MATH519 — Complex Analysis

Based on the lectures by Jeff Achter

Notes written by Ignacio Rojas

Spring 2023

This course is an introduction to analytic functions of a single complex variable. The subject is beautiful.– it turns out that a function with a complex derivative is highly structured – and enjoys a give and take with many other areas of mathematics.

Requirements

Knowledge of convergence of sequences, series: limits, continuity, differentiation, integration of one-variable functions is required.

Contents

Chapter 1

First Midterm

1.1 Interim | HW1

Exercise 1.1.1 (1.1 Stein & Shakarchi)**.** Describe geometrically the sets of points z in the complex plane defined by the following relations:

  (a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$.
  (b) $1/z = \overline{z}$.
  (c) $\mathrm{Re}(z) = 3$
  (d) $\mathrm{Re}(z) > c$, (resp.,$\geqslant c$) where $c \in \mathbb{R}$.
  (e) $\mathrm{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$.
  (f) $|z| = \mathrm{Re}(z) + 1$.
  (g) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$.

> **Answer**
>
>   i) The first set is the set of points at the same distance from z_1 and z_2. If we consider the line segment $z_1 z_2$, then the set in question is the bisector of that line segment.
>   ii) Note that
>
> $$1/z = \overline{z} \iff 1 = \overline{z}z \iff 1 = |z|^2 \iff 1 = |z|,$$
>
>   thus the set is the unit circle.
>   iii) The set is a perpendicular line to the real axis at $z = 3$.
>   iv) This infinite set is an infinite half plane to the right (but not including) of the line $z = c$. In the other case, we do include the line in question.

v) Let us rephrase this inequality in terms of real numbers. Call $a = a_1 + ia_2$, $b = b_1 + ib_2$ and $z = x + iy$. Then

$$\mathrm{Re}(az + b) = \mathrm{Re}[a_1 x - a_2 y + b_1 + i(a_2 x + a_1 y + b_2)],$$

thus our desired inequality is true whenever $a_1 x - a_2 y + b_1 > 0$. Solving for y we get $y > (a_1 x + b_1)/a_2$, which is the half plane located above the line $y = (a_1 x + b_1)/a_2$.

vi) The equation in question is equivalent to

$$\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 = (\mathrm{Re}(z) + 1)^2.$$

To ease the notation, assume $z = x + iy$. Then the equation reads

$$x^2 + y^2 = x^2 + 2x + 1 \iff y^2 = 2x + 1 \iff x = (y^2 - 1)/2.$$

It holds the the parabola in question contains the points which satisfy the equation.

vii) This set is a line parallel to the real axis at $z = c$

Exercise 1.1.2. Do the following:

i) Show that the complex conjugation map $\kappa : \mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$ is an involution, i.e., a ring homomorphism such that $\kappa \circ \kappa = \mathrm{id}$.

ii) Suppose $a \in \mathbb{R}$, $z \in \mathbb{C}$. Show that

$$\mathrm{Re}(az) = a\,\mathrm{Re}(z), \quad \text{and} \quad \mathrm{Im}(az) = a\,\mathrm{Im}(z).$$

> **Answer**
>
> Let us take $z = x + iy$ with $x, y \in \mathbb{R}$.
>
> i) We have $\overline{z} = x + i(-y) = x - iy$. Once more we get $\overline{\overline{z}} = x - i(-y) = x + iy = z$. Thus $\overline{\overline{z}} = z$ for any $z \in \mathbb{C}$. In conclusion $\overline{\overline{\cdot}} = \mathrm{id}$.
>
> ii) It holds that
>
> $$\mathrm{Re}(az) = \mathrm{Re}(ax + aiy) = ax = a\,\mathrm{Re}(z),$$
> $$\mathrm{Im}(az) = \mathrm{Im}(ax + aiy) = ay = a\,\mathrm{Im}(z).$$

Exercise 1.1.3. Do the following:

   i) Prove that $|z + w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w})$.

   ii) Use this to prove the parallelogram rule: $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$.

Answer

  i) Note that

$$|z + w|^2 = (z + w)\overline{(z + w)} = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + w\overline{z} + z\overline{w} + w\overline{w}.$$

     The number $w\overline{z}$ is the conjugate of $z\overline{w}$, and summing a number and its conjugate returns twice its real part. Thus we get the desired identity.

  ii) As the past identity holds for all complex numbers, it holds when $w = -w$. This means that $|z-w|^2 = |z|^2 + |-w|^2 + 2\operatorname{Re}(z(\overline{-w})) = |z|^2 + |w|^2 - 2\operatorname{Re}(z\overline{w})$ and summing this together with the first identity gives us the parallelogram law.

Exercise 1.1.4 (1.5 Stein & Shakarchi). A set Ω is said to be pathwise connected if any two points in Ω can be joined by a (piecewise-smooth) curve entirely contained in Ω. The purpose of this exercise is to prove that an open set Ω is pathwise connected if and only if Ω is connected.

   i) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a parametrization $z : [0, 1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let

$$t_* = \sup_{0 \leqslant t \leqslant 1} \{\, t : \forall s[(0 \leqslant s < t) \Rightarrow (z(s) \in \Omega_1)] \,\}.$$

     Arrive at a contradiction by considering the point $z(t_*)$.

  ii) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subseteq \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subseteq \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega_1$ as desired.

Answer

  i) We will proceed using a topological argument instead of a metric one. As the function γ is continuous, it pulls back Ω_1 and Ω_2 into $[0, 1]$ as open sets. However, as the sets are disjoint, their inverse images are disjoint as well.

In other words, we have found two open disjoint sets which separate $[0, 1]$:

$$[0, 1] = \gamma^{-1}[\Omega_1] \uplus \gamma^{-1}[\Omega_2].$$

But this is impossible because $[0, 1]$ is a connected set. Thus, our assumption that Ω was disconnected must be false. We conclude that path-connectedness implies connectedness.

ii) Take Ω_1, Ω_2 as in the statement. Then Ω_1 is non-empty as $w \in \Omega_1$ because it's connected to itself through a trivial path. Suppose now that $z \in \Omega_1$ and that $d(z, \partial\Omega_1) > r > 0$. Take $x \in B(z, r)$, then there exists a line-segment between z and x and there's a smooth curve which connects $z \in \Omega_1$ with w. Thus the piecewise-continuous path from x to z and from z to w is a path which connects x and w. As x is arbitrary, it follows that $B(z, r) \subseteq \Omega_1$, and thus Ω_1 is open.

Formally, if $\gamma : [0, 1] \to \Omega_1$ is the map which parametrizes the curve between z and w and $r : [0, 1] \to B(z, r)$ is the map $t \mapsto tz + (1 - t)x$, then the curve from x to w is parametrized by the function

$$f = \begin{cases} 2tz + (1 - 2t)x, \ t \in [0, 1/2], \\ \gamma(2t - 1), \ t \in [1/2, 1]. \end{cases}$$

On the other hand take a point $z \in \Omega_2$ and let $d(z, \partial\Omega_2) > r > 0$. Consider a point $x \in B(z, r)$ and assume by way of contradiction that such x can be connected to w by a curve which can be parametrized by a smooth function γ. As the ball is convex, we can connect z to x and then to w creating a path between z and w. This is impossible as z cannot be connected to w by a path, thus our assumption must be false. It holds that x cannot be connected to w by a path and thus $x \in \Omega_2$. Therefore Ω_2 is also open. We conclude that $\Omega = \Omega_1 \cup \Omega_2$ is a union of two disjoint open sets, and since Ω is connected, it must hold that Ω_2 is empty. We conclude that Ω is path-connected.

Exercise 1.1.5 (1.7 Stein & Shakarchi). The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

i) Let $z, w \in \mathbb{C}$ such that $\overline{z}w \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \overline{w}z} \right| < 1$$

if $|z| < 1$ and $|w| < 1$, and also that

$$\left| \frac{w - z}{1 - \overline{w}z} \right| = 1$$

if $|z| = 1$ or $|w| = 1$. [[Hint: Why can one assume that z is real? I then suffices to prove that $(r - w)(r - \overline{w}) \leqslant (1 - rw)(1 - r\overline{w})$ with equality for appropriate r and $|w|$.]][[Here is an alternate approach, which you may use if you like. Fix $w \in \mathbb{C}$ with $w < 1$, and consider the function $z \mapsto \frac{w-z}{1-\overline{w}z}$. What is $\overline{f(z)}$? By computing $f(z)\overline{f(z)}$, show that $|z| = 1$ implies $|f(z)| = 1$. Find a point z with $|z| < 1$ such that $|f(z)| < 1$. Since f is continuous, this shows that f takes the unit disc to itself. (Why?)]]

ii) Prove that for a fixed $w \in \mathbb{D}$, the mapping $F : z \mapsto \frac{w-z}{1-\overline{w}z}$ satisfies the following:

     a) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.
     b) F interchanges 0 and w.
     c) $|F(z)| = 1$ if $|z| = 1$.
     d) F is bijective. [[Hint: Calculate $F \circ F$.]]

