Exercise 1. In class, you have seen examples of infinite-dimensional spaces: Notably, (infinite) sequences of numbers and function spaces. But one can come up with many other sets of objects that

(i) satisfy the vector space axioms, and

(ii) are infinite-dimensional.

Come up with your own example of an infinite-dimensional space that doesn't fit the examples you have seen in class. Show that it is a vector space (if you define scalar multiplication and vector addition appropriately) and why you think that the set is infinite-dimensional.

Answer

Consider a set A, its power set $\mathcal{P}(A)$ and the operation $\triangle$ as symmetric difference. Observe the following:

- The symmetric difference of two subsets of A is yet again a subset of A.

$$X, Y \subseteq A \Rightarrow X \cup Y \subseteq A \quad \text{and} \quad X \triangle Y = (X \cup Y)\backslash(X \cap Y) \subseteq X \cup Y.$$

  This means that, as a binary operation, the symmetric difference is closed in A,

- As an operation, it is associative: For $X, Y, Z \subseteq A$ we have

$$(X \triangle Y) \triangle Z = X \triangle (Y \triangle Z).$$

  The proof of this fact is attached at the end of this exercise. For now, this allows us to say that "$3X$" is well defined, because if it wasn't associative, then the expression $X \triangle X \triangle X$ would be ambiguous.

- There is an additive identity for this operation, recall that the empty set is a subset of all sets. Observe then that for all $X \subseteq A$ we have

$$X \triangle \varnothing = (X \cup \varnothing)\backslash(X \cap \varnothing) = X \backslash \varnothing = X.$$

- Finally observe that every element has an inverse. This is, there is an element Y for each X such that $X \triangle Y = \varnothing$. In this case, Y is the same as X because

$$X \triangle X = (X \cup X)\backslash(X \cap X) = X \backslash X = \varnothing.$$

Now arises the question, about uniqueness of solutions to the equation $X \triangle Y = \varnothing$. Once we have existance of inverses, uniqueness is guaranteed because of commutativity.

The previous statements show that $(\mathcal{P}(A), \triangle)$ is a group. From the last fact we also deduce that every element has order 2. Now, observe that our operation is commutative:

$$X \triangle Y = (X \backslash Y) \cup (Y \backslash X) = (Y \backslash X) \cup (X \backslash Y) = Y \triangle X.$$

Thus this is an Abelian group where every element has order 2. Let us now define a scalar multiplication on this set via $\mathbb{F}_2$. We declare that

$$0 \cdot X = \varnothing, \quad \text{and} \quad 1 \cdot A = A.$$

This makes sense as $2 \equiv 0 \pmod 2$ and $2A = A \triangle A = \varnothing$. The preceding operation satisfies all four axioms of scalar multiplication:

- $1 \cdot X = X$ by definition.
- Scalar multiplication is associative with the field multiplication: $c(dX) = (cd)X$. Observe that

  $$\begin{aligned} &0(0X) = 0(X) = \varnothing,\ (0 \cdot 0)X = (0)X = \varnothing \quad \text{and} \\ &1(1X) = 1(X) = X,\ (1 \cdot 1)X = (1)X = X \end{aligned}$$

  so when both scalars are the same we get the desired associative property. When they are different:

  $$\begin{aligned} &0(1X) = 0(X) = \varnothing,\ (0 \cdot 1)X = (0)X = \varnothing \quad \text{and} \\ &1(0X) = 1(\varnothing) = \varnothing,\ (1 \cdot 0)X = (0)X = \varnothing. \end{aligned}$$

  So even when scalars are different, the result is true. These are all the possibilities so associativity is guaranteed.
- Scalar multiplication distributes with respect to field multiplication: $(c + d)X = cX + dX$. Checking this is analogous to the last item.
- Finally scalar multiplication distributes with respect to vector space addition: $c(X + Y) = cX + cY$. This we can verify in two cases:

