Exercise 1 (Exercise 1.a)**.** Compute $s_2(T)$ where T is the tableau below:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\cline{1-2} 3 & 4 \\ \cline{1-5} 2 & 2 & 2 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ \hline \end{array}.$$

Answer

We have

$$\mathrm{rw}(T) = 34222331111222233.$$

With this, we can pair 3 and 2's as follows:

$$\underline{342}22\overline{3\underline{311112}2}2233$$

and we can see we have 4 unpaired 2's and 2 unpaired 3's. By applying s_2 we wish to reflect this about the corresponding $\mathfrak{sl}_2$ chain to get 2 unpaired 2's and 4 3's. We have

$$34222331111222233 \xrightarrow{F_2} 34222331111222{\color{red}3}33 \xrightarrow{F_2} 342223311112 2{\color{red}33}33$$

and this is the desired word. The corresponding tableau is

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\cline{1-2} 3 & 4 \\ \cline{1-5} 2 & 2 & 2 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ \hline \end{array}$$

Exercise 2 (Exercise 1.b)**.** Explain why it suffices to show that $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ when acting on tableaux, for all i.

Answer

Checking a braid relation of the form $s_i s_j s_i$ where $|j - i| \neq 1$ is unnecessary because in those cases s_i commutes with s_j.

Exercise 3 (Exercise 1.c)**.** Show that, using the compatibility of JDT slides with crystal operators (which you may use as a fact), it suffices to show part 1 for $i = 1$, and therefore it suffices to work with $\mathfrak{sl}_3$ crystals, that is, tableaux whose letters are all $1, 2, 3$.

Answer

Consider the following diagram:

$$\begin{tikzcd} T \arrow[r, "\text{rem}"] \arrow[d, "\text{braid}"'] & \mathsf{skew}(T) \arrow[r, "\text{rect}"] & T' \arrow[d, "\text{braid}"] \\ S \arrow[r, "\text{rem}"'] & \mathsf{skew}(S) \arrow[r, "\text{rect}"'] & S' \end{tikzcd}$$

where rem is the operation which removes all letters but $i, i+1, i+2$. This diagram commutes because the braid operation is a combination of raising and lowering operators and the removal plus rectification is a JDT.
So from this, we may relabel $i, i+1, i+2$ to $1, 2, 3$ and then we can work on the tableau because the diagram commutes.

Exercise 4 (Exercise 1.d)**.** Show that one can further reduce to the case that the tableau shape has two rows.

Answer

Given the previous facts, we can reduce our case to a 3-row tableau on the alphabet $\{1, 2, 3\}$. Observe that it can't have more than 3 rows due to semistandardness. Now the third row must be comprised of 3's who should be paired with 2's below them and further those with 1's.
As s_1, s_2 act by raising/lowering, there's no way that elements on columns with more than two rows could get affected by the s_i's. So it suffices to work on tableau with only two rows as it will be the only part affected by s_i's.

Exercise 5 (Exercise 1.e)**.** Show that, using the symmetry in the relation $(s_1 s_2)^3 = 1$, it suffices to consider the case when the tableau has partition weight, that is, its content is (a, b, c) for some $a \geqslant b \geqslant c$.

Answer

Observe that the possible contents for our words are the 6 permutations of the word abc arranged into a content vector, given the condition that $a \geqslant b \geqslant c$. Applying s_1 will switch the number of 1's and 2's whereas s_2 does that 2's and 3's.

From this we have

$$\to (a,b,c) \xrightarrow{s_1} (b,a,c) \xrightarrow{s_2} (b,c,a) \xrightarrow{s_1} (c,b,a) \xrightarrow{s_2} (c,a,b) \xrightarrow{s_1} (a,c,b) \xrightarrow{s_2} (a,b,c) \to$$

which means that all possible contents are obtained after applying s_i's. Since starting from any point gets us any content, it suffices to only consider those tableau with content (a, b, c).

Exercise 6 (Exercise 1.f)**.** Show that the reading word of such tableau must be of the form $2^d3^e1^a2^f3^g$ where $d + f = b$ and $e + g = c$.

Answer

Given that our tableau has two rows and content (a, b, c) it must occur that it looks like

$$\begin{array}{|c|ccc}\hline ? & & & \\ \hline 1 & 1 \ldots & 1 & ? \\ \hline\end{array}$$

where there are a ones in a row. Twos can be placed on top of our row of ones or to the side:

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 2 & \ldots & 2 & ? & & & & \\ \hline 1 & 1 & \ldots & 1 & 2 & \ldots & 2 & ? \\ \hline\end{array}.$$

There can be no third row on top with threes which means that the reading word doesn't start with 3. Our three's can only be placed in following the twos as

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 2 & \ldots & 2 & 3 & \ldots & 3 & & & & \\ \hline 1 & 1 & \ldots & 1 & 2 & \ldots & 2 & 3 & \ldots & 3 \\ \hline\end{array}$$

which means that the reading word should be a string of twos, one of threes, the a string of one and a couple more of twos and threes.
The given conditions, $d + f = b$ and $e + g = c$ guarantee that there's no overlap between the rows or that the tableau looks like

$$\begin{array}{|c|c|c|c|}\hline 3 & 3 & 3 & 3 \\ \hline 1 & 1 & 2 & 2 \\ \hline\end{array}$$

as that will not have content (a, b, c) with $a \geqslant b \geqslant c$.

Exercise 7 (Exercise 2)**.** Compute the chromatic symmetric function of the triangle graph, that is, the complete graph K_3, and express it in terms of elementary symmetric functions and in terms of Schur functions.

Answer

Observe that $\chi(K_3) = 3$ which means that there's no proper colorings with 1 or 2 colors. Thus we must color vertices $1, 2, 3$ with colors $i, j, k \in \mathbb{N}$. However there's $3!$ ways of doing this, so that each monomial $x_i x_j x_k$ is accounted $3!$ times. We thus have that

$$X_{K_3} = 3! m_{(1,1,1)} = 3! s_{(1,1,1)} = 3! e_3.$$