Answer

i) The inequality in question is equivalent to

$$0 \leqslant |w - z| < |1 - \overline{w}z|.$$

Since the quantities are positive, we can square them and preserve the order. It holds that

$$0 \leqslant |w - z|^2 < |1 - \overline{w}z|^2 \iff 0 \leqslant (w - z)\overline{(w - z)} < (1 - \overline{w}z)\overline{(1 - \overline{w}z)},$$

Simplifying this expression we get

$$(w - z)(\overline{w} - \overline{z}) < (1 - \overline{w}z)(1 - w\overline{z})$$
$$\iff w\overline{w} - w\overline{z} - z\overline{w} + z\overline{z} < 1 - w\overline{z} - \overline{w}z + \overline{w}zw\overline{z}$$
$$\iff |w|^2 + |z|^2 < 1 + |w|^2|z|^2$$
$$\iff 0 < (1 - |w|^2)(1 - |z|^2).$$

> The inequality is true whenever both moduli are less than one, and whenever either is one equality is achieved.
>
> ii) Now we suppose $w \in \mathbb{D}$ which means that $|w| < 1$. Taking $z \in \mathbb{D}$ and applying F gives us the quantity $\frac{w-z}{1-\overline{w}z}$ which by the previous argument, has modulus less than 1 whenever w, z do.
>
> The function F is holomorphic because it is a quotient of holomorphic functions. The denominator is never zero inside the domain because that would mean that $1 = \overline{w}z$. And taking moduli in both sides of the equation gives us
>
> $$1 = |1| = |w||z| < 1$$
>
> which is impossible.
>
> Now $F(0) = \frac{w-0}{1-0} = w$ and $F(w) = \frac{w-w}{1-|w|^2} = 0$. The denominator in the last expression is never zero because $|w| < 1$.
>
> By the second part of the previous argument it holds that $|z| = 1$ immediately gives us $|F(z)| = 1$. And finally we will see that F is an involution:
>
> $$F(F(z)) = F\left(\frac{w-z}{1-\overline{w}z}\right) = \frac{w - \left(\frac{w-z}{1-\overline{w}z}\right)}{1 - \overline{w}\left(\frac{w-z}{1-\overline{w}z}\right)}.$$
>
> Homogenizing and clearing denominators we get
>
> $$\frac{w(1-\overline{w}z) - w + z}{1 - \overline{w}z - \overline{w}(w-z)} = \frac{-w\overline{w}z + z}{1 - \overline{w}w} = \frac{(-w\overline{w}+1)z}{1 - \overline{w}w} = z.$$
>
> This means that F is it's own inverse and therefore, F is bijective.

1.2   Day 1 | 20230120

The Complex Numbers

To construct the complex numbers we take the real numbers, adjoin a variable and mod out by $\langle x^2 + 1 \rangle$. We can also define $\mathbb{C}$ as $\{\, a + bi \, : \, a, b \in \mathbb{R} \,\}$ with the property $i^2 = -1$. This means that we can multiply complex numbers in the following way:

$$(a + bi)(c + di) = ac + (bc + ad)i + bdi^2 = (ac - bd) + (ad + bc)i.$$

Also as $x^2 + 1$ is irreducible in $\mathbb{R}[x]$, $\mathbb{C}$ is a finite field extension of $\mathbb{R}$ of degree 2. As a 2-dimensional vector space $\{\, 1, i \,\}$ is a basis for $\mathbb{C}$.

The map $a + bi \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$ is not a ring homomorphism, it's a bijection with a bit of structure. The map $z \mapsto \alpha z$, when $\alpha = a + bi$, is a linear map with the following action over the basis

$$\alpha \cdot 1 = \alpha \Rightarrow [\alpha] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\alpha \cdot i = -b + ai \Rightarrow [\alpha] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

which means that $[\alpha] = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. The converse, if we have a $\mathbb{R}$-linear transformation, then it's $\mathbb{C}$-linear if and only if it looks like $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Definition 1.2.1. The complex conjugation map is $a + bi \mapsto a - bi$, or $z \mapsto \overline{z}$.

This map is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

Example 1.2.2. For $\alpha = a + bi$, we have

$$\overline{2\alpha} = \overline{2(a + bi)} = \overline{2a + 2bi} = 2a - 2bi = 2\overline{al}.$$

Whereas if instead

$$\overline{i\alpha} = \overline{ai - b} = -b - ai \neq i\overline{\alpha} = b + ai.$$

As a $\mathbb{R}$-linear map, we can identify with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By looking at the shape of this matrix we can see that it is not $\mathbb{C}$-linear.

Lemma 1.2.3. *The map $z \mapsto \overline{z}$ is a ring homomorphism*

> **Proof**
>
> $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\overline{w}$.

With the complex conjugation we can pick out the real and imaginary parts of $\alpha = a + bi$.

$$\alpha + \overline{\alpha} = 2\operatorname{Re}(\alpha), \quad \alpha - \overline{\alpha} = 2i\operatorname{Im}(\alpha)$$

A Notion of Size

Can't do geometry without one. Notice that for $z = a + bi$

$$z\bar{z} = a^2 + b^2 > 0.$$

From a complex number we have extracted a positive quantity.

Definition 1.2.4. The complex modulus of z is $|z| = \sqrt{z\bar{z}}$.

The fact that every number has n roots is very important in complex analysis. As a vector in the plane, the norm of z is $|z|$

INC FIG

This means that $a + bi \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$ is an isometry. In this sense the distance between two complex numbers is $d(z, w) = |z - w|$.

Polar Coordinates (*ad hoc*)

For $\theta \in \mathbb{R}$, define

$$\exp(i\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta) \Rightarrow |\exp(i\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

Every point in the unit circle is of the form $e^{i\theta}$ and vice-versa.

INC FIG

For non-zero complex numbers, $z = |z|e^{i\theta}$ for some θ.

Definition 1.2.5. For a complex number $z = re^{i\theta}$, an argument of z is θ.

To have a well defined function, we mod out by multiples of 2π:

$$\arg : \mathbb{C}\backslash\{\,0\,\} \to \left.\mathbb{R}\middle/2\pi\mathbb{Z}\right.,$$

and we obtain a group isomorphism. In general, "lengths multiply, angles add."
For inverses if $z = re^{i\theta}$, then $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$.

Definition 1.2.6. The upper-half plane is $\mathbb{H} = \{\,\text{Im}(z) > 0\,\}$.

Lemma 1.2.7. *If H is a half plane* $\text{Im}(z - \beta/\gamma) > 0$

1.3   Day 2 | 20230123

Recall the complex conjugation map and the modulus of a complex number. This gives us an isometry between $\mathbb{R}^2$ and $\mathbb{C}$. Let us prove the lemma from last time.

Lemma 1.3.1. *If $H \subseteq \mathbb{C}$ is a half plane, then there exist $\beta, \gamma \in \mathbb{C}$ such that*

$$H = \left\{ z : \ \mathrm{Im}\left(\frac{z - \beta}{\gamma} \right) > 0 \right\}.$$

INC FIG

Pick a point $\beta \in H$, then translate H to the origin by $z \mapsto z - \beta$. The plane is now rotated by θ at the origin so we should rotate every point. Then $z \in H - \beta$ whenever $ze^{-i\theta} \in \mathbb{H}$. REDO

Let us see an application, for a polynomial, the coefficients determine the roots. The following lemma is a technical lemma.

