

# Tropical Geometry

MATH 676

Fall 2023

These notes arose from Tropical Geometry with Dr. Renzo Cavaleri during the Fall of 2023 at CSU. They come from his lectures.

## Contents

<b>1</b>	<b>Intro</b>	<b>3</b>
1.1	Current State of Literature . . . . .	3
1.2	Tropical Geometry . . . . .	3
1.2.1	Tropical Smei-Field . . . . .	3
1.2.2	Valued Fields . . . . .	4
1.2.3	Amoebas . . . . .	4
1.2.4	Degenerations of Algebraic Varieties . . . . .	5
<b>2</b>	<b>Motivation of Tropical Geometry</b>	<b>5</b>
2.1	Tropical Arithmetics . . . . .	5
2.1.1	Minimization Problems . . . . .	5
2.1.2	Forget the Phase . . . . .	6
2.1.3	Puiseux Series . . . . .	6
<b>3</b>	<b>The Tropical Semifield</b>	<b>7</b>
3.1	Weird/fun facts . . . . .	8
3.2	Algebraic Geometry: Tropical Univeraitate polynomials and their roots . . . . .	8
<b>4</b>	<b>Root finding</b>	<b>10</b>
4.1	Combinatorilization of tropical root finding . . . . .	11
4.2	Grobner . . . . .	12

<b>5</b>	<b>More Variables</b>	<b>13</b>
5.1	Lines . . . . .	13
<b>6</b>	<b>Sufficient conditions for tropical curves</b>	<b>19</b>
<b>7</b>	<b>Intersection of Tropical Curves</b>	<b>20</b>
<b>8</b>	<b>Tropical Toric Varieties</b>	<b>22</b>
<b>9</b>	<b>Geometric Tropicalization</b>	<b>25</b>
<b>10</b>	<b>Student Presentations</b>	<b>27</b>
10.1	Joel, Gfan . . . . .	27
10.2	Kristina, Tropical Geometry of Deep Neural Networks . . . . .	27
10.3	Jacob, The Joswig Algorithm . . . . .	28
10.4	Natalie, Group Theory and Tropical Geometry . . . . .	29
10.5	Kylie, Cryptography . . . . .	30

# 1 Intro

## 1.1 Current State of Literature

There are current books for Tropical Geometry. This includes

- *Tropical Geometry* by Maclagan-Sturmfels, which has a very algebraic take on the subject.
- *Tropical Geometry*, Which is in progress, being written by Mikhalkin-Rau, which has a geometric and intersection theoretic take.
- There are also various expository articles for Tropical Geometry

## 1.2 Tropical Geometry

Tropical Geometry is sometimes called a ‘combinatorial’ shadow of algebraic geometry. We take as inputs algebraic varieties, and receive as an output a piecewise linear object.

**Example 1.1.** The input can be a line in the plane  $\mathbb{C}^2$ , i.e.  $az + bw = d$ . Then the output of the construction can be a tripod/tropical  $Y$ , i.e. three lines connecting a vertex.

The input could also be an elliptic curve in  $\mathbb{C}^2$ , and the output can be another more complicated connection of vertices and lines (a tropical cubic)

Finally, we can consider an abstract nodal curve (a sphere and tori connected at vertices), with the corresponding piecewise linear object being a dual graph, which has a vertex at each component, an edge for each node, and a label for each part.

The questions that naturally arise are

1. What algebraic information about the initial object is carried over in the simplified object?
2. How do we extract the information carried by the simplified object?
3. Does the lifting problem have a solution?

There are four ways to tropicalize the algebraic variety.

### 1.2.1 Tropical Semi-Field

We do “algebraic geometry” over the tropical semi-field. The tropical semi-field is  $(\mathbb{T}, \oplus, \odot) = (\mathbb{R} \cup \{-\infty\}, \max, +)$ . We can take a polynomial  $p(x_1, \dots, x_n)$ . Our variety is the roots of the

polynomial. If we consider the tropical  $p(x_1, \dots, x_n) : (\mathbb{R} \cup \{-\infty\})^n \rightarrow \mathbb{R} \cup \{-\infty\}$ , this is an affine piecewise linear function. For example, We can take  $p(x_1, x_2)$ .

The tropical hypersurface is the locus of non-linearity.

A tropical polynomial would be something of the form  $p(x, y) : x \oplus y \oplus 0 = \max(x, y, 0)$

By the corner locus, we take the graph of our function.

Why does this work to make piecewise linear? We get expressions such as  $3 \odot x^2 = 3 + 2x$ .

### 1.2.2 Valued Fields

our second perspective will be of valued fields. We let  $K$  be a field with a valuation, i.e. let  $K = \mathbb{C}(t)$  be the field of rational functions in 1 variables. A valuation  $val_0$  is a function  $val_0 : \mathbb{C}(t) \rightarrow \mathbb{R} \cup \{\infty\}$ , where  $0 \mapsto \infty$ , else  $f \mapsto$  order of vanishing (or pole) as you approach  $t \rightarrow 0$ .

We can ask what happens to the order of vanishing when you add 2 functions. In this case the order of vanishing is the minimum of the two orders. i.e. if  $f_1 = t^2$ ,  $f_2 = t^3$ , then  $val_0 f_1 = 2$ ,  $val_0 f_2 = 3$ , and then  $val_0(f_1 + f_2) = 2$ . So in this context,  $val_0(f_1 + f_2) \geq \min(val_0 f_1, val_0 f_2)$ , and  $val_0(f_1 f_2) = val_0 f_1 + val_0 f_2$ . Let  $p(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$ . Take the variety  $X = \{x_1, \dots, x_n \mid p(x_1, \dots, x_n) = 0\} \subset K^n$ . We can take the map  $K^n \rightarrow (\mathbb{R} \cup \{\infty\})^n$  by taking the valuation at every coordinate, i.e. apply  $val_0$  to the result. Each point of  $K$  is some rational function of  $t$ , so looking at the order of vanishing makes sense. So the tropicalization of  $X$  is the image via this map.

For example, we can take  $p(x, y) = tx + y + t^2$ . Then  $X = \{(x, y) \mid tx + y + t^2 = 0\} = \{(x, y) \mid y = -tx - t^2\}$ . So  $(0, -t^2)$  is an option. So the point  $(\infty, 2)$  is a point in  $Trop(X)$ .

### 1.2.3 Amoebas

We consider  $p(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ . It defines  $X = \{(z_1, \dots, z_n) \mid p(z_1, \dots, z_n) = 0\} \subset \mathbb{C}^n$ . We can consider the map from  $\mathbb{C}^n$  to the image  $\mathbb{C}^n \rightarrow (\mathbb{R} \cup \{-\infty\})^n$  via  $(z_1, \dots, z_n) \mapsto (\log_t |z_1|, \dots, \log_t |z_n|)$ . The image of  $X$  via this map  $Im(X)$  is the  $t$ -Amoeba of  $X$ .

**Example 1.2.**  $p(x, y) = x + y + 1$  We can take the variety, points such as  $x = i$ ,  $y = -1 - i$ , and  $p(x, y) \in X$ . We then consider  $(\log_t 1, \log_2 \sqrt{2})$

We can take  $\lim_{t \rightarrow \infty} (t - Amoeba)$ , called the spine of the amoeba or  $Trop(X)$ .

### 1.2.4 Degenerations of Algebraic Varieties

This is a complicated way to get tropics

## 2 Motivation of Tropical Geometry

### 2.1 Tropical Arithmetics

#### 2.1.1 Minimization Problems

An important question is where tropical numbers come from. One example is toll minimization problems. We let every city be a vertex, and every directed edge is a tollway. Let  $A$ ,  $B$ , and  $C$  be the three cities/vertices. The incidence matrix would be

$$M = \begin{bmatrix} 0 & \infty & 2 \\ x & 0 & y \\ \infty & 1 & 0 \end{bmatrix}$$

where the rows represent the starting point and the columns represent the ending location, so there is a toll road from  $A$  to  $B$  which costs 2. There is no road from  $A$  to  $C$ . In other words,  $M_{i,j}$  records the (minimum) price of going from city  $i$  to city  $j$  in at most one trip (one trip being moving across one toll road, i.e. traversing one edge). We can now ask how to compute the best strategy of going from  $i$  to  $j$  in at most two trips.

**Example 2.1.** If I want to go from  $A$  to  $B$  in 2 steps, We can do  $A \rightarrow A \rightarrow B$ ,  $A \rightarrow B \rightarrow B$ , or  $A \rightarrow C \rightarrow B$ . The associated costs are  $0 + \infty$ ,  $\infty + 0$ , and  $2 + 1$ , respectively. The best strategy is the minimum of these three, which is  $A \rightarrow C \rightarrow B$ , which gives a cost of 3. However, these sums can be thought of as coming from entries of the matrix. The associated sums are  $a_{11} + a_{12}$ ,  $a_{12} + a_{22}$ , and  $a_{13} + a_{32}$ .

Where do these values come from? The  $b_{12}$  entry of  $M^2$  is  $\sum_{j=1}^n a_{1j}a_{j2}$ . So the best strategy is equal to the 12 entry of  $M^2$ , so long as you interpret  $+$  as the minimum, and  $*$  as  $+$ . In that sense,  $b_{12}$  entry is  $M^2$  is  $\min(a_{11} + a_{12}, a_{12} + a_{22}, a_{13} + a_{32})$ .

If we make the assumption that there are no negative tolls, this arithmetic for higher powers of  $n$  gives the best solution for up to  $n$  trips. Eventually we stabilize, as for a high enough  $N$  the solutions will say to stop moving for a certain number of steps. (Eventually we get something like idempotence).

The minimization problem eventually becomes a linear algebra problem over  $(\mathbb{T}, \oplus, \odot) = (\mathbb{R} \cup \{\infty\}, \min, +)$ .

### 2.1.2 Forget the Phase

Another context is in physics/electromagnetism. If we take a complex number, we can write it in polar coordinates, i.e.  $z \in \mathbb{C}$  can be written as  $z = r\theta e^{i\theta}$ . Maybe we don't particularly care about the phase ( $\theta$ ), or our work works on a logarithmic scale. So our function would be  $T_t : \mathbb{C} \rightarrow \mathbb{R} \cup \{-\infty\}$ , where  $z \mapsto \log_t |z|$ . We add the point at negative infinity to be able to define  $T_t(0)$ .

We have that  $\mathbb{C}$  has its own natural operations, and we can ask if we can induce operations on  $\mathbb{R} \cup \{-\infty\}$  utilizing the map  $T_t$ . If we want to define addition on  $\mathbb{R} \cup \{-\infty\}$ , we could define it as  $x \boxplus y$ . We could try  $T_t(T_t^{-1}(x) + T_t^{-1}(y))$ . The problem this is not well defined, as different inverse images can have different absolute values. This is a hyper operation, This is a function  $X \times X \rightarrow \mathcal{P}(X)$ , to the power set of  $X$ . So this function outputs an interval.

