Problem 9

Given $\mathbf{r}(t) = \langle 1+t^2, 1-t^2, e^t \rangle$, $-1 \leqslant t \leqslant 1$, and $\delta(x,y,z) = z^2$. The mass is

$$\begin{aligned} m &= \int_C \delta(x,y,z)\,ds = \int_{-1}^{1} \delta(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,dt, \\ \mathbf{r}'(t) &= \langle 2t, -2t, e^t \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{8t^2 + e^{2t}}, \\ \delta(\mathbf{r}(t)) &= e^{2t}. \end{aligned}$$

Thus,

$$m = \int_{-1}^{1} e^{2t}\sqrt{8t^2 + e^{2t}}\,dt.$$

Problem 13

Given $\mathbf{F}(x,y) = \langle x^2 y, y \rangle$ and $y = x^2$, $0 \leqslant x \leqslant 1$.

$$\begin{aligned} \mathbf{r}(x) &= \langle x, x^2 \rangle, \quad \mathbf{r}'(x) = \langle 1, 2x \rangle, \\ \mathbf{F}(\mathbf{r}(x)) &= \langle x^4, x^2 \rangle, \\ \mathbf{F}(\mathbf{r}(x)) \cdot \mathbf{r}'(x) &= x^4 + 2x^3. \end{aligned}$$

Hence,

$$W = \int_0^1 (x^4 + 2x^3)\,dx = \frac{7}{10}.$$

Problem 20

$\mathbf{F}(x,y) = \langle x e^{x^2}, \frac{1}{2\sqrt{y}} \rangle$.

$$\begin{aligned} \frac{\partial P}{\partial y} &= 0, \quad \frac{\partial Q}{\partial x} = 0 \Rightarrow \text{conservative.} \\ f_x &= x e^{x^2} \Rightarrow f = \frac{1}{2}e^{x^2} + g(y), \\ f_y &= g'(y) = \frac{1}{2\sqrt{y}} \Rightarrow g(y) = \sqrt{y} + C. \end{aligned}$$

$$f(x,y) = \frac{1}{2}e^{x^2} + \sqrt{y} + C.$$

Problem 25

$\mathbf{F}(x, y) = \langle \sin y, x \cos y + e^y \rangle$, $C : (0,0) \to (1,1)$.

$$\begin{aligned}
&\frac{\partial P}{\partial y} = \cos y, \quad \frac{\partial Q}{\partial x} = \cos y \Rightarrow \text{conservative.} \\
&f_x = \sin y \Rightarrow f = x \sin y + g(y), \\
&f_y = x \cos y + g'(y) = x \cos y + e^y \Rightarrow g'(y) = e^y \Rightarrow g(y) = e^y + C.
\end{aligned}$$

$$f(x, y) = x \sin y + e^y, \quad \int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1) - f(0,0) = \sin 1 + e - 1.$$

Problem 28

$\mathbf{F}(x, y) = \langle x^2, xy \rangle$

$$\frac{\partial P}{\partial y} = 0, \quad \frac{\partial Q}{\partial x} = y \Rightarrow \text{not conservative.}$$

For $C_1 : (0,0) \to (1,1)$, $\mathbf{r}(t) = \langle t, t \rangle, 0 \leqslant t \leqslant 1$:

$$\begin{aligned}
\mathbf{r}'(t) &= \langle 1, 1 \rangle, \quad \mathbf{F}(\mathbf{r}(t)) = \langle t^2, t^2 \rangle, \\
\mathbf{F} \cdot \mathbf{r}' &= 2t^2, \quad \int_0^1 2t^2 dt = \frac{2}{3}.
\end{aligned}$$

For $C_2 : \mathbf{r}(t) = \langle t, t^2 \rangle, 0 \leqslant t \leqslant 1$:

$$\begin{aligned}
\mathbf{r}'(t) &= \langle 1, 2t \rangle, \quad \mathbf{F}(\mathbf{r}(t)) = \langle t^2, t^3 \rangle, \\
\mathbf{F} \cdot \mathbf{r}' &= t^2 + 2t^4, \quad \int_0^1 (t^2 + 2t^4) dt = \frac{11}{15}.
\end{aligned}$$

Since $\frac{2}{3} \neq \frac{11}{15}$, the Fundamental Theorem does not hold for this non-conservative field.