Homework 2

Due: Friday, February 3

1. *This is another way to derive the Cauchy-Riemann equations* Suppose $S \subset \mathbb{C}$ is a domain, and that $f : S \to \mathbb{C}$ is a function with a (complex) derivative at $z_0 \in S$.

   As usual, let $z = x + iy$, and write $f(x + iy) = u(x, y) + iv(x, y)$, where u, v are $\mathbb{R}$-valued functions.

   (a) By computing $f'(z_0)$ along the trajectory $x_0 + \Delta x + iy_0$, where $\Delta x \to 0$, show that

   $$f'(z_0) = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})|_{z_0}.$$

   We will often write this as $f'(z) = \frac{\partial}{\partial x} f(z) = \partial_x f(z)$.

   (b) By computing $f'(z_0)$ along the trajectory $x_0 + i(y_0 + \Delta y)$ as $\Delta y \to 0$, show that

   $$f'(z) = \frac{1}{i}(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}).$$

   (c) Deduce the Cauchy-Riemann equations.

2. [SS]1.13. (HINT: *Let $f = u + iv$; in each case, write the function in question in terms of u and v.*)

3. [SS]1.19(a),(b).

4. [SS]1.19(c). (HINT: *You may use the result of [SS]1.14. If $z \neq 1$, what is $\sum_{n=1}^{N} z^n$?*)

5. *Don't start this until after class on Monday, January 30.* For $\alpha \in \mathbb{C}$ and $r > 0$, let $\gamma_r(\alpha)$ be the arc given by

   $$[0, 2\pi] \xrightarrow{z} \mathbb{C}$$

   $$t \longmapsto r\exp(it) + \alpha.$$

   Let n be an integer. Calculate

   $$\int_{\gamma_1(0)} z^n \, dz.$$