MATH676 — Tropical Geometry

Based on the lectures by Renzo Cavalieri

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Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This is a topics course on this stuff

Requirements

Knowledge on stuff

TO DO:

- Write info on course description and requirements.
- Polish notes from TG11 about primitive vectors and pages 6-8
- Write Interim about grobner complexes

Contents

Chapter 1

Combinatorial Shadow of Algebraic Geometry

1.1 Day 1 | 20230821

Think of an algorithm where the input is an algebraic variety and the output is a combinatorial object, a piecewise linear object.

Example 1.1.1. Consider as an input a line in the plane. Say $V(x + y - 1)$, then an output would be a tropical line. If we remain in the plane and consider a higher degree polynomial, say an elliptic curve, as an output we obtain a tropical cubic.

Leaving the plane behind and thinking of abstract nodal curves, we can think of a sphere attached to a torus which is attached to a genus 2 torus, then the corresponding object is what we call the dual graph.

Right now we do not know the specific algorithm, but we can observe that the outputs are *more simple* than the inputs. So the important question is:

What algebraic information does the simplified object remember? How do we extract the information the object remembers? And once we know how to work with this objects, can we return to algebraic geometry from any kind of these objects?

Observe that the number of ends which go to infinity corresponds with the degree.

1.2 Day 2 | 20230823

Algebraic Geometry on $\mathbb{T}$

Let us talk about ways to get into tropical geometry. We will first define the tropical semifield which the base set over which we will do algebraic geometry.

Definition 1.2.1. The tropical semifield is the set $(\mathbb{R} \cup \{-\infty\})$ equipped with tropical addition and multiplication:

$$\begin{cases} x \oplus y = \max(x, y) \\ x \odot y = x + y \end{cases}$$

With this set we can make multivariable polynomials

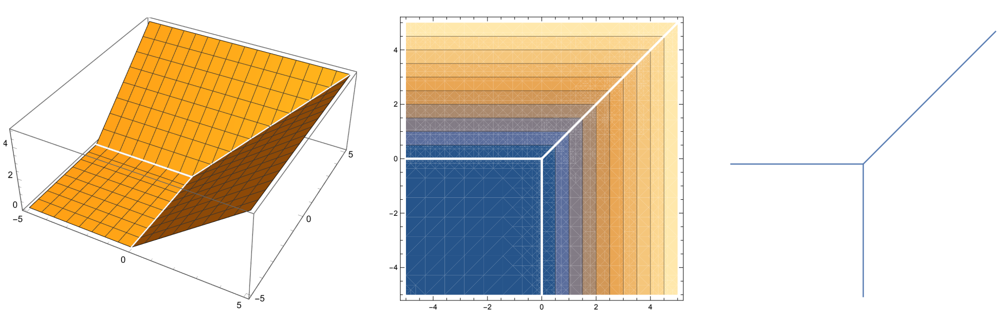
$$p(x_1, \ldots, x_n) : (\mathbb{R} \cup \{-\infty\})^n \to \mathbb{R} \cup \{-\infty\}$$

which gives rise to their *tropicalization*, a piecewise linear function $\mathrm{Trop}(p) : \mathbb{R}^n \to \mathbb{R}$.

Example 1.2.2. Consider the polynomial

$$p(x, y) = x \oplus y \oplus 0 \in (\mathbb{R} \cup \{-\infty\})[x, y].$$

When tropicalizing we get the piecewise linear function $\mathrm{Trop}(p)(x, y) = \max(x, y, 0)$ which goes from $\mathbb{R}^2$ to $\mathbb{R}$. Observe that the surface is not smooth where the planes



(a) $x \oplus y \oplus 0$ (b) Projection onto xy-plane (c) Corner locus

meet, this is what we will call the *corner locus* or *tropical hypersurface*.

Definition 1.2.3. The tropical hypersurface $V(\mathrm{Trop}(p))$ is the codimension 1 locus in $\mathbb{R}^n$ where the function is non-linear (corner locus).

Example 1.2.4. If we consider higher degree tropical polynomials, they will become linear in the usual sense. Consider

$$p(x) = 3x^2 = 3 \odot x \odot x = 3 + x + x = 3 + 2x$$

which is indeed linear with respect to usual sum and product.

Valued fields

Definition 1.2.5. The field of <u>Puiseux series</u> or rational functions over $\mathbb{C}$ is $\mathbb{C}(t)$ where the elements are of the form

$$f(t) = \sum_{i=k_0}^{\infty} a_i t^{i/n}.$$

The lower bound k_0 could be negative and the exponents, are rational with bounded denominators.

If we have not seen Puiseux series before it would be useful to consider a quick example:

Example 1.2.6. Entirely by definition, consider the Puiseux series

$$f(t) = \sum_{i=-12}^{\infty} \frac{-i}{12} t^{i/6}.$$

Expanding out the first few terms we get

$$\frac{12}{12}t^{-12/6} + \frac{11}{12}t^{-11/6} + \frac{10}{12}t^{-10/6} + \frac{9}{12}t^{-9/6} + \cdots = t^{-2} + \frac{11}{12}t^{-11/6} + \frac{5}{6}t^{-5/3} + \frac{3}{4}t^{-3/2} + \ldots$$

This field can be equipped with a valuation

$$\mathrm{val}_0 : \mathbb{C}(t) \to \mathbb{R} \cup \{\, \infty \,\}, \begin{cases} 0 \mapsto \infty \\ f \mapsto \text{order of vanishing at } 0. \end{cases}$$

This order of vanishing is the value α such that f/t^α approaches a finite non-zero value. Formally we can express this as

$$\mathrm{val}_0(f) = \min\left\{ \alpha \leqslant \infty : \lim_{t\to 0} \frac{f}{t^\alpha} \in \,]0, \infty[\, \right\}.$$

The corresponding coefficient in the series expansion of f for this value is called the valuation coefficient.

Example 1.2.7. In our original example, we can see that if we divide f by t^{-2} (or similarly multiply by t^2) we get

$$t^2 f(t) = 1 + \frac{11}{12}t^{1/6} + \frac{5}{6}t^{1/3} + \frac{3}{4}t^{1/2} + \dots \xrightarrow[t\to 0]{} 1 + 0 + 0 + \dots$$

If we were to divide by a lower power of t, say t^{-3}, we would get

$$t^3 f(t) = t + \frac{11}{12}t^{7/6} + \frac{5}{6}t^{4/3} + \frac{3}{4}t^{3/2} + \dots \xrightarrow[t\to 0]{} 0 + 0 + \dots$$

and even if zero is a finite value, we have stated that the order of vanishing makes f/t^α approach a finite **non-zero** value.

While on the other hand if we divide by a higher value than -2, say -1, then we get

$$tf(t) = t^{-1} + \frac{11}{12}t^{-5/6} + \frac{5}{6}t^{-2/3} + \frac{3}{4}t^{-1/2} + \dots \xrightarrow[t\to 0]{} \infty.$$

So in this case, we get a **non-finite** value. At least with this example, it gives us the intuition that the order of vanishing is a unique value.

We now ask, how does the order of vanishing behave when operating functions algebraically? What happens to the order of vanishing when you add two functions? Or when we multiply them?

Example 1.2.8. Consider two small functions $f(t) = t^2$ and $g(t) = t^3$, then $f+g = t^2+t^3$ which has order of vanishing 2. Observe that $2 = \min(2, 3)$.

In general what happens is that

$$\mathrm{val}_0(f+g) \geqslant \min(\mathrm{val}_0 f, \mathrm{val}_0 g), \quad \text{and} \quad \mathrm{val}_0(fg) = \mathrm{val}_0(f) + \mathrm{val}_0(g).$$

We can do algebraic geometry over this field! Let $\mathbb{K}$ be the field of Puiseux series, if $p(x_1, \dots, x_n) \in \mathbb{K}[x_1, \dots, x_n]$ then we consider the algebraic variety

$$X = V(p) = \{ \mathbf{x} \in \mathbb{K}^n : p(\mathbf{x}) = 0 \} \subseteq \mathbb{K}^n.$$

Each entry of $\mathbf{x}$ is a Puiseux series we can take the valuation of. So the image through the n-fold valuation of X will be a set in $(\mathbb{R} \cup \{ \infty \})^n$. We will call the tropicalization of X the image through this map. This is the tropical hypersurface for p.

Example 1.2.9. Consider the polynomial in $\mathbb{K}[x, y]$

$$p(x, y) = tx + y + t^2,$$

then the variety is $X = \{ (x, y) : tx + y + t^2 = 0 \}$. We can solve the equation to $y = -tx - t^2$.

If we choose $x = 0$ then y becomes $-t^2$. Now we take the valuation of $(0, -t^2)$ and so $(\infty, 2)$ is a point in $\mathrm{Trop}(X)$.

$$\begin{array}{ccc} \mathrm{val}_0 : \mathbb{K}^n & \longrightarrow & (\mathbb{R} \cup \{ \infty \})^n \\ \cup\!| & & \cup\!| \\ V(p) & \longmapsto & \overline{\mathrm{val}_0(V(p))} \\ \| & & \| \\ \{\mathbf{x} : p(\mathbf{x}) = 0\} & & \mathrm{Trop}(V(p)) \end{array}$$

Figure 1.2: Diagram on the tropical hypersurface

Amoebas

Let us return to the usual stage and consider $p \in \mathbb{C}[x_1, \dots, x_n]$ which defines an algebraic variety $X = V(p) \subseteq \mathbb{C}^n$. Now consider the map which sends every coordinate's modulus to its logarithm in base t:

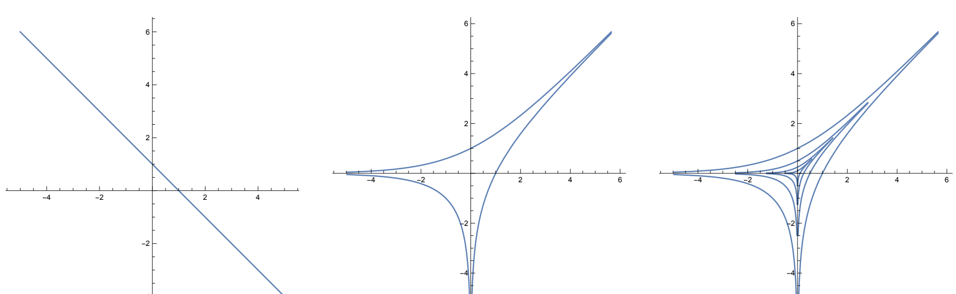
$$\mathbb{C}^n \to (\mathbb{R} \cup \{ -\infty \})^n, \quad (z_1, \dots, z_n) \to (\log_t |z_1|, \dots, \log_t |z_n|).$$

The image of X under this map, $\log_t(X)$, is the t-amoeba of X. If we take the limit as $t \to \infty$ then we get the *spine* of the amoeba.

Example 1.2.10. When $p(x, y) = x+y-1$ then we can describe $V(p)$ via the parametrization $(x, 1-x)$. So the corresponding t-amoeba in the real case is

$$\{ (\log_t |x|, \log_t |1-x|) : x \in \mathbb{R} \}$$

and we ordinarily take the limit $\lim_{t\to\infty} \frac{\log |x|}{\log t}$, we see that the functions converge to zero point-by-point. But the set is actually approaching the spine!



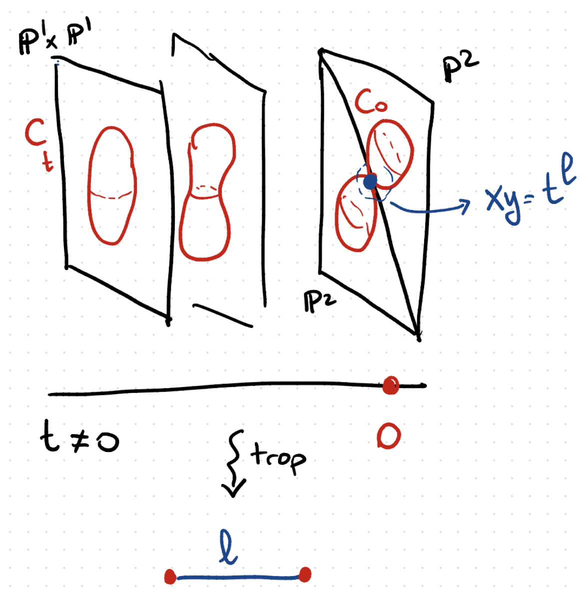
(a) $X = V(x+y-1)$ (b) 2-amoeba of X (c) Sequence of amoebas as $t \to \infty$

Observe that the spine approaches the tropical hypersurface associated to p. In other words we have that the tropical hypersurface is $\lim_{t\to\infty} \log_t(V(p))$.

Degenerations

We may parametrize any algebraic variety with a time variable, then converting the information to a graph, edges code the information about how fast the node forms related to the length.

Consider a family of of what, what is this family of?! Stuff? Curve in P1xP1 which eventually becomes P2?



It is too early to understand this point of view. We will set everything up to get to it.

In general, the big idea will be to explore and understand these perspectives in the case of plane curves. We want to show how they are equivalent and then recover classical algebraic geometry results in terms of tropical geometry.

1.3 Day 3 | 20230825

Recall that the last time we discussed the classical (25 to 30 years old) ways to get to tropical geoemtry. We now would like to answer the question

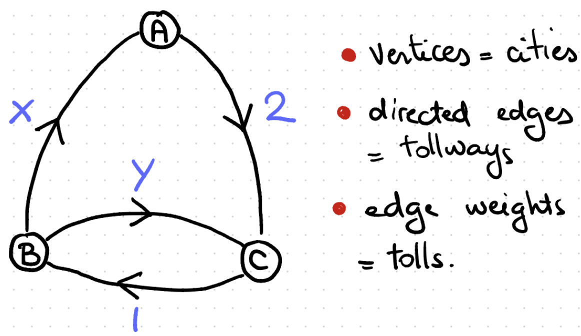
Where do tropical numbers come from?

So let us begin with an applications problem and see how the tropical numbers arise from the context of the problem.

Tropical Arithmetics

Minimizing Tolls

Consider a set of cities connected by a network of toll-ways: If we only care about



minimizing toll expenses when traveling, what would be the cheapest way to go from one given city to another? Let us record the information as an incidence matrix:

$$M_{ij} = \text{price of going from city } i \text{ to city } j \text{ in at most one trip} \Rightarrow M = \begin{pmatrix} 0 & \infty & 2 \\ x & 0 & y \\ \infty & 1 & 0 \end{pmatrix}$$

In this matrix, the rows determine the outbound city, while the columns are the destination. Each entry records the cost of a toll and tolls are considered to be infinite when the road does not exist. We can also think of M as recording the cheapest toll to go from one city to another with at most one move.

How would we compute the best strategy of going from city i to j in *at most two trips*? If for example we want to find trips from A to B in two steps then we have three choices:

$$AAB, \quad ABB, \quad ACB.$$

The costs of each one are

$$(0, \infty), \quad (\infty, 0), \quad (2, 1)$$

so we sum them and take the minimum. That will be the optimal route from A to B in two steps. In fact, if we relate this to the entries of the matrix M, we could use M^2. However we must redefine our basic operations as follows:

$$+ = \min, \quad \cdot = +$$

So we have the identification

$$(1,2) \text{ entry of } M^2 = \sum_{j=1}^{3} M_{1k}M_{k2} = \min(M_{11}+M_{12}, M_{12}+M_{22}, M_{13}+M_{32}).$$

In general:

$$\begin{pmatrix} 0 & \infty & 2 \\ x & 0 & y \\ \infty & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} \min\begin{pmatrix} 0+0 \\ \infty+x \\ 2+\infty \end{pmatrix} & \min\begin{pmatrix} 0+\infty \\ \infty+0 \\ 2+1 \end{pmatrix} & \min\begin{pmatrix} 0+2 \\ \infty+y \\ 2+0 \end{pmatrix} \\ \min\begin{pmatrix} x+0 \\ 0+x \\ y+\infty \end{pmatrix} & \min\begin{pmatrix} x+\infty \\ 0+0 \\ y+1 \end{pmatrix} & \min\begin{pmatrix} x+2 \\ 0+y \\ y+0 \end{pmatrix} \\ \min\begin{pmatrix} \infty+0 \\ 1+x \\ 0+\infty \end{pmatrix} & \min\begin{pmatrix} \infty+\infty \\ 1+0 \\ 0+1 \end{pmatrix} & \min\begin{pmatrix} \infty+2 \\ 1+y \\ 0+0 \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 3 & 2 \\ x & \min(0, y+1) & \min(x+2, y) \\ 1+x & 1 & \min(0, 1+y) \end{pmatrix}.$$

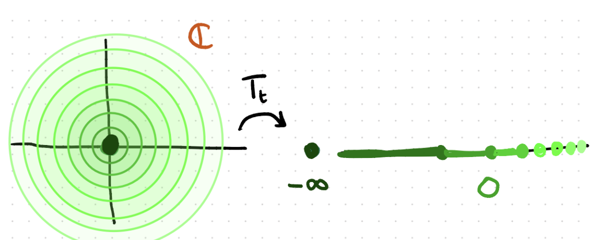
Observe that $1+y$ can be the minimum in the diagonal when we allow *negative tolls*.

Remark 1.3.1. If we disallow negative tolls, the products M^n eventually stabilize to a matrix whose entries record the cheapest way to get from one city to another in n steps.

This gives us an intuition that minimization problems correspond to linear algebra problems over $(\mathbb{T}, +, \cdot)$ which is precisely $(\mathbb{R} \cup \{\infty\}, \min, +)$.