We first wish to understand  $T_t^{-1}(y)$ . This is  $T_t^{-1}(y) = \{t^y e^{-\theta}\}$ . adding and subtracting is thus given by the interval of the possible extremes, so we get the interval.

$$T_t(T_t^{-1}(x) + T_t^{-1}(y)) = [\log_t |t^x - t^y|, \log_t (t^x + t^y)]$$

But we don't want a hyperoperation, we want an operation. To pick something that always works, we can either pick the max or the minimum. We can also see if in a limiting process, we get one answer. We start with  $x \boxplus_t y \rightarrow x +_t y$ , with some consistent choice. we can also ask for  $\lim_{t \rightarrow \infty}$  to get a new operation. We define  $x \boxplus y = \lim_{t \rightarrow \infty} x \boxplus_t y$ . Why is this better? If we take  $t^2 e^{i\theta_1} + t^4 e^{i\theta_2}$ . So what really matters at the end is that our expression is equal to  $t^4(t^{-2}e^{i\theta} + e^{i\theta})$ . Taking  $\log_t$ , we have  $4 + \log_2 |t^{-2}e^{i\theta} + e^{i\theta}|$ . Taking the limit as  $t \rightarrow \infty$ , we have  $4 + 0 = 4$ . So this is not a hyperoperation if  $x = 2$   $y = 4$ . But, if we have  $x = 2$  and  $y = 2$ , we get issues. We have

$$x \boxplus y = \begin{cases} \max(x, y) & x \neq y \\ (\infty, \max(x, y)) & x = y \end{cases}$$

This suggests that the nice consistent choice to define  $x \oplus y$  is max. This is fully consistent when  $x \neq y$  regardless of any decision, and it makes the interval  $(\infty, \max(x, y))$  work nicer. To define multiplication, we would say  $x * y = T_t(T_t^{-1}(x)T_t^{-1}(y)) = T_t(t^x e^{i\theta_1} t^y e^{i\theta_2}) = x + y$ . This leads to the tropical numbers, where  $(\mathbb{T}, \oplus, \odot) = \mathbb{R} \cup \{-\infty\}, \max, +$ .

### 2.1.3 Puiseux Series

In cacl 1, we find ways to approximate. In approximation, we lose some information. we learn that  $\lim_{t \rightarrow 0} \frac{\sin(t)}{t} = 1$ , which essentially says  $\sin(t) = t + (t)$ , and  $\frac{1}{t} = t^{-1} + 0$ . In otherwords,  $\lim_{t \rightarrow 0} (t + (t))t^{-1} = 1 + (1)$ , where  $(t)$  goes to zero as  $t$  goes to zero. All of this is to say that we have a concept of order of vanishing of functions.

A Prototypical example of a field with a valuation is the field of Puiseux series. Denoted  $\mathbb{C}\{\{t\}\}$ , this is the Laurent series in  $t$  with rational exponents, and all exponents of terms with nonzero coefficients have a common denominator. For example,  $\sum_{i=-37}^{\infty} t^{i/42}$  works, but  $\sum_{i=1}^{\infty} t^{1/i}$  does not work. In other words,  $\mathbb{C}\{\{t\}\} = \bigcup_{n \in \mathbb{N}} \text{Laur}(t^{1/n})$ , that is to say it is the union of all Laurent series in the variable  $t^{1/n}$ . This has a valuation

$$\text{val} : \mathbb{C}\{\{t\}\} \rightarrow \mathbb{R} \cup \{\infty\} \quad (1)$$

$$(a_N \neq 0) \sum_i a_i t^{q_i} \mapsto q_N \quad (2)$$

$$0 \mapsto \infty \quad (3)$$

This map technically has as its image  $\mathbb{Q} \cup \{\infty\}$ .

Once again, the properties/axioms of a valuation field is that

- $\text{val}(\alpha + \beta) \geq \min\{\text{val}(\alpha) + \text{val}(\beta)\}$ ,
- $\text{val}(\alpha * \beta) = \text{val}(\alpha) + \text{val}(\beta)$ .

If we want to induce a sum on the image, we would have a hyper operation. We define  $x \boxplus y := \text{val}(\text{val}^{-1}(x) + \text{val}^{-1}(y)) = \begin{cases} [\min(x, y), \infty] & x = y \\ \min(x, y) & x \neq y \end{cases}$ . For example, we would have  $0 \boxplus 0 := \text{val}(a_0 + t^{q_1} a_1 + \dots) + (-a_0 + t^{r_1} b_1 + \dots)$ . Then we can let  $q_1$  and  $r_1$  equal whatever we want.

To avoid issues, we say  $x + y := \min(x, y)$ . This takes us to  $(\mathbb{R} \cup \infty, \min, +)$ .

### 3 The Tropical Semifield

**Definition 3.1.** The *tropical semifield* is one of  $(\{-\infty\} \cup \mathbb{R}, \max, +)$ , or  $(\mathbb{R} \cup \{\infty\}, \min, +)$ , either one of which is denoted by  $(\mathbb{T}, \oplus, \odot)$ . The operations are associative, and the distributive law holds.

The two definitions are isomorphic via  $x \mapsto -x$ . In the following writing, we will tend to utilize  $\max$  more often than  $\min$ . We have the following properties of this semifield

1.  $\{-\infty\}$  is the additive identity,
2.  $\{-\infty\}$  is the only element that has an additive inverse.
3.  $\oplus$  is an idempotent operation  $x \oplus x = x$ .
4. We cannot add inverses, not even formally.

5. 0 is the multiplicative identity.
6. every element  $x \neq -\infty$  has a multiplicative inverse (namely  $-x$ ).

**Example 3.1.** Let us solve  $x \oplus y = -\infty$ . This is to say  $\max(x, y) = -\infty$ , which necessitates  $x = y = -\infty$ . To see that inverses are not definable, We take  $x \in \mathbb{T}$ . Let us attempt to construct a formal inverse  $y$ , defined by  $x \oplus y = -\infty$ . This will force  $x = -\infty$ . We see this by considering  $x \oplus x \oplus y = (x \oplus x) \oplus (y) = -\infty$ . As this operation is associative, we also have  $x \oplus (x \oplus y) = x \oplus -\infty = x$ , and so  $x = -\infty$ .

### 3.1 Weird/fun facts

Pascal's triangle in tropical arithmetic looks like all zeros. Furthermore, the freshman's dream holds  $(x \oplus y)^n = x^n \oplus y^n$ . This is because  $(x \oplus y)^n = n * (\max(x, y)) = \max(nx, ny)$ . However, the fact that  $x^2 \oplus (x \odot y) \oplus y^2 = x^2 \oplus y^2$  does not imply cancellation, i.e. we do not automatically have that  $x \odot y = -\infty$ .

The Tropical determinant of a matrix (call it permanent) gives a solution to the assignment problem: We have  $n$ -jobs to  $n$ -workers.  $x_{ij}$  is the profitability of worker  $i$  in job  $j$ , and the goal is to find the best assignment to maximize profits. This is the tropical determinant of the matrix  $X$ . We define  $\text{tropdet}(X) = \sum_{\sigma \in S_d} \prod_{\sigma(i)} x_{i, \sigma(i)}$  where the sum is the tropical sum and the product is the tropical product. We do not get a signed determinant as tropical geometry does not have subtraction.

### 3.2 Algebraic Geometry: Tropical Univariate polynomials and their roots

**Definition 3.2.** A tropical univariate (Laurent) monomial is an expression of the form  $a \odot x^{\odot m}$ , where  $a \in \mathbb{T}$  and  $m \in \mathbb{Z}$ .

In ordinary algebra, a tropical monomial corresponds to an affine linear function with integer slope.

**Example 3.2.**  $\sqrt{5} \odot x^{\odot 3} \sqrt{5} + 3x = y$ . This is an affine linear transformation. It is affine because we can shift via  $a$ , and it has integer slope as the slope is  $m$ . Furthermore,  $\{-\infty\} \odot x^{\odot m} = -\infty + mx = -\infty$ , so we maintain that multiplying by the additive identity gives the additive identity.

**Definition 3.3.** A tropical univariate (Laurent) polynomial is the finite sum of monomials.



The tropical univariate polynomial corresponds to a continuous, piecewise affine linear function with  $\mathbb{Z}$ -slopes

**Example 3.3.**  $p(x) = -5 \odot x^{\odot 2} \oplus (-2) \odot x^{\odot -3} \oplus 0 = \max(-5 + 2x, -2 - 3x, 0)$

By construction, tropical polynomials give rise to convex functions. In the univariate case, the map from tropical  $L$  polynomials to convex  $\mathbb{Z}$  affine piecewise linear (with finitely many distinct regions of linearity) functions is a surjective map. However, this map is not surjective.

**Example 3.4.** Consider  $p_1(x) = x^{-1} \oplus x$ , i.e.  $y = |x|$ . Then consider  $p_2(x) = x^{-1} \oplus x \oplus 0$ . Then we once again get  $y = |x|$ . Furthermore, we can take  $p_3 = x^{-1} \oplus x \oplus -8$ , this works for any negative number.

In general,  $p(x) = p(x) \oplus \{-\infty\}$ . However,  $p(x) \oplus$  any function which is smaller than the minimum value attained by  $p(x)$  does not change the output of  $p(x)$ , i.e.  $p(x) = -5 \odot x^{\odot 2} \oplus (-2) \odot x^{\odot -3} \oplus 0 = p(x) = -5 \odot x^{\odot 2} \oplus (-2) \odot x^{\odot -3} \oplus 0 \oplus (-4) \odot x$ .

Now that we have defined polynomials, we wish to make an interpretation of roots. It does not make sense to say "values of  $x$  for which  $p(x) = -\infty$ , as that typically won't happen (We deal with max). Solving for 0 also doesn't help, as in this context 0 is just another number/function.

**Definition 3.4.** Let  $p(x) \in \mathbb{T}[x]$  (an honest polynomial, no negative exponents, i.e. only positive slopes). Then

- $(-\infty)$  is a root of  $p$  if the slope of the corresponding affine piecewise linear function is  $\neq 0$  for  $x \ll 0$ .
- We allow  $r \in \mathbb{R}$  to be a root of  $p$  if  $f'_p(r)$  (The piecewise linear function arising from  $p$ , and the derivative of THAT function). is not defined.

In other words, roots will be where the function changes slopes. Now we can discuss multiplicities.

**Definition 3.5.** If  $-\infty$  is a root, its multiplicity is equal to the slope of  $f_p(x)$  for  $x \ll 0$ . If  $r \in \mathbb{R}$  is a root, its multiplicity is the difference in slopes across  $r$ .

**Example 3.5.** Take  $p(x) = x^2 \oplus 1 \odot x \oplus 0 = \max(2x, 1 + x, 0)$  has two simple roots, one at  $x = 0$  and one at  $x = 1$ . On the other hand, for  $p(x) = x^2 \oplus 1 \odot x$ , we get two simple roots, one at  $x = 1$  and one at  $-\infty$ .

Now, take  $p(x) \oplus (-1) \otimes x \oplus 0$ . Then there is a double root at  $x = 0$ , as we go from slope 0 to slope 2.

**Note 1.** If we accept Laurent polynomials, then the multiplicity of  $-\infty$  is equal to the slope towards  $-\infty$ .

**Example 3.6.** Take  $q(x) = (-5) \otimes x^2 \oplus (-2) \otimes x^{-3} \oplus 0$ , Then  $-\infty$  has a pole of order 3, there is a root of order 3 at  $x = -5/2$ , and a root of order 2 at  $x = 2$

**Lemma 3.1.** *Only  $-\infty$  can have a pole (root with negative multiplicity) due to concavity.*

**Lemma 3.2.**  *$r \neq -\infty$  is a root for  $p(x)$  iff when you write down  $f_p(x) = \max(f_{m_0}(x), \dots, f_{m_d}(x))$  at  $r$  the maximum value is obtained at least twice.*

*Furthermore, the multiplicity of the root is equal to the difference in the two extremal positions where the max is attained for  $r$ .*

## 4 Root finding

In previous lectures, we had the following theorem

**Theorem 4.1.** *Let  $q(X) = \sum A_i X^i$ ,  $A_i \in \mathbb{C}\{\{t\}\}$ ,  $a_i = \text{val}(A_i)$ , and consider  $p(x) \text{trop}(q(X)) = \sum a_i \odot x^{\odot i}$ . Then  $R$  is a root of  $q$  implies  $r = \text{val}(R)$  is a root of  $p$ . Furthermore, if  $r$  is a root of  $p$ , then there exists a root  $R$  of  $q$  where  $r = \text{val}(R)$ .*

We prove this inductively.

