This is my doctoral notebook where I will add clean information regarding whatever I'm learning about at the moment. It should serve as a starting point for writing. ¿Writing what? You may ask, I don't know.

TO DO 1

- Finish reading stratification of $M_{0,n}$ (DONE b4 20241120)
- Write ex 6.2
- Re-read Zvonkine paper now with new knowledge
- Summarize Borcherds Chow Ring and Chern Classes DONE 20240921

TO DO 2

- Read Theorem 3.1 in MattAndRenzo plus examples afterwards. Understand and write summary.↓
- Add new bibliographies, Chow's paper, Borcherds lectures.

TO DO 3

- Understand $\mathcal{G}^1$ self intersection (ALMOST) and read new examples. (DONE b4 20241120)
- Write down Hodge bundle info on distributivity. (DONE see master thesis SM2025)
- Do exercises Zvonkine. (DONE b4 20241120)

TO DO 4

- Read chap 2 green book on stable maps (kinda, B4 SM25)
- Read intros on stable map/kontsevich spaces papers.
- Finish Mumford formula understanding.(DONE b4 20241120)

TO DO 5

- Read equivariant cohomology of $\mathbb{P}^1$. (DONE by end of II2024)
- Write down examples of E.C. for a point by torus and point by integers. (DONE around 202411)
- Polish chapter 4 on strata, products, psi and lambda classes. (WORKING ON THIS RN)(DONE see TM SM25)
- Write down understanding of T3.1 MattAndRenzo.
- Write down info on forgetful, pullback, pushforward (DONE see TM SM25)
- Write down statements of string and dilaton equation. (DONE see TM SM25)
- Write down integrals as constant map pushforward. (DONE see TM SM25)

TO DO 6 20241215

- ⋄ Finish writing examples on intersection products. (DONE)
- ⋄ Add examples on psi and lambda classes.
- ⋄ Polish and pass to DN E.C. of point by T and Z. (DONE see TM SM25, not passed here but ok)
- ⋄ Write and polish E.C. of proj line. (DONE see TM SM25)
- ⋄ Apply Atiyah Bott to E.C. of proj line to get the same result. (DONE see TM SM25)
- ⋄ Read somewhere about E.C. of proj plane and write it down. (DONE see TM SM25)
- ⋄ Apply A.B. to proj plane and find the intersection of two lines. (DONE)
- ⋄ Write down info on different linearizations of line bundles.
- ⋄ Understand what is the hyperplane class H in equivariant cohomology.

TO DO 7 20250311 Phew it's been a long time

- ⋄ Understand and write what is a linearization of line bundles. (see Mumford original paper)
- ⋄ Understand the space of maps' stratification.
- ⋄ Compute integrals of cohomology classes in the space of stable maps via localization.
- ⋄ Write thesis. (DONE Summer2025)

TO DO 8 20251015

- ⋄ Read up on the Kodaira-Spencer map between deformations and cohomology [16] p. 76 and mse/1322093

TO DO 9 20251126

- ⋄ Neatly write a proposition on the tangent normal sequence. **Exercise**: Prove that for *a map* $f: X \to Y$ the sequence

$$0 \to TX \to f^*TY \to {f^*TY}\big/{TX} \to 0$$

  is an exact sequence. Here TX denotes X's tangent sheaf. Find enough conditions on f which guarantee that this sequence is indeed exact.
- ⋄ Answer question **??**.

TO DO 10 20260113

- ⋄ Figure out the business about cusps deformations and whatever $\underline{\mathbb{C}x}$ is. In general normalization sequences.
- ⋄ prove a result about the product $(\lambda^{\mathrm{ps}})(\prod \psi' s)$
- ⋄ prove the general case of $C^{\mathrm{ps}}(g,d)$

Contents

Chapter 1

A Study of the Introduction and Prologue of *The Green Book*

1.1 Really quickly: The Introduction

The main objective of the green book is to prove the formula for the number N_d of rational curves of degree d passing through $3d-1$ points in general position in $\mathbb{P}^1_{\mathbb{C}}$. Let's begin by unwrapping some concepts:

Definition 1.1.1. A projective curve $\mathcal{C}$ is the zero locus of points in $\mathbb{P}^2_k$ which satisfy a homogeneous polynomial equation. Formally, for a homogeneous polynomial $f \in k[X,Y,Z]$, the projective curve determined by f is

$$V(f) = \{p \in \mathbb{P}^2_k : f(p) = 0\}.$$

If f has degree d, then the curve $\mathcal{C}$ is said to be a curve of degree d.

Example 1.1.2. Consider the polynomial

$$f(X,Y,Z) = X - Y - Z.$$

Inside the affine plane $\{Z = 1\}$, this contains all the points of the form $(X : X-1 : 1)$. This is the line $y = x-1$ in $\mathbb{A}^2$. But it also contains the point at infinity $(1:1:0)$. The degree 1 curve being described here is a projective line.

Example 1.1.3. The degree 2 curve described by the equation $XY - Z^2 = 0$ is an affine hyperbola containing two points at infinity $(1:0:0)$ and $(0:1:0)$.

Definition 1.1.4. A parametrization of a curve $\mathcal{C}$ is a generically injective function

$$\phi : \mathbb{P}^1_k \to \mathbb{P}^2_k,\ (S:T) \mapsto (P(S:T) : Q(S:T) : R(S:T)), \quad P,Q,R \in k[S,T]_h.$$

A projective plane curve admitting a parametrization is called a rational curve.

Example 1.1.5. The line $X - Y - Z = 0$ can be parametrized with $\phi(S:T) = (S:T:S-T)$. ¿Is the other curve rational?

Example 1.1.6. The family of curves $XY - aZ^2 = 0$ for $a \in \mathbb{A}^1 \setminus \{0\}$ can be parametrized by

$$\phi_1[S:T] = [aS^2 : T^2 : ST], \quad \text{or} \quad \phi_2[S:T] = [S^2 : aT^2 : ST].$$

Observe that indeed: $(aS^2)(T^2) - a(ST)^2 = 0$. ¿Is there any difference between both parametrizations?

Example 1.1.7. A family of cubics $bF + G$ where F is the nodal cubic and G is the union of 3 lines:

$$F = Y^2Z - X^2(X - Z), \quad \text{and} \quad G = X(X+Y)(X-Y)$$

can be parametrized by intersecting with pencil of lines $SX + TY$. ¿How do Kock and Vainsencher find this parametrization nicely?

Example 1.1.8. Degree d curves with a $d-1$-tuple point are rational. As they can be parametrized by a line passing through the singular point.

The dimension of maps from $\mathbb{P}^1$ to $\mathbb{P}^2$ of degree d

The number $3d-1$ sounds like an arbitrary number. It certainly did to me at least; this number corresponds to the dimension of the space of maps from $\mathbb{P}^1$ to $\mathbb{P}^2$ of degree d. There's this very important question,

¿which vector space is the space of maps from $\mathbb{P}^1$ to $\mathbb{P}^2$ of degree d?

Proposition 1.1.9. *The aforementioned space has dimension $3d-1$.*

Proof

A map $F : \mathbb{P}^1 \to \mathbb{P}^2$ is defined via homogeneous, degree d polynomials. This means that

$$F(s:t) = (X:Y:Z) = (F_1(s:t), F_2(s:t), F_3(s:t)),$$

where each F_i is a homogeneous degree d polynomial. Explicitly we may write

$$F_j(s:t) = \sum_{i=0}^{d} a_i s^{d-i} t^i = a_0 s^d + a_1 s^{d-1} t + \cdots + a_{d-1} s t^{d-1} + a_d t^d$$

which allows us to see that every F_j has $d+1$ degrees of freedom. But we have to take of changes in the input and output spaces:

- ◇ 3 dimensions off for $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$.
- ◇ 1 dimension off for projective quotients: $(X:Y:Z) = \lambda(X:Y:Z)$.

This leaves us with $3d + 3 - 3 - 1 = 3d - 1$ dimensions.

There's another way to prove this by counting the general number of degree d curves and then making sure they are rational. For this we need the genus-degree formula.

Proposition 1.1.10. *A projective curve of degree* d *has genus* $\binom{d-1}{2}$.

The proof of the genus-degree formula will be written down at a later point when we have to talk about Bézout's theorem. For now, the second proof of the dimension question:

Proof

Consider a general degree d curve defined by a homogeneous polynomial F. Such a polynomial can be written as a combination of monomials $X^aY^bZ^c$ where $a+b+c=d$. So to count the number of monomials, we must find the number of triples (a,b,c) of non-negative integers whose sum is d. This is precisely

$$\left(\!\!\binom{3}{d}\!\!\right)=\binom{3+d-1}{d}=\binom{d+2}{d}=\binom{d+2}{2},$$

and we have to take off 1 dimension due to projective quotients. Thus the dimension of the space of degree d curves is precisely $N:=\binom{d+2}{2}-1$.

Consider now the universal curve over $\mathbb{P}^N$

$$\mathcal{U}=\{F=\sum_{i+j+k=d} a_I x^i y^j z^k=0\}\subseteq \mathbb{P}^N\times\mathbb{P}^2.$$

such that the curve $\{F=0\}$ is the fiber above the point $a_I\in\mathbb{P}^N$. The family's dimension is precisely $N+1$, there's two ways to see this:

- ◇ We have all the dimensions of $\mathbb{P}^N$ plus the dimension of the curve so that's $N+1$.
- ◇ The universal curve lives inside a $(N+2)$-dimensional space and it's a hypersurface, as it is defined via one equation. So we add one codimension to get $N+1$.

Inside $\mathcal{U}$ we have the singular locus

$$\mathrm{Sing}=\mathcal{U}\cap\{\partial_x F=0\}\cap\{\partial_y F=0\},$$

where each of this conditions impose a dimension 1 restriction so this whole singular locus has dimension $N+1-2=N-1$. Projecting down to $\mathbb{P}^N$ we get the set of points in $\mathbb{P}^N$ whose fibers correspond to curves with *at least* one singularity $\pi(\mathrm{Sing})$. This set still has dimension $N-1$ inside of $\mathbb{P}^N$ which means that adding a singularity adds one codimension.

Continuing this process we find smaller subsets corresponding to curves with more nodes and adding each node means adding one codimension. For the curve to be rational, we must add $g = \binom{d-1}{2}$ nodes in total, so that we add g codimensions. The set of curves with g singularities forms an open set inside of the rational curves which means that the dimension of the set of rational curves is

$$\binom{d+2}{2} - 1 - \binom{d-1}{2} = (d+1) + d + (d-1) - 1 = 3d - 1.$$

Remark 1.1.11. Recall $\left(\!\binom{n}{k}\!\right)$ is the number of ways that I can distribute k cookies amongst n friends.

Exercise 1.1.12. Learn why removing geometric genus reduces the dimension of the space of curves.

It has to do with resolution or normalization of singularities. Clarify and add the picture

The whole idea is to use the moduli space of maps from $\mathbb{P}^1$ to $\mathbb{P}^r$, $\overline{M}_{0,3d-1}(\mathbb{P}^r, d)$, to show the formula. Isomorphism classes inside this set look like classes of bundles. And the formula is derived from intersection theory of this space.

1.2 Quadruplets of Points

I consider myself lucky to already know what $M_{0,4}$ is (it's the set of genus 0 Riemann surfaces with 4 distinct marked points). The notion of

$$Q = \{\text{quadruplets of distinct points in } \mathbb{P}^1\}$$

is introduced to alleviate the posterior definition of the moduli space. The set of quadruplets of points can actually be viewed as

$$Q = (\mathbb{P}^1)^4 \backslash \Delta,$$

where Δ is the set of diagonals. This means that if we have $\mathbf{x} = (x_1, \ldots, x_4)$, then $\Delta = \{x_i = x_j\}$ for some i, j. So indeed Q is the set of distinct quadruplets.

Remark 1.2.1. When I find it convenient, points in $(\mathbb{P}^1)^n$ will be denoted $\mathbf{x}$, but I'll mostly forget and I'll just call them x without acknowledging that they are arrays.

Exercise 1.2.2. Show that Q is an affine algebraic variety. [Hint: It's a similar argument to proving that $\mathbb{A}^1\backslash\{0\}$ is an algebraic variety by considering $\mathbb{C}[x,y]/\mathrm{gen}(xy-1)$.]

The set Δ is a *divisor*, we can see that it is

$$\Delta = V(x_1-x_2)\cup V(x_1-x_3)\cup\cdots\cup V(x_3-x_4) = V\left(\prod_{\substack{i,j\in[4]\\ i<j}}(x_i-x_j)\right).$$

Call this polynomial f, then $Q=\mathbb{A}^4\backslash V(f)$ which can be seen as $V(tf-1)\subseteq\mathbb{A}^{n+1}$.

Remark 1.2.3. Recall affine algebraic varieties are those who are Spec of someone. In particular, Q is the spectrum of the quotient of $\mathbb{C}[x_1,\ldots,x_4]$ by the ideal generated by the product.

Ahh, you've dug the hole for yourself in this one…

Exercise 1.2.4. Show that indeed $\mathbb{A}^1\backslash\{0\}$ is an affine algebraic variety.

The set Q is *tautologically* a moduli space for quadruplets. In the easiest of terms, every element in Q corresponds to a quadruplet of distinct points.

In the same way you go up to $M_{0,4}\simeq\mathbb{P}^1\backslash\{0,1,\infty\}$, look at a point $[\lambda:\mu]$ and find a $\mathbb{P}^1$ with 4 marked points $(0,1,\infty,\lambda/\mu)$, you can go up to Q and look at a point $\mathbf{p}=(p_1,\ldots,p_4)$ and see that it *tautologically* corresponds to the point $\mathbf{p}=(p_1,\ldots,p_4)$. ¡The parameter is the quadruplet itself!

It is claimed that Q is a *fine* moduli space *because* it carries a universal family. In my mind, this notion of *fineness*[1] is the same as representability of the moduli space as a functor.

Exercise 1.2.5. ¿Is the fineness the same as representability as a functor? Also, ¿does having a universal family guarantee that a moduli space as a functor be representable?

The Family Business

Intuitively, a family of pointed Mickies is a diagram: where B is

[1]**What is the difference between finesse and fineness?** Finesse refers to the skill and cleverness someone shows in the way they deal with a situation or problem. Fineness refers to a thing's quality of being fine—for example, the fineness of print (that is, how small the letters are) or the fineness of one wire in comparison to another (that is, how thin they are).

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ B \arrow[u, bend right, "\sigma_i"'] \end{tikzcd}$$

called the base variety, E is more usually than not $\{\ \} \times B$ in genus 0 (But in higher genus and some genus 0 cases like the Hirzebruch surface, it's not) and σ_i are sections which single out the important points in each Mickey. For each $b \in B$, the fiber over b, $\pi^{-1}(b)$ is isomorphic to a particular Mickey.

For a family to be *universal*, it is my understanding that the base variety should be the moduli space of Mickies itself. So the universal family should be

$$\begin{tikzcd} U \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "s_i"'] \end{tikzcd}$$

such that every fiber $\pi^{-1}(x)$ is the corresponding Mickey and $s_i(x)$ is the i^{th} characteristic of the corresponding Mickey but seen in the upstairs Mickey.

Remark 1.2.6. The idea behind universality is that this family $U \to M$ contains *every possible Mickey*. All the possible Mickeys are coded as equivalence classes into points of M, so when looking at fibers upstairs in U there's not a Mickey we're missing.

Definition 1.2.7. A family of quadruplets in $\mathbb{P}^1$ over a base variety B is a family of pointed $\mathbb{P}^1$'s with 4 sections σ_i singling out the points in each $\mathbb{P}^1$. Diagramatically:

$$\begin{tikzcd} B \times \mathbb{P}^1 \arrow[d, "\pi"'] \\ B \arrow[u, bend right, "\sigma_i(\times 4)"'] \end{tikzcd}$$

so a fiber over a point $b \in B$ is a copy of $\mathbb{P}^1$ with four points marked via the map $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_4)$.

From this, the universal family over Q is the family of quadruplets over Q as a base. The section σ_i is given by the i^{th} projection mapping $\pi_i : Q \to \mathbb{P}^1$ which singles out the i^{th} point of the quadruple.

Exercise 1.2.8. The *universal* family enjoys the *universal* property that any other family of quadruples is induced from it via pullback. Explain how this happens and prove that the universal family indeed has this universal property.

Let us begin by considering a family of quadruplets $\pi : B \times \mathbb{P}^1 \to B$ along with its four sections σ_i. We can build a map $\boldsymbol{\sigma}$ which is the $\kappa : B \to Q$ map we are looking for in this case from B to Q by considering all the sections:

$$\boldsymbol{\sigma} : B \to Q,\; b \mapsto (\sigma_1(b), \sigma_2(b), \sigma_3(b), \sigma_4(b)).$$

In order to create the pullback family, we look at the fiber of $\boldsymbol{\sigma}(b)$ on the universal family of Q: $Q \times \mathbb{P}^1$. To construct the pullback, we build it fiber by fiber.

For every $b \in B$, the fiber will be a copy of the fiber of $\boldsymbol{\sigma}(b)$ but pasted on top of B and the sections will be the pullback of the sections of Q via $\boldsymbol{\sigma}$:

$$\begin{tikzcd}
B \times \mathbb{P}^1 \arrow[rr, dashed, "\exists ? \phi"] \arrow[dr, "\pi"] & & B \times_Q (Q \times \mathbb{P}^1) \arrow[dl, "\pi"'] \\
& B \arrow[ul, bend left, "\sigma_i (\times 4)"] \arrow[ur, bend right, "\boldsymbol{\sigma}^*(s_i)(\times 4)"'] &
\end{tikzcd}$$

Finally, we are in need of the base morphism ϕ. Observe that this ϕ we are looking for is the identity map on the fibers. It takes fibers to fibers, the points of the quadruple to the corresponding *same* points but in the other fiber, and it's invertible. It follows that ϕ is an isomorphism of families over B which means that the original family and the one induced via pullback are equivalent.

In terms of the diagram for fibered products what we have is the following:

$$\begin{tikzcd}
B \times \mathbb{P}^1 \arrow[dr, dashed, "\exists ? \phi"] \arrow[drr, bend left] \arrow[ddr, bend right, "\pi"'] & & \\
& B \times_Q (Q \times \mathbb{P}^1) \arrow[r] \arrow[d, "\pi"'] & Q \times \mathbb{P}^1 \arrow[d, "\pi"] \\
& B \arrow[r, "\kappa"'] & Q
\end{tikzcd}$$

$B\times\mathbb{P}^1$ play the role of the new object which has morphisms to the already existing ones, and the pullback or fibered product is the universal object with this property.

Remark 1.2.9. Now, returning to our previous observation 1.2.6, the universal family enjoys the universal property *because it contains all of the possible fibers*. The κ map literally looks at a point b in the base, asks which is the fiber above it and then points to that fiber's equivalence class in the moduli space. ¡Such a map's existence is guaranteed because the moduli space has (the equivalence class of) all the fibers!

