Exercise 1 (Exercise 3)**.** Let c_n be the number of sequences of length n in which:

- ⋄ Each number is one of $0, 1, 2, 3$.
- ⋄ No two 3's are consecutive.

For instance 0221030132 is a valid sequence but 033112333 is not.

i) Find a recursion for c_n (this should be similar to the Fibonacci recurrence). Remember to include the initial conditions!
ii) Use the recursion to find a closed form for the generating function of c_n.
iii) Use the formula discussed in class for solving linear recurrences to find an explicit formula for c_n in terms of n.

Answer

i) First, we can observe that the total number of sequences of length n with characters $0, \dots, 3$ is 4^n. Dividing this amount into conditioned sequences and forbidden sequences we get the following

$$4^n = c_n + f_n,$$

where c_n is the quantity we are looking for and f_n is the number of *forbidden* sequences of length n. This is, sequences which do have 33 as a substring.

After considering the initial conditions:

$$f_0 = 0,\ f_1 = 0,\ f_2 = 1,\ f_3 = 7,\ f_4 = 40,$$

it is possible to conjecture that

$$f_{n+2} = 3f_{n+1} + 3f_n + 4^n.$$

To prove this recurrence we will append a digit to a length $(n+1)$ sequence. There are several ways to this:

- ⋄ If the digit we are appending is either 0, 1 or 2, we are not adding any more forbidden substrings. So for each of those digits, our count of forbidden sequences goes up by f_{n+1}. Right now we have $3f_{n+1}$ forbidden sequences.
- ⋄ If the digit we are appending is 3, there are two cases:
  - Either the last digit of the $(n+1)$ sequence is 0, 1 or 2 in which case we are not adding more forbidden strings. Each of those possibilities accounts for f_n forbidden sequences. This adds up to our past total to get $3f_{n+1} + 3f_n$.

– If the last digit of the $(n+1)$ sequence is 3, then we just added a forbidden substring. Counting this is the same as counting all $(n+2)$ strings which end in 33. This amount is 4^n.

In total, we have $3f_{n+1} + 3f_n + 4^n$ forbidden sequences of length n. Then by our initial relation we have

$$c_n = 4^n - f_n = -(3f_{n+1} + 3f_n),\ f_0 = 0,\ f_1 = 0.^{a}$$

ii) Let us derive a generating function for f_n and from that we will obtain c_n's generating function.

Call F, (f_n)'s generating function, then the recurrence

$$f_{n+2} = 3f_{n+1} + 3f_n + 4^n$$

translates to the equation

$$\frac{F(x) - f_0 - f_1 x}{x^2} = \frac{3(F(x) - f_0)}{x} + 3F(x) + \frac{1}{1-4x}.$$

Applying the initial conditions we get

$$\frac{F(x)}{x^2} = \frac{3F(x)}{x} + 3F(x) + \frac{1}{1-4x}.$$

We can solve for F to obtain

$$\begin{aligned} F(x)\left(\frac{1}{x^2} - \frac{3}{x} - 3\right) &= \frac{1}{1-4x}, \\ \Rightarrow F(x)\left(\frac{1-3x-3x^2}{x^2}\right) &= \frac{1}{1-4x}, \\ \Rightarrow F(x) &= \frac{1}{1-4x}\left(\frac{x^2}{1-3x-3x^2}\right). \end{aligned}$$

Now let us factor $1 - 3x - 3x^2$ by taking $x = \frac{-3\pm\sqrt{21}}{6}$, where α is the root with $+$ while β, the one with negative. Then

$$\begin{aligned} (1-3x-3x^2) &= -3(x-\alpha)(x-\beta) = -3\alpha\beta\left(1 - \frac{x}{\alpha}\right)\left(1 - \frac{x}{\beta}\right) \\ &= \frac{-3}{ab}(1-ax)(1-bx), \end{aligned}$$

where a, α and b, β are pairs of reciprocals. We can now continue to solve F as a sum of partial fractions as follows

$$F(x) = \frac{-abx^2}{3(1-4x)(1-ax)(1-bx)} = \frac{A}{1-ax} + \frac{B}{1-bx} + \frac{C}{1-4x}.$$

Homogenizing the denominator on the equation to the right we get

$$\frac{-ab}{3}x^2 = A(1-4x)(1-bx) + B(1-4x)(1-ax) + C(1-ax)(1-bx).$$

Since this equation holds for any value of x, we might substitute certain values to get cleaner equations for A, B and C:

$$\begin{cases} (x=\alpha) \Rightarrow \dfrac{-ab}{3}\alpha^2 = A(1-4\alpha)(1-b\alpha) = A\alpha^2(4-a)(b-a), \\ \qquad\qquad \Rightarrow \dfrac{-ab}{3(4-a)(b-a)} = A, \\ (x=\beta) \Rightarrow \dfrac{-ab}{3}\beta^2 = B(1-4\beta)(1-a\beta) = B\beta^2(4-b)(a-b), \\ \qquad\qquad \Rightarrow \dfrac{-ab}{3(4-b)(a-b)} = B, \\ (x=1/4) \Rightarrow \dfrac{-ab}{3\cdot 16} = C(1-a/4)(1-b/4) = \dfrac{C}{16}(4-a)(4-b), \\ \qquad\qquad \Rightarrow \dfrac{-ab}{3(4-a)(4-b)} = C. \end{cases}$$

Comparing coefficients $(1-ax)(1-bx) = 1-(a+b)x+abx^2$, we have that $a+b=3$ and $ab=-3$. Expanding $(4-a)(4-b) = 16-4(a+b)+ab = 16-12-3=1$. From this we get $\underline{C=1}$. Also, using the polarization identity it holds that

$$|a+b|^2 - |a-b|^2 = 4ab \Rightarrow |a-b|^2 = -(-12-9) = 21 \Rightarrow |a-b| = \sqrt{21}.$$