Forgetting phases

Consider the map $T_t : \mathbb{C} \to \{-\infty\} \cup \mathbb{R}, \quad z \mapsto \log_t(|z|)$. This map is surjective, and



this we can see by checking it. Any $x \in \mathbb{R}$ is

$$T_t(t^x e^{i\theta}) = \log_t |t^x e^{i\theta}| = \log_t t^x = x, \quad \theta \in \mathbb{R}.$$

This means that the inverse image of a point contains a plethora of points, in fact:

$$\begin{cases} T_t^{-1}(x) = \{ t^x e^{i\theta} \} \subseteq \mathbb{C}, & \text{for} \quad x \in \mathbb{R}, \\ T_t^{-1}(-\infty) = 0. \end{cases}$$

With this in hand, we wish to define an exotic addition and multiplication on $\{ -\infty \} \cup \mathbb{R}$ using T_t. We will dequantize!

We begin with **hyper-addition**, the output will be a subset of $\{ -\infty \} \cup \mathbb{R}$ so it's not a binary operation by itself.

$$x \diamondplus_t y := T_t(T_t^{-1}(x) + T_t^{-1}(y)) = \left[\log_t(|t^x - t^y|), \log_t(t^x + t^y)\right].$$

This is an interval in $\{ -\infty \} \cup \mathbb{R}$, in order to make $\diamondplus_t$ into an operation we take a limit:

$$\begin{tikzcd} x \diamondplus_t y \arrow[r, "\lim\limits_{t\to\infty}"] \arrow[d, "\max"] & x \diamondplus y = \lim_{t\to\infty} x \diamondplus_t y \arrow[d, "\max"] \\ x +_t y \arrow[r, "\lim\limits_{t\to\infty}"] & x + y = \max(x, y) \end{tikzcd}$$

Remark 1.3.2. Note that $\diamondplus$ is still a hyperoperation. Its output is not a singleton *only* when adding a number to itself:

$$x \diamondplus y = \begin{cases} \max(x, y), & x \neq y \\ [-\infty, x], & x = y \end{cases}$$

Formally this process, taking a limit of a family of operations, is known as *dequantization*.

In the case of multiplication, the process goes a lot smoother when defining it:

$$x \cdot y = T_t \left[T^{-1}(x) \cdot T^{-1}(y)\right] = \log_t \left[|(t^x e^{i\theta})(t^y e^{i\varphi})|\right] = \log_t t^{x+y} = x + y$$

Example 1.3.3. Let us consider a small example like summing 2 and 4. Observe that

$$4 \diamondplus_t 2 = T_t(T_t^{-1}(4) + T_t^{-1}(2)) = T_t \left|t^4 e^{i\theta} + t^2 e^{i\varphi}\right|$$

and the term on the inside can be simplified to $t^4(e^{i\theta}+t^{-2}e^{i\varphi})$. T_t takes that expression to

$$4+\log_t|e^{i\theta}+t^{-2}e^{i\varphi}| = 4+\frac{\log|e^{i\theta}+t^{-2}e^{i\varphi}|}{\log t}.$$

What happens if we take the limit as $t\to\infty$? We get an independent from t result! The term on the right vanishes and we are left with $4=\max(4,2)$. So it got a tad bit better, but it's still a hyperoperation!

Exercise 1.3.4. Check how the definition of $+$ and $\cdot$ extend to the *number* $-\infty$.

The point of this exercise is to operate $-\infty$ with finite numbers and itself.
For a finite x we will find $x+(-\infty)$. This is the limit of the previous hyperoperation:

$$x \diamondplus_t (-\infty) = T_t(T_t^{-1}(x)+T_t^{-1}(-\infty)) = T_t(T_t^{-1}(x)+0) = T_t(T_t^{-1}(x)) = x.$$

If we let t grow, the result doesn't change and so this goes according to $\max(x,-\infty)=x$.
On the other hand when taking the product:

$$x\cdot(-\infty) = T_t\left[T^{-1}(x)\cdot T^{-1}(-\infty)\right] = T_t\left[T^{-1}(x)\cdot 0\right] = T_t(0) = \log_t(0)\to-\infty$$

which is also similar to the notion of $x+(-\infty)=-\infty$.
We can now proceed to operate $-\infty$ with itself:

$$(-\infty)\diamondplus_t(-\infty) = T_t(0) = \log_t(0) = -\infty = \max(-\infty,-\infty),$$

and when taking the product:

$$(-\infty)\cdot(-\infty) = T_t(0)\log_t(0) = -\infty = (-\infty)+(-\infty)$$

where the last sum is a sum in the usual sense.

So, summarizing this process:

- We forgot about the phase of the complex numbers and only looked at them radially.
- The modulus of these numbers was scaled logarithmically.
- Finally we took the limit of these operations and obtained the desired (somewhat) result.

This is known as Maslov[1] dequantization and with this we can see $(\mathbb{T}, +, \cdot)$ as $(\{ -\infty \} \cup \mathbb{R}, \max, +)$. Also, we will abbreviate $\lim_{t\to\infty} T_t$ with $T_{t\to\infty}$

1.4 Interim 1 | Valuations

Valuations

In essence a valuation provides a measure of the *size* (or multiplicity) of elements in a field.

Definition 1.4.1. If K is a field, a valuation on K is a mapping

$$\mathrm{val} : K \to \mathbb{R} \cup \{ \infty \}$$

with the properties:

i) $\mathrm{val}(x) = \infty \iff x = 0$,
ii) $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$,
iii) $\mathrm{val}(x + y) \geqslant \min(\mathrm{val}(x), \mathrm{val}(y))$, with equality if $\mathrm{val}(x) \neq \mathrm{val}(y)$.

In this case, we say that K is a valued field.

Previous discussion has shown us that the order of vanishing, val_0 is a valuation of the field of Puiseux series $\mathbb{K}$. The properties can be shown to be true by writing out two Puiseux series and showing that indeed they obey the properties.

Exercise 1.4.2. Verify that the field of Puiseux series, is indeed a valued field with valuation val_0.

There's also a another common example coming from number theory which is the p-adic valuation.

Example 1.4.3. We first define the valuation on $\mathbb{Z}$ as

$$v_p(a) = \max\{ k \in \mathbb{Z} : \ p^k \mid a \},$$

where p is a prime number. For the rational numbers the valuation is defined as $v_p(m/n) = v_p(m) - v_p(n)$.

This valuation can be used to study the field of p-adic numbers which is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value $|r|_p = p^{-v_p(r)}$.

Exercise 1.4.4. In a similar fashion, verify that v_p is a valuation over $\mathbb{Q}$.

[1] Viktor Pavlovich Maslov (1930615-20230803)

Chapter 2

The Tropical Numbers

2.1 Day 4 | 20230828

We have seen where our ideas come from. Certain kinds of minimization problems give rise to our tropical numbers. Also by expressing complex numbers in a logarithmic scale without phase then when inducing a sum we actually get a hypersum. The way we converted into an operation is by taking a limit. Then the algebraic structure we obtained was once again the tropical numbers. Let us talk about the perspective of valued fields.

Puiseux series

Recall from our times in Calculus 1 that when resolving indeterminate limits, the relevant information is contained in the order of vanishing of the function.

Example 2.1.1. Consider the limit $\lim_{t\to 0} \frac{\sin(x)}{x} = 1$. Near $t = 0$ we have

$$\sin(t) = t + o(t) \sim t^1 \quad \text{and} \quad \frac{1}{t} = t^{-1} \quad \text{so} \quad t^1 t^{-1} = t^0 = 1.$$

From this, we care to study the orders of zeroes and poles of Laurent series. In order to extend the class of functions to an algebraically closed field, we consider Puiseux series, or rational functions. We can identify Puiseux series as

$$\mathbb{C}\{\{t\}\} = \bigcup_{n\in\mathbb{N}} \mathbb{C}(t^{1/n}).$$

Concretely, elements here are Laurent series with rational exponents and the exponents of terms with non-zero coefficients have a common denominator.

Example 2.1.2. The series $\sum_{k=-37}^{\infty} t^{k/42}$ is a Puiseux series while $\sum_{k=1}^{\infty} t^{1/k}$ is not because the exponents keep getting smaller and smaller.

This is the most natural algebraically closed field with a *canonical* valuation. This is the function:

$$\mathrm{val} : \mathbb{C}\{\{\, t \,\}\} \to \mathbb{R} \cup \{\, \infty \,\}, \begin{cases} 0 \mapsto \infty \\ t^{p/q} + \text{higher order} \mapsto p/q \end{cases}$$

In other words the valuation sends $\sum_{k=k_0}^{\infty} a_k t^{q_k}$ to q_{k_0}.

Proposition 2.1.3. *For $\alpha, \beta \in \mathbb{C}\{\{\, t \,\}\}$, the valuation enjoys the following properties*:

i. $\mathrm{val}(\alpha \cdot \beta) = \mathrm{val}(\alpha) + \mathrm{val}(\beta)$.

ii. $\mathrm{val}(\alpha + \beta) \geqslant \min(\mathrm{val}(\alpha), \mathrm{val}(\beta))$.

Equality holds when $\mathrm{val}(\alpha) \neq \mathrm{val}(\beta)$.

So if we decide to define operations on $\mathbb{R} \cup \{\, \infty \,\}$ by inducing them from the operations on $\mathbb{C}\{\{\, t \,\}\}$, then we obtain

$$x \oplus y = \mathrm{val}\left(\mathrm{val}^{-1}(x) + \mathrm{val}^{-1}(y)\right), \qquad x \cdot y = \mathrm{val}\left(\mathrm{val}^{-1}(x) \cdot \mathrm{val}^{-1}(y)\right).$$

Now $\cdot$ coincides with usual addition and $+$ is the hyperoperation

$$x \oplus y = \begin{cases} \min(x, y) & \text{when} \quad x \neq y, \\ [\min(x, y), \infty] & \text{when} \quad x = y. \end{cases}$$

Example 2.1.4. If we try to sum 0 with itself, we get

$$0 \oplus 0 = \mathrm{val}\left((a_0 + a_1 t^{q_1} + \dots) + (-a_0 + b_1 t^{r_1} + \dots)\right)$$

and this could be either q_1 or r_1 because the constant terms cancel!

The only natural way to turn this into an operation is to define $x + y = \min(x, y)$. In conclusion, the field of Puiseux series with the order of vanishing and poles is congruent to $(\mathbb{T}, +, \cdot)$ which in this case is $(\mathbb{R} \cup \{\, \infty \,\}, \min, +)$.

The Tropical Semifield

Definition 2.1.5. The tropical semifield is $(\mathbb{T}, \oplus, \odot)$ where we can choose:

- $\mathbb{T} = \mathbb{R} \cup \infty$, $\oplus$ to be $\min$ and $\odot$ is $+$, the min convention.
- $\mathbb{T} = \{-\infty\} \cup \mathbb{R}$, $\oplus = \max$ and $\odot = +$, the max convention.

There is a natural isomorphism between the two choices given by $x \mapsto -x$. As we have mentioned, different contexts may be more natural than the other when using certain conventions. We will typically use the $\max$ convention.

Proposition 2.1.6. *The following algebraic properties hold for* $(\mathbb{T}, +, \cdot)$:

i) $0_{\mathbb{T}} = -\infty$.
ii) $1_{\mathbb{T}} = 0$.
iii) $x \oplus y = 0_{\mathbb{T}}$ *only has the solution* $x = y = 0_{\mathbb{T}}$. *This means that only* $-\infty$ *has an additive inverse.*
iv) *Addition is idempotent*: $x \oplus x = x$.
v) *Every non-zero element has a multiplicative inverse*: $1/x = -x$.

Proof

i) Observe that $x \oplus 0_{\mathbb{T}} = \max(x, -\infty) = x$.
ii) $x \odot 1_{\mathbb{T}} = x + 0 = x$.
iii) $x \oplus y = 0_{\mathbb{T}} \iff \max(x, y) = -\infty \Rightarrow x = y = -\infty$.
iv) $x \oplus x = \max(x, x) = x$.
v) $x \cdot (1/x) = x + (-x) = 0 = 1_{\mathbb{T}}$.

Observe that it is not possible to adjoin formal additive inverses. Suppose that for $x \in \mathbb{T}$ there exists a y such that $x + y = 0_{\mathbb{T}}$, then

$$(x \oplus x) \oplus y = x \oplus y = 0_{\mathbb{T}} \quad \text{and} \quad x \oplus (x \oplus y) = x \oplus 0_{\mathbb{T}} = x \quad \text{but} \quad x \neq 0_{\mathbb{T}}.$$

This means that any invertible element necessarily has to be $-\infty$.

Exercise 2.1.7 (2-). Which other algebraic properties do these operations enjoy? We have claimed for example that $\oplus$ is associative. Prove this.

Are the operations commutative? Do they distribute with respect to each other?

Let us assume without losing generality that

$$x < y < z$$

and then

$$\begin{aligned} x \oplus (y \oplus z) &= \max(x, \max(y, z)) = \max(x, z) = z \\ (x \oplus y) \oplus z &= \max(\max(x, y), z) = \max(y, z) = z \end{aligned}$$

which shows us that $\oplus$ is associative. We can see that it's commutative by realizing that $\max$ is also commutative.
Finally we ask if distribution occurs, but let us observe first that

$$y < z \Rightarrow x + y < x + z,$$

now we can see that

$$\begin{aligned} & x \odot (y \oplus z) = x \odot y \oplus x \odot z \\ \iff & x + \max(y, z) = \max(x + y, x + z) \\ \iff & x + z = x + z \end{aligned}$$

which proves distributivity. Also it is not necessary to check distribution on the other side as our operations are commutative.

Proposition 2.1.8 (Weird Fun Facts)**.** *Recall that the usual Pascal Triangle is built by adding the previous two elements to get the next one. In the tropical case we have*

$$\begin{matrix} & & 1_{\mathbb{T}} & & \\ & 1_{\mathbb{T}} & & 1_{\mathbb{T}} & \\ 1_{\mathbb{T}} & & 1_{\mathbb{T}} & & 1_{\mathbb{T}} \end{matrix} \quad = \quad \begin{matrix} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \end{matrix}$$

and this extends downwards with the same pattern.

In the case of the tropical binomial theorem, we have a "freshman's dream" type of identity:

$$(x \oplus y)^n = x^n \oplus y^n.$$

Observe that in other words, what the binomial theorem is telling us is a restatement of the identity $n \max(x, y) = \max(nx, ny)$.

Exercise 2.1.9 (2)**.** Recall that the coefficients in the expansion for the binomial theorem are the corresponding elements in the rows of the Pascal Triangle. Verify if the coefficients agree in the tropical case for the binomial theorem.

We must verify that the coefficient of every monomial in the expansion of $(x \oplus y)^n$ does match the n^{th} row of Pascal's triangle. To that effect note that

$$\begin{aligned}(x \oplus y)^n &= (x \oplus y) \odot (x \oplus y) \odot \cdots \odot (x \oplus y) \\ &= x^n \oplus x^{n-1}y \oplus \cdots \oplus xy^{n-1} + y^n \\ &= x^n \oplus y^n\end{aligned}$$

and in the second line, remember we would usually have $\binom{n}{k}$ terms of the form $x^k y^{n-k}$. However, as addition is idempotent here, all those terms become just one term.
Also, observe that the coefficient (tropically) multiplying each term is $1_{\mathbb{T}}$. This is because multiplication by one is just adding by zero. So it is indeed the case that all coefficients in the binomial expansion are $1_{\mathbb{T}}$.
Finally observe that for any k, $kx + (n-k)y \leqslant \max(nx, ny)$ which means that the only terms that survive are the power n monomials in the expansion.

The Optimal Assignment Problem

Suppose we have n jobs for n workers. Each worker can only work one job and once the job is taken, no one else can do it. We wish to assign a job to each worker in order to maximize our company's profit.

Example 2.1.10. As a little example consider Alice and Bob's hydroponics farm. When working with the weeds Alice produces 5 credits while working with the water she produces 6. On the other hand Bob produces 3 and 5 respectively.

It is easy to see that Alice should be assigned to to the weed and Bob to the water in order to maximize. But let us apply what we know with tropical arithmetics.

Call

$$M_{ij} = \text{amount of credits work } i \text{ produces when doing job } j.$$

Then we can summarize the previous information in a matrix

$$M = \begin{pmatrix} 5 & 6 \\ 3 & 5 \end{pmatrix}$$

and if we take the tropical determinant (which is really a permanent since we lack subtraction) we get

$$\operatorname{Trop}\det M = 5 \cdot 5 + 6 \cdot 3 = \max(5+5, 6+3) = 10$$

which is the maximal profit we can make by assigning our workers.

Exercise 2.1.11. Do the following:

(1-) Construct a 3×3 matrix with non-permuted entries such that there's more than one possible assignment for the optimal jobs.

(1) Use the combinatorial definition of permanent to show that the tropical determinant of M is indeed the maximal profit. ⟦Hint: The definition of permanent is the same as the determinant but without the $(-1)^{\operatorname{sgn}\sigma}$.⟧

(1) Assuming you know the tropical determinant of a matrix, devise a way to identify one job combination which reaches the optimum value. ⟦Actually! It is not necessary to know the value of the determinant.⟧

i) Consider a matrix $A \in \mathbb{R}^{3\times 3}$. For a given $n \in \mathbb{N}$, the profit, we may build an infinite family of matrices which satisfy the required conditions.

The conditions our matrix must satisfy are sums of permuted entries.