Lemma 1.3.2. *Suppose $p \in \mathbb{C}[z]$ and H is a half plane which contains all the roots of p. Then H contains all the roots of p'.*

> **Proof**
>
> We can assume p is monic, so suppose $\alpha_1, \ldots, \alpha_d$ are the roots of $\mathbb{C}$. This means that
>
> $$p(z) = \prod_{k=1}^{d}(z - \alpha_k) \Rightarrow p'(z) = \sum_{k=1}^{d} \frac{p(z)}{z - \alpha_k} \Rightarrow \frac{p'(z)}{p(z)} = \sum_{k=1}^{d} \frac{1}{z - \alpha_k}.$$
>
> Now suppose that H contains all α_k and suppose $z_0 \notin H$, if we show $p'(z_0) \neq 0$ we are done because all the points which make p' vanish won't be outside H. Describe H by the previous lemma, there exist β, γ such that points in H satisfy the inequality $\mathrm{Im}\left(\frac{z - \beta}{\gamma} \right) > 0$. As z_0 is not in H, then $\mathrm{Im}\left(\frac{z_0 - \beta}{\gamma} \right) < 0$. For each $k \in [d]$, we have that
>
> $$z_0 - \alpha_k = z_0 - \beta + \beta - \alpha_k = (z_0 - \beta) - (\alpha_k - \beta)$$
>
> so by taking imaginary parts
>
> $$\mathrm{Im}\left(\frac{z - \alpha_k}{\gamma} \right) = \mathrm{Im}\left(\frac{z - \alpha_k}{\gamma} \right) - \mathrm{Im}\left(\frac{z - \alpha_k}{\gamma} \right)$$
>
> The quantity on the right is negative because it's a negative number minus a

positive. So it holds that $\mathrm{Im}\left(\frac{\gamma}{z-\alpha_k}\right) > 0$. With this we can calculate the following:

$$\mathrm{Im}\left(\gamma\frac{p'(z_0)}{p(z_0)}\right) = \mathrm{Im}\left(\sum_{k=1}^{d}\frac{\gamma}{z_0 - \alpha_k}\right) > 0$$

so in particular this number is non-zero. Thus $p'(z_0) \neq 0$

Definition 1.3.3. A set $S \subseteq \mathbb{R}^n$ is <u>convex</u> if for any two points $x, y \in S$, the line segment between x and y is also contained in S. This is

$$\{\, ty + (1-t)x : \ x, y \in S \,\} \subseteq S.$$

The <u>convex hull</u> of S is the intersection of all convex sets containing S.

In the case of a finite set of complex numbers, the convex hull can be found by intersecting half-planes which contain them.

Corollary 1.3.4 (Gauss-Lucas). *The roots of $p'(z)$ are contained in the convex hull of the roots of $p(z)$.*

Metric Spaces

Definition 1.3.5. A <u>metric space</u> is a set with a distance function.

Example 1.3.6. $\mathbb{R}^n$ is a metric space with $d(x, y) = \|x - y\|$. Subsets of metric spaces with an induced distance are metric spaces.

⋄ nbhd

⋄ open and closed

⋄ Cauchy

Definition 1.3.7. Cauchy sequence

1.4 Day 3 | 20230125

The defining property of $\mathbb{R}$ is that it is complete. In that sense it is possible to prove that $\mathbb{R}^n$ is also complete.

Derivatives

Recall a real function g is differentiable at x_0 if there exists a real number a such that

$$g(x) = g(x_0) + a(x - x_0) + \psi(x), \quad \frac{\psi(x)}{x - x_0} \xrightarrow[x \to x_0]{} 0.$$

In the same sense a multivariable function is differentiable when there exists a linear transformation such that a similar condition holds.

Definition 1.4.1. f has complex derivative iff real derivative and Cauchy-Riemann equations

Example 1.4.2. The map $z \mapsto \bar{z}$ is not complex-differentiable. First by matrix definition and second with limit.

1.5 Day 4 | 20230127

Lemma 1.5.1. *If $\sum_{n \geqslant 0} z_n$ is absolutely convergent, then it's convergent.*

> **Proof**
>
> If s_n is a partial sum, then
>
> $$\left| s_n - s_m \right| = \left| \sum_{i=m+1}^{n} z_i \right| \leqslant \sum_{i=m+1}^{n} |z_i| < \varepsilon$$
>
> because $\sum |z_n|$ is Cauchy.

Power Series

Definition 1.5.2. A <u>power series</u> (centered at 0) is an expression of the form $\sum_{n \geqslant 0} a_n z^n$.

Example 1.5.3. The power series for the exponential function is $e^z = \sum_{n \geqslant 0} \frac{z^n}{n!}$.

Theorem 1.5.4 (Cauchy-Hadamard). *Suppose $\sum_{n \geqslant 0} a_n z^n$ has radius of convergence $\frac{1}{r} = \limsup |a_n|^{\frac{1}{n}}$. Then the series converges for $|z| < r$ and diverges for $|z| > r$.*

> **Proof**

1.6 Day 5 | 20230130

Last time with Hadamard's criterion we learned something that we *already know*. Recall that for radii less than the radius of convergence, power series converge.

As a corollary we can prove the following:

Corollary 1.6.1. *Suppose $f(z) = \sum_{n \geqslant 0} a_n z^n$ has radius of convergence R. Then the following holds*:

i) *The formal derivative of f,*
$$g(z) = \sum_{n \geqslant 1} n a_n z^{n-1}$$
converges absolutely and uniformly on $B(0, R)$.

ii) *$f'(z) = g(z)$.*

Proof

Notice that
$$\lim_{n \to \infty} \sqrt[n]{n} = 1 \Rightarrow g \text{ converges.}$$

This is because $\limsup |n a_n|^{1/n} = \limsup |a_n|^{1/n}$.

Call S_N the Nth partial sum of f. For $r < R$, suppose $|z - z_0| < r$. Then

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - g(z) \right| = \left| \frac{S_n(z) - E_n(z) - S_n(z_0) + E_n(z_0) - g(z_0)(z - z_0)}{z - z_0} \right|.$$

Let us now add zero carefully and apply the triangle inequality. The previous term is less than

$$\left| \frac{S_N(z) - S_N(z_0)}{z - z_0} - S_N'(z_0) \right| + |S_N'(z_0) - g(z_0)| + \left| \frac{E_N(z) - E_N(z_0)}{z - z_0} \right|.$$

The last term which contains the errors can be written as

$$\left| \sum_{n \geqslant N} \frac{a_n(z^n - z_0^n)}{z - z_0} \right| \leqslant \sum_{n \geqslant N} n |a_n| r^{n-1}$$

and for large N, this quantity is small. With a similar reasoning we get that

$$|S_N'(z_0) - g(z_0)| \leqslant \sum_{n \geqslant N} n |a_n z^{n-1}|.$$

For z close to z_0, the first term is small as well.

Corollary 1.6.2. *A complex power series is infinitely differentiable.*

Lemma 1.6.3. *The power series of the exponential function satisfies the equality $e^{z+w} = e^z e^w$.*

> **Proof**
>
> $$
> \begin{aligned}
> e^z e^w &= \left(\sum_{n \geqslant 0} \frac{z^n}{n!} \right) \left(\sum_{n \geqslant 0} \frac{w^n}{n!} \right) \\
> &= \sum_{n \geqslant 0} \sum_{k+\ell=n} \frac{z^k}{k!} \frac{w^\ell}{\ell!} \\
> &= \sum_{n \geqslant 0} \frac{1}{n!} \sum_{k+\ell=n} \frac{n!}{k! \ell!} z^{k+\ell} \\
> &\sum_{n \geqslant 0} \frac{1}{n!} (z+w)^n = e^{z+w}.
> \end{aligned}
> $$

Theorem 1.6.4. $e^{i\theta} = \cos(\theta) + i \sin(\theta)$

Lemma 1.6.5. $\theta \in \mathbb{R}$, *then $e^{i\theta} = 1$ iff $\theta \in 2\pi\mathbb{Z}$.*

Proposition 1.6.6. *If $z = re^{i\alpha}$, then β is an argument of z iff $\alpha - \beta \in 2\pi\mathbb{Z}$.*

Corollary 1.6.7. *There is a grp isom $\mathbb{C}^\times \to \mathbb{R}_{\geqslant 0} \times \mathbb{R}/2\pi\mathbb{Z}$.*

1.7   Interim 2 | HW2

Exercise 1.7.1. Suppose $S \subseteq \mathbb{C}$ is a domain and $f : S \to \mathbb{C}$ is differentiable at $z_0 \in S$.

  i) Compute $f'(z_0)$ along a a trajectory $z_0 + \Delta x$ where $\Delta x \to 0$. Show that

$$f'(z_0) = u_x(z_0) + iv_y(z_0).$$

  ii) Compute $f'(z_0)$ along a a trajectory $z_0 + i\Delta y$ where $\Delta y \to 0$. Show that

$$f'(z_0) = (1/i)(u_y(z_0) + iv_x(z_0)).$$

  iii) Conclude that f satisfies the Cauchy-Riemann equations.