We have mentioned the idea of base morphisms, but formally...

Definition 1.2.10. A base morphism between two families $E \xrightarrow{\pi} B$, $F \xrightarrow{\lambda} B$ is a map $\phi: E \to F$ which makes the following diagram commute.

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\lambda"] \\ & B & \end{tikzcd}$$

This does not add intuition to our understanding, but let us unravel the definition. The diagram commutes when

$$\pi = \lambda\phi,$$

and we would like to see how fibers behave. So take y in the fiber $\pi^{-1}(x)$,

$$y \in \pi^{-1}(x) \xrightarrow{\phi} \phi(y).$$

Now observe that if we map this element down to B we get, via the commuting relationship

$$\lambda\phi(y) = \pi(y) = x.$$

This means that $\phi(y)\in\lambda^{-1}(x)$. And as our element was arbitrary, every fiber gets mapped to another fiber. In the case of base isomorphisms we have correspondence among the fibers, and so an automorphism of families is basically a rearrangement of fibers. Now in the case of sections:

$$\begin{tikzcd} E \arrow[rr, "\sigma(x)\mapsto\phi(\sigma(x))"] \arrow[dr, "\pi"] & & F \arrow[dl, "\lambda"'] \\ & B \arrow[ul, bend left, "\sigma"] \arrow[ur, bend right, "s"'] & \end{tikzcd}$$

This diagram commutes when $s=\phi\sigma$. So, ¿does a point $\sigma(x)\in\pi^{-1}(x)$ get sent to another *special point* on a fiber of x, or does it go to another fiber? Observe that if we map $\sigma(x)$ through ϕ we get

$$\phi(\sigma(x))=s(x)$$

but also $\lambda(s(x))=x$, so it happens that $s(x)\in\lambda^{-1}(x)$ and not in another fiber.

Intuitively, isomorphisms of families are morphisms of families which also happen to be bijective. This helps us now because the next step is looking at…

1.3 Quadruplets along with $\mathbb{P}^1$

The Green Book goes along with the notion of quadruplets but now up to equivalence. Two quadruplets are equivalent when they differ by a $\mathbb{P}^1$ automorphism term-by-term. Families of quadruplets will be projectively equivalent when there is a base-isomorphism between them.

Example 1.3.1. The following quadruplets are equivalent:

- ◇ $(0,1,\infty,8)$ and $(0,2,\infty,16)$ via the Möbius transformation $z\mapsto 2z$.
- ◇ $(-7/8,0,\infty,1/8)$ and $(0,7/8,\infty,1)$ via $z\mapsto z+7/8$.
- ◇ $(\infty,1,0,1/9)$ and $(0,3,\infty,27)$ via $z\mapsto 3/z$.

Observe that their cross-ratio

$$(a,b,c,d)\mapsto\frac{(a-b)(c-d)}{(a-d)(c-b)}$$

is presevered by the transformation. These respectively are: $1/8,7/8$ and $1/9$.

The book's definition of $M_{0,4}$ is then

$$M_{0,4}:=\{\text{quadruplets of }\mathbb{P}^1\}/\text{projective equivalence}.$$

In a similar fashion to Q, we can construct a universal family for $M_{0,4}$. The fibers are once again copies of $\mathbb{P}^1$ imbued with the corresponding quadruplet, it's *tautological*. The family is thus

$$\begin{tikzcd} M_{0,4}\times\mathbb{P}^1 \arrow[d, "\pi"'] \\ M_{0,4} \arrow[u, bend right, "\tau_i(\times 4)"'] \end{tikzcd}$$

where τ_1,τ_2 and τ_3 single out $0,1$ and ∞ in each fiber. The last section is *like* a diagonal section. It is given by the inclusion map of $M_{0,4}$ seen as the diagonal (z,z) of $M_{0,4}\times\mathbb{P}^1$.

Theorem 1.3.2. *The universal family over $M_{0,4}$ also induces every other family of quadruplets up to projective equivalence via pullback.*

Proof

Indeed as in the case of the exercise 1.2.8, we start with a family of quadruplets

$$\begin{tikzcd} B\times\mathbb{P}^1 \arrow[d, "\pi"'] \\ B \arrow[u, bend right, "\sigma_i(\times 4)"'] \end{tikzcd}$$

and show that there exists a base isomorphism between this family and the pullback of the universal family via a certain map.

Step 1 is to construct the map κ:

$$\begin{tikzcd} B\times\mathbb{P}^1 \arrow[d, "\pi"'] & \\ B \arrow[r, "\exists ?\kappa"] & M_{0,4} \end{tikzcd}$$

Intuitively, the map looks at a point $b\in B$, asks the family for its corresponding quadruplet and then normalizes it via a Möbius transformation.

Formally, for $b\in B$ we map

$$b\mapsto \boldsymbol{\sigma}(b):=(\sigma_1(b),\sigma_2(b),\sigma_3(b),\sigma_4(b))$$

which gives us the associated quadruplet. To normalize it we take the Möbius transformation

$$\lambda(z)=\frac{(z-\sigma_2(b))(\sigma_3(b)-\sigma_4(b))}{(z-\sigma_4(b))(\sigma_3(b)-\sigma_2(b))}$$

and apply it repeatedly to our quadruplet as

$$\boldsymbol{\lambda}(p,q,r,s)=(\lambda(p),\lambda(q),\lambda(r),\lambda(s)).$$

Call $t:=\lambda(\sigma_1(b))$ the cross-ratio of our quadruplet. The map κ is thus

$$\kappa(b):=(\boldsymbol{\lambda}\circ\boldsymbol{\sigma})(b)=(t,0,1,\infty).$$

Actually, this is still a quadruplet, we are missing the quotient map which sends that quadruplet to its equivalence class inside $M_{0,4}$. But for all the effects, that is our desired map.

As this is a quadruplet inside $M_{0,4}$, we now have the *canonical* map from B to $M_{0,4}$. With it, we will pullback the universal family and show it's isomorphic to the original family over B.

To construct a fiber of the pullbacked family we take the fiber over $\kappa(b)$ and attach it to our original point b. In this case the fiber is

$$\pi^{-1}(\kappa(b)) = \{[t,0,1,\infty]\} \times \mathbb{P}^1$$

so that'll be the same fiber over $b \in B$. In the same way that I can define a bundle via fibers and transition functions, is it possible to define a family via fibers and...and...and what?.

Chapter 2

A Project on Sheaf Cohomology of Line Bundles over $\mathbb{P}^1$

2.1 Is Bundle like a Family?

The notion of a family of lines over a base variety B is a map $\pi: B\times\mathbb{C}\to\mathbb{C}$ where π is the projection, so that over each point $b\in B$, the fiber of π is a copy of the fixed $\mathbb{C}$.

Definition 2.1.1. A line bundle over a base B is a map $\pi: L\to B$ with the following properties:

(a) There's an open cover $(U_i)_{i\in I}$ of B such that
$$\pi^{-1}(U_i)\simeq U_i\times\mathbb{C}$$
where we call $\phi_i:\pi^{-1}(U)\to U\times\mathbb{C}$ the isomorphism. This means that the fiber is isomorphic to $\mathbb{C}$.

(b) For $b\in U_i\cap U_j$, the composition
$$\{b\}\times\mathbb{C}\xrightarrow{\phi_i^{-1}}\pi^{-1}(b)\xrightarrow{\phi_j}\{b\}\times\mathbb{C}$$
is a linear isomorphism. This map, $\phi_j\phi_i^{-1}$, is multiplication by a nonzero scalar λ_b.

Comparing this with the notion of family, we have *local triviality* and the vector space structure between fibers is compatible. This means a line bundle is a locally trivial family of complex lines.

Definition 2.1.2. A section of a line bundle L over B is a map $s: B\to L$ with $\pi s=\mathrm{id}_B$.

Sections can be defined locally on open sets $U\subseteq B$ or globally when they are defined everywhere on B. Intuitively, a section singles out points in fibers. For every $b\in B$, $s(b)$ is a point on the fiber $\pi^{-1}(b)$.

The Fun Part is not the sets, it's the…

Base morphisms of a families are maps between total spaces which carry fibers onto fibers and distiguished points to distinguished points. For the case of line bundles we have just a tiny bit more.

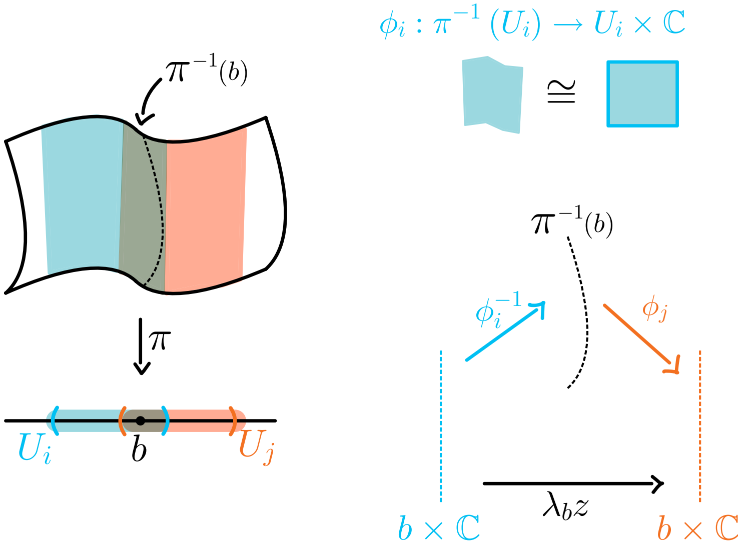


Figure 2.1: Line Bundle with the two properties

Definition 2.1.3. A <u>morphism of line bundles</u> is a map $f : L_1 \to L_2$ which makes the following diagram commute:

$$\begin{tikzcd}
L_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & L_2 \arrow[dl, "\pi_2"] \\
& B &
\end{tikzcd}$$

It must hold that on fibers, the map f restricts to a **linear map**.

Unwrapping the definition a bit, we have that the diagram commutes when $\pi_1 = f\pi_2$. So the question is, where does a point in a fiber, $x \in \pi_1^{-1}(b)$, map to? We would like it to be in the corresponding fiber of L_2.

For $f(x) \in \pi_2^{-1}(x)$, it must occur that $\pi_2(f(x)) = b$. But we have

$$\pi_1(x) = \pi_2(f(x)) \quad \text{and} \quad \pi_1(x) = b,$$

so it holds that

$$f\left(\pi_1^{-1}(b)\right) \subseteq \pi_2^{-1}(b).$$

Similarly for sections for sections, the diagram commutes when $s_2 = f s_1$. But this means that for $b \in B$,

$$s_2(b) = f(s_1(b)),$$

so distinguished points of fibers get sent to the corresponding distinguished points.

2.2 Line Bundles over $\mathbb{P}^1$

We begin by introducing a family of complex manifolds.

Definition 2.2.1. The manifold $\mathcal{O}_{\mathbb{P}^1}(d)$ is defined by two charts and a transition function:

$$(\mathbb{C}^2,(x,u)) \xrightarrow[v=u/x^d]{y=1/x} (\mathbb{C}^2,(y,v)).$$

This transition function is $(y,v) = \left(\frac{1}{x},\frac{u}{x^d}\right)$ with inverse $\left(\frac{1}{y},\frac{y}{v^d}\right)$.

We could also regard this set as $\mathbb{C}^2$ under the equivalence relation described via the transition function.

No other line bundles

$\mathcal{O}_{\mathbb{P}^1}(d)$ comes with a natural projection onto $\mathbb{P}^1$

$$(x,u) \mapsto x$$

This allows us to see $\mathcal{O}_{\mathbb{P}^1}(d)$ as a line bundle, because every fiber $\pi^{-1}(x)$ is isomorphic to $\mathbb{C}$. When x is non-zero we get a copy of $\mathbb{C}$ on both charts, but when $x = 0$ or ∞, the line is only on one of the charts.

Spoiling ourselves of the fun[1], we claim that all line bundles over $\mathbb{P}^1$ are of the form $\mathcal{O}_{\mathbb{P}^1}(d)$ for an integer d.

[1] Because it'd be so much fun to prove this are all the line bundles.

Chapter 3

Premier on the Chow Ring by *Richard E. Borcherds*

3.1 The Chow ring

This summary is based on Richard E. Borcherds' introduction to the Chow ring in Youtube.

For a non-singular variety V, we define the *Chow ring* $A^*(V)$, whose elements *correspond* to subvarieties of V, and the product reflects the intersection of these subvarieties. The ring is graded by codimension:

$$A^*(V) = \bigoplus_i A^i(V)$$

where $A^i(V)$ consists of classes of subvarieties with codimension i. Ideally, the intersection of a codimension m subvariety X and a codimension n subvariety Y would yield a subvariety of codimension $m+n$. However, this does not always hold. Imagine a hyperplane H intersected with itself. So there's a complication in defining this product.

Cycles and Intersection Numbers

An initial attempt to resolve starts by defining *cycles*, which are formal sums of subvarieties. Specifically, we define the group of codimension i cycles as:

$$A^i(V) = \langle X \mid X \subseteq V, \text{closed subvariety}, \operatorname{codim}(X) = i \rangle$$

For two cycles $X \in A^i(V)$ and $Y \in A^j(V)$, we aim to define their product as:

$$X \cap Y = \sum_Z i(X,Y;Z)Z$$

where the sum runs over the irreducible components Z of $X \cap Y$, and $i(X,Y;Z)$ is an *intersection number*, representing the multiplicity of the intersection at Z. However, defining this intersection number precisely poses significant challenges.

Rational Equivalence and the Chow Group

To address the ambiguities in defining intersections, particularly when subvarieties do not intersect transversely, we use the notion of *rational equivalence*. Cycles are considered equivalent if their difference is the divisor of a rational function on a subvariety of dimension $j+1$. This leads to the following definition:

Definition 3.1.1. The i-th *Chow group* $A^i(V)$ of a non-singular variety V consists of equivalence classes of codimension i cycles, where two cycles are equivalent if their difference is a principal divisor, i.e., the zero set of a rational function.

The *Chow ring* is the direct sum over all Chow groups:
$$A^*(V) = \bigoplus_i A^i(V)$$
The intersection product on the Chow ring is well-defined:
$$[X] \cap [Y] = \sum_{[Z]} i(X,Y;Z)[Z]$$
where $[X]$, $[Y]$, and $[Z]$ denote rational equivalence classes of cycles.

Example and Further Considerations

Lemma 3.1.2 (Chow's moving lemma (1956))**.** *Given two algebraic cycles X,Y in V a non-singular variety, there is another cycle Y' rationally equivalent to Y on V such that X and Y' intersect properly. Look for better reference than wikipedia.*

Let's illustrate this by considering the next example.

Example 3.1.3. Consider the surface $S = \mathcal{B}\ell_{\text{pt}}\mathbb{P}^2$. We have a copy of $\mathbb{P}^1$ as the exceptional divisor E. If we ask what is $E \cap E$, then we have to move E slightly.

This is impossible as E can't be deformed. This is not a counterexample to Chow's lemma but to see this we have be a bit subtle. Consider a line A which we deform to pass about the exceptional divisor, call it B. This means that
$$A \sim B \cup E \Rightarrow E \sim A - B.$$
$A-B$ has a well-defined intersection with E because A doesn't meet B and B has a transversal intersection with E. Thus $(A-B)\cap E$ is well defined as we've deformed E into a cycle with negative coefficients.

Observe that if we deform E to something which has transversal intersection with itself, we will acquire negative coefficients because
$$E \cap E = (-1)[\text{pt}].$$

So we will acquire negative coefficients when turning E into a well-behaved cycle.

In essence, Chow's lemma doesn't say we can deform subvarieties, it says we can deform cycles and even if we start with one with positive coefficients, we may end up with one with negative coefficients.

Now if V is a singular variety, we may end up with extra complications.

Example 3.1.4. Suppose we take V to be a cone and two subvarieties X,Y being a hyperbola and a line through the singularity. Add figure If we intersect $X \cap Y$ we get just one point without troubles, but let us slide X so that X becomes a *double line* through the singularity. So it might the case that $X \sim 2Z$ where Z is the corresponding line. It must happen as well that

$$2Z \cap Y = [\mathrm{pt}] \Rightarrow Z \cap Y = \frac{1}{2}[\mathrm{pt}].$$

We can still get intersections on varieties with singularities but with rational coefficients.

This means that it certainly does make sense to consider the Chow ring of $\overline{M}_{g,n}$.

Question. ¿Is this why there appears a $\frac{1}{24}$ in some places when doing calculations with λ-classes?

> [RQ 241212] It's actually not the λ class, ¡but the ψ class! The result I was remembering at that moment was
> $$\int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}$$

Finally, lets consider examples of Chow groups and Chow rings. In general finding the Chow ring is very hard, but we have

- ⋄ The 0^{th} Chow group, $A^0(V) \simeq \mathbb{Z}$, is generated by $[V]$. This is the class of the whole variety or *the fundamental class*. This is the multiplicative identity in the Chow ring as $X \cap V = X$.
- ⋄ $A^1(V)$ contains hypersurfaces modulo linear equivalence. This are basically divisors up to linear equivalence which amounts to the Picard group of V, $\mathrm{Pic}(V)$. Already for an elliptic curve, $\mathrm{Pic}(V)$ is uncountable.

Example 3.1.5. If $V = \mathbb{P}^2$, then $A^i(V) \simeq \mathbb{Z}$ for $0 \leqslant i \leqslant 2$. This is because we can decompose $\mathbb{P}^2$ as

$$\mathbb{P}^2 = \mathrm{pt.} \cup \mathrm{line} \cup \mathrm{plane}$$

In general $A^*(\mathbb{P}^n) \simeq \mathbb{Z}[H]/\langle H \rangle^{n+1}$ where H represents the class of a hyperplane in $A^1(\mathbb{P}^n)$.

¡This is the same case for the Grassmannian! Its Chow ring is also easy to describe as we may decompose the Grassmannian into affine spaces.

Question. ¿What is the Chow ring of the Grassmannian?

Remark 3.1.6. Some of this intersection numbers can also be calculated with Schubert calculus, and they happen to coincide with Littlewood-Richardson coefficients.

Exercise 3.1.7. Consider the rational normal curve in $\mathbb{P}^2, C = V(XY - Z^2)$. As a closed subvariety of $\mathbb{P}^2$, $[C] \in A^1(\mathbb{P}^2)$, which means $[C] = d[H]$ where $[H]$ is the class of a hyperplane. In $\mathbb{P}^2$, H is the class of a line.

(a) ¿Can we form a conjecture about what the multiple of the hyperplane class will $[C]$ be intuitively?
(b) Show that $d = 2$ by explicitly describing a rational map between C and 2 lines.

