Exercise 1. Let Ω be a domain containing the unit circle $C_1(0)$. Show that there is no function $F(z)$ holomorphic on Ω such that $e^{F(z)} = z$ on Ω. ⟦Hint: What would $F'(z)$ be? Can you prove that $F'(z)$ admits no primitive on Ω?⟧

Answer

Suppose by way of contradiction that F is a holomorphic function which satisfies the equation. Differentiating we get

$$e^{F(z)}F'(z) = 1 \to F'(z) = \frac{1}{e^{F(z)}} = \frac{1}{z}.$$

In $\mathbb{C}\setminus\,]-\infty, 0]$ we have that $\frac{1}{z}$ admits $\log(z)$ as a primitive. So let us define $G(z) = F(z) - \log(z)$, differentiating G we obtain $\frac{1}{z} - \frac{1}{z} = 0$ so this means that G is constant.

From this we deduce that $F(z) = \log(z) + C$ where C is constant. However F is defined in a domain which contains the complete unit circle and the logarithm can only be defined as a multi-valued function in that region, not as a particular branch. This contradicts the fact that F is a function.

Therefore our assumption that F is holomorphic and satisfies the equation is false. In conclusion, there does not exist such a function.

Exercise 2. Let Ω be a domain with $0 \notin \Omega$.

(a) Suppose f, g are continuous branches of the logarithm on Ω. Show that there is some integer n such that $g(z) = f(z) + 2\pi i n$. ⟦Hint: Ω is connected.⟧

(b) Suppose $f(z)$ is a continuous branch of the logarithm. Show that $f(z)$ is holomorphic. ⟦Hint: Ω can be covered by simply connected domains.⟧

Answer

(a) If f, g are continuous branches of the logarithm then

$$f'(z) = g'(z) = \frac{1}{z}, \quad \text{for} \quad z \in \Omega.$$

With this $f - g = c$ where c is a constant. Now

$$z = e^{g(z)} = e^{g(z)+c} = e^c z \Rightarrow e^c = 1 \Rightarrow c = 2\pi i n$$

for some $n \in \mathbb{Z}$.

(b) Let f be a continuous branch of the logarithm. This is, $e^{f(z)} = z$, so we may apply the inverse function theorem to e^z on the domain Ω. For any $w_0 \in \Omega$, let $f(w_0) = z_0$. Then we have that $\frac{\mathrm{d}e^z}{\mathrm{d}z} = e^{z_0} \neq 0$ so by the inverse function theorem there exists a neighborhood $U \ni z_0$ such that e^z is locally invertible. Also it must occur that the inverse is holomorphic, so f is holomorphic at every point of Ω as our point was arbitrary.

Exercise 3 (5.1 Stein & Shakarchi)**.** Give another proof of Jensen's formula in the unit disc using the functions (called Blaschke factors)

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

⟦Hint: The function $f/(\psi_{z_1} \dots \psi_{z_n})$ is nowhere vanishing.⟧

Answer

Following the proof outline we will consider the function

$$g = \frac{f}{\psi_1 \dots \psi_n}$$

where $\psi_j(z) = \frac{z - z_j}{1 - \overline{z}_j z}$ is the Blaschke factor associated to the root z_j of f. In this case the roots of f lie inside the unit circle and g's zeroes lie outside the unit circle as they are of the form $\frac{1}{\overline{z}_j}$.
Now, we know that the formula holds for nowhere vanishing functions, and also that the formula is multiplicative. So it suffices to prove it for the Blaschke factors.
Consider $\psi_j(z)$, the formula states

$$\log|\psi_j(0)| = \sum_{z \in Z} \log\left(\frac{|z|}{1}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|\psi_j(1 \cdot e^{i\theta})| \mathrm{d}\theta$$

where the set Z is the set of zeroes of ψ_j. In this case, we only have one zero: z_j, so the sum becomes a unique term.
For the integral, we ought to remember that Blaschke factors map the unit circle to the unit circle. This means that there is a φ such that $e^{i\varphi} = \psi_j(e^{i\theta})$. Replacing this into the integral we get

$$\int_0^{2\pi} \log|\psi_j(e^{i\theta})| \mathrm{d}\theta = \int_0^{2\pi} \log|e^{i\varphi}| \mathrm{d}\theta = \int_0^{2\pi} \log(1) \mathrm{d}\theta = 0.$$

So what the formula tells us is that

$$\log|\psi_j(0)| = \log|z_j|$$

but this is true because ψ_j exchanges 0 and z_j.
Finally as the formula is multiplicative, it holds for $f = g\psi_1 \dots \psi_n$ and so we have proven it in the case of the unit circle.