MATH502 — Combinatorics 2

Based on the lectures by Maria Gillespie

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Spring 2023

Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

Requirements

Contents

Chapter 1

Symmetric functions

1.1 Recall

Definition 1.1.1. $f(x_1, x_2, \dots)$ is symmetric if it's fixed under permutations of variables.

Example 1.1.2. $f(x_1, \dots, x_4) = x_1^5 + \cdots + x_4^5$. This is known as p_5 or $m_{(5)}$, where p is the power-sum symmetric function and m, the monomial symmetric function.

Example 1.1.3. Consider $g = x_1^4 x_2 + x_1^4 x_3 + \cdots + x_i^4 x_j + \cdots + 3x_1 + \cdots = m_{(4,1)} + 3m_{(1)}$.

Let us recall some **notation**:

i) $\Lambda_R(x_1, \dots, x_n)$ is the ring of symmetric polynomials over R. In *infinitely* many variables we have $\Lambda_R(\underline{x})$.

In the case $R = \mathbb{Q}$, then $\dim \Lambda_Q(\underline{x})_{(d)}$, where every monomial has degree d, is $p(d)$. This is the number of partitions of d. Because for every partition we can form monomials and monomials form a basis.

Bases of Λ_Q

Suppose $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geqslant \dots \geqslant \lambda_k$.

- Monomial: $m_\lambda = \sum_{i_1 \neq \dots i_k} x_{i_1}^{\lambda_1} \dots x_{i_k}^{\lambda_k}$.
- Elementary: $e_\lambda = \prod e_{\lambda_i}$ where $e_d = m_{(1,1,\dots,1)}$ (d ones).
- Homogenous: $h_\lambda = \prod h_{\lambda_i}$ and $h_d = x_1^d + \cdots + x_1^{d-1} x_2 + \cdots + x_1^{d-2} x_2^2 + x_1^{d-2} x_2 x_3 + \dots$. In general $h_d = \sum_{\lambda \vdash d} m_\lambda$.
- Power sum: $p_\lambda = \prod p_{\lambda_i}$ and $p_d = \sum x_i^d$.

For Schur basis recall SSYT

Example 1.1.4. Consider $\lambda = (5, 4, 1)$, rows $\leqslant\rightarrow$ and columns $<$, we associate the monomial $x_1^2 x_2^3 x_3^3 x_4^2 := x^T$.

- Schur: $s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$ but also $\sum K_{\lambda\mu} m_\mu$ where the sum is over SSYT of shape λ, content μ.

Schur function motivation (preview)

The first place they showed up is in the representation theory of Lie group. The function $s_\lambda(x_1, \ldots, x_n)$ is a character of irreducible polynomial representations of GL_n. In theoretical physics we have matrix groups acting on particles, representations are smaller matrix groups of things that they are mapping to. We want to take tensor product and direct sums of representations, the tensor product is related to multiplication of Schur function while direct sum into sum of Schur functions.

There's also the Schur-Weyl duality which takes representations into the Weyl group. Under the *Frobenius map*, s_λ corresponds to irreducible representations of S_n.

A more modern application of Schur function goes into geometry, s_λ correspond to Schubert varieties in Grassmannians. Multiplication corresponds to interesections and sum to unions.

There's also context in Probability Theory. But in the end, Schur positivity is important because of this connections.

Definition 1.1.5. $f \in \Lambda$ is Schur-positive if $f = \sum c_\lambda s_\lambda$, $c_\lambda \geqslant 0$.

Example 1.1.6. $3s_{(2,1)} + 2s_{(3)}$ schur pos but change 2 to $-\frac{1}{2}$ then not.

1.2 day 2

Alg defn Schur fncs

Definition 1.2.1. A function is antisymmetric if for $\pi \in S_n$,

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \operatorname{sgn}(\pi) f(x_1, \ldots, x_n).$$

Example 1.2.2. The following functions are antisymmetric:

(a) $f(x, y) = x - y$ then $f(y, x) = -f(x, y)$.
(b) $g(x, y) = (x - y)(x + y)$.
(c) $h(x, y) = x^2 y - y^2 x$.

