

**Exercise 1.** Draw the  $\mathfrak{sl}_3$  crystal for weight  $(3, 3, 0)$ .

**Exercise 2.** Prove that the elements of the hyperoctahedral group, written in cycle notation as a permutation on  $\{\pm 1, \dots, \pm n\}$ , has all of its cycles coming in either pairs of the form  $(a_1 \dots a_k)(-a_1 \dots -a_k)$ , or of the form  $(a_1 \dots a_k - a_1 - a_2 \dots - a_k)$ .

**Exercise 3.** Define the Lie algebra  $\mathfrak{so}_{2n+1}$  as  $\{X : X^T S + SX = 0\}$  where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0_n & I_n \\ 0 & I_n & 0_n \end{pmatrix}$$

and  $I_n$  is the  $n \times n$  identity matrix and 1 is in the upper left corner. Write down what an arbitrary element  $X$  looks like, and using the fact that with respect to this setup the torus is simply the set of diagonal matrices  $X$  satisfying these conditions, explain how one obtains the type  $B$  root system.

#### Answer

First observe that the matrix  $S$  is a permutation matrix which acts by row permutation when applied as left multiplication. The permutation it applies is a product of disjoint transpositions of the form  $(i \ n+i)$  for  $i \in [n+1] \setminus \{1\}$ .

**Exercise 4.** What is the dimension of the adjoint representation of  $\mathfrak{so}_7$ ?

#### Answer

Via the isomorphism  $X \mapsto [X, -]$  we have that the dimension of the adjoint representation is the same as  $\dim \mathfrak{so}_7$  which is  $\binom{7}{2} = 21$ .

**Exercise 5.** Explain why the set of 5<sup>th</sup> roots of unity in the plane don't form a root system. Which axioms of root systems does it satisfy?

#### Answer

The axioms we should check are:

- (a) The roots span our vector space.
- (b) The reflections across hyperplanes are still roots.
- (c) Projections onto the span of a single root are an integer multiple or a half-integer multiple of the root.

(d) If  $\alpha, \beta$  are roots such that  $\beta = \lambda\alpha$  then  $\lambda = \pm 1$ .

The first axiom is satisfied as any non-zero complex number spans  $\mathbb{C}$ . The last axiom is satisfied vacuously.

The second axiom isn't satisfied as reflections across hyperplanes send the 5<sup>th</sup> roots to 10<sup>th</sup> (primitive) roots of unity. Projections are also not integer multiples nor half-integer multiples of other roots.

**Exercise 6.** Compute the evacuation of the Young tableau below, and then evacuate again, and show you have returned to the starting tableau.

5			
2	7	8	
1	3	4	6

### Answer

We switch entries following the rule  $k \mapsto n + 1 - k$  and then rotating 180°:

5			
2	7	8	
1	3	4	6

 $\longrightarrow$ 

4			
7	2	1	
8	6	5	3

 $\longrightarrow$ 

3	5	6	8
×	1	2	7
			4

Here we have already marked the first inner corner we will move. This leads us to

3	5	8	
1	2	6	7
	×		4

 $\longrightarrow$ 

3	5	8	
1	2	6	
	×	4	7

 $\longrightarrow$ 

3	8		
1	5	6	
×	2	4	7

 $\longrightarrow$ 

8			
3	5	6	
1	2	4	7

where every **green** character moved when clearing out the inner corner in the previous step. Redoing the process we obtain the skew tableau

8			
3	5	6	
1	2	4	7

 $\longrightarrow$ 

1			
6	4	3	
8	7	5	2

 $\longrightarrow$ 

2	5	7	8
×	3	4	6
			1

With the first inner corner marked, we move it out and continue the process:

5	7	8	
2	3	4	6
	×		1

 $\longrightarrow$ 

5	7	8	
2	3	4	
	×	1	6

 $\longrightarrow$ 

5	7		
2	3	8	
×	1	4	6

 $\longrightarrow$ 

5			
2	7	8	
1	3	4	6

As we have returned to our original tableau we conclude that the process is correct.

**Exercise 7.** Compute the Hall-Littlewood polynomial  $\tilde{H}_{(2,1,1)}(x; q)$ .

**Exercise 8.** Let  $w = w_1 \dots w_n$  be a word of partition content, and suppose  $w_1 \neq 1$ . Let  $w' = w_2 \dots w_n w_1$  be formed by cycling  $w_1$  around to the end of the word. Show that  $c(w') = c(w) - 1$  where  $c$  is cocharge. This operation is called *cyclage*.

#### Answer

Observe that it suffices to view this on standard words. This is because we may separate a word into standard subwords and calculate cocharge<sup>a</sup>. Consider the subword  $\tilde{w}$  of  $w$  which contains  $w_1$  in the previous decomposition sense, as  $w$  has partition content so does  $\tilde{w}$ .

When cycling  $w_1$  to the end of  $\tilde{w}$ , cocharge is reduced by 1 as there is a element in  $\tilde{w}$  smaller than  $w_1$  which was to the right of  $w_1$ . After cycling, it's to the *left* and so the cocharge labeling drops by one.

<sup>a</sup>Ah! Inadvertently **you** helped me with this problem as the decomposition idea was written on your thesis!

**Exercise 9.** Give a counterexample showing that the formula in the above problem does not hold in general when  $w_1 = 1$ .

#### Answer

The word 121 has cocharge labeling 000 giving it a cocharge of 0 whereas 211 has cocharge labeling 100 with cocharge 1.