In this case the solution is given by 5 parameters including the profit:

$$(n-f-h, n-g-g, n-f-g-h+i, f+g-i, f+h-i, f, g, h, i)$$

so a valid matrix could be

$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & 5 & 3 \\ 2 & 4 & 2 \end{pmatrix}.$$

ii) The permanent by definition is

$$\bigoplus_{\sigma \in S_n} \bigodot_{i=1}^{n} M_{i\sigma_i} = \max_{\sigma \in S_n} \left(\sum_{i=1}^{n} M_{i\sigma_i} \right).$$

What the last expression says is, out all the possible permutations, which is the highest sum over all possible job assignments. So the permanent will indeed find the maximal profit.

iii) We can proceed with a greedy algorithm. Row by row, choose the largest element. Then eliminate the column the found element was in and repeat the process.

For example, pick $A_{1k} = \max(\text{row } 1)$, then throw out column k and repeat the process with the $(1, k)$ minor of A.⟦ This doesn't actually prove that the greedy algorithm works. ⟧

Remark 2.1.12. Observe that the first problem can be solved in any dimension d, because in total we have $2d$ equations while having d^2 indeterminates. As $2d < d^2$ for $d \geqslant 2$, we have that the problem will always be under-determined. So there's always more than one possible optimal assignment.

We now have another question,

Is there an instance where the greedy algorithm fails to find an optimal assignment for the jobs?

Exercise 2.1.13 (5)**.** Prove or disprove, the greedy algorithm will find an optimal assignment for the jobs given the conditions above. You may assume to know the value of the permanent of the matrix.

2.2 Day 5 | 20230830

The last time we talked about the algebraic structure of the value group of the Puiseux series. We now have plenty of motivation of why would we define the tropical numbers.

Tropical Polynomials and Roots

An univariate,tropical, (Laurent) monomial is equivalent to an affine linear function with integer coefficients. Such a monomial is an expression of the form

$$a \odot x^{\odot m}, \quad a \in \mathbb{T}, \quad m \in \mathbb{Z}.$$

Example 2.2.1. We have for example:

$$5x^2 \leftrightarrow 5 + 2x, \quad 2x^{-3} \leftrightarrow 2 - 3x.$$

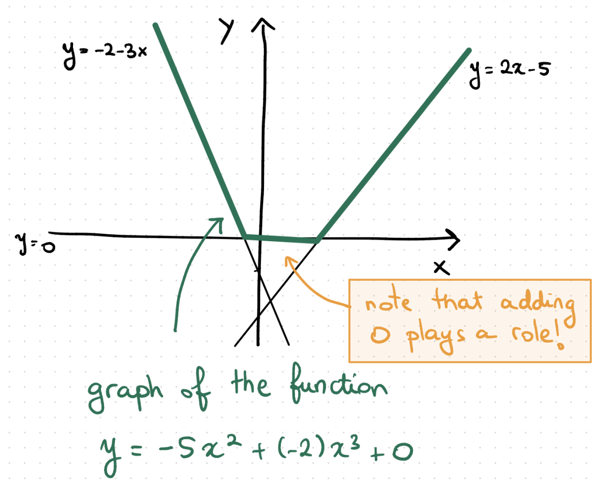
The second one is a Laurent monomial because of the negative power. Also consider $\sqrt{5} \odot x^{\odot 3}$ which corresponds to $y = \sqrt{5} + 3x$. Notice how the slope is always an integer, meanwhile the translation can be any number.

An univariate tropical (Laurent) polynomial is a finite sum of monomials which give rise to a *convex*, continuous, piecewise, affine linear function with integer slopes.

Example 2.2.2. Consider the function $-5 \odot x^{\odot 2} \oplus (-2) \odot x^{\odot -3} \oplus 0$ which corresponds to

$$\max(-5 + 2x, -2 - 3x, 0).$$

If we graph this functions we obtain Observe that this function is indeed convex, and



fulfills all of the previous properties from before.

In fact the map from $\mathbb{T}[x]$ to convex, affine piecewise linear functions with *finitely* many distinct regions of linearity is surjective. If we don't want to take the finiteness condition into consideration, we have to amplify the domain to tropical Laurent series.

A small measure of care should be taken because there are multiple tropical polynomials which map to the same function.

Example 2.2.3. Consider the functions

$$p_1 = x + \frac{1}{x} + 0, \quad p_2 = x + \frac{1}{x} - 2.$$

When converting we get

$$\max(x, -x, 0), \quad \max(x, -x, -2)$$

which produce $|x|$ in both cases. Adding something which is smaller than the minimum value of the function doesn't change it in general. It also doesn't have to be a constant in general. In the previous example, the the monomial $(-4) \odot x^{\odot 1}$ is smaller than any of the linear functions, so adding it changes nothing.

To talk about the roots, we will start with a purely combinatorial definition.

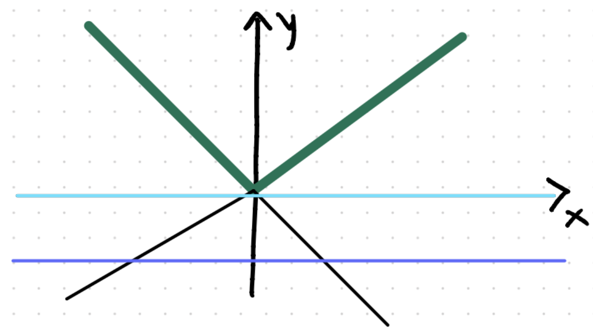


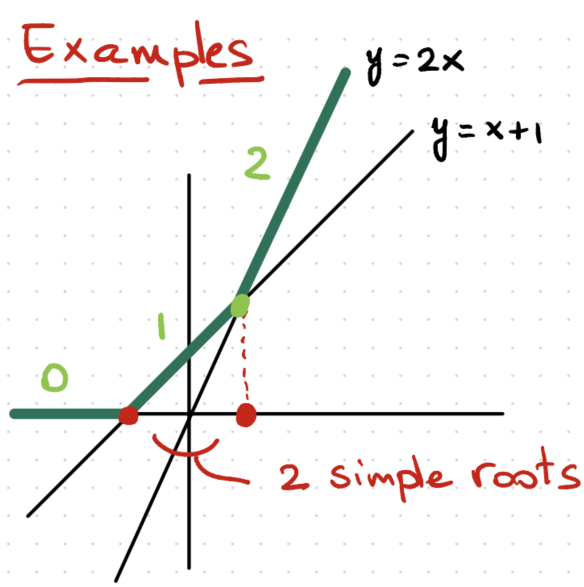
Figure 2.1: Failure of injectivity as both functions map to $|x|$ with $y = 0$ and $y = -2$ shown.

Definition 2.2.4. Given a polynomial $p \in \mathbb{T}[x]$ of degree d we say the following:

- $-\infty$ is a root of p if the slope of the piecewise linear function is non-zero for $x \ll 0$.
- $x_0 \in \mathbb{R}$ is a root of p if $p'(x_0)$ is undefined. Observe that the derivative is undefined only when there's a change in slope.

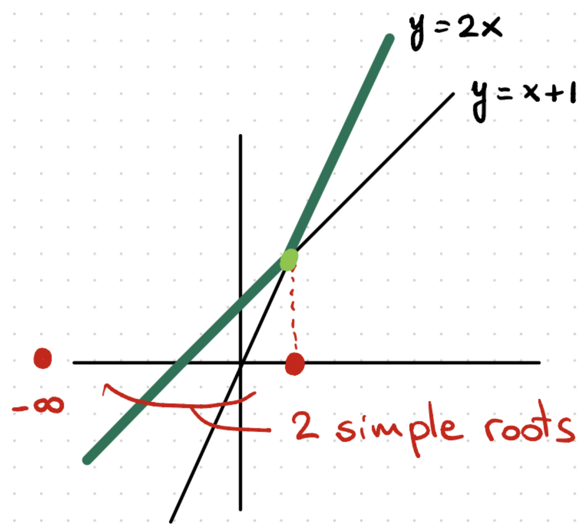
We say that the multiplicity of x_0 is the difference between slopes across x_0. If $-\infty$ is a root, then its multiplicity is equal to the slope of the associated function for $x \ll 0$.

Example 2.2.5. Consider the polynomial $x^{\odot 2} \oplus 1 \odot x^1 \oplus 0 = \max(2x, x, 0)$. We can see



that there are changes in slope at $x_1 = -1$ and $x_2 = 1$. The number of roots coincides with the degree of the polynomial as in the usual sense.

Example 2.2.6. Let's remove the zero, recall zero isn't the additive identity, so the polynomial we have is $x^{\odot 2} \oplus 1 \odot x^1 = \max(2x, x)$. Now one of the roots is still $x = 1$,



but remember that if the slope is non-zero when $x \ll 0$, then $-\infty$ is a root of p. This is the case here because the slope is 1 as $x \to -\infty$. Once again there's two roots $x_1 = -\infty$ and $x_2 = 1$.

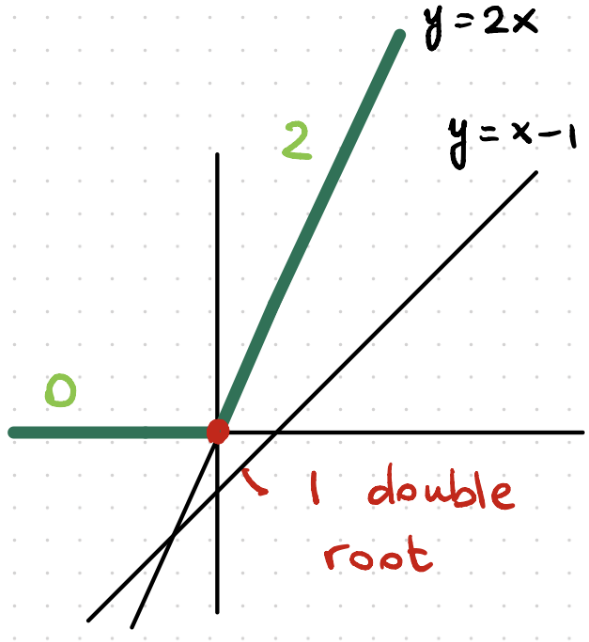
Example 2.2.7. Let us change a sign in a coefficient, take $x^2 - 1 \cdot x^1 + 0$. But what is tropical subtraction? It's not that, let's convert this slowly into what it's supposed to be:

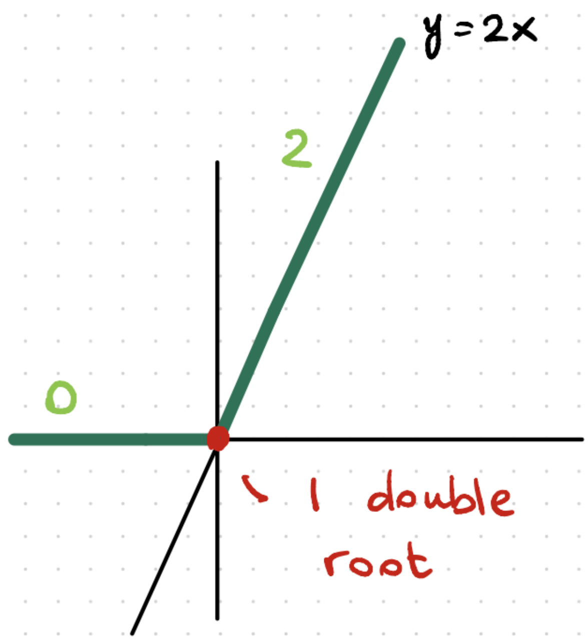
$$x^2 - 1 \cdot x^1 + 0 = (x \cdot x) + (-1) \cdot x + 0 = (2x) + (x + (-1)) + 0 = \max(2x.x - 1, 0).$$

Observe that because the line $y = x - 1$ is below our graphs, it doesn't interfere with the calculation of zeroes. So the only place where there occurs a change in sign is $x = 0$. The slope on the right is 2 and on the left is 0 so the multiplicity is $2 - 0 = 2$.

Example 2.2.8. In a similar fashion, $x^2 + 0$ also has a double root at $x = 0$. There is only one change in slope once again at $x = 0$ and the difference in slopes is 2.

Lemma 2.2.9. *For a tropical polynomial p, a finite x_0 is a root of f if and only if when we write the function as a* max *of linear functions, at x_0 the maximum value is obtained at least twice.*





The multiplicity of the root is equal to the difference in the two extremal positions where the max *is attained.*

This should be more or less obvious. Being a root means that we are an intersection of two lines which are above all the others. It's pretty useful to have this notions around.

Questions arise:

Which functions have only one simple zero at $-\infty$? What would a function with an order 2 zero at $-\infty$ look like?

Exercise 2.2.10. Do the following:

(5) Is it possible for a function to have only a simple zero at $-\infty$? Provide an example of function with one simple zero at $-\infty$ or prove that such function cannot exist.

(5) Do functions with zeroes at $-\infty$ have infinite order at such zero or is it arbitrarily high? If a function has a finite order zero at $-\infty$ provide an example of one with a double zero at $-\infty$. Else, prove that such functions have infinite order at that zero.

2.3 Day 6 | 20230901

How do we know that the notions of roots are natural or useful?

Factorization of Tropical Polynomials

Suppose a polynomial $p \in \mathbb{T}[x]$ has roots a_k with multiplicity m_k. Then we may factor p as a product of linear polynomials

$$p(x) = c_0 \bigodot (x \oplus a_k)^{m_k}.$$

This p is the affine piecewise-linear function, not the formal object. And so, in a sense, $\mathbb{T}$ is algebraically closed. But instead of proving this, we will sketch the proof to get an idea of how things *work* with a couple of examples.

The idea of the proof is that we check that product does define a P.L. function with the right slopes and then c_0 gives the translation factor.

Example 2.3.1. First lets deal with the case where $-\infty$ is not a root. Consider the polynomial

$$p(x) = (-1) \oplus (-1) \odot x \oplus (-4) \odot x^4 = \max(-1, x-1, 4x-4).$$

Remember, as in the case of real polynomials, the square and cube terms are still there. The coefficient that foes along them is just $-\infty$. We can graph the polynomial in order to see the roots: The points where there is a change in slope are $a_1 = 0$ and $a_2 = 1$.

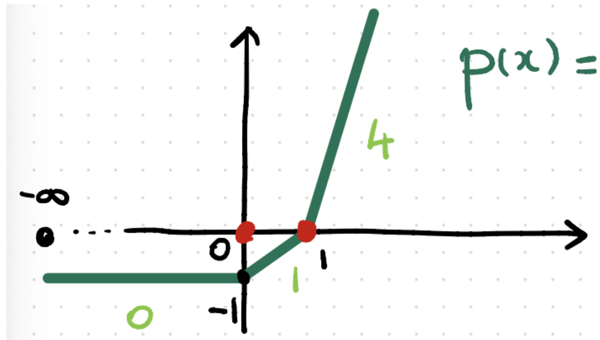


Figure 2.2: Graph of $p(x)$ with roots shown

Then their multiplicities are $1 - 0 = 1$ and $4 - 1 = 3$ respectively. We may write p as

$$p(x) = c_0 \odot (x \oplus 0) \odot (x \oplus 1)^3 = c_0 + \max(x, 0) + \max(3x, 3).$$

Whatever function we have, we can write as the sum of three terms. So let us subdivide the tropical line in order to see which terms goes where. The constant can be

$x \leqslant 0$	$0 \leqslant x \leqslant 1$	$1 \leqslant x$
c_0	c_0	c_0
0	x	x
3	3	$3x$
$c_0 + 3$	$c_0 + 3 + x$	$c_0 + 4x$

Behavior of $p(x)$ across $\mathbb{T}$

determined by plugging in $x = -\infty$. We can see that

$$\begin{aligned} p(-\infty) &= (-1) \oplus (-1) \odot (-\infty) \oplus (-4) \odot (-\infty)^4 = -1 \\ &= c_0 \odot (-\infty \oplus 0) \odot (-\infty \oplus 1)^3 = c_0 \odot 0 \odot 1^{\odot 3}. \end{aligned}$$

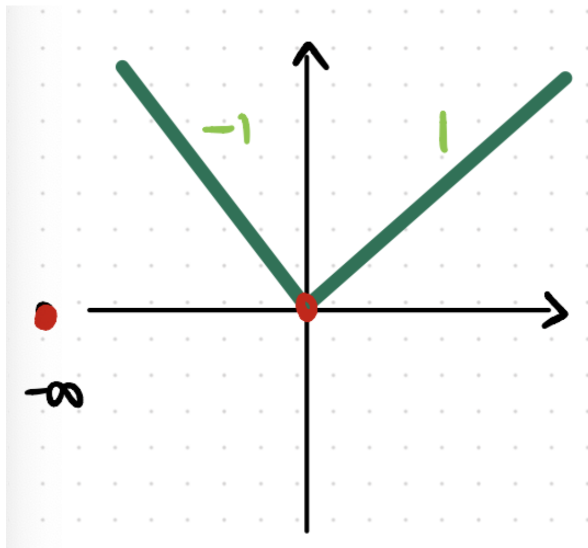
This gives us the equation $c_0 + 0 + 3 = -1$ which leads us to $c_0 = -4$. With this we verify that

$$p(x) = \begin{cases} -1 & x \leqslant 0 \\ x - 1 & 0 \leqslant x \leqslant 1 \\ 4x - 4 & 1 \leqslant x \end{cases}$$

So in this case $c_0 = p(-\infty) - \sum m_k a_k \in \mathbb{R}$.

Example 2.3.2. We now explore the case where $-\infty$ is a root or a pole. The argument will essentially be the same with a small modification.