Notice that the last function can factor as $h = -xy(x-y)$. We claim that this is always the case.

Lemma 1.2.3. *Every antisymmetric polynomial f in two variables x, y can factor as $f(x,y) = (x-y)g(x,y)$ where g is symmetric.*

Proof

Suppose f is antisymmetric, then $f(x,x) = 0$ by taking $y = x$. This means that $(x-y) \mid f$. Thus $f(x,y) = (x-y)g(x,y)$ and we now need to show that g is symmetric.

$$g(y,x) = \frac{f(y,x)}{y-x} = \frac{-f(x,y)}{-(x-y)} = \frac{f(x,y)}{x-y} = g(x,y).$$

Monomial Antisymmetric Functions

Definition 1.2.4. Given a strict partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 > \cdots > \lambda_k$, we define

$$a_\lambda(x_1, \ldots, x_n) = x_1^{\lambda_1} \cdots x_k^{\lambda_k} \pm \text{similar terms} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_k x_{\pi(k)}^{\lambda_k}.$$

This a_λ can be zero.

Example 1.2.5. For two variables we've seen some antisymmetric polynomials. Let us calculate

$$a_{(3,1)}(x,y) = x^3y - y^3x.$$

The smallest possible example in 3 variables is

$$a_{(2,1,0)}(x,y,z) = x^2y + y^2z + z^2x - y^2x - z^2y - x^2z.$$

This can be factored as $(x-y)(y-z)(x-z)$. A similar construction gives us

$$a_{(4,2,0)}(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - y^4x^2 - z^4y^2 - x^4z^2,$$

but how does this factor? We get

$$a_{(4,2,0)}(x,y,z) = (x^2-y^2)(y^2-z^2)(x^2-z^2) = a_{(2,1,0)}(x,y,z)(x+y)(y+z)(x+z).$$

Lemma 1.2.6. *The set $\{ a_\lambda \}_{\lambda \text{ strict}}$ is a basis of the antisymmetric polynomials over $\mathbb{Q}$, $A_\mathbb{Q}$. Even more any a_λ is divisible by a_ρ where $\rho = (n-1, n-2, \ldots, 2, 1, 0)$.*

As an algebra generator, a_ρ is a generator.

Proof

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Proposition 1.2.7. *The a_ρ antisymmetric function is also the Vandermonde determinant:*

$$a_\rho = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \dots & x_1^2 & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \dots & x_2^2 & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \dots & x_n^2 & x_n & 1 \end{pmatrix}$$

Schur Polynomials

Definition 1.2.8. The Schur polynomial of $\lambda \in \mathrm{Par}$ is

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})}.$$

Here $\lambda + \rho$ is the pointwise sum as arrays.

Remark 1.2.9. This is the Weyl character proof.

The following proof is due to Proctor(1987) find ref

Lemma 1.2.10. *Any a_λ can be seen as a determinant in the following way:*

$$a_\lambda(\underline{x}) = \det \begin{pmatrix} x_1^{\lambda_1} & x_1^{\lambda_2} & \dots & x_1^{\lambda_n} \\ x_2^{\lambda_1} & x_2^{\lambda_2} & \dots & x_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \dots & x_n \end{pmatrix}$$

Proof

We want to see that

$$\frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})} = \sum x^T$$

where the sum ranges through T's which are SSYT(la) with max entry n.

(a) We will show a recursion for the combinatorial definition that the character formula will also satisfy. It holds that

$$s_\lambda(\underline{x}) = \sum s_\mu(\underline{x}) x_n^{|\lambda|-|\mu|}$$

where μ has $n-1$ parts with $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \ldots$.

(b) We also show that the ratio of determinants satisfies the same recursion.

Example 1.2.11. Consider $\lambda = (8, 8, 4, 1, 1)$ and $\mu = (8, 5, 2, 1)$, then $\lambda \backslash \mu$ is a skew-table in which we can fill in n's

Corollary 1.2.12. *The Schur polynomials are a basis of* $\Lambda_{\mathbb{Q}}$.