Exercise 1 (Exercise 2)**.** Compute the following:

i) $s_{(5,3,3,1)}s_{(4)}$.

ii) $s_{(2,1)}s_{(2,1)}$.

iii) The decomposition of $V_{(2,1)} \otimes_o V_{(2,1)}$ into irreducible S_6 representations.

Answer

Recall that the product of Schur functions can be calculated using Littlewood-Richardson coefficients as follows:

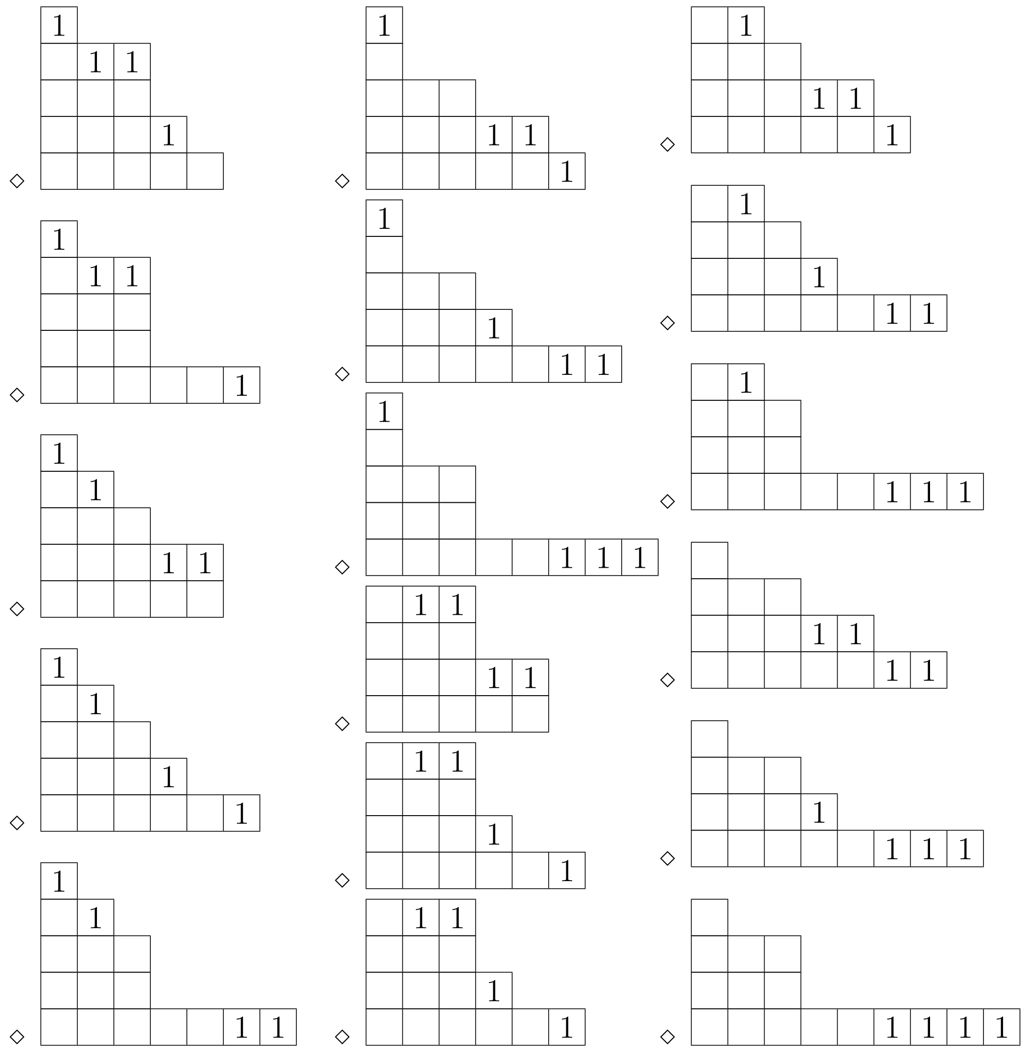
$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

where $c^\lambda_{\mu\nu}$ is the number of Littlewood-Richardson tableaux of shape λ/μ with content ν.
In order to fill out λ/μ with content ν it must hold that $|\lambda| - |\mu| = |\nu|$.

i) In this first case we have $\mu = (5, 3, 3, 1)$ and $\nu = (4)$ which means that λ should partition 16.

This means that we must append 4 new blocks to our partition $(5, 3, 3, 1)$ to obtain a partition of 16 which we will fill with only ones.

