

Exercise 1. (Exercise 3.12.11) Show that

$$\mathcal{Fl}(d_1, \dots, d_k) \cong O(n)/(O(n_1) \times \cdots \times O(n_k)),$$

where $n_1 = d_1$ and $n_i = d_i - d_{i-1}$ for $i = 2, \dots, k$. (In other words, the n_i are the jumps in dimension as we go up the flag.)

Exercise 2. Let M be a manifold with an affine connection ∇ . Suppose $\alpha : I \rightarrow M$ is a constant curve; that is, $\alpha(t) = p$ for all $t \in I$. Let V be a vector field along α , meaning that $V(t) \in T_{\alpha(t)}M = T_pM$ just gives a curve in the tangent space T_pM . Show that $\frac{DV}{dt} = V'(t)$; that is, the covariant derivative agrees with the usual derivative in this case, regardless of what ∇ is.

Answer

Exercise 3. (Exercise 4.3.4) Show that an affine connection ∇ is compatible with a Riemannian metric g on M if and only if, for any vector fields V and W along a smooth curve $\alpha : I \rightarrow M$, we have

$$\left. \frac{d}{dt} \right|_{t=t_0} g_{\alpha(t)}(V(t), W(t)) = g_{\alpha(t_0)} \left(\frac{DV}{dt}, W \right) + g_{\alpha(t_0)} \left(V, \frac{DW}{dt} \right).$$

In other words, for compatible connections we can use the usual product rule to differentiate the inner product.

Answer

Let us suppose first that ∇ is compatible with g . If α is a curve, we may take an orthonormal basis of $T_{\alpha(t_0)}M$:

$$\{u_1(t_0), \dots, u_n(t_0)\}.$$

As ∇ is compatible with g , we may parallel-transport this basis throughout all the curve α . This means that for any $t \in I$,

$$\langle u_1(t), \dots, u_n(t) \rangle = T_{\alpha(t)}M.$$

Now, our vector fields V, W may be expressed as linear combinations of these

basic elements in the following way:

$$\begin{cases} V(t) = \sum_{k=1}^n \alpha_k u_k(t) \\ W(t) = \sum_{k=1}^n \beta_k u_k(t) \end{cases} \Rightarrow \begin{cases} \frac{DV}{dt} = \sum_{k=1}^n \alpha'_k u_k(t) \\ \frac{DW}{dt} = \sum_{k=1}^n \beta'_k u_k(t) \end{cases}$$

where α_k, β_k are smooth functions. Now if we compute the quantity of the left, we have that

$$\begin{aligned} & \left. \frac{d}{dt} \right|_{t=t_0} g_{\alpha(t)}(V(t), W(t)) \\ &= \sum_{k=1}^n \sum_{\ell=1}^n \left. \frac{d}{dt} \right|_{t=t_0} \alpha_k \beta_\ell g_{\alpha(t)}(u_k(t), u_\ell(t)) \\ &= \sum_{k=1}^n \left. \frac{d}{dt} \right|_{t=t_0} \alpha_k \beta_k \\ &= \sum_{k=1}^n \left. \frac{d\alpha_k}{dt} \right|_{t=t_0} \beta_k + \sum_{k=1}^n \alpha_k \left. \frac{d\beta_k}{dt} \right|_{t=t_0} \end{aligned}$$

and then readding indices by multiplying $\delta_{k\ell}$ and a sum through ℓ we recover the final expression:

$$\sum_{\ell=1}^n \sum_{k=1}^n \left. \frac{d\alpha_k}{dt} \right|_{t=t_0} \beta_k g_{\alpha(t_0)}(u_k(t), u_\ell(t)) + \sum_{\ell=1}^n \sum_{k=1}^n \alpha_k \left. \frac{d\beta_k}{dt} \right|_{t=t_0} g_{\alpha(t_0)}(u_k(t), u_\ell(t)).$$

Condensing everything by linearity we recover

$$g_{\alpha(t_0)} \left(\frac{DV}{dt}, W \right) + g_{\alpha(t_0)} \left(V, \frac{DW}{dt} \right).$$