Applications of Integrals

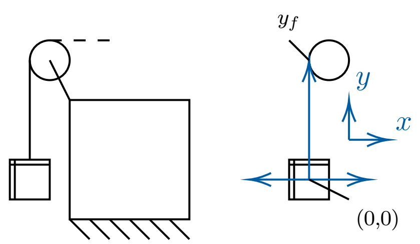
Work

When a force is applied to an object, energy is transferred from or into it. *Work* is transferred energy and *doing work* is the act of transferring that energy.

Definition. If $F = F(x)$ is a force moves an object moved along a path with endpoints x_i and x_f, then the work done by F is

$$\int_{x_i}^{x_f} F(x)\mathrm{d}x.$$

Example 1. Suppose a box weighs 19 N. It is hanging by an infinitesimally thin rope which is 6 m long. We will find the work necessary to pull the box up.



We place the (0,0) coordinate at the center of box assuming its mass is concentrated at this point. Displacement goes from $y_i = 0$ m to $y_f = 6$ m.

As only the gravitational force acts of the box, the work required to move the box from y_i to y_f is equal to:

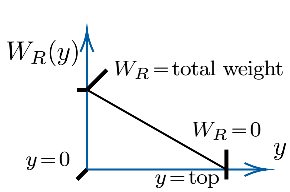
$$W = \int_{y_i=0}^{y_f=6} (19 \text{ N})(\mathrm{d}y \text{ m}) = \int_0^6 19\mathrm{d}y \quad \text{Nm} = \underline{114 \text{ J}}.$$

Remark. We will not worry about units, in this example all units were included to illustrate.

Example 2. Suppose the rope is **not weightless**. The *linear density* of the rope is $\rho_R = 5 \text{ Nm}^{-1}$, and it's pulled upwards at a speed of 1 ms^{-1}. **As it's pulled up, mass decreases. Therefore less work is required to pull the rope**. We can summarize:

$$\begin{cases} \text{Weight box} = F_B = 7 \text{ N} \\ \text{Length rope} = \ell_R = 10 \text{ m} \\ \text{Weight rope} = F_R = ??? \end{cases}$$

At the starting point, the whole weight of the rope contributes to the work. And at the end, there's no more rope. Since we are pulling with constant velocity we can model the situation as follows:



The total weight of the rope is

$$W_{R,i} = \rho_R \cdot \ell_R = 50 \text{ N}.$$

And at the end $W_{R,f} = 0$ N. The speed at which the rope moves is 1 ms^{-1} so the rate of change in speed is the same as in distance.

We can model the weight at any height y using the linear equation

$$W_R(y) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) y + b = \left(\frac{0-50}{10-0}\right) y + 50 = -5y + 50.$$

Notice that this can also be modeled by the equation

$$W_R(y) = \rho_R(\ell_R - y) = 5(10 - y).$$

Thus the work is

$$W = \underbrace{\int_0^{10} 7\mathrm{d}y}_{\text{box}} + \underbrace{\int_0^{10} (-5y+50)\mathrm{d}y}_{\text{rope}} = 70 + 250 = \underline{320 \text{ J}}.$$

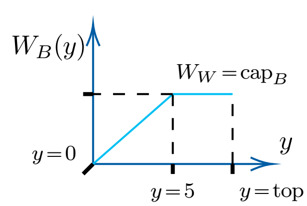
Example 3. Suppose it's now raining at a rate of 2 Ns^{-1}, we are now raising a **open** box at 1 ms^{-1} with the following conditions:

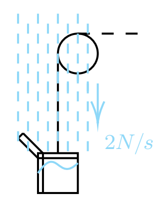
$$\begin{cases} \text{Weight box} = F_B = 12 \text{ N, Box capacity} = \text{cap}_B = 10 \text{ N} \\ \text{Length rope} = \ell_R = 8 \text{ m, Density rope} = \rho_R = 4 \text{ Nm}^{-1} \end{cases}$$

Since it's **raining**, the box is getting filled with water whose weight will contribute to the work. We have to model this as with the weight of the rope.

Initially, $W_{W,0} = 0$ N since the box is empty, and the box can't hold more than 10 N, then that's $W_{W,f}$.

However the box takes $t_{\text{fill}} = \frac{\text{cap}_B}{v_{\text{fill}}} = \frac{10}{2} = 5$ s to fill. And by that time, the box has been raised just 5 m. The final 3 m will be with the full box of water. We can model as follows:





By using a linear relation we can see that the water's weight is

$$\begin{cases} W_W(y) = \dfrac{10-0}{5-0} y + b = 2y, \ 0 \leqslant y \leqslant 5 \\ W_W(y) = 10, \ 5 \leqslant y \leqslant 8 \end{cases}$$

Finally the work is

$$W = \underbrace{\int_0^8 12\mathrm{d}y}_{\text{box}} + \underbrace{\int_0^8 4(8-y)\mathrm{d}y}_{\text{rope}} + \underbrace{\int_0^5 2y\mathrm{d}y + \int_5^8 10\mathrm{d}y}_{\text{water}} = \underline{279 \text{ J}}.$$

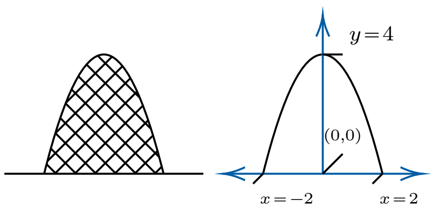
> **Practice**
>
> Instead of raining, the box is leaking (2 Ns^{-1}), what is the work required to raise it?