> **Answer**
>
> By definition, for $h \in \mathbb{C}$, we have
>
> $$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$
>
> whenever f is differentiable at z_0.
>
> i) Take $h = \Delta x$, a number with no imaginary part. Then separating f into its real and imaginary parts we have
>
> $$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ &= \frac{\partial}{\partial x}u(x_0, y_0) + i\frac{\partial}{\partial x}v(x_0, y_0) \end{aligned}$$
>
> ii) On the flipside, take $h = i\Delta y$ with $\Delta y \to 0$. We once again separate f as follows:
>
> $$\begin{aligned} f'(z_0) &= \lim_{\Delta y \to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i\lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \\ &= \frac{1}{i}\frac{\partial}{\partial y}u(x_0, y_0) + \frac{\partial}{\partial y}v(x_0, y_0) \\ &= \frac{\partial}{\partial y}v(x_0, y_0) - i\frac{\partial}{\partial y}u(x_0, y_0) \end{aligned}$$
>
> iii) As the derivatives along both trajectories should match, we have that
>
> $$\frac{\partial}{\partial x}u(x_0, y_0) + i\frac{\partial}{\partial x}v(x_0, y_0) = \frac{\partial}{\partial y}v(x_0, y_0) - i\frac{\partial}{\partial y}u(x_0, y_0).$$
>
> Two complex numbers are equal whenever both the real and imaginary parts coincide, so it must hold that
>
> $$\frac{\partial}{\partial x}u(x_0, y_0) = \frac{\partial}{\partial y}v(x_0, y_0), \quad \frac{\partial}{\partial x}v(x_0, y_0) = -\frac{\partial}{\partial y}u(x_0, y_0).$$

> If a function is holomorphic for every z, then this translates to $u_x = v_y$ and $v_x = -u_y$.

Exercise 1.7.2 ([1] 1.13). Suppose f is holomorphic in an open set Ω. Prove that in any one of the following cases:

$$\mathrm{Re}(f) \text{ is constant;} \quad \mathrm{Im}(f) \text{ is constant;} \quad |f| \text{ is constant;}$$

one can conclude that f is constant.

Answer

As f is holomorphic, f satisfies the Cauchy-Riemann equations. This means that if $f = u + iv$, then

$$u_x = v_y, \quad v_x = -u_y.$$

◇ If either u or v are constant, then u_x, u_y or v_x, v_y are both zero. In either of those case, by the Cauchy-Riemann equations we can conclude that the other pair of derivatives is zero respectively.

◇ If the complex modulus is constant, then $|f|^2 = u^2 + v^2$ is constant as well. Differentiating the expression with respect to both variables gives us

$$\begin{cases} 2uu_x + 2vv_x = 0 \\ 2uu_y + 2vv_y = 0 \end{cases} \Rightarrow \begin{cases} uu_x + vv_x = 0 \\ uu_y + vv_y = 0 \end{cases}$$

Now, for sake of argument suppose u isn't zero. Applying the Cauchy-Riemann equations we can restate the first equation as follows:

$$\begin{cases} uv_y + v(-u_y) = 0 \\ uu_y + vv_y = 0 \end{cases} \Rightarrow \begin{cases} v_y = \frac{v}{u}u_y \\ uu_y + vv_y = 0 \end{cases}$$

Substituting the first equation into the second we obtain

$$uu_y + v\left(\frac{v}{u}u_y\right) = \left(\frac{u^2 + v^2}{u}\right)u_y = 0$$

from which follows that either $u^2 + v^2 = 0$ or $u_y = 0$. In the first case, as u is a non-zero real function, it is impossible for the sum to be zero. So it must hold that $u_y = 0$.

Doing a similar process by solving for u_y on the second equation we reach the condition that $v_y = 0$ as well. From here, using the Cauchy-Riemann

equations we see that all partial derivatives of u and v are zero as we wished.

In the case that $u = 0$, we refer to the first case, where u is a constant.

Finally we conclude that f is constant in any case.

Exercise 1.7.3. Prove the following:

i) The power series $\sum_{n \geqslant 0} nz^n$ doesn't converge for any point on the unit circle.
ii) The power series $\sum_{n \geqslant 0} \frac{z^n}{n^2}$ converges for *every* point in the unit circle.
iii) The power series $\sum_{n \geqslant 0} \frac{z^n}{n}$ converges for every point in the unit circle, *except $z = 1$*.

Answer

i) We will prove that the series in question isn't Cauchy. Consider S_m, the mth partial sum, then
$$|S_{m+1} - S_m| = m + 1$$
because z has complex modulus 1. Recall that a sequence of complex numbers (z_n) is a Cauchy sequence whenever
$$\forall \varepsilon \exists N \left[\forall m \forall n (m \geqslant N \wedge n \geqslant N \wedge \varepsilon > 0 \Rightarrow |z_m - z_n| < \varepsilon)\right].$$

In order to prove that (S_m) isn't Cauchy we must contradict this statement. Thus we must find an $\varepsilon_0 > 0$ such that for all N, there are m, n for which $|S_m - S_n| > \varepsilon_0$.

Take $\varepsilon_0 = 1$, m any sufficiently large natural number and $n = m + 1$ as we did before. Thus we have that $m + 1 > 1$ which lets us conclude that (S_m) isn't Cauchy. There are no non-Cauchy convergent sequences in $\mathbb{C}$ so it must hold that our series diverges given the condition that $|z| = 1$.

ii) Recall the Weierstrass M-test which states that if $(f_n(z))$ is a sequence of functions and there are $M_n > 0$ such that $|f_n(z)| \leqslant M_n$ and $\sum M_n$ is a convergent series, then $\sum f_n$ converges uniformly.

In this case, pick $M_n = \frac{1}{n^2}$. The series $\sum \frac{1}{n^2}$ converges as it is a p-series. Then
$$|z| = 1 \Rightarrow \left|\frac{z^n}{n^2}\right| \leqslant \frac{1}{n^2}$$
and thus we can conclude that $\sum \frac{z^n}{n^2}$ converges uniformly for points in the unit circle.

iii) The series in question is the harmonic series when $z = 1$ so it diverges. We will prove that when $|z| = 1$, but $z \neq 1$ this series is Cauchy. So let us fix z with $|z| = 1$ and call $S_m = \sum_{k=0}^{m} \frac{z^k}{k}$, then let $\varepsilon > 0$. Assume $n > m$ for sake of argument and then

$$|S_n - S_{m-1}| = \left| \sum_{k=m}^{n} \frac{z^k}{k} \right|$$

$$= \left| \frac{1}{n} \sum_{k=1}^{n} z^k - \frac{1}{m} \sum_{k=1}^{m-1} z^k - \sum_{k=m}^{n-1} \left(\frac{1}{k+1} - \frac{1}{k} \right) \sum_{j=1}^{k} z^j \right|$$

$$\leqslant \frac{1}{n} \left| \frac{z^{n+1} - z}{z - 1} \right| + \frac{1}{m} \left| \frac{z^m - z}{z - 1} \right| + \sum_{k=m}^{n-1} \left| \frac{z^{n+1} - z}{(z - 1)(k^2 + k)} \right|$$

$$\leqslant \frac{2}{n} \left| \frac{1}{z - 1} \right| + \frac{2}{m} \left| \frac{1}{z - 1} \right| + \sum_{k=m}^{n-1} \left| \frac{1}{z - 1} \right| \frac{2}{k^2 + k}$$

Now let us state a couple of facts:

◇ $\left| \frac{1}{z-1} \right|$ might be arbitrarily large, but z is fixed. This means that $\left| \frac{1}{z-1} \right|$ is finite.

◇ Call $\widetilde{S}_r = \sum_{k=1}^{r} \frac{2}{k^2+k}$, it is important to note that this a sequence of *positive numbers*. $\widetilde{S}_\infty$ converges after comparing with $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

We will name $M = \left| \frac{1}{z-1} \right|$ so that the last expression can be written as follows:

$$\frac{2M}{n} + \frac{2M}{m} + M(\widetilde{S}_{n-1} - \widetilde{S}_{m-1}).$$

Now, as $\frac{1}{n}$ converges to zero, there exists $N_1 \in \mathbb{N}$ such that

$$n \geqslant N_1 \Rightarrow \frac{1}{n} < \frac{\varepsilon}{6M}, \ \varepsilon > 0$$

On the other hand as $\widetilde{S}_r$ converges, there exists an $N_2 \in \mathbb{N}$ such that

$$m, n \geqslant N_2 \Rightarrow |\widetilde{S}_m - \widetilde{S}_n| < \frac{\varepsilon}{3M}, \ \varepsilon > 0$$

Pick $N = \max N_1, N_2$ and let $\varepsilon > 0$, then whenever $m, n \geqslant N$, the following holds

$$\frac{2M}{n} + \frac{2M}{m} + M(\widetilde{S}_{n-1} - \widetilde{S}_{m-1}) \leqslant 2M \frac{\varepsilon}{6M} + 2M \frac{\varepsilon}{6M} + M \frac{\varepsilon}{3M} = \varepsilon.$$

Therefore the series in question is Cauchy and we can conclude that it converges.