## 4.1 Combinatorilization of tropical root finding

We momentarily convert to the max convention. We take  $p(x) = \sum_{i=0}^d a_i \odot x^{\odot i}$ . Indeed,  $p(x) = \max_i \{a_i + ix\}$ . There are  $d + 1$  lines  $y = a_i + ix$ . It is a finite process to intersect every pair of these lines, and then to compare the corresponding heights of each function to find the max. To make root finding more efficient, we start from left to right. To find the left most root, consider the left most intersection. It is thus the minimum of the x-value of intersection of the line  $y = a_0$  and  $y = a_i + ix$ , when there exist a horizontal line (otherwise start with the line with the smallest slope, or factor out powers of  $x$ ). This is the  $\min\{x = (a_0 - a_i)/i\} = -\max\{\frac{a_i - a_0}{i}\}$ .  $\frac{a_i - a_0}{i}$  is now reminiscent of slopes. We view this as rise over run, we look at all the lines through  $(0, a_0)$  connecting to  $(i, a_i)$ , and we are looking for the largest such slope. This gives us  $a_j$  for our first root. Our next root is to the right of  $a_j$ . This is because the corresponding line has a larger slope than the preceding lines, and the preceding lines have a later intersection, and are thus no longer considered in our root finding. So we use that as our next starting point, and we continue on.

We get a description of how to find the roots utilizing these points. This results in the following algorithm.

1. The segment  $[0, d] \subset \mathbb{R}$  (aka the Newton polytope of the polynomial  $p(x)$ ) is the convex hull of the  $i$  such that  $a_i \neq -\infty$  (i.e. assume degree  $d$  polynomial with constant term)s.
2. Find the convex hull of the points  $(i, a_i) \in [0, d] \times \mathbb{T}$  for  $i \in \mathbb{N} \cap [0, d]$ .
3. We construct the line from  $(0, a_0)$  to  $(d, a_d)$ , and we disregard the section of the convex hull below this line. We call the section above this line  $\Sigma^+$ .
4. Project the vertices of  $\Sigma^+$  back onto  $[0, d]$ . This gives a regular subdivision of the Newton polytope.

Following this algorithm, we have that

- A) The roots of  $p(x)$  are in bijection with the complement of the projections, i.e. the subdivisions of the Newton polytope.
- B) The value of the root corresponding to a given segment  $(i, j)$  is found by solving the equation  $a_i + ix = a_j + jx$ .
- C) The multiplicity of the root is equal to the length of the segment.

**Example 4.1.** Let  $p(x) = 0 \oplus (1 \odot x) \oplus (1 \odot x^2) \oplus x^3 \oplus (2 \odot x^4) \oplus (1 \odot x^5)$ . When graphing the convex hull, the points above the line connecting  $a_0$  and  $a_5$  are  $a_1, a_2, a_4$ . The vertices are  $a_0, a_1, a_4$ , and  $a_5$ . Thus, we expect to have two simple roots  $r_1$  and  $r_3$ , and one root of multiplicity of 3  $r_2$ . To find the root  $r_1$ , we solve  $0 = 1 + x$ ,  $x = -1$ . To find  $r_2$ , we solve  $1 + x = 2 + 4x$ , which is  $x = \frac{-1}{3}$ . To find  $r_3$ , we solve  $2 + 4x = 1 + 5x$ , which has as solution  $x = 1$ .

## 4.2 Grobner

Let  $\mathbb{K}$  be a field with a valuation, such as  $\mathbb{C}\{\{t\}\}$ . Then let  $R_{\mathbb{K}}$  be the set of all elements with non-negative valuation, i.e.  $\bigcup_{n \geq 0} \mathbb{C}[[t^{1/n}]]$ . In particular, we can consider ideals and maximal ideals. and we define  $M_{R_{\mathbb{K}}}$  to be the maximum ideal, the set of all elements with positive valuation. and thus  $\bigcup_{n > 0} t^{1/n} \mathbb{C}[[t^{1/n}]]$ .  $k$  is the residue field  $R_{\mathbb{K}}/M_{R_{\mathbb{K}}} = \mathbb{C}$ . The subset relation is  $M_{R_{\mathbb{K}}} \subset R_{\mathbb{K}} \subset \mathbb{K}$ .

**Example 4.2.** Suppose we had  $q(x) = t^{-4}\sqrt{2}x + 3t^2x^2$ . Suppose we picked a valuation  $val(x) = -3$ . We can thus ask for the valuation of the individual terms. Then  $t^{-4} \rightarrow -4$ ,  $\sqrt{2}x \rightarrow 0 + (-3)$ , and  $3t^2x^2 \rightarrow 2 - 6 = -4$ . Now, we decide that the lowest order terms are from  $t^{-4}$  and  $3t^2x^2$ . Now, we no longer care about the value of  $t$ , so we only keep the coefficients. We say we have an initial form of  $q$  for a valuation of  $x$ .  $In_{w=-3}(q(x)) = 1 + 3x^2$ . With a polynomial in a value field, we have an equivalence relation based on the initial forms. Breaking up  $\mathbb{R}$  into initial forms gives roots of  $q(x)$ . The roots are where the valuations give initial forms which are not monomials.

We say  $\mathbb{K}$  is our valued field, such as  $\mathbb{C}\{\{t\}\}$ .  $R_{\mathbb{K}}$  is the set of all series that start with  $t \geq 0$ , while  $M_{\mathbb{K}}$  is the set of all series that start with  $t > 0$ . The residue field  $R/M = \mathbb{C}$ . If we start with a polynomial  $q(X) \in \mathbb{K}[x]$  and  $w = val(x) \in \mathbb{R}$ , we get an initial form  $In_w q$  to then get a polynomial with coefficients in the residue field  $\mathbb{C}$ . We look at the valuation of each monomial assuming  $val(x) = w$ , then we save only the monomials with the smallest valuation, and we only keep the coefficient in front of  $t^*$  (where  $t^*$  is the smallest term).

**Example 4.3.** Consider  $q(X) = (t^{-4} + t^2) + \sqrt{2}x + 2t^2x^2$ . The individual valuations are  $-4$ ,  $-3$ , and  $(2 - 6) = -4$ . Since  $-4$  is the lowest valuation, we only consider the first and third form. So we only consider  $t^{-4} + t^2$ , and  $3t^2x^2$ . Then the limit with the limit of  $t$ , we get  $In_{-3}q = 1 + 3x^2$ . We define  $W := tropq(w)$ . Then  $In_w q[t^{-W}q(t^w x)]|_{t=0}$

( $\sqrt{2}$  is ignored because valuations of products are added, and  $\sqrt{2} = \sqrt{2}t^0$ , which has valuation 0)

We now consider  $(\mathbb{R}, w)$ , with some fixed  $q(X)$ . We can define an equivalence relation by  $w_1 \equiv w_2 \iff In_{w_1} q = In_{w_2} q$ . This equivalence relation decomposes  $\mathbb{R}$  into equivalence classes of two types. The equivalence classes are either single points or open intervals. The open intervals correspond to when the initial form is a monomial. The single points are otherwise.

**Example 4.4.** Consider  $q(X) = t^2 + \sqrt{2}x + 3x^2x^2$ . At

**Definition 4.1.** The complement of the locus of  $w$  such that  $In_W q$  is a monomial is called the Grobner complex of  $q(X)$ .

The Grobner complex of  $q(X)$  is equal to the roots of  $trop(q(x))$ .

## 5 More Variables

Let  $p(x, y) = \sum a_{ij} \odot x^i \odot y^j$  be a tropical polynomial in two variables.

**Definition 5.1.** Define a tropical curve  $V(p)$  to be either

1. The locus in the domain of piecewise linear  $p$  where  $p$  is not linear, or  $(x, y) \mid \max(a_{ij} + ix + jy)$  is attained  $> 1$ .

We have a correspondence theorem.

**Theorem 5.1.** If  $q(x, y)$  is a polynomial with coefficients over a valued field  $\mathbb{C}\{\{t\}\}$ , and  $trop(q) = p$ , then the tropical curve  $V(p)$  is equal to the closure of the valuation of the points  $\{(val(x), val(y)) \mid (x, y) \in V(q)\}$ .

We can then study structural properties of tropical curves. We get correspondence statements with subdivisions of Newton polygon, and we get balancing and edge weights. We will also see the tropical versions of classical plane curve theorems. In particular, we get a tropical Bezout theorem (two projective curves of degree  $d$  and  $e$  intersect in  $d * e$  points) and a tropical deg/genus formula.

To recap  $p(x, y)$  a tropical polynomial, we can define the variety of  $p$   $V(p)$ , which is either the locus of non-linearity of  $p$ , or the locus where the maximum is attained more than once. Then for  $q(X, Y)$  a polynomial, we can tropicalize it.

### 5.1 Lines

Lines are  $V(p)$  such that  $deg(p) = 1$ . Now, to see what happens with tropical lines, consider  $p(x, y) = (a \odot x) \oplus (b \odot y) \oplus c$ . Assume  $-\infty < a < b < c$ . Then  $p(x, y) = \max\{a + x, b + y, c\}$ .

Setting any two equal to each other, we get  $a + x = b + y$ ,  $a + x = c$ , and  $b + y = c$ . We get three lines  $y = x + (a - b)$ ,  $x = c - a$ , and  $y = c - b$ .

Every tropical line is of the form of a tripod. Even if we only keep  $-\infty < a, b, c$ , we still keep the tripod, and the corresponding regions of maxima are maintained. This is because the locus are found by setting (constant plus variable) = constant which gives a vertical or horizontal line, or (constant plus variable) = (constant plus variable), which gives a line of slope one

**Example 5.1.** What happens when some of the coefficients are  $-\infty$ .

As a second perspective, let's let  $q(X, Y)$  be a degree 1 polynomial with coefficients from the Puiseux series. We thus have  $q(X, Y) = t^a X + t^b Y + t^c$ . If we tropicalize  $q$ , we get  $\text{trop}(q) = (a \odot x) \oplus (b \odot y) \oplus c$ . If we take  $V(\text{trop}(q))$  we get the thing we had before modulo adjusting for switch between min and max conventions. In this context,  $V(q) = \{(X, Y) \mid q(X, Y) = 0\} \subset (\mathbb{K}^*)^2$

The great thing about lines is that they can be parameterized. We can write  $V(q) = \{(\alpha, -t^{a-b}\alpha - t^{c-b}) \mid \alpha \in \mathbb{K}\}$ . Now, for any point in  $V(q)$ , we want to take the valuation  $\{(val(\alpha), val(-t^{a-b}\alpha - t^{c-b})) \mid \alpha \in \mathbb{K}^*\} \subset \mathbb{R}^2$ . We now study the valuation of  $-t^{a-b}\alpha - t^{c-b}$ , which is done by studying the valuation of the individual terms. The first term has valuation  $a - b + val(\alpha)$ , and the valuation on the right equals  $c - b$ . Interesting things happen when the valuations are equal, when  $val(\alpha) = c - a$ .

