MATH502 — Combinatorics 2

Based on the lectures by Maria Gillespie

Notes written by Ignacio Rojas

Spring 2023

Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This is the second semester of an introductory graduate-level course on combinatorics. We will be covering symmetric function theory, Young tableaux, counting with group actions, designs, matroids, finite geometries, and not-so-finite geometries.

The goal of this class is to give an overview of the wide variety of topics and techniques in both classical and modern combinatorial theory.

Requirements

Knowledge on theory of enumeration, generating functions, combinatorial species, the basics of graph theory, posets, partitions and tableaux, and basic symmetric function theory is required.

Contents

Chapter 1

Symmetric functions

1.1 Day 1 | 20230120

Definition 1.1.1. $f(x_1, x_2, \dots)$ is symmetric if it's fixed under permutations of variables. For a permutation σ this is,

$$f(x_{\sigma(1), x_{\sigma(2)}}, \dots) = f(x_1, x_2, \dots).$$

Example 1.1.2. The function

$$f(x_1, \dots, x_4) = x_1^5 + \cdots + x_4^5$$

is known as p_5 or $m_{(5)}$, where p is the power-sum symmetric function and m, the monomial symmetric function.

We can have the function defined on infinitely many variables. Consider the function g defined as

$$g = x_1^4 x_2 + x_1^4 x_3 + \cdots + x_i^4 x_j + \cdots + 3x_1 + \cdots + 3x_i + \cdots = m_{(4,1)} + 3m_{(1)}.$$

Let us recall some **notation**,

$$\begin{cases} \Lambda_R(x_1, \dots, x_n) \to \text{symmetric functions on } n \text{ variables over } R, \\ \Lambda_R(\underline{x}) \to \text{symmetric functions on } \textit{infinitely} \text{ many variables over } R. \end{cases}$$

In our case $R = \mathbb{Q}$, so the object of study is $\Lambda_{\mathbb{Q}}$.

Proposition 1.1.3. *The space $\Lambda_{\mathbb{Q}}^n$ is the space of symmetric functions of degree n. Its dimension is $p(n)$, the number of partitions of n.*

This is because, for every such function we can decompose it into monomials and the monomial symmetric functions form a basis.

Bases of Λ_Q

Suppose $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ with $\lambda_1 \geqslant \dots \geqslant \lambda_k$.

Monomial Symmetric Functions

The function $m_\lambda(\underline{x})$ is the smallest symmetric function which contains the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_k^{\lambda_k}$ as a term. In general

$$m_\lambda = \sum_{i_1 \neq \dots \neq i_k} x_{i_1}^{\lambda_1} \dots x_{i_k}^{\lambda_k}.$$

Example 1.1.4. Consider the partition $(5,3) \vdash 8$. The function $m_{(5,3)}$ will be different depending on the number of variables:

- ◇ In one variable we can't have monomials of the form $x_i x_j$, so $m_{(5,3)} = 0$.
- ◇ In two variables we have $m_{(5,3)}(x,y) = x^5y^3 + y^5x^3$.
- ◇ In three variables the function is

$$m_{(5,3)}(x,y,z) = x^5y^3 + y^5z^3 + z^5x^3 + y^5x^3 + z^5y^3 + x^5z^3.$$

Considering some special cases, take the partition $(1,1,1,1) \vdash 4$, then

$$\begin{aligned} m_{(1,1,1,1)}(u,v,x,y,z) &= uvxy + vxyz + xyzu + yzuv + zuvx \\ &= uvxy + uxyz + uvyz + uvxz + vxyz. \end{aligned}$$

For cases with less than 4 variables the function is zero and in exactly four, it has 1 term. The partition $(4) \vdash 4$ returns the function

$$m_{(4)}(x) = x^4,\ m_{(4)}(x,y) = x^4 + y^4,\ m_{(4)}(x,y,z) = x^4 + y^4 + z^4,$$

and so on with any number of variables.

