Exercise 1 (Weak derivatives and membership in $W^{1,2}$)**.** In talking about the definition of weak derivatives, we have discussed that functions with kinks and, in some cases, functions that are discontinuous can have weak derivatives even though they clearly are not differentiable in the "classical" sense.

But not all discontinuous functions have a weak derivative. For example, the following function of one variable,

$$u(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x \geqslant 0, \end{cases}$$

does not have a weak derivative. Convince yourself that that is true, and explain your thinking.

Next, consider the function $u : \mathbb{R}^2 \to \mathbb{R}$ of two arguments,

$$u(\mathbf{x}) = \sin(\arctan(x_2/x_1))$$

that is discontinuous at the origin?[1] Can you guess a weak gradient for this function u (which is of course a two-dimensional vector field) and prove that it really is *the* weak gradient?

If so, what spaces $W^{1,p}(B_1(0))$ is u in if we take the unit ball in $\mathbb{R}^2$ as the domain?

(Hint: Plot the function. Then think about whether there is possibly a coordinate system better suitable to the task.)

I sadly, was unable to tackle this exercise.

Exercise 2. The dual space X' of a vector space X is the set of all linear, continuous functionals $\varphi : X \to \mathbb{R}$.

We say that a functional φ is bounded if it satisfies the condition

$$|\varphi(x)| \leqslant c\|x\|_X$$

for all $x \in X$ and with some constant $c = c(\varphi) < \infty$.

Show that (i) if $\varphi \in X'$, then it is bounded; and (ii) that if a linear functional is bounded, then it is also continuous. (In other words, a linear functional φ is in X' *if and only if* it is also bounded.)

[1]For the given expression for u to make sense, we need to make sure that you define $\arctan$ with the right branch cut. It is here to be understood as $\arctan(x_2/x_1) = \theta$ where θ is the angle a point $\mathbf{x} \in \mathbb{R}^2$ makes against the positive x_1 axis.

Answer

Suppose φ is bounded, then there is a $c > 0$ such that

$$|\varphi(x)| \leqslant c\|x\|.$$

For $\varepsilon > 0$, let $\delta = \varepsilon/c$ and suppose $\|x - y\| \leqslant \delta$ for $x, y \in X$. Then

$$|\varphi(x) - \varphi(y)| = |\varphi(x - y)| \leqslant c\|x - y\| \leqslant c\delta = \varepsilon$$

and so φ is continuous.
If φ were continuous, it would be continuous at $y = 0$. So for all ε, in particular $\varepsilon = 1$, there exists a $\delta > 0$ such that $\|x\| \leqslant \delta$ implies

$$|\varphi(x)| = |\varphi(x) - \varphi(0)| \leqslant 1.$$

About zero, φ is bounded by any value as $\varphi(0) = 0$, so now take $x \neq 0$. The vector $\delta(x/\|x\|)$ has norm δ, so by our previous observation it holds that

$$\varphi\left(\delta \frac{x}{\|x\|}\right) \leqslant 1 \Rightarrow \varphi(x) \leqslant \frac{1}{\delta}\|x\|.$$

Thus φ is bounded and therefore, boundedness is equivalent to continuity here.

Exercise 3. Every $a \in L^q$ induces a bounded linear functional $\varphi : L^p \to \mathbb{R}$ of the form

$$\varphi(u) = \int_\Omega a(x)u(x)\,\mathrm{d}x.$$

Here and below, we will always assume that $\frac{1}{p} + \frac{1}{q} = 1$, and that $\Omega \subset \mathbb{R}^d$ is a bounded domain.

Show in a first step that this functional is linear and bounded (and consequently continuous), i.e., that indeed we have $\varphi \in (L^p(\Omega))'$.

It is of course conceivable that X' is indeed larger than just the functionals introduced above. One of the possibilities would be that one could choose a larger class of functions a than just the L^q functions above. Show that this is not the case, i.e., that a function $a \notin L^q$ (for example if $a \in L^r$ with $r < q$ but $a \notin L^q$) does not induce a functional $\varphi \in X'$.

This statement is not easy to show in its full generality – though as often, it's all about finding the right approach. Here, it means showing that for such an a, φ can not be linear and continuous (or linear and bounded). If you don't see how to do this, create an example: Pick a particular p and corresponding q, then choose a specific $a \in L^r \backslash L^q$ (for example, a function with a singularity) and show that the corresponding φ is either not linear, not continuous, or not bounded by playing with functions $u \in L^p$ and investigating what $\varphi(u)$ is.

