Exercise 1 (Exercise 1)**.** Prove that all three definitions of representations of finite groups given in the lecture notes are equivalent. Then, for the examples of the groups G and H from Examples 2.1 and 2.2 in the lecture notes, express these representations as a vector space with an action, and as a module.

The definitions in question are:

Definition 1. A representation of a group G over a field $\mathbb{F}$ is a homomorphism

$$\rho : G \to \mathrm{GL}_n(\mathbb{F})$$

where $\mathrm{GL}_n(\mathbb{F})$ is the group of invertible $n \times n$ matrices over $\mathbb{F}$.

Definition 2. A representation of a group G over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space V along with an action $G \rhd V$ by linear transformations, i.e. a homomorphism $\rho : G \to \mathrm{GL}(V)$.

Definition 3. A representation of a group G over a field $\mathbb{F}$ is an $\mathbb{F}G$-module V. (Here $\mathbb{F}G$ is the group ring consisting of formal linear combinations of elements of G over $\mathbb{F}$. A module is essentially a "vector space over a ring".)

Answer

We begin by showing that the first definition implies the second. Let $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ be a representation and take the vector space $V = \mathbb{F}^n$. As $\mathrm{GL}_n(\mathbb{F}) = \mathrm{GL}(\mathbb{F}^n)$, we define the action via

$$G \times V \to V,\ (g, v) \mapsto \rho(g)v.$$

This map is an action by linear transformations because:

- Every $\rho(g)$ is a linear transformation.
- $(\mathrm{id}, v) \mapsto \rho(\mathrm{id})v = I_{n\times n}v$. Because ρ is a group homomorphism and it sends the identity to the identity.
- $(g, (h, v)) = (g, \rho(h)v) = \rho(g)\rho(h)v = \rho(gh)v = (gh, v)$ by virtue of ρ being a homomorphism.

Now assume we would like to create a module over the group ring $\mathbb{F}G$. Recall every element of $\mathbb{F}G$ is of the form

$$\sum_{g\in G} a_g g$$

as a formal linear combination. Given an action $G \rhd V$, we may extend it to a $\mathbb{F}G$ action via

$$\rho\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g \rho(g)$$