

**Math 670 HW #4**  
Due 11:59 PM Friday, May 2

1. (Exercise 3.12.11) Show that

$$\mathrm{Fl}(d_1, \dots, d_k) \cong \mathrm{O}(n) / (\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k)),$$

where  $n_1 = d_1$  and  $n_i = d_i - d_{i-1}$  for  $i = 2, \dots, k$ . (In other words, the  $n_i$  are the jumps in dimension as we go up the flag.)

2. Let  $M$  be a manifold with an affine connection  $\nabla$ . Suppose  $\alpha: I \rightarrow M$  is a constant curve; that is,  $\alpha(t) = p$  for all  $t \in I$ . Let  $V$  be a vector field along  $\alpha$ , meaning that  $V(t) \in T_{\alpha(t)}M = T_pM$  just gives a curve in the tangent space  $T_pM$ . Show that  $\frac{DV}{dt} = V'(t)$ ; that is, the covariant derivative agrees with the usual derivative in this case, regardless of what  $\nabla$  is.
3. (Exercise 4.3.4) Show that an affine connection  $\nabla$  is compatible with a Riemannian metric  $g$  on  $M$  if and only if, for any vector fields  $V$  and  $W$  along a smooth curve  $\alpha: I \rightarrow M$ , we have

$$\left. \frac{d}{dt} \right|_{t=t_0} g_{\alpha(t)}(V(t), W(t)) = g_{\alpha(t_0)} \left( \frac{DV}{dt}, W \right) + g_{\alpha(t_0)} \left( V, \frac{DW}{dt} \right).$$

In other words, for compatible connections we can use the usual product rule to differentiate the inner product.