Exercise 1.7.4. Let $\alpha \in \mathbb{C}$, $r > 0$ and $\gamma_r : [0, 2\pi[\to \mathbb{C}$ given by $t \mapsto re^{it} + \alpha$. Let $n \in \mathbb{N}$, calculate the integral $\int\limits_{\gamma_1(0)} z^n \mathrm{d}z$.

Answer

We can parametrize with $t \mapsto e^{it}$ with $r \in [0, 2\pi[$ so that

$$\int\limits_{\gamma_1(0)} z^n \mathrm{d}z = \int\limits_0^{2\pi} (e^{it})^n (ie^{it}\mathrm{d}t) = \int\limits_0^{2\pi} ie^{i(n+1)t}\mathrm{d}t = \frac{e^{i(n+1)t}}{n+1}\bigg|_0^{2\pi} = \frac{e^{2\pi i(n+1)}}{n+1} - \frac{1}{n+1} = 0.$$

Exercise 1.7.5. Consider the following three groups:

⋄ $\mathbb{C}^\times$ with multiplication as binary operation.
⋄ $\mathbb{R}_{>0}$ with multiplication as binary operation.
⋄ $\mathbb{R}/_{2\pi\mathbb{Z}}$ with addition as binary operation.

Show that

$$\alpha : \mathbb{C}^\times \to \mathbb{R}_{>0} \oplus \mathbb{R}/_{2\pi\mathbb{Z}}, \; z \mapsto (|z|, \arg(z))$$

is a group isomorphism as follows:

i) Show that α is a group homomorphism. ⟦Hint: This comes down to show that $|zw| = |z||w|$ and $\arg(zw) = \arg(z) + \arg(w)$.⟧
ii) Show that α is surjective. ⟦Hint: For $r \in \mathbb{R}_{>0}$ and representative $\theta \in \mathbb{R}$ show that there is some $\mathbb{C}^\times$ such that $|z| = r$ and $\arg(z) = \theta$.⟧
iii) Show that α is injective. ⟦Hint: Suppose $\alpha(z) = (1, 0)$, then show that $z = 1$.⟧

Answer

i) The function α is a homomorphism because

$$\alpha(wz) = (|wz|, \arg(wz)) = (|w||z|, \arg(w) + \arg(z))$$
$$= (|w|, \arg(w)) \circ (|z|, \arg(z)) = \alpha(w) \circ \alpha(z)$$

where $\circ$ is the group operation in the direct product. To prove the equalities hold, take $wz = r_1 e^{i\theta_1}$, $w = r_2 e^{i\theta_2}$ and $z = r_3 e^{i\theta_3}$. Then

$$wz = (r_2 e^{i\theta_2})(r_3 e^{i\theta_3}) = (r_2 r_3)e^{i(\theta_2 + \theta_3)} = r_1 e^{i\theta_1},$$

and as the polar representation of a complex number is unique we have that $r_1 = r_2 r_3$ and $\theta_1 = \theta_2 + \theta_3$.

---

ii) Take (r, θ) in the codomain of α. As $r > 0$, we can write it as $x^2 + y^2$ for $x, y \in \mathbb{R}$. Given that condition we may find the angle by the relation $\tan(\theta) = \frac{y}{x}$. Taking r and θ as given lets us construct a complex number $z = x + iy$ such that $\alpha(z) = (r, \theta)$.

iii) If it happened that $r = 1$ and $\theta = 0$, then the complex number in question could be represented as $1 \cdot e^0 = 1$. Thus $z = 1$. This means that $\ker(\alpha) = \{\,\mathrm{id}\,\}$ and thus, as α is a morphism, it's also injective.

1.8   Day 6| 20230201

Definition 1.8.1. A parametrization of a curve is a function $z : [a, b] \to \mathbb{C}$.

It is smooth if it's differentiable and piecewise smooth if for a partition of $[a, b]$, z is smooth on the parts.

Example 1.8.2. The function $z : [0, 2\pi] \to \mathbb{C}$, $t \mapsto e^{it}$ is a parametrization of the unit circle.

Definition 1.8.3. Two parametrizations w, z are equivalent if there exists a bijection $[a, b] \to [c, d]$ such that $w(s) = z(s(t))$.

Example 1.8.4. An equivalent parametrization of e^{it} is $[0, 1] \to \mathbb{C}$, $t \mapsto e^{2\pi it}$.

The reverse parametrization of $z : [a, b] \to \mathbb{C}$ is $z^- : [-b, -a] \to \mathbb{C}$, $t \mapsto z(-t)$. A curve is closed if it starts where it ends. Simple curves don't cross themselves.

Definition 1.8.5. The integral over a curve γ is

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t$$

where $z : [a, b] \to \gamma$ parametrizes the curve.

Example 1.8.6. Consider the integral of $\bar{z}$ over the unit circle. This is

$$\int_{\{|z|=1\}} \bar{z}\mathrm{d}z = \int_0^{2\pi} \overline{e^{it}}ie^{-it}\mathrm{d}t = \int_0^{2\pi} i\mathrm{d}t = 2\pi i.$$

Integrals over the complex numbers obey the same properties as over the real numbers. The arc length of a curve is the same as in multivariable calculus. The integral also obeys the triangle inequality.

Definition 1.8.7. A <u>domain</u> is a non-empty, open and connected subset of $\mathbb{C}$.

Lemma 1.8.8. *If F, f are functions defined on Ω, a domain, with $F' = f$, and $w, z \in \Omega$, then*

$$\int_\gamma f(z)\mathrm{d}z = F(z) - F(w)$$

where $\gamma \subseteq \Omega$ is a curve connecting w to z.

Corollary 1.8.9. *If the curve is closed the integral is zero.*

As a consequence, the function $\bar{z}$ has no antiderivative in any ball around the origin.

Lemma 1.8.10. *Suppose f is holomorphic on Ω and $f' = 0$ on Ω. Then f is constant.*

Proof

If $w, z \in \Omega$, then

$$0 = \int f' = f(z) - f(w) \Rightarrow f(z) = f(w)$$

so f must be constant.

Next time: Goursat's theorem.

1.9 Day 7 | 20230203

1.10 Day 8 | 20230206

Last time we proved Morera's theorem. Recall Goursat's theorem, which tells us that along a contour a holomorphic function has zero integral. From this it is extracted that holomorphic functions have primitives.

Corollary 1.10.1. *If f is holomorphic on an open disk D and $\gamma \subseteq D$ is a closed contour, then $\int_\gamma f(z)\mathrm{d}z = 0$.*

Proof

As f admits a primitive F on D, the integral in question is $F(end) - F(begin) = 0$.

Toy Contours

A toy contour is a closed curve with well defined interior and exterior and it's *easy to describe*^{TM}.

Example 1.10.2. Squares are toy contours. A hollow circle with a rectangle is a toy contour.

Theorem 1.10.3. *If γ is a toy contour, f is holomorphic on an open set containing γ in its interior, then $\int f(z)\mathrm{d}z = 0$.*

Example 1.10.4. Let us calculate $\int\limits_0^\infty \frac{1-\cos(x)}{x^2}\mathrm{d}x$.

For that, consider the function $z \mapsto \frac{1-e^{iz}}{z^2}$. DRAW FIG. The integral can be separated into

$$\int\limits_{-R}^{-\varepsilon} f + \int\limits_{\gamma_\varepsilon} f + \int\limits_{\varepsilon}^{R} f + \int\limits_{\gamma_R} f.$$

We can parametrize γ_R by $t \mapsto R^{it}$ with $t \in [0, \pi]$. Computing the integral we get the function

$$\frac{1 - \exp(iRe^{it})}{(Re^{it})^2} \Rightarrow |f(t)| \leqslant \frac{1 + |\exp(iRe^{it})|}{|(Re^{it})^2|} \leqslant \frac{2}{R^2}.$$

1.11   Interim 3 | HW3

Exercise 1.11.1 (Exercise 2)**.** Evaluate the integral $\int\limits_{\gamma_1(0)} \mathrm{Re}(z)\mathrm{d}x$ in two ways:

   i) Directly using the definition. ⟦Hint: You can model your calculation on the work we did in class to compute the integral of $\overline{z}$.⟧

   ii) Using the fact that $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$.