(It is worth noting that this argument only shows that linear functionals of the form shown above with $a \in L^r \backslash L^q$ do not give rise to functionals φ in $(L^p(\Omega))'$. It does not show that there are no *completely different* ways to construct linear and continuous functionals. It is the Riesz representation theorem that states that such other ways do not, in fact, exist.)

Answer

To show that φ is linear, consider $u, v \in L^p$ and $c \in \mathbb{R}$, then

$$\begin{aligned}
\varphi(cu+v) &= \int_\Omega a(x)(cu+v)(x)\mathrm{d}x \\
&= \int_\Omega ca(x)u(x) + a(x)v(x)\mathrm{d}x \\
&= c\int_\Omega a(x)u(x)\mathrm{d}x + \int_\Omega a(x)v(x)\mathrm{d}x \\
&= c\varphi(u) + \varphi(v)
\end{aligned}$$

which shows φ is linear. On the other hand observe that using Hölder's inequality we have

$$\begin{aligned}
&|\varphi(u)| = \left|\int_\Omega a(x)u(x)\mathrm{d}x\right| \leqslant \|a\|_{L^q}\|u\|_{L^p} \\
\Rightarrow &\frac{|\varphi(u)|}{\|u\|_{L^p}} \leqslant \|a\|_{L^q} \\
\Rightarrow &\sup_{u\neq 0}\frac{|\varphi(u)|}{\|u\|_{L^p}} \leqslant \|a\|_{L^q}
\end{aligned}$$

and this last quantity is the operator norm of φ. This means that $\|\varphi\|_{\text{op}} \leqslant \|a\|_{L^q}$ and so φ is a bounded functional.

Let us consider the function $a(x) = 1/x^{2/3a}$ which is in $L^1([0,1])$, its integral has a value of 3, but not in $L^2([0,1])$. Observe that linearity doesn't depend on our choice of function, what should fail is boundedness of the functional. So let us pick a function in $u \in L^2$ like $u(x) = \frac{1}{x^{1/3}}$, observe that when we apply the function φ_a to u we get

$$\int_0^1 \left(\frac{1}{x^{2/3}}\right)\left(\frac{1}{x^{1/3}}\right) \mathrm{d}x = \int_0^1 \frac{1}{x}\mathrm{d}x = \infty.$$

In order to show φ_a is unbounded we can construct a sequence of functions $(u_n) \subseteq L^2$ which converged to $u(x)$ and then show that the operator norm of φ_a does indeed grow without bound.

[a] Thanks for helping me out with the example.

Indeed as mentioned, this is not sufficient to show that all functionals are of the desired form. Also a non-explicit way to construct an unbounded functional would be to invoke Zorn's lemma. It is indeed with the Riesz representation theorem that we can show how all functionals look.

But quick question, how does one represent the evaluation functionals? I believe that those functionals are unbounded so we can't represent them in the desired way, is that correct?

Exercise 4. The previous problem claimed that every (dual) functional $\varphi \in (L^p(\Omega))'$ can be written in the form

$$\varphi(u) = \int_\Omega a(x)u(x)\,\mathrm{d}x$$

for some $a \in L^q$. Furthermore, we have defined the norm on the dual space as

$$\|\varphi\|_{(L^p(\Omega))'} = \sup_{u \in L^p(\Omega)} \frac{|\varphi(u)|}{\|u\|_{L^p(\Omega)}}.$$

Prove that

$$\|\varphi\|_{(L^p(\Omega))'} = \|a\|_{L^q(\Omega)}.$$

(Hint: It is not difficult to show that $\|\varphi\|_{(L^p(\Omega))'} \leqslant \|a\|_{L^q(\Omega)}$ using Hölder's inequality. To complete the proof, find a $u \in L^p$ so that $|\varphi(u)| = \|a\|_{L^q(\Omega)}\|u\|_{L^p(\Omega)}$ for which a good approach is to try $u(x) = |a(x)|^s \operatorname{sign}(a(x))$ with some exponent s to be chosen conveniently. You'll then have to check that this u is in L^p, as well as what $\varphi(u)$ is.)

Answer

Indeed consider the function u as hinted with exponent $s = q/p$. We claim that the $\sup$ is reached for such a u. This is, $\|\varphi\|_{\text{op}} = \|a\|_{L^q}$. Observe that

$$|\varphi(u)| = \left|\int_\Omega a(x)|a(x)|^{q/p}\operatorname{sgn}(a(x))\mathrm{d}x\right|$$

and $a(x)\operatorname{sgn}(a(x)) = |a(x)|$ so the integrand is $|a(x)|^{q/p+1}$. The exponent in question is in fact q, because

$$\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{q}{p} + 1 = q.$$

From this we have $|\varphi(u)| = \|a\|_{L^q}^q$.
On the other hand the L^p norm of u is

$$\int_\Omega ||a(x)|^{q/p}\operatorname{sgn}(a(x))|^p\mathrm{d}x = \int_\Omega |a(x)|^q\mathrm{d}x = \|a\|_{L^q}^q.$$

I feel that I'm really really close. I just can't finish the argument. How does one finish the argument?

Exercise 5. Define the function w on $[0,1]$ as

$$w(x) = \begin{cases} \alpha & \text{if } x \leqslant \frac{1}{2} \\ \beta & \text{if } x > \frac{1}{2}, \end{cases}$$

and let $\bar{w}$ be its periodic extension to all of $\mathbb{R}$.

Next consider the following sequence of functions on the set $\Omega = [0,1]$:

$$u_n(x) = \bar{w}(nx).$$

In other words, $u_n(x)$ takes the periodic extension $\bar{w}(x)$, compresses it by a factor of n in x direction, and then restricts consideration to the domain $[0,1]$.

Show the following statements

(a) The sequence u_n does not converge strongly to any u in any of the L^p spaces, $1 \leqslant p \leqslant \infty$.

(b) We have

$$u_n(x) \rightharpoonup \frac{\alpha+\beta}{2} \qquad \text{in } L^p$$

for any $1 \leqslant p < \infty$.

(c) We have

$$u_n(x) \xrightarrow{*} \frac{\alpha + \beta}{2} \qquad \text{in } L^\infty.$$

(A similar result is actually true for more general cases: If you start with *any* bounded function $w : [0,1] \to \mathbb{R}$ and its periodic extension $\bar{w}$, then the u_n defined above converges weakly or weak-* to the *mean value* $\int_0^1 w(x)\,\mathrm{d}x$. The proof is not much different but does not shed any further light on the issue, so we are content with the simpler case above.)

Answer

(a) Let us assume that u_n converges weakly to $1/2(\alpha + \beta)$. Observe that if u_n were to converge strongly, then it should converge to its weak limit and for sake of argument assume $\beta = -\alpha$, in this case the limit is zero.

If we let $m = \min(\alpha, -\alpha)$ then taking the L^p norm of u_n we get

$$\|u_n\|_{L^p}^p = \int_0^1 |u_n(x)|^p \mathrm{d}x = \int_0^1 |\overline{w}(nx)|^p \mathrm{d}x \geqslant \int_0^1 |m|^p \mathrm{d}x = |\alpha|^p$$

which means that if $\alpha \neq 0$ (and we are assuming ($\beta = -\alpha$)), then the sequence is always bounded below and thus can't go to zero in L^p norm. On the other hand if it were the case that $\alpha = 0$ then the result would be true and u_n would converge to zero.

(b) We must show that $\int_0^1 u_n v \mathrm{d}x \to 0$ for $v \in L^q$ where $1/p + 1/q = 1$. Let $0 = a_0 < a_1 < \cdots < a_r = 1$, by density of step functions in L^q we have that

$$s(x) = \sum_{k=1}^{r} \alpha_k \mathbf{1}_{A_k}(x), \quad \text{where} \quad A_k =]a_{k-1}, a_k[^{a}$$

approximates v: $\|v - s\|_{L^q} \leqslant \varepsilon$ for $\varepsilon > 0$. Then

$$\int_0^1 u_n v \mathrm{d}x = \int_0^1 u_n(v - s + s)\mathrm{d}x = \int_0^1 u_n(v - s)\mathrm{d}x + \int_0^1 u_n s \mathrm{d}x.$$

We may bound this quantity using the triangle inequality as follows:

$$\left|\int_0^1 u_n v \mathrm{d}x\right| \leqslant \int_0^1 |u_n||v - s|\mathrm{d}x + \left|\int_0^1 u_n s \mathrm{d}x\right|.$$

The first integral we bound using Hölder's inequality as follows:

$$\int_0^1 |u_n||v-s|\mathrm{d}x \leqslant \|u_n\|_{L^p}\|v-s\|_{L^q}$$

and observe that from our first discussion about u_n we may deduce that its norm is actually w's norm. We may use a change of variables to see that

$$\int_0^1 |\overline{w}(nx)|^p \mathrm{d}x = \int_0^n |\overline{w}(y)|^p (\mathrm{d}y/n), \quad \text{where} \quad \left(\substack{y=nx \,\Rightarrow\, \mathrm{d}y=n\mathrm{d}x \\ x=0, x=1 \,\Rightarrow\, y=0, y=n}\right).$$

But the last integral contains a periodic function of period 1. So this is actually

$$\int_0^n |\overline{w}(y)|^p (\mathrm{d}y/n) = n\int_0^1 |\overline{w}(y)|^p (\mathrm{d}y/n) = \int_0^1 |\overline{w}(y)|^p \mathrm{d}y = \|w\|_{L^p}^p.$$

This lets us conclude that $\|u_n\|_{L^p} = \|w\|_{L^p}$ and as w is a step function, we can find its integral to be

$$\|w\|_{L^p}^p = \int_0^{1/2} |\alpha|^p \mathrm{d}x + \int_{1/2}^1 |\beta|^p \mathrm{d}x = \frac{|\alpha|^p + |\beta|^p}{2} < \infty.$$

So in this way we may finish bounding our original first integral as $\int_0^1 |u_n||v-s|\mathrm{d}x \leqslant \varepsilon\|w\|_{L^p}$.

The second integral we rewrite as

$$\int_0^1 u_n s \mathrm{d}x = \sum_{k=1}^r \alpha_k \int_0^1 u_n \mathbf{1}_{A_k} \mathrm{d}x = \sum_{k=1}^r \alpha_k \int_{A_k} u_n \mathrm{d}x$$

and we treat the integrals over A_k as follows:

$$\begin{aligned} \int_{A_k} u_n \mathrm{d}x &= \int_{a_{k-1}}^{a_k} u_n \mathrm{d}x \\ &= \int_{a_{k-1}}^{a_k} \overline{w}(nx)\mathrm{d}x \\ \left(\substack{y=nx \,\Rightarrow\, \mathrm{d}y=n\mathrm{d}x \\ y=na_{k-1}, y=na_k}\right) &= \int_{na_{k-1}}^{na_k} \overline{w}(y)(\mathrm{d}y/n). \end{aligned}$$

This last integral we chop into pieces:

$$\begin{aligned}&\int_{na_{k-1}}^{na_k} \overline{w}(y)(\mathrm{d}y/n)\\ =&\frac{1}{n}\left(\int_{na_{k-1}}^{\lfloor na_{k-1}\rfloor+1} \overline{w}(y)\mathrm{d}y + \int_{\lfloor na_{k-1}\rfloor+1}^{\lfloor na_k\rfloor} \overline{w}(y)\mathrm{d}y + \int_{\lfloor na_k\rfloor}^{na_k} \overline{w}(y)\mathrm{d}y\right)\end{aligned}$$

and observe that the quantities

$$[(\lfloor na_{k-1}\rfloor + 1) - na_{k-1}] \quad \text{and} \quad (na_k - \lfloor na_k\rfloor)$$

are both between 0 and 1. Also the second integral is between two integer values. We may apply periodicity of $\overline{w}$ to conclude that

$$\left|\int_0^1 u_n s \mathrm{d}x\right| \leqslant \frac{2}{n}\int_0^1 |\overline{w}|\mathrm{d}y + \frac{\lfloor na_k\rfloor - \lfloor na_{k-1}\rfloor + 1}{n}\left|\int_0^1 \overline{w}\mathrm{d}y\right|.$$

The first integral is $(2/n)\|\overline{w}\|_{L^1}$ and the second one is zero because we have assumed that $\beta = -\alpha$. In total, we have

$$\left|\int_0^1 u_n v \mathrm{d}x\right| \leqslant \varepsilon\|w\|_{L^p} + \frac{2}{n}\|\overline{w}\|_{L^1}$$

and letting n grow we get the desired result.

(c) Observe that the previous argument also applies for weak * convergence as we may take $v \in L^1$.

[a] I know this is the other notation, but I just get too confused when reading it with parenthesis because I read it as an ordered pair.