When $c = 0$ we have

$$\varnothing = \varnothing \triangle \varnothing$$

and $\varnothing \triangle \varnothing = \varnothing$. In the other case when $c = 1$ we have

$$1(X \triangle Y) = 1X \triangle 1Y \Rightarrow X \triangle Y = X \triangle Y.$$

Thus this operation is a well defined scalar multiplication over $\mathcal{P}(A)$. This can be seen also in another way by recalling that any Abelian group is a $\mathbb{Z}$-module. In this case, because every element has order 2, it's a $\mathbb{Z}/2\mathbb{Z}$-module which means its an $\mathbb{F}_2$-vector space.
Let us now consider two different non-empty elements $X, Y \subseteq A$ and the equation

$$aX + bY = 0$$

If either a, b are non-zero then the equation has no solutions:

- $X + Y = 0$ can't occur as $Y \neq X$.
- $X = 0$ also can't occur as X is non-empty, similarly for Y.

So the only solution is $a = b = 0$. This means that any two distinct elements are linearly independent.
Observe now that singleton sets are a generating set for our vector space as any set A can be seen as

$$A = \bigtriangleup_{x \in A} \{x\}.$$

Singletons in particular are all linearly independent from one another. Observe that this doesn't necessarily occur when we have 3 different arbitrary sets, as we could have

$$X + Y + (X + Y) = 0.$$

If we assume that A is uncountably infinite, then singletons are a set as big as A which generates our vector space and is linearly independent. This means that our space is infinite-dimensional.

Exercise 2. Defining what the "dimension" of a space is is intuitively obvious, but *technically* perhaps not quite as much.

For $\mathbb{R}^n$ and other finite-dimensional spaces, if you have a basis of the space with n

elements, then we say that the space has dimension n[1]. Importantly, every other basis you can find will then also have exactly n elements. This also means that the operation that converts one basis to another can be written as a square matrix/operator that is invertible. This all will turn out to be more complicated for infinite-dimensional spaces.

(i) Take $V = \mathbb{R}^3$. Provide a basis $\{\, \mathbf{a}_i \,\}_{i=1}^3$ (that is, a set of three vectors) for this space. Then provide another basis $\{\, \mathbf{b}_i \,\}_{i=1}^3$.

(ii) There is an operator R (here, a 3×3 matrix) that converts from one basis to another. That is, if I give you a vector $x \in \mathbb{R}^3$, it can be written as $\mathbf{x} = \sum_{i=1}^3 \alpha_i \mathbf{a}_i$ and as $\mathbf{x} = \sum_{i=1}^3 \beta_i \mathbf{b}_i$. The operator R is then the one that translates between expansion coefficients:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = R \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Provide the form of R for your choice of basis and show that it is invertible.

(iii) Repeat the previous two steps if V is the space of symmetric 2×2 matrices.

Answer

(i) Consider the vectors

$$\mathbf{a}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{a}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

which form a basis because the matrix whose columns are the $\mathbf{a}_i$'s is invertible. The other basis we will pick is the canonical basis $\mathbf{b}_i = (\delta_{ij})_{j=1}^3$.

(ii) Suppose we have a vector

$$\mathbf{x} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3$$

which means $\mathbf{x}$ is written in $\mathbf{a}_i$ coordinates. Implicitly we claiming that we know the $\mathbf{a}_i$'s coordinates in canonical basis. If we wish to write $\mathbf{x}$ in

[1] Recall: A basis of a space V is a set of vectors $\{\, \mathbf{a}_i \,\}$ so that every vector $\mathbf{v} \in V$ can be written as a unique linear combination $v = \sum_i \alpha_i \mathbf{a}_i$. Note that the basis vectors do not need to be normalized (we are only working with a vector space, no norms so far) and they do not have to be orthogonal (again, we are only working with a vector space, no inner products have been defined so far).

canonical coordinates, then it suffices to expand the $\mathbf{a}_i$'s in terms of the canonical basis as follows:

$$\mathbf{x} = \begin{pmatrix} 0 \\ \alpha_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_2 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} -\alpha_3 \\ \alpha_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

This means that the matrix whose columns are the $\mathbf{a}_i$'s is the change of basis matrix which goes from $\mathbf{a}_i$ coordinates to $\mathbf{b}_i$ or canonical coordinates.