Consider the function $\frac{1}{x} \oplus x$. We have $-\infty$ as a pole of order 1 and 0 is a root of



order $1 - (-1) = 2$. So this can be factored as

$$p(x) = c_0 \odot (x^{-1}) \odot (x + \oplus 1)^2$$

and even if $-\infty$ doesn't give us a particular value for the function, we can still find $c_0 = 0$ from the equation $p(0) = 0$.

If on the other hand we have a negative slope then we have a zero at $-\infty$. Consider the function $p(x) = x + x^2$: This function has two simple roots at $-\infty$ and 0. We may factor it as

$$p(x) = c_0 \odot (x \oplus -\infty) \odot (x \oplus 0)$$

and even if $p(-\infty) = -\infty$ we can plug in 0 to get 0 back in order to get $c_0 = 1$.

Correspondence Theorems

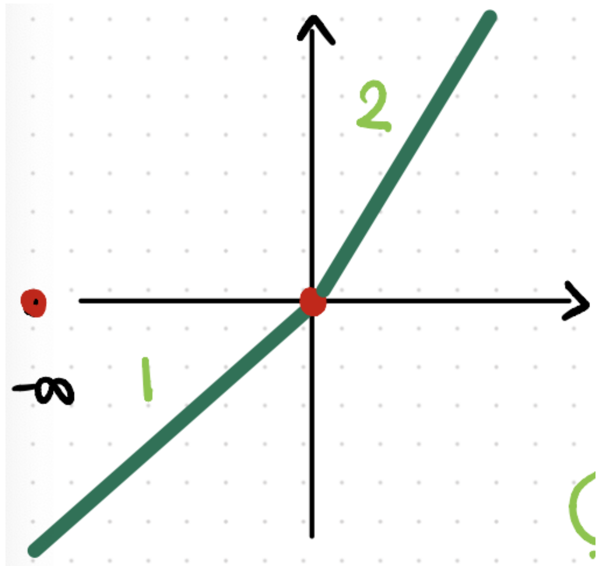
Recall the maps

$$\begin{cases} T_t : \mathbb{C} \to \mathbb{T} & \text{(with } \max), \\ \operatorname{val} : \mathbb{C}\{\{t\}\} \to \mathbb{T} & \text{(with } \min). \end{cases}$$

If we consider a polynomial

$$p(X) \in \mathbb{C}[X] \quad \text{or} \quad p(x) \in \mathbb{C}\{\{t\}\}[X]$$

then we can produce a tropical polynomial as follows:



i. Apply T_t or val to the coefficients, and

ii. Perform tropical operations.

We expect that if $r \in \mathbb{C}$ or $r \in \mathbb{C}\{\{t\}\}$ is a root of p, then $\lim_{t\to\infty} T_t(r)$ will be a root of the new polynomial.

Or the other way around, given $p \in \mathbb{T}[x]$, we can lift the coefficients to $\mathbb{C}$ or the Puiseux series via the above maps. We can find the roots of the corresponding polynomials in $\mathbb{C}[x]$ or $\mathbb{C}\{\{t\}\}[x]$ and then the image of those roots via T_t or val are the tropical roots of $p(x)$.

Example 2.3.3. Consider the polynomial $p(x) = 2 \odot x \oplus 3 \in \mathbb{T}[x]$. We wish to construct a polynomial in $\mathbb{C}[x]$ which tropicalizes to p. Take the polynomial

$$q(x) = t^2 X + t^3 \in \mathbb{C}[x], \quad t > 0$$

We could certainly add phase as $e^{i\theta}$ to the t^k's, but that won't change anything. Taking the logarithm of the coefficients we get

$$t^2 \mapsto 2 \quad \text{and} \quad t^3 \mapsto 3.$$

Then switching the operations to tropical operations we have

$$t^2 X + t^3 \xrightarrow{\text{Trop}} 2 \odot X \oplus 3$$

which was our original polynomial p.

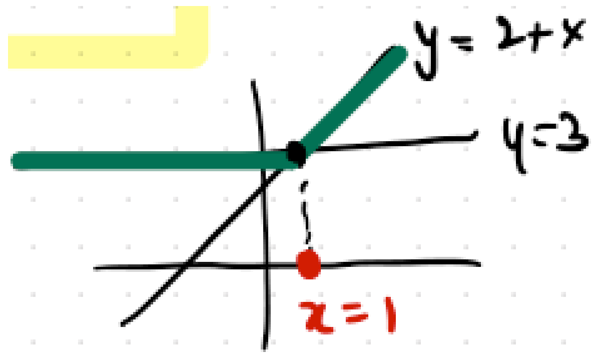


Figure 2.3: Root of $p(x)$ in correspondence with $-t$ of $q(x)$

Additionally if we solve the equation $q = 0$ we obtain the root $X = -t^3/t^2 = -t$. Now $\log_t |-t| = 1$. Lo and behold, this is the same root of $p(x)$.

We should be skeptical because this was only an example of a linear polynomial. Lets increase the degree and see what happens. Eventually this correspondence must be shown to hold in its entirety.

Example 2.3.4. Consider the polynomial

$$q(X) = X^2 + t^2 X + 1 \in \mathbb{C}[X] \xrightarrow{\text{Trop}} p(x) = x^2 \oplus 2 \odot x \oplus 0.$$

We can identity the roots of p as -2 and 2. However, we may find it difficult to interpret the roots of q as roots of p. Observe that using the quadratic formula we may derive those to be:

$$X_{1,2} = \frac{-t^2}{2} \pm \frac{\sqrt{t^4 - 4}}{2} = \frac{-t^2}{2}\left(1 \pm \sqrt{1 - \frac{4}{t^4}}\right).$$

Even if taking the logarithm seems hard, we are not interested in the logarithm itself, just the limit! Observe that

$$\lim_{t\to\infty} \log_t \left|\frac{-t^2}{2}\left(1 + \sqrt{1 - \frac{4}{t^4}}\right)\right| = 2 + \lim_{t\to\infty} \frac{1}{\log(t)} \log\left|\frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{t^4}}\right)\right|$$

and the quantity on the right tends to $1/\infty$ which collapses to zero and then the logarithm only has 1 as its argument. So overall we find one our original roots: 2! The next limit has a different sign so it is not as direct. We may calculate that limit as follows:

$$\lim_{t\to\infty} \log_t \left|\frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{t^4}}\right)\right| \approx \lim_{t\to\infty} \log_t \left|\frac{1}{2}\left(1 - \left(1 - \frac{4}{2t^4}\right)\right)\right| = \lim_{t\to\infty} \log_t \frac{1}{t^4} = -4.$$

So for the negative root we would actually obtain $2 - 4 = -2$ which is the other root of our polynomial.

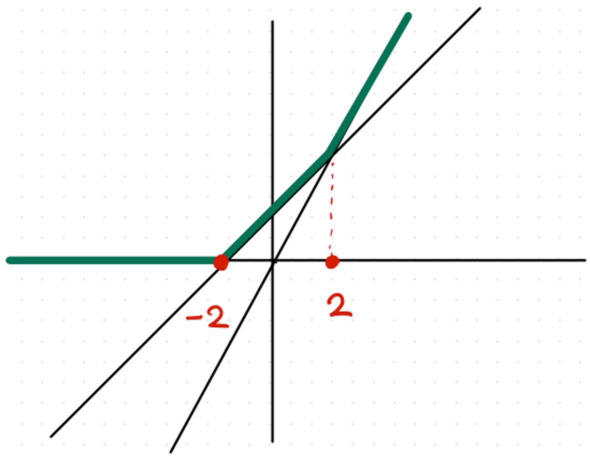


Figure 2.4: Indeed the roots of q correspond with p's

2.4 Interim 2

Definition 2.4.1. If $q(x) = \sum a_k x^k \in \mathbb{C}[X]$ or $\mathbb{C}\{\{t\}\}[x]$, then the tropicalization of q is

$$\mathrm{Trop}(q) = \sum T_{t\to\infty}(a_k)x^k$$

or respectively with the valuation. In this case we omit the notation for tropical operations but the sum and product are tropical.

Theorem 2.4.2. *For a polynomial q, r_k is a root of $q(x)$ with multiplicity m_k if and only if $T_{t\to\infty}(r_k)$ is a root of $\mathrm{Trop}(q)$ of multiplicity m_k.*

In the univariate case, we may prove the theorem using the following lemmas.

Lemma 2.4.3. Trop *is a multiplicative function on polynomials. That is*

$$\mathrm{Trop}(pq) = \mathrm{Trop}(p)\,\mathrm{Trop}(q) \quad \textit{for} \quad p, q \in \mathbb{C}[x].$$

Lemma 2.4.4. *The roots of $\mathrm{Trop}(p)\,\mathrm{Trop}(q)$ are the union of the roots of the factors. If a root is repeated then the multiplicities are added.*

Exercise 2.4.5 (2). Prove the preceding lemmas and then conclude the theorem as a result.

Otherwise, we may prove the correspondence theorem in a different way. This is more conducive to a higher number of variables. This is helpful, as in higher dimensions we don't have a fundamental theorem of algebra. But, in this case, the most convenient perspective is the valued field perspective. So let us swtich to that point of view and interpret

$$x \oplus y = \min(x, y).$$

Theorem 2.4.6. *Let* $q \in \mathbb{C}\{\{t\}\}[x]$, *then* $r \in \mathbb{C}\{\{t\}\}$ *is a root of* q *if and only if* $\mathrm{val}(r) \in \mathbb{T} \cap \mathbb{Q}$ *is a root of* $\mathrm{Trop}(q)$.

Proof

Let us begin by considering a root r of q, then $q(r) = 0$ which means that

$$a_0 + a_1 r + \cdots + a_d r = 0.$$

This is formal sum of monomials which in order to vanish, at least two of the monomials must reach a minimum order of vanishing to cancel. This is equivalent to $\mathrm{val}(r)$ being a root of $\mathrm{Trop}(q)$.
The other directior is substantially more difficult. This will be an instance of a realizability question. We have two cases, r is a finite root or $r = \infty$. We will assume that r is finite and do a proof by example.

Example 2.4.7. Consider the polynomial

$$q(x) = tx^3 + x^2 + x + t \Rightarrow \mathrm{Trop}\, q(x) = 1 \cdot x^3 + x^2 + x + 1$$

The roots of this polynomial are $-1, 0$, and 1. We will now find a root $r_1 \in \mathbb{C}\{\{t\}\}$ of q with $\mathrm{val}(r_1) = r_1$. For this to happen we requiere

$$r_1 = yt^{-1} + z \quad \text{where} \quad y \in \mathbb{C} \quad \text{and} \quad z \in \mathbb{C}\{\{t\}\},\ \mathrm{val}\, z > r_1.$$

We now plug in r_1 into q and we obtain

$$\begin{aligned} q(r_1) &= t(yt^{-1} + z)^3 + (yt^{-1} + z)^2 + (yt^{-1} + z) + t \\ &= \underline{y^3 t^{-2}} + 3y^2 z t^{-1} + 3yz^2 + z^3 t + \underline{y^2 t^{-2}} + 2yzt^{-1} + z^2 + yt^{-1} + z + t \end{aligned}$$

Extracting the coefficients we get $y^3 + y^2 = 0$ which means that $y = -1$. Plugging this back into our expression as y we get

$$3zt^{-1} - 3z^2 + z^3 t - 2zt^{-1} + z^2 - t^{-1} + z + t = tz^3 - 2z^2 + (t^{-1} + 1)z + (-t^{-1} + t).$$

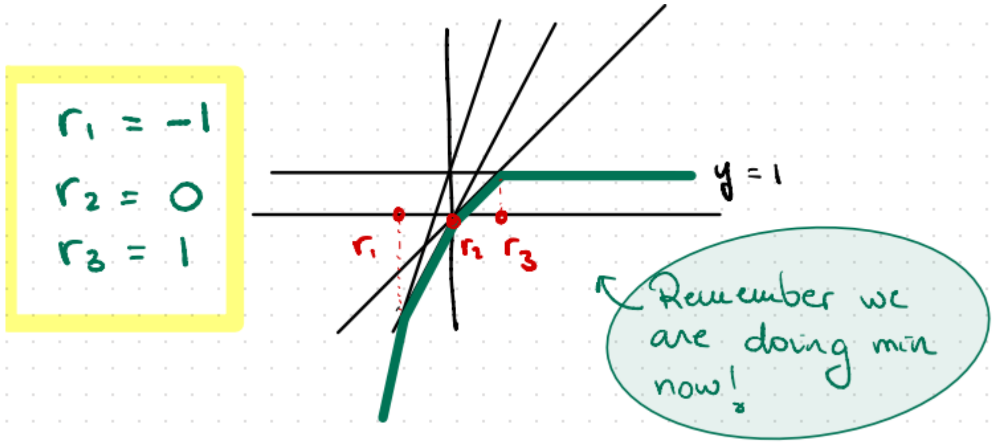


Figure 2.5: Tropicalization of q in $\min$ convention

Tropicalizing (is it actually or is it the reverse operation?) we get

$$1 \cdot z^3 + z^2 + (-1)z + (-1)$$

which has as a root $1 > -1$ So

$$z = y + z_1 \quad \text{with} \quad y \in \mathbb{C}, \quad z_1 \in \mathbb{C}\{\{t\}\}.$$

ASK MAPLE CODE

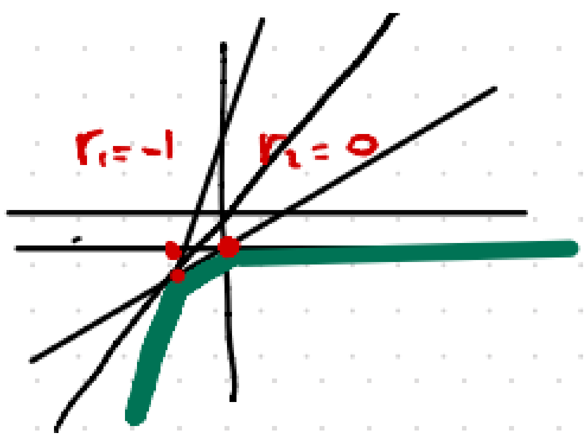


Figure 2.6: I don't know what this is

The question now is: how do we turn this idea into a formal proof?

i. We do one root at a time, starting with the rightmost one.

ii. Observe that if r is a tropical root and $\alpha = yt^r$ with y chosen so cancellation happens, then denoting $\tilde{q}$, q without the x^0 term:

$$\operatorname{Trop}(q(x+\alpha)) > \operatorname{Trop}(\tilde{q}) \oplus \operatorname{Trop}(q(\alpha)).$$

iii. Finally we iterate and check that the sequence of r_i's goes to ∞.

Combinatorialization of Root Finding

We will be using the $\max$ convention now. So let us consider $p(x) = \sum_{k=0}^{d} a_k x^k$. Can we a systematic and simple way to say how many roots, with what multiplicity, and what equations to solve?

The left-most root can be found via

$$\min\left(\frac{a_0 - a_k}{k}\right) = \text{achieved by } k \text{ such that } \frac{a_0 - a_k}{k} \text{ is maximized.}$$

In other words we are looking for the largest slope: We may repeat this argument for

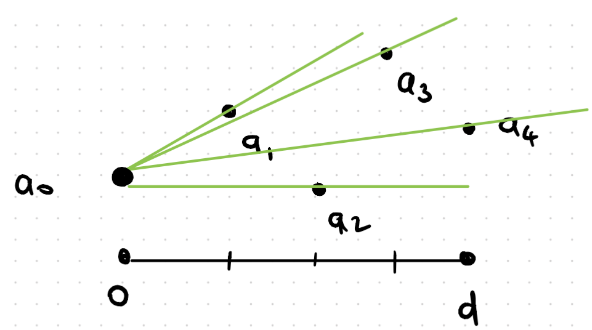


Figure 2.7: Difference of coefficients as slopes

the following roots to get the following algorithm:

i. Let $p_k = (k, a_k) \in [0, d] \times \{-\infty\} \cup \mathbb{R}$.

ii. Now Σ is the convex hull of the points $\{p_k : k \in [d]\}$. We may divide the region into Σ^+ and Σ^-.

iii. Call $q_i = \pi(p_i)$ for p_i's that for the vertices of Σ^+.

The roots will be in bijection with the connected components of $[0, d] \setminus \{q_i\}_{i \in I}$ and the multiplicity is the length of the segment.