> **Answer**
>
> Both integrals will use the parametrization $t \mapsto e^{it}$ with $t \in [0, 2\pi[$.

i) The first integral is

$$\int_0^{2\pi} \mathrm{Re}(e^{it})(ie^{it})\mathrm{d}t = \int_0^{2\pi} ie^{it}\cos(t)\mathrm{d}t = \int_0^{2\pi}(i\cos^2(t) - \sin(t)\cos(t))\mathrm{d}t$$

$$= i\int_0^{2\pi} cos^2(t)\mathrm{d}t - \int_0^{2\pi} \sin(t)\cos(t)\mathrm{d}t = i\pi + 0 = i\pi.$$

ii) The second integral is

$$\int_{\gamma_1(0)} \frac{z+\overline{z}}{2} = \frac{1}{2}\int_{\gamma_1(0)} z\mathrm{d}z + \frac{1}{2}\int_{\gamma_1(0)} \overline{z}\mathrm{d}z = 0 + \frac{1}{2}\int_0^{2\pi} e^{-it}(ie^{it})\mathrm{d}t = \frac{1}{2}(2\pi i) = i\pi.$$

Both calculations coincide in the value of $i\pi$.

Exercise 1.11.2. Suppose f is defined on a domain Ω with $\gamma \subseteq \Omega$, a closed contour. Additionally, suppose that for $\varepsilon > 0$, there exists a polynomial $P_\varepsilon(z)$ such that $|f(z) - P_\varepsilon(z)| < \varepsilon$ for $z \in \gamma$. Show that $\int_\gamma |f(z)|\mathrm{d}z = 0$ and thus conclude that $\int_\gamma f(z)\mathrm{d}z = 0$.

Consider γ an arbitrary, but fixed contour inside Ω. For $\varepsilon/\ell(\gamma) > 0$, there exists $P_{\varepsilon/\ell(\gamma)}(z)$ such that

$$|f(z) - P_{\varepsilon/\ell(\gamma)}(z)| < \frac{\varepsilon}{\ell(\gamma)}.$$

Then, applying the triangle inequality we have

$$\int_\gamma f(z)\mathrm{d}z \leqslant \int_\gamma |f(z)|\mathrm{d}z \leqslant \int_\gamma |f(z) - P_{\varepsilon/\ell(\gamma)}(z)|\mathrm{d}z + \int_\gamma |P_{\varepsilon/\ell(\gamma)}(z)|\mathrm{d}z$$

The first integral can be bounded in the contour γ by hypothesis as follows:

$$\int_\gamma |f(z) - P_{\varepsilon/\ell(\gamma)}(z)|\mathrm{d}z \leqslant \sup_{z\in\gamma}|f(z) - P_{\varepsilon/\ell(\gamma)}(z)|\int_\gamma \mathrm{d}z \leqslant \frac{\varepsilon}{\ell(\gamma)}\ell(\gamma) = \varepsilon.$$

The other integral can't be shown to be zero using the theorems we have at hand.

Lemma 1.11.3. *Suppose f is holomorphic, then there are two possibilities:*

i) *Either $|f|$ is holomorphic and therefore constant (from which we conclude that f is constant).*

ii) *Or $|f|$ is not holomorphic.*

Proof

The function $|f|$ is a real valued complex function. This means that

$$|f(z)| = g(z) + ih(z), \quad \text{with} \quad h = 0.$$

If $|f(z)|$ was holomorphic, it would satisfy the Cauchy-Riemann equations which means that $g_x = g_y = 0$. This can be used to conclude that f is also constant (by a previous homework exercise.)

In the other case, $|f(z)|$ is simply not holomorphic.

By the previous lemma, assuming P_ε is not a constant polynomial, $|P_\varepsilon|$ is not holomorphic. We cannot state results at this moment about the integral $\int_\gamma |P_{\varepsilon/\ell(\gamma)}(z)|\mathrm{d}z$.

As an alternative approach without showing that $\int_\gamma |f(z)|\mathrm{d}z$ we have the following:

Answer

Recall the reverse triangle inequality:

$$|x + y| \leqslant |x| + |y| \Rightarrow |x + y| - |y| \leqslant |x| \xrightarrow{x = \tilde{x} - y} |\tilde{x}| - |y| \leqslant |\tilde{x} - y|.$$

Now take $P_{\varepsilon/\ell(\gamma)}(z)$ as before, then

$$\left| \int_\gamma (f(z) - P_{\varepsilon/\ell(\gamma)}(z))\mathrm{d}z \right| \geqslant \left| \int_\gamma f(z)\mathrm{d}z \right| - \left| \int_\gamma P_{\varepsilon/\ell(\gamma)}(z)\mathrm{d}z \right|$$

and the rightmost integral is zero as $P_{\varepsilon/\ell(\gamma)}$ is a polynomial and therefore, a holomorphic function. Then, applying the integral triangle inequality we have

$$\left| \int_\gamma f(z)\mathrm{d}z \right| \leqslant \left| \int_\gamma (f(z) - P_{\varepsilon/\ell(\gamma)}(z))\mathrm{d}z \right| \leqslant \int_\gamma |f(z) - P_{\varepsilon/\ell(\gamma)}(z)|\mathrm{d}z \leqslant \varepsilon.$$

The last integral is smaller than ε by the previous attempted argument. As $\varepsilon > 0$ is arbitrary and $\left| \int_\gamma f(z)\mathrm{d}z \right|$ is a real number, the only possibility is that it's equal to

zero. The only complex number with zero modulus is the origin, so we conclude that $\int_\gamma f(z)\mathrm{d}z = 0$.

Exercise 1.11.4. Prove that $\int_0^\infty \sin(x^2)\mathrm{d}x = \int_0^\infty \cos(x^2)\mathrm{d}x = \frac{\sqrt{2\pi}}{4}$. These are the Fresnel integrals. Here, $\int_0^\infty$ is interpreted as $\lim_{R\to\infty} \int_0^R$. [[Hint: Integrate the function e^{-z^2} over the path in Figure 14. Recall that $\int_{-\infty}^\infty e^{-x^2}\mathrm{d}x = \sqrt{\pi}$.]]

Answer

The path in question is boundary of the circular sector of fixed radius from $\theta = 0$ to $\theta = \frac{\pi}{4}$. Call γ the curve which bounds the sector. Now notice that

$$\cos\left(x^2\right) + i\sin\left(x^2\right) = e^{ix^2},$$

so we will work with the function e^{iz^2} through the circular sector in question. We have that $\int_\gamma e^{iz^2}\mathrm{d}z$ is zero because the exponential function is holomorphic, but also that integral can be broken down into three pieces as follows:

$$\int_0^R e^{ix^2}\mathrm{d}x + \int_0^{\pi/4} \exp\left[i(Re^{it})^2\right](iRe^{it}) + \int_0^1 \exp\left[i((1-t)Re^{i\frac{\pi}{4}})^2\right](-Re^{i\frac{\pi}{4}})\mathrm{d}t.$$

The second integrand can be bounded as follows:

$$\begin{aligned}
\left|\exp\left[i(Re^{it})^2\right](iRe^{it})\right| &\leqslant R|\exp\left[i(Re^{it})^2\right]| = R|\exp\left[iR^2(\cos(2t) + i\sin(2t))\right]| \\
&= R|\exp\left[iR^2\cos(2t) - R^2\sin(2t)\right]| \\
&= Re^{-R^2\sin(2t)}|e^{i(R^2\cos(2t))}| \\
&= Re^{-R^2\sin(2t)} \xrightarrow[R\to\infty]{} 0
\end{aligned}$$

So, as R grows, we can bound the second integral by small quantity which decreases to zero. The third integral can be manipulated as follows, first consider the exponent in the integrand:

$$i((1-t)Re^{i\frac{\pi}{4}})^2 = i(1-t)^2R^2e^{\frac{i\pi}{2}} = -(1-t)^2R^2.$$

Taking the substitution $u = (1-t)R$ we have $\mathrm{d}u = -R\mathrm{d}t$ and as $t \to 0$, $u \to R$ while $t \to 1 \Rightarrow u \to 0$. The third integral can be written as

$$\int\limits_R^0 e^{-u^2} e^{i\frac{\pi}{4}} \mathrm{d}u = -e^{\frac{i\pi}{4}} \int\limits_0^R e^{-u^2} \mathrm{d}u.$$

We have the following equation at this point

$$\int\limits_0^R e^{ix^2} \mathrm{d}x - e^{\frac{i\pi}{4}} \int\limits_0^R e^{-u^2} \mathrm{d}u = 0$$

where we now take the limit as $R \to \infty$. From here we get

$$\int\limits_0^\infty e^{ix^2} \mathrm{d}x = e^{\frac{i\pi}{4}} \frac{\sqrt{\pi}}{2}.$$

Taking real and imaginary parts of the integral in question gives us the desired result.