The operator is invertible because $\{\, \mathbf{a}_i \,\}_{i=1}^{3}$ is a basis of $\mathbb{R}^3$. We can also see it is invertible because the matrix has non-zero determinant:

$$\det \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = -1.$$

(iii) Now let us consider the space of symmetric 2×2 matrices:

$$\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : \ a, b, c \in \mathbb{R} \right\}.$$

The canonical basis in this space is the following set of matrices:

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Any symmetric matrix can be written as a linear combination of these matrices and the only way to get the zero matrix is to have $a = b = c = 0$. So it is indeed a basis. On the other hand, we can also consider the basis given by

$$B_1 = E_2, \quad B_2 = E_2 + E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad B_3 = -E_1 + E_3 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

To check this is a basis, we pile up the matrices into a 3×4 matrix and see that

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{\texttt{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

has rank 3 so it generates a space of dimension 3.

This means that we can find an isomorphism φ between this space and $\mathbb{R}^3$. In fact, the matrices B_i were not chosen randomly, the coordinates have been picked to match up with the $\mathbf{a}_i$ vectors from the first item. Assume $\varphi(B_i) = \mathbf{a}_i$ for all i and in that case, the linear transformation R' which switches basis in terms of symmetric matrices is the composition $(\varphi^{-1} \circ R \circ \varphi)$ where R is the original change of basis matrix from the first item.

Exercise 3. Let's see how this looks like for the infinite-dimensional case. The upshot of this problem is that infinite-dimensional spaces, obviously, do not have a finite basis but that a space can have both countable and uncountable bases!

As an example, let's consider the vector space of sequences, i.e.,

$$V = \{ (q_1, q_2, q_3, ...) : q_i \in \mathbb{R} \}.$$

Let us think about bases of this space, i.e., sets of vectors $\mathbf{a}_i \in V$ so that every $v \in V$ can again be written as $\mathbf{v} = \sum_i \alpha_i \mathbf{a}_i$[2]

(i) Convince yourself that the set $\{ \mathbf{a}_i \}_{i=1}^{\infty}$ where

$$\mathbf{a}_1 = (1, 0, 0, \dots), \quad \mathbf{a}_2 = (0, 1, 0, \dots), \quad \mathbf{a}_3 = (0, 0, 1, \dots), \quad \text{and so on}$$

is a basis of V . (To "convince" yourself, look up the formal properties of a basis.) It is obviously countable.

(ii) Create a second countable basis of your choice.

(iii) Can you somehow describe the operator R that translates between these two bases, in the same way as was done in the previous problem?

(iv) Now convince yourself that the set of vectors $\{ \mathbf{b}_\lambda \}_{\lambda \in [0,1]}$ where

$$\mathbf{b}_\lambda = (1, \lambda, \lambda^2, \lambda^3, \dots)$$

is also a basis. This is not a countable basis because the set is indexed by the real number λ!]

[2] This may not be obvious at first: Being able to write $\mathbf{v} = \sum_i \alpha_i \mathbf{a}_i$ with an infinite sum requires that the infinite sum makes sense - which we will interpret as saying that $\lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \mathbf{a}_i \to \mathbf{v}$. This in turn requires that we can measure convergence in V, which requires that we have a *norm*. That is, bases in infinite-dimensional spaces inherently only make sense if the vector space V is a *normed* vector space! For the case here, let us assume that the norm on V is $\|\mathbf{v}\| = \sup_i |v_i|$. That is, we take $V = \ell_\infty$.

