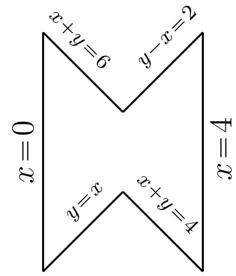
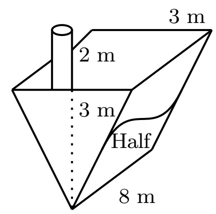
Consider the following figures:





Exercise 1. The figure on the left describes a cross-section of a solid of revolution bounded by the curves

$$\{y=x, x+y=6\},\ x\in[0,2], \text{ and } \{x+y=4, y-x=2\},\ x\in[2,4].$$

The density for such a cross section is given by the equation $\rho(y)=1-3y$. Express the mass of the solid of revolution obtained after rotating about the axis $y=8$ as a sum of integrals.

[`You may use any method, in any case your answer will involve more than one integral.`]

By using the method of rings, we identify two sections of our solid, from 0 to 2 and 2 to 4.
The first section obeys the equations

$$r_1=(8)-(6-x),\ R_1=(8)-(x)$$

while the second one obeys

$$r_1=(8)-(2+x),\ R_2=(8)-(4-x).$$

We must find the density in terms of x, $1-3y=\rho(y)=x$ gives us $y=(1-x)/3$. Then the mass will be

$$\begin{aligned} m &= \int_0^2 \pi(R_1^2-r_1^2)(1-x)/3\,\mathrm{d}x + \int_2^4 \pi(R_2^2-r_2^2)(1-x)/3\,\mathrm{d}x \\ &= \int_0^2 \pi[(8-x)^2-(2+x)^2](1-x)/3\,\mathrm{d}x + \int_2^4 \pi[(4+x)^2-(6+x)^2](1-x)/3\,\mathrm{d}x. \end{aligned}$$

Doing this problem by cylinders can be done creatively, we can find the mass of the "holes". In both of them the radius will be $r=8-y$, and the heights will be

$$h_1=(4-y)-(y),\ h_2=(y-2)-(6-y).$$

So by subtracting this mass from the total mass of our region we get

$$m=\int_0^6 2\pi(8-y)(4)(1-3y)\,\mathrm{d}y - \int_0^2 2\pi(8-y)(4-2y)(1-3y)\,\mathrm{d}y - \int_4^6 2\pi(8-y)(2y-8)(1-3y)\,\mathrm{d}y.$$

Exercise 2. The figure on the right describes a tank half-full (filled up to half of the total height) of water ($\rho=1000$ kgm^{-3}). Find the work required to pump out the water from the tank.

A *very thin* slice of the water in the tank can be realized as a rectangular prism with dimensions

$$\ell=x \text{ m},\ d=8 \text{ m},\ h=\mathrm{d}y.$$

When taking a cross section of the tank parallel to the y-axis, we can see that the length ℓ is governed by the equation $x=2y$ so the volume of a slice is $V=\ell\cdot d\cdot h=(2y)(8)\mathrm{d}y$. The weight is obtained by multiplying the density and the gravitational acceleration to that amount.
The distance that the slice of water must travel is $(3-y)+2=5-y$. The first slice appears at $y=0$ and the last one at half the total height of the tank $y=3/2$. Then the work will be

$$\int_0^{3/2} \rho g[(2y)(8)](5-y)\,\mathrm{d}y=72\rho g.$$