1.12 Day 9| 20230208

The Cauchy Integral Formula

Suppose f is holomorphic on a domain Ω which contains a disk D with boundary C. Then for $z \in D$
$$f(z) = \frac{1}{2\pi i} \oint\limits_C \frac{f(w)}{w-z} \mathrm{d}w.$$

Proof

Consider the function $g(w) = \frac{f(w)}{w-z}$ and a keyhole contour around z. We know that inside the contour g is holomorphic, so this means that $\oint\limits_{\Gamma(\delta,\varepsilon)} g(w)\mathrm{d}w = 0$.

Away from z, g is continuous, so this means that

$$\lim_{\delta \to 0} \oint\limits_{\Gamma(\delta,\varepsilon)} g(w)\mathrm{d}w = \oint\limits_{-B(z,\varepsilon)} g(w)\mathrm{d}w + \oint\limits_C g(w)\mathrm{d}w$$

where $B(z, \varepsilon)$ is the ball of radius ε centered at z. We can now write

$$\oint_{-B(z,\varepsilon)} g(w)\mathrm{d}w = \oint_{-B(z,\varepsilon)} \frac{f(w) - f(z)}{w - z}\mathrm{d}w + \oint_{-B(z,\varepsilon)} \frac{f(z)}{w - z}\mathrm{d}w$$

so as f is holomorphic we can bound $\frac{f(w)-f(z)}{w-z}$ with $\sup_B |f'(z)|$ (WATCH OUT).
On the other hand

$$\oint_{-B(z,\varepsilon)} \frac{f(z)}{w - z}\mathrm{d}w = f(z)\int_0^{2\pi} \frac{(-i\varepsilon e^{-it})}{z + \varepsilon e^{-it} - z}\mathrm{d}t = f(z)\int_0^{2\pi} -i\mathrm{d}t = f(z)(-2\pi i).$$

We conclude that

$$\oint_C g(w)\mathrm{d}w = 2\pi i f(z)$$

which we can rearrange to the desired equality.

Example 1.12.1. With the formula we can compute

$$\oint_{B(0,1)} \frac{z}{2z + 1} = \frac{-i\pi}{2}.$$

Theorem 1.12.2. *Suppose f is holomorphic on Ω, then*

   i) *f is infinitely differentiable.*
  ii) *If C is a curve inside Ω,*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{n+1}}\mathrm{d}w.$$

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Exercise 1.13.1. In this problem, we'll use the Cauchy integral formula to show that an analytic function has a power series representation.

Suppose f is analytic on an open set containing $\overline{B}(0, R)$. We will show that, on $B(0, R)$, there is an equality of functions

$$f(z) = \sum_{n \geqslant 0} a_n z^n, \quad \text{where} \quad a_n = \frac{f^{(n)}(0)}{n!}.$$

Let $\mathcal{C} = \partial B(0, R)$ be the circle of radius R centered at the origin oriented in positive direction. Fix z with $|z| < r = R$.

i) Show that

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} g(w) f(w) \mathrm{d}w, \quad \text{where} \quad g(w) = \frac{1}{w} \frac{1}{1 - z/w}.$$

ii) Let $N \in \mathbb{N}$. Show that

$$g(w) = \sum_{n=0}^{N-1} \frac{z^n}{w^{n+1}} + \frac{z^N}{(w-z)w^N}.$$

iii) Show that

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z), \quad \text{where} \quad \rho_N(z) = \frac{z^N}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z)w^N} \mathrm{d}w.$$

iv) Let $M = \sup_{z \in \mathcal{C}} |f(z)|$, show that

$$|\rho_N(z)| \leqslant \frac{rM}{R-r} \left(\frac{r}{R}\right)^{N-1}.$$

⟦ Hint: If $w \in \mathcal{C}$, then $|w - z| \geqslant R - r$. ⟧

v) Show that $\lim_{N \to \infty} \rho_N(z) = 0$.

Answer

i) By Cauchy's formula for $z \in B(0, R)$ we have

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{w - z} \mathrm{d}z.$$

Taking the $\frac{1}{w-z}$ and factoring a w on the bottom we get

$$\frac{1}{w - z} = \frac{1}{w(1 - z/w)} = \frac{1}{w} \frac{1}{1 - z/w}.$$

Replacing inside the integral we get the desired equality.

---

ii) Consider the geometric series with common term $\frac{z}{w}$, we have the following:

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n = \frac{1}{1-z/w} \\ \displaystyle\sum_{n=0}^{N-1} \left(\frac{z}{w}\right)^n = \frac{1-(z/w)^N}{1-(z/w)} \end{cases}$$

Now notice that

$$\frac{1}{1-z/w} - \frac{1-(z/w)^N}{1-(z/w)} = \frac{(z/w)^N}{1-(z/w)}$$

$$\Rightarrow g(w) - \frac{1}{w}\sum_{n=0}^{N-1}\left(\frac{z}{w}\right)^n = \frac{(z/w)^N}{w-z}$$

$$\Rightarrow g(w) = \sum_{n=0}^{N-1}\frac{z^n}{w^{n+1}} + \frac{z^N}{(w-z)w^N}.$$

iii) Replacing the last value on the identity of the first item we get

$$f(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}}\left(\sum_{n=0}^{N-1}\frac{z^n}{w^{n+1}} + \frac{z^N}{(w-z)w^N}\right)f(w)\mathrm{d}w$$

and as the sum is finite, it commutes with the integral without any restrictions. Exchanging the integral we get

$$\sum_{n=0}^{N-1}\frac{1}{2\pi i}\oint_{\mathcal{C}}\left(\frac{z^n}{w^{n+1}}f(w)\mathrm{d}w\right) + \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{z^N}{(w-z)w^N}f(w)\mathrm{d}w.$$

Using Cauchy's integral formula for derivatives we have

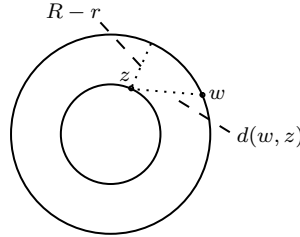$$f^{(n)}(0) = \frac{n!}{2\pi i}\oint_{\mathcal{C}}\frac{f(w)}{(w-0)^{n+1}}\mathrm{d}w$$

$$\Rightarrow \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{f(w)}{w^{n+1}}\mathrm{d}w.$$

So when factoring out the z's from the previous expressions we obtain

$$\sum_{n=0}^{N-1}z^n\frac{f^{(n)}(0)}{n!} + \frac{z^N}{2\pi i}\oint_{\mathcal{C}}\frac{f(w)}{(w-z)w^N}\mathrm{d}w.$$

This is the desired expression.

iv) For this item, it is paramount to remember that $|z| = r < R$ and points $w \in \mathcal{C}$ satisfy $|w| = R$. Now look at the following diagram:

Recall that for a closed set C $d(z, C) = \min_{w \in C} \{ d(w, z) \}$. In the case of the disk of radius R we have that

$$d(z, \mathcal{C}) = R - r \geqslant d(z, w), \quad \text{where} \quad w \in \mathcal{C}.$$

This means that

$$|w - z| \geqslant R - r \implies \frac{1}{|w - z|} \leqslant \frac{1}{R - r}.$$

Now taking the complex modulus of ρ_N for z with $|z| = r$, we have

$$|\rho_N(z)| \leqslant \frac{r^n}{2\pi} \oint_{\mathcal{C}} \frac{|f(z)|}{|w - z||w|^N} \mathrm{d}w \leqslant \frac{r^N}{2\pi} \frac{M}{(R - r)R^N} (2\pi R) = \frac{Mr}{R - r} \left(\frac{r}{R} \right)^{N-1}.$$

v) It suffices to show that $|\rho_N| \to 0$ as this implies $\rho_N \to 0$. Notice that $\frac{r}{R} < 1$ because $r < R$. So this means that $\left(\frac{r}{R} \right)^{N-1} < \frac{\varepsilon(R-r)}{rM}$ where $\varepsilon > 0$ and N is large enough. Thus for such an N we have

$$|\rho_N(z)| \leqslant \frac{rM}{R - r} \left(\frac{r}{R} \right)^{N-1} < \frac{rM}{R - r} \frac{\varepsilon(R - r)}{rM} = \varepsilon.$$

We thus conclude that $|\rho_N(z)| \to 0$ as $N \to \infty$ and therefore $\rho_N \to 0$ as well.

Exercise 1.13.2. For $0 \leqslant r \leqslant n$, the binomial coefficient $\binom{n}{r}$ is defined by $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

i) Let $\gamma = \partial B(0, 1)$, the positive, circular arc around 0 with radius 1. Show that

$$\binom{n}{r} = \frac{1}{2\pi i} \oint_{\gamma} \frac{(1 + z)^n}{z^{r+1}} \mathrm{d}z.$$

⟦In fact, this works for any simple closed curve around the origin.⟧

ii) Use the previous item to show that $\binom{n}{r} \leqslant 2^n$. *This can also be shown directly by computing* $(1+1)^n$.

