Exercise 1 (Exercise 2)**.** Compute the following:

i) $s_{(5,3,3,1)}s_{(4)}$.

ii) $s_{(2,1)}s_{(2,1)}$.

iii) The decomposition of $V_{(2,1)} \otimes_o V_{(2,1)}$ into irreducible S_6 representations.

Answer

Recall that the product of Schur functions can be calculated using Littlewood-Richardson coefficients as follows:

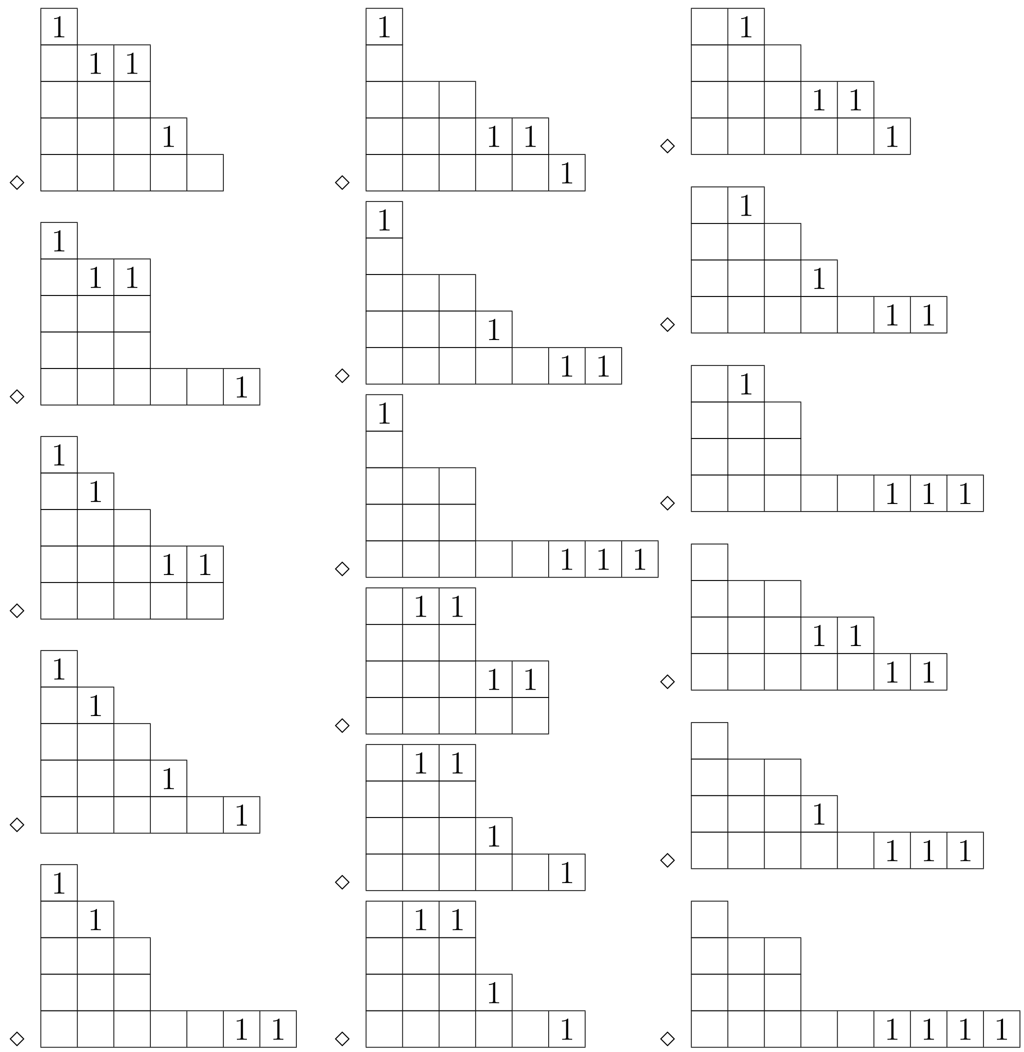
$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

where $c^\lambda_{\mu\nu}$ is the number of Littlewood-Richardson tableaux of shape λ/μ with content ν.
In order to fill out λ/μ with content ν it must hold that $|\lambda| - |\mu| = |\nu|$.

i) In this first case we have $\mu = (5, 3, 3, 1)$ and $\nu = (4)$ which means that λ should partition 16.

This means that we must append 4 new blocks to our partition $(5, 3, 3, 1)$ to obtain a partition of 16 which we will fill with only ones.