Finally, we consider  $val(\alpha) = c - a$  our claim is that we can obtain any value for  $Y$ , but it has to be  $\geq c - b$

*Proof.* Let  $\gamma \geq 0$ , and let  $\alpha = -t^{c-a}(1 + t^\gamma)$ . We need  $\gamma \geq 0$ , so that the valuation of  $1 + t^\gamma$  equals zero, and so the valuation of  $\gamma$  remains  $c - a$ . Then  $val(Y(\alpha)) = val(-t^{a-b}(-t^{c-a}(1 + t^\gamma)) - t^{c-b})$ . This equals  $val(t^{c-b}(1 + t^\gamma) - t^{c-b}) = val(t^{\gamma+c-b}) = \gamma + c - b$ , and so we can make  $y$  take valuation any value greater than or equal to  $c - b$ .  $\square$

If we were to send  $a \mapsto -a$ ,  $b \mapsto -b$ ,  $c \mapsto -c$ ,  $X \mapsto -X$ , and  $Y \mapsto -Y$ .

This process can be repeated for the Amoeba perspective. Take a family of polynomials  $q_t(X, Y)$ . The coefficients are functions of  $t$ , but we specify  $q_t(X, Y) = t^a X + t^b Y + t^c$ . We now want to consider  $q_t = 0 \subset \mathbb{C}^2$ . For every  $(X, Y) \in L$ , we consider  $(\log_t |X|, \log_t |Y|)$ . We first study the real trace of this object, i.e. when  $X, Y \in \mathbb{R}$ . We now have three cases to consider.

We can consider real image, where  $X, Y \in \mathbb{R}$ , and further when  $0 < X, Y$ . Then we can take  $(\log_t X, \log_t Y)$ . We can once again parameterize to get  $X = t^\alpha$ ,  $Y = -t^{a-b+\alpha} - t^{c-b}$ . For each of the three cases, we pick up asymptotes.

To recap, we start with a family of lines indexed by  $t \in \mathbb{R}_{>1}$ , denoted  $L_t = \{t^a X + t^b Y - t^c = 0\} \subset \mathbb{C}^2$ . We denote the function  $T_t$ , which makes  $x = \log_t |X|$ , and  $y = \log_t |Y|$ . We can solve for this line, and get  $Y = -t^{a-b}X + t^{c-b}$ .

We focus momentarily at when  $X, Y \in \mathbb{R}$ , so we focus on  $\mathbb{R}^2$ . In the case where  $X, Y$  are both positive is our first case. In particular,  $0 < X < t^{c-a}$ . In particular,  $-\infty < x < c - a$ .

We define a path  $X_s := e^{i\pi s} X_0$ , where  $s \in [0, 1]$ . Then for each  $x_s$ , we have a corresponding  $Y_s$  which is the corresponding equation of  $L$  for  $X_s$ . Now, we can ask about what happens to  $T_t(|X_s|, |Y_s|)$ . In this case, phase changes are irrelevant to the  $X$  coordinate, so  $T_t(|X_s|, |Y_s|) = (\log_t(X_0), f(s))$ , where  $f(s)$  is continuous. This traces a full interval. which allows us to take advantage of the other cases where  $X$  and  $Y$  can be complex numbers.

We now take  $q(X, Y) \in \mathbb{K}[X, Y]$  (Consider Puiseux series for  $\mathbb{K}$ ). We then consider the variety  $V(q) = \{(X, Y) \mid q(X, Y) = 0\} \subset (\mathbb{K}^*)^2$ . We can also take the tropicalization  $p(x, y) = \text{trop}(q(X, Y))$ . From here we can define the variety of  $p$  to be  $V(p)$  which is the locus where  $p$  fails to be linear.  $V(p) \subset \mathbb{R}^2$ . We also have the function  $(\mathbb{K}^*)^2 \rightarrow \mathbb{R}^2$  defined by  $(\text{val}, \text{val})$ , which takes the valuation for the coordinates. We hope to call  $(\text{val}, \text{val})$  *trop*.

**Theorem 5.2** (Kapranou's).  $\overline{\text{trop}(V(q))} = V(\text{trop}(q))$ , where the closure is with respect to the euclidean topology of  $\mathbb{R}^2$ .

*Proof.*  $\subset$  still the same idea. If  $(x_0, y_0) \in \text{trop}(V(q))$ , that means that there exists  $(X_0, Y_0) \in (\mathbb{K}^*)^2$  such that  $\text{val}(X_0) = x_0$ ,  $\text{val}(Y_0) = y_0$ , and  $q(X_0, Y_0) = 0$ . Let  $q = \sum a_{ij} X^i Y^j$ , let  $m_{ij} = a_{ij} X^i Y^j$  be the monomial. We can then consider the collection  $\{m_{ij}(X_0, Y_0)\}_{ij}$  which is a collection of elements of  $\mathbb{K}^*$  with the property that their sum = 0. We let  $\mu = \min \text{val}\{m_{ij}(X_0, Y_0)\}_{ij}$ . The claim is that there are at least two monomials whose valuation is  $\mu$ . This implies that  $(x_0, y_0) \in V(p)$ . We have shown  $\text{trop}(V(q)) \subset V(\text{trop}(q))$ . However,  $V(\text{trop}(q))$  is closed in the Euclidean topology (since the variety comes from equalities and inequalities).

$\supset$  This direction is a bit tougher. We will prove the claim in dimension 0 and proceed by induction. We first want to show that  $V(\text{trop}(q)) \cap \mathbb{Q}^2$  is dense in  $V(\text{trop}(q))$ . This is true because all monomials  $m_{ij}$  correspond to affine linear functions with integer slopes and rational coefficients. We had  $\text{trop}(m_{ij}) = \text{val}(a_{ij}) \odot x^i \odot y^j = \text{val}(a_{ij}) + ix + jy$ , where  $\text{val}(a_{ij}) \in \mathbb{Q}$ , and  $ix + jy \in \mathbb{N}$ .

We can thus focus on rational points, checking that  $V(\text{trop}(q)) \cap \mathbb{Q}^2$  lives in  $\text{trop}(V(q))$ , and then it will follow that the closure gives  $V(\text{trop}(q)) = \overline{V(\text{trop}(q)) \cap \mathbb{Q}^2} \subset \text{trop}(V(q))$ .

We proceed by the following assumption. If we have a polynomial  $q(X, Y)$ , we can consider it a polynomial in  $Y$  with coefficients in  $\mathbb{K}[X]$ , i.e.  $q(X, Y) = r_0(X) + r_1(X)Y + \cdots + r_d(X)Y^d$ , with  $r_i(X) \in \mathbb{K}[X]$ . We assume that  $r_i(X)$  is a monomial for every  $i$ . We have  $(x_0, y_0) \in V(\text{trop}(q))$ , and we want to find corresponding  $(X_0, Y_0) \in (\mathbb{K}^*)^2$  such that

1.  $q(X_0, Y_0) = 0$ , and
2.  $\text{val}(X_0) = x_0$ , and  $\text{val}(Y_0) = y_0$

We choose  $X_0$  arbitrarily, so long as  $\text{val}(X_0) = x_0$ . Because we have made our assumption of  $r_i(X)$  being a monomial, no matter how we choose  $X_0$ ,  $X_0$  is not a root of these monomials, and so this

implies that  $r_i(X_0) \neq 0$  for all  $i$ . Let us now consider the polynomial  $q(X_0, Y_0) = \sum r_i(X_0)Y^i \in \mathbb{K}[Y]$ . Let  $\tilde{p}(y) = \text{trop}(q(X, Y)) = \bigoplus_{i=1}^d \text{val}(r_i(X_0)) \odot y^i$ . Furthermore, we claim  $y_0$  is a root of  $\tilde{p}(y) = \min(\text{val}(r_i(X_0) + iy) = \min(\text{val}(a_{ij} + jx_0 + iy)$  recall that  $a_{ij}$  is a Puiseux series. Then

$$\tilde{p}(y) = \min(\text{val}(a_{ij} + jx_0 + iy) = \text{trop}(q(x, y)|_{x=x_0})$$

As we started with  $(x_0, y_0)$  is in  $V(\text{trop}(q))$ , then clearly  $y_0$  is a root of our  $\tilde{p}(y)$ .

By this univariate case, there exist  $Y_0 \in \mathbb{K}^*$  such that  $Y_0$  is a root of  $q(X_0, Y)$ , and  $\text{val}(Y_0) = y_0$ .

We now need to show that the polynomial case proves the general case. To see why the assumption that  $r_i(x)$  is not monomial in  $X$  isn't too restrictive, consider  $q(X, Y) = XY + X^2Y = (X + X^2)Y$ . This is not of the form  $\sum r_i(X)Y^i$ . However, we can consider  $\tilde{q}(X, Y) = q(XY, Y) = XY^2 + X^2Y^3$ . This satisfies the assumption of monomials. If  $(\tilde{X}_0, \tilde{Y}_0)$  is a solution for  $\tilde{q} = 0$ , then  $(\frac{\tilde{X}_0}{Y_0}, \tilde{Y}_0)$  is a solution for  $q = 0$ . The key point is that  $\tilde{q}$  is obtained from  $q$  by an invertible transformation in  $(\mathbb{K}^*)^2$ .

Given  $q(X, Y)$  of degree  $d$ , define  $\tilde{q}(X, Y) = q(XY, Y^{d-1})$ , this satisfies the assumptions we have made. If we have  $q(X, Y) = \sum r_{ij}x^iY^j$ , then  $\tilde{q}(X, Y) = \sum r_{ij}X^iY^{(d+1)j+1}$ . Now, if we ask if it's possible to have conflicting  $i, j$ , i.e.  $(d+1)j_1 + i_1 = (d+1)j_2 + i_2$ ? This says we need  $(d+1)j_1 - j_2 = i_2 - i_1$ . However,  $0 \leq i_1, i_2 \leq d$ , so their difference is  $\leq d$ . Furthermore,  $j_1 - j_2 \geq d+1$  when  $j_1 = j_2$ . (Easier proof, sifting powers of  $Y$  first, then assure powers of  $X$  cannot cause overlap).

□

We can ask which polynomial with Puiseux valued coefficients has as its tropical polynomial  $p(x, y) = xy \oplus x \oplus y \oplus 0$ . valuations of zero only occurs with the constant complex numbers. We set the coefficients associated with  $X$  and  $Y$ 's to be 1, so we let  $q(X, Y) = XY + X + Y + C$ , where  $C \in \mathbb{C}$ .  $q$  is a polynomial in  $\mathbb{K}[X, Y]$  with the property that  $\text{trop}(q) = p$ . Now, for particular  $q$ , we can ask for  $V(q)$ . For  $q = (X+1)(Y+1) + (C-1) = 0$ , we can ask for the set  $\{(X+1)(Y+1) = C\}$ . This is a translation of the  $\mathbb{C}^2$  hyperbola where the asymptotes are the axes  $X = -1$  and  $Y = -1$ .

We can ask to describe this curve in the projective plane. We homogenize with  $Z$ , to then describe the points at infinity. It is odd that the curve hits infinity twice, considering that two of the three special points of the projective plane of  $\mathbb{C}^2$ , the other being the origin. Instead, we can compactify via  $\mathbb{P}^1 \times \mathbb{P}^1$ , the product of  $\mathbb{C}^2 \cup \{\infty\}$ . We do this by making the polynomial bi-homogeneous (homogeneous on  $X$ , and homogeneous on  $Y$ ). So we have  $\tilde{q} = X_1Y_1 + X_1Y_0 + Y_1X_0 + X_0Y_0C = 0$ . This is done by treating  $Y$  as a constant, then we get homogeneous in  $X$ , and vice versa. This is bi-homogeneous of degree 1. Now we do not intersect the four special points at all, and we intersect each of the special lines once. This gives us general behavior (transversal intersection with the boundary).