As the tableaux must be semi-standard, it can't happen that we append more than one block in the same column. In other words, we must append a horizontal strip of ones. The following tableaux correspond to partitions λ such that λ/μ is a horizontal strip:



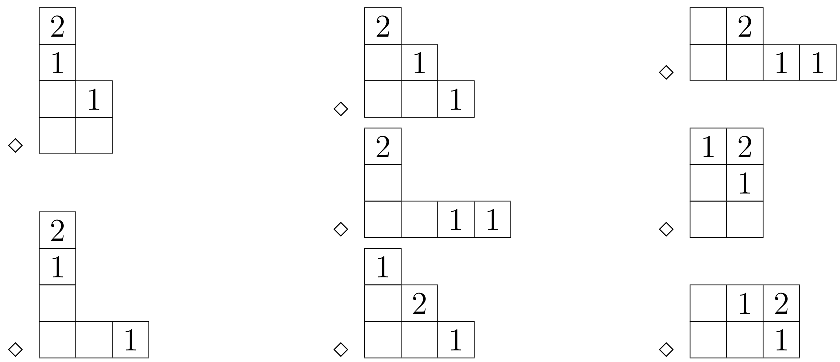
As there is only one possible tableau for each shape we have that each coefficient is one. Therefore we have

$$
\begin{aligned}
s_{(5,3,3,1)}s_{(4)} =& s_{(5,4,3,3,1)} + s_{(6,3,3,3,1)} + s_{(5,5,3,2,1)} + s_{(6,4,3,2,1)} + s_{(7,3,3,2,1)} \\
&+ s_{(6,5,3,1,1)} + s_{(7,4,3,1,1)} + s_{(8,3,3,1,1)} + s_{(5,5,3,3)} + s_{(6,4,3,3)} + s_{(7,3,3,3)} \\
&+ s_{(6,5,3,2)} + s_{(7,4,3,2)} + s_{(8,3,3,2)} + s_{(7,5,3,1)} + s_{(8,4,3,1)} + s_{(9,3,3,1)}.
\end{aligned}
$$

ii) For this next case we have $\mu = \nu = (2,1)$. This means that $\lambda \vdash 6$ and we must append two 1's and one 2. As our tableau must be Littlewood-Richardson, the reading word must be one of either

$$211, \quad \text{or} \quad 121.$$

The following tableaux correspond to $\lambda \vdash 6$ such that λ/μ is a Littlewood-Richardson tableau with content $(2,1)$:



We see that $\mathrm{sh}(\lambda) = (3,2,1)$ is repeated once so we account for a coefficient of 2 with that tableau. The decomposition is thus:

$$s_{(2,1)}^2 = s_{(2,2,1,1)} + s_{(3,1,1,1)} + 2s_{(3,2,1)} + s_{(4,1,1)} + s_{(4,2)} + s_{(2,2,2)} + s_{(3,3)}.$$

iii) By the correspondence of the outer tensor product with the product of Schur functions we have

$$V_{(2,1)} \otimes_o V_{(2,1)} = V_{(2,2,1,1)} \oplus V_{(3,1,1,1)} \oplus 2V_{(3,2,1)} \oplus V_{(4,1,1)} \oplus V_{(4,2)} \oplus V_{(2,2,2)} \oplus V_{(3,3)}.$$

Exercise 2 (Exercise 3)**.** Use the Straightening Algorithm (Garnir Relations) to express the Garnir polynomial F_T where T is the filling

$$\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 1 \\ \hline \end{array}$$

in terms of F_S's where each S is a standard Young tableau. Write out each polynomial in your formula and check that your answer works.

Answer

We first column straighten T by ordering its columns, we obtain

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & 1 \\ \hline \end{array}.$$

To row-straighten we find the topmost row with a descent and consider rightmost decrease. The block in question is

$$\begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

and for this block we will find permutations of S_3 which preserve the columns-increasing condition withing the blocks. We have the permutations

$$\text{id}, \quad (12), \quad (123)$$

which correspond to

$$\begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array}, \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}.$$

This means that

$$(-1)^0(x_3 - x_2)(x_4 - x_1) + (-1)^1(x_3 - x_1)(x_4 - x_2) + (-1)^2(x_2 - x_1)(x_4 - x_3) = 0$$

and from this relation we obtain

$$(x_3 - x_2)(x_4 - x_1) = (x_3 - x_1)(x_4 - x_2) - (x_2 - x_1)(x_4 - x_3).$$

Similarly just by calculating the Garnir polynomial of T we obtain $(x_2 - x_3)(x_4 - x_1)$ which corresponds to our formula, just with a sign change.