Answer

i) Notice that the integral in question, by the binomial theorem is:

$$\frac{1}{2\pi i} \oint_\gamma \frac{1}{z^{r+1}} \left(\sum_{k=0}^n \binom{n}{k} z^k \right) \mathrm{d}z = \frac{1}{2\pi i} \sum_{k=0}^n \binom{n}{k} \oint_\gamma z^{k-r-1} \mathrm{d}z.$$

Notice that most of the terms in the sum are actually zero because

$$\oint_\gamma z^{k-r-1} \mathrm{d}z = (2\pi i)\delta_{kr}.$$

With this, we have the desired equality as the expression is equal to

$$\frac{1}{2\pi i} \binom{n}{r} \oint_\gamma z^{-1} \mathrm{d}z = \binom{n}{r}.$$

ii) Now bounding terms in the integral we have that

$$\binom{n}{r} \leqslant \frac{1}{2\pi}(2^n)\mathrm{len}(\gamma) = 2^n.$$

Exercise 1.13.3 ([1] 2.7). Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies $2|f'(0)| \leqslant d$.

Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = az + b$. ⟦In connection with this result, see the relationship between the diameter of a curve and Fourier series described in Problem 1, Chapter 4, Book I.⟧ ⟦Hint: $2f'(0) = \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{f(\zeta)-f(-\zeta)}{\zeta^2} \mathrm{d}\zeta$ when $0 < r < 1$.⟧

Answer

To show the desired identity for $f'(0)$ it suffices to use Cauchy's formula:

$$f'(z) = \frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{(w-z)^2} \mathrm{d}w \Rightarrow f'(0) = \frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{w^2} \mathrm{d}w.$$

Now take the change of variable $w = -u \Rightarrow \mathrm{d}w = -\mathrm{d}u$. The curve through which we are integrating remains the same, as $|w| = |-u| = |u|$. Thus

$$f'(0) = \frac{1}{2\pi i} \oint_{|u|=r} \frac{-f(u)}{u^2}\,\mathrm{d}u.$$

Renaming variables and adding the last two results we get

$$2f'(0) = \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2}\,\mathrm{d}\zeta.$$

Taking the complex modulus we see that

$$2|f'(0)| \leqslant \frac{1}{2\pi} \oint_{|\zeta|=r} \frac{|f(\zeta) - f(-\zeta)|}{|\zeta|^2}\,\mathrm{d}\zeta \leqslant \frac{1}{2\pi r^2} \sup_{|\zeta|=r} |f(\zeta) - f(-\zeta)|(2\pi r)$$

As $\{\,|\zeta| = r\,\} \subseteq \mathbb{D}$ we have that

$$\sup_{|\zeta|=r} |f(\zeta) - f(-\zeta)| \leqslant d \Rightarrow 2|f'(0)| \leqslant \frac{d}{r}.$$

As the last inequality holds for all $0 < r < 1$, in particular we have that

$$|f'(0)| \leqslant \inf_{0<r<1} \frac{d}{r} = d$$

which is the desired inequality.

Exercise 1.13.4. Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then φ is linear. ⟦ Hint: Why can one assume that $z_0 = 0$? Write $z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\varphi_k = \varphi \circ \varphi \circ \cdots \circ \varphi$ (where φ appears k times), then $\varphi_k(z) = k a_n z^n + O(z^{n+1})$. Apply the Cauchy inequalities and let $k \to \infty$ to conclude the proof. ⟧

Answer

Let us begin by making some observations:

⋄ As φ is holomorphic inside Ω, it is analytic inside Ω and thus we may write

it as a power series centered at z_0

$$\varphi(z) = \sum_{n \geqslant 0} a_n (z - z_0)^n.$$

The conditions in the problem imply that $a_0 = z_0$ and $a_1 = 1$ which means that

$$\varphi(z) = z_0 + (z - z_0) + \sum_{n \geqslant 2} a_n (z - z_0)^n.$$

$\diamond$ We can thus consider the function $\varphi(z) - z_0$ and move the whole domain to the origin by translating by z_0. This way we can reinterpret

$$\varphi(z) = z + \sum_{n \geqslant 2} a_n z^n.$$

$\diamond$ Suppose $m \geqslant 2$ is the smallest integer such that $a_m \neq 0$ in the expansion of φ, we this we have

$$\varphi(z) = z + a_m z^m + O(z^{m+1}).$$

$\diamond$ φ is a bounded function because $\mathrm{im}(\varphi) \subseteq \Omega$ and as Ω is bounded, there exists $r > 0$ such that $\Omega \subseteq B(0, r)$. this means that $\|\varphi\|_\infty \leqslant r$.

With this in hand we are ready to proceed. Consider the composition of φ with itself. We have

$$\begin{aligned}
&\varphi\left(z + a_m z^m + O(z^{m+1})\right) \\
&= \left(z + a_m z^m + O(z^{m+1})\right) + a_m \left(z + a_m z^m + O(z^{m+1})\right)^m + O(z^{m+1}) \\
&= z + a_m z^m + a_m z^m + O(z^{m+1}) \\
&= z + 2a_m z^m + O(z^{m+1}),
\end{aligned}$$

where the second-to-last equality comes after expanding the mth power and realizing that all the other terms in the expansion belong in $O(z^{m+1})$. Inductively we have

$$\begin{aligned}
&\varphi\left(z + (k-1)a_m z^m + O(z^{m+1})\right) \\
&= \left(z + (k-1)a_m z^m + O(z^{m+1})\right) + a_m \left(z + (k-1)a_m z^m + O(z^{m+1})\right)^m + O(z^{m+1}) \\
&= z + (k-1)a_m z^m + a_m z^m + O(z^{m+1}) \\
&= z + k a_m z^m + O(z^{m+1}),
\end{aligned}$$

so in general φ_k is what we expect it to be. Notice that $v f_k$ is still a function from Ω to Ω which means it's uniformly bounded independent of k. Now by using the Cauchy inequality we have

$$|D^m \varphi_k(z)| = km!|a_m| \leqslant \frac{m!}{R^m}\|\varphi_k\|_\infty \Rightarrow |a_m| \leqslant \frac{\|\varphi_k\|_\infty}{kR^m} \xrightarrow{k\to\infty} 0.$$

As m is the smallest index such that a_m isn't zero, we conclude that there can be no such smallest index. All the a_m are zero except for the linear coefficient and thus we conclude that φ is a linear function.

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We have used the Cauchy integral formula for derivatives. As a corollary we have that

Corollary 1.14.1. *If f is holomorphic about $z_0 \in \Omega$, then f is analytic at z_0.*

Theorem 1.14.2. *Suppose f is holomorphic on a domain Ω with $(z_n) \subseteq \Omega$ such that $z_n \to z_0 \in \Omega$. Suppose that $f(z_n) = 0$ for all n, then f is identically zero.*

Proof

About z_0 we have

$$f(z) = \sum_{n \geqslant m} a_n(z - z_0)^n = (z - z_0)^m \sum_{n \geqslant m} a_n(z - z_0)^{n-m} = (z - z_0)^m g(z),\ a_m \neq 0.$$

We also have that $g(z_0) = a_m \neq 0$. As g is holomorphic, $|z - z_0| < \varepsilon$ implies $g(z) \neq 0$. But for large enough k, if $|z_k - z_0| < \varepsilon$, if $f(z_k) = 0$, then

$$f(z_k) = (z_k - z_0)^m g(z_k).$$

Theorem 1.14.3. *Suppose f, g are holomorphic on a domain Ω. Suppose also that there is a non-constant sequence $(z_n) \subseteq \Omega$ such that $z_n \to z \in \Omega$ and $f(z_n) = g(z_n)$. Then $f = g$.*

Proof

Letting $h = f - g$, we see that h is identically zero with the previous result so $f = g$.

Definition 1.14.4. We say a that a sequence of functions (f_n) converges to f pointwise on S, if

$$\forall z (f_n(z) \xrightarrow{n \to \infty} f(z)).$$

Meanwhile the convergence is uniform if

$$\|f_n - f\|_\infty \xrightarrow{n \to \infty} 0.$$

Example 1.14.5. The sequence of functions (z^n) converges pointwise to δ_{z0} which is not continuous. Uniform convergence preserves continuity so (z^n) is not uniformly convergent.

Power series converge uniformly.

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Bibliography

[1] Lawrence C. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010.