Somehow, the shape of the tropical curve tells us that it is tropicalization of some plane curve, but it should be compactified in  $\mathbb{P}^1 \times \mathbb{P}^1$ , not  $\mathbb{P}^2$ .



**Theorem 5.3.** *Bummer* Let  $q(X, Y) = \sum_{i+j \leq d} a_{ij} X^i Y^j$  be a polynomial of degree  $d$  in  $\mathbb{C}[X, Y] \subset \mathbb{C}\{\{t\}\}[X, Y]$ , and all coefficients are  $\neq 0$ , i.e.e  $a_{ij} \neq 0$  for all  $i, j$ . Then  $\overline{\text{trop}(v(q))}$  looks like a tropical line with vertex at  $(0, 0)$ .

This occurs for trivially valued field, where  $0 \mapsto \infty$ , and everything else maps to 0.

*Proof.* The tropicalization of  $q$  is  $\bigoplus x^i \odot y^j$  (the  $a_{ij}$  are all nonzero  $\mathbb{C}$ , so their valuation is 0). Then  $\text{trop}(q) = \min\{ix + jy\}_{i+j \leq d}$ . Then the minimum is always obtained by 0 in the first quadrant when  $i = j = 0$ ,  $dx$  in the section containing the second quadrant, and  $dy$  in the section containing the fourth quadrant.  $\square$

**Definition 5.2.** Let  $p(x, y) = \bigoplus a_{ij} \odot x^i \odot y^j$  be a tropical polynomial. The *Newton polygon* of  $p$  is the convex hull of  $(i, j)$  such that  $a_{ij} \neq -\infty$ .

**Definition 5.3.** Let  $\Sigma$  be the convex hull of the points  $(i, j, a_{ij}) \in \text{vert}(NP) \times \mathbb{R}$ ,  $\Sigma$  ( $\text{vert}(NP)$  is the vertex set of the newton polygon) is a convex polytope in  $\mathbb{R}^2 \times \mathbb{R}$ . We consider  $\pi_z : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$  defined by  $(x, y, z) \mapsto (x, y)$ , and let  $\tilde{N}$  be the subdivision of the newton polytope by projecting the corners of  $\Sigma$  you can see from above (positive  $z$  coordinate). Then the tropical curve  $V(p)$  is *dual* to such a subdivision, i.e.

1. There is a bijection between vertices/edge of a tropical curve  $\leftrightarrow$  faces/edges of  $\tilde{N}$ ,
2. reversing poset structure given by inclusion into the closure,
3. every edge of  $V(p)$  is  $\perp$  to the edge of  $\tilde{N}$  it corresponds to, and
4. The coordinates of a vertex are found by solving the linear system of equations obtained by setting equal the linear functions corresponding to monomials corresponding to vertices of the face of the  $\tilde{N}$  dual to  $V$ .

**Example 5.2.** The Newton polynomial of  $p(x, y) = 0 \oplus x^2 \oplus y^2 \oplus 1x \oplus 1y \oplus (1 + xy)$  has six points in a triangle, the three vertices and the three midpoints of the line segments. Each point gets a height corresponding to the value of the coefficient of the monomial ( $y^2$  has coefficient 0, and thus weight 0, while  $1xy$  has coefficient 1, and thus weight 1). We then get the three dimensional polytope  $(a, b, \text{wt}(a, b))$ , drape a curtain over the top, and we get a distinguished top corresponding to the triangle with vertices (the midpoints of the line segments which had weight 1).

**Theorem 5.4.** *If  $q(X, Y) \in \mathbb{K}[X, Y]$  such that  $\text{trop}(q) = p$  (modulo adapting for min/max). Then the subdivision of the Newton polytope keep track of the initial forms of  $q$ , in the sense that for any cell in the Newton polygon subdivision the initial form is given by the monomials corresponding to the lattice points in the cell.*

**Example 5.3.** Consider  $q(X, Y) = 7 + 3X^2 + Y^2 + t^{-1}X + 2t^{-1}Y + t^{-1}XY$ .

The silly but crucial observation to prove this theorem is

**Lemma 5.5.** *Evaluating a tropical monomial at a point  $(x_0, y_0)$  can be done as a dot product*

*Proof.* A tropical monomial is of the form  $m = a \odot x^i \odot y^j$ . Then  $m(x_0, y_0) = a + ix_0 + jy_0 = (i, j, a) \cdot (x_0, y_0, 1)$ .  $\square$

*Proof.* When we construct the subdivision of the Newton polygon, we consider all points with coordinates  $(i, j, a_{ij})$  as  $i, j$  range where  $a_{ij} \neq -\infty$ . So evaluating at  $(x_0, y_0)$  amounts to searching for the maximum of the dot product of the vector  $(x_0, y_0, 1)$  with all points  $(i, j, a_{ij})$ .

Evaluating the tropical polynomial at the normal vector to the plane at the top of the polytope for vectors on the edges of that face, we get zero, so the evaluation at the vertices of the vector are equal, so the evaluation of the point at the two monomials is equal. We want the vertex to be on the face of the tropical curve. For any other dot products, the evaluations are negative (the other vertices are below the plane, so the evaluations of the dot product will be negative). We construct the vectors from the plane down, so the vertex on the face is larger than the ones below. So we have  $m_{ij}(n_x, y) < \tilde{m}_{ij}(n_x, n_y)$ . When  $m_{ij}$  correspond to vertex not in face, and  $\tilde{m}_{ij}$  correspond to vertex in face.

This is done for every face of the Newton Polygon. then  $(n_x, n_y) \in \mathbb{R}^2$  is the vertex of the tropical curve dual to the particular face we were considering.

Precisely, we consider the faces of the convex whose outward pointing normal has positive  $z$  coordinate. In higher dimension, we say the last coordinate of the normal vector of the face has positive value.

For the edges, we focus on a particular edge  $e$  of the Newton polygon.  $e$  bounds two faces  $F_1$  and  $F_2$ .  $F_1$  and  $F_2$  have their respective normal vectors  $n_1$  and  $n_2$ .  $e$  dots to zero with  $n_1$  and  $n_2$ . The same is true for all linear combinations of  $n_1$  and  $n_2$ . In particular, it is true on the segment connecting  $n_1$  and  $n_2$ . So every point in the segment in  $\mathbb{R}^2$  joining  $(n_x, n_y)_1$  and  $(n_x, n_y)_2$  has the

property that the vector  $(n_x, n_y, 1)_i(m_1) = (n_x, n_y, 1)_i(m_2) > (n_x, n_y, 1)_i m_{other}$ . As the maximum is obtained twice, those points belong to the tropical curve.

□

## 6 Sufficient conditions for tropical curves

**Definition 6.1.** Any edge of a tropical plane curve  $V(p)$  is given weight  $\omega_e$  equal to the lattice length of the segment of the Newton Polytope Subdivision dual to the edge.

**Definition 6.2.** A *primitive vector* of the direction vector  $p$  is the first integral vector (vector with integer coordinates) after the origin which lies on the line defined by  $p$  in the direction of  $p$ .

**Theorem 6.1.** Tropical plane curves are balanced, i.e. at every vertex, the summation  $\sum_{v \in e} \omega_e \vec{p}_e = 0$ , where  $\omega_e$  is the weight of the edge, and  $\vec{p}_e$  is the primitive vector in the direction of  $e$ .

*Proof.* Any vertex  $v$  is dual to a face  $F_v$  of the Newton polygon subdivision. For every edge bounding  $F_v$ , the vector  $\omega_e p_e$  is obtained by the vector tracing the dual edge via the linear transformation  $(x, y) \mapsto (y, -x)$ . Now,  $\sum_{v \in e} \omega_e \vec{p}_e = 0$  is equivalent to the polygon being a closed polygon. □

Now that we have defined weights, we want to ask what they represent.

If we are just looking for solutions of  $x^2 y P(x^3 y) = 0$  in the torus  $(\mathbb{C}^*)^2$  or asymptotically ( $|x|, |y| \gg 0$ ), then (A) the monomial part  $x^2 y$  is irrelevant (Only vanish at zero or infinity), and (B) We have  $\deg(P) = \text{lattice length of the segment}$ . On the torus, we have 1-parameter subgroup orbits of the form  $x^3 y = r_i$ , where  $r_i$  is a root of  $p$  and is counted with multiplicity.

So if we take a point in the tropical curve  $w$ . We think of the Puiseux plane as many planes assembled by valuation. Now that we have fixed valuations, we single out a particular  $\mathbb{C}^2$ . When  $x, y$  are very large, the polynomial is approximated by the initial form, so we get (eight of edge) number of torus orbits. The direction of the edge in the subdivision tells us what we get for our 1-parameter subgroup. The primitive vector is  $(3, 1)$ , so we get  $x^3 y$ .

The following are things to consider about tropical curves. We can draw all topological types of tropical plane conics and cubics. We can experiment with various tropical plane curves and seek a conjecture to compute their  $b_1$ . Finally, we can study a pencil of tropical conics, i.e. draw a conic, pick four points on it in general position, then find all conics through those four points.

When following four parameter conics through the four points, we find interesting points between leaves and edges.

## 7 Intersection of Tropical Curves

**Definition 7.1.** Two tropical curves intersect *transversally* if they intersect in finitely many points which are not vertices of either curve.

Our second kind of intersection will be stable intersection. Let  $v \in \mathbb{R}^2 - \{(0,0)\}$ , and define  $\Gamma_1 \cap_v \Gamma_2 = \lim_{t \rightarrow 0} \Gamma_1 \cap (\Gamma_2 + tv)$ , i.e. translate  $\Gamma_2$  by the vector  $v$ . If we were to pick another  $v$ , our translations would be different. We are forced to ask ourselves if distinct choices of  $v$  may lead to different limits.

**Note 2.** Points of intersections should be weighted by multiplicities of the edges they belong to.

**Example 7.1.** How do the complex curves  $C_1 = \{x^a = y^b\}$  and  $C_2 = \{x^c = y^d\}$  intersect in  $\mathbb{C}^2$ ?  
*Solution:* We can parameterize  $x = t^b$ ,  $y = t^a$ , then we get  $t^{bc}(t^{ad-bc} - 1) = 0$  🌈

**Definition 7.2.** Let  $p$  be a point of transversal intersection of  $\Gamma_1, \Gamma_2$ . We let the multiplicity of the point of intersection  $p$  to be

$$M_p(\Gamma_1, \Gamma_2) := w_{e_1} w_{e_2} \left| \det \begin{bmatrix} P_{1,x} & P_{2,x} \\ P_{1,y} & P_{2,y} \end{bmatrix} \right| = [\mathbb{Z}^2 \mid p_1 \mathbb{Z} + p_2 \mathbb{Z}]$$

Where  $p_i$  are the direction vectors of the edges for the intersection,  $p_{1,x}$  is the  $x$  coordinate of the direction vector for edge 1, and  $w_{e_i}$  is the weight of the corresponding edge

Last week: Fans  $\Sigma$  lead to toric varieties  $X_\Sigma$ . Maximal cones lead to affine charts, and faces lead to transition functions.  $\Sigma \subset N_{\mathbb{R}}$ .

we get two neat consequences.

1. There is an inclusion into closure reversing bijection between cones of  $\sigma$  and the torus orbits of  $X_\Sigma$  (also, exchanging dimension with co dimension).
2.  $T$ -equivariant maps of toric varieties correspond to maps of fans.