> There was a way to go about the second item via the affine case, say the curve is $xy - 1 = 0$ and we can take the two axes x,y and then form the family of curves $xy = t$ with t varying from 0 to 1. Then this, in some fashion, formed the desired rational map. ASK RENZO

Exercise 3.1.8. Generalize the previous exercise to the case of a degree d curve in $\mathbb{P}^2$. Also, ¿what happens if we consider the sum of different degree cycles? Say something like $3U + 2V$, ¿what will be the multiple of the hyperplane class there?

3.2 Chern classes

A quick cheat sheet

Let us begin with the Chern class cheat sheet™. This based off of my meeting with Renzo on 20240917. First assume $E \xrightarrow{\pi} B$ is a rank r vector bundle. We have the following:

- $c_i(E) \neq 0$ whenever $0 \leqslant i \leqslant r$.
- $c_i(E)$ has degree i in the Chow ring. This means $c_i(E) \in A^i(B)$.
- $c_0(E) = 1$, or in words, it's the fundamental class. It's usually the case that we rescale in order for this to be exactly 1.

◇ If we define
$$c_{\text{tot}} = c_0 + c_1 + \cdots + c_r$$
and we have a short exact sequence of vector bundles
$$0 \to F \to E \to Q \to 0,$$
then $c_{\text{tot}}(E) = c_{\text{tot}}(F) \cdot c_{\text{tot}}(Q)$. In particular we have
$$c_1(E) = c_1(F) + c_1(Q).$$
◇ If E is a line bundle L, i.e. rank 1, then
$$c_1(L) = [\mathrm{div}(s)],$$
this is the class of the divisor of a meromorphic section.
◇ c_1 commutes with pullbacks.

For more, check out this math.se post[1] or here[2].

A Historical Note

Continuing with Richard Borcherds' lecture, we focus on how to obtain Chern classes from a vector bundle $E \xrightarrow{\pi} V$ over a non-singular variety V. These are certain elements of $A^i(V)$

On the side of historical notes, in differential geometry, Chern classes take values in the cohomology ring of V when seen as a complex manifold. The Chow ring is related to the cohomology ring via a homeomorphism
$$A^i(V) \to H_{2n-2i}(V) \to H^{2i}(V)$$
where the first map is taking a cycle to a cycle, and then applying Poincaré duality.

Remark 3.2.1. Personally, I believe it would be more accurate to say that the Chow ring is more closely related to homology groups than to cohomology. This is because cohomology deals with functions mapping cycles to $\mathbb{R}$ while homology is primarily concerned with formal groups of cycles.

This map is not injective in general as $A^i(V)$ is hopelessly huge in general. For an elliptic curve E, it happens that
$$A^1(E) \quad \text{is uncountable, but} \quad H^2(E) = \mathbb{Z}.$$
It is also not onto as nothing in the Chow ring maps to $H^{2i+1}(V)$ for any i. We may refine our question to
$$\text{¿is} \quad A^i(V) \to H^{2i}(V) \quad \text{onto?}$$
And the answer remains negative. The image of this map is subtle, and Hodge established that
$$H^{2i}(V,\mathbb{R}) \simeq \bigoplus H^{p,q}(X) \quad \text{and} \quad \mathrm{Im}(A^i) \subseteq H^{i,i}(V).$$
So the previous question can be further refined to,
$$\text{¿is} \quad \mathrm{Im}(A^i) = H^{i,i}(V) \cap H^*(X,\mathbb{Z})?$$

[1] `math.stackexchange.com/q/989147/`

[2] `https://rigtriv.wordpress.com/2009/11/03/chern-classes-part-1/`

Once again not positive. Atiyah and Hirzebruch found a torsion element in the cohomology that is not in the image of $A^i(V)$. This leads us to the Hodge conjecture which remains unsolved.

Characteristic Classes

Example 3.2.2. For a line bundle $L \xrightarrow{\pi} B$, take a section f. So at each point $b \in B$, f picks out a point in the fiber $\pi^{-1}(b)$. Usually the set of zeroes of f is a codimension 1 cycle of B.

add drawing

This cycle lives in $H_{n-1}(B)$ where $\dim B = n$. Via Poincaré duality, this gives us an element in $H^1(B)$. This is the first Stiefel-Whitney class of L.

If we take $B = S^1$, we have two obvious line bundles, $S^1 \times \mathbb{R}$ and the Möbius bundle.

add drawing

In the case of the trivial bundle, the zeroes of a non-zero section will always be even, while in the case of the Möbius bundle, it'll always be odd. This means that at the level of $H^1(S^1, \mathbb{Z}/2\mathbb{Z})$ we always get the 0 and 1 elements respectively. So we can distinguish this line bundles by counting the zeroes of a generic section.

The idea of characteristic classes is a generalization of this ideas. If we have a vector bundle, we can look at a section's zeroes and they will give us homology (or cohomology) classes.

A First Step into Complex Land

Chern classes on a complex vector bundle will take values in $A^*(V)$ (or $H^*(V)$ if we are doing differential geometry).

Consider a complex line bundle $L \xrightarrow{\pi} V$ over a non-singular variety or complex manifold V. We take the set of zeroes of a section f. This has complex codimension 1 in V and means that we have an element of $A^1(V)$, or in the real case, codimension 2 with it being an element of $H_{2n-2}(V) \simeq H^2(V)$.

So the first Chern class is given by the cycle of zeroes of a section. We denote it $c_1(V)$.

Example 3.2.3. ¿What is c_1 of $\mathcal{O}(-1)$, the tautological bundle of $\mathbb{P}^1$? We know that $\mathcal{O}(-1)$ possesses no non-zero sections. But this is not a problem, as we may use the fact that

$$c_1(L \otimes L') = c_1(L) + c_1(L')$$

so we can tensor $\mathcal{O}(-1)\otimes\mathcal{O}(d)$ for a sufficiently large d. This will be a line bundle with enough sections (*hopefully*) so that c_1 makes sense here. In this case we define

$$c_1(\mathcal{O}(-1)) = c_1(\mathcal{O}(-1)\otimes\mathcal{O}(d)) - c_1(\mathcal{O}(d)).$$

More analytically, but not algebraically

Another way to get c_1, we have the sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 1$$

which only works in complex analytic geometry. So in terms of the long exact sequence of cohomology we have

$$\cdots \to H^1(V,\mathcal{O}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to \dots$$

where the first term *classifies* line bundles and the connecting morphism is more or less the first Chern class.

Higher Chern classes

Returning to our vector bundle $E \xrightarrow{\pi} V$ of rank $r > 1$, we follow Grothendieck's approach. We can associate E with the projective bundle $P(E)$ where the typical fiber is isomorphic to $\mathbb{P}^{r-1}$. Now $P(E)$ has itself a line bundle $\mathcal{O}(1)$ leading us to a Chern class $c_1(\mathcal{O}(1)) \in A^1(P(E))$ which is not in $A^1(V)$. Thus, we must relate $A^*(P(E))$ with $A^*(V)$.

Call $H = c_1(\mathcal{O}(1))$, the hyperplane class, to simplify notation. The Chow ring $A^*(P(E))$ forms a free, rank r $A^*(V)$-module with basis $\{1, H, \dots, H^{r-1}\}$. This modularity arises from a similar reasoning to the construction of the Chow ring of $\mathbb{P}^n$, which corresponds to the case when V is a point. Now the element H^r can be expressed as a linear combination of the basic elements:

$$H^r = c_1 H^{r-1} - c_2 H^{r-2} + \cdots \pm c_r, \quad \text{for some} \quad c_i \in A^i(V).$$

¡These coefficients are the Chern classes in $A^*(V)$! The class c_r can be described easily. It is represented by a cycle of the zeroes of a section of E. The zeroes of that section will generally have codimension r and we will be able to represent it by c_r.

Anticlimatically, we will call it the Euler class of our vector bundle. The reason we name it is because it's the only other Chern class that we can naturally give a description of.

Definition 3.2.4. For a vector bundle $E \xrightarrow{\pi} V$ of rank $r \geqslant 1$, the Euler class is

$$e(E) := c_r(E) = [\mathrm{div}(s)],$$

the class of a divisor of a section.

Remark 3.2.5. ¡This is familiar! It generalizes the case of line bundles when $r=1$ as $e=c_1$. Recall that $c_1(\mathcal{O}(-1))$ is $-[\text{pt.}]$ but we could pick a section and the cycle will change. If we pick a section with 20 poles and 19 zeroes then the divisor of that section is also c_1.

3.3 Intersections and the role of the Euler class

This will summarize my meeting with Renzo from two weeks ago, on 20241001.

Transverse Intersections

Definition 3.3.1. Let $X,Y \subseteq Z$. We say that X and Y intersect transversely at a point $p \in X \cap Y$ if

$$T_pZ = T_pX + T_pY.$$

Example 3.3.2. Consider X as a line and Y as a plane in $\mathbb{R}^3$ such that X is not contained in Y (but not parallel to it). In this case, X and Y intersect transversely since their tangent spaces, $\mathbb{R}$ and $\mathbb{R}^2$ respectively, sum to $\mathbb{R}^3$ at the intersection point.

Note that the sum of the tangent spaces need not be a direct sum, as illustrated in the following example:

Example 3.3.3. Let π_1 and π_2 be two planes in $\mathbb{R}^3$. Their intersection is a line. Each plane contributes two independent directions to their tangent spaces. Together, these directions span all of $\mathbb{R}^3$, meaning that the intersection is transverse.

We also have the following result:

Proposition 3.3.4. *If X and Y intersect transversely, then*

$$[X\cap Y] = [X]\cdot[Y] \quad \textit{and} \quad \operatorname{codim}(X\cap Y) = \operatorname{codim}(X) + \operatorname{codim}(Y).$$

Non-transverse intersection

Let's begin with an example which lies in our minds to introduce non-transverse intersections.

Example 3.3.5. Consider the plane and line defined by

$$\begin{cases} \pi = \{3x+4y+6z=12\} = \{s(4,0,-2)+t(0,3,-2)+(0,0,2)\}. \\ \ell = \{t(4,0,-2)+(0,0,2)\}. \end{cases}$$

We have that $\ell \subseteq \pi$. So to follow the previous idea in transverse intersections we have two choices:

- ◇ Think like a topologist and wiggle ℓ until becomes transverse with X, but watch out, we wouldn't like to wiggle it completely inside of X.
- ◇ Think like an algebraic geometer and replace ℓ with an equivalent transverse cycle. This idea uses Chow's moving lemma!

The smart topologist wishes to wiggle ℓ outside of π, this is done through the normal bundle of π inside of $\mathbb{R}^3$. Observe quickly that this bundle is the collection of normal spaces of points of the plane. For $P \in \pi$ we have

$$N_P(\pi) = \{x \in T_P(\mathbb{R}^3) : \langle x|y\rangle = 0, \quad y \in T_P(\pi)\}$$

where $T_P(\mathbb{R}^3)$ is the whole of $\mathbb{R}^3$ and $T_P(\pi)$ is π but parametrized as

$$T_P(\pi) = \{u(4,0,-2) + v(0,3,-2) + P\}.$$

In any case the normal space is spanned by the vector $(3,4,6)$ so that at point P, the normal space is $\{t(3,4,6) + P\}$. Restricting it to ℓ we obtain the wiggle room, the normal bundle

$$N_{\pi \subseteq \mathbb{R}^3}|_{\ell} = \{N_P(\pi) : P \in \ell\}.$$

Wiggling ℓ across the normal bundle gives us a *section* of the normal bundle. First of all, finish

Chapter 4

Graphs and Chapter 1 of *The Green Book*

4.1 Dual graphs

Let us recall an important definition:

Definition 4.1.1. A <u>stratum</u> in $\overline{M}_{g,n}$ is the closure of the set of all curves with the same topological type. Strata are classified by dual trees.

Example 4.1.2. In $\overline{M}_{0,6}$, the fundamental class $1 \in A^0(M_{0,6})$ is the class of all curves homeomorphic to a 6-pointed $\mathbb{P}^1$. It can be represented by the dual graph:

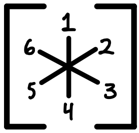


Figure 4.1: Fundamental class representative of $\overline{M}_{0,6}$

The top stratum in $\overline{M}_{0,6}$ contains only one topological type of curve as all other 6-pointed $\mathbb{P}^1$'s will be homeomorphic to this one.

Example 4.1.3. Going into the boundary of $\overline{M}_{0,6}$ we can find two different types of strata, for example From the stratum on the left,

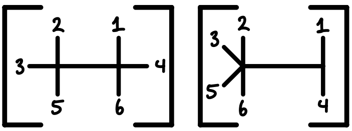


Figure 4.2: Examples of boundary strata of $\overline{M}_{0,6}$

we can find 9 other strata with the same graph design but different labelings. This is because there are

$$\frac{1}{2}\binom{6}{3,3} = \frac{6!}{2\cdot 3!\cdot 3!} = 10$$

ways to label the tree. The factor of $\frac{1}{2}$ accounts for the symmetry in the tree.

For the one on the right we can find $\binom{6}{4,2} = 15$ labelings.

Definition 4.1.4. Boundary cycles of codimension one in $\overline{M}_{g,n}$ are called <u>boundary divisors</u>. These can be denoted $D(I\,|\,J)$ where I,J is a partition of $[n]$ and $|I|,|J| \geqslant 2$.

Example 4.1.5. For simplicity, the previous boundary divisors can be named $D(235\,|\,146)$ and $D(2356\,|\,14)$.

Proposition 4.1.6. *There are*

$$\frac{1}{2}\sum_{k=2}^{n-2}\binom{n}{k} = 2^{n-1} - n - 1$$

irreducible boundary divisors in $\overline{M}_{0,n}$.

Proof

A partition into two sets of $[n]$ of size k and $n-k$ can be done in $\binom{n}{k,n-k}$ ways. Observe that this coefficient is precisely $\binom{n}{k}$. We need to divide by 2 because $\binom{n}{k} = \binom{n}{n-k}$. And as we need $2 \leqslant k \leqslant n-2$ for the curve to be stable, we must sum over all of those possibilities.

Products of cycles as graphs

In order to find the intersection product of two cycles we do the following:

(a) Find the dual graphs corresponding to the cycles.
(b) Paint the edges of both graphs with *oriented* colorings c_1, c_2.
(c) Compute the codimension of the product, recall codimension is the number of edges.
(d) The resulting graphs, should have no more edges than the expected codimension.
(e) Consider all graphs which contain the factors' dual graphs as minors[1] at the same time.

[1]The current lingo in geometry is specialize.

(f) Color the edges of the resulting graphs such that when collapsing edges, the original coloring of a factor is obtained. No edge of a resulting graph may be left **uncolored**.
(g) If an edge is bicolored multiply the corresponding ψ-classes.
(h) Count number of automorphisms of the graph and divide the class by that number.
(i) Count the number of possible colorings which respect the conditions and multiply the class by that number.

Remark 4.1.7. When we think we have a graph but it doesn't match the conditions in the end, we say that it is a non-generic intersection. That might occur because it leaves in a lower codimensional strata.

In summary, the formula for the intersection product is as follows:

Proposition 4.1.8. *For two curves C_1,C_2 whose dual graphs are Γ_1,Γ_2 we have that the class of their intersection product has dual graph:*

$$\Gamma_1 \cdot \Gamma_2 = \sum_{\Gamma \supseteq \Gamma_1, \Gamma_2} \frac{(\#\text{ colorings})}{|\mathrm{Aut}(\Gamma)|} \cdot \Gamma \cdot \prod_i (-\psi_{\bullet i} - \psi_{\star i}),$$

where the sum runs through graphs Γ containing Γ_1, Γ_2 as minors, the number of colorings is the amount of colorings which respect the edge-contraction process, and the product of ψ-classes runs through the amount of bicolored edges.

Computations of Intersection Products

This becomes clearer when doing examples:

What I hear, I forget;
When I see, I remember;
When I do, I learn.
-Some Chinese fella

Example 4.1.9. Let's begin with a sanity check. Consider the product in $\overline{M}_{0,4}$ of the fundamental class with $D(ac|bd)$. The classes in graph form are the following:Add drawing.

Let's proceed by following the algorithm:

- ◇ We have found the graphs and now we color. The fundamental class has no edges, we do not color. Whereas $D(ac|bd)$ has one edge which we choose to color as we showed.
- ◇ The codimension of this product should be 1. There's only one edge in total and the result shouldn't have more than one edge.

- Many graphs contain our factors as minors:Add drawing.
  But, as in the case of this one, they might not be stable. And even if they were, observe that a graph like this one has 3 edges, 2 of which we can't color. Recall, the cases when the graph is *actually stable* but it can't be colored it's a non-generic intersection.
  The only graph which contains then, both of our factors is the dual graph of $D(ac\,|\,bd)$. It can only be colored in the way that respects the coloring of the factor.
- No edges are multicolored and in this case there's no automorphisms nor different colorings.

Therefore it happens that:

$$1 \cdot D(ac\,|\,bd) = D(ac\,|\,bd)$$

as expected.

Example 4.1.10. Consider now the product $D(ac\,|\,bd)^2$ in $\overline{M}_{0,4}$. As this boundary divisor has codimension 1, its square should have codimension 2.

¡But $\overline{M}_{0,4}$ has only one dimension! It must follow that the product is empty.

To see this via the algorithm, we consider all graphs containing $D(ac\,|\,bd)$ as a minor. There's no graphs with two edges as that violates the stability condition (all vertices should have at least 3 special points), and so the only possible graph is a copy of $D(ac\,|\,bd)$ with its only edge bicolored.Add drawing.

Label the marks $*$ and $\cdot$ on the edge of $D(ac\,|\,bd)$ so that we may see this class as the image of the gluing morphism

$$gl : M_{0,\{a,c,*\}} \times M_{0,\{b,d,\bullet\}} \to M_{0,4}.$$

Then the ψ-classes we muliply are $(-\psi_{\bullet} - \psi_*)$ so that the result is $D(ac\,|\,bd) \cdot (-\psi_{\bullet} - \psi_*)$.Add drawings

> *Question.* How was the process of saying that this intersection was zero by distributing the psi class into the $M_{0,3}$'s or something?

Let's go up some dimensions

Example 4.1.11. Consider the intersection product in $\overline{M}_{0,6}$:

$$D(abc\,|\,def)D(ab\,|\,cdef)^2$$

The corresponding dual graphs are Add drawings

The expected codimension of the result is 3 as every factor contributes 1 codimension.

We first multiply a copy of $D(ab\,|\,cdef)$ with $D(abc\,|\,def)$:add drawing with edges colored.

Graphs which contain these two as minors are scarce. In fact, ¡there's only one: $D(ab|c|def)$!add drawing

We get only one copy of our resulting graph with the coloring already determined. This graph has no non-trivial automorphisms and so we only get $D(ab|c|def)$ which we now have to multiply by another copy of $D(ab|cdef)$.add drawing with edges colored.