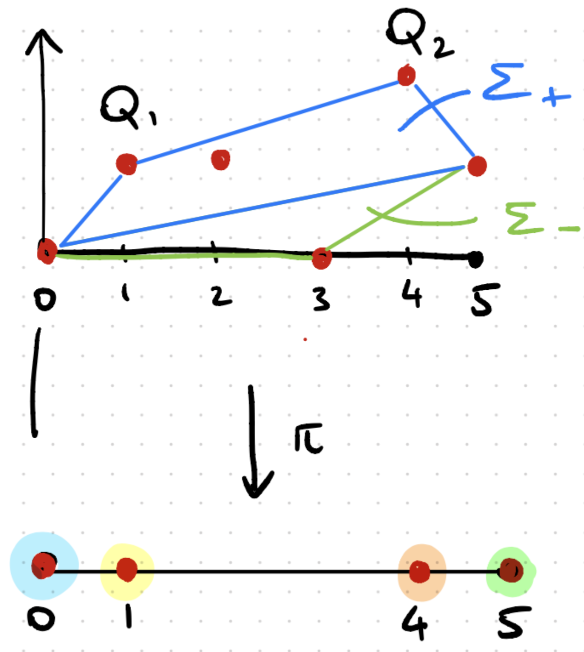


Figure 2.8: Root finding for $p(x)$

Example 2.4.8. Take for example the polynomial

$$p(x) = 0 + 1 \cdot x + 1 \cdot x^2 + x^3 + 2 \cdot x^4 + 1 \cdot x^5.$$

We now place the points in our diagram and project: From this we deduce that there are 2 simple roots and 1 triple root. This come from the equations

$$\begin{cases} 0 = x + 1 & \Rightarrow x = -1 \\ x + 1 = 2 + 4x & \Rightarrow x = -1/3 \\ 2 + 4x = 1 + 5x & \Rightarrow x = 1 \end{cases}$$

Gröbner Complexes

If K is a field with a valuation, then call

$$\begin{cases} R_K \subseteq K = & \text{elements with non-negative valuation} \\ \mathfrak{m} \subseteq R_K = & \text{elements with positive valuation} \end{cases}$$

so ${}^{R_K}/_{\mathfrak{m}}$ is a residue field. In the case of tropical polynomials, they form a Gröbner complex[1].

$$\mathfrak{m} = \bigcup_n t^{1/n}\mathbb{C}\left[\left[t^{1/n}\right]\right] \subseteq R_K = \bigcup_n \mathbb{C}\left[\left[t^{1/n}\right]\right] \subseteq \mathbb{C}\{\{t\}\}, \quad \text{and} \quad {}^{R_K}\Big/_{\mathfrak{m}} = \mathbb{C}.$$

Definition 2.4.9. Given $q \in K[x]$ and $w \in \mathbb{T}$, the <u>initial form</u> of $q(x)$ with respect to w is a polynomial in $k[x]$that records the part of q that has lowest order when $\mathrm{val}(x) = w$.

Example 2.4.10. Let us consider the polynomial

$$q(x) = t^{-4} + \sqrt{2}x + 3t^2x^2,$$

Here[2]

$$t^{-4} \to -4, \quad \sqrt{2}x \to -3, \quad 3t^2x^2 \to -4, \quad \text{so} \quad w = -3^3.$$

We may construct the initial form as $I_w q = 1 + 3x^2$ but formally this is $\left[t^4(q(t^{-3}x))\right]_{t=0}$ and in general if $W = \mathrm{Trop}\, q(w)$ then

$$I_w q = \left[t^{-W}(q(t^w x))\right]_{t=0}.$$

Gröbner Complex of $q(x)$

Polyhedral decomposition of $\mathbb{R}$ (in the case of a valuation space if we want, we can also add in ∞ but it usually is left out.) induced by the equivalence relation

$$w_1 \sim w_2 \iff In_{w_1}q = In_{w_2}q^4$$

Example 2.4.11. Consider the polynomial

$$t^2 + \sqrt{2}x + 3t^2x^2$$

and each monomial maps[5] to $2, w$ and $2 + 2w$ respectively. So the tropical roots are the locus where the initial form is not a monomial.

[1] What are Gröbner comlpexes? To see in interim.
[2] What does this mean?
[3] I srsly don't understand
[4] Does this refer to initial form?
[5] Through what? The valuation?

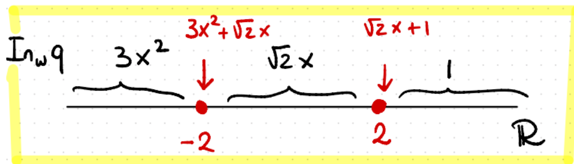


Figure 2.9: Initial form determination and roots

2.5 Day 7 | 20230906

Our First Correspondence Theorem

Definition 2.5.1. Given a family of polynomials

$$q_t = \sum A_k(t)x^k \in \mathbb{C}[x] \quad \text{with} \quad t > 1$$

then the tropicalization of q_t is

$$\mathrm{Trop}(q_t)(x) = \sum a_k \odot x^{\odot k}, \quad \text{where} \quad a_k = \lim_{t\to\infty} T_t(A_k).$$

We may also use the $\min$ convention by exchanging the field to Puiseux series and T_t by the valuation.

Theorem 2.5.2 (Correspondence). *For a polynomial q_t, R_t is a root of q_t if and only if $\mathrm{Trop}(R_t) = \lim_{t\to\infty} T_t(R_t)$ is a root of $\mathrm{Trop}(q_t)$.*

This is saying that we have an object in algebraic geometry, a polynomial. Tropical geometry will somehow knowing about its roots by degenerating it. Then its easy to find the tropical roots and then there must be certain algebraic roots which should map to them. It may not be easy to understand this last map but at least we have some qualitative information.

We will use the fundamental theorem of algebra to reduce to the linear case. So the first step is to prove the theorem for the case of linear polynomials. We have a couple of lemmas to finish the proof and expand it to the general case:

Lemma 2.5.3. Trop *is a multiplicative function on polynomials. That is*

$$\mathrm{Trop}(pq) = \mathrm{Trop}(p) \odot \mathrm{Trop}(q) \quad \textit{for} \quad p, q \in \mathbb{C}[x].$$

This first lemma doesn't add anything weird because the tropical product is just the usual addition.

Lemma 2.5.4. *The roots of $\mathrm{Trop}(p) \odot \mathrm{Trop}(q)$ are the union of the roots of the factors. If a root is repeated then the multiplicities are added.*

Essentially what this is saying is that if we have two piecewise linear functions which change slope at the same place, then the sum will also change slope at the same place. As the functions are convex, a root can never be cancelled. Except possibly $-\infty$.

Higher Dimension

We will go back to the Puiseux series convention now:

$$P(X) \in \mathbb{C}\{\{t\}\}[X],\ P(R) = 0 \iff \mathrm{Trop}(P)(\mathrm{val}(R)) = 0.$$

The easy direction is to begin with a root of our Puiseux polynomial. Let

$$P(X) = \sum A_i(t)X^i, \quad \text{and} \quad \mathrm{Trop}(P)(X) \sum a_i \odot x^i$$

where $a_i = \mathrm{val}(A_i)$. Let $R = R(t)$ be a root of $P(X)$.

We know $\mathrm{val}(P(R)) = \infty$ because $P(R) = 0$. Formally $\mathrm{val}(P(R))$ should greater or equal than the minimum of the valuation of each of the monomials evaluated at R. In other words

$$\min(\mathrm{val}(A_i(t)R^i)) = \min_i(a_i + i\,\mathrm{val}(R)) = Trop(P)(R).$$

Since we know that strict inequality holds, the terms in the formal evaluation with lowest order must cancel, in other words, the minimum is attained at least twice by two different monomials.

Last week we mentioned attaining the minimum twice is the same as being a root.

Example 2.5.5. Consider the polynomial $(t^2 + 7t^3)X + (t^5 + t^{27}) = Q(X)$. The root here is $R = -\frac{t^5+t^{27}}{t^2+7t^3}$ and its valuation is $5 - 2 = 3$. If we plug in something of this form instead of X we get

$$Q(-t^3 + O(t^4)) = (t^2 + 7t^3)(-t^3 + O(t^4)) + (t^5 + t^{27}) = (-t^5) + t^5 + O(t^6)$$

In particular the first thing that will cancel is the lowest order term: t^5. So *two* monomials must have lower order term.

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The next question is if this process makes sense if we instead begin with a Puiseux series polynomial. If the process ends up being the same, does this mean that tropical geometry over a trivial valued field is uninteresting? That's not the case, it's only because we are in dimension zero.

Gröbner Complexes

These types of complexes arise in commutative algebra. The setup begins with a valuated field, in our case Puiseux series $\mathbb{C}\{\{t\}\}$. We can find the ring of integers, the positive valuated elements, in our field. These types of functions are regular at $t = 0$. Inside this ring we have the maximal ideal of functions which vanish at zero. If we wish we can take a quotient to find the residue field which is a copy of $\mathbb{C}$.

Everytime we are given the data of polynomial q in $\mathbb{C}\{\{t\}\}[x]$ plus a choice of a valuation, we can recover the initial form of q which is a polynomial with coefficients in the residue field.

The way to find it is to look at the valuation of each monomial assuming $\mathrm{val}(x) = w$ and then save only the monomials with the smallest valuation and only keep the coefficient in front of the smallest term.

Example 2.6.1. Consider the polynomial

$$q(x) = t^{-4} + t^2 + \sqrt{2}x + 3t^2x^2$$

and take $w = -3$. This means that $\mathrm{val}(x) = -3$. Let us now consider the valuation monomial by monomial:

The term $(t^{-4} + t^2)$ has valuation -4 because there's no x, next for $\sqrt{2}x$ we have

$$\mathrm{val}(\sqrt{2}x) = \mathrm{val}(\sqrt{2}) + \mathrm{val}(x) = 0 + (-3) = -3$$

so it has valuation -3 and $3t^2x^2$ has valuation $2 - 6 = -4$. We now consider only the first and last terms as they have the smallest valuation and extract the coefficients of the smallest terms. In the case of $t^{-4} + t^2$ its the 1 accompanying the t^{-4} and a 3 accompanying the last term. So the initial form is

$$\mathrm{In}_{-3}(q) = 1 + 3x^2.$$

FORMULA for initial form

We now define an equivalence relation over $(\mathbb{R}, w)$:

$$w_1 \sim w_2 \iff In_{w_1}q = In_{w_2}q$$

which separates $\mathbb{R}$ into two types of equivalence classes:

- Single points in which the initial form is not a monomial.
- Open intervals where the initial is a monomial.

Example 2.6.2. Consider the polynomial

$$t^2 + \sqrt{2}x + 3t^2x^2$$

and each monomial maps[6] to $2, w$ and $2 + 2w$ respectively. So the tropical roots are

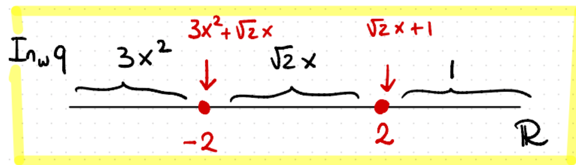

Figure 2.10: Initial form determination and roots

the locus where the initial form is not a monomial.

Definition 2.6.3. The complement of the locus where the initial form is a monomial is called the Gröbner complex of $q(x)$.

The Gröbner complex of $q(x)$ is equal to the roots of $\mathrm{Trop}(q)(x)$. This is indeed in correspondence with Gröbner basis, which is very interesting in higher dimension.

1-dimensional Tropical Geometry

If we have $p(x, y)$ a tropical polynomial in two variables, then we can define its tropical variety to be $V(p)$:

- The locus in the domain where the piecewise linear function where p is not linear.
- The locus of points (x, y) where the $\max$ associated to each monomial is obtained more than once.

We will have a correspondence theorem which says that if $q(x, y)$ is a polynomial with coefficients over a valued field and the tropicalization of q is p, then

$$V(p) = \overline{\{ (\mathrm{val}(x), \mathrm{val}(y)) : (x, y) \in V(q) \}}.$$

Exercise 2.6.4. Show that pairs of rational numbers are dense here. ⟦ It has to do with the valuation only taking rational numbers. ⟧

In two dimensions we have way more features, the study of tropical curves will enclose the correspondence statement with subdivisions of Newton Polygon and balancing edge weights. Our objective now is to see the tropical versions of tropical curve theorems. For example, the tropical Bézout and tropical degree/genus formula.

[6] Through what? The valuation?

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Tropical Lines

The max convention

If we have a tropical polynomial of degree 1,

$$p(x,y) = a \odot x \oplus b \odot y \oplus c$$

and assume for the sake of drawing pictures, that $-\infty \neq a, b, c$. This corresponds to the piecewise linear function

$$\max(a+x, b+y, c)$$

and if we set any of these two equations equal to each other, we can see that there are three lines that play a role:

$$\begin{aligned}
&a + x = b + y \Rightarrow y = x + (a-b)\\
&a + x = c \Rightarrow x = c - a\\
&b + y = c \Rightarrow y = c - b
\end{aligned}$$

So this is the locus where two functions are equal to each other. In each of the regions the maximum is attained by a particular linear function, the boundary between them is the locus of non-linearity. The point in the middle is $(c-a, c-b)$.

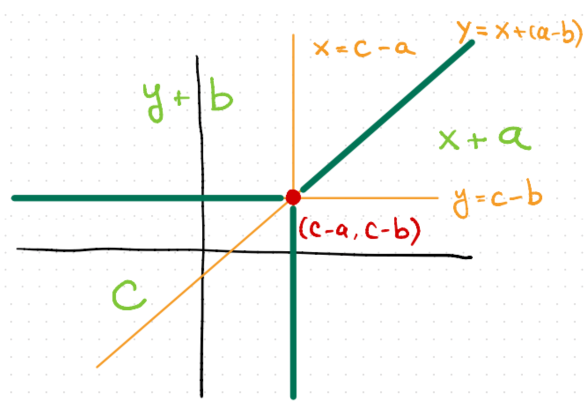

Figure 2.11: Graph of $p(x,y) = 0$ in $\mathbb{R}^2$

So in general, tropical lines look like this "tripod" and changing the a, b, c shifts the graph.

Exercise 2.7.1 (2). Figure out what happens when a coefficient is $-\infty$.

This is analogous to what we have done with tropical univariate polynomials. The tropical line $V(p)$ is the locus of non-linearity of our function:

$$V(p) = \{\, (x, y) \in \mathbb{R}^2 : \ df_p|_{(x,y)} \text{ isn't defined} \,\}.$$

The Case of Puiseux Series

In this case, lines will we the zero loci of polynomials of the form

$$p(X, Y) = A(t)X + B(t)Y + C(t)$$

with

$$a = \mathrm{val}(A), \quad b = \mathrm{val}(B) \quad \text{and} \quad c = \mathrm{val}(C).$$

We let $L = \{\, (X, Y) \in \mathbb{K}^2 : \ p(X, Y) = 0 \,\}$ be the zero locus and then define

$$\mathrm{Trop}(L) = \overline{\{\, (\mathrm{val}(X), \mathrm{val}(Y)) : \ (X, Y) \in L \,\}} \subseteq \mathbb{T}^2.$$

We may parametrize p in the following way, we let $X = \gamma(t)$ with an arbitrary valuation and then we solve for Y:

$$Y = \underbrace{\frac{-A(t)}{B(t)}\gamma(t)}_{a-b+\mathrm{val}\,\gamma} - \underbrace{\frac{C(t)}{B(t)}}_{c-b}$$

ASK RENZO Depending on the value of $\mathrm{val}\,\gamma(t)$, then we may get different values for $\mathrm{val}\,Y(\gamma(t))$.

- If $\mathrm{val}\,\gamma(t) > c - a$, then $\mathrm{val}\,Y(\gamma(t)) = c - b$.
- If $\mathrm{val}\,\gamma(t) < c - a$, then $\mathrm{val}\,Y(\gamma(t)) = x + a - b$.
- And if $\mathrm{val}\,\gamma(t) = c - a$ then $\mathrm{val}\,Y(\gamma(t))$ can be anything above $c - b$. We thus set

$$\gamma(t) = \left(-\frac{C(t)}{A(t)}\right)(1 + t^{\odot m}), \quad \text{where} \quad m > 0$$

The first two items represent a graph of $(a - b)x + (c - a)$[7]

Remark 2.7.2. If we send a, b, c, X and Y to their negatives then $\mathrm{Trop}(L)$ agrees with the previous perspective. Alternatively, we can check that $\mathrm{Trop}(L)$ agrees with the previous perspective but using the $\min$ convention.

[7] ask Renzo because I don't understand, page 4 of TG7.

