbb{C}, (a, b, c) \mapsto \frac{a+ib}{1+c}.$$

$$\diamond \psi_S : U_S \rightarrow \mathbb{C}, (a, b, c) \mapsto \frac{a-ib}{1+c}.$$

Do the following:

i) The inverse of ϕ_N is

$$\phi_N^{-1}(z) = \left(\frac{2 \operatorname{Re}(z)}{|z|^2 + 1}, \frac{2 \operatorname{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

Calculate ϕ_S^{-1} and ψ_S^{-1} .

ii) Among the three charts $\{(U_N, \phi_N), (U_S, \phi_S), (U_S, \psi_S)\}$, one pair is compatible and the other two are not. Which is which? Why?

[[Hint: Remember a function is holomorphic if and only if $\partial_{\bar{z}} f = 0$.]]

Answer

We claim that

$$\phi_S^{-1}(z) = \left(\frac{2 \operatorname{Re}(z)}{|z|^2 + 1}, \frac{2 \operatorname{Im}(z)}{|z|^2 + 1}, \frac{1 - |z|^2}{|z|^2 + 1} \right).$$

When composing this function with ϕ_S we obtain

$$\phi_S^{-1}(\phi_S(a, b, c)) = \phi_S^{-1}\left(\frac{a + bi}{1 + c}\right)$$

To ease our calculations we may calculate the modulus of this complex number beforehand:

$$\left| \frac{a + bi}{1 + c} \right|^2 = \frac{a^2 + b^2}{(1 + c)^2} = \frac{1 - c^2}{(1 + c)^2} = \frac{1 - c}{1 + c}.$$

From this we can also see

$$\frac{1 - c}{1 + c} + 1 = \frac{2}{1 + c}, \quad \text{and} \quad 1 - \frac{1 - c}{1 + c} = \frac{2c}{1 + c}.$$

Applying this to our calculation we obtain

$$\phi_S^{-1}\left(\frac{a + bi}{1 + c}\right) = \left(\frac{(2a)/(1 + c)}{2/(1 + c)}, \frac{(2b)/(1 + c)}{2/(1 + c)}, \frac{(2c)/(1 + c)}{2/(1 + c)} \right) = (a, b, c).$$

In a similar fashion we have

$$\begin{aligned}\phi_S(\phi_S^{-1}(a, b, c)) &= \phi_S\left(\frac{2\operatorname{Re}(z)}{|z|^2+1}, \frac{2\operatorname{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right) \\ &= \frac{(2\operatorname{Re}(z))/(|z|^2+1) + i(2\operatorname{Im}(z))/(|z|^2+1)}{1 + (1-|z|^2)/(|z|^2+1)} \\ &= \frac{2z}{|z|^2+1+1-|z|^2} = z.\end{aligned}$$

Therefore ϕ_S^{-1} is indeed the inverse map of ϕ_S . Now, observe that $\psi_S = \overline{\phi_S}$ from which we may conclude that $\psi_S^{-1}(z) = \phi_S^{-1}(\bar{z})$, this is

$$\psi_S^{-1}(z) = \left(\frac{2\operatorname{Re}(z)}{|z|^2+1}, \frac{-2\operatorname{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right).$$

Finally considering the transition maps we may see after calculating that

$$\phi_S \circ \phi_N^{-1} = \frac{1}{\bar{z}}, \quad \psi_S \circ \phi_N^{-1} = \frac{1}{z}, \quad \text{and} \quad \psi_S \circ \phi_S^{-1} = \bar{z}.$$

Among these three, the only holomorphic transition map is $\psi_S \circ \phi_N^{-1}$. From this, we see that \mathbb{CP} with the atlas $\{(U_N, \phi_N), (U_S, \psi_S)\}$ is a complex manifold.

Exercise 3. If f is meromorphic on Ω and $z_0 \in \Omega$, we define the order of f by

$$\operatorname{ord}_{z_0}(f) = \begin{cases} 0 & \text{when } f \text{ is holomorphic at } z_0 \text{ and } f(z_0) \neq 0, \\ m & \text{when } f \text{ has a zero of order } m \text{ at } z_0, \\ -m & \text{when } f \text{ has a pole of order } -m \text{ at } z_0. \end{cases}$$

Do the following:

- i) Let $p(z)$ be a polynomial of degree d , thought of as a meromorphic function $\hat{C} \rightarrow \hat{C}$. Use the definition of a pole at infinity ([SS, p. 87]) to show that $\operatorname{ord}_\infty p = -d$.
- ii) Show that if $p(z)$ is a polynomial, then

$$\sum_{z_0 \in \hat{C}} \operatorname{ord}_{z_0}(f) = 0.$$

[[Hint: Use the fundamental theorem of algebra.]]

iii) Show that if $f(z) = \frac{p(z)}{q(z)}$ is a rational function, then

$$\sum_{z_0 \in \hat{\mathbb{C}}} \text{ord}_{z_0}(f) = 0.$$

Answer

i) The behavior of p at infinity is the behavior of $p\left(\frac{1}{z}\right)$ at the origin. Observe that if p had degree d then

$$p(z) = a_0 + a_1z + \cdots + a_dz^d, \quad \text{where } a_d \neq 0$$

$$\Rightarrow p\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \cdots + \frac{a_d}{z^d} = \frac{1}{z^d}(a_0z^d + a_1z^{d-1} + \cdots + a_d).$$

Observe that at $z = 0$, the function $a_0z^d + a_1z^{d-1} + \cdots + a_d$ doesn't vanish because $a_d \neq 0$ and it's holomorphic. Then we see that the order of the pole at the origin is $-d$. Thus for p , $-d = \text{ord}_\infty p$.

ii) We may factor p as

$$p(z) = a \prod_{k=1}^r (z - z_k)^{\alpha_k}$$