Mass

Recall that the density ρ of a solid with mass m and volume V is given by $\rho = m/V$. In our planar case we will consider area A instead of volume V. This means that $\rho = m/A$, and in consequence $m = \rho A$.

The problem of finding the mass of an object with *variable density* can be solved by integrating.

Example 4. Suppose we have a parabolic plate $y = 4 - x^2$ bounded by the x-axis.



If the plate's density was $\rho(x) = 2x+1$ then

$$\begin{aligned} m &= \int_{-2}^{2} \rho(x)(\text{up}-\text{down})\,\mathrm{d}x \\ &= \int_{-2}^{2} (2x+1)[(4-x^2)-0]\,\mathrm{d}x = 32/3. \end{aligned}$$

Remark. We know that

$$\text{Area} = \text{width}\cdot\text{height}.$$

Sometimes we will use up-down orientation, where we will use

$$\text{Area} = \mathrm{d}x\cdot(\text{up}-\text{down}) = (g(x)-f(x))\,\mathrm{d}x,$$

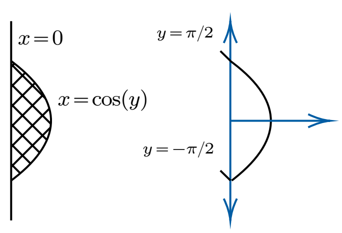
where $g(x) \geqslant f(x)$.

Or in the other case

$$\text{Area} = (\text{right}-\text{left})\cdot\mathrm{d}y = (g(y)-f(y))\,\mathrm{d}y$$

where $g(y) \geqslant f(y)$.

Example 5. Suppose that our plate is bounded by the curve $x = \cos(y)$ and the y-axis. If the plate has density $\rho(y) = y+1$, then we can find its mass by setting up the integral in the other way.



In this case

$$\begin{cases} \text{left} = x \text{ axis} = 0, \\ \text{right} = \text{curve} = \cos(y) \end{cases}$$

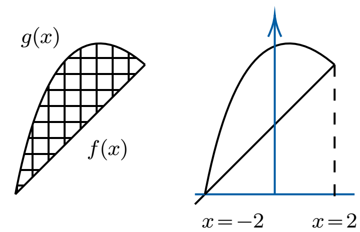
the $\mathrm{d}y$ is the height which goes from $-\pi/2$ to $\pi/2$ and so

$$m = \int_{-\pi/2}^{\pi/2} \rho(y)\cos(y)\,\mathrm{d}y = \int_{-\pi/2}^{\pi/2} (y+1)\cos(y)\,\mathrm{d}y = 2.$$

It might be the case that our plate might not have an axis as its boundary. Consider the following example:

Example 6. Let us find the mass of the plate with density $\rho(x) = 6-20x$ bounded by the equations

$$\begin{cases} f(x) = x+2, \\ g(x) = -x^2+x+6. \end{cases}$$



To find the limits in the x-axis we have to equate both expressions.

$$\begin{aligned} f(x) = g(x) &\iff x+2 = -x^2+x+6 \\ &\iff 0 = -x^2+4 \iff x = \pm 2. \end{aligned}$$

It follows that the integral runs from -2 to 2 and since the upper limit of the plate is g and the lower is f, we obtain the following expression for the mass

$$\begin{aligned} m &= \int_{-2}^{2} \rho(x)(g(x)-f(x))\,\mathrm{d}x \\ &= \int_{-2}^{2} (6-20x)[(-x^2+x+6)-(x+2)]\,\mathrm{d}x = 64. \end{aligned}$$

Practice

Consider a plate bounded by the horizontal lines $y=1$ and $y=4$ and the oblique lines $y=2x+1$ and $y=2x+5$.

I) Draw the enclosed region.
II) If the density of the plate is $\rho(x) = x^2$, find its mass.
III) Now assume that ρ is in terms of y, $\rho(y) = 2y-1$. To calculate mass, does the setup of the integral change?

Exercise 7. Suppose you are pushing a box which weighs 20 N across a path of length 15 m, at a speed of 2 ms^{-1}. If it's raining at a speed of 4 Ns^{-1} and the box has a capacity of 10 N, what is the work done on the box after displacing it?

Exercise 8. Find the mass of a plate with density $\rho(x) = 1$ bounded below by the curves $y = 2x$ and $y = 8-2x$, and above by the curve $y = 12x - 3x^2$.