Example 2.7.3. Let us explicitly choose A, B and C:

$$q(X,Y) = t^a X + t^b Y + t^c.$$

Note that q's tropicalization is $p(X,Y)$. Points in $V(q)$ can be parametrized as

$$X = \alpha, \quad Y = \frac{-t^a}{t^b}\alpha - \frac{t^c}{t^b} = -t^{a-b}\alpha - t^{c-b}, \quad \text{where} \quad \alpha \in \mathbb{K}^*.$$

Taking valuations of X and Y we get $\operatorname{Trop}(L)$ (but without closing it). Specifically we are looking at the set

$$\operatorname{Trop}(L) = \left\{ \big(\operatorname{val}(\alpha), \operatorname{val}(-t^{a-b}\alpha - t^{c-b})\big) \in \mathbb{T}^2 : \ \alpha \in \mathbb{K}^* \right\}$$

We can let α have any valuation we want and depending on that, we determine the valuation of the binomial Y. The possible valuations are

$$\operatorname{val} Y = a - b + \operatorname{val}(\alpha) \quad \text{or} \quad \operatorname{val} Y = c - b$$

which are equal when $\operatorname{val}(\alpha) = c - a$. ASK RENZO ABOUT THIS HOW TO DETERMINE THE CRITERION ABOUT VAL AND STUFF

- What happens if $\operatorname{val}(\alpha) < c - a$ then $-t^{a-b}$
- Something I fell asleep

Claim: We can obtain any value for y but is to be greater than $c - b$. Let $r \geqslant 0$ and $\alpha = -t^{c-a}(1 + t^r)$

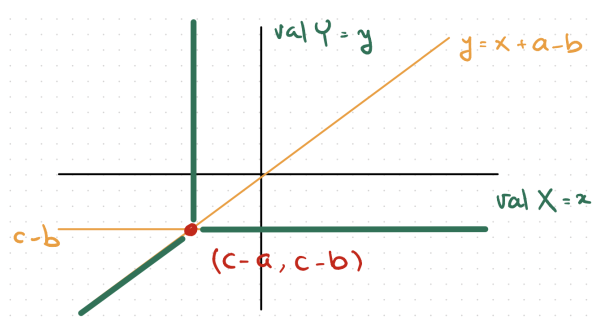

Figure 2.12: The tropical line from the Puiseux perspective

Glimpse into Amoebas

Recall that what matters the most is the logarithm base t of our function. So let us continue with

$$q(x, y) = t^a x + t^b y + t^c$$

and play the same as before. Look for solutions to the equation $q_t = 0$ in $\mathbb{C}^2$ which is a line intersecting the x axis at $-t^{c-a}$ and the y at $-t^{c-b}$. Every pair of points (x, y), gives us a pair $(\log_t |x|, \log_t |y|)$. The real trace of this, when $x, y \in \mathbb{R}$ can be parametrized with $x = t^\alpha$ and $y = -t^{a-b+\alpha}$. We analyze the trace in three intervals,

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The Amoeba Perspective

Our goal is to understand the image of the line

$$L_t = \{\, t^a x + t^b y - t^c = 0 \,\} \subseteq \mathbb{C}^2$$

via the map $(x, y) \mapsto (\log_t |x|, \log_t |y|)$. The line L_t has three sections, where both x, y are positive and one section corresponding to each x and y being negative. For ease of calculation we may solve the equation as $y = -t^{a-b}x + t^{c-b}$

Let us consider the case where x, y are both positive. We can see that $0 < x < t^{c-a}$, this traces an x in the parameter space such that $-\infty < x < c - a$. Via the solution for y we may write

$$\log_t |y| = \log_t(t^{c-b} - t^{a-b+x})$$

where we have solved the equation for x which is why we have an x exponent and y is positive as we have assumed. We can simplify this as

$$\log_t \left[t^{c-b} \left(1 - t^{a-c+x}\right)\right] = (c - b) + \log_t(1 - t^{a-c+x}).$$

This can be traced as a function of t and in particular

$$\lim_{x \to -\infty} (c - b) + \log_t(1 - t^{a-c+x}) = c - b \quad \text{and} \quad \lim_{x \to (c-a)^-} (c - b) + \log_t(1 - t^{a-c+x}) = -\infty.$$

With this information we see two asymptotes for our function, $y = c - b$ and $x = c - a$.

Arbitrary Degree d

Recall that for a polynomial $q \in \mathbb{K}[x, y]$, we may describe its algebraic variety in $\mathbb{K}^2$. We may think of that field as Puiseux series. Along it, we may tropicalize it to p and we get its tropical hypersurface, the set of non-linearity.

Kapranov's theorem allows us to see a correspondence as follows:

$$\overline{\mathrm{Trop}(V(q))} = V(\mathrm{Trop}(q)).$$

Left-to-right is still the same idea as the correspondence theorem. If $(x_0, y_0) \in \mathrm{Trop}(V(q))$ then, there exists $(X_0, Y_0) \in \mathbb{K}^2$ such that $\mathrm{val}(X_0) = x_0,\ \mathrm{val}(Y_0) = y_0$ and $q(X_0, Y_0) = 0$. Let

$$q = \sum a_{ij} X^i Y^j$$

If we call m_{ij} each monomial, then $\{\, m_{ij}(X_0, Y_0)\,\}_{i,j}$ is a set of elements of $\mathbb{K}^*$ with the property that their sum is zero. Now call

$$\mu = \min\{\, m_{ij}(X_0, Y_0)\,\}_{i,j}$$

we claim that there are at least two monomials whose valuation is μ. If there was only one monomial with valuation μ, then that power of μ *cannot* be cancelled. This means that (x_0, y_0) is in $V(p)$. Now we use minimality of closure and we are done.

Now the harder direction will use the fact that we have proven this in dimension zero and proceed by induction. First we want to show that $V(\mathrm{Trop}(q)) \cap \mathbb{Q}^2$ is dense in $V(\mathrm{Trop}(q))$. This is true because all monomials m_{ij} correspond to all linear functions with integer slopes of rational coefficients.

$$a_{ij} X^i Y^j = \mathrm{Trop}(m_{ij}) = \mathrm{val}(a_{ij} \odot x^i \odot y^j) = \mathrm{val}(a_{ij}) + ix + jy$$

It suffices to check that ERASED TOO QUICK

Now we wish to proceed by induction. For example a polynomial $q(X, Y)$ can be seen as

$$q(X, Y) = r_0(X) + r_1(X)Y + \cdots + r_d(X)Y^d \quad \text{with} \quad r_i(X) \in \mathbb{K}[X].$$

We do not lose generality when assuming that all r_i's are monomials[8]. So we have $(x_0, y_0) \in\in V(\mathrm{Trop}(q))$ and we want to find $(X_0, Y_0) \in (\mathbb{K}^*)^2$ such that

$$q(X_0, Y_0) = 0 \quad \text{and} \quad (\mathrm{val}(X_0), \mathrm{val}(Y_0)) = (x_0, y_0).$$

Choose X_0, however we want as long as we have the valuation condition. Given our assumption, this implies that $r_i(X_0)$ is non-zero for all i. Now consider the polynomial

$$q(X_0, Y) = \sum r_i(X_0) Y^i \in \mathbb{K}[Y]$$

and its tropicalization $\tilde{p}(y) = \mathrm{Trop}(q(X_0, Y)) = \sum \mathrm{val}(r_i(X_0)) y^i = \min(\mathrm{val}\, R_i(X_0) + iy) = \min$ They are hidden in terms of unknown,

[8] to see next time

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Theorem 2.9.1. $V(\text{Trop}(q)) = \overline{\text{Trop}(V(q))}$

If we start with a polynomial in valued field, we can tropicalize or look at ... and then take the image of coordinates of points and back in $\mathbb{R}^2$ then take closure, we end in the same place. The fact that every point of the algebraic curve lies somewhere is an argument of cancellation of Lowest Order Terms. In particular when plugging the value for the Puiseux solution two terms must cancel.

Proof

We have shown that right-to-left is easy, cancellation of L.O.T.
The other direction is trickier, it's a lifting problem. Given $(x_0, y_0) \in V(\text{Trop}(q)) \subseteq \mathbb{R}^2$, then we must find

$$(X_0, Y_0) \in V(q) \subseteq \mathbb{K}^{*2}, \quad \text{val}(X_0) = x_0 \quad \text{val}(Y_0) = y_0$$

If we write q then we will assume that we can write

$$q(X, Y) = \sum r_i(X) Y^i, \quad r_i(X) \text{ monomials}$$

If we first plug in $X = X_0$ (which is any Puiseux series we want with valuation x_0 [We have picked such X_0]),

$$q(X_0, Y) = \sum r_i(X_0) Y^i$$

is a polynomial in Y with Puiseux series coefficients, now tropicalize this q we get

$$\tilde{p}(y) = \sum \text{val}\, r_i(X_0) y^i \quad \text{(tropical sum and product now)}.$$

We claim that y_0 is a root of $\tilde{p}(y)$. HEre's where we are using the monomial assumption.
What is the linear function associated to $\tilde{p}(y)$:

$$\tilde{p}(y) = \min(\text{val}\, r_i(X_0) + iy)$$

and as r_i is monomial, call it $r_i(X_0) = A_{ij} X^j$ where A_{ij} is a Puiseux series. So this $\tilde{p}$ becomes:

$$\tilde{p}(y) = \min(\text{val}(A_{ij}) + jx_0 + iy)$$

which is exactly the tropicalization of $q(x, y)$ and plug in x_0. This is a univariate polynomial, which allows to apply the univariate case. So there exists a Y_0, Puiseux series, such that Y_0 is a root of $q(X_0, Y)$ with $\text{val}(Y_0) = y_0$.
It remains to see that our monomial condition is not a restriction.

This allows us not only to lift, but to pick one coordinate freely and then the other one is determined!

Example 2.9.2. Consider the polynomial

$$q(X,Y) = XY + X^2Y = (X + X^2)Y, \quad \text{and} \quad \tilde{q}(X,Y) = q(XY,Y) = XY^2 + X^2Y^3$$

and $\tilde{q}$ does satisfy the previous assumption. If $(\tilde{X}_0, \tilde{Y}_0)$ is a solution to our problem for $\tilde{q} = 0$, then $\left(\frac{\tilde{X}_0}{\tilde{Y}_0}, \tilde{Y}_0\right)$ is a solution for $q = 0$.

The key point is that $\tilde{q}$ is obtained q by an invertible transformation in the torus $(\mathbb{K}^*)^2$.

Given $q(X,Y)$ of degree d, then picking

$$\tilde{q}(X,Y) = q(XY, Y^{d+1})$$

satisfies the monomials assumption. This is because *we are giving enough space*.

$$q(X,Y) = \sum r_{ij}X^iY^j \Rightarrow \tilde{q}(X,Y) = \sum r_{ij}X^iY^{(d+1)j+i}$$

where if we wished to find …then

$$(d+1)j_1 + i_1 = (d+1)j_2 + i_2 \Rightarrow (d+1)(j_1 - j_2) = i_2 - i_1$$

$j_1 - j_2 \geqslant d+1$ when $j_1 = j_2$ and $i_2 - i_1 \leqslant d$.

Example 2.9.3. Compute $V(p)$ for the following polynomials

- $p_1 = 0 + x + y + xy$
- $p_2 = 0 + x + y - xy$
- $p_3 = 0 - x - y + xy$

For each of this polynomials there are $\binom{4}{2} = 6$ line possibilities so we check each one.

2.10 Day 14 | 20230922

Consider the polynomial

$$q(X,Y) = XY + X + Y + c, \quad c \in \mathbb{C}$$

as a polynomial in $\mathbb{K}[X,Y]$ such that $\operatorname{Trop} q = p$. We can factor q as

$$(X+1)(Y+1)+(c-1)=0 \Rightarrow (X+1)(Y+1)=\tilde{c}$$

which looks hyperbolic. The real locus is PICTURE and we would like to compactify. However in $\mathbb{P}^2$ we get an anomalous curve. PICTURE So we would like to compactify instead in $\mathbb{P}^1 \times \mathbb{P}^1$. We bi-homogenize q to obtain

$$\tilde{q} = X_1Y_1 + X_1Y_0 + Y_1X_0 + X_0Y_0c = 0$$

In this case PICTURE we won't intersect the special points. Instead just general points. The point we want to emphazise is the fact that the shape of the tropical curve tells us that this could be the tropicalization of a curve but that the plane curve wants to be compactified in $\mathbb{P}^1 \times \mathbb{P}^1$ instead of $\mathbb{P}^2$.

Bummer Theorem

Theorem 2.10.1. *Let $q \in \mathbb{C}[X,Y]$ considered as a subset of Puiseux series polynomials where no coefficient is zero. We can write $q(X,Y) = \sum_{i+j\leqslant d} a_{ij}X^iY^j$ with $a_{ij} \neq 0$.*

Then $\overline{\operatorname{Trop}(V(q))}$ looks like a tropical line with a vertex at zero.

This is a bummer because any polynomial of this form will look like a tripod. But what happens with *lines off to infinity matches the degree*? We may able to endow lines with information about the degree, but nonetheless, we lose every other piece of information. So choosing coefficients in a trivially valued field reduces all information.

To prove this theorem we will unravel the definitions.

Proof

We have that the tropicalization of q is

$$\operatorname{Trop} q = \bigoplus (x^i \odot y^j) = \min_{i+j\leqslant d}(ix+jy)$$

and so we can see that the minimum is attained by PICTURES

Working with Puiseux series is not only because its fancy, its because we wish to have non-trivial objects.

Structure Theorem for Tropical Curves

Let $p(x,y) = \bigoplus a_{ij} \odot x^i \odot y^j$ be a tropical polynomial. The Newton polygon of p is the convex hull of (i,j) such that $a_{ij} \neq 0$.

Let Σ be the convex hull of the points $(i,j,a_{ij}) \subseteq V(NP) \times \mathbb{R}$, which is for every (i,j) point consider the height a_{ij} so Σ is a convex polytope in $\mathbb{R}^2 \times \mathbb{R}$. Seeing the polytope from the top and projecting down we get a subdivision of the Newton polygon.

Consider $\pi_z((x,y),z) = (x,y)$ and let $\tilde{N}$Ñ be the subdivision of the N.P. obtained by projecting the corners of Σ you can see from above. If we want to say this in fancier words, a polytope is a finite intersection of half-spaces and then we look at intersections of planes with outward normal vector (z postive coordinate) then project down to xy plane.

Then the tropical curve is **DUAL** to such subdivision, meaning that:

- Vertices of the tropical curve map to faces of Ñ, edges map to edges of Ñ.
- There is a reversing structure given by inclusion into the closure.
- Every edge of $V(p)$ is perpendicular to the edges of Ñ, it corresponds to.
- Coordinates of vertices v are found by solving the linear system obtained by setting equal the linear functions corresponding to monomials corresponding to vertices of the face of Ñ dual to v.

Example 2.10.2. Consider the polynomial

$$p(x,y) = 0 \oplus x^2 \oplus y^2 \oplus 1x \oplus 1y \oplus xy,$$

each monomial corresponds to a vertex in our triangle. Somehow now we know that our tropical curve has 4 vertices, then we can correspond edges.

To get the coordinates of the lowe left vertex we look at the vertices surrounding the corresponding triangle. The linear system we ought to solve is

$$0 = 1 + x = 1 + y \Rightarrow x = -1,\ y = -1$$

and in this fashion we obtained the coordinates.

2.11 Day 15 | 20230925

We were talking about how tropical plane curves are dual to the subdivisions of the Newton polygon of the tropical polynomial. Let's recall this with an example.

Example 2.11.1. Consider the polynomial $0 + x^2 + y^2 + x + y + xy$. All coefficients are non infinity so we get 6 points corresponding to all terms, the convex hull is a triangle. We give each point a height, $0, x^2$ and y^2 have 0 coefficient while x, y and xy

have coefficient 1. We imagine throwing a drape over that figure and then divide the faces, edges of tropical curve are perpendicular so the curve should look similar to the following figure.

FIGURE

Now for example, the central vertex is dual to the face F_2 which corresponds to the monomials $1+x$, $1+y$ and $1+x+y$. The solution to the system

$$\begin{cases} 1+x=0 \\ 1+y=0 \\ 1+x+y=0 \end{cases} \Rightarrow (x,y)=(0,0)$$

so $(0,0)$ is the coordinate of the central vertex.

Remark 2.11.2. There's not metric duality between the triangle and the curve! It is possible to make a Newton polygon which doesn't fit in the curve as well.

Exercise 2.11.3 (5)**.** Find such an example!

If we did this in the case of Puiseux series, then the subdivision of the Newton polytope keeps track of the initial forms of q in the sense that for any cell in the Newton subdivision, the initial form is given by the monomials corresponding to the lattice points in this set.

Example 2.11.4. Consider the polynomial

$$q(X,Y)=7+3X^2+2Y^2+t^{-1}X+2t^{-1}Y+t^{-1}XY,$$

this polynomial tropicalizes via $-\operatorname{val}$ to the polynomial p from last example.

FIGURE

The initial form in the face F_1 corresponds to the monomials $XY+2Y+2Y^2$, then $XY+2Y+X$ in F_2, $X+2Y+7$ in F_3 and $3X^2+X+XY$ in the last F_4. The Gröbner fan is still the same as in the tropical curve.

Along the edge 01 of F_1 the initial form is $2Y^2+XY=In_w(q)$ where $w=(w_1,w_2)=-(\operatorname{val}X,\operatorname{val}Y)$ for w in the blue edge of the tropical curve (corresponds to stuff which has that initial form).

We will prove this fact by making a crucial observation. Evaluating a tropical monomial at a point (x_0,y_0) can be done as a dot product. Take $m=a\odot x^i\odot y^j$ so the evaluation is at (x_0,y_0) is

$$\langle (i,j,a)|(x_0,y_0,1)\rangle\,.$$

When we construct the subdivision of the N.P. we consider all points with coordinates (i, j, a_{ij}) as i, j ranges over $\{ a_{ij} \neq -\infty \}$ So evaluating $p(x_0, y_0)$ amounts to looking for the maximum of the dot products of $(x_0, y_0, 1)$ with all (i, j, a_{ij}).

FIGURE

In other words $m_{ij}(n_x, n_y)$ is equal for all monomials corresponding to vertices of the green face. For vertices below, it occurs that $\langle \mathbf{n} | \mathbf{v} \rangle < 0$ so

$$m_{ij}(n_x, n_y) < m_{\tilde{i}\tilde{j}}(n_x, n_y)$$

when m_{ij} corresponds to vertices not in the face and $m_{\tilde{i}\tilde{j}}(\dots)$ in face.

We have identified why the vertices correspond to tropical subdivisions, but the edges? If we focus on one, it bounds two faces F_1 and F_2 which span two planes with normal vectors $\mathbf{n}_1, \mathbf{n}_2$ with their respective z coordinates equal to 1. So any vector between these two, i.e. any one in with first two coordinates in the segment n_1 to n_2 has the property that

$$\langle n | m_1 \rangle = \langle n | m_2 \rangle > \langle n | m_{\text{other}} \rangle$$

2.12 Day 16 | 20230927

We have discussed that tropical plane curves are dual to a subdivision of the Newton polytope. There's a combinatorial algorithm that will allow us to divide the N.P.

We will study a couple more characteristics to discern between stick figures and tropical curves. We need to introduce the fact that each *stick* gets a weight.