**Definition 7.3.** Given  $\Sigma_1 \subset N_{1\mathbb{R}}$  and  $\Sigma_2 \subset N_{2\mathbb{R}}$  a *map of fans* is a  $\mathbb{Z}$ -linear map  $L : N_{1\mathbb{R}} \rightarrow N_{2\mathbb{R}}$  such that for every cone  $\tau$  of  $\Sigma_1$ ,  $L(\tau) \subset$  a cone of  $\Sigma_2$

with the embedding, we do not necessarily want to subdivide fans, as that introduces non trivial orbits.

If a toric variety is made of toric varieties, torus are homotopic to circles, which have genus 0, so they contribute trivially. To see how this bijection works

1. For every cone  $\tau$  of  $\Sigma$ , look at the limits as  $t \rightarrow 0$  of torus orbits of 1-parameter subgroups  $[\gamma]$  with  $\gamma \in \tau^0$  (in the interior of the cone  $\tau$ ).
2. For every affine patch dual to a cone, set all the coordinates that you can set to zero to zero

Toric varieties can be viewed as a generalization of projective space (we basically have homogeneous coordinates). We consider the Quotient Construction. Given a fan  $\Sigma$ , the toric variety  $X_\Sigma$  can be obtained as a quotient space of the form  $\mathbb{C}^N - \{\text{Irrelevant}\}/G$ , where  $N$  is the number of rays in the fan  $\Sigma$  (for every ray we get a homogeneous coordinate in the toric variety), the irrelevant stuff is the locus determined by sets of rays that do not span cones of the fan, and  $G$  is given by linear relations among rays.

Toric varieties have an orbit cone correspondence. This is a bijection between the cones of a toric variety, and the toric orbits. We have that good maps of toric varieties correspond to maps of fans. We also have a quotient construction  $X_\Sigma = \mathbb{C}^N - \{\text{stuff}\}/G$ , where  $N$  is the number of rays, the stuff is the collection of rays not spanning a cone, and  $G$  is a linear relation among rays.

**Example 7.2.**  $\mathbb{P}^2$  has three rays, denoted by  $P_X$ ,  $P_Y$ , and  $P_Z$ . The only subset of rays NOT spanning a cone is the subset  $\{P_X, P_Y, P_Z\}$  (any two span a cone). so we throw away the locus  $\{X = Y = Z = 0\}$ , i.e.e the origin. The relation we have between the three is that  $1P_X + 1P_Y + 1P_Z = 0$  (the vectors are  $P_X = (1, 0)$ ,  $P_Y = (0, 1)$ , and  $P_Z = (-1, -1)$ ). all of the 1's tell us we have a one dimensional torus, and our action will be  $t(X, Y, Z) = (t^1 X, t^1 Y, t^1 Z)$  With these relations, we get  $\mathbb{C}^*$

Now, we can take the fan, and declare that any ray of the fan is generated by a basis vector of a new vector space. so  $P_X = (1, 0, 0)$ ,  $P_Y = (0, 1, 0)$ , and  $P_Z = (0, 0, 1)$ . Then our three corresponding rays in  $\mathbb{R}^3$  become the three positive half lines of the X, Y, Z axis. Now we want to lift the cones in our toric variety, so we get octant planes. Now, everything we make is contained in the first octant of  $\mathbb{R}^3$ , so this is a subset of all the cones of the first octant, but the first octant with removal. But the first octant is  $\mathbb{C}^n$ , and we removed something. We remove orbits corresponding to not being centered in the octant, i.e.e we remove the three dimensional cone, so the toric variety is  $\mathbb{C}^3$  minus the orbit corresponding to the center cone, which is  $(0, 0, 0)$ .

In tropical geometry,  $k$  is identified with  $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ , then  $k^*$  corresponds to  $\mathbb{R}_+$ . Then  $T = (k^*)^n$  corresponds to  $\mathbb{R}^n$ , and the action  $*$  corresponds to  $+$ .

How to make tropical  $\mathbb{P}^2$

## 8 Tropical Toric Varieties

From the fan  $\Sigma$  we use tropical numbers and tropical operations for transition functions, i.e.  $\mathbb{TP}^2$  provides us with  $\mathbb{T}^2, x_i, y_i$ , for  $i = 1, 2, 3$ . We get relations from  $i = 1$  and  $i = 2$  via  $x_i = -x_2$  &  $y_1 = y_2 - x_2$ , from 2 to 3 via  $x_2 = y_3 - x_3$  &  $y_2 = -x_3$ , and finally from 1 and 3 via  $x_1 = x_3 - y_3$  &  $y_1 = -y_3$ .

We get invertible in GLZ, which is our notion of automorphism of the torus. We can always do this translation. We observe that the tropical toric variety  $\mathbb{TX}_\Sigma$  has a stratification into “ $\mathbb{R}^k$ ” strata, which has a natural poset isomorphism with the stratification of the complex toric variety  $X_\Sigma$ . Furthermore,  $\mathbb{TX}_\Sigma$  has the structure of an “ $\infty$ ” polytope, i.e. scale the polytope so that lengths go to infinity, which is the normal/dual polytope to the fan of  $X_\Sigma$ .

A stratification is a disjoint union into locally closed spaces. We get 0-dim, 1-dim, and 2-dim spaces, each is a tropical tori. This is similar as how  $\mathbb{CP}^2$  has a stratification as points, lines, and the surface of the triangle.

Now, the shortcut to get the tropical toric variety via the fan of  $\mathbb{P}^2$ , we make rays orthogonal to the rays of  $\mathbb{P}^2$ , introduce fans corresponding to each of the 2-dim pieces of the fan, and set the lines to infinity.

The role taken by  $\lim -t \rightarrow 0t * p$  in  $\mathbb{C}$ -toric land is replaced by  $\lim -T \rightarrow \infty T + p$  in tropical toric land. So if we take a one parameter subgroup, say  $(1, 1)$ , and we ask for the orbits of this subgroup, the orbits are found via  $T(x_1, y_1) = (T + x_1, T + y_1)$ . So regardless of the starting point, the orbit gets to the point at infinity. For the one parameter subgroup  $(2, 1)$ , we have  $T(x_1, y_1) = (2T + x_1, T + y_1)$ .

We can also show how the quotient construction works tropically. We begin at  $\mathbb{P}^1$ . In our quotient construction, this is  $\mathbb{C}^2 - (0, 0) / \mathbb{C}^*(t(x, y) = (tx, ty))$ , where  $t*(x, y) = (tx, ty)$ . The one parameter subgroup comes from the linear relation on the fans of  $\mathbb{P}^1$ , the relation being  $p_1 + p_2 = 0$ . Now we do the same from the fan to get us into tropical space.  $\mathbb{TP}^1 = \mathbb{T}^2 - (\infty, \infty) / \mathbb{R}(T(X, Y) = (T + X, T + Y))$ , where  $\mathbb{R}$  is the torus on  $\mathbb{T}$ . We expect  $\mathbb{TP}^1$  to be a line segment, as we have  $\mathbb{P}^1$  is two rays connected. We get a point for the first ray, a point for the second, and a line connecting the two. This is related to initial forms (See other texts).

We have parallel lines connecting to the singular point at infinity which we remove. This means that each parallel ray hits a different point at infinity. We can create an orthogonal line to these rays, and we say this ray hits two other points at infinity (at the limits of identifying the x and y axis).

We get some kind of orthogonal line to these parallel rays, and this line intersects each orbit exactly

once. This line connects at points which represent  $(\infty, 0)$  and  $(0, \infty)$ .

We have our original construction of tropical curves. Now, assume we have  $Y \subset \text{Torus} \subset X_\Sigma$ , so a curve in a torus in a toric variety. Now we have  $\text{Trop}(Y) \subset \mathbb{R}^n \subset \mathbb{T}X_\Sigma$ .

**Definition 8.1.** The *extended Tropicalization* of  $Y$  inside of  $X_\Sigma$  is the closure of  $\text{Trop}(Y)$  inside of  $\mathbb{T}X_\Sigma$ .

So we start with a line  $L \subset (\mathbb{C}^*)^2 \subset \mathbb{P}^2$ . This gives rise to a tropical line  $\text{Trop}(L) \subset \mathbb{R}^2$ . The closure of  $\text{Trop}(L)$  gives three additional points, one for each section of the tropical  $L$  which hits lines at infinities.

Next time, we will compare the combinatorics of tropical fans in varieties to toric varieties.

**Theorem 8.1.**  $\bar{Y} \cap O_\sigma \neq \emptyset$  and dimensional transversality  $\text{codim}_{\bar{Y}}(\bar{Y} \cap O_\sigma) = \text{codim}_{X_\Sigma}(O_\sigma)$  iff  $\text{Trop}Y = |\Sigma|$

*Proof.*  $\implies$  we know that  $\text{trop}(Y) \subset |\Sigma|$ . We want to prove  $\supset$ . First, we have that any  $\sigma \in \Sigma$  must intersect  $\text{trop}Y$ , as  $\bar{Y} \cap O_\sigma \neq \emptyset$ . Second, we have that  $\dim(\Sigma)$  is at most  $\dim(Y)$ , by the dimensional transversality. Third, we have that a top dimensional cone of  $\Sigma$  cannot intersect  $\text{trop}(Y)$  in a positive codimension locus, otherwise  $\text{Trop}(Y)$  would intersect  $(d+1)$  dimensional cones of  $\Sigma$  (which don't exist). Finally, a top dimensional cone of  $\sigma$  cannot be partially covered by  $\text{trop}(Y)$ , as either this would violate the balancing condition for  $\text{Trop}(Y)$  (balancing condition in higher dimensions is done by quotienting), or  $\text{trop}Y$  has to intersect at a face of  $\Sigma$ , not the of a face.

With these four conditions, we get that every  $\sigma \in \Sigma$  must intersect  $\text{trop}(Y)$ , and the intersections are all showing  $\text{trop}()$  is covering  $\sigma$ , so we must have that  $|\Sigma| \subset \text{trop}(Y)$ .

$\Leftarrow$  Now we are assuming that  $\text{trop}(Y) = |\Sigma|$ . We need some algebraic geometry black boxes.

1. There are no positive dimensional compact subvarieties of tori.
2. For toric varieties, an orbit  $O_\sigma$  of codimension  $k$  is (locally) cut out by exactly  $k$  equations
3. For any subvariety  $Z \subset X_\Sigma$  and any hypersurface  $HS \subset X_\Sigma$ ,  $Z \cap HS$  either remains the same dimension of  $Z$ , the dimension goes down by exactly one, or the intersection is empty.

With these three facts we can begin to show the other implication. We begin with  $\sigma_d$  being a top dimensional cone in  $\Sigma$ . Then  $O_{\sigma_d}$  is a codimension  $d$  orbit. Furthermore, we have  $O_{\sigma_d} \cap \bar{Y} \neq \emptyset$ , and that  $O_{\sigma_d}$  is isomorphic to some torus. Because  $\bar{Y}$  is complete, the intersection  $O_{\sigma_d} \cap \bar{Y}$  is complete, and so it has a finite number of points.

All together, this says we can obtain the orbit  $O_\sigma$  by cutting down by  $k$  hypersurfaces. In particular, if we have our cone  $\sigma$ , we can take a chain of subsequent surfaces  $r_1, \text{span}(r_1, r_2), \text{span}(r_1, r_2, r_3)$ , and corresponding to  $r_1$  we have a hypersurface  $H_1$  (orbit of  $r_1$ ,  $H_1 \cap H_2$  is orbit of  $\sigma_2$ , and so on), we can take  $\bar{Y} \cap H_1$ , then  $\bar{Y} \cap H_1 \cap H_2$ , and then  $\bar{Y} \cap H_1 \cap H_2 \cap H_3$ . We know  $\bar{Y}$  has dimension 3, at the end we have just points at the end, and we drop dimensions by exactly one at every step.