As the tableaux must be semi-standard, it can't happen that we append more than one block in the same column. In other words, we must append a horizontal strip of ones. The following tableaux correspond to partitions λ such that λ/μ is a horizontal strip:



As there is only one possible tableau for each shape we have that each coefficient is one. Therefore we have

$$\begin{aligned} s_{(5,3,3,1)}s_{(4)} =& s_{(5,4,3,3,1)} + s_{(6,3,3,3,1)} + s_{(5,5,3,2,1)} + s_{(6,4,3,2,1)} + s_{(7,3,3,2,1)} \\ &+ s_{(6,5,3,1,1)} + s_{(7,4,3,1,1)} + s_{(8,3,3,1,1)} + s_{(5,5,3,3)} + s_{(6,4,3,3)} + s_{(7,3,3,3)} \\ &+ s_{(6,5,3,2)} + s_{(7,4,3,2)} + s_{(8,3,3,2)} + s_{(7,5,3,1)} + s_{(8,4,3,1)} + s_{(9,3,3,1)}. \end{aligned}$$

ii) For this next case we have $\mu = \nu = (2,1)$. This means that $\lambda \vdash 6$ and we must append two 1's and one 2. As our tableau must be Littlewood-Richardson, the reading word must be one of either

$$211, \quad \text{or} \quad 121.$$

The following tableaux correspond to $\lambda \vdash 6$ such that λ/μ is a Littlewood-Richardson tableau with content $(2, 1)$:

$\diamond\ \begin{ytableau} 2 \\ 1 \\ {} & 1 \\ {} & {} \end{ytableau}$ $\quad$ $\diamond\ \begin{ytableau} 2 \\ {} & 1 \\ {} & {} & 1 \end{ytableau}$ $\quad$ $\diamond\ \begin{ytableau} {} & 2 \\ {} & {} & 1 & 1 \end{ytableau}$

$\diamond\ \begin{ytableau} 2 \\ 1 \\ {} \\ {} & {} & 1 \end{ytableau}$ $\quad$ $\diamond\ \begin{ytableau} 2 \\ {} \\ {} & {} & 1 & 1 \end{ytableau}$ $\quad$ $\diamond\ \begin{ytableau} 1 & 2 \\ {} & 1 \\ {} & {} \end{ytableau}$

$\diamond\ \begin{ytableau} 1 \\ {} & 2 \\ {} & {} & 1 \end{ytableau}$ $\quad$ $\diamond\ \begin{ytableau} {} & 1 & 2 \\ {} & {} & 1 \end{ytableau}$

We see that $\text{sh}(\lambda) = (3, 2, 1)$ is repeated once so we account for a coefficient of 2 with that tableau. The decomposition is thus:

$$s_{(2,1)}^2 = s_{(2,2,1,1)} + s_{(3,1,1,1)} + 2s_{(3,2,1)} + s_{(4,1,1)} + s_{(4,2)} + s_{(2,2,2)} + s_{(3,3)}.$$

iii) By the correspondence of the outer tensor product with the product of Schur functions we have

$$V_{(2,1)} \otimes_o V_{(2,1)} = V_{(2,2,1,1)} \oplus V_{(3,1,1,1)} \oplus 2V_{(3,2,1)} \oplus V_{(4,1,1)} \oplus V_{(4,2)} \oplus V_{(2,2,2)} \oplus V_{(3,3)}.$$

Exercise 2 (Exercise 3)**.** Use the Straightening Algorithm (Garnir Relations) to express the Garnir polynomial F_T where T is the filling

$$\begin{ytableau} 2 & 4 \\ 3 & 1 \end{ytableau}$$

in terms of F_S's where each S is a standard Young tableau. Write out each polynomial in your formula and check that your answer works.

Answer

We first column straighten T by ordering its columns, we obtain

$$\begin{ytableau} 3 & 4 \\ 2 & 1 \end{ytableau}.$$

To row-straighten we find the topmost row with a descent and consider rightmost decrease. The block in question is

$$\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 1 \\ \hline \end{array}$$

and for this block we will find permutations of S_3 which preserve the columns-increasing condition withing the blocks. We have the permutations

$$\text{id}, \quad (12), \quad (123)$$

which correspond to

$$\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 3 \\ \hline \end{array}.$$

This means that

$$(-1)^0(x_3 - x_2)(x_4 - x_1) + (-1)^1(x_3 - x_1)(x_4 - x_2) + (-1)^2(x_2 - x_1)(x_4 - x_3) = 0$$

and from this relation we obtain

$$(x_3 - x_2)(x_4 - x_1) = (x_3 - x_1)(x_4 - x_2) - (x_2 - x_1)(x_4 - x_3).$$

Similarly just by calculating the Garnir polynomial of T we obtain $(x_2 - x_3)(x_4 - x_1)$ which corresponds to our formula, just with a sign change.

Exercise 3 (Exercise 5)**.** Decompose the inner tensor product $V(2,1) \otimes_i V(2,1)$ into irreducible representations of S_3.

Answer

Each of the factors in the product is generated by the pair of polynomials

$$(x_2 - x_1), (x_3 - x_1), \quad \text{and} \quad (y_2 - y_1), (y_3 - y_1)$$

and therefore the tensor product is generated by the pairwise products of these polynomials. Let us call

$$\begin{cases} v_1 = (x_2 - x_1)(y_2 - y_1) = x_2y_2 - x_2y_1 - x_1y_2 + x_1y_1 \\ v_2 = (x_2 - x_1)(y_3 - y_1) = x_2y_3 - x_2y_1 - x_1y_3 + x_1y_1 \\ v_3 = (x_3 - x_1)(y_2 - y_1) = x_3y_2 - x_3y_1 - x_1y_2 + x_1y_1 \\ v_4 = (x_3 - x_1)(y_3 - y_1) = x_3y_3 - x_3y_1 - x_1y_3 + x_1y_1 \end{cases}$$

We now consider the action of S_3 on each of our basic elements. It suffices to consider the action per representative of cycle type, so we only consider id, (12) and (123).
Applying the identity to each of our elements returns it as it was. So we have

$$M_{\mathrm{id}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, we apply the transposition (12) to our elements to obtain

- ◇ $(12)v_1 = (x_1 - x_2)(y_1 - y_2) = v_1$
- ◇ $(12)v_2 = (x_1 - x_2)(y_3 - y_2) = x_1y_3 - x_1y_2 - x_2y_3 + x_2y_2$
- ◇ $(12)v_3 = (x_3 - x_2)(y_1 - y_2) = x_3y_1 - x_3y_2 - x_2y_1 + x_2y_2$
- ◇ $(12)v_4 = (x_3 - x_1)(y_3 - y_1) = x_3y_3 - x_3y_2 - x_2y_3 + x_2y_2$

On the other hand notice that

$$\begin{aligned} v_1 - v_2 &= x_2y_2 - x_2y_1 - x_1y_2 + x_1y_1 - (x_2y_3 - x_2y_1 - x_1x_3 + x_1y_1) \\ &= x_2y_2 - x_2y_3 - x_1y_2 + x_1y_3 \\ v_1 - v_3 &= x_2y_2 - x_2y_1 - x_1y_2 + x_1y_1 - (x_3y_2 - x_3y_1 - x_1y_2 + x_1y_1) \\ &= x_2y_2 - x_2y_1 - x_3y_2 + x_3y_1 \\ v_1 - v_2 - v_3 + v_4 &= x_2y_2 - x_2y_3 - x_1y_2 + x_1y_3 - (x_3y_2 - x_3y_1 - x_1y_2 + x_1y_1) \\ &\quad + x_3y_3 - x_3y_1 - x_1y_3 + x_1y_1 \\ &= x_2y_2 - x_2y_3 - x_3y_2 + x_3y_3 \end{aligned}$$

We can identity the image of the transposition with this elements to build the matrix

$$M_{(12)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

And in a similar fashion we may calculate the image of our basic elements through the 3-cycle:

- ◇ $(123)v_1 = (x_3 - x_2)(y_3 - y_2) = v_1 - v_2 - v_3 + v_4$
- ◇ $(123)v_2 = (x_3 - x_2)(y_1 - y_2) = v_1 - v_3$
- ◇ $(123)v_3 = (x_1 - x_2)(y_3 - y_2) = v_1 - v_2$
- ◇ $(123)v_4 = (x_1 - x_2)(y_1 - y_2) = v_1$

With this information we may build

$$M_{(123)} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The traces of these matrices allow us to build character table for $V(2,1) \otimes_i V(2,1)$:

$$\chi_{\text{inner}} = \begin{pmatrix} 4 & 0 & 1 \end{pmatrix}$$

To find the decomposition we solve a system of linear equations by row reducing the following matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

which means that

$$V(2,1) \otimes_i V(2,1) = V_{(3)} \oplus V_{(1,1,1)} \oplus V_{(2,1)}.$$

Exercise 4 (Exercise 6)**.** Use the Murnaghan-Nakayama rule to compute $\langle s_{(3,3,1)} | p_{(3,3,1)} \rangle$.

Answer

Recall that the Murnaghan-Nakayama rule has the following statement:

$$\chi_\mu(\lambda) = \langle s_\mu | p_\lambda \rangle = \sum_{(*)} \prod_{i=1}^{\ell(\lambda)} (-1)^{\mathrm{ht}(\mathrm{strip}(i))}$$

where the sum runs through border strip tableaux T with shape μ and content λ. In this case we have $\mu = \lambda = (3,3,1)$ which means we must fill the diagram

$$\begin{array}{|c|cc}\hline & \multicolumn{1}{c}{} & \\ \hline & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline \end{array}$$

with the word 1112223 only forming border strips. The only possible fillings are

$$\begin{array}{|c|c|c|}\cline{1-1} 3 & \multicolumn{2}{c}{} \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 2 & 2 & 3 \\ \hline 1 & 1 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\cline{1-1} 3 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 2 \\ \hline 1 & 1 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\cline{1-1} 1 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & 2 \\ \hline \end{array}$$

Also, recall that the height of the strip is one less than the *actual height*. With this we have the corresponding height vectors:

$$(0,0,0), \quad (0,1,0), \quad (1,1,0), \quad (2,1,0)$$

and we may calculate $(-1)^{\mathrm{v}}$ for each of these to obtain

$$\left\langle s_{(3,3,1)} \middle| p_{(3,3,1)} \right\rangle = 1 - 1 + 1 - 1 = 0.$$

Then $s_{(3,3,1)}$ and $p_{(3,3,1)}$ form a dual pair. *Is this true in general for any λ?*