Definition 2.12.1. Any edge of a tropical curve $V(p)$ is given weight w_e equal to the lattice length of the segment of the N.P. subdivision dual to the edge.

Before even defining what things mean, let see an example.

Example 2.12.2. Consider the tropical cubic with subdivision edges $1 \to yx^2$ and $yx^2 \to y^2$. We may identify weights of edges with black in the coming figure:

Theorem 2.12.3. *Tropical plane curves are balanced, meaning that at every vertex*

$$\sum_{v \in e} w_e \mathbf{p}_e = 0.$$

Here w_e is the weight of the edge $\mathbf{p}_e$ outgoing primitive vector in the direction of e.

Continuing the line of the previous example, at the vertex we were looking at we have outgoing primitive vectors

$$\begin{pmatrix}1\\2\end{pmatrix}, \quad \begin{pmatrix}-1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\-2\end{pmatrix}.$$

Observe that when taking the weighted sum at the vertex we get

$$2\begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

We will prove the theorem next.

Proof

Any vertex is dual to a face F_v of N.P. subdivision. For every edge bounding F_v, the vector $w_e\mathbf{p}_e$ is obtained by the vector tracing the dual edge via a 90^circ rotation. We claim to be done, the fact that the equation is satisfied is equivalent to the fact that the N.P. is a closed polygon.

So imagine a stick figure with weighted edges comes up to you, then we can check the compatibility condition. If it doesn't have weights, does there exist an assignment of weights in order to form a tropical curve?

What do the weights mean?

Example 2.12.4. Suppose we have a subdivision with edges

$$x^2y \to x^5y^2 \to x^8y^3 \to x^11y^4$$

and the corresponding edge in the tropical curve. We want to see the initial form. This subdivision *remembers* the monomials that appear in the initial form! It will be a linear combination of the monomials in the edge!

$$\mathrm{In}_\omega(q) = Ax^2y+Bx^5y^2+Cx^8y^3+Dx^{11}y^4 = x^2y(A+Bx^3y+Cx^6y^2+Dx^9y^3) = x^2yP(x^3y)$$

This polynomial factors so nicely because they lie on the same line! If we are just looking for solutions in $(\mathbb{C}^*)^2$ or asymptotically $|x|, |y| \gg 0$ what happens is that the monomial part is irrelevant. Also $\deg(P)$ is equal to the lattice length of the segment. A 1-parameter subgroup orbits $x^3y = r_i$ where the r_i is a root of P counted with multiplicity.

2.13 Day 17 | 20230929

Last time we talked about edges weighted by lattice length determines the segments of the Newton Polygon it is dual to. Today we will draw topological types of tropical plane curves and cubics. The objective is to experiment with various tropical curves and seek a conjecture to compute their b_1. Study a pencil of tropical conics, in other words draw a conic and pick 4 points on it in general position. Then find all conics through those 4 points.

2.14 Day 18 | 20231002

For any interior point of the Newton Polygon, we can make sure that we can find as many cycles can a tropical curve have. There is a correspondence theorem.

The solution to the parameter space of conics is that we get a trivalent tree with 6 edges. This corresponds to $\mathbb{P}^5$ which has 6 boundary divisors.

Intersections of tropical curves

Definition 2.14.1. Two tropical curves have a transversal intersection if they intersect in finitely many points which are not vertices of either curve.

Example 2.14.2. Two tropical lines seen as tripods which touch just at a point intersect transversally.

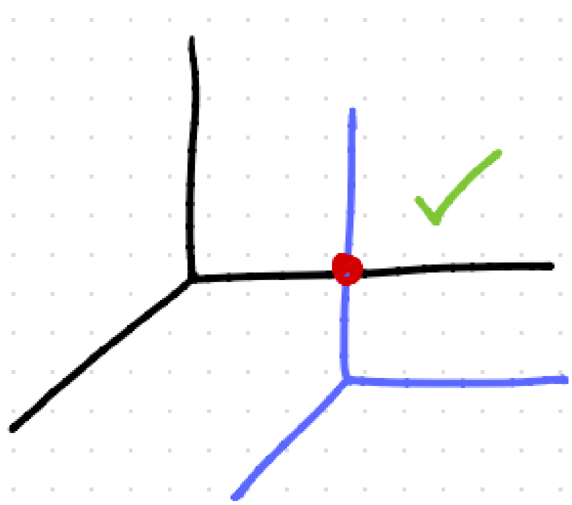

Figure 2.13: Example of a Transversal Intersection

Example 2.14.3. If the line is placed right on top of the other one, they could also intersect non-transversally on the whole edge.

If for example we had a conic, it could intersect the vertex of another tropical line non-transversally. Or two vertices could intersect!

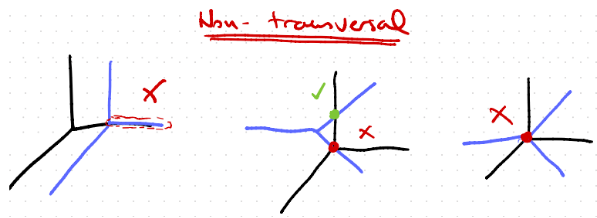

Figure 2.14: Example of Non-Transversal Intersections

It looks like if the intersection is transversal then the set of intersection is just a point. Otherwise let's be very topological and talk about stable intersections.

Definition 2.14.4. For a vector $\mathbf{v} \in \mathbb{R}^2 \backslash \{ 0 \}$, call the <u>vector intersection</u> of Γ_1, Γ_2

$$\Gamma_1 \cap_{\mathbf{v}} \Gamma_2 = \lim_{t \to 0} \left(\Gamma_1 \cap (\Gamma_2 + t\mathbf{v}) \right).$$

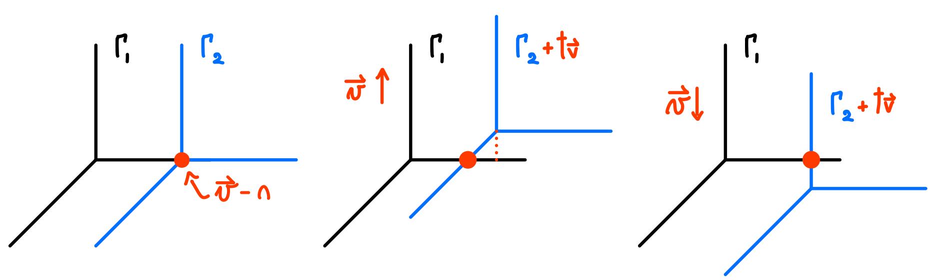

Figure 2.15: Example of Vector Intersections

Observe that this definition may be troublesome because of the choice of the vector $\mathbf{v}$. What happens if we get another intersection point with another $\mathbf{v}$?

From this we deduce that intersection points should be weighted by multiplicity of the edges they belong to. We need more than just the edge weights to define this.

Before we jump into the theory, let's have an aside question, let us ask how the curves

$$C_1 = \{ x^a = y^b \}, \quad \text{and} \quad C_2 = \{ x^c = y^d \}$$

intersect in $\mathbb{C}^2$. Observe that we may parametrize C_1 as $(x, y) = (t^b, t^a)$ and substitute into C_2's to get

$$t^{bc} = t^{ad} \Rightarrow t^{bc}(t^{ad-bc} - 1) = 0$$

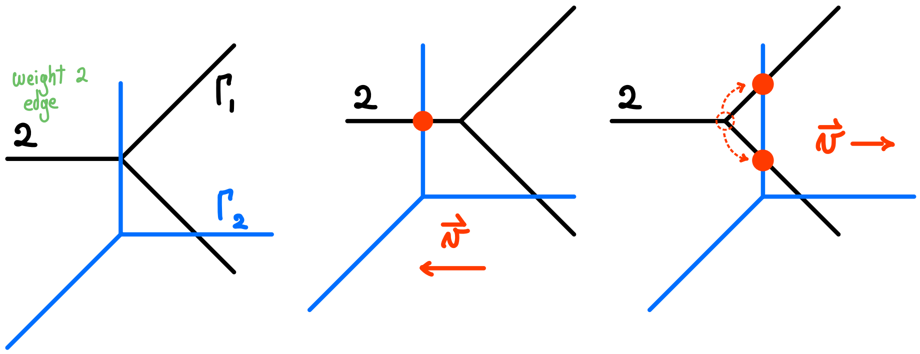

Figure 2.16: Weighted Vector Intersections

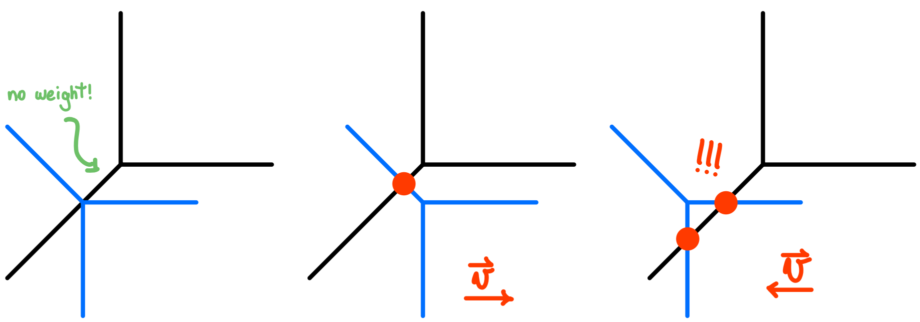

Figure 2.17: Problematic Vector Intersections

which means that at zero we have a point of high multiplicity and $(ad - bc)$ points away from the origin. The interesting question is how does this relate to our problem. At a glance there doesn't seem to be a connection, but if we look at the *valuation* we might just start to see some hints of it.

Definition 2.14.5. Let q be a point of transversal intersection of two tropical curves Γ_1, Γ_2. Then the multiplicity of the intersection is determined by the primitive[9] directions $\mathbf{p}_1, \mathbf{p}_2$ of the intersection as

$$m_q(\Gamma_1, \Gamma_2) = w_1 w_2 |\det(\mathbf{p}_1, \mathbf{p}_2)| = w_1 w_2 \left[\mathbb{Z}^2 : p_1\mathbb{Z} \oplus p_2\mathbb{Z}\right].$$

Where the last quantity is the subgroup index.

[9]Primitive lattice vectors are the shortest lattice vectors possible

2.15 Day 19 | 20231004

Our objective last time was to motivate the idea of intersections with certain multiplicity. With out new definition of multiplicity, does this resolve the issue we had?

Example 2.15.1. FIGURE In the first case we have intersections at Q_1, Q_2, where the first intersection has multiplicity

$$m_{Q_1} = \left|\det \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\right|$$

and the other point also has multiplicity 1. But if we move to the non-transversal intersection we get multiplicity 2. This goes according to the definition.

Lemma 2.15.2. *The amount of intersections of two tropical curves is invariant under generic translation.*

The proof follows from the balancing conditions and it suffices to analyze cases locally.

Proof

Consider an edge of a tropical curve and a local intersection. The edge partitions the plane into two half planes H^+, H^-. Then the contribution to a $\mathbf{v}$ deformed intersection is the same when $\mathbf{v}$ points in the direction of H^+ or H^-.
Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are two vectors pointing in different directions, if we raise the intersection, we get the lower edges intersecting. Their primitive vectors are all in the clockwise direction form p_e. So the weight of the $\mathbf{v}_1$-deformed intersection is

$$\sum_{e' \in E^-} w_e w_{e'} |\det(p_1, p_2)|$$

Tropical Bézout

From this lemma the result follows immediately!

Theorem 2.15.3. *Let Γ_1, Γ_2 be two tropical curves of degree d_1, d_2. Then $|\Gamma_1 \cap \Gamma_2| = d_1 d_2$.*

A curve has degree d if its dual to a subdivision of 0, $(d, 0)$ and $(0, d)$. We know that if we have a tropical curve of degree d_1 then we can move one of degree d_2 in such a way that just so many edges intersect. Counting down we get the product of the multiplicities.

2.16 Day 20 | 20231009

Last time we started talking about the idea behind Bézout's theorem. Recall, that a curve with degree d_i is dual to the polygon with vertices 0, $(0, d_i)$ and $(d_i, 0)$. Intersections, as in the ordinary case, are counted with multiplicities.

The quantity $|\Gamma_1 \cap \Gamma_2|$ is invariant under translation, so we can move two curves so that only unbounded ends intersect. Looking at this picture from far away and squinting our eyes what we see is two tripods with degree d_i. This means that all the information needed to compute the intersection is independent of the subdivisions of the Newton Polygon.

Definition 2.16.1. In $\mathbb{P}^1 \times \mathbb{P}^1$, coordinates are $([x_0 : x_1], [y_0 : y_1])$, a bi-degree (a, b) curve is the zero locus of a bi-homogenous polynomial in the aforementioned variables which is

- ◇ homogenous of degree a is x_i,
- ◇ homogenous of degree b is y_i.

Tropically, this means that Γ is dual to a subdivision of a rectangle with sides a, b.

Example 2.16.2. The polynomial $x_0y_0 + x_1x_0y_1 = 0$ is a bidegree $(2, 1)$ curve. On the tropical side we have FIGURE which is a $(1, 1)$ curve.

The tropical Bézout theorem for $\mathbb{P}^1 \times \mathbb{P}^1$ says that if we have Γ_i with bidegree (a_i, b_i), then

$$|\Gamma_1 \cap \Gamma_2| = a_1b_2 + a_2b_1.$$

But what if we wanted to intersect a degree d curve with a bidegree (a, b) curve? We just draw the stick figure and notice that degree of the intersection is $d(a + b)$. We can ask how to generalize it and the answer is precisely Bernstien's theorem.

Theorem 2.16.3. *Let Γ_i be tropical curves of degree Δ_i. Then*

$$|\Gamma_1 \cap \Gamma_2| = \text{MixedArea}(\Delta_1, \Delta_2).$$

In this case the degree of a tropical curve is the Newton polygon of its equation. In other words a lattice polygon.

Minkowski Sum of Polytopes

Once a long time ago we were told that a degree is a number, but a degree is actually a polygon.

Definition 2.16.4. Consider $\Delta_i \subseteq \mathbb{R}^2$ lattice polygon, then the Minkowski sum is

$$\Delta_1 + \Delta_2 = \{\, (x_1, y_1) + (x_2, y_2) \in \mathbb{R}^2 : \ (x_i, y_i) \in \Delta_i \,\}.$$

This definition is **compatible** with translations! At this moment we can choose how to put two polygons in $\mathbb{R}^2$ and sum them.

Example 2.16.5. The Minkowski sum of a square and a right triangle is a gem-shaped pentagon. The idea is that we decide one of the vertices to be the origin and then make the polygon travel through the perimeter of the other one.

Exercise 2.16.6. The Minkowski sum appears to be commutative, prove it! Which other properties does the Minkowski sum enjoy?

Example 2.16.7. FIGURE ABOUT MIXED SUDVISIONS

Definition 2.16.8. The mixed area of Δ_1, Δ_2 is either of

i) $A(\Delta_1 + \Delta_2) - A(\Delta_1) - A(\Delta_2)$,

ii) the area of the mixed cells in any mixed subdivision of $\Delta_1 + \Delta_2$,

iii) the $\lambda\mu$ coefficient in $A(\lambda P + \mu Q)$.

Chapter 3

Toric Varieties

3.1 Day 21 | 20231011

IT says that if we have two tropical curves of degree Δ_1, Δ_2 (polytopes), and we wish to compute the intersections, then this is counted via the mixed area. This correspondence is very tight, satisfying. When taking stuff, we get the mixed cells (parallelograms with sides parallel to our original polygons). Having the polygons in positions means that the curves have been moved and the mixed cells correspond to the intersections.

The weight of the edges is the length of sides of the polygon, in particular the Minkowski sum of two lattice polytopes is itself a lattice polytope.

Torus actions

A space X with a torus action $T \cdot X$ and an open set isomorphic to the torus on which T acts by multiplication is a toric variety. The geometry of a torus is fairly simple or trivial, which means that *most* of the geometry of X can be recovered from the complement of this dense open set (the one isomorphic to T). The fact that we have a torus action allows for a *combinatorialization* of the geometric information.

If we don't know what a torus is, it's a group, and the space is just the space. The action moves the points in a certain way. Geometry is connected to the representation theory of the group.

Definition 3.1.1. The algebraic torus of rank k is G_m^k (think of it as $(\mathbb{C}^*)^k$) $[\![$ [$]\!]$If we have a different field than $\mathbb{C}$, actually G is a multiplicative group *scheme*]. This object is a group under pointwise multiplication and in general we can view it as the scheme

$$\operatorname{Spec}\left(\mathbb{C}\left[x_1, \frac{1}{x_1}, \ldots, x_k, \frac{1}{x_k}\right]\right).$$

If we are more familiar with differential geometry, local coordinates are similar but the coordinate is a generalization. We can imagine elements on the scheme as regular functions from the torus to $\mathbb{C}$. We will usually denote the torus by T.

Recall a group acts on itself via multiplication and in a similar faction the torus action on a space X will be a map $T \times X \to X$. We can thus relate the characters of T to invertible regular functions on T and if we are choosing coordinates, then they are also related to monomial functions in x_i's.

The characters of T are the group homomorphisms from T to $\mathbb{C}^*$. It is actually the trace of an irreducible representation of T, but they are one dimensional and... Monomial functions on the x_i's are precisely regular functions vanish. Those places are on the coordinate axis so nowhere near the torus. As a $\mathbb{C}$-vector space, monomials are a basis of our ring. That is gonna play an important role.