We also have that $\beta < -\frac{1}{2} < \alpha$, so $b > -2 > a$. This means that $a-b = -\sqrt{21}$. From this we can replace in A and B's expressions:

$$A = \frac{1}{(4-a)\sqrt{21}},\ B = \frac{-1}{(4-b)\sqrt{21}}$$

so in the end we have the expression for F:

$$F(x) = \left(\frac{1}{(4-a)\sqrt{21}}\right)\frac{1}{1-ax} + \left(\frac{-1}{(4-b)\sqrt{21}}\right)\frac{1}{1-bx} + \frac{1}{1-4x}.$$

If $C(x)$ is (c_n)'s generating function, then the following equation holds $C(x) + F(x) = 1/(1-4x)$ which means that

$$C(x) = \left(\frac{1}{(4-b)\sqrt{21}}\right)\frac{1}{1-bx} + \left(\frac{-1}{(4-a)\sqrt{21}}\right)\frac{1}{1-ax}.$$

iii) Combining the terms in F's generating function we get

$$F(x) = \sum_{n=0}$$

[a]Even though the recurrence is not in terms of c's, it's still a recursive formula. The derivation for c_n's recursive formula lies below this answer.

We can also construct the recurrence in terms of the allowed sequences c_n. Take any length n allowed sequence, then there are two possibilities:

- ◇ The last digit is 0, 1 or 2, then the rest of the sequence is a length $(n-1)$ allowed sequence. For each digit we count c_{n-1} allowed sequences. So in total we have $3c_{n-1}$ allowed sequences.
- ◇ If the last digit is 3, then the second-to-last digit can't be three. There are only 3 other possibilities: 0, 1 or 2. For each of these the remaining length n sequence has to fullfil the condition. Which means we count c_{n-2} allowed sequences per digit.

This total amounts to $c_n = 3c_{n-1} + 3c_{n-2}$. With this recurrence we have counted all the possibilities since the only options for the last digit are the ones mentioned above.

Exercise 2 (Exercise 5)**.** Carlitz defined the *q-Stirling numbers* of the second kind as follows:

Given a set partition B of $[n]$ into k blocks, let its blocks be $B_1, B_2, \ldots, B_k$ where the blocks are written in order by their minimum element from least to greatest. Then an inversion of B is a pair (b, B_j) where b is in some block B_i to the left of B_j (that is, $i < j$) and $b > \min(B_j)$.

For instance, in the set partition

$$\{1,3,6\},\{2,4\},\{5,7\},$$

we have the inversions $(3, \{2, 4\})$, $(6, \{2, 4\})$ and $(6, \{5, 7\})$. We write $\operatorname{inv}(B)$ to denote the total number of inversions in B, and define

$$S_q(n, k) = \sum_B q^{\operatorname{inv}(B)}$$

with B ranging over all set partitions of $[n]$ into k blocks.

Prove that the q-Stirling numbers satisfy the recursion

$$S_q(n, k) = S_q(n-1, k-1) + (1 + q + q^2 + \cdots + q^{k-1}) S_q(n-1, k).$$

Answer

It is paramount to understand that even if S_q is "$q$ounting" the set partitions of $[n]$ into k blocks by inversions, in the end S_q is a polynomial. So to prove this identity we will separate the set of set partitions into partitions where n is its own block, and partitions where $\exists j(n \in B_j)$. We have the following

$$S_q(n, k) = \sum_{B \in \mathcal{O}} q^{\operatorname{inv}(B)} + \sum_{B \in \mathcal{I}} q^{\operatorname{inv}(B)}$$

where we have defined sets $\mathcal{O}, \mathcal{I}$ in the following manner:

- $\mathcal{O}$ is the set of *the set of set partitions of* $[n]$ *into* k *blocks* such that n is OUT of other blocks, in this sense $\exists j(B_j = \{n\})$ for $B \in \mathcal{O}$.
- On the other hand $\mathcal{I}$ is *the set of set partitions of* $[n]$ *into* k *blocks* such that n is IN one of the k blocks.
- We can further decompose $\mathcal{I} = \dot{\bigcup}_{j=1}^{k} \mathcal{I}_j$ where $\mathcal{I}_j$ is *the set of set partitions of* $[n]$ *into* k *blocks* such that $n \in B_j$ (IN block j). (Recall that the B_j's are ordered by $\min(B_j)$.)

We can find a bijection between $\mathcal{O}$ and *the set of set partitions of* $[n-1]$ *into* $k-1$ *blocks* given by

$$F : \mathcal{O} \to (\text{aforementioned set}),\ A \mapsto A \backslash \{n\}$$

with its inverse being $B \mapsto B \cup \{n\}$. This function is well defined since every $A \in \mathcal{O}$ can be seen as a set $\widetilde{A} \cup \{n\}$. Thus

$$\sum_{B \in \mathcal{O}} q^{\operatorname{inv}(B)} = \sum_{F(B) \in F[\mathcal{O}]} q^{\operatorname{inv}(F(B))} = \sum_B q^{\operatorname{inv}(B)} = S_q(n-1, k-1),$$

where the third sum runs B through *the set of set partitions of* $[n-1]$ *into* $k-1$ *blocks* and this, by definition, is $S_q(n-1, k-1)$.

Let us now consider a partition $B \in \mathcal{I}_\ell$, we can associate B to a partition of $[n-1]$ with k blocks by removing n from B_j.