We expect the resulting graphs to have three edges as that would have codimension 3. But take for example $D(ab \mid c \mid d \mid ef)$. add drawing with edges colored.

Such graph has three edges but **they can't be colored**. When collapsing edges, the colors disappear so that we would have to leave the edge between d and ef uncolored to respect the coloring.Add drawing

This cannot happen as is the case for any other possibilities, they will be non-generic intersections. So $D(ab|c|def)$ is the only possible factor. The edge between ab component and the c component will be bicolored in a way that we have to multiply ψ-classes. Labeling the vertices attached to the edge as $\cdot$ and $\star$ we get

$$D(ab|cdef)^2 D(abc|def) = D(ab|c|def)(-\psi_{\bullet} - \psi_{\star}).$$

This class has the correct codimension as the graph has two edges and the ψ class adds another codimension.

Question. ¿Is there a way to concretely determine whether a strata belongs in another one only by looking at their dual graphs?

Example 4.1.12 ($(4,0)$ instead of $(0,4)$)**.** Consider the curves represented by the following dual graphs:add drawing

We have already colored them and now, very carefully, look at graphs which contain these ones as minors.

(a) The first graph itself has the second factor as a minor. We can color it in the following way:add drawing with edges colored.

(b) The following is a path graph which might appear to have many symmetries, but when imposing the coloring, we are left with only one copy of its class: add drawing

This is because the two possible colorings and the reflection symmetry across the vertical axis cancel each other out. So in total we get a coefficient of $\frac{1}{2} \cdot 2 = 1$ accompanying it.

(c) Finally we get a star graph with a reflection across the axis which has the higher genus vertex.add drawing

In the same fashion as before, that symmetry cancels the two possible colorings leaving us with only one copy of its class.

There are no more graphs satisfying the required conditions[2]. As the first term had a bicolored edge, we multiply by the corresponding ψ class and write down the answer as follows: add drawing.

Example 4.1.13 (Unexpectedly zero...)**.** The following graph represents a curve in $\overline{M}_{1,2}$:add drawing

Let us take its second power and color each factor's edge green and blue respectively:add drawing w edges colored

Following the process we now look for graphs which have this graph as a minor.

- The same graph certainly contains itself. But we must multiply by a ψ class to get the correct codimensionadd drawing w edges colored

  Counting possible colorings we get 4 (ask) colorings and 2 automorphisms for this graph.
- The other curve consists of a torus twice-pinched with a mark on each of the isomorphic $\mathbb{P}^1$ components. The dual graph looks like this:add drawing w edges colored

  It has quite a bunch of colorings, 8 and only two automorphisms.

So in total, the square of this curve can be represented by the sum of classes:add drawings

¿Why did I say that the previous class in example 4.1.13 is unexpectedly zero? It all has to do with...

¿Do I move the lambda class example from MattAndRenzo here? Do I add a "other cohomology classes" section and then the MattAndRenzo example?

4.2 Integration

My reference for this definition is Fulton and Pandharipande[9].

Definition 4.2.1 ([9] pg. 2)**.** For a complete In which sense? Cauchy sequences? Is it a Cauchy space[3] variety, $c \in A^*(X)$ and $\beta \in A_k(X)$ then

$$\int_\beta c = \deg(c_k, \beta)$$

where c_k is the component of c in $A^k(X)$ and (c_k, β) is the evaluation of c_k on β giving us a zero cycle. When V is a closed, pure-dimensional

[2] And if you claim to find one, try checking if you actually left edges uncolored or if it has the correct codimension.

[3] insane (clicky clicky)

can a class β not be closed or pure-dimensional? subvariety of X, then we write

$$\int_V c \quad \text{instead of} \quad \int_{[V]} c.$$

Let us at once clear the idea of the squares-to-zero example 4.1.13:

Example 4.2.2. Let α by the curve in question for example 4.1.13. We are interested in computing

$$\int_{\overline{M}_{1,2}} \alpha^2.$$

We have that

$$\alpha^2 = 2\alpha(-\psi_{\bullet 1} - \psi_{\star 1}) + 4\beta.$$

Recall that the dimension of $\overline{M}_{1,2}$ is

4.3 Incursion into Pseudo-Stable Land

Theorem 3.1 in MattAndRenzo [7] deals with the Mumford formula. This is the product of the total Chern class of the Hodge bundle over $\overline{M}_{g,n}$ with the one of the dual. The formula itself is

$$(1+\hat{\lambda}_1+\cdots+\hat{\lambda}_g)(1-\hat{\lambda}_1+\cdots+(-1)^g\hat{\lambda}_g) = \sum_{i=0}^{g} \frac{1}{i!} \mathcal{G}_*^i \left(\prod_{j=1}^{i} (\psi_{\star j} - \psi_{\bullet j}) \right).$$

To prove the formula we will do examples first. The first case that we deal with is in $\overline{M}_{1,n}$ for a fixed $n \geqslant 1$.

Example 4.3.1. The pseudo stable Mumford formula in this case states:

$$(1+\hat{\lambda}_1)(1-\hat{\lambda}_1) = \frac{1}{0!}\mathcal{G}_*^0(\mathrm{id}) + \frac{1}{1!}\mathcal{G}_*(\psi_{\star 1} - \psi_{\bullet 1}).$$

Let us analyze the left side. By definition, the pseudo-stable lambda class is

$$\hat{\lambda}_j = \mathcal{T}^*(\lambda_j) = \sum_{i=0}^{j} \frac{1}{i!} \mathcal{G}_*^i(p_0^*(\lambda_{j-i}))$$

so in the case of $\hat{\lambda}_1$ we have

$$\hat{\lambda}_1 = \sum_{i=0}^{1} (\ldots) = \mathcal{G}_*^0(p_0^*(\lambda_{1-0})) + \mathcal{G}_*^1(p_0^*(\lambda_{1-1}))$$

where $\mathcal{G}^0$ is the identity map and λ_0 is the fundamental class of $A^*(\overline{M}_{g,n})$. So using this we have

$$\begin{aligned} \hat{\lambda}_1 &= \mathcal{G}_*^0(p_0^*(\lambda_1)) + \mathcal{G}_*^1(p_0^*(\lambda_0)) \\ &= \lambda_1 + \mathcal{G}_*^1(\mathrm{id}) \end{aligned}$$

and expanding the product we get

$$\begin{aligned}(1+\hat{\lambda}_1)(1-\hat{\lambda}_1) &= (1+\lambda_1+\mathcal{G}^1_*(\mathrm{id}))(1-\lambda_1-\mathcal{G}^1_*(\mathrm{id}))\\ &= (\Lambda_1(1)+\mathcal{G}^1_*(\mathrm{id}))(\Lambda_1(-1)-\mathcal{G}^1_*(\mathrm{id}))\\ &= \Lambda_1(1)\Lambda_1(-1)-\Lambda_1(1)\mathcal{G}^1_*(\mathrm{id})+\mathcal{G}^1_*(\mathrm{id})\Lambda_1(-1)-\left(\mathcal{G}^1_*(\mathrm{id})\right)^2.\end{aligned}$$

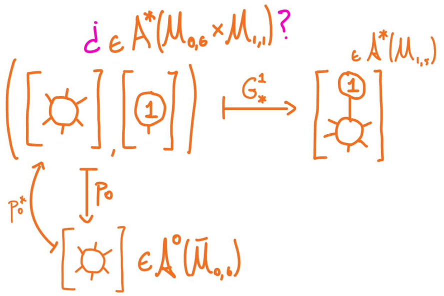
Here, Λ_1 represents the total Chern class. We may analyze term by term this expression:

- ◇ The product $\Lambda_1(1)\Lambda_1(-1)=1$ is the usual Mumford formula.
- ◇ For the case $\Lambda_1(1)\mathcal{G}^1_*(\mathrm{id})$ what we have is $(1+\lambda_1)\mathcal{G}^1_*(\mathrm{id})$.

We now arrive to the question of what is $\mathcal{G}^1_*(\mathrm{id})$ so let's take a step back and remember how $\mathcal{G}^1$ works as a map:

$$\mathcal{G}^1 : \overline{M}_{(1-1),n+1}\times\overline{M}_{1,1}\to\overline{M}_{1,n},$$

and recall that the id in the argument is $p_0^*(\lambda_0)$ where the λ_0 comes from the Chow ring of $\overline{M}_{0,n+1}$. In particular, it is the fundamental class of the space. Pulling it back just makes it part of an ordered pair and then pushing it forwards attaches it to a copy of an elliptic curve. Graphically: This means that the pushforward of the fundamental



class through $\mathcal{G}^1$ is the class of curves described by the dual graph on the right of the diagram.[4]

From this, we have

$$\begin{aligned}(1+\lambda_1)\mathcal{G}^1_*(\mathrm{id}) &= \mathcal{G}^1_*(\mathrm{id})+\lambda_1\mathcal{G}^1_*(\mathrm{id})\\ &= \mathcal{G}^1_*(\mathrm{id})+\mathcal{G}^1_*(p_1^*(\lambda_1))+\mathcal{G}^1_*(p_0^*(\lambda_1))\\ &= \mathcal{G}^1_*(\mathrm{id})+\mathcal{G}^1_*(p_1^*(\lambda_1))+0.\end{aligned}$$

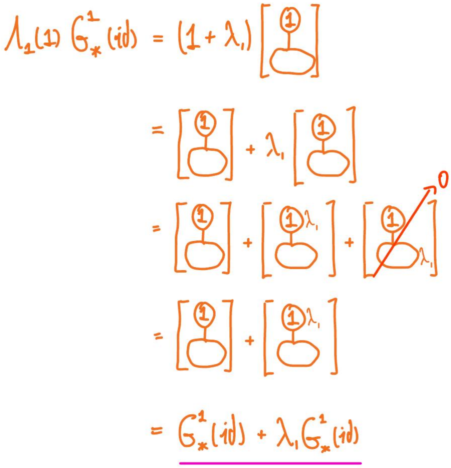
This quantity should be equal to $\mathcal{G}^1_*(p_1^*(\Lambda_1(1)))$, and expanding this we have

$$\begin{aligned}\mathcal{G}^1_*(p_1^*(\Lambda_1(1))) &= \mathcal{G}^1_*(p_1^*(1+\lambda_1))\\ &= \mathcal{G}^1_*(p_1^*(1))+\mathcal{G}^1_*(p_1^*(\lambda_1))\\ &= \mathcal{G}^1_*(1)+\mathcal{G}^1_*(p_1^*(\lambda_1)),\end{aligned}$$

so both quantities are equal.

[4] This sentence sounds a bit fishy. I just remember what you said about "I can't pick a concrete representative".

Diagramatically we have



The calculation for $\mathcal{G}^1_*(\mathrm{id})\Lambda_1(-1)$ is similar but different signs. Finally calculating $\mathcal{G}^1_*(\mathrm{id})^2$ amounts to finding a self intersection number.

Question. I remember how to find $E \cap E$ in $\mathcal{B}\ell_{\mathrm{pt}}\mathbb{P}^2$ by deforming a cycle that goes through E. ¿Is the technique to find $\mathcal{G}^1_*(\mathrm{id}) \cap \mathcal{G}^1_*(\mathrm{id})$ similar?

4.4 A reconsideration on the Mumford formula

Chapter 5

A bit of Algebra

5.1 Schemes

I'm lacking a bit on the side of what schemes are. The definition we have is as follows:

Definition 5.1.1. An affine scheme is a topological space X which is isomorphic to an irreducible algebraic subset of k^n, together with a sheaf of k-valued functions $\mathcal{O}_X$.

All of the words here make sense, right? So let's do some examples:

Example 5.1.2. Consider the space $\mathrm{Spec}\mathbb{Z}$

Chapter 6

II2025 - The space of maps

6.1 The Faber-Pandharipande formula for multiple covers

We are interested in calculating the Gromov-Witten invariant[1]

$$\int_{[\overline{\mathcal{M}}_{g,0}(X,d\beta)]^{\mathrm{vir}}} 1.$$

Here, X is a Calabi-Yau threefold and β represents the class of a line. The moduli space then parametrizes maps from $\overline{\mathcal{M}}_g$ to a Calabi-Yau which land on a line and such map has degree d.

To this effect, we first deal with the examples in genus 0 known as the Aspinwall-Morrison formula. We illustrate this via examples.

Degree 3

In degree 3 and genus 0 we have

$$\int_{[\overline{\mathcal{M}}_{0,0}(X,3\beta)]^{\mathrm{vir}}} 1 = (\text{deg. } 3 \text{ inmersions}) + (\text{deg. } 3 \text{ covers of } \beta).$$

The part that we may calculate consists of the triple covers of the line β. Such integral is actually

$$\int_{\overline{M}_{0,0}(\mathbb{P}^1,3)} 1$$

and the (virtual) fundamental class is $e(\mathrm{Ob})$ as our moduli space is non-singular. This comes from section 26.1.2 of the Mirror Symmetry book [13]. For a map $[(C,f)] \in \mathcal{M}$ (we now abbreviate the moduli space into $\mathcal{M}$), we have the following tangent-normal sequence

$$0 \to TC \hookrightarrow f^*TX \twoheadrightarrow N_{C|X} \to 0$$

[1]Whatever a GW invariant is, one of its many manisfestations is as an integral.

which, after applying $H^*(C,-)$, becomes

$$\begin{tikzcd} H^0(C,TC) \arrow[r] & H^0(C,f^*TX) \arrow[r] & H^0(C,N_{C|X}) \arrow[dll, rounded corners] \\ H^1(C,TC) \arrow[r] & H^1(C,f^*TX) \arrow[r] & H^1(C,N_{C|X}) \arrow[dll, rounded corners] \\ H^2(C,TC)=0 & & \end{tikzcd}$$

Via deformation theory, this sequence is actually

$$\begin{tikzcd} \mathrm{Aut}(C) \arrow[r] & \mathrm{Def}(f) \arrow[r] & \mathrm{Def}(C,f) \arrow[dll, rounded corners] \\ \mathrm{Def}(C) \arrow[r] & \mathrm{Ob}(f) \arrow[r] & \mathrm{Ob}(C,f) \end{tikzcd}$$

And lemma 24.4.3 of [13] guarantees that for our map $f:C\to\mathbb{P}^1$ we have $h^1(C,f^*T\mathbb{P}^1)=0$ because C is a genus 0 curve. In consequence

$$\mathrm{Ob}(f)\simeq H^1(C,N_{C|X})$$

and in order for this to be a bundle over our source, we pull it back via f.

Proposition 6.1.1. *Given our conditions, X being Calabi-Yau, the bundle $N_{C|X}$ is a rank 2 vector bundle of degree -2.*

Proof

The rank can be seen inmediately to be two as the dimension of C inside X is 1, so it has codimension 2. For the degree, using the same tangent-normal sequence for $f:C\to X$ we have

$$0\to TC\hookrightarrow f^*TX\twoheadrightarrow N_{C|X}\to 0$$

and $TC\simeq\mathcal{O}(2)$. Taking first Chern class we get

$$c_1(f^*TX)=c_1(TC)+c_1(N_{C|X}).$$

As X is Calabi-Yau, $c_1(TX)=0$ and so

$$0=2+c_1(N_{C|X})$$

so that the degree of $N_{C|X}$ is -2, the degree of its first Chern class.

Such a bundle splits into a direct sum

$$N_{C|X}\simeq L_1\oplus L_2=\mathcal{O}(d_1)\oplus\mathcal{O}(d_2)$$

and being of degree -2 means that $d_1+d_2=-2$. A theorem due to Grothendieck[2] states that such distribution must be as balanced as possible so that $d_1=d_2=-1$. In conclusion

$$e(\mathrm{Ob})=e(H^1(C,f^*N_{C|X}))=e(H^1(C,f^*\mathcal{O}(-1)^2)).$$

[2] God knows which

Our target integral along with the expected result via Aspinwall-Morrison's result is

$$\int_{\overline{M}_{0,0}(\mathbb{P}^1,3)} e(H^1(C,f^*\mathcal{O}(-1)^2)) = \frac{1}{3^3}.$$

Via localization, as the torus action $t \cdot x = tx$ extends naturally from $\mathbb{P}^1$ to $\mathcal{M}$, we have 7 fixed loci:

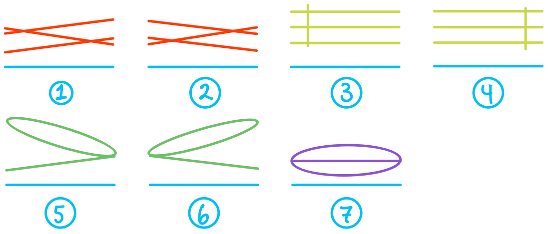


Figure 6.1: Fixed loci of $\mathcal{M}$ (uhhhh)

The calculation for odd numbered fixed loci is the same as even numbered so we only do the calculation for the odd ones.

Fixed locus 1: Consider the first fixed loci representing a rational curve with two nodes, each $\mathbb{P}^1$ maps with degree 1 to the base. The normalization sequence for this curve is

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \oplus \mathcal{O}_{C_3} \to \mathbb{C}_{n_1} \oplus \mathbb{C}_{n_2} \to 0$$

where $\mathcal{O}_{C_i}$ is the trivial sheaf of the i^{th} component of the curve C. The $\mathbb{C}_{n_i}$ correspond to skyscraper sheaves over the nodes. The numbering of the normalization sequence goes from bottom to top.

Question. What are the maps in this normalization sequence?

Tensoring the sequence via $- \otimes f^*\mathcal{O}(-H)$ we obtain

$$0 \to f^*\mathcal{O}(-H) \to \mathcal{O}(-H)^3 \to \langle(0)\rangle \oplus \langle(-t)\rangle \to 0$$

and then taking the long exact sequence in cohomology $H^*(C,-)$ we get

$$\begin{tikzcd} 0 \arrow[r] & H^0(f^*\mathcal{O}(-H)) \arrow[r] & H^0(\mathcal{O}(-H))^3 \arrow[r] & H^0(\langle(-t)\rangle) \oplus H^0(\langle(0)\rangle) \arrow[dll, rounded corners] \\ & H^1(f^*\mathcal{O}(-H)) \arrow[r] & H^1(\mathcal{O}(-H))^3 \arrow[r] & 0 \end{tikzcd}$$

Observe that the first two terms are zero because the bundle is negative. By Serre duality:

$$h^1(\mathcal{O}(-1)) = h^0(K_{\mathbb{P}^1} \otimes \mathcal{O}(-1)^\vee) = h^0(\mathcal{O}(-2) \otimes \mathcal{O}(1)) = h^0(\mathcal{O}(-1)) = 0.$$

Finally the last term corresponding to the H^1 of the skyscrapers is zero because of dimension vanishing. The sequence becomes

$$0 \to H^0(\langle(-t)\rangle) \oplus H^0(\langle(0)\rangle) \to H^1(f^*\mathcal{O}(-H)) \to 0$$

and taking Euler classes returns

$$e(H^1(f^*\mathcal{O}(-H))) = e(H^0(\langle(-t)\rangle))e(H^0(\langle(0)\rangle)) = (-t)(0).$$

Immediately the integral is zero for this and the second fixed loci. It is possible to calculate its normal bundle's Euler class, but we reserve that for the next maps.