Remark 1.1.5. The number of terms in $m_\lambda(x_1, \dots, x_d)$ is I actually don't know, while the degree of m_λ is $|\lambda| = n$.

Elementary Symmetric Functions

Definition 1.1.6. For any $r \in \mathbb{N}$, the elementary symmetric function e_r is $m_{(1,1,\dots,1)}$ (r ones). For λ, a partition, $e_\lambda = \prod e_{\lambda_i}$. As an alternative for $m_{(1,1,\dots,1)}$ we can also write

$$e_r(x_1, \dots, x_d) = \sum_{1 \leqslant i_1 < \dots < i_r \leqslant n} x_{i_1} \dots x_{i_r}.$$

Example 1.1.7. Let us calculate $e_{(2,1)}$ for 1 through 3 variables. When we have $e_{(2,1)}(x) = e_2(x)e_1(x)$, we can't compute $e_2(x)$ because there are no two-term monomials with only one variable. On two variables we have the following

$$e_{(2,1)}(x,y) = e_2(x,y)e_1(x,y) = (xy)(x+y) = x^2y + y^2x$$

and when talking about 3 variables the following happens:

$$\begin{aligned} e_{(2,1)}(x,y,z) &= e_2(x,y,z)e_1(x,y,z) \\ &= (xy+yz+zx)(x+y+z) \\ &= x^2y + y^2z + z^2x + y^2x + z^2y + x^2z + 2xyz. \end{aligned}$$

Consider now the partitions $(2,2,2,2)$ and (5). Then

$$e_{(2,2,2,2)} = e_2^4 \Rightarrow e_{(r,r,\ldots,r)} = e_r^{m_r(\lambda)}$$

where $m_i(\lambda)$ is number of parts of λ equal to i. For the partition (5) we have that $e_{(5)} = e_5$ and in general $e_{(n)} = e_n$.

Remark 1.1.8. As before we don't know how many terms per function, but knowing m implies knowing e. As for the degree, it holds that $\deg(e_\lambda) = |\lambda|$.

Homogenous Symmetric Functions

- Homogenous: $h_\lambda = \prod h_{\lambda_i}$ and $h_d = x_1^d + \cdots + x_1^{d-1}x_2 + \cdots + x_1^{d-2}x_2^2 + x_1^{d-2}x_2x_3 + \ldots$. In general $h_d = \sum_{\lambda \vdash d} m_\lambda$.
- Power sum: $p_\lambda = \prod p_{\lambda_i}$ and $p_d = \sum x_i^d$.

For Schur basis recall SSYT

Example 1.1.9. Consider $\lambda = (5,4,1)$, rows $\leqslant \rightarrow$ and columns $<$, we associate the monomial $x_1^2x_2^3x_3^3x_4^2 := x^T$.

- Schur: $s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$ but also $\sum K_{\lambda\mu} m_\mu$ where the sum is over SSYT of shape λ, content μ.

Schur function motivation (preview)

The first place they showed up is in the representation theory of Lie group. The function $s_\lambda(x_1, \ldots, x_n)$ is There's also the Schur-Weyl duality which takes representations into the Weyl group. Under the *Frobenius map*, s_λ corresponds to irreducible representations of S_n.

A more modern application of Schur function goes into geometry, s_λ correspond to Schubert varieties in Grassmannians. Multiplication corresponds to interesections and sum to unions.

There's also context in Probability Theory. But in the end, Schur positivity is important because of this connections.

Definition 1.1.10. $f \in \Lambda$ is Schur-positive if $f = \sum c_\lambda s_\lambda$, $c_\lambda \geqslant 0$.

Example 1.1.11. $3s_{(2,1)} + 2s_{(3)}$ schur pos but change 2 to $-\frac{1}{2}$ then not.

1.2 day 2

Alg defn Schur fncs

Definition 1.2.1. A function is antisymmetric if for $\pi \in S_n$,

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \operatorname{sgn}(\pi) f(x_1, \ldots, x_n).$$

Example 1.2.2. The following functions are antisymmetric:

(a) $f(x, y) = x - y$ then $f(y, x) = -f(x, y)$.
(b) $g(x, y) = (x - y)(x + y)$.
(c) $h(x, y) = x^2 y - y^2 x$.

