Exercise 1. Let $H \subseteq \mathbb{P}^n$ be a hyperplane. H is defined as the zero locus of a linear equation

$$H = \mathbb{V}(a_0 x_0 + \cdots + a_n x_n),\ a_0, \ldots, a_n \in \mathbb{C}.$$

Prove that $H \simeq \mathbb{P}^{n-1}$. (Also, be comfortable with the fact that $\mathbb{P}^n \backslash L \simeq \mathbb{A}^n$.)

Answer

Exercise 2. Let $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$ be the Veronese embedding of degree 2. Write out the equations defining the image of ν_2.

Answer

Recall that this particular Veronese map takes a point in $\mathbb{P}^2$ to all possible monomials of degree 2 in $\mathbb{P}^5$. This means that

$$\nu_2([u : v : w]) = \left[u^2 : v^2 : w^2 : uv : vw : wu\right].$$

Recall that the image of the Veronese map is defined using a multi-indexed array, so let us relabel in that sense:

$$\left[u^2 : v^2 : w^2 : uv : vw : wu\right] = \left[z_{2,0,0} : z_{0,2,0} : z_{0,0,2} : z_{1,1,0} : z_{0,1,1} : z_{1,0,1}\right].$$

The defining equations for the image are given by

$$z_I z_J = z_K z_L,\ I + J = K + L,$$

where $I, \ldots, L$ are our multi-indices. The only ways to sum our multi-indices in a non-trivial manner are:

$$\begin{cases} (2,0,0) + (0,1,1) = (1,1,0) + (1,0,1) \\ (0,2,0) + (1,0,1) = (1,1,0) + (0,1,1) \\ (0,0,2) + (1,1,0) = (1,0,1) + (0,1,1) \end{cases} \begin{cases} (2,0,0) + (0,2,0) = (1,1,0) + (1,1,0) \\ (2,0,0) + (0,0,2) = (1,0,1) + (1,0,1) \\ (0,2,0) + (0,0,2) = (0,1,1) + (0,1,1) \end{cases}$$

If we name $[a : b : c : d : e : f]$ the coordinates of $\mathbb{P}^5$, then we have the following system of equations

$$\begin{cases} ae = df \\ bf = de \\ cd = ef \end{cases} \begin{cases} ab = d^2 \\ bc = e^2 \\ ca = f^2 \end{cases}$$