Exercise 1. Do the following:

I) Give a simple description of the closed sets in $\mathbb{A}^1$ (with respect to the Zariski topology).
II) Use your previous answer to prove that $\mathbb{A}^1$ is not Hausdorff.

Answer

I) If we consider $\mathbb{A}^1$ over an algebraically closed field k of characteristic zero then every closed set is of the form $V(I)$ where $I \in \operatorname{Spec}(k[x])$. Since $k[x]$ is a PID, then $I = \operatorname{gen}(p)$ for a polynomial $p \in k[x]$. Then $V(I)$ would be the set of roots inside k of p. Since p is arbitrary, every closed set $V(I)$ of $\mathbb{A}^1$ is a finite set.

   This means that the open sets are the complement of the finite sets. In essence, the Zariski topology coincides with the cofinite topology over $\mathbb{A}^1$.
II) The cofinite topology is not Hausdorff, so it follows that the Zariski topology isn't Hausdorff as well.

Exercise 2. Show that the Zariski topology on $\mathbb{A}^2$ is not the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$. (Hint: Consider the diagonal.)

Answer

Recall the following topological facts:

I) If X, Y are not Hausdorff, it follows that $X \times Y$ is not Hausdorff.
II) X is Hausdorff if and only if the diagonal set is closed.

To show that the Zariski topology and the product topology are different on $\mathbb{A}^2$ we will show that the diagonal set D in $\mathbb{A}^2$ is closed. Then the argument is as follows:

$$\left(D \text{ is closed in } \mathbb{A}^2 \text{ with } \mathcal{Z}_2\right) \wedge \left(D \text{ is open in } \mathbb{A}^2 \text{ with } \mathcal{Z}_1 \times \mathcal{Z}_1\right) \Rightarrow \mathcal{Z}_2 \neq \mathcal{Z}_1 \times \mathcal{Z}_1.$$

To show that D is closed in $\mathbb{A}^2$, we can see that it is the zero locus for the polynomial $x - y$. This is $D = V(x - y)$.
On the other hand, the product topology $\mathcal{Z}_1 \times \mathcal{Z}_1$ is not Hausdorff since $\mathcal{Z}_1$ is not Hausdorff. Therefore D is open in $\mathcal{Z}_1 \times \mathcal{Z}_1$.
We conclude that both topologies are different on $\mathbb{A}^2$.

Exercise 3. Let $F : V \to W$ be a morphism of affine algebraic varieties. Prove that F is continuous in the Zariski topology.

Answer

Suppose $\mathcal{Z}$ is the Zariski topology. Recall that

$$F \text{ is continuous } \iff \forall U \in \mathcal{Z}(F^{-1}[U \backslash W] \notin \mathcal{Z}).$$

This means that F sends closed sets back to closed sets. Since a morphism is a vector of polynomials we can easily check this result.
Consider an arbitrary closed set $V_0 \subseteq W$. Then there exists an ideal $I \trianglelefteq k[\mathbf{x}]$ such that $V_0 = V(I)$. Now consider the following observation

$$\mathbf{x} \in V(I) \iff \forall p \in I(p(\mathbf{x}) = 0) \iff \forall p \in I(\mathbf{x} \in V(p)) \iff \mathbf{x} \in \bigcap_{p \in I} V(p).$$

Since the inverse image behaves well with intersections, it suffices to prove that for any polynomial p, $F^{-1}[V(p)]$ is also a closed set.
By definition

$$F^{-1}[V(p)] = \{ \mathbf{x} \in V : F(\mathbf{x}) \in V(p) \}, \; F(\mathbf{x}) \in V(p) \iff p(F(\mathbf{x})) = 0.$$

This means that this set is precisely $V(p \circ F)$ and therefore is a closed set. Since p was arbitrary, it holds for any closed set. Therefore F is continuous in the Zariski topology.

Exercise 4. Show that the twisted cubic V of Figure 1.5 is isomorphic to the affine line by constructing an explicit isomorphism from $\mathbb{A}^1$ to V. (Hint: See Exercise 1.2.3)

Exercise 5. Show that if $F : X \to Y$ is a surjective morphism of affine algebraic varieties, then the dimension of X is at least as large as the dimension of Y.

Answer

Recall that the dimension of an algebraic variety is the length of the longest chain of irreducible proper subvarieties. This is:

$$\dim X = \max_n \{ X \supsetneq V_n \supsetneq V_{n-1} \supsetneq \cdots \supsetneq V_0 = \{ \mathbf{x}_0 \} \}.$$