All of these interpretations are groups under multiplication. Given a monomial we can read a vector of exponents:

$$x^3y^{-5} \mapsto (3, -5)$$

so this is also related to the lattice of exponents of the monomial functions. This gives a multiplicative to additive isomorphism. This lattice is called M, it is also called the character lattice and is isomorphic to $\mathbb{Z}^k \subseteq \mathbb{R}^k$ which we will also called $M_{\mathbb{R}}$. This is actually a *tensor product*. In this sense we can match

$$(3, -5) \mapsto \varphi_{(3,-5)} : T \to \mathbb{C}^*,\ (t_1, t_2) \mapsto t_1^3 t_2^{-5}.$$

We also have a relationship between co-characters (linear functions on characters), one parameter subgroups of T (group homomorphisms $\mathbb{C}^* \to T$, $t \mapsto (t^a, t^b)$). The co-character lattice N is the dual of the character lattice is isomorphic to $\mathbb{Z}^k \subseteq \mathbb{R}^k = N_{\mathbb{R}}$. There is a natural pairing $M \times N \to \mathbb{Z}$ which turns out to be the usual dot product. Slightly more satisfying, the natural way to get a pairing is to imagine we have a character χ and a one parameter subgroup γ we get a map

$$\mathbb{C}^* \xrightarrow{\chi} T \xrightarrow{\gamma} \mathbb{C}^*,\ t \mapsto (t^a, t^b) \mapsto t^{aA+bB}$$

so the exponent is precisely $\langle (a,b) | (A, B) \rangle$.

After talking about so many ingredients, let us talk about one more thing. If we have one parameter subgroups, we can think of them as acting on the torus so we have torus orbits. Let's see some torus orbits we get with different actions.

Example 3.1.2. Consider the one parameter subgroup $\gamma_{(1,1)}(t) = (t, t)$. Acting on a point (a, b) we get the point (ta, tb). So the orbits are lines through the origin. And notice the in fact, $\gamma_{(-1,-1)}(t) = (t^{-1}, t^{-1})$, even if we get the same orbits, on the first case we get flow inwards while on the second one we get *outwards* flow.

We could get $\gamma_{(1,0)}(t) = (t, 1)$, the orbits are horizontal lines. Finally $\gamma_{(1,-1)}(t) = (t, t^{-1})$, then the orbits are hyperbolas.

Now we are gonna say ok, we are gonna allow ourselves to grow the torus a little bit by saying that we want to add limit points of one parameter or allow/decide that some monomial functions are regular and some are not and then add the points. In the case of $\gamma_{(1,1)}$ we are going to need to fill in the point of the origin.

3.2 Day 22 | 20231013

Last time we talked about the setup: the toric variety is a space X. One is that it contains the torus as a dense open set and the other is that the torus acts on X and in particular in the open set it acts by multiplication. We linked objects from the algebraic geoemtry and representation theory of the torus. The way we think of characters of the torus are monomial functions on the coordinates of the torus. These are regular functions because there are no poles on the torus. On the other hand N is a lattice of one-parameter subgroups, a map from a one dimensional torus to a k-dimensional one. In particular inside the torus, if we take a point and act on it via a one parameter subgroup, we get a one-dimensional orbit. It is always the case that if we let $t = 0$ then the point falls off of the torus. These objects are dual to each other via dot product.

Example 3.2.1. An example of a toric variety is $\mathbb{P}^1$, the set of homogenous points $[x : y]$. Take out zero and infinity and we get a one-dimensional torus.

We can also think of points in $\mathbb{P}^1$ as points of $\mathbb{C}^\times$ as $[x : 1]$ or $[1 : y]$. Our objective is to build toric varieties from simpler varieties.

Definition 3.2.2. A <u>cone</u> in $N_\mathbb{R}$ will tell us how to add points to a torus in two ways:

i) It would tell us what orbits of one-parameter subgroups acquire a limit point when $t \to 0$.

ii) Or it would tell us what monomial functions are regular on such limit points.

Let us illustrate with an example taking two dimensional $N_\mathbb{R}$:

Example 3.2.3. A cone is what we think is a cone.FIGURE This cone σ_1 contains all lattice points corresponding to one-parameter subgroups of the form

$$t \mapsto (t^a, t^b), \quad a, b > 0$$

Now let's assume that we have a point of the two dimensional torus $P = (r_p, s_p)$, $r_p, s_p \neq 0$. We now look at the orbit of P via the coordinate-wise multiplication action:

$$t \cdot P = (t^a r_p, t^b s_p)$$

and now taking the limit $t \to 0$ of $t \cdot P$ and we want it to exist, then we introduce three cases:

- Both a, b are non-zero.
- $a = 0$ but b not, then we get points of the form $(r_p, 0)$.
- The other way around for b: $(0, s_p)$.

$$\lim_{t \to 0} t \cdot P = \begin{cases} (0,0), \ a, b \neq 0 \\ (r_p, 0), \ a = 0 \\ (0, s_p), \ b = 0 \end{cases}$$

The cone contains all the ways to pick the map. All the dots in the *interior* go to zero! If a, b are both zero the action is trivial. This cone is telling us to add the axes to T to get $\mathbb{C}^2$.

Notice that if we had taken the lower cone we would've obtained a similar situation but instead of adding zeroes we would get infinities. We would still get $\mathbb{C}^2$ but adding different axes.

If we instead were algebraic geometers we would be interested on regular functions at the limit points. Take $f = x^\alpha y^\beta$, when is it regular at the $t \to 0$ limit point of the orbit of a 1.p.s. $t \mapsto (t^a, t^b)$, then

$$f(t) = t^{a\alpha + b\beta} r_p s_p,$$

the limit exists as long as $a\alpha + b\beta$ is non-negative. This is precisely the dot product, so we are basically saying that we want all vectors in M whose dot products with vectors in N is non-negative. This defines the notion of *dual cone*. This combinatorial object acquires the meaning of telling us what are the regular functions when adding the limit points. In this case $\sigma_1^\vee$ looks exactly like σ_1 PICTURE. This also tells us that x, y are regular functions so every polynomial in x, y is a regular function. So in particular $\mathbb{C}[x, y]$ is the ring of regular functions.

Let us focus our attention now on the projective plane.

Definition 3.2.4. A fan is a collection of cones living *nicely* in N. This means that we are not just wacking at random at some cones. All cones will have the vertex at the origin and the only overlaps occur at boundary edges. PICTURE

This particular fan Σ is the fan of $\mathbb{P}^2$. To construct it we follow this strategy

(a) Each top dimensional cone produces an affine open chart in the sense of manifolds.

(b) Each face, or ray in common, provides transition functions.

If we understand how it works in $\mathbb{P}^2$, generalizing is pretty simple. We will look at dual cones, monomial functions there give us regular functions.

Example 3.2.5. The dual to the red cone is itself, the dual to the blue cone is $\{ x < 0, 0 < y < -x \}$ and for green its $\{ y < 0, 0 < x < -y \}$. Call two monomial functions x_1, y_1 on red$^\vee$ so the regular functions are $\mathbb{C}[x_1, y_1]$.

On the blue we take the points $(-1, 0)$ and $(-1, 1)$, they correspond to the functions $\frac{1}{t_1}$ and $\frac{t_2}{t_1}$. But actually let's just call them x_2, y_2 so the coordinate ring is $\mathbb{C}[x_2, y_2]$ so our affine chart is a copy of $\mathbb{C}^2$ with coordinates x_2, y_2. In a similar fashion we will get $\mathbb{C}[x_3, y_3]$ from the green dual.

To make them talk to each other, we take a ray in Σ and ask for its dual. The dual of the positive y-axis is the upper half plane. The regular functions for this space are $\mathbb{C}[\tilde{x}, \tilde{y}, \tilde{x}^{-1}]$ so it is $\mathbb{C}^\times \times \mathbb{C}$ with coordinates $\tilde{x}, \tilde{y}$. Now notice that this black cone contains the red and blue dual cones. So in particular we can express x_1, y_1 in terms of $\tilde{x}, \tilde{y}$. The key point is that $\mathbb{C}^\times \times \mathbb{C}$ is contained in the two copies of $\mathbb{C}^2$ we had before. We get the following relations

$$\tilde{x} = x_1 = x_2^{-1}, \quad \tilde{y} = y_1 = y_2 x_2^{-1}.$$

These are the transition functions which allow us to glue this two patches!

Cones select subsets and produces regular monomials, this produces coordinate ring, take the algebraic variety corresponding to this ring. This variety contains the torus. This can be done in various ways so that's how we build the atlas.

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Then the combinatorial date of the fan gives us a way to construct a toric variety a s a amanigold. Every face gives us the transition functions. The way to construct this chart is the dual cone. Then we take the algebra spanned by the monomials and we ask what variety's coordinate ring is this span.

Consequences of our construction

(a) There is an inclusion into closure reversing bijection between cones of Σ and the torus orbits of X_Σ, the toric variety. And in particular, also exchanging dimension with codimension.

In action, the fan of $\mathbb{P}^2$ has 3 two dimensional cones R, G, B and likewise $\mathbb{P}^2$ has 3 two-codimensional points. Similarly the fan has 3 one dimensional cones. $\mathbb{P}^2$ has 3 one dimensional torus orbits. From the fan of $\mathbb{P}^2$ we can immediately read the Euler characteristic of $\mathbb{P}^2$. The only things that will contribute are the zero dimensional cones.

In general, to see how the bijection works you can take two different approach:

- The geoemtric approach is: for every cone τ of Σ, look at the limits as $t \to 0$ of the torus orbits of 1-parameter subgroups γ with $\gamma \in \tau^\circ$. (stuff) and instead we pick a 1 parameter subgroup which lies on a ray, take for example $b = 0$, the y coordinate doesn't change and x goes to zero.
- The algebraic perspective is that for every affine patch dual to a cone, set all the coordinates that you can set to zero to zero.

  What we mean by that is that if we look at the spec of the algebra dual to the red cone we get $\mathbb{C}[x_1, y_1]$ that's why it's affine. Setting both to zero we get the origin. When looking for transition functions, the ray has an affine chart, the algebra generated by the dual cone $\mathbb{C}[\hat{x}, \frac{1}{\hat{x}}, \hat{y}]$. In this case $\hat{x}$ can't be set to zero. Doing it, we get any non-zero complex number so we get a one dimensional something. And the origin is itself a cone, the dual cone corresponding to the trivial cone is the whole thing. The corresponding algebra is $\mathbb{C}[x, \frac{1}{x}, y, \frac{1}{y}]$ and that's why we can't set anything equal to zero.

  Coordinates here correspond to local coordinates in the affine patch. There's a lot of things with the same name.

  Even if in blue we get the same algebra, setting coordinates to zero gives us a different point.

(b) The second useful thing is very natural. T-equivarant maps of toric varieties correspond to maps of fans. (Maps of fans: Given $\Sigma_k \subseteq N_{k,\mathbb{R}}$, a map of fans is a $\mathbb{Z}$-linear map $L : N_{1,\mathbb{R}} \to N_{2,\mathbb{R}}$ such that for τ cone of Σ, $L(\tau) \subseteq$ cone of Σ_2.)

Consider the fan of $\mathbb{P}^1 \times \mathbb{P}^1$, a 2 dimensional toric variety, for $\Sigma,\ X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$. We have a projection map $\pi : X_{\Sigma_1} \to \mathbb{P}^1$ which is equivalent to the map of fans $\tilde{\pi}$ form Σ_1 to Σ_2. It is also the case for $\mathbb{P}^2 \backslash \{ pt \}$ to $\mathbb{P}^1$ which is projecting from that point.

The proof of this facts is left to the reader. Even without the proof this is a natural statement which makes sense.

$L(\tau)$ doesn't have to be a full cone. In the previous examples it is, but that's just an accident. We would allow it to not be a full cone because we can allow mor maps.

For example if we wanted to send a line in a plane as a map of.ko

Quotient construction

Given a fan Σ, the toric variety X_Σ can be obtained as a quotient space of the form:

We start with affine space, cheack the irrelevant locus and then we take something that if we are lucky is a torus:

$$\mathbb{C}^N \backslash \{ irrelevant \} \Big/ G$$

N is the numnber of rays in Σ. The irrelevant set is the set determind by rays that dont span cones.And G is given by lienar relations among maps.

Example 3.3.1. In $\mathbb{P}^2$ we have the tripod fan with three coordinates. So we have a $\mathbb{C}^3$. We throw away the rays which don't span a cone. In this case th irrelevant. This is $\mathbb{C}^3 \backslash$ FELL ASLEEP

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Last time we talked about orbit-cone correspondence for toric varieties. This is a poset-reversing bijection, every biggest cone corresponds to a smallest orbit and vice-versa. In particular, this helps us do really funky things. An orthant corresponds to affine space, the dual cone is the same orthant and those are the coordinates of affine space. The fan with only the axes is $\mathbb{C}^2 \backslash \{ (0,0) \}$ and so on.

Maps of fans corresponds to good maps of toric varieties. They both have combinatorial descriptions, maps beteween fans are the restriciton of an integral linear map between vector spaces. Cones of the first fan move to cones of the second fan. Integral in the sense that integer coefficients.

With that now we have a quotient construction: take $\mathbb{C}^N$ remove *stuff* which is obtained from equations coming from rays that do not span a cone and then we mod out by linear relations among rays.

Example 3.4.1. In the case of $\mathbb{P}^2$ we have the fan which looks like a tripod. As it has 3 rays we begin with $\mathbb{C}^3$. Each ray corresponds to a coordinate of $\mathbb{C}^3$: x, y and z. The only subset of rays that does not span a cone of the fan is the three rays together $\{\rho_x, \rho_y, \rho_z\}$. We take the coordinates corresponding to the rays and set them equal to zero. Then we take that locus and throw it away: $\{x = y = z = 0\}$. The way to determine the group is that we see the primitive vectors: $\mathbf{p}_x = \mathbf{1}, \dots\ p_z = -e_1 - e_2$ and the fact that we have one relation

$$1 \cdot p_x + 1 \cdot p_y + 1 \cdot p_z = 0$$

means that the torus is one dimensional. t acts on (x, y, z) as $(t^1 x, t^1 y, t^1 z)$. The quotient we obtain is

$$\mathbb{C}^3 \backslash 0 \Big/ \mathbb{C}^\times.$$

Observe that the exponents in the action are the coefficients in the relation.

If the coefficients were $2, 3, 4$ *the action would be* $(t^2 x, \dots)$

If we were working with $\mathbb{P}^1 \times \mathbb{P}^1$ the quotient would be by $(\mathbb{C}^\times)^2$ and the action would be $(s, t) \cdot (x_1, x_2, y_1, y_2) = (sx_1, sx_2, ty_1, ty_2)$. More relations means more copies of $\mathbb{C}^\times$. The real question is *why is this true*? What if we take a fan and declare that any ray of the fan becomes a basis of a new vector space? Artificially we create a vector space.

Example 3.4.2. In the case of the $\mathbb{P}^2$ fan, $\mathbf{p}_x$ becomes $(1, 0, 0)$ and so on. Now we want to lift the other cones as cones generated by the vectors. The space we just made is the first quadrant. All the cones that we get are in the totally positive quadrant. We are actually missing a three dimensional cone. We have *removed* from this fan that 3-cone. The orbits we are removing are *not* lifts of cones back in the tripod. Now what we want is a map of fans that will make the quadrant go into the tripod. This will be a map of vector spaces. In particular in this case the map is projection by $(1, 1, 1)$. This is a one-parameter subgroup of the torus. Basically the orbits of our one-parameter subgroup get identified, so the projection is the quotient map of the action whose exponent vector is the coefficients of the relation.

Tropicalizing Toric

In tropical geometry, the *field* is the tropical numbers $\mathbb{R} \cup \{\infty\}$, then $k^\times$ is $\mathbb{R}$. So the torus which is $(k^\times)^n$ is $\mathbb{R}^n$, since multiplication is the action, then tropical multiplication corresponds to addition.

Example 3.4.3. How do we make the tropical projective plane $\mathbb{P}^2$? Remember we had the fan of $\mathbb{P}^2$ (the tripod inside N) and then from here we got the dual cones inside M. Each of the dual cones gave us a copy of $\mathbb{C}^2$ and then we had transtition functions of the form $x_2 = \frac{1}{x_1}$, and $y_2 = \frac{y_1}{x_1}$. Also between red and green $x_3 = \frac{x_1}{y_1}$ and $y_3 = \frac{1}{y_1}$. This is what we did a couple of days ago.

To do it tropically we do it exactly the same, but instead of $\mathbb{C}^2$ we get $\mathbb{T}^2$. Specifically three copies with coordinates:

$$\mathbb{T}^2, x_1, y_1, \quad \mathbb{T}^2, x_2, y_2, \quad \mathbb{T}^2, x_3, y_3.$$

These sets are basically copies of $\mathbb{R}^2$ with lines at infinity so we get points like (∞, ∞), (∞, r) and (s, ∞). Now we have to glue this things together to obtain one space. Every time we see a times we see a plus actually:

$$x_2 = -x_1, \quad y_2 = y_1 - x_1, \quad x_3 = x_1 - y_1, \quad y_3 = -y_1$$

To see what this objects glue to, take for example the line (s, ∞), in the second $\mathbb{T}^2$ we get $x_2 = -s$ and $y_2 = \infty - s = \infty$ (this is ordinary algebra). So the (s, ∞) line maps to line at infinity *in the reverse direction*. For the $1 \to 3$ transitions we get $x_3 = s - \infty = -\infty$ and $y_3 = -\infty$.Finish discussing.

What happens is that we get an infinite triangle.

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Bibliography

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