Fixed locus 3:

The normalization sequence is now

$$0 \to \mathcal{O}_C \to \bigoplus_{i=1}^{4} \mathcal{O}_{C_i} \to \bigoplus_{i=1}^{3} \mathbb{C}_{n_i} \to 0$$

Here C_4 is the curve connecting the other 3 $\mathbb{P}^1$'s. Tensoring $- \otimes f^*\mathcal{O}(-H)$ we obtain

$$0 \to f^*\mathcal{O}(-H) \to \mathcal{O}(-H)^3 \oplus \mathcal{O}(-t) \to \langle(-t)\rangle^3 \to 0$$

where $\mathcal{O}(-t)$ is the copy of the fiber above 0 at every point of the collapsed $\mathbb{P}^1$. The pullback takes a copy of the fiber over zero and puts it over every point of that $\mathbb{P}^1$ giving us the trivial sheaf in question.

Taking now long exact sequence in cohomology $H^*(C,-)$ we obtain

$$\begin{aligned} 0 \to H^0(f^*\mathcal{O}(-H)) \to H^0(\mathcal{O}(-H))^3 \oplus H^0(\mathcal{O}(-t)) \to H^0(\langle(-t)\rangle)^3 \\ \to H^1(f^*\mathcal{O}(-H)) \to H^1(\mathcal{O}(-H))^3 \oplus H^1(\mathcal{O}(-t)) \longrightarrow 0 \end{aligned}$$

Cancelling some terms we are left with

$$0 \to H^0(\mathcal{O}(-t)) \to H^0(\langle(-t)\rangle)^3 \to H^1(f^*\mathcal{O}(-H)) \to 0.$$

The first cohomology of the trivial bundle cancels because by Serre duality it has dimension $h^0(\mathcal{O}(-2)) = 0$. The Euler class is thus

$$H^1(f^*\mathcal{O}(-H)) = \frac{(-t)^3}{-t} = t^2$$

however when linearizing with $\mathcal{O}(-H+t)$ all the weights in question become zero. So that the numerator ends up being zero altogether. Same reasoning applies to the fourth fixed loci.

Fixed locus 5:

The normalization sequence is

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathbb{C}_n \to 0$$

and observe that when pulling back $\mathcal{O}(-H)$ to C_2 it pulls back with the same weights $-t$ and 0. With respect to C_2's hyperplane class, say $\widetilde{H}$ whose weights for $\mathcal{O}(-\widetilde{H})$ are $\frac{-t}{2}$ and 0, the pullback becomes $\mathcal{O}(-2\widetilde{H})$. So the result of applying $- \otimes f^*\mathcal{O}(-H)$ is

$$0 \to f^*\mathcal{O}(-H) \to \mathcal{O}(-H) \oplus \mathcal{O}(-2\widetilde{H}) \to \langle(0)\rangle \to 0.$$

We then take long exact sequence in cohomology $H^*(C,-)$:

$$\begin{aligned} 0 \to H^0(f^*\mathcal{O}(-H)) \to H^0(\mathcal{O}(-H)) \oplus H^0(\mathcal{O}(-2\widetilde{H})) \to H^0(\langle(0)\rangle) \\ \to H^1(f^*\mathcal{O}(-H)) \to H^1(\mathcal{O}(-H)) \oplus H^1(\mathcal{O}(-2\widetilde{H})) \longrightarrow 0 \end{aligned}$$

Immediately the first two terms cancel out because of negative degree and one H^1 cancels due to Serre duality. We are left with the sequence
$$0 \to H^0(\langle(0)\rangle) \to H^1(f^*\mathcal{O}(-H)) \to H^1(\mathcal{O}(-2\widetilde{H})) \to 0.$$
Observe that the middle section of $H^1(\mathcal{O}(-2\widetilde{H}))$ has weight $\frac{-t}{2}$. This comes from the formula
$$\begin{cases} w_0 - w_\infty = (\text{deg.})w_{0,TX} \Rightarrow w_0 - (\text{deg.})w_{0,TX} \\ -t - 0 = (-2)\left(\dfrac{t}{2}\right) \Rightarrow -t - (-2)\left(\dfrac{t}{2}\right) = 0. \end{cases}$$
The middle section then has weight
$$-t - (-1)\left(\frac{t}{2}\right) = -t + \frac{t}{2} = \frac{-t}{2}$$
Thus the whole Euler class for our desired bundle is
$$e(H^1(f^*\mathcal{O}(-H))) = (0)\left(\frac{-t}{2}\right) = 0.$$

Exercise 6.1.2. The Euler class for the other linearization $\mathcal{O}(-H+t)$ ends up being
$$e(H^1(f^*\mathcal{O}(-H+t))) = (t)\left(\frac{t}{2}\right).$$

Multiplying both classes returns 0 and thus the integral is zero. The remaining fixed locus is the only one with a non-zero contribution to the integral.

Fixed locus 7:

There's no need to apply a normalization sequence argument as there's only one component. We directly ask for $H^1(f^*(\mathcal{O}(-H)))$. In this case $\mathcal{O}(-H)$ pulls back to a $\mathcal{O}(-3\widetilde{H})$ where $\widetilde{H}$ is the hyperplane class of the curve C_7. $H^1(\mathcal{O}(-3\widetilde{H}))$ has **2** middle sections with weights
$$w_0 - w_{\tan}, \quad \text{and} \quad w_0 - 2w_{\tan}.$$
These are
$$-t - (-1)\left(\frac{t}{3}\right), \quad \text{and} \quad -t - (-2)\left(\frac{t}{3}\right)$$
so that the resulting Euler class is
$$e(H^1(f^*(\mathcal{O}(-H)))) = \left(\frac{-2t}{3}\right)\left(\frac{-t}{3}\right) = \frac{2t^2}{9}.$$
In an analogous fashion, $f^*\mathcal{O}(-H+t)$ coincides with C_7's $\mathcal{O}(-3\widetilde{H}+t)$. It's two middle sections have weights
$$w_\infty + w_{\tan}, \quad w_\infty + 2w_{\tan} \quad \text{which are} \quad t + (-1)\frac{t}{3}, \quad t + (-2)\frac{t}{3}$$

so that
$$i_7^*\big(e(H^1(f^*(\mathcal{O}(-H))))e(H^1(f^*(\mathcal{O}(-H+t))))\big)=\frac{2t^2}{9}\cdot\frac{2t^2}{9}=\frac{4t^4}{81}.$$

Remark 6.1.3. The key difference between this locus and the others is that the H^1 contribution only considers the *middle* sections which have non-zero weights. In the other cases, the extremes are taken, and one of those weights is zero.

We now proceed by considering the normal bundle of F_7 inside $\mathcal{M}$. The dual graph of C_7 consists of an edge with label 3 connecting two vertices. Such valence 1 vertices contribute each their corresponding tangent weight to automorphism bundle's Euler class:
$$e(\mathrm{Aut}(C_7))=\left(\frac{t}{3}\right)\left(\frac{-t}{3}\right)=\frac{-t^2}{9}.$$
The curve C_7 has no deformations as it is smooth. And finally looking at the deformations of the map
$$\mathrm{Def}(f)=H^0(C_7,f^*T\mathbb{P}^1)$$
which we identify to $\mathcal{O}(2\cdot3\widetilde{H}-t)$ with respect to $\widetilde{H}$. This $\mathcal{O}(6)$ has 7 global sections with weights $w_0-kw_{\tan}$ with $0\leqslant k\leqslant\deg\mathcal{O}(6)$, these are
$$t,\frac{2t}{3},\frac{t}{3},0,-\frac{t}{3},-\frac{2t}{3},-t.$$
The one with zero weight corresponds to a fixed section so we don't take that into account of the *moving part* of the Euler class. We get
$$e(\mathrm{Def}(f_7))=\frac{-4t^6}{81}\Rightarrow e(N)=\frac{4t^6}{81}\left(\frac{-t^2}{9}\right)^{-1}=\frac{4t^4}{27}.$$

Remark 6.1.4. We have that $e(N)$'s degree coincides with the codimension of F_7 inside of $\mathcal{M}$. The dimension of $\overline{M}_{0,0}(\mathbb{P}^1,3)$ is
$$(1+1)(3+1)+(0-3)-1=8-3-1=4$$
which coincides with our observation that F_7 is a point (or a zero-dimensional scheme) inside $\mathcal{M}$.

Finally putting this together we have that
$$\int_{\overline{M}_{0,0}(\mathbb{P}^1,3)}e(\mathrm{Ob})=0+\cdots+0+\frac{1}{3}\int_{F_7}\frac{i_7^*(e\cdot e)}{e(N_{F_7|\mathcal{M}})}=\frac{4t^4/81}{4t^4/27}=\frac{1}{3}\cdot\frac{1}{9}=\frac{1}{27}$$
as expected.

6.2 My (?) theorem

In the pseudostable case, we can calculate the same GW invariant, just that now the calculation is a bit different because of cusps.

$C^{\mathrm{ps}}(3,d)$

Let us compute the genus 3, degree d GW invariant:

$$\int_{[\overline{\mathcal{M}}^{\mathrm{ps}}_{3,0}(X,d\beta)]^{\mathrm{vir}}} 1.$$

Ross [17] conjectures that the fundamental class localizes in a similar fashion as to the non-pseudostable case. We have

$$\int_{[\overline{\mathcal{M}}^{\mathrm{ps}}_{3,0}(X,d\beta)]^{\mathrm{vir}}} 1 = \int_{[\overline{\mathcal{M}}^{\mathrm{ps}}_{3,0}(\mathbb{P}^1,d)]^{\mathrm{vir}}} e(H^1 f^*\mathcal{O}(-H))e(H^1 f^*\mathcal{O}(-H+t)).$$

The fixed maps consist of a $\mathbb{P}^1$ mapping as $z \mapsto z^d$ together with positive genus components collapsing down to either 0 or ∞. Such maps can be enumerated by ordered partitions of 3 with 2 parts, we have the fixed loci

$$F_{(3,0)}, \quad F_{(2,1)}, \quad F_{(1,2)}, \quad \text{and} \quad F_{(0,3)}.$$

The fixed locus $F_{(i,j)}$ denotes the nodal curve consisting of three components: one of genus i mapping to zero, the smooth component $\mathbb{P}^1$ and the genus j component mapping to infinity; this, along the information of the map. Let us analyze the contribution of $F_{(3,0)}$.

The $(3,0)$ contribution

Instead of pulling back $\mathcal{O}(-H)$, we pullback $\mathcal{O}(-H+s+t)$ and then specialize to the cases $s=0$ and $s=-t$. Let C be the curve in question, its normalization sequence is

$$0 \to \mathcal{O}_C \to \mathcal{O}_{g=3} \oplus \mathcal{O}_{\mathbb{P}^1} \to \underline{\mathbb{C}}_{\text{node}} \to 0$$

where $\mathcal{O}_{g=3}$ is the trivial sheaf over the genus 3 component. Tensoring with $f^*\mathcal{O}(-H+s+t)$ returns

$$0 \to f^*\mathcal{O}(-H+s+t) \to \mathcal{O}(s) \oplus \mathcal{O}(-d\widetilde{H}+s+t) \to \langle (s) \rangle \to 0.$$

Taking long exact sequence in cohomology via $H^*(C,-)$ gives us

$$\begin{tikzcd} 0 \arrow[r] & H^0(f^*\mathcal{O}(-H+s+t)) \arrow[r] & H^0(\mathcal{O}(s)) \oplus H^0(\mathcal{O}(-d\widetilde{H}+s+t)) \arrow[r] & H^0(\langle (s \arrow[dll, rounded corners] \\ & H^1(f^*\mathcal{O}(-H+s+t)) \arrow[r] & H^1(\mathcal{O}(s)) \oplus H^1(\mathcal{O}(-d\widetilde{H}+s+t)) \arrow[r] & 0 \end{tikzcd}$$

Observe that the two negative bundles have zero H^0, then taking Euler classes and cancelling $H^0(\mathcal{O}(s))$ with $H^0(\langle (s) \rangle)$'s we obtain

$$\begin{aligned} e\big(H^1(f^*\mathcal{O}(-H+s+t))\big) &= e\big(H^1(\mathcal{O}(s))\big) e\big(H^1(\mathcal{O}(-d\widetilde{H}+s+t))\big) \\ &= \big(s^3 + (-\lambda_1^{\mathrm{ps}})s^2 + \lambda_2^{\mathrm{ps}} s + (-\lambda_3^{\mathrm{ps}})\big) \left(\prod_{k=1}^{d-1} s + \frac{kt}{d} \right) \end{aligned}$$

where we used that

$$H^1(\mathcal{O}(s)) \simeq \mathbb{E}^\vee \otimes H^0(\mathcal{O}(s)),$$

via Serre duality. Specializing this quantity to $s=-t$ and $s=0$ and taking the product recovers the initial desired product pulledback to $F_{(3,0)}$. We have

$$\begin{aligned}
&i^*_{(3,0)}(e(H^1 f^*\mathcal{O}(-H))e(H^1 f^*\mathcal{O}(-H+t)))\\
&=(-\lambda_3^{\mathrm{ps}})\left(\prod_{k=1}^{d-1}\frac{kt}{d}\right)(-t^3-\lambda_1^{\mathrm{ps}}t^2-\lambda_2^{\mathrm{ps}}t-\lambda_3^{\mathrm{ps}})\left(\prod_{k=1}^{d-1}-t+\frac{kt}{d}\right)\\
&=\lambda_3^{\mathrm{ps}}\left(t^3+\lambda_1^{\mathrm{ps}}t^2+\lambda_2^{\mathrm{ps}}t+\lambda_3^{\mathrm{ps}}\right)(-1)^{d-1}\left(\frac{t}{d}\right)^{2d-2}(d-1)!^2.
\end{aligned}$$

Remark 6.2.1. We would like to use the notation $\Lambda_g^{\mathrm{ps}}(k)$ from [7] which is

$$\Lambda_g^{\mathrm{ps}}(k)=\sum_{i=0}^{g}k^i\lambda_i^{\mathrm{ps}}.$$

Evaluating at $k=1$ returns the total Chern class of the Hodge bundle and at -1, the dual's. Observe that the second quantity in our result is not exactly the total Chern class, but the *homogeneous* version.

Perhaps we should introduce new notation to be

$$\Lambda_g^{\mathrm{ps}}(k,\ell)=\sum_{i=0}^{g}k^i\lambda_i^{\mathrm{ps}}\ell^{g-i}$$

so that our quantity is $\Lambda_g^{\mathrm{ps}}(1,t)$. The quantity $\Lambda_g^{\mathrm{ps}}(k,\ell)$ can even be expressed as a convolution of two sequences

$$\Lambda_g^{\mathrm{ps}}(k,\ell)=\left((k^i\lambda_i^{\mathrm{ps}})_i*(\ell^i)_i\right)(g)$$

In particular, we may express the series whose general term is $\Lambda_g^{\mathrm{ps}}(k,\ell)$ as the product of two series:

$$\sum_{g=0}^{\infty}\Lambda_g^{\mathrm{ps}}(k,\ell)=\left(\sum_{g=0}^{\infty}k^g\lambda_g^{\mathrm{ps}}\right)\left(\sum_{g=0}^{\infty}\ell^g\right)=\frac{\Lambda_g^{\mathrm{ps}}(k)}{1-\ell}.$$

Indeed, the sum $\Lambda_g^{\mathrm{ps}}(k)$ is finite as there's no lambda class above the genus of the curve.

Remark 6.2.2 (20251202, after meeting). Observe the lack of need to use the notation $\Lambda_g^{\mathrm{ps}}(k,\ell)$. It suffices to redefine $\Lambda_g^{\mathrm{ps}}(k)$ as

$$\Lambda_g^{\mathrm{ps}}(k):=\sum_{i=0}^{g}k^{g-i}\lambda_i^{\mathrm{ps}}.$$

Then the old $\Lambda_g^{\mathrm{ps}}(1,t)$ is just the new $\Lambda_g^{\mathrm{ps}}(t)$. However the $\Lambda_g^{\mathrm{ps}}(-1,t)$ doesn't match up quite nicely with $\Lambda_g^{\mathrm{ps}}(-t)$. This last quantity is

$$(-1)^g t^g+(-1)^{g-1}t^{g-1}\lambda_1^{\mathrm{ps}}+\cdots+(-1)^0\lambda_g^{\mathrm{ps}}$$

so that multiplying the whole quantity by $(-1)^g$, we change this expression to

$$t^g-t^{g-1}\lambda_1+\cdots+(-1)^g\lambda_g=\Lambda_g^{\mathrm{ps}}(-1,t)$$

as desired. We will thus stick with this notation

The simplification of the products is done in the following fashion.

$$\prod_{k=1}^{d-1}\left(-t+\frac{kt}{d}\right)\left(\frac{kt}{d}\right)$$

$$=\left(\frac{t}{d}\right)^{2d-2}(d-1)!\prod_{k=1}^{d-1}-d+k.$$

Substituting $-\ell$ as $-d+k$ we note that the product becomes

$$\prod_{k=1}^{d-1}-d+k=\prod_{\ell=d+1}^{1}-\ell=(-1)^{d-1}(d-1)!$$

so that combining with the previous amount we get

$$(-1)^{d-1}\left(\frac{t}{d}\right)^{2d-2}(d-1)!^2$$

We are left with analyzing the normal bundle to $F_{(3,0)}$ inside the moduli space. As in the stable case, going through the tangent-normal sequence and using deformation theory to relate cohomology with the particular bundles we get

$$e(N_{(3,0)})=\frac{e(\mathrm{Def}(f))e(\mathrm{Def}(C))}{e(\mathrm{Aut}(C))e(\mathrm{Ob}(f))}$$

and each equivariant Euler class only contributes its moving part.

