Exercise 1. Do the following:

I) Give a simple description of the closed sets in $\mathbb{A}^1$ (with respect to the Zariski topology).

II) Use your previous answer to prove that $\mathbb{A}^1$ is not Hausdorff.

Answer

I) If we consider $\mathbb{A}^1$ over an algebraically closed field k of characteristic zero then every closed set is of the form $V(I)$ where $I \in \mathrm{Spec}(k[x])$. Since $k[x]$ is a PID, then $I = \mathrm{gen}(p)$ for a polynomial $p \in k[x]$. Then $V(I)$ would be the set of roots inside k of p. Since p is arbitrary, every closed set $V(I)$ of $\mathbb{A}^1$ is a finite set.

This means that the open sets are the complement of the finite sets. In essence, the Zariski topology coincides with the cofinite topology over $\mathbb{A}^1$.

II) The cofinite topology is not Hausdorff, so it follows that the Zariski topology isn't Hausdorff as well.

Exercise 2. Show that the Zariski topology on $\mathbb{A}^2$ is not the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$. (Hint: Consider the diagonal.)

Exercise 3. Let $F : V \to W$ be a morphism of affine algebraic varieties. Prove that F is continuous in the Zariski topology.

Answer

Recall that a function is continuous if the inverse image of a closed set is once again a closed set.
Suppose $V_0 \subseteq W$ is a closed set, we would like to see that $F^{-1}[V_0] \subseteq V$ is a closed set as well. Since V_0 is closed, then there exists an ideal I such that $V_0 = V(I)$. We can decompose $V(I) = \bigcap_{G \in I} V(G)$ and use the fact that the inverse image of an intersection is the intersection of inverse images to show our result.
FINISH

Exercise 4. Show that the twisted cubic V of Figure 1.5 is isomorphic to the affine line by constructing an explicit isomorphism from $\mathbb{A}^1$ to V. (Hint: See Exercise 1.2.3)

Exercise 5. Show that if $F : X \to Y$ is a surjective morphism of affine algebraic varieties, then the dimension of X is at least as large as the dimension of Y.