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8. (3-) [8 points] Prove that all conics in the projective plane $\mathbb{P}^2_{\mathbb{R}}$ are equivalent up to a projective transformation. That is, prove that any two conics defined by quadratic equations of the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$$

can be transformed into one another by applying a projective transformation to the homogeneous coordinates $(x : y : z)$.

First assume we are working over $\mathbb{R}^3$, then we'll adjust to $\mathbb{P}^2$.

$\rightarrow$ Consider two quadratic forms $\begin{cases} ax^2 + by^2 + \ldots + fyz \\ a'x^2 + b'y^2 + \ldots + f'yz \end{cases}$

We may write them as $\vec{x}^T A \vec{x}$ where $\vec{x} = (x, y, z)$ and

$$A = \begin{pmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{pmatrix}$$

As A is symmetric, the spectral theorem guarantees the existance of $P \in O_3(\mathbb{R})$ s.t. $D = P^T A P$ where D is diagonal.

Furthermore, we can normalize the columns of P and rearrange them so that $P \in SO_3(\mathbb{R})$.

↳ Rearranging the columns just switches the positions of the eigenvalues in D, while normalizing rescales the columns of P. This process doesn't change the basis of eigenvectors.

$\rightarrow$ Also, as $SO_3 \leq GL_3$, when taking the quotient by $M \sim \lambda M$, we have that $SO_3/\sim \; \leq GL_3/\sim \; = PGL_2$ which means that these matrices also define projective changes of coordinates.

↳ In fact $SO_3 \cong SO_3/\sim \; = SO_3/\{I\}$

We may change coordinates by taking $\vec{x} = P\vec{u}$, $\vec{u} = (u,v,w)$. So that

$$\vec{x}^T A \vec{x} = (P\vec{u})^T A (P\vec{u}) = \vec{u}^T P^T A P \vec{u} = \vec{u}^T D \vec{u} = \lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2$$

And in a similar fashion we can orthogonally diagonalize A' to obtain $P' \in SO_3$ and D' diagonal, such that $P'^T A' P' = D'$.

This gives rise to the quadratic form $\lambda_1' p^2 + \lambda_2' q^2 + \lambda_3' r^2$.

Now we wish to find a way to transform between the 2 quadratic forms without mixed terms.

But beware, (u,v,w) and (p,q,r) are different coordinates.

In fact, this will help us to actually change coordinates.

If we take $u = \frac{\sqrt{\lambda_1'}}{\sqrt{\lambda_1}} p$, $v = \frac{\sqrt{\lambda_2'}}{\sqrt{\lambda_2}} q$, $w = \frac{\sqrt{\lambda_3'}}{\sqrt{\lambda_3}} r$ which corresponds to the matrix equation $\vec{u} = \operatorname{diag}\left(\frac{\sqrt{\lambda_1'}}{\sqrt{\lambda_1}}, \frac{\sqrt{\lambda_2'}}{\sqrt{\lambda_2}}, \frac{\sqrt{\lambda_3'}}{\sqrt{\lambda_3}}\right)\vec{p} =: C\vec{p}$

So from the quadratic form $\lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2$ we have

$C = C^T$ Subtle reference of Center for Disease Control

$$\vec{u}^T D \vec{u} = (C\vec{p})^T D (C\vec{p}) = \vec{p}^T (CDC) \vec{p}$$ and

