Exercise 1 (4.1.A Vakil)**.** Show that the natural map $A_f \to \mathcal{O}_{\mathrm{Spec}(A)}(D(f))$ is an isomorphism. ⟦ Hint: Exercise 3.5.E Vakil. ⟧

First let us recall that Exercise 3.5.E is the following:

Lemma 1. *The next statements are equivalent*:

i) $D(f) \subseteq D(g)$.
ii) $\exists n (n \geqslant 1 \Rightarrow f^n \in \mathrm{gen}(g))$.
iii) g *is an invertible element of* A_f.

We have proven this in class so let us make a quick recapitulation.

The first two statements are equivalent because

$$\begin{aligned} D(f) \subseteq D(g) &\iff V(g) \subseteq V(f) \\ &\iff \{\, \mathfrak{p} : \mathrm{gen}(g) \subseteq \mathfrak{p} \,\} \subseteq \{\, \mathfrak{p} : \mathrm{gen}(f) \subseteq \mathfrak{p} \,\} \end{aligned}$$

The last statement can be rephrased as *if a prime contains* g, *then it also contains* f. In particular this equivalent to saying

$$\begin{aligned} f \in \bigcap_{g \in \mathfrak{p}} \mathfrak{p} = \sqrt{\mathrm{gen}(g)} \\ \iff \exists n (n \geqslant 1 \Rightarrow f^n \in \mathrm{gen}(g)). \end{aligned}$$

For the last two statements, we first assume g is invertible in A_f. This means that there exists an n such that

$$\left(\frac{g}{1}\right)\left(\frac{a}{f^n}\right) = \frac{1}{1}.$$

Recall that the equality condition in the localization means that there exists and element f^m with $m \geqslant 1$ which is invertible in A_f such that

$$f^m(ag - f^n) = 0 \Rightarrow agf^m = f^{m+n}.$$

This last equation is in A without localizing, and the term on the right, agf^m, is in $\mathrm{gen}(g)$. Thus the power we were searching for is $m+n$ and $f^{m+n} \in \mathrm{gen}(g)$. On the other direction, if $f^n \in \mathrm{gen}(g)$ for some $n \geqslant 1$, then there is an $a \in A$ such that

$$f^n = ag,$$

and localizing at f turns this equation into $\frac{1}{g} = \frac{a}{f^n}$.

Answer

We begin by recalling the definition of $\mathcal{O}_{\mathrm{Spec}(A)}(D(f))$, we have

$$\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) = S^{-1}A, \quad \text{where} \quad S = \{\, g \in A : D(f) \subseteq D(g) \,\}.$$

By the lemma we can rewrite S as

$$S = \{\, g \in A : \exists n (f^n \in \mathrm{gen}(g)) \,\}.$$

Now notice that when localizing at S we are able to invert f^n for some n. From this we have that f is also invertible in $S^{-1}A$ because

$$f^n g = u \Rightarrow f(f^{n-1}g) = u \Rightarrow f \text{ is invertible.}$$

This means that localizing at S is a further localization of A at f because we have already inverted all powers of f.
Notice however that this isn't adding anything new to A_f, because of the last equivalence of the lemma. Every g such that $D(f) \subseteq D(g)$ is already invertible in A_f. We conclude that the inclusion is actually an isomorphism.

Exercise 2 (Restrictions)**.** Do the following:

i) Explain, using Definition 4.1.1 (and not exercise 4.1.A) what the restriction map is.
ii) Explain, using exercise 4.1.A what the restriction map is.

Answer

i) Recall that

$$\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) = (S^f)^{-1}A, \quad \text{where} \quad S^f = \{\, h \in A : D(f) \subseteq D(h) \,\}$$

and on the same vein the set associated to $D(g)$ is the localization at $S^g = \{\, h \in A : D(g) \subseteq D(h) \,\}$. So if we take $D(f) \subseteq D(g)$ then the restriction map is a function

$$\underset{D(g),D(f)}{\mathrm{res}} \mathcal{O}_{\mathrm{Spec}(A)}(D(g)) \to \mathcal{O}_{\mathrm{Spec}(A)}(D(f)).$$

ii) Using the previous exercise we have the isomorphism between localizing at S^f and localizing at powers of f. So once again let us assume that

$D(f) \subseteq D(g)$, then the restriction map is a function

$$\underset{D(g),D(f)}{\text{res}} A_f \to A_g.$$

In this case we have an element $\frac{a}{f^n}$ which is being mapped to

Exercise 3 (4.1.D Vakil)**.** Suppose M is an A-module. Show that the following construction describes a sheaf $\widetilde{M}$ on the distinguished base. Define $\widetilde{M}(D(f))$ to be the localization of M at the multiplicative set of all functions that do not vanish outside of $V(f)$.

Define restriction maps $\text{res}_{D(f),D(g)}$ in the analogous way to $\mathcal{O}_{\text{Spec}(A)}$.

Show that this defines a sheaf on the distinguished base, and hence a sheaf on $\text{Spec}(A)$. Then show that this is an $\mathcal{O}_{\text{Spec}(A)}$-module.

Answer

Exercise 4. Let $A = \mathbb{C}[x, y]$ and let $\mathfrak{p} = \text{gen}(y)$, viewed as a point of $X = \text{Spec}(A)$. What is $\mathcal{O}_{X,p}$?

Recall that $\mathcal{O}_{X,p}$ is a local ring, that is, it has a unique maximal ideal, $\mathfrak{m}_p$.

What is the residue field $\kappa_{\mathfrak{p}} = \mathcal{O}_{X,p}/\mathfrak{m}_p$?

Answer

Exercise 5 (4.4.A Vakil)**.** Show that you can glue an arbitrary collection of schemes together. Suppose we are given:

- schemes X_i (as i runs over some index set I, not necessarily finite),
- open subschemes $X_{ij} \subseteq X_i$ with $X_{ii} = X_i$,
- isomorphisms $f_{ij} : X_{ij} \to X_{ji}$ with f_{ii} the identity

such that

the isomorphisms "agree on triple intersections", i.e.,

$$f_{ik} \mid_{X_{ij} \cap X_{ik}} = f_{jk} \mid_{X_{ji} \cap X_{jk}} \circ f_{ij} \mid_{X_{ij} \cap X_{ik}} \circ$$

(so implicitly, to make sense of the right side, $f_{ij}(X_{ik} \cap X_{ij}) \subseteq X_{jk}$).

This *cocycle condition* ensures that f_{ij} and f_{ji} are inverses. In fact, the hypothesis that f_{ii} is the identity also follows from the cocycle condition.

Show that there is a unique scheme X (up to unique isomorphism) along with open subsets isomorphic to the X_i respecting this gluing data in the obvious sense.⟦Hint: what is X as a set? What is the topology on this set? In terms of your description of the open sets of X, what are the sections of this sheaf over each open set?⟧

Answer