Exercise 1. Find an example of two curves in $\mathbb{P}^2$ that have the same degree but are not isomorphic.

Answer

Exercise 2. Do the following:

(a) Find the Hilbert polynomial P of a k-dimensional linear subvariety of $\mathbb{P}^n$.

(b) Describe the Hilbert scheme of varieties in $\mathbb{P}^n$ with Hilbert polynomial P.

Answer

Exercise 3. Assume that the variety $V \subseteq \mathbb{P}^n$ has the Hilbert polynomial $P(n)$. Calculate the Hilbert polynomial of the image variety $\nu_d(V) \subseteq \mathbb{P}^{\binom{n+d}{d}-1}$ of the Veronese map. ⟦Hint: Do the case of $V = \mathbb{P}^1$ first.⟧

Answer

Recall that the Hilbert function for $\mathbb{P}^1$ is the dimension of, R_m, the m^{th} graded piece of $\mathbb{C}[x, y]$. The homogenous polynomials in $\mathbb{C}[x, y]$ have

$$\{ x^m, x^{m-1}y, \ldots, xy^{m-1}, y^m \}$$

as a basis. So in this case $m \mapsto \dim(R_m) = m + 1$ is the Hilbert function of $\mathbb{P}^1$. Let us now consider the image of $\mathbb{P}^1$ through the d^{th} Veronese embedding.

Exercise 4. Using the theorem describing the defining equations for T_pV in terms of the equations for V, compute the tangent spaces of the curves in examples (1), (2), and (3) at the origin.

Answer

(a) The curve in question is $\mathbb{V}(y - x^2)$, our function is $P_1(x, y) = y - x^2$ then $\nabla P_1(x, y) = (-2x, 1)$. The tangent space at the origin is the zero locus of

$$\langle \nabla P_1(0, 0) | (x, y) - (0, 0) \rangle = \langle (0, 1) | (x, y) \rangle = y.$$

This coincides with our original finding because $\mathbb{V}(y)$ is precisely the x-axis which is tangent to the parabola at the origin.

(b) Now we are working with $\mathbb{V}(y^2 - x^2 - x^3)$, then $P_2(x, y) = y^2 - x^2 - x^3$. The differential in this case is

$$\nabla P_2(x,y) = (-2x - 3x^2, 2y) \xrightarrow{\varepsilon_0} \nabla P_2(0,0) = (0,0)$$

and so the variety in question is the zero locus of the zero function. As the whole of $\mathbb{A}^2$ is such set, we can see that this makes sense because the origin is a singular point of our variety.

(c) Finally let us consider $\mathbb{V}(y^2 - x^3)$. In this case

$$\langle \nabla P_3(0,0) | (x,y) - (0,0) \rangle = \Big\langle (-3x^2, 2y) \Big| \Big\rangle$$

Exercise 5. Let $V \subseteq \mathbb{P}^n$ be a hypersurface defined by a homogeneous irreducible polynomial F. Find an explicit description of the tangent space to V at a point p. What conditions on p ensure that the tangent space to V at p has dimension $n - 1$?

Answer