We now reiterate the previously stated algebraic geometry black boxes.

1. The only complete/compact subvariety of a torus are 0-dimensional
2. In a smooth toric variety, every orbit of sigma  $O_\sigma$  of codimension  $k$  is locally cut out by  $k$ -equations.
3. If  $Y \subset X_\Sigma$ , and  $Y \cap \text{Hypersurface} \neq \emptyset$ , then the dimension of the intersection goes down at most by 1.

Now we assume  $\text{Trop}(Y) = |\Sigma|$ . We can consider some face  $\sigma \in \Sigma$ , and let  $\tilde{\sigma}$  to be a top dimensional cone of  $\Sigma$  containing  $\sigma$  as a face. We choose an ordering of the rays of  $\tilde{\sigma}$  such that the first  $k$  rays belong to  $\sigma$ . (Each ray corresponds to a codimension 1 orbit, after the  $k$  steps we get to an intersection with  $O_\sigma \cap Y$ ). We let  $H_i$  be the closure  $\overline{O_{\rho_i}}$ , i.e. the hypersurface in  $X_\sigma$  corresponding to the ray  $\rho_i$ . Now,  $\dim(\bar{Y} \cap H_1) \geq \dim(\bar{Y}) - 1$ . Then,  $\dim((\bar{Y} \cap H_1) \cap H_2) \geq \dim(\bar{Y}) - 2$ . Eventually we get to  $\dim(\bar{Y} \cap H_1 \cap \dots \cap H_k) \geq \dim(\bar{Y}) - k$ . To check that we get minimal dimensions, we terminate at  $\dim(\bar{Y} \cap \dots \cap H_d) = \dim(\bar{Y} \cap O_{\tilde{\sigma}})$ . Note  $\bar{Y} \cap \overline{O_{\tilde{\sigma}}} = \bar{Y} \cap O_{\tilde{\sigma}}$  (inherently the closure of  $Y$  misses any missing limit points of the orbit). So we know that  $\bar{Y} \cap \overline{O_{\tilde{\sigma}}}$  is compact, compact living in the torus  $O_{\tilde{\sigma}}$ , but by our black box we get that subvariety of a torus is 0 dimension, at every step we needed to reduce our dimension by exactly once, so at no point would our dimension ever stay the same.

□

**Definition 8.2.**  $\bar{Y} \subset T \subset X_\Sigma$  is a *tropical compactification* when  $\text{trop}(Y) = |\Sigma|$

This definition says that  $Y$  thinks  $X_\Sigma$  is a good place to be compactified in. This is because  $X_\Sigma$  doesn't waste any orbits, and all orbit of  $X_\Sigma$  are dimensionally transverse to  $\bar{Y}$ .

In particular, the nice properties of the left hand side of the theorem hold. So far we have not talked about the singularity of  $\bar{Y}$  when we compactify it. It is possible that  $\bar{Y}$  could have gotten singular in this process. Via a technical process and analysis with a toolkit from a third course in algebraic geometry, we notice that the statement of tropical compactification is discussing the support of  $\Sigma$ , not the fan itself.

In toric geometry if we take a fan and subdivide into cones, this corresponds to blow ups in strata of toric varieties (which are used to resolve singularities). So maybe we had singularities, but we use blow ups to resolve singularities.



1. One can always refine  $\Sigma$  so that  $\overline{Y}$  is Cohen-Macaulay (read “not too badly singular”)
2. In Char 0,  $X_\Sigma$  projective, then we can find an open subset such that  $\overline{Y} \cap O_\sigma$  is smooth for all  $\sigma$ .

Now,  $K$ -trivially valued, tropical compactification tells us that if  $Y \subset T$ , we get that  $\text{trop}(Y)$  determines a toric variety inside which  $Y$  compactifies nicely!

## 9 Geometric Tropicalization

If  $Y \subset X_\Sigma$  is a subset of a toric variety, and  $Y$  sits “nicely” in  $X_\Sigma$ , then the toric variety allows us to know  $\text{trop}(Y)$ . We illustrate this in an example, and then generalize.

**Example 9.1.** We take the line in  $\mathbb{P}^2$ ,  $\{X + Y + Z = 0\} \subset \mathbb{P}^2$ . The strata of the line is defined via a boundary complex, where smaller stratifications lead to larger structures in the boundary complex. In the case of  $\mathbb{P}^2$  and the boundary complex of  $L$ , we get three points  $\{a, b, c\}$ . We can construct the cone over the boundary complex  $cx$  by taking a point in an independent dimension from  $a, b, c$ , and join the points with half lines. The toric variety has 3 divisors (codimension 1 subvarieties, in  $\mathbb{P}^2$  it is the lines, not lying within the torus) on  $\mathbb{P}^2$  that induce (via intersection) 3 divisors on  $L$  (the points of intersection of  $L$  with the toric invariant lines of  $\mathbb{P}^2$ ). Divisors give rise to “divisorial valuation”  $\text{val}_D : K(L) \rightarrow \mathbb{Z} \cup \infty$ , where for every function  $f \in K(L)$  it defines the order of vanishing or pole of  $f$  along  $D$ . So as long as the space is not too singular, locally and divisor has one local equation, take the rational function where on this local set is valid, and see how many times we can factor the equation with our function.

Notice that  $\text{val}_a : K(L) \rightarrow \mathbb{Z} \cup \infty$ , where we get a rational function on  $\mathbb{P}^2$ , restrict to  $L$ , then do the evaluation, but we really like monomials as rational functions. So we actually take  $\text{val}_a : M_T \rightarrow K(L) \rightarrow \mathbb{Z} \cup \infty$ , where  $M_T \rightarrow K(L)$  is via restriction, and  $K(L) \rightarrow \mathbb{Z} \cup \infty$  is the order of vanishing. So we assign an element of  $M$  a valuation, which is dual to  $M$ , so  $\text{image}(\text{val}_a) \subset N_T$ , which is connected to the co-character lattice.

$M_T$  is the character space of the torus i.e. monomials, and  $N_T$  is the co-character 1-parameter subgroups.

For our recipe, we use  $\text{val}_a$ ,  $\text{val}_b$ , and  $\text{val}_c$  to get three points in  $N_T$ . But we think of each of these points as points of height one in the cone of the boundary complex, and we take the cone over these point in  $N^T$ . We then get  $\text{Trop}(Y)$ .

Now we do computations, we take the point  $a = (0 : 1 : 1)$ . We can parameterize the line  $L$  around the neighborhood of  $a$  via  $t$  by defining affine coordinates in  $\mathbb{P}^2$ , so we take  $x = \frac{X}{Z}$ ,  $y = \frac{Y}{Z}$ , as the line we parameterize as  $x(t) = t$ , and  $y(t) = 1 - t$ . We now need to compute  $\text{val}_a$ . In this case,  $M_T$  is  $\mathbb{Z}_2$ , so we can test on the generating set  $x$  and  $y$ . So  $\text{val}_a(x) = 1$ , as the ordering of vanishing of  $t$  at 0 is 1.  $\text{val}_a(y) = 0$ , as the order of vanishing of  $1 - t$  at  $t = 0$  is 0. Similarly,  $b = (1 : 0 : 1)$ , this switches the role of  $x$  and  $y$ , so  $x(t) = 1 - t$  and  $y(t) = t$ , so  $\text{val}_b(x) = 0$  and

$val_a(y) = 1$ . The trickier point is  $c$ , which is the point at infinity. We can choose a parameterization of  $L$  near this point centered at  $c$ , which becomes  $x(t) = \frac{1}{t}$  and  $y(t) = 1 - \frac{1}{t}$ . Now  $val_c(x) = -1$  and  $val_c(y) = -1$ . We thus define  $val_a = (1, 0)$ ,  $val_b = (0, 1)$ , and  $val_c = (-1, -1)$ . This gives us  $Trop(L)$ .

In general, we start with a very affine variety with divisorial boundary. A divisorial boundary, we start with a space  $Y$ , with codimension 1 subvarieties, very affine means that  $Y - \partial Y$  is isomorphic to a closed subvariety of a torus (closed inside the torus, typically the complement lies in a torus).

**Definition 9.1.** We say  $\partial Y$  has combinatorial normal crossings if every time  $n$  divisors intersect they intersect in codimension  $n$ . If the intersection is transversal, we say this is a simple normal crossing

Simple normal crossing means that the boundary locally looks like hyperplane coordinates in our space.

**Definition 9.2.** A normal crossing is locally SNC

We can ask for divisorial evaluations. We embed the curve minus its boundary into a torus. In practice we can embed our boundary into any toric variety we choose. When  $Y - \partial Y \subset T$ , and toric variety containing  $T$  as their dense torus gives rise to divisorial valuations.

**Lemma 9.1.** 1.  $Trop(Y) = \{c * val_D \mid c \in \mathbb{R}_{\geq 0}, D \text{ is any divisor coming from and } X_\Sigma \supset T\} \subset N_T$

2. If  $\partial Y$  is combinatorial normal crossing, then there exists a map  $\pi : C\Delta_{\partial Y} \rightarrow N_T$ ,  $Y$  was a variety with some divisor, so we use those varieties, so that  $\pi(D_i, 1) \mapsto val_{D_i}$ , and we extend by linearity. With combinatorial normal crossings, we can guarantee that  $Trop(Y) \subset Im(\pi)$ . If we want equality, we ensure simple normal crossing boundaries, or (if char 0 is fine), combinatorial normal crossing with characteristic 0 is sufficient.

We get a map from the torus  $T^2$  to  $\mathbb{P}_{\mathbb{K}}^3$  via  $(s : u) \mapsto (1 : u : tsu)$ . We get the conic  $x_0x_3 = tx_1x_2$ . For  $t \neq 0$  we get the smooth quadratic, for  $t = 1$  we get the intersection of planes. The two copies of  $\mathbb{P}^2$  which have the line is also given by the lines of Puiseux valued points corresponding to the fact that the vertices of the curve are  $(0, 0)$  and  $(1, 1)$ . The tropical curve tells us that the central fiber of the family of curves corresponds to  $x + y = 0$ , when  $x_0$  and  $x_3$  both equal 0.

When the valuation of  $x$  and  $y$  are both zero, then the valuation for  $txy$  is irrelevant.

We get a fan in  $N_T \times \mathbb{R}$  and a ray dealing with  $t$  depending on expanding on  $t$  from  $1+x+y+txy=0$ . We have the inverse image of a fan  $\mathbb{P}^1 \times \mathbb{P}^1$ , under the special./generic fiber, we have the actual tropical curve.

## 10 Student Presentations

### 10.1 Joel, Gfan

Gfan is a software package for computing Grobner fans and tropical varieties. Overview of installation. Gfan runs on the kernel. Gfan consists of various programs. Gfan can be run by creating scripts to pass to the Gfan library. Gfan natively supports  $\mathbb{Q}$  and finite fields. Integers can be used with some work. For example, we can begin with an ideal, such as  $\mathbb{Q}[x, y, z]$  and the ideal  $\{xxy - z, yyz - x, zzx - y\}$ . The library can compute a Grobner basis for any such ideal.

There is a relationship between Grobner basis and tropical varieties. A Grobner basis can give a Grobner cone. A collection of these Grobner cones construct a Grobner fan (distinct from tropical fan). A tropical variety is a union of Grobner cones, and is thus a subfan of a Grobner fan. Gfan does not have the most advanced visualization techniques. Gfan can visualize figure files (only with xfig). We can imagine the Grobner fan exists in  $\mathbb{R}^3$  in the positive quadrant. The Grobner cone is 2-D cones. The unit vectors in  $\mathbb{R}^3$  form a triangle, and the image is the intersection of these cones with that triangle. Given an ideal and a permutation in a permutation group, it can compute a cone in an orbit of the permutation group.

