Partial Fractions

Integrals involving quadratic polynomials can be solved in many different ways, not only trigonometric substitution. The partial fractions method involves separating a polynomial into its <u>irreducible</u> factors.

Example 1. Consider the integral
$$\int \frac{\mathrm{d}x}{(x-5)(x-4)}.$$
We will split the integrand into a couple of fractions whose denominators are the factors of the original denominator.
$$\frac{1}{(x-5)(x-4)} = \frac{A}{x-5} + \frac{B}{x-4},$$
where A,B are constants. To find these constants we multiply both side of the equation by the original denominator $(x-5)(x-4)$:
$$\begin{aligned} 1 = \frac{(x-5)(x-4)}{(x-5)(x-4)} &= \frac{A(x-5)(x-4)}{x-5} + \frac{B(x-5)(x-4)}{x-4} \\ &= A(x-4)+B(x-5) \\ \Rightarrow 1 &= A(x-4)+B(x-5). \end{aligned}$$
Expanding the expression in the right we obtain the equality
$$1 = Ax-4A+Bx-5B \Rightarrow 1 = (A+B)x+(-4A-5B)$$
and since the expression in the left is also a polynomial we can equate coefficients.
$$\begin{cases} 1 = -4A-5B \\ 0 = A+B \end{cases}$$
The second equation tells us that $B=-A$ and therefore we can replace that in the first equation
$$1 = -4A-5(-A) \Rightarrow 1 = A \Rightarrow B = -1.$$
Thus it follows that
$$\frac{1}{(x-5)(x-4)} = \frac{(1)}{x-5} + \frac{(-1)}{x-4},$$
and we can separate this expression using linearity of the integral:
$$\begin{aligned} \int \frac{\mathrm{d}x}{(x-5)(x-4)} &= \int \left(\frac{1}{x-5} + \frac{-1}{x-4} \right) \mathrm{d}x \\ &= \int \frac{\mathrm{d}x}{x-5} - \int \frac{\mathrm{d}x}{x-4} \\ &= \underline{\log(x-5)-\log(x-4)} \end{aligned}$$

We will not always find ourselves with a factored polynomial. Consider the following example:

Example 2. Compute the integral of $\frac{x+1}{(x-1)(x-2)}$.