$$\begin{pmatrix} \frac{\sqrt{\lambda_1'}}{\sqrt{\lambda_1}} & & \\ & \frac{\sqrt{\lambda_2'}}{\sqrt{\lambda_2}} & \\ & & \frac{\sqrt{\lambda_3'}}{\sqrt{\lambda_3}} \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{\lambda_1'}}{\sqrt{\lambda_1}} & & \\ & \frac{\sqrt{\lambda_2'}}{\sqrt{\lambda_2}} & \\ & & \frac{\sqrt{\lambda_3'}}{\sqrt{\lambda_3}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\lambda_1'}}{\sqrt{\lambda_1}} & & \\ & \frac{\sqrt{\lambda_2'}}{\sqrt{\lambda_2}} & \\ & & \frac{\sqrt{\lambda_3'}}{\sqrt{\lambda_3}} \end{pmatrix} \begin{pmatrix} \lambda_1 \frac{\sqrt{\lambda_1'}}{\sqrt{\lambda_1}} & & \\ & \lambda_2 \frac{\sqrt{\lambda_2'}}{\sqrt{\lambda_2}} & \\ & & \lambda_3 \frac{\sqrt{\lambda_3'}}{\sqrt{\lambda_3}} \end{pmatrix} = \begin{pmatrix} \lambda_1 \frac{\lambda_1'}{\lambda_1} & & \\ & \lambda_2 \frac{\lambda_2'}{\lambda_2} & \\ & & \lambda_3 \frac{\lambda_3'}{\lambda_3} \end{pmatrix} = \begin{pmatrix} \lambda_1' & & \\ & \lambda_2' & \\ & & \lambda_3' \end{pmatrix}$$

Which means that the product CDC is the diagonal matrix D'.

Our quadratic form is now $\vec{p}^T D' \vec{p}$ which we may transfer back to $\vec{x}$ coordinates by the change of variables $\vec{x} = P'\vec{p} \implies P'^T \vec{x} = \vec{p}$

$$\implies \vec{p}^T D' \vec{p} = (P'^T \vec{x})^T D' (P' \vec{x}) = x^T (P' D' P'^T) \vec{x} = \vec{x}^T A' \vec{x}$$

In summary, the desired change of coordinates is

$\vec{x} = P\vec{u}$, $\vec{u} = C\vec{p}$, $\vec{p} = P'^T \vec{x} \implies P'C^{-1}P^T$ is our change of coords.

Observe that in all the steps of the argument, the matrices are still projective changes of coordinates.

* Also, if it happens that one of the matrices has zero as an eigenvalue, it suffices to set the respective entry of the matrix $C = \operatorname{diag}\left(\frac{\sqrt{\lambda'_1}}{\sqrt{\lambda_1}}, \frac{\sqrt{\lambda'_2}}{\sqrt{\lambda_2}}, \frac{\sqrt{\lambda'_3}}{\sqrt{\lambda_3}}\right)$ to zero when doing the change of basis and when going backwards, instead of dividing we just multiply.

7. (2-) [3 points] Prove that a projective transformation of $\mathbb{P}^1$ is uniquely determined by where it sends three points (say, $(0:1), (1:0), (1:1)$).

$\longrightarrow$ Suppose $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ is a projective transformation with

$$\begin{cases} Te_1 = p = [p_1 : p_2] \\ Te_2 = q = [q_1 : q_2] \\ Te = r = [r_1 : r_2] \end{cases}$$

Here $e_1 = [1:0]$, $e_2 = [0:1]$ and $e = [1:1]$

We wish to determine $t_{11}, \dots, t_{22}$.

These equations give us the following:

as affine pts.

$$Te_1 = \begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix} = [p_1 : p_2] \implies \exists\, m_1 \neq 0 \left((t_{11}, t_{21}) = m_1 (p_1, p_2) \right)$$

$$\wedge\ Te_2 = \begin{pmatrix} t_{12} \\ t_{22} \end{pmatrix} = [q_1 : q_2] \implies \exists\, m_2 \neq 0 \left((t_{12}, t_{22}) = m_2 (q_1, q_2) \right)$$

$\hookrightarrow$ In other words, columns are proportional to images.

