Exercise 1 (5.1.A Vakil)**.** Show that $\mathbb{P}^n_k$ is irreducible.

Answer

Let us take two closed sets in $\mathbb{P}^n_k$ generated by homogeneous ideals I, J. Then

$$V(I) \cup V(J) = V(IJ) = \{\, \mathfrak{p} : \mathfrak{p} \text{ is homogeneous}, \; \mathfrak{p} \supseteq IJ \,\}.$$

For any $\mathfrak{p} \in V(IJ)$, either $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$.
If it were the case that $\mathbb{P}^n_k = V(I) \cup V(J)$ then every ideal $\mathfrak{p}$ would contain either I or J. I can't seem to finish it here, but I'm feeling I'm very close.

Exercise 2 (5.1.G Vakil)**.** Show that affine schemes are quasi-separated. ⟦Hint: 5.1.F Vakil⟧

To solve this exercise we will use the equivalent condition in the following lemma:

Lemma 1 (5.1.F Vakil)**.** *Show that a scheme is quasi-separated if and only if the intersection of any two affine open subsets is a finite union of affine open subsets.*

Answer

To show that an affine scheme X is quasi-separated it suffices to show that the intersection of $U, V \subseteq X$, affine and open, is a finite union of affine and open subsets. As U, V are affine and open we have that they are isomorphic to Spec of something. This means that

$$U \simeq \operatorname{Spec} A, \quad \text{and} \quad V = \operatorname{Spec} B$$

and as the distinguished open sets form a basis we have

$$U \subseteq \bigcup_{f \in A} D(f) \Rightarrow U \subseteq \bigcup_{i=1}^{n} D(f_i)$$

because of quasi-compactness. Similarly $V = \bigcup_{j=1}^{m} D(g_j)$ and this way we have

$$U \cap V = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} D(f_i) \cap D(g_j) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} D(f_i g_j).$$

The double union in question is a finite union of affine open sets, we conclude that X is quasi-separated.

Exercise 3 (5.2.D Vakil)**.** Show that $\left({}^{k[x,y]}/_{\langle y^2,xy\rangle}\right)_x$ has no nonzero nilpotent elements. [[Hint: Show that it is isomorphic to another ring, by considering the geometric picture. Exercise 3.2.L may give another hint.]]

Show that the only point of $\operatorname{Spec} {}^{k[x,y]}/_{\langle y^2,xy\rangle}$ with a non-reduced stalk is the origin.

Answer

Algebraically the ring in question contains elements of the form

$$\frac{ay + b_0 + b_1 x + \cdots + b_n x^n}{x^m}$$

but we must observe that here $x \neq 0$. So the relation $xy = 0$ implies that $y = 0$. Therefore our ring only contains elements of the form $p(x)/x^m$. For a power of this to be zero, we require the original p to have already been null. In conclusion, our ring is reduced.

Geometrically, the picture that we have when taking the quotient by $\langle y^2, xy\rangle$ is the xy axes intersected with a *fuzzy* x axis. What we are left with is an x axis with a fuzzy origin. But localizing removes the origin, and the origin carries the *fuzziness* with it. What we are left with is a *fuzziless* line without an origin. And recall, no *fuzziness* is equivalent to being reduced.

The stalk of $\operatorname{Spec} {}^{k[x,y]}/_{\langle y^2,xy\rangle}$ at the origin is

$$\mathcal{O}_{\operatorname{Spec}(\dots),\langle x-0,y-0\rangle} = \left({}^{k[x,y]}\Big/_{\langle y^2,xy\rangle}\right)_{\langle x,y\rangle}$$

In this ring we do have $y \neq 0$, but $y^2 = 0$.

Exercise 4 (5.2.I Vakil)**.** Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} A$ is any nonempty affine open subset of X. Show that $\mathcal{O}_{X,\eta}$ (the stalk of $\mathcal{O}_X$ at η) is naturally identified with $K(A)$, the fraction field of A.

This is called the function field $K(X)$ of X. It can be computed on any nonempty open set of X, as any such open set contains the generic point. The elements of $K(X)$ are called rational functions on X (to be generalized further in Definition 6.5.35).

Answer

As η is generic, we may identify it with a prime ideal in A and so $\eta \in \operatorname{Spec} A$. Now, as $\operatorname{Spec} A$ is an open set, we have

$$\mathcal{O}_X(\operatorname{Spec} A) = A$$

and as X is integral, A is an integral domain. This implies that A is reduced and therefore

$$\langle 0\rangle = \mathrm{Nil}(A) = \bigcap_{\mathfrak{p}\in\operatorname{Spec} A} \mathfrak{p}.$$

We have that η is generic in X so it's generic in A. Taking closures inside A we get

$$\overline{\eta} = \{\, \mathfrak{p} \in \operatorname{Spec} A : \ \mathfrak{p} \supseteq \eta \,\} = \operatorname{Spec} A.$$

This means that any prime ideal is a prime ideal containing η, thus

$$\eta \subseteq \bigcap_{\mathfrak{p}\in\operatorname{Spec} A} \mathfrak{p} = \langle 0\rangle$$

and so we conclude that

$$\mathcal{O}_{X,\eta} \simeq \mathcal{O}_{\operatorname{Spec} A,\langle 0\rangle} = (A\backslash\langle 0\rangle)^{-1}A = K(A)$$

as we desired.