Exercise 1. Consider the following expressions: $\sqrt{4+9y^2}$ and $\frac{16}{\sqrt{4t^2-9}}$. What are the trigonometric substitutions required to simplify each of the expressions?

The first expression has a plus which indicates it's a tangent substitution. As it has constants different than 1, we must multiply or divide by their square roots. The 9 multiplying the y^2 indicates we should divide by $\sqrt{9}=3$, while the 4 away from the y^2 indicates we must multiply by 2. Thus the substitution is $y=\frac{2}{3}\tan(\theta)$. A similar reasoning gives us $t=\frac{3}{2}\sec(\theta)$ for the second expression.

Exercise 2. Find an anti-derivative of $\sin^3(t)\cos(t)$.

Even though it might seem like a trigonometric integral, it suffices to do the following u-substitution: $u=\sin(t)$ and $\mathrm{d}u=\cos(t)\mathrm{d}t$. Then

$$\int \sin^3(t)\cos(t)\mathrm{d}t=\int u^3\mathrm{d}u=\frac{u^4}{4}+C=\frac{1}{4}\sin^4(t)+C.$$

Exercise 3. Manipulate the expression $\tan^6(r)\sec^4(r)$ in order to find an antiderivative of this function.

Separating a $\sec^2(r)$ for a possible substitution we have that

$$\tan^6(r)\sec^4(r)=\tan^6(r)\sec^2(r)\sec^2(r)=\tan^6(r)(\tan^2(r)+1)\sec^2(r).$$

Now taking the integral of this expression and making the u-substitution $u=\tan(r)$ we obtain

$$\int \tan^6(r)(\tan^2(r)+1)\sec^2(r)\mathrm{d}r=\int u^6(u^2+1)\mathrm{d}u=\frac{1}{9}u^9+\frac{1}{7}u^7+C=\frac{1}{9}(\tan(r))^9+\frac{1}{7}(\tan(r))^7+C$$

Exercise 4. Evaluate the integral $\int \frac{\mathrm{d}x}{\sqrt{(2x-1)^2-4}}$. *Your answer must be a function depending on x.*[It is helpful to know that an anti-derivative of $\sec(\theta)$ is $\log(\sec(\theta)+\tan(\theta))$.]

We first take a u-substitution, $u=2x-1$ and $\mathrm{d}u=2\mathrm{d}x$. This converts our integral to

$$\frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{u^2-4}}.$$

The negative sign indicates sine or secant substitution, but since the u^2 is positive, we choose secant. The 4 indicates we must multiply that secant by $\sqrt{4}=2$. So we take $u=2\sec(\theta)$ and $\mathrm{d}u=2\sec(\theta)\tan(\theta)$. The integral becomes

$$\frac{1}{2}\int \frac{2\sec(\theta)\tan(\theta)}{\sqrt{(2\sec(\theta))^2-4}}\mathrm{d}\theta=\int \frac{\sec(\theta)\tan(\theta)}{\sqrt{4(\sec^2(\theta)-1)}}\mathrm{d}\theta=\frac{1}{2}\int \frac{\sec(\theta)\tan(\theta)}{\sqrt{\tan^2(\theta)}}\mathrm{d}\theta=\frac{1}{2}\int \sec(\theta)\mathrm{d}\theta$$

By the result in the statement we have that this integral is

$$\begin{aligned}\frac{1}{2}\log(\sec(\theta)+\tan(\theta))&=\frac{1}{2}\log(\sec(\operatorname{arcsec}(u/2))+\tan(\operatorname{arcsec}(u/2)))\\&=\frac{1}{2}\log(\sec(\operatorname{arcsec}((2x-1)/2))+\tan(\operatorname{arcsec}((2x-1)/2)))\end{aligned}$$

Also, knowing $\sec(\theta)=\frac{u}{2}$ means (in SOHCAHTOA) $H=u$, $A=2$ and then $O=\sqrt{u^2-4}$. The result is then

$$\frac{1}{2}\log\left(\frac{u}{2}+\frac{\sqrt{u^2-4}}{2}\right)=\frac{1}{2}\log\left(\frac{2x-1}{2}+\frac{\sqrt{(2x-1)^2-4}}{2}\right).$$