- ⋄ The dual graph of the curve has one valence one vertex, it corresponds to the only smooth point of the curve. It lies over infinity so its contribution to the automorphisms bundle is

$$e(\mathrm{Aut}(C))=-\frac{t}{d}.$$

- ⋄ In this case, the deformations of the curve come from the curve's normalization. Let P,Q be the points in positive genus and smooth components respectively which map to the node. The Euler class then is

$$e(\mathrm{Def}(C))=-\psi_P-\psi_Q=-\psi_P+\frac{t}{d}$$

  as the negative psi class corresponds to c_1 of the *tangent* bundle. So it is the tangent weight at the point in question.
- ⋄ The map's deformations and obstructions come from the same sequence. Tensoring the normalization sequence with $f^*T\mathbb{P}^1$ we get

$$0\to f^*T\mathbb{P}^1\to\mathcal{O}(t)\oplus\mathcal{O}(2d\widetilde{H}-t)\to\langle(t)\rangle\to 0.$$

Now the long sequence in cohomology contains the desired bundles as

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Def}(f) \arrow[r] & H^0(\mathcal{O}(t))\oplus H^0(\mathcal{O}(2d\widetilde{H}-t)) \arrow[r] & H^0(\langle(t)\rangle) \arrow[dll, rounded corners] \\
& \mathrm{Ob}(f) \arrow[r] & H^1(\mathcal{O}(t))\oplus H^1(\mathcal{O}(2d\widetilde{H}-t)) \arrow[r] & 0
\end{tikzcd}$$

Taking Euler classes, cancelling out terms and realizing the H^1 of a positive bundle is zero gives us

$$\frac{e(\mathrm{Def}(f))}{e(\mathrm{Ob}(f))} = \frac{e\big(H^0(\mathcal{O}(2d\widetilde{H}-t))\big)}{e\big(H^1(\mathcal{O}(t))\big)}.$$

$H^0(\mathcal{O}(2d\widetilde{H}-t))$ has $2d+1$ middle sections and the one right in the middle is fixed, so we get $2d$ contributions as

$$e\big(H^0(\mathcal{O}(2d\widetilde{H}-t))\big) = \prod_{\substack{k=0\\k\neq d}}^{2d} t-\frac{kt}{d} = \left(\frac{t}{d}\right)^{2d} d!^2(-1)^d.$$

The simplification for this product goes along similar lines:

$$\begin{aligned}
&\prod_{\substack{k=0\\k\neq d}}^{2d} t-\frac{kt}{d}\\
&=\left(\frac{t}{d}\right)^{2d}\prod_{\substack{k=0\\k\neq d}} d-k\\
&=\left(\frac{t}{d}\right)^{2d}\big(d\cdot(d-1)\cdots 1\cdot(-1)\cdot(-2)\cdots(-d)\big)\\
&=\left(\frac{t}{d}\right)^{2d} d!^2(-1)^d.
\end{aligned}$$

Again via Serre duality,

$$e\big(H^1(\mathcal{O}(t))\big) = t^3+(-\lambda_1^{\mathrm{ps}})t^2+\lambda_2^{\mathrm{ps}}t+(-\lambda_3^{\mathrm{ps}}) = (-1)^3\Lambda_3^{\mathrm{ps}}(-t).$$

We wrap everything up into

$$e(N_{(3,0)}) = \frac{\big[(t/d)^{2d}d!^2(-1)^d\big]\big[-\psi_P+t/d\big]}{\big[(-1)^3\Lambda_3^{\mathrm{ps}}(-t)\big]\big[-t/d\big]}.$$

and putting it together with the pullback of the Euler classes up top, we get

$$\frac{\big[(-1)^3\Lambda_3^{\mathrm{ps}}(-t)\big]\big[-t/d\big]\big[\lambda_3^{\mathrm{ps}}\Lambda_3^{\mathrm{ps}}(t)(-1)^{d-1}(t/d)^{2d-2}(d-1)!^2\big]}{\big[(t/d)^{2d}d!^2(-1)^d\big]\big[-\psi_P+t/d\big]}$$
$$=\frac{\big[(-1)^3\Lambda_3^{\mathrm{ps}}(-t)\big]\big[-t/d\big]\big[\lambda_3^{\mathrm{ps}}\Lambda_3^{\mathrm{ps}}(t)\big]}{\big[-t^2\big]\big[(t/d)(1-d\psi_P/t)\big]}$$
$$=\frac{\lambda_3^{\mathrm{ps}}\Lambda_3^{\mathrm{ps}}(t)(-1)^3\Lambda_3^{\mathrm{ps}}(-t)}{t^2(1-d\psi_P/t)}.$$

Remark 6.2.3. Let us observe that the degree of the class in question is
$$3+3+3-2=7$$
which coincides with $\dim\overline{\mathcal{M}}_{3,1}^{\mathrm{ps}}=3(3)-3+1=7$, the dimension of the fixed locus $F_{(3,0)}$.

In the previous observation we have used the fact that $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ has the same dimension as its stable counterpart. This is mentioned in [5], pp.5. We won't limit ourselves to only using this fact, we need two more results to further the calculation.

Theorem 6.2.4 ([5], T.2.4). *If $\mathcal{T}:\overline{\mathcal{M}}_{g,n}\to\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$ is the "contracting-eliiptic-tails" morphism, then*
$$\mathcal{T}^*(\lambda_j)=\sum_{k=0}^{j}\frac{1}{k!}\mathrm{gl}_*^k\big(p_0^*(\lambda_{j-k})\big)$$

Theorem 6.2.5 ([7], T.3.1). *The following relationship holds*:
$$\Lambda_g^{\mathrm{ps}}(1)(-1)^g\Lambda_g^{\mathrm{ps}}(-1)=\sum_{k=0}^{g}\frac{1}{k!}\mathrm{gl}_*^k\left(\prod_{j=1}^{k}\psi_{\star j}-\psi_{\bullet j}\right).$$

In the previous result, homogenizing the total lambda classes, homogenizes the right hand side as well. This means that
$$\Lambda_g^{\mathrm{ps}}(t)(-1)^g\Lambda_g^{\mathrm{ps}}(-t)=\sum_{k=0}^{g}\frac{t^{2g-2k}}{k!}\mathrm{gl}_*^k\left(\prod_{j=1}^{k}\psi_{\star j}-\psi_{\bullet j}\right).$$

Now, we just expand the pseudostable Mumford relation to obtain
$$\lambda_3^{\mathrm{ps}}\Lambda_3^{\mathrm{ps}}(t)(-1)^3\Lambda_3^{\mathrm{ps}}(-t)$$
$$=\lambda_3^{\mathrm{ps}}\cdot\big(t^6+t^4\mathrm{gl}_*^1(\psi_\star-\psi_\bullet)+\tfrac{t^2}{2}\mathrm{gl}_*^2((\psi_{\star1}-\psi_{\bullet1})(\psi_{\star2}-\psi_{\bullet2}))+\tfrac{1}{6}\mathrm{gl}_*^3(\dots)\big)$$

After distributing, it's sufficient to just analyze terms of the form
$$\lambda_3^{\mathrm{ps}}\cdot\mathrm{gl}_*^i\left(\prod_{j=1}^{i}(\psi_{\star j}-\psi_{\bullet j})\right)$$
$$=\big(\lambda_3+\mathrm{gl}_*^1(p_0^*(\lambda_2))+\tfrac{1}{2}\mathrm{gl}_*^2(p_0^*(\lambda_1))+\tfrac{1}{6}\mathrm{gl}_*^3(1)\big)\cdot\mathrm{gl}_*^i\left(\prod_{j=1}^{i}(\psi_{\star j}-\psi_{\bullet j})\right)$$

where i ranges from 0 to g. This means that the first term of the resulting sum is just λ_3^{ps}.

Let us work with

$$\lambda_3^{\mathrm{ps}} \cdot \mathrm{gl}_*^1(\psi_\star - \psi_\bullet).$$

For the first term:

$$\begin{aligned}\lambda_3 \cdot \mathrm{gl}_*^1(\psi_\star - \psi_\bullet) &= \mathrm{gl}_*^1\left(p_0^*(\lambda_2) p_1^*(\lambda_1)(\psi_\star - \psi_\bullet)\right)\\ &= \mathrm{gl}_*^1\left(p_0^*(\lambda_2) p_1^*(\lambda_1)\psi_\star\right)\end{aligned}$$

where we lose the term with $\psi_\bullet$ because on $\overline{\mathcal{M}}_{1,1}$ we have a $\lambda_1 \psi_\bullet$.

Now the second:

$$\begin{aligned}&\mathrm{gl}_*^1(p_0^*(\lambda_2)) \cdot \mathrm{gl}_*^1(\psi_\star - \psi_\bullet)\\ =&\mathrm{gl}_*^1(p_0^*(\lambda_2)(\psi_\bullet^2 - \psi_\star^2)) + \mathrm{gl}_*^2\left(p_0^*(\lambda_1) p_2^*(\lambda_1)(\psi_{\star 2} - \psi_{\bullet 2})\right)\\ =&-\mathrm{gl}_*^1\left(p_0^*(\lambda_2)\psi_\star^2\right) + \mathrm{gl}_*^2\left(p_0^*(\lambda_1) p_2^*(\lambda_1)\psi_{\star 2}\right)\end{aligned}$$

where we once again lose the corresponding $\psi_\bullet$'s because of codimension reasons.

Question. In the second term of the expansion

$$\mathrm{gl}_*^1(p_0^*(\lambda_2)) \cdot \mathrm{gl}_*^1(\psi_\star - \psi_\bullet),$$

in the transverse intersection, ¿why doesn't that gl^2 carry a 2 factor?

With **Kelsey**, we developed the formula for the coefficient to be

$$\binom{\text{Edges of result}}{\text{LCS}}\binom{\text{SCS}}{\text{uncolored}}\binom{\text{LCS}}{\text{SCS} - \text{already colored}}$$

where LCS/SCS stands for largest/smallest color set. The issue is that in this case the factor accompanying the gl^2 would be a 2 which is not the real answer. Revisions must be made to the coefficient formula.

The next term is

$$\begin{aligned}&\mathrm{gl}_*^2(p_0^*(\lambda_1)) \cdot \mathrm{gl}_*^1(\psi_\star - \psi_\bullet)\\ =&2\mathrm{gl}_*^2(p_0^*(\lambda_1)(\psi_{\bullet 1}^2 - \psi_{\star 1}^2)) + \mathrm{gl}_*^3(p_3^*(\lambda_1)(\psi_{\star 3} - \psi_{\bullet 3}))\\ =&-2\mathrm{gl}_*^2(p_0^*(\lambda_1)\psi_{\star 1}^2) + \mathrm{gl}_*^3(p_3^*(\lambda_1)\psi_{\star 3})\end{aligned}$$

where we'll need to add a factor of $\frac{1}{2}$ because the lambda term carried it from it's own expansion. Different from this one, the next term doesn't need it because

$$\mathrm{gl}_*^3(1) \cdot \mathrm{gl}_*^1(\psi_\star - \psi_\bullet) = 3\mathrm{gl}_*^3(\psi_{\bullet 1}^2 - \psi_{\star 1}^2) = 0.$$

Gathering everything gives the equality

$$\begin{aligned}\lambda_3^{\mathrm{ps}}\cdot \mathrm{gl}^1_*(\psi_\star-\psi_\bullet)=&\mathrm{gl}^1_*\big(p_0^*(\lambda_2)p_1^*(\lambda_1)\psi_\star\big)\\ &-\mathrm{gl}^1_*\big(p_0^*(\lambda_2)\psi_\star^2\big)+\mathrm{gl}^2_*\big(p_0^*(\lambda_1)p_2^*(\lambda_1)\psi_{\star 2}\big)\\ &-\mathrm{gl}^2_*(p_0^*(\lambda_1)\psi_{\star 1}^2)+\frac{1}{2}\mathrm{gl}^3_*(p_3^*(\lambda_1)\psi_{\star 3})\end{aligned}$$

In a similar fashion we compute the product

$$\lambda_3^{\mathrm{ps}}\cdot \mathrm{gl}^2_*\big((\psi_{\star 1}-\psi_{\bullet 1})(\psi_{\star 2}-\psi_{\bullet 2})\big).$$

As before, after expanding λ_3^{ps} we see that

$$\begin{aligned}&\lambda_3\cdot \mathrm{gl}^2_*\big((\psi_{\star 1}-\psi_{\bullet 1})(\psi_{\star 2}-\psi_{\bullet 2})\big)\\ =&\mathrm{gl}^2_*\big(p_0^*(\lambda_1)p_1^*(\lambda_1)p_2^*(\lambda_1)(\psi_{\star 1}-\psi_{\bullet 1})(\psi_{\star 2}-\psi_{\bullet 2})\big)\\ =&\mathrm{gl}^2_*\big(p_0^*(\lambda_1)p_1^*(\lambda_1)p_2^*(\lambda_1)\psi_{\star 1}\psi_{\star 2}\big)\end{aligned}$$

The next term is

$$\begin{aligned}&\mathrm{gl}^1_*(p_0^*(\lambda_2))\cdot \mathrm{gl}^2_*\big((\psi_{\star 1}-\psi_{\bullet 1})(\psi_{\star 2}-\psi_{\bullet 2})\big)\\ =&2\mathrm{gl}^2_*\big(p_0^*(\lambda_1)p_2^*(\lambda_1)(\psi_{\bullet 1}^2-\psi_{\star 1}^2)(\psi_{\star 2}-\psi_{\bullet 2})\big)\\ &+\mathrm{gl}^3_*\big(p_2^*(\lambda_1)p_3^*(\lambda_1)(\psi_{\star 2}-\psi_{\bullet 2})(\psi_{\star 3}-\psi_{\bullet 3})\big)\end{aligned}$$

Observe that this term is immediately zero. On the first summand, no $\psi_\bullet$ can survive on the elliptic tails so the only possibility is to have the $\psi_\star$'s on the main component. This adds 3 degrees to the already loaded $M_{1,3}$ with a λ_1. In total, codimension 4 on dimension 3 space makes the element vanish. Similarly, no $\psi_\bullet$ survives the next term and the $M_{0,4}$ main component would have a $\psi_{\star 2}\psi_{\star 3}$ accompanying it, thus annihilating it.

Now multiplying the 2 gl^2's together:

$$\begin{aligned}&\mathrm{gl}^2_*(p_0^*(\lambda_1))\cdot \mathrm{gl}^2_*\big((\psi_{\star 1}-\psi_{\bullet 1})(\psi_{\star 2}-\psi_{\bullet 2})\big)\\ =&2\mathrm{gl}^2_*\big(p_0^*(\lambda_1)(\psi_{\bullet 1}^2-\psi_{\star 1}^2)(\psi_{\bullet 2}^2-\psi_{\star 2}^2)\big)\\ &+4\mathrm{gl}^3_*\big(p_3^*(\lambda_1)(\psi_{\bullet 2}^2-\psi_{\star 2}^2)(\psi_{\star 3}-\psi_{\bullet 3})\big)\end{aligned}$$

Any term with a $\psi_\bullet^2$ immediately vanishes and also any $\psi_\bullet$ accompanying a lambda class. This leaves us with

$$2\mathrm{gl}^2_*\big(p_0^*(\lambda_1)\psi_{\star 1}^2\psi_{\star 2}^2\big)-4\mathrm{gl}^3_*\big(p_3^*(\lambda_1)\psi_{\star 2}^2\psi_{\star 3}\big)$$

The first term carries codimension 5 on its $M_{1,3}$ component while the second one has codimension 3 on its $M_{0,4}$ component. This immediately annihilates those terms making the product zero at once.

Remark 6.2.6. Notice that the excess codimension on the "0^{th} component" comes from the lack of codimension on the elliptic tails. The first term has two elliptic tails with no decorations, adding two codimensions to the otherwise perfectly fine 3 dimensions, and the second term as well carries two more classes than needed.

We come to the final term which is $\mathrm{gl}_*^3(1)$. The product results in
$$6\mathrm{gl}_*^3\big((\psi_{\bullet 2}^2-\psi_{\star 2}^2)(\psi_{\bullet 3}^2-\psi_{\star 3}^2)\big).$$
As before, the $\psi_\bullet^2$ terms vanish. This leaves us with an $M_{0,4}$ carrying 4 psi classes. The three excess codimensions correspond to the three elliptic tails without decorations.

We have verified that

$$\begin{aligned}&\lambda_3^{\mathrm{ps}}\cdot\mathrm{gl}_*^2\big((\psi_{\star 1}-\psi_{\bullet 1})(\psi_{\star 2}-\psi_{\bullet 2})\big)\\ =&\lambda_3\cdot\mathrm{gl}_*^2\big((\psi_{\star 1}-\psi_{\bullet 1})(\psi_{\star 2}-\psi_{\bullet 2})\big)\\ =&\mathrm{gl}_*^2\big(p_0^*(\lambda_1)p_1^*(\lambda_1)p_2^*(\lambda_1)\psi_{\star 1}\psi_{\star 2}\big)\end{aligned}$$

For the final calculation we're looking at

$$\lambda_3^{\mathrm{ps}}\cdot\mathrm{gl}_*^3\big((\psi_{\star 1}-\psi_{\bullet 1})(\psi_{\star 2}-\psi_{\bullet 2})(\psi_{\star 3}-\psi_{\bullet 3})\big).$$

The term with psi classes can be simplified to a linear combination of $\mathrm{gl}_*^3(\psi_{\star a}\psi_{\bullet b}\psi_{\bullet c})$ for $1\leqslant a,b,c\leqslant 3$ distinct plus a $-\mathrm{gl}_*^3(\psi_{\bullet 1}\psi_{\bullet 2}\psi_{\bullet 3})$.

Against the ordinary lambda class, none of those terms survive as they all carry at least one $\psi_\bullet$.
Multiplying the $\mathrm{gl}_*^1(p_0^*(\lambda_2))$ term still returns a $\mathrm{gl}_*^3(1)$ graph with one edge bicolored. The lambda class distributes into two tails as λ_1's. At least one of the $\psi_\bullet$'s is in the same component as a λ_1 in any of the cases, so the product is zero.
Similarly for $\mathrm{gl}_*^2(p_0^*(\lambda_1))$, the only case where the product could be non-zero is when the λ_1 gets distributed into the elliptic tail corresponding to the $\psi_{\star a}$. However, the other two edges would be bicolored giving $\psi_\bullet^2$ terms which annihilate the elliptic tails or adding more $\psi_\star$'s to the $M_{0,4}$ component. Both of these cases are zero.
Finally multiplying a $\mathrm{gl}_*^3(1)$ repeats all edges. The terms get annihilated either by excess $\psi_\bullet^2$'s or $\psi_\star$'s on the main component.