For cases like this, one has to think about what it means to expand a vector in this basis. Before, we had that for every vector $\mathbf{v} \in V$, we can write $\mathbf{v} = \sum_i \alpha_i \mathbf{a}_i$. With the uncountable basis here, this has to be replaced by $\mathbf{v} = \int_0^1 \beta_\lambda \mathbf{b}_\lambda \mathrm{d}\lambda$.

(v) Can you come up with a description of the basis transformation operator R for these two bases?

Answer

(i) Intuitively we may think of the $\mathbf{a}_i$'s as a basis for the space of sequences. This is because we can decompose a sequence into its components:

$$(q_1, q_2, q_3, \dots) = q_1(1, 0, 0, \dots) + q_2(0, 1, 0, \dots) + q_3(0, 0, 1, \dots) + \dots$$

And the $\mathbf{a}_i$'s are linearly independent because the only way to get the zero sequence as a linear combination of them is to have $q_i = 0$ for all i.

This type of basis is not a Hamel basis nor a Schauder basis, as we need finite linear combinations for the first type and a notion of convergence for the second one. Observe that if we were to restrict the space to the sequences which are eventually zero then we do have a formal Schauder basis.

(ii) Another "basis" could be the sequences

$$\mathbf{b}_1 = \mathbf{a}_1,\ \mathbf{b}_2 = (1, 1, 0, 0, \dots),\ \mathbf{b}_3 = (0, 1, 1, 0, \dots),\ \mathbf{b}_4 = (0, 0, 1, 1, \dots). \dots$$

Once again we have a linearly independent set because we can induct on the sets of vectors of the form $e_{i-1} + e_i$ on finite dimension in order to see they are l.i. and transfer the arguemnt inductively to this set of vectors. We can show that this set is a generating set for the space of sequences by expanding the $\mathbf{a}_i$'s as a linear combination of the $\mathbf{b}_i$'s and then expanding the sequence normally.

(iii) The operator which transfers from the $\mathbf{b}$ basis to the $\mathbf{a}$ basis can be "represented" as an infinite matrix whose columns are the $\mathbf{b}$ sequences. This operator can be explicity described as

$$\mathbf{a}_1 \mapsto \mathbf{b}_1, \quad \mathbf{a}_i \mapsto \mathbf{a}_{i-1} + \mathbf{a}_i,\ i \geqslant 2.$$

(iv) The sequences $\mathbf{b}_\lambda$ can be seen to be eigenvectors of the operator

$$L(a_0, a_1, a_2, a_3, \dots) = (a_1, a_2, a_3, a_4, \dots).$$

Each one has a different eigenvalue $\lambda \in [0,1]$, so as eigenvectors corresponding to different eigenvalues of an operator are l.i., we have that the $\mathbf{b}_\lambda$'s are l.i.

There's an infinite number of such eigenvectors and they generate a space with uncountable dimension. With my current tools, I'm unable to prove that sequence can be written as a (possibly) infinite linear combination of such sequences. Observe that the operator L is not compact so we are not able to deduce, using the spectral theorem, that the eigenvectors of L form a basis. The most we can conclude is that all eigenvectors are l.i.

(v) A basis transformation operator has to be bijective. As $\{ \mathbf{a}_i \}_{i\in\mathbb{N}}$ is countable and $\{ \mathbf{b}_\lambda \}_{\lambda\in[0,1]}$ is uncountable, no such operator exists.

Exercise 4. Let's repeat the previous problem once more for spaces of functions. Concretely, take

$$V = C^0 = \{ f : [0,1] \to \mathbb{R},\ f \quad \text{is continuous} \},$$

with the norm $\|f\| = \sup_{[0,1]} |f(x)|$.

i) Is $\mathbf{a}_n = \sin(\pi n x)$ a countable basis?

ii) Is

$$\mathbf{b}_\lambda = \begin{cases} 1 & \text{if} \quad x = \lambda \\ 0 & \text{otherwise} \end{cases}$$

for $\lambda \in [0,1]$ an uncountable basis? (If you think about exapnding a function as $\mathbf{v} = \int_0^1 \beta_\lambda \mathbf{b}_\lambda \mathrm{d}\lambda$, you will probably want to replace $\mathbf{b}_\lambda$ by a delta function, $\mathbf{b}_\lambda(x) = \delta(x-\lambda)$. If you don't know what that means, just ignore the difference.)

