Exercise 1. Recall $\mathbb{P}^n$ is defined as (a set) ${}^{\mathbb{C}^{n+1}\setminus\{\mathbf{0}\}}/_{\sim}$ where $\mathbf{x} \sim \lambda\mathbf{x}$ for $\lambda \neq 0$. For $d \in \mathbb{Z}$ $\mathcal{O}_{\mathbb{P}^n}(d)$ is defined as ${}^{\mathbb{C}^{n+1}\setminus\{\mathbf{0}\}\times\mathbb{C}}/_{\sim}$ where $(\mathbf{x}, t) \sim (\lambda\mathbf{x}, \lambda^d t)$ and $\lambda \neq 0$.

The map $\pi : \mathcal{O}_{\mathbb{P}^n}(d) \to \mathbb{P}^n$ forgets the last coordinate, what are the fibers of the map π? What is another way of writing this space when $d = 0$?

Answer

The fibers of the map are

$$\pi^{-1}(\mathbf{x}) = \{ (\mathbf{x}, t) : \pi(\mathbf{x}, t) = \mathbf{x} \} = \{ (\mathbf{x}, t) : t \in \mathbb{C} \} \simeq \mathbb{C}.$$

When $d = 0$ the relation in $\mathcal{O}_{\mathbb{P}^n}(0)$ is $(\mathbf{x}, t) \sim (\lambda\mathbf{x}, t)$ for $\lambda \neq 0$. Such points lie in $\mathbb{P}^n \times \mathbb{C}$.

Exercise 2. Show that the map $\pi : \mathcal{O}_{\mathbb{P}^n}(d) \to \mathbb{P}^n$ is a vector bundle by finding a local trivialization (U_i, ϕ_i).

Using this trivialization, what are the maps

$$psi_{ij} : U_i \cap U_j \times \mathbb{C} \to U_i \cap U_j \times \mathbb{C}$$

where recall that ψ_{ij} is defined to be $\phi_j \circ \phi_i^{-1} \mid_{\pi^{-1}(U_i \cap U_j)}$.

Answer

Take an open chart of P_n, then

$$\pi^{-1}(U_i) = {}^{\{ (\mathbf{x}, t) : x_i \neq 0 \}}\Big/_{\sim}$$

where $(\mathbf{x}, t) \sim (\lambda\mathbf{x}, \lambda^d t)$ for $\lambda \neq 0$. Then the maps ϕ_i are

$$\begin{cases} \phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}, \ (\mathbf{x}, t) \mapsto ([\mathbf{x}], t/x_i^d), \\ \phi_i^{-1} : U_i \times \mathbb{C} \to \pi^{-1}(U_i), \ ([\mathbf{x}], t) \mapsto (\mathbf{x}, t x_i^d). \end{cases}$$

These maps are well defined on equivalence classes. For ϕ_i, take another representative $(\lambda\mathbf{x}, \lambda^d t)$, then

$$\phi_i(\lambda\mathbf{x}, \lambda^d t) = ([\lambda\mathbf{x}], \lambda^d t/(\lambda x_i)^d) \sim ([\mathbf{x}], t/x_i^d).$$

On the other hand

$$\phi_i^{-1}([\lambda\mathbf{x}], t) \mapsto (\lambda\mathbf{x}, (\lambda x_i)^d t) \sim (\mathbf{x}, x_i^d t).$$

Finally the transition maps are

$$\begin{aligned}\psi_{ij} : U_i \cap U_j \times \mathbb{C} &\to \pi^{-1}(U_i \cap U_j) \to U_i \cap U_j \times \mathbb{C},\\ ([\mathbf{x}], t) &\mapsto (\mathbf{x}, x_i^d t) \mapsto ([\mathbf{x}], (x_i/x_j)^d t).\end{aligned}$$

Exercise 3. A *section* of $\mathcal{O}_{\mathbb{P}^n}(d)$ is a morphism $s : \mathbb{P}^n \to \mathcal{O}_{\mathbb{P}^n}(d)$ such that $\pi \circ s$ is the identity map on $\mathbb{P}^n$. The space of sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ is a finite dimensional vector space for each d. Find a basis for the space of sections of $\mathcal{O}_{\mathbb{P}^n}(d)$.

Answer

Any section is of the form

$$s : \mathbb{P}^n \to \mathcal{O}_{\mathbb{P}^n}(d), \; [\mathbf{x}] \mapsto (\mathbf{x}, \tilde{s}(\mathbf{x}))$$

where $\tilde{s} : \mathbb{P}^n \to \mathbb{C}$ is homogeneous of degree d because of the defining relationship of $\mathcal{O}_{\mathbb{P}^n}(d)$:

$$(\lambda\mathbf{x}, \tilde{s}(\lambda\mathbf{x})) = (\lambda\mathbf{x}, \lambda^d \tilde{s}(\mathbf{x})).$$

This means that the space of sections must have all the degree d monomials in variables $x_0, \ldots, x_n$ as a basis.

Exercise 4. Fix some $d > 1$. Consider the map from $\mathbb{P}^n$ to a (larger) projective space defined by the complete linear series $|\mathcal{O}_{\mathbb{P}^n}(d)|$. Prove that this map is the same as the Veronese embedding.

Answer

I sadly was not able to do exercise 4, nor 5 :(