

Exercise 1 (Exercise 3). Let c_n be the number of sequences of length n in which:

- ◇ Each number is one of 0, 1, 2, 3.
- ◇ No two 3's are consecutive.

For instance 0221030132 is a valid sequence but 033112333 is not.

- i) Find a recursion for c_n (this should be similar to the Fibonacci recurrence). Remember to include the initial conditions!
- ii) Use the recursion to find a closed form for the generating function of c_n .
- iii) Use the formula discussed in class for solving linear recurrences to find an explicit formula for c_n in terms of n .

Answer

- i) First, we can observe that the total number of sequences of length n with characters 0, ..., 3 is 4^n . Dividing this amount into conditioned sequences and forbidden sequences we get the following

$$4^n = c_n + f_n,$$

where c_n is the quantity we are looking for and f_n is the number of *forbidden* sequences of length n . This is, sequences which do have 33 as a substring.

After considering the initial conditions:

$$f_0 = 0, f_1 = 0, f_2 = 1, f_3 = 7, f_4 = 40,$$

it is possible to conjecture that

$$f_{n+2} = 3f_{n+1} + 3f_n + 4^n.$$

To prove this recurrence we will append a digit to a length $(n+1)$ sequence. There are several ways to this:

- ◇ If the digit we are appending is either 0, 1 or 2, we are not adding any more forbidden substrings. So for each of those digits, our count of forbidden sequences goes up by f_{n+1} . Right now we have $3f_{n+1}$ forbidden sequences.
- ◇ If the digit we are appending is 3, there are two cases:
 - Either the last digit of the $(n+1)$ sequence is 0, 1 or 2 in which case we are not adding more forbidden strings. Each of those possibilities accounts for f_n forbidden sequences. This adds up to our past total to get $3f_{n+1} + 3f_n$.

- If the last digit of the $(n + 1)$ sequence is 3, then we just added a forbidden substring. Counting this is the same as counting all $(n + 2)$ strings which end in 33. This amount is 4^n .

In total, we have $3f_{n+1} + 3f_n + 4^n$ forbidden sequences of length n . Then by our initial relation we have

$$c_n = 4^n - f_n = -(3f_{n+1} + 3f_n), \quad f_0 = 0, \quad f_1 = 0.^a$$

- ii) Let us derive a generating function for f_n and from that we will obtain c_n 's generating function.

Call F , (f_n) 's generating function, then the recurrence

$$f_{n+2} = 3f_{n+1} + 3f_n + 4^n$$

translates to the equation

$$\frac{F(x) - f_0 - f_1x}{x^2} = \frac{3(F(x) - f_0)}{x} + 3F(x) + \frac{1}{1 - 4x}.$$

Applying the initial conditions we get

$$\frac{F(x)}{x^2} = \frac{3F(x)}{x} + 3F(x) + \frac{1}{1 - 4x}.$$

We can solve for F to obtain

$$\begin{aligned} F(x) \left(\frac{1}{x^2} - \frac{3}{x} - 3 \right) &= \frac{1}{1 - 4x} \\ \Rightarrow F(x) \left(\frac{1 - 3x - 3x^2}{x^2} \right) &= \frac{1}{1 - 4x} \\ \Rightarrow F(x) &= \frac{1}{1 - 4x} \end{aligned}$$

iii) formula

^aEven though the recurrence is not in terms of c 's, it's still a recursive formula. The derivation for c_n 's recursive formula lies below this answer.

We can also construct the recurrence in terms of the allowed sequences c_n . Take any length n allowed sequence, then there are two possibilities:

- ◇ The last digit is 0, 1 or 2, then the rest of the sequence is a length $(n - 1)$ allowed sequence. For each digit we count c_{n-1} allowed sequences. So in total we have $3c_{n-1}$ allowed sequences.
- ◇ If the last digit is 3, then the second-to-last digit can't be three. There are only 3 other possibilities: 0, 1 or 2. For each of these the remaining length n sequence has to fulfill the condition. Which means we count c_{n-2} allowed sequences per digit.

This total amounts to $c_n = 3c_{n-1} + 3c_{n-2}$. With this recurrence we have counted all the possibilities since the only options for the last digit are the ones mentioned above.