Answer

i) Via the Fourier series theorem, we know that the sine waves form a basis of $L^2[0,1]$. Observe also that any function in C^0 is bounded since its a continuous function defined on a compact set. If $p > 0$ then

$$\|f\|^p_{L^p[0,1]} = \int_0^1 |f|^p \mathrm{d}x \leqslant (\sup_{[0,1]} |f|)^p \int_0^1 \mathrm{d}x < \infty$$

which in particular means that $f \in V$ implies f being in L^2. As sine waves can represent any function in L^2, they can also represent any continuous function as desired.

ii) Observe that the function $\mathbf{b}_\lambda$ is not continuous for any x so it cannot form a basis of C^0. Intuitively it should be the case that those functions do form a basis, as we could "decompose" a function pointwise for all $\lambda \in \mathbb{R}$ and then write it down as

$$f(x) = f(0)\mathbf{b}_0(x) + \cdots + f(\lambda)\mathbf{b}_\lambda(x) + \cdots + f(1)\mathbf{b}_1(x) \text{``}=\text{''} \int_0^1 f(t)\mathbf{b}_t(x)\mathrm{d}t$$

for $\lambda \in [0,1]$. Moreover observe that if wanted to decompose a function in terms of delta functions it is true that

$$\int_a^b f(t)\delta(x-t)\mathrm{d}t = f(x)\mathbf{1}_{[a,b]}(x).$$

However we cannot say that the delta functions form a basis of L^2 and even less of C^0, this is because delta functions do not belong in L^2.

Exercise 5 (Operators on ℓ_2). Recall that the set of sequences $V = \{ (q_1, q_2, \dots) : q_i \in \mathbb{R} \}$. When equipped with the norm $\|\mathbf{v}\| = \sum_i |q_i|^2$ then this defines the normed vector space

$$\ell_2 = (V, \|\cdot\|) = \{ \mathbf{v} = (v_1, v_2, \dots) : \ v_i \in \mathbb{R}, \ \|\mathbf{v}\| < \infty \}.$$

Now consider the set of linear operators $\ell_2 \to \ell_2$, i.e., linear operators that map a sequence in ℓ_2 to another sequence in ℓ_2.

Provide three examples of such operators. Argue that the set of linear operators $\ell_2 \to \ell_2$ is (i) a vector space, (ii) infinite dimensional.

Answer

We can consider the shift operators on ℓ_2:

$$T(v_1, v_2, v_3, \dots) = (v_2, v_3, v_4, \dots), \quad \text{and} \quad T_n = T \circ T \circ \cdots \circ T \quad n \text{ times}.$$

We could also consider a type of "matrix multiplication" operator. For $(c_{mn})_{m,n\in\mathbb{N}} \subseteq \mathbb{R}$ with the property that all "rows" and "columns" are uniformly in ℓ_1, i.e.

$$\sup_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} |c_{mn}| < \infty, \quad \text{and} \quad \sup_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} |c_{mn}| < \infty$$

we can define the operator $T\mathbf{v}$ pointwise as:

$$(T\mathbf{v})_k = \sum_{\ell\in\mathbb{N}} c_{k\ell} v_\ell.$$

Also, summing a particular sequence or multiplying a sequence by a particular scalar define families of operators on l_2.
Defining sum and scalar product of operators over l_2 is analogous to defining it for functions. For two operators S, T their sum $(S + T)$ is the pointwise sum. Likewise for $c \cdot T$, this is just multiplying the scalar after applying the operator to a sequence.
The previous families of operators are all infinite, between families of operators we have linear independence and so the dimension of $\{ T : \ell_2 \to \ell_2 \}$ must be infinite.