Collecting everything together we get

$$\begin{aligned}
&\lambda_3^{\mathrm{ps}}\cdot\left(t^6+t^4\mathrm{gl}_*^1(\psi_\star-\psi_\bullet)+\tfrac{t^2}{2}\mathrm{gl}_*^2((\psi_{\star1}-\psi_{\bullet1})(\psi_{\star2}-\psi_{\bullet2}))+\tfrac{1}{6}\mathrm{gl}_*^3(\ldots)\right)\\
=&t^6\lambda_3^{\mathrm{ps}}\\
&+t^4\Big(\mathrm{gl}_*^1\left(p_0^*(\lambda_2)p_1^*(\lambda_1)\psi_\star\right)-\mathrm{gl}_*^1\left(p_0^*(\lambda_2)\psi_\star^2\right)+\mathrm{gl}_*^2\left(p_0^*(\lambda_1)p_2^*(\lambda_1)\psi_{\star2}\right)\\
&\qquad-\mathrm{gl}_*^2(p_0^*(\lambda_1)\psi_{\star1}^2)+\frac{1}{2}\mathrm{gl}_*^3(p_3^*(\lambda_1)\psi_{\star3})\Big)\\
&+\frac{t^2}{2}\mathrm{gl}_*^2\left(p_0^*(\lambda_1)p_1^*(\lambda_1)p_2^*(\lambda_1)\psi_{\star1}\psi_{\star2}\right)+0
\end{aligned}$$

We may divide by the t^2 term and then use the power series

$$\frac{1}{1-(d\psi_P/t)}=\sum_{k\geqslant0}(d\psi_P/t)^k.$$

The only power that will survive is that which cancels the corresponding t in the expansion. This is because of codimension reasons: there's either too little or too much. Our expression becomes

$$\begin{aligned}
&d^4\lambda_3^{\mathrm{ps}}\psi_P^4+d^2\mathrm{gl}_*^1\left(p_0^*(\lambda_2)p_1^*(\lambda_1)\psi_\star\right)\psi_P^2+d^2(0)\\
&+\frac{d^0}{2}\mathrm{gl}_*^2\left(p_0^*(\lambda_1)p_1^*(\lambda_1)p_2^*(\lambda_1)\psi_{\star1}\psi_{\star2}\right)
\end{aligned}$$

where the remaining terms accompanied by the d^2 degree get cancelled after multiplying the psi class.

We may now calculate the integral corresponding to $F_{(3,0)}$'s contribution as

$$\begin{aligned}
I_{(3,0)}=&d^4\int_{\overline{M}_{3,1}^{\mathrm{ps}}}\lambda_3^{\mathrm{ps}}\psi_P^4+d^2\left(\int_{\overline{M}_{2,2}}\lambda_2\psi_P^2\psi_\star\right)\left(\int_{\overline{M}_{1,1}}\lambda_1\right)\\
&+\frac{1}{2}\left(\int_{\overline{M}_{1,3}}\lambda_1\psi_{\star1}\psi_{\star2}\right)\left(\int_{\overline{M}_{1,1}}\lambda_1\right)^2
\end{aligned}$$

Citing a couple more results is the way to calculate this integrals. the first result is about the behaviour of linear Hodge integrals.

Theorem 6.2.7 ([5], P.3.2). *For any $j\in[g]$ and $F\in\mathbb{Z}[\underline{x}]$,*

$$\int_{\overline{M}_{g,n}^{\mathrm{ps}}}\lambda_j^{\mathrm{ps}}F(\underline{\psi})=\int_{\overline{M}_{g,n}}\lambda_jF(\underline{\psi}).$$

This tells us that the result of the first integral is

$$d^4b^3,\quad\text{where}\quad b_g=\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}$$

and B_g is the g^{th} Bernoulli number.

For the integrals with one lambda class, we use the result previously known as the λ_g-conjecture.

Theorem 6.2.8 ([10], T.3). *For $k_1,\dots,k_n$ summing to $2g+n-3$,*
$$\int_{\overline{M}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g = \binom{2g+n-3}{k_1,\dots,k_n} b_g.$$

This allows us to conclude that
$$I_{(3,0)} = d^4 b_3 + d^2\binom{4+2-3}{2,1} b_2\cdot\frac{1}{24} + \frac{1}{2}\binom{2+3-3}{1,1} b_1\cdot\frac{1}{24^2}$$
$$= d^4 b_3 + \frac{3d^2 b_2}{24} + \frac{2b_1}{2\cdot 24^2}$$

The $(2,1)$ contribution

The calculation proceeds in a similar fashion. The curve $C_{(2,1)}$ has a normalization sequence given by

$$0\to\mathcal{O}_C\to\mathcal{O}_{g=2}\oplus\mathcal{O}_{\mathbb{P}^1}\to\underline{\mathbb{C}}_{\text{node}}\oplus\underline{\mathbb{C}}_{\text{cusp}}x\to 0.$$

In this case, $\underline{\mathbb{C}}x$ is the structure sheaf of $\operatorname{Spec}(\langle x\rangle/(x^2))$ MUST ADD GOOD EXPLANATION ON WHY THIS IS THIS SHEAF, ¿why is the normalization sequence of cusp that? ¿why does the cusp normalize into such sheaf?. Skipping my lack of understanding of cusps, we may tensor by $\star := f^*\mathcal{O}(-H+s+t)$ to get
$$0\to\star\to\mathcal{O}(s)\oplus\mathcal{O}(-d\widetilde{H}+s+t)\to\langle(s)\rangle\oplus(\underline{\mathbb{C}}x\otimes\mathcal{O}(s+t))\to 0.$$
After simplifying the long exact sequence in cohomology we get
$$0\to H^0(\mathcal{O}(s))\to H^0(\langle(s)\rangle)\oplus H^0(\underline{\mathbb{C}}x\otimes\mathcal{O}(s+t))$$
$$\to\star\to H^1(\mathcal{O}(s))\oplus H^1(\mathcal{O}(-d\widetilde{H}+s+t))\to 0$$
Taking Euler class returns
$$e(\star) = e(H^1(\mathcal{O}(s)))e(H^1(\mathcal{O}(-d\widetilde{H}+s+t)))e(H^0(\underline{\mathbb{C}}x\otimes\mathcal{O}(s+t)))$$
$$=(s^2+(-\lambda_1^{\text{ps}})s+\lambda_2^{\text{ps}})\left(\prod_{k=1}^{d-1}\left(s+\frac{kt}{d}\right)\right)\left(-\frac{-t}{d}+s+t\right)$$

Remark 6.2.9. The Euler class of the tensor product is the sum of both classes, no question there. However, the reason as to why the Euler class of $\underline{\mathbb{C}}x$ is not clear. The section x carries the dual tangent weight at the corresponding point. Since the cusp is located at ∞ and the tangent weight there is $\frac{-t}{d}$, we have that the Euler class is $-\frac{-t}{d}$.

Specializing into $s=-t$ and $s=0$ gives us the numerator
$$i^*_{(2,1)}(e(H^1 f^*\mathcal{O}(-H))e(H^1 f^*\mathcal{O}(-H+t)))$$
$$=\Lambda_2^{\text{ps}}(t)\left(\prod_{k=1}^{d-1}\left(-t+\frac{kt}{d}\right)\right)\left(\frac{t}{d}\right)\lambda_2^{\text{ps}}\left(\prod_{k=1}^{d-1}\left(\frac{kt}{d}\right)\right)\left(\frac{t}{d}+t\right)$$
$$=\lambda_2^{\text{ps}}\Lambda_2^{\text{ps}}(t)\left((-1)^{d-1}(t/d)^{2d-2}(d-1)!^2\right)\left((t/d)(t/d+t)\right)$$

For the normal bundle we follow a similar procedure.

- ◇ There's no smooth points, so there's no automorphisms. In this case, $e(\mathrm{Aut}(C))=0$, but since this is not part of the *moving* part of the Euler class, we don't consider it into the calculation.
- ◇ We get a contribution from the node at 0, connecting the rational part with the positive genus part, and one from the cusp.
$$e(\mathrm{Def}(C))=\left(-\psi_P+\frac{t}{d}\right)\cdot 24\left(\frac{-t}{d}\right)^2$$
- ◇ Tensoring the normalization sequence with $f^*T\mathbb{P}^1$ we get
$$0\to f^*T\mathbb{P}^1\to \mathcal{O}(t)\oplus\mathcal{O}(2d\widetilde{H}-t)\to\langle(t)\rangle\oplus\big(\mathbb{C}x\otimes\mathcal{O}(-t)\big)\to 0.$$
Now the long sequence in cohomology contains the desired bundles as
$$0\ \to\ \mathrm{Def}(f)\ \to\ H^0(\mathcal{O}(t))\oplus H^0(\mathcal{O}(2d\widetilde{H}-t))\ \to\ H^0(\langle(t)\rangle)\oplus H^0\big(\underline{\mathbb{C}}x\otimes\mathcal{O}$$
$$\hookrightarrow\ \mathrm{Ob}(f)\ \to\ H^1(\mathcal{O}(t))\oplus H^1(\mathcal{O}(2d\widetilde{H}-t))\ \longrightarrow\ 0$$
Taking Euler classes:
$$\frac{e(\mathrm{Def}(f))}{e(\mathrm{Ob}(f))}=\frac{e\big(H^0(\mathcal{O}(2d\widetilde{H}-t))\big)}{e\big(H^0\big(\underline{\mathbb{C}}x\otimes\mathcal{O}(-t)\big)\big)e\big(H^1(\mathcal{O}(t))\big)}$$
$$=\frac{(t/d)^{2d}d!^2(-1)^d}{\big(-(-t/d)+(-t)\big)\big((-1)^2\Lambda_2^{\mathrm{ps}}(-t)\big)}$$

Condensing this with the deformations of the curve we get
$$e(N_{(2,1)})=\frac{\big[(t/d)^{2d}d!^2(-1)^d\big]\big[(-\psi_P+t/d)\cdot 24(-t/d)^2\big]}{\big(t/d-t\big)\big((-1)^2\Lambda_2^{\mathrm{ps}}(-t)\big)}$$
whose degree is
$$(2d+1+2)-(1+2)=2d$$
which should match with the codimension of $\overline{M}_{2,1}$ inside the space of maps. The way to check this without computing the dimension of the space of maps is by considering the whole quotient:
$$\frac{\lambda_2^{\mathrm{ps}}\Lambda_2^{\mathrm{ps}}(t)\big((-1)^{d-1}(t/d)^{2d-2}(d-1)!^2\big)(t/d)(t/d+t)(t/d-t)\Lambda_2^{\mathrm{ps}}(-t)}{\big[(t/d)^{2d}d!^2(-1)^d\big]\big[(-\psi_P+t/d)\cdot 24(-t/d)^2\big]}$$
$$=\frac{\lambda_2^{\mathrm{ps}}\Lambda_2^{\mathrm{ps}}(t)\Lambda_2^{\mathrm{ps}}(-t)\big((1/d)(1/d+1)(1/d-1)t^3\big)}{(-t^2)\big(24(t/d)^3(1-d\psi_P/t)\big)}$$
$$=\frac{d^2-1}{24}\cdot\frac{\lambda_2^{\mathrm{ps}}\Lambda_2^{\mathrm{ps}}(t)\Lambda_2^{\mathrm{ps}}(-t)}{t^2\big(1-d\psi_P/t\big)}$$

which has degree 4, same as the dimension of $\overline{M}_{2,1}$, $3\cdot 2-3+1=4$.

Using Matt's theorem 6.2.5 we simplify this class as

$$\begin{aligned}&\frac{\lambda_2^{\mathrm{ps}}}{\left(1-d\psi_P/t\right)}\sum_{k=0}^{2}\frac{t^{4-2k-2}}{k!}\mathrm{gl}_*^k\left(\prod_{j=1}^{k}\psi_{\star j}-\psi_{\bullet j}\right)\\ =&\sum_{k=0}^{2}\frac{d^{2-2k}}{k!}\lambda_2^{\mathrm{ps}}\mathrm{gl}_*^k\left(\prod_{j=1}^{k}\psi_{\star j}-\psi_{\bullet j}\right)\psi_P^{2-2k}\\ =&d^2\lambda_2^{\mathrm{ps}}\psi_P^2+\lambda_2^{\mathrm{ps}}\mathrm{gl}_*^1(\psi_\star-\psi_\bullet)+0\end{aligned}$$

Remark 6.2.10. The last term is zero before even expanding the psi class.

Exercise 6.2.11. Show that $\lambda_g^{\mathrm{ps}}\mathrm{gl}_*^g\left(\prod_{j=1}^{k}\psi_{\star j}-\psi_{\bullet j}\right)$ is zero in general. ¿Can this be done by induction on the number of elliptic tails of λ_g^{ps}'s terms?

We end this segment by calculating the product of the pseudostable lambda class with the 1 elliptic tail graph:

$$\begin{aligned}&\left(\lambda_2+\mathrm{gl}_*^1(p_0^*(\lambda_1))+\tfrac{1}{2}\mathrm{gl}_*^2(1)\right)\mathrm{gl}_*^1(\psi_\star-\psi_\bullet)\\ =&\mathrm{gl}_*^1(p_0^*(\lambda_1)p_1^*(\lambda_1)\psi_\star)\\ &+\mathrm{gl}_*^1\left(p_0^*(\lambda_1)(\psi_\bullet^2-\psi_\star^2)\right)+\mathrm{gl}_*^2\left(p_2^*(\lambda_1)(\psi_{\star 2}-\psi_{\bullet 2})\right)\\ &+\frac{1}{2}\left(2\mathrm{gl}_*^2\left(\psi_{\bullet 1}^2-\psi_{\star 1}^2\right)\right)\\ =&\mathrm{gl}_*^1(p_0^*(\lambda_1)p_1^*(\lambda_1)\psi_\star)\end{aligned}$$

Here, most terms vanish for codimension reasons. Putting this together and integrating we get

$$\begin{aligned}&d^2\int_{\overline{M}_{2,1}^{\mathrm{ps}}}\lambda_2^{\mathrm{ps}}\psi_P^2+\left(\int_{\overline{M}_{1,2}}\lambda_1\psi_\star\right)\left(\int_{\overline{M}_{1,1}}\lambda_1\right)\\ =&d^2b_2+1/24^2\end{aligned}$$

where we used theorem 6.2.7 once again for the first integral.

Mutiplying by the coefficient we left behind we get a final result for the contribution of $F_{(2,1)}$

$$I_{(2,1)}=\frac{d^2-1}{24}\left(d^2b_2+\frac{1}{24^2}\right).$$

The $(1,2)$ contribution

Analogous to the previous case, the normalization sequence is

$$0\to\mathcal{O}_C\to\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{g=2}\to\underline{\mathbb{C}}_{\text{cusp}}y\oplus\underline{\mathbb{C}}_{\text{node}}\to 0.$$

Tensoring by $\star:=f^*\mathcal{O}(-H+s+t)$ returns

$$0 \to \star \to \mathcal{O}(-d\widetilde{H}+s+t) \oplus \mathcal{O}(s+t) \to (\underline{\mathbb{C}y} \otimes \mathcal{O}(s)) \oplus \langle (s+t) \rangle \to 0.$$

As before, we simplify with some minor changes

$$\begin{aligned} 0 \to H^0(\mathcal{O}(s+t)) \to H^0(\underline{\mathbb{C}y} \otimes \mathcal{O}(s)) \oplus H^0(\langle (s+t) \rangle) \\ \to \star \to H^1(\mathcal{O}(-d\widetilde{H}+s+t)) \oplus H^1(\mathcal{O}(s+t)) \to 0 \end{aligned}$$

Taking Euler classes we get

$$\begin{aligned} e(\star) &= e(H^1(\mathcal{O}(-d\widetilde{H}+s+t)))e(H^1(\mathcal{O}(s+t)))e(H^0(\underline{\mathbb{C}y} \otimes \mathcal{O}(s))) \\ &= \left(\prod_{k=1}^{d-1} \left(s + \frac{kt}{d} \right) \right) ((s+t)^2 + (-\lambda_1^{\mathrm{ps}})(s+t) + \lambda_2^{\mathrm{ps}}) \left(-\frac{t}{d} + s \right) \end{aligned}$$

Plugging in $s=-t$ and $s=0$ gives

$$\begin{aligned} & i^*_{(1,2)}(e(H^1 f^* \mathcal{O}(-H))e(H^1 f^* \mathcal{O}(-H+t))) \\ &= \prod_{k=1}^{d-1} \left(-t + \frac{kt}{d} \right) \lambda_2^{\mathrm{ps}} \left(-\frac{t}{d} - t \right) \prod_{k=1}^{d-1} \left(\frac{kt}{d} \right) ((-1)^2 \Lambda_2^{\mathrm{ps}}(-t)) \left(-\frac{t}{d} \right) \\ &= \lambda_2^{\mathrm{ps}} \Lambda_2^{\mathrm{ps}}(-t) \big((-1)^{d-1} (t/d)^{2d-2} (d-1)!^2 \big) \big((t/d)(t/d+t) \big) \end{aligned}$$

On the side of the normal bundle we don't have automorphisms as there's not smooth points. For curve deformations we get

$$e(\mathrm{Def}(C)) = 24 \left(\frac{t}{d} \right)^2 \left(-\psi_P + \frac{-t}{d} \right)$$

whereas for map deformations and obstructions we repeat the previous process of tensoring the normalization sequence and then taking cohomology. First we get

$$0 \to f^* T\mathbb{P}^1 \to \mathcal{O}(2d\widetilde{H}-t) \oplus \mathcal{O}(-t) \to \big(\underline{\mathbb{C}y} \otimes \mathcal{O}(t) \big) \oplus \langle (-t) \rangle \to 0.$$

This gets us

$$\begin{aligned} 0 \to \mathrm{Def}(f) \to H^0(\mathcal{O}(2d\widetilde{H}-t)) \oplus H^0(\mathcal{O}(-t)) \to H^0\big(\underline{\mathbb{C}y} \otimes \mathcal{O}(t) \big) \oplus H^0(\langle (\\ \hookrightarrow \mathrm{Ob}(f) \to H^1(\mathcal{O}(2d\widetilde{H}-t)) \oplus H^1(\mathcal{O}(-t)) \longrightarrow 0 \end{aligned}$$

Taking Euler classes:

$$\begin{aligned}\frac{e(\mathrm{Def}(f))}{e(\mathrm{Ob}(f))} &= \frac{e\big(H^0(\mathcal{O}(2d\widetilde{H}-t))\big)}{e\big(H^0\big(\underline{\mathbb{C}}y\otimes\mathcal{O}(t)\big)\big)e\big(H^1(\mathcal{O}(-t))\big)}\\ &= \frac{(t/d)^{2d}d!^2(-1)^d}{\big(-(t/d)+t\big)\Lambda_2^{\mathrm{ps}}(t)}\end{aligned}$$

Our integrand then is

$$\frac{\lambda_2^{\mathrm{ps}}\Lambda_2^{\mathrm{ps}}(-t)\big((-1)^{d-1}(t/d)^{2d-2}(d-1)!^2\big)(t/d)(t/d+t)(-t/d+t)\Lambda_2^{\mathrm{ps}}(t)}{\big[(t/d)^{2d}d!^2(-1)^d\big]24(t/d)^2(-\psi_P-t/d)}$$

$$=\frac{\lambda_2^{\mathrm{ps}}\Lambda_2^{\mathrm{ps}}(-t)\Lambda_2^{\mathrm{ps}}(t)\big((t/d)^3(d^2-1)\big)}{(-t^2)24(t/d)^2(-t/d)\big(1-(-d\psi_P/t)\big)}$$

$$=\frac{d^2-1}{24}\cdot\frac{\lambda_2^{\mathrm{ps}}\Lambda_2^{\mathrm{ps}}(-t)\Lambda_2^{\mathrm{ps}}(t)}{t^2\big(1-(-d\psi_P/t)\big)}$$