Notice that the last function can factor as $h = -xy(x - y)$. We claim that this is always the case.

Lemma 1.2.3. *Every antisymmetric polynomial f in two variables x, y can factor as $f(x, y) = (x - y)g(x, y)$ where g is symmetric.*

Proof

Suppose f is antisymmetric, then $f(x, x) = 0$ by taking $y = x$. This means that $(x - y) \mid f$. Thus $f(x, y) = (x - y)g(x, y)$ and we now need to show that g is symmetric.

$$g(y,x) = \frac{f(y,x)}{y-x} = \frac{-f(x,y)}{-(x-y)} = \frac{f(x,y)}{x-y} = g(x,y).$$

Monomial Antisymmetric Functions

Definition 1.2.4. Given a strict partition $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda_1 > \cdots > \lambda_k$, we define

$$a_\lambda(x_1, \dots, x_n) = x_1^{\lambda_1} \cdots x_k^{\lambda_k} \pm \text{similar terms} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_k x_{\pi(k)}^{\lambda_k}.$$

This a_λ can be zero.

Example 1.2.5. For two variables we've seen some antisymmetric polynomials. Let us calculate

$$a_{(3,1)}(x,y) = x^3y - y^3x.$$

The smallest possible example in 3 variables is

$$a_{(2,1,0)}(x,y,z) = x^2y + y^2z + z^2x - y^2x - z^2y - x^2z.$$

This can be factored as $(x-y)(y-z)(x-z)$. A similar construction gives us

$$a_{(4,2,0)}(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - y^4x^2 - z^4y^2 - x^4z^2,$$

but how does this factor? We get

$$a_{(4,2,0)}(x,y,z) = (x^2-y^2)(y^2-z^2)(x^2-z^2) = a_{(2,1,0)}(x,y,z)(x+y)(y+z)(x+z).$$

Lemma 1.2.6. *The set $\{ a_\lambda \}_{\lambda \text{ strict}}$ is a basis of the antisymmetric polynomials over $\mathbb{Q}$, $A_\mathbb{Q}$. Even more any a_λ is divisible by a_ρ where $\rho = (n-1, n-2, \dots, 2, 1, 0)$.*

As an algebra generator, a_ρ is a generator.

Proof

WRITE

Proposition 1.2.7. *The a_ρ antisymmetric function is also the Vandermonde determinant:*

$$a_\rho = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \dots & x_1^2 & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \dots & x_2^2 & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \dots & x_n^2 & x_n & 1 \end{pmatrix}$$

Schur Polynomials

Definition 1.2.8. The Schur polynomial of $\lambda \in \mathrm{Par}$ is

$$s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})}.$$

Here $\lambda + \rho$ is the pointwise sum as arrays.

Remark 1.2.9. This is the Weyl character proof.

The following proof is due to Proctor(1987) find ref

Lemma 1.2.10. *Any a_λ can be seen as a determinant in the following way*:

$$a_\lambda(\underline{x}) = \det \begin{pmatrix} x_1^{\lambda_1} & x_1^{\lambda_2} & \ldots & x_1^{\lambda_n} \\ x_2^{\lambda_1} & x_2^{\lambda_2} & \ldots & x_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \ldots & x_n \end{pmatrix}$$

Proof

We want to see that
$$\frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})} = \sum x^T$$
where the sum ranges through T's which are SSYT(la) with max entry n.

(a) We will show a recursion for the combinatorial definition that the character formula will also satisfy. It holds that

$$s_\lambda(\underline{x}) = \sum s_\mu(\underline{x}) x_n^{|\lambda|-|\mu|}$$

where μ has $n-1$ parts with $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \ldots$.

(b) We also show that the ratio of determinants satisfies the same recursion.

Example 1.2.11. Consider $\lambