Exercise 1. We have proven rigorously that any succinctly smooth minimizer (say, if the minimizer happens to be in C^2) of the functional

$$I[u] = \int_a^b f(x, u(x), u'(x)) \mathrm{d}x$$

has to satisfy the Euler-Lagrange equations

$$f_u(x, u(x), u'(x)) = \frac{\mathrm{d}}{\mathrm{d}x} f_\xi(x, u(x), u'(x))$$

for all $x \in]a, b[$. The energy related to the vertical defection of a thin (one-dimensional) beam subject to a vertical gravity geld with strength g is given by

$$I[u] = \int_a^b \left[\mu\left(\sqrt{1 + u'(x)^2} - 1\right) - gu(x)\right] \mathrm{d}x$$

where μ is related to the elasticity constants of the material.

i) Derive the Euler-Lagrange equations (including boundary conditions) for $u(x)$ for this problem if you try to minimize the energy over the set of functions

$$D = \{\, \varphi \in C^2 : \ \varphi(a) = \varphi(b) = 1 \,\}.$$

ii) Can you prove that the solution $\overline{u}$ of the (Euler-Lagrange) deferential equations is (i) a minimizer of $I[u]$, and (ii) is unique?

Answer

Exercise 2. In general, the Euler-Lagrange equations will be a (system of) nonlinear ordinary deferential equation. Most often, they will not be exactly solvable. But occasionally, we can solve simplices problems.

If you take Problem 1, consider the case of stir materials that do not deform very much. In that case, u' will be small, and we can use the approximation

$$\sqrt{1 + y} \approx 1 + \frac{1}{2} y.$$

(This is just Taylor expansion around $y = 0$.) Use this to dene an approximate energy functional $\overline{I}[u]$.

i) Derive the Euler-Lagrange equations (including boundary conditions) for $u(x)$ for this approximate problem with the same D as before.

ii) Can you prove that the solution $\overline{u}$ of the (Euler-Lagrange) deferential equations is the unique minimizer of $\overline{I}\,[u]$? (In other words, that the solution of the Euler-Lagrange equation is not just a stationary point of $\overline{I}\,[u]$, but in fact a minimizer?)

iii) Actually solve this problem, i.e., find $\overline{u}$ that satires the Euler-Lagrange equations.

Answer

Exercise 3. We have started the semester by considering Newton's minimal resistance problem and Bernoulli's brachystochrone problem (the "bead on a wire"). The former is a bit more complicated because the right integration bound b depends on the solution. But for the former, the case is easy: We have

$$I\,[u] = \int_0^L \sqrt{\frac{\frac{1}{2}(1+u'(x)^2)}{gH - gu(x)}}\mathrm{d}x$$

and

$$D = \{\, \varphi \in C^2 : \ \varphi(0) = H, \ \varphi(L) = 0 \,\}.$$

State the Euler-Lagrange equations and boundary conditions any scent smooth minimizer $\overline{u}$ would have to satisfy. Is the solution unique?

Answer

Exercise. If you can, also solve the Euler-Lagrange equations. The solution can of course be found on the internet or in any number of books, but if you want to get these bonus points, you will need to show step by step how you solve the equations - this is going to be non-trivial.

Exercise 4. Everything we have done in class was based on functions $u(x)$ of a single argument $x \in [a,b] \subseteq \mathbb{R}$. But in reality, it is not very dict to derive the same kind of Euler-Lagrange equations also for functions of multiple arguments $u(\mathbf{x}),\ \mathbf{x} \in \Omega \subseteq \mathbb{R}^n$. To this end, let us assume that we want to find a minimizer of

$$I\,[u] = \int_\Omega f(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x}))\mathrm{d}x$$

and

$$D = \{\, \varphi \in C^2(\Omega) :\ \varphi(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \partial\Omega \,\}.$$

Go through the one-dimensional derivation of the Euler-Lagrange equations and adapt it as appropriate to derive the (now partial) deferential equation any succinctly smooth minimizer $\overline{u} \in D$ has to satisfy. The key step is to remember your integration-by-parts rules and use what you know about the boundary values of the variations. You will also have to keep in mind that you now really have $f(x, u, \xi_1, \ldots, \xi_n)$. State both the Euler-Lagrange equations and boundary conditions.

Exercise 5. The generalization to higher dimensions of the erst problem is to look for the minimizer of the functional

$$I\,[u] = \int\limits_{\Omega} \left[\mu\left(\sqrt{1 + |\nabla u(\mathbf{x})|^2} - 1\right) - g u(\mathbf{x})\right] \mathrm{d}x,$$

state the Euler-Lagrange equations for this problem. Next, apply the same simplification we considered in Problem 2 and again derive the corresponding Euler-Lagrange equations.