Simplifying again via theorem 6.2.5 we notice that the negative coefficient on the psi class cancels due to the even powers it is raised to

$$\begin{aligned}&\frac{\lambda_2^{\mathrm{ps}}}{\big(1-(-d\psi_P/t)\big)}\sum_{k=0}^{2}\frac{t^{4-2k-2}}{k!}\mathrm{gl}_*^k\left(\prod_{j=1}^{k}\psi_{\star j}-\psi_{\bullet j}\right)\\ &=\sum_{k=0}^{2}\frac{(-d)^{2-2k}}{k!}\lambda_2^{\mathrm{ps}}\mathrm{gl}_*^k\left(\prod_{j=1}^{k}\psi_{\star j}-\psi_{\bullet j}\right)(-\psi_P)^{2-2k}\\ &=d^2\lambda_2^{\mathrm{ps}}\psi_P^2+\lambda_2^{\mathrm{ps}}\mathrm{gl}_*^1(\psi_\star-\psi_\bullet)\end{aligned}$$

Thus

$$I_{(1,2)}=I_{(2,1)}=\frac{d^2-1}{24}\left(d^2b_2+\frac{1}{24^2}\right)$$

The $(0,3)$ **contribution**

We may expedite the calculation following the first case. First the normalization sequence

$$0\to\mathcal{O}_C\to\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{g=3}\to\underline{\mathbb{C}}_{\text{node}}\to 0$$

then tensoring with $f^*\mathcal{O}(-H+s+t)$

$$0\to f^*\mathcal{O}(-H+s+t)\to\mathcal{O}(-d\widetilde{H}+s+t)\oplus\mathcal{O}(s+t)\to\langle(s+t)\rangle\to 0.$$

Further, the long exact sequence in cohomology is

$$\begin{aligned}0\to H^0(\mathcal{O}(s+t))\to H^0(\langle(s+t)\rangle)\\ \to\bigstar\to H^1(\mathcal{O}(s+t))\oplus H^1(\mathcal{O}(-d\widetilde{H}+s+t))\to 0\end{aligned}$$

so that the desired Euler class is the product

$$
\begin{aligned}
&e(H^1(\mathcal{O}(s+t)))e(H^1(\mathcal{O}(-d\widetilde{H}+s+t)))\\
=&\left((s+t)^3+(s+t)^2(-\lambda_1^{\mathrm{ps}})+(s+t)\lambda_2^{\mathrm{ps}}-\lambda_3^{\mathrm{ps}}\right)\left(\prod_{k=1}^{d-1}\left(s+\frac{kt}{d}\right)\right)
\end{aligned}
$$

Plugging $s=-t$, $s=0$ returns

$$
\begin{aligned}
&i^*_{(3,0)}(e(H^1f^*\mathcal{O}(-H))e(H^1f^*\mathcal{O}(-H+t)))\\
=&\left(-\lambda_3^{\mathrm{ps}}\right)\left(\prod_{k=1}^{d-1}\frac{kt}{d}\right)\left(t^3+t^2(-\lambda_1^{\mathrm{ps}})+t\lambda_2^{\mathrm{ps}}-\lambda_3^{\mathrm{ps}}\right)\left(\prod_{k=1}^{d-1}-t+\frac{kt}{d}\right)\\
=&(-\lambda_3^{\mathrm{ps}})(-1)^3\Lambda_3^{\mathrm{ps}}(-t)(-1)^{d-1}\left(\frac{t}{d}\right)^{2d-2}(d-1)!^2.
\end{aligned}
$$

For the normal bundle:

- The smooth point above zero gives us
$$e(\mathrm{Aut}(C))=t/d.$$
- The deformations of the curve give
$$e(\mathrm{Def}(C))=-\psi_P+\frac{-t}{d}.$$
- For deformations and obstructions of the map we tensor the normalization sequence
$$0\to f^*T\mathbb{P}^1\to\mathcal{O}(2d\widetilde{H}-t)\oplus\mathcal{O}(-t)\to\langle(-t)\rangle\to 0.$$
Taking cohomology gets us
$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Def}(f) \arrow[r] & H^0(\mathcal{O}(2d\widetilde{H}-t))\oplus H^0(\mathcal{O}(-t)) \arrow[r] & H^0(\langle(-t)\rangle) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (\tikztotarget)}] \\
 & \mathrm{Ob}(f) \arrow[r] & H^1(\mathcal{O}(-t)) \arrow[r] & 0
\end{tikzcd}
$$
which gives
$$\frac{e(\mathrm{Def}(f))}{e(\mathrm{Ob}(f))}=\frac{e(H^0(\mathcal{O}(2d\widetilde{H}-t)))}{e(H^1(\mathcal{O}(-t)))}=\frac{(t/d)^{2d}d!^2(-1)^d}{(-1)^3\Lambda_3^{\mathrm{ps}}(t)}$$

so that putting all together gets

$$e(N_{(0,3)})=\frac{\left[(t/d)^{2d}d!^2(-1)^d\right]\left[-\psi_P-t/d\right]}{\left[(-1)^3\Lambda_3^{\mathrm{ps}}(t)\right]\left[t/d\right]}.$$

The integrand is then

$$\frac{\big[(-\lambda_3^{\mathrm{ps}})(-1)^3\Lambda_3^{\mathrm{ps}}(-t)(-1)^{d-1}(t/d)^{2d-2}(d-1)!^2\big]\big[(-1)^3\Lambda_3^{\mathrm{ps}}(t)\big]\big[t/d\big]}{\big[(t/d)^{2d}d!^2(-1)^d\big]\big[-\psi_P-t/d\big]}$$
$$=\frac{\big(\lambda_3^{\mathrm{ps}}\Lambda_3^{\mathrm{ps}}(t)(-1)^3\Lambda_3^{\mathrm{ps}}(-t)\big)(t/d)}{(-t^2)(-t/d)\big(1-(-d\psi_P/t)\big)}$$
$$=\frac{\lambda_3^{\mathrm{ps}}\Lambda_3^{\mathrm{ps}}(t)(-1)^3\Lambda_3^{\mathrm{ps}}(-t)}{t^2\big(1-(-d\psi_P/t)\big)}.$$

Because the psi class only obtains even powers, this calculation is exactly the same as in the $(3,0)$ case. We conclude that

$$I_{(0,3)}=I_{(3,0)}=d^4b_3+\frac{3d^2b_2}{24}+\frac{2b_1}{2\cdot 24^2}.$$

The final result is

$$C^{\mathrm{ps}}(3,d)=\frac{2}{d}\left(d^4b_3+\frac{3d^2b_2}{24}+\frac{2b_1}{2\cdot 24^2}\right)+\frac{2}{d}\left(\frac{d^2-1}{24}\left(d^2b_2+\frac{1}{24^2}\right)\right)$$
$$=\frac{d^3}{6048}+\frac{d}{2880}=C(3,d)\left(1+\frac{21}{10d^2}\right)$$

The general case, $C^{\mathrm{ps}}(g,d)$

Contributing maps here are enumerated by pairs (g_1,g_2) with $g_1+g_2=g$ and $g_1,g_2\geqslant 0$. These are ordered partitions of g. We study the general case g_1,g_2 with $g_1\neq g-1,0$ and we treat these ones separately.

The (g_1,g_2) contribution

The curve $C_{(g_1,g_2)}$ has the normalization sequence
$$0\to\mathcal{O}_C\to\mathcal{O}_{g_1}\oplus\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{g_2}\to\underline{\mathbb{C}}_{n_1}\oplus\underline{\mathbb{C}}_{n_2}\to 0.$$
Tensoring with $\text{🐭}:=f^*\mathcal{O}(-H+s+t)$ gives
$$0\to\text{🐭}\to\mathcal{O}(s)\oplus\mathcal{O}(-d\widetilde{H}+s+t)\oplus\mathcal{O}(s+t)\to\langle(s)\rangle\oplus\langle(s+t)\rangle\to 0.$$
After taking the long exact sequence in cohomology and simplifying, we're left with
$$0\to H^0(\mathcal{O}(s))\oplus H^0(\mathcal{O}(s+t))\to H^0(\langle(s)\rangle)\oplus H^0(\langle(s+t)\rangle)$$
$$\to H^1(\text{🐭})\to H^1(\mathcal{O}(s))\oplus H^1(\mathcal{O}(-d\widetilde{H}+s+t))\oplus H^1(\mathcal{O}(s+t))\to 0$$
Taking Euler classes gives us
$$e(H^1(\text{🐭}))=e\big(H^1(\mathcal{O}(s))\big)e\big(H^1(\mathcal{O}(-d\widetilde{H}+s+t))\big)e\big(H^1(\mathcal{O}(s+t))\big)$$
$$=(-1)^{g_1}\Lambda_{g_1}^{\mathrm{ps}}(-s)\left(\prod_{k=1}^{d-1}\left(s+\frac{kt}{d}\right)\right)(-1)^{g_2}\Lambda_{g_2}^{\mathrm{ps}}(-(s+t))$$

We find the numerator by plugging in $s=-t$, $s=0$ and multiplying.

$$
\begin{aligned}
&i^*_{(g_1,g_2)}(e(H^1 f^*\mathcal{O}(-H))e(H^1 f^*\mathcal{O}(-H+t)))\\
&=(-1)^{g_1}\Lambda^{\mathrm{ps}}_{g_1}(t)\prod_{k=1}^{d-1}\left(-t+\frac{kt}{d}\right)(-1)^{g_2}\lambda^{\mathrm{ps}}_{g_2}(-1)^{g_1}\lambda^{\mathrm{ps}}_{g_1}\prod_{k=1}^{d-1}\left(\frac{kt}{d}\right)(-1)^{g_2}\Lambda^{\mathrm{ps}}_{g_2}(-t)\\
&=\left((-1)^{d-1}(t/d)^{2d-2}(d-1)!^2\right)\left(\lambda^{\mathrm{ps}}_{g_1}\Lambda^{\mathrm{ps}}_{g_1}(t)\right)\left(\lambda^{\mathrm{ps}}_{g_2}\Lambda^{\mathrm{ps}}_{g_2}(-t)\right)
\end{aligned}
$$

There's no automorphisms as there's no smooth points so that the normal bundle only considers deformations and obstructions. In this case

$$
e(\mathrm{Def}(C))=(-\psi_{P_1}-\psi_{Q_1})(-\psi_{P_2}-\psi_{Q_2})=\left(-\psi_{P_1}+\frac{t}{d}\right)\left(-\psi_{P_2}+\frac{-t}{d}\right)
$$

For the map, we tensor the normalization sequence by $f^*T\mathbb{P}^1$ to get

$$
0\to f^*T\mathbb{P}^1\to\mathcal{O}(t)\oplus\mathcal{O}(2d\widetilde{H}-t)\oplus\mathcal{O}(-t)\to\langle(t)\rangle\oplus\langle(-t)\rangle\to 0.
$$

Taking cohomology gets us

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Def}(f) \arrow[r] & H^0(\mathcal{O}(t))\oplus H^0(\mathcal{O}(2d\widetilde{H}-t))\oplus H^0(\mathcal{O}(-t)) \arrow[r] & H^0(\langle(t)\rangle)\oplus \arrow[dll, rounded corners] \\
& \mathrm{Ob}(f) \arrow[r] & H^1(\mathcal{O}(t))\oplus H^1(\mathcal{O}(-t)) \arrow[r] & 0
\end{tikzcd}
$$

thus the desired quotient of Euler classes is

$$
\frac{e(\mathrm{Def}(f))}{e(\mathrm{Ob}(f))}=\frac{e(H^0(\mathcal{O}(2d\widetilde{H}-t)))}{e(H^1(\mathcal{O}(t)))e(H^1(\mathcal{O}(-t)))}=\frac{(t/d)^{2d}d!^2(-1)^d}{(-1)^{g_1}\Lambda^{\mathrm{ps}}_{g_1}(-t)(-1)^{g_2}\Lambda^{\mathrm{ps}}_{g_2}(t)}.
$$

We end by condensing everything into the integrand.

$$
\begin{aligned}
&\frac{\left((-1)^{d-1}(t/d)^{2d-2}(d-1)!^2\right)\left(\lambda^{\mathrm{ps}}_{g_1}\Lambda^{\mathrm{ps}}_{g_1}(t)\right)\left(\lambda^{\mathrm{ps}}_{g_2}\Lambda^{\mathrm{ps}}_{g_2}(-t)\right)(-1)^{g_1}\Lambda^{\mathrm{ps}}_{g_1}(-t)(-1}{(t/d)^{2d}d!^2(-1)^d(-\psi_{P_1}+t/d)(-\psi_{P_2}+(-t/d))}\\
&=\frac{\left(\lambda^{\mathrm{ps}}_{g_1}\Lambda^{\mathrm{ps}}_{g_1}(t)(-1)^{g_1}\Lambda^{\mathrm{ps}}_{g_1}(-t)\right)\left(\lambda^{\mathrm{ps}}_{g_2}\Lambda^{\mathrm{ps}}_{g_2}(-t)(-1)^{g_2}\Lambda^{\mathrm{ps}}_{g_2}(t)\right)}{(-t^2)(-\psi_{P_1}+t/d)(-\psi_{P_2}+(-t/d))}\\
&=\frac{1}{d^2}\left(\frac{\lambda^{\mathrm{ps}}_{g_1}\Lambda^{\mathrm{ps}}_{g_1}(t)(-1)^{g_1}\Lambda^{\mathrm{ps}}_{g_1}(-t)}{t^2(1-d\psi_{P_1}/t)}\right)\left(\frac{\lambda^{\mathrm{ps}}_{g_2}\Lambda^{\mathrm{ps}}_{g_2}(-t)(-1)^{g_2}\Lambda^{\mathrm{ps}}_{g_2}(t)}{t^2(1-(-d\psi_{P_2}/t))}\right).
\end{aligned}
$$

We've seen before the minus sign on the psi class has no effect because of even powers. In this case, we just treat a general term of genus g. Expanding via Matt's relation (theorem 6.2.5) gets us to

$$\frac{\lambda_g^{\mathrm{ps}}\Lambda_g^{\mathrm{ps}}(t)(-1)^g\Lambda_g^{\mathrm{ps}}(-t)}{t^2(1-d\psi_P/t)}$$
$$=\frac{\lambda_g^{\mathrm{ps}}}{t^2(1-d\psi_P/t)}\sum_{k=0}^{g}\frac{t^{2g-2k}}{k!}\mathrm{gl}_*^k\left(\prod_{j=1}^{k}\psi_{\star j}-\psi_{\bullet j}\right)$$
$$=\sum_{k=0}^{g}\frac{d^{2g-2k-2}}{k!}\lambda_g^{\mathrm{ps}}\mathrm{gl}_*^k\left(\prod_{j=1}^{k}\psi_{\star j}-\psi_{\bullet j}\right)\psi_P^{2g-2k-2}$$
$$=\sum_{k=0}^{g}\frac{d^{2g-2k-2}}{k!}\mathrm{gl}_*^k\left(p_0^*(\lambda_{g-k})\prod_{j=1}^{k}\psi_{\star j}p_j^*(\lambda_1)\right)\psi_P^{2g-2k-2}$$

Exercise 6.2.12. The product
$$\lambda_g^{\mathrm{ps}}\mathrm{gl}_*^k\left(\prod_{j=1}^{k}\psi_{\star j}-\psi_{\bullet j}\right)\psi_P^{2g-2k-2}$$
results in
$$\mathrm{gl}_*^k\left(p_0^*(\lambda_{g-k})\prod_{j=1}^{k}\psi_{\star j}p_j^*(\lambda_1)\right)\psi_P^{2g-2k-2}.$$
However on its own,
$$\lambda_g^{\mathrm{ps}}\mathrm{gl}_*^k\left(\prod_{j=1}^{k}\psi_{\star j}-\psi_{\bullet j}\right)\neq\mathrm{gl}_*^k\left(p_0^*(\lambda_{g-k})\prod_{j=1}^{k}\psi_{\star j}p_j^*(\lambda_1)\right).$$

Taking the integral over $\overline{M}_{g,1}$ gets us to

$$\sum_{k=0}^{g}\frac{d^{2g-2k-2}}{k!}\left(\int_{\overline{M}_{g-k,1+k}}\lambda_{g-k}\prod_{j=1}^{k}\psi_{\star j}\psi_P^{2g-2k-2}\right)\left(\int_{\overline{M}_{1,1}}\lambda_1\right)^k$$
$$=\sum_{k=0}^{g}\frac{d^{2g-2k-2}}{k!}\frac{(2g-2-k)!}{(2g-2-2k)!}b_{g-k}\frac{1}{24^k}$$

Here we applied the λ_g-conjecture 6.2.8 again to find the value of the first integral. Call this amount I_g. Then our desired (g_1,g_2) contribution is
$$I_{(g_1,g_2)}=\frac{1}{d^2}I_{g_1}I_{g_2}.$$

The $(g,0)$ contribution

The only difference with the previous calculation lies in the automorphisms bundle. In this case we get an
$$e(\mathrm{Aut}(C))=-t/d$$
which cancels with the t/d from the curve deformations, and the minus sign cancels out the minus in the $-t^2$. On the $(0,g)$ case, signs

are inverted but the only change is in the psi class in the denominator which doesn't alter the result of the integral. Thus

$$I_{(g,0)} = I_{(0,g)} = I_g$$

Exercise 6.2.13. Write this neatly.

The $(g-1,1)$ contribution

Exercise 6.2.14. Prove that

$$I_{(g-1,1)} = I_{(1,g-1)} = \frac{d^2-1}{24} I_{g-1}.$$

The whole contribution and generating functions

The full quantity we are looking for is

$$\begin{aligned} C^{\mathrm{ps}}(g,d) &= \frac{1}{d}\big(I_{(g,0)} + I_{(g-1,1)} + \cdots + I_{(g_1,g_2)} + \cdots + I_{(1,g-1)} + I_{(0,g)}\big) \\ &= \frac{1}{d}\left(I_g + \frac{d^2-1}{24} I_{g-1} + \cdots + \frac{1}{d^2} I_{g_1} I_{g_2} + \cdots + \frac{d^2-1}{24} I_{g-1} + I_g\right) \end{aligned}$$

Question. If such integrals can be extended beyond their just obtained definition, what are I_0 and I_1? Can we fit them into the sum?

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