If we want to consider tropical varieties, we can give Gfan a principle ideal  $(x + y + z + w)$  within  $\mathbb{Q}[x, y, z, w]$ . If we want to ask more of the tropical varieties, we would run tropical intersection to compute the rays, the cones generated by the rays. This is a tropical hyperplane in 4-dim.

Generically, Gfan does trivially valued fields. It is possible to add nontrivial valuations, but that takes longer to compute.

### 10.2 Kristina, Tropical Geometry of Deep Neural Networks

There is an equivalence between feedforward neural networks with ReLU activation and tropical rational functions.

(Discussion of the cat vs dog pictures for neural networks) Orientation, identifiable features, replicate human brain processing. Neural networks have hidden layers. Each layer is a matrix product (weight assignment) and vector addition (bias). We then introduce cost functions to compare results, back propagation (gradient descent) to adjust weights.

We have activation functions. The first is the sigmoid  $\sigma(Ax + b) = \frac{1}{1+e^{-(Ax+b)}}$  we also have the rectified linear unit ReLU  $\sigma(Ax + b) = \max(0, Ax + b)$ . ReLU makes data more sparse, and looks

like tropical geometry. First we make assumptions about our  $L$ -layer network. We assume that the weight matrices  $A^{(1)}, \dots, A^{(l)}$  are integer values, the biasvectors  $b^{(1)}, \dots, b^{(l)}$  are real valued, and the activation functions take the form  $\sigma(Ax + b) = \max(0, Ax + b)$ .

To build our equivalence, we first consider the output from the first layer in the neural network  $\nu(x) = \max\{Ax + b, t\}$ , where  $t \in (\mathbb{R} \cup \infty)^l$ . So we can rewrite  $\max\{Ax + b, t\} = \max\{A_+x + b, A_-x + t\} - A_-x$ . So every coordinate of a one layer network is the difference of two tropical polynomials. For networks with multiple layers, apply this decomposition recursively.

**Theorem 10.1.** *A feedforward neural network under the assumptions is a function  $\nu : \mathbb{R}^d \rightarrow \mathbb{R}^p$  whose coordinates are tropical rational functions of the input, i.e.  $\nu(x)F(x) \odot G(x) = F(x) - G(x)$ , where  $F$  and  $G$  are tropical rational functions.*

We can use this equivalence to consider decision boundaries of a neural network. The input space of a neural network is partitioned into disjoint subsets, where each subset determines a final decision (what is a dog, what is a cat). So our input space might be a tropical curve, and the 2-cells give decision boundaries. We can bound the number of linear regions of a NN by bounding vertices in the dual subdivision of the Newton polytope. This number of linear regions measures complexity of a neural network. These don't make better bounds, but it shows that tropical geometries can do the same work.

### 10.3 Jacob, The Joswig Algorithm

We let  $\mathbb{T} := (\mathbb{R} \cup \{\infty\}, \min, +)$ . We begin by saying that networks can be modeled using graphs. We have Dijkstra's Algorithm which gives us shortest path (including weights). This is relevant to traversing cities with weights on roads. Sometimes, fixed edge weights are too limiting. We let  $x, y \in \mathbb{T}$  be parameters. We have separate graphs, only one copy of each variable associated to each edge (no two edges have the same variable). We then have a parametric shortest path. We let  $\odot$  be a concatenation of edges/paths, and  $\oplus$  a comparison. We interpret  $\infty$  as being an edge that doesn't exist. We can consider the two parametric equations to be  $4 + y \leq 5 + x$  to go down the left path, or  $4 + y \geq 5 + x$  to go down the right path, so we have  $\min\{4 + y, 5 + x\} = (4 \odot y) \oplus (5 \odot x)$ .

The key observation is that the regions of optimal solutions are separated by tropical varieties. Now, what if our graph has multiple tropical polynomials. Each tropical polynomial corresponds to a different destination node. We can instead consider  $(A \rightarrow B, 2)$ ,  $(A \rightarrow B, y)$ ,  $(B \rightarrow D, 1)$ ,  $(A \rightarrow D, 2)$ ,  $(A \rightarrow C, x)$ ,  $(A \rightarrow C, 2)$ , and  $(C \rightarrow D, 1)$ . We then have polynomials with residues when how we end at  $D$ . The Joswig Algorithm is reducing these polynomials. A selection of a solution is selecting a path for each destination node. In each region we select a path for that destination.

We decompose our parameter space into cells where within each cell we have an optimal solution. If we have a path through our  $x$ - $y$  plane, we can segment that path into optimal segments.

This decomposition of a parameter space came from three tropical polynomials. We can ask if we

can describe the decomposition as the tropical varieties of some polynomials. We cannot! We have a proof by pictorial contradiction. We have vertices on the corners, which have coefficients  $\infty$ , but that would cause kinds and additional cells (new vertices).

You are guaranteed convex cells (for any two solutions in a cell, any solution in between is a solution). The computation are also doable and efficient. In summary, the Joswig algorithm produces a decomposition of parameter space into convex cells via tropical varieties. The algorithm works on the order of  $10^3$  parameters, as the parametric shortest path problem is (probably) NP-complete (or hard).

## 10.4 Natalie, Group Theory and Tropical Geometry

We let  $\xi$  be a nonzero real number, and denote  $G_\xi$  by the group generated by  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  and  $X = \begin{bmatrix} 1 & \xi \\ 0 & 1 \end{bmatrix}$ . We then ask if  $G_\xi$  is finitely presented. If  $\xi$  is transcendental, the answer is no. However, let  $\xi$  be the root of some irreducible polynomial  $f(x) \in \mathbb{Z}[x]$ . Then  $G_\xi$  is finitely presented iff  $\xi$  or  $\frac{1}{\xi}$  is an algebraic integer over  $\mathbb{Q}$ .

The statement that  $\xi$  or  $\frac{1}{\xi}$  is an algebraic integer over  $\mathbb{Q}$  implies that highest order term or lowest order term of  $f(x)$  is  $\pm 1$ .

Consider the Laurent Polynomial ring  $S = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ . Units are monomials  $\pm x_1^{a_1} \cdots x_n^{a_n}$ .

**Definition 10.1.** The *initial form* of  $f(\vec{x})$  wrt  $\vec{w}$  is a polynomial that records the terms that admit the minimum when tropicalized with  $\vec{w} = \text{val}(\vec{x})$ .

**Example 10.1.** Consider  $f(x) = 4x^3 + 3x + 2$ . Then  $\text{Trop}(f) = \min(3\text{val}(x), \text{val}(x), 0)$ . Now, let  $w = 1 = \text{val}(x)$ . Then  $\text{trop}(f) = \min\{3, 1, 0\}$ , the initial form  $\in_{w=1}(f) = 2$ , as 2 is where the valuation of zero comes from. Now, let  $w = 0$ . Then the minimum is the same, so  $\text{in}_{w=0}(f) = 4x^3 + 3x + 2$ , as everything hits the minimum.

**Definition 10.2.** Let  $I$  be a proper ideal of  $S$ . Then the *initial ideal*  $\text{in}_w(I)$  is the ideal generated by all initial forms  $\text{in}_w(f)$  where  $f$  runs through  $I$ .

**Definition 10.3.** The *Tropical variety* of  $I$  is  $V_{\text{trop}_{\mathbb{Z}}}(I) = \{w \in \mathbb{R}^n \mid \text{in}_w(I) \neq S\}$ , or  $V_{\text{trop}_{\mathbb{Z}}}(I) = \{w \in \mathbb{R}^n \mid \in_w(I) \text{ does not contain a unit of } S\}$ .

**Example 10.2.** Let  $I$  be the principal ideal  $I = \langle x_1 + x_2 + 3 \rangle = \{sx_1 + sx_2 + 3s \mid s \in S\}$ . We want to find  $V_{trop_{\mathbb{Z}}}(I)$ . First we find  $in_w(f)$  for possible values of  $w \in \mathbb{R}^2$ . Note that  $trop(f) = \min\{val(s) + val(x_1), val(s) + val(x_2), val(s)\}$ . Now we let  $w = (a, b)$ , and so  $val(x_1) = a$  and  $val(x_2) = b$ . Now, if  $a < 0, b$ , then  $\in_w(f) = sx_1$ . If we let  $x_1$  have valuation 1, we would get a unit, so it is not in the variety. If  $0 < a, b$ , then  $\in_w(f) = 3s$ , not a unit, so  $w$  is in the variety.

**Example 10.3.** Let  $\xi$  be a root of an irreducible polynomial  $p(X) = a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x^{\pm 1}]$ . Then  $trop(p(x)) = \min\{2val(x), val(x), 0\}$ . We have  $\in_{w < 0} = a_2x^2$ ,  $in_{w=0} = a_2x^2 + a_1x + a_0$ , and  $\in_{w > 0} = a_0$ .

## 10.5 Kylie, Cryptography

Cryptography is a format for sending secret methods that can't be read by anyone besides the recipient. The two major schools are public and private key cryptography. Within Public key cryptography, there are two keys: one public and one private. The public key is used to encrypt information, while the private key is used to decrypt information. In Private Key cryptography, there is a single private key used for both encryption and decryption.

Public key is more secure, while private key is faster. In the discussed paper, a key exchange protocol is used. This is a secure way to exchange a key using public methods between parties.

We will take the min convention. So  $\begin{bmatrix} 1 & 2 \\ \infty & -1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & -1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 2 \\ \infty & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$   
 $\otimes$  is matrix multiplication (and then using plus and min) is not commutative, and the tropical identity matrix is the matrix with zero on the diagonal, *inf* elsewhere. Diagonal matrices have something on the diagonal.

We want our public key to be  $n \times n$  matrices  $A$  and  $B$  with tropical entries, with the requirement that  $A \otimes B \neq B \otimes A$ .

Alice has a secret: two tropical polynomials  $p_1$  and  $p_2$  with integer coefficients. Bob does the same with  $q_1$  and  $q_2$ . Alice computes  $p_1(A) \otimes p_2(B) = P_{Alice}$ , while Bob computes  $q_1(A) \otimes q_2(B) = Q_{Bob}$ . Then Alice sends Bob her  $P_{Alice}$ , Bob sends Alice  $Q_{Bob}$ . Alice knows  $P_1(A)$  and  $P_2(B)$ . Then Alice computes  $p_1(A) \otimes Q_{Bob} \otimes p_2(B) = K_{Alice}$ , while Bob computes  $q_1(A) \otimes P_{Alice} \otimes q_2(B) = K_{Bob}$ . It turns out that

**Lemma 10.2.**  $K_{Alice} = K_{Bob}$

*Proof.* Tropical matrices commute with themselves, so  $q_1(A) \otimes p_1(A) = p_1(A) \otimes q_1(A)$ . □

The key will then be the matrix  $K_a = K_b = K$ . We can then use these matrices to code and decode matrices via vector multiplication. PROBLEM! Tropical invertible matrices are rare.

**Lemma 10.3.** *The only tropical invertible matrices are permutations of a diagonal matrix.*

*Proof.* That permutations of a diagonal matrix is easy. To see that invertibles must be permutation of a diagonal, we note that  $A \otimes B = I$ , we have  $\oplus a_{ik} \otimes b_{kj}$  which is 0 if  $i = j$  and  $\infty$  if  $i \neq j$ . Keep going to see issues.  $\square$