Exercise 1 (Exercise 3)**.** Let c_n be the number of sequences of length n in which:

- Each number is one of $0, 1, 2, 3$.
- No two 3's are consecutive.

For instance 0221030132 is a valid sequence but 033112333 is not.

i) Find a recursion for c_n (this should be similar to the Fibonacci recurrence). Remember to include the initial conditions!
ii) Use the recursion to find a closed form for the generating function of c_n.
iii) Use the formula discussed in class for solving linear recurrences to find an explicit formula for c_n in terms of n.

Answer

i) First, we can observe that the total number of sequences of length n with characters $0, \ldots, 3$ is 4^n. Dividing this amount into conditioned sequences and forbidden sequences we get the following

$$4^n = c_n + f_n,$$

where c_n is the quantity we are looking for and f_n is the number of *forbidden* sequences of length n. This is, sequences which do have 33 as a substring.

After considering the initial conditions:

$$f_0 = 0,\ f_1 = 0,\ f_2 = 1,\ f_3 = 7,\ f_4 = 40,$$

it is possible to conjecture that

$$f_{n+2} = 3f_{n+1} + 3f_n + 4^n.$$

To prove this recurrence we will append a digit to a length $(n+1)$ sequence. There are several ways to this:

- If the digit we are appending is either 0, 1 or 2, we are not adding any more forbidden substrings. So for each of those digits, our count of forbidden sequences goes up by f_{n+1}. Right now we have $3f_{n+1}$ forbidden sequences.
- If the digit we are appending is 3, there are two cases:
  - Either the last digit of the $(n+1)$ sequence is 0, 1 or 2 in which case we are not adding more forbidden strings. Each of those possibilities accounts for f_n forbidden sequences. This adds up to our past total to get $3f_{n+1} + 3f_n$.

  - If the last digit of the $(n+1)$ sequence is 3, then we just added a forbidden substring. Counting this is the same as counting all $(n+2)$ strings which end in 33. This amount is 4^n.

In total, we have $3f_{n+1} + 3f_n + 4^n$ forbidden sequences of length n. Then by our initial relation we have

$$c_n = 4^n - f_n = -(3f_{n+1} + 3f_n),\ f_0 = 0,\ f_1 = 0.^a$$

ii) Let us derive a generating function for f_n and from that we will obtain c_n's generating function.

Call F, (f_n)'s generating function, then the recurrence

$$f_{n+2} = 3f_{n+1} + 3f_n + 4^n$$

translates to the equation

$$\frac{F(x) - f_0 - f_1 x}{x^2} = \frac{3(F(x) - f_0)}{x} + 3F(x) + \frac{1}{1-4x}.$$

Applying the initial conditions we get

$$\frac{F(x)}{x^2} = \frac{3F(x)}{x} + 3F(x) + \frac{1}{1-4x}.$$

We can solve for F to obtain

$$\begin{aligned} F(x)\left(\frac{1}{x^2} - \frac{3}{x} - 3\right) &= \frac{1}{1-4x}, \\ \Rightarrow F(x)\left(\frac{1-3x-3x^2}{x^2}\right) &= \frac{1}{1-4x}, \\ \Rightarrow F(x) &= \frac{1}{1-4x}\left(\frac{x^2}{1-3x-3x^2}\right). \end{aligned}$$

Now let us factor $1 - 3x - 3x^2$ by taking $x = \frac{-3\pm\sqrt{21}}{6}$, where α is the root with $+$ while β, the one with negative. Then

$$\begin{aligned} (1-3x-3x^2) &= -3(x-\alpha)(x-\beta) = -3\alpha\beta\left(1 - \frac{x}{\alpha}\right)\left(1 - \frac{x}{\beta}\right) \\ &= \frac{-3}{ab}(1-ax)(1-bx), \end{aligned}$$

where a, α and b, β are pairs of reciprocals. We can now continue to solve F as a sum of partial fractions as follows

$$F(x) = \frac{-abx^2}{3(1-4x)(1-ax)(1-bx)} = \frac{A}{1-ax} + \frac{B}{1-bx} + \frac{C}{1-4x}.$$

Homogenizing the denominator on the equation to the right we get

$$\frac{-ab}{3}x^2 = A(1-4x)(1-bx) + B(1-4x)(1-ax) + C(1-ax)(1-bx).$$

Since this equation holds for any value of x, we might substitute certain values to get cleaner equations for A, B and C:

$$\begin{cases} (x=\alpha) \Rightarrow \dfrac{-ab}{3}\alpha^2 = A(1-4\alpha)(1-b\alpha) = A\alpha^2(4-a)(b-a), \\ \qquad\qquad \Rightarrow \dfrac{-ab}{3(4-a)(b-a)} = A, \\ (x=\beta) \Rightarrow \dfrac{-ab}{3}\beta^2 = B(1-4\beta)(1-a\beta) = B\beta^2(4-b)(a-b), \\ \qquad\qquad \Rightarrow \dfrac{-ab}{3(4-b)(a-b)} = B, \\ (x=1/4) \Rightarrow \dfrac{-ab}{3\cdot 16} = C(1-a/4)(1-b/4) = \dfrac{C}{16}(4-a)(4-b), \\ \qquad\qquad \Rightarrow \dfrac{-ab}{3(4-a)(4-b)} = C. \end{cases}$$

Comparing coefficients $(1-ax)(1-bx) = 1-(a+b)x+abx^2$, we have that $a+b=3$ and $ab=-3$. Expanding $(4-a)(4-b) = 16-4(a+b)+ab = 16-12-3 = 1$. From this we get $\underline{C=1}$. Also, using the polarization identity it holds that

$$|a+b|^2 - |a-b|^2 = 4ab \Rightarrow |a-b|^2 = -(-12-9) = 21 \Rightarrow |a-b| = \sqrt{21}.$$

We also have that $\beta < -\frac{1}{2} < \alpha$, so $b > -2 > a$. This means that $a-b = -\sqrt{21}$. From this we can replace in A and B's expressions:

$$A =$$

iii) formula

[a] Even though the recurrence is not in terms of c's, it's still a recursive formula. The derivation for c_n's recursive formula lies below this answer.

We can also construct the recurrence in terms of the allowed sequences c_n. Take any length n allowed sequence, then there are two possibilities:

- $\diamond$ The last digit is 0, 1 or 2, then the rest of the sequence is a length $(n-1)$ allowed sequence. For each digit we count c_{n-1} allowed sequences. So in total we have $3c_{n-1}$ allowed sequences.
- $\diamond$ If the last digit is 3, then the second-to-last digit can't be three. There are only 3 other possibilities: 0, 1 or 2. For each of these the remaining length n sequence has to fullfil the condition. Which means we count c_{n-2} allowed sequences per digit.

This total amounts to $c_n = 3c_{n-1} + 3c_{n-2}$. With this recurrence we have counted all the possibilities since the only options for the last digit are the ones mentioned above.