Exercise 1 (Exercise 3)**.** Prove that a word w has highest weight (i.e., $E_i(w) = 0$ for all i) if and only if w is Yamanouchi

Answer

First suppose $w \in [n]^k$ is a word of length k on the alphabet $[n]$. Now suppose additionally that w is Yamanouchi. This means that for every $s \leqslant k$, the suffix

$$w_{k-s+1} \ldots w_{k-1} w_k$$

contains at least as many i's after the $(i+1)$'s. In particular this holds when $s = k$. So when applying the raising E_i operator we pair $(i+1)$ with an i to its right as a parenthesis. There are as much i's as $(i+1)$'s so every $(i+1)$ is paired and so the E_i operator can't convert any $(i+1)$ to an i.
As i is arbitrary, we can't apply any E_i to w which means that w has highest weight.
On the other hand[a] suppose $w \in [n]^k$ has highest weight. Then for all i, we can't apply E_i to w. This means that in w, it is possible to match all the $(i+1)$'s with i's that succeed them.
The previous fact lets us see that when reading w from right-to-left we will find at least as many i's as we find $(i+1)$'s. In other words, this means that w is Yamanouchi.

[a]Once again, **Kelsey**, **Trent**, and myself have talked about this problem in order to under the idea.

Exercise 2 (Exercise 4)**.** Formulate and prove a Yamanouchi-type condition for w to be lowest weight, that is, $F_i(w) = 0$ for all $i \in [n]$. Such a word is called <u>anti-ballot</u>.

Answer

We will define anti-ballot by remembering the definition of ballot. Recall a ballot word w in $[n]^k$ has the property that for every $p \leqslant k$, the prefix

$$w_1 w_2 \ldots w_p$$

contains at least as many i's *before* the $(i+1)$'s. So an anti-ballot word will contain as many $\underline{(i+1)}$'s before the $\underline{i}$'s. The claim is as follows:

A word w has lowest weight if and only if w is anti-ballot.

To prove this we first assume a word w is anti-ballot. Before applying F_i we notice that all the latter i's must already be paired with a previous $(i+1)$ because the word is anti-ballot.
This property holds for all i so it happens that we can't apply F_i to w, thus w has lowest weight.
On the flip-side, if we assume w has lowest weight, it means that all i's in w are matched with $(i+1)$ that precede them. So it must hold that there are at least as many $(i+1)$'s as i's in w. This condition holds for every i which means that w is anti-ballot.

Remark 1. It is important to notice that an anti-ballot word is not necessarily Yamanouchi even if it contains all the possible letters of the alphabet in question. The word

$$33\dots321$$

is anti-ballot but not Yamanouchi.

The examples from the crystal graphs contain Yamanouchi words on the top and anti-ballot words on the bottom.

Exercise 3 (Exercise 7)**.** How many ballot words of length n have only 1's and 2's?

Answer

Let us begin by counting the small cases:

- When $k=1$ we only have the word 1.
- When $k=2$ we have both 11 and 21.
- When $k=3$ we have $111, 211$ and 121. We can't add 221 because it stops being Yamanouchi.
- For $k=4$ the words are $1111, 2111, 1211, 2211, 1121$ and 2121.
- The next case is $k=5$ with

  $$11111, 21111, 12111, 22111, 11211, 21211, 12211, 11121, 21121 \quad \text{and} \quad 12121.$$

- Notice now that we will get 20 possibilities for $k=6$ because Yamanouchi words on $k=5$ all have three 1's and two 2's so it's possible to add a 1 or a 2 on each possibility which brings our total to 20.