* Observe that just knowing the ratios columnwise doesn't determine T because we could have

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \longrightarrow \begin{array}{cc} \times 2 & \times 3 \\ \end{array} \begin{pmatrix} 2 & 6 \\ 6 & 12 \end{pmatrix},$$ but no λ satisfies $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \lambda \begin{pmatrix} 2 & 6 \\ 6 & 12 \end{pmatrix}$

Now the preceding relations imply that $T = \begin{pmatrix} m_1 p_1 & m_2 q_1 \\ m_1 p_2 & m_2 q_2 \end{pmatrix}$ where p_1, p_2, q_1, q_2 are known. So, to determine T, it suffices to determine m_1, m_2.

Observe also that $T = (MP^T)^T$ where

$$M = \operatorname{diag}(m_1, m_2), \quad P = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}$$

$$\left(\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}^T \right)^T = \left(\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \right)^T = \begin{pmatrix} m_1 p_1 & m_1 p_2 \\ m_2 q_1 & m_2 q_2 \end{pmatrix}^T = \begin{pmatrix} m_1 p_1 & m_2 q_1 \\ m_1 p_2 & m_2 q_2 \end{pmatrix}$$

The last condition is $Te = r$, so we have

$$Te = r \Rightarrow \begin{pmatrix} m_1 p_1 & m_2 q_1 \\ m_1 p_2 & m_2 q_2 \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} r_1 \\ \vdots \\ r_2 \end{pmatrix} \Rightarrow \begin{cases} p_1 m_1 + q_1 m_2 = r_1 \\ p_2 m_1 + q_2 m_2 = r_2 \end{cases} \Rightarrow P\begin{pmatrix} m_1 \\ \vdots \\ m_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ \vdots \\ r_2 \end{pmatrix}$$

Observe that P's columns are lin. indep. because, in particular, T is a linear transformation. Thus, P^{-1} exists, and so:

$$P\vec{m} = \vec{r} \Rightarrow \vec{m} = P^{-1}\vec{r}$$

This determines m_1, m_2 up to projective equivalence because we know P and $\vec{r}$.

Ex.: Assume $Te_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $Te_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $Te = \begin{pmatrix} 11 \\ 12 \end{pmatrix}$ then

Then $\exists m_1, m_2 \neq 0$ s.t. $\begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix} = m_1 \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5m_1 \\ 3m_1 \end{pmatrix}$, $\begin{pmatrix} t_{12} \\ t_{22} \end{pmatrix} = \begin{pmatrix} 3m_2 \\ 2m_2 \end{pmatrix}$

Now $Te = \begin{pmatrix} 11 \\ 12 \end{pmatrix} \Rightarrow \begin{pmatrix} t_{11} + t_{12} \\ t_{21} + t_{22} \end{pmatrix} = \begin{pmatrix} 5m_1 + 3m_2 \\ 3m_1 + 2m_2 \end{pmatrix} = \begin{pmatrix} 11m_3 \\ 12m_3 \end{pmatrix}$, $m_3 \neq 0$

$$\Rightarrow \begin{cases} 5m_1 + 3m_2 - 11m_3 = 0 \\ 3m_1 + 2m_2 - 12m_3 = 0 \end{cases} \Rightarrow \begin{cases} m_1 = -14m_3 \\ m_2 = 27m_3 \end{cases}$$

$$\Rightarrow T = \begin{pmatrix} -70m_3 & 81m_3 \\ -42m_3 & 54m_3 \end{pmatrix} = \begin{bmatrix} -70 & 81 \\ -42 & 54 \end{bmatrix}$$

We are left with checking T is unique, so assume that

$$\begin{cases} Te_1 = p \\ Te_2 = q \\ Te = r \end{cases} \quad \text{and} \quad \begin{cases} \tilde{T}e_1 = p \\ \tilde{T}e_2 = q \\ \tilde{T}e = r \end{cases}$$

Call $S = T^{-1}\tilde{T}$, then e_1, e_2 and e are fixed by S. The conditions $Se_i = e_i$ implies S is a diagonal matrix. And $Se = e$ now gives us that S is a scalar multiple of id. which means that T is unique up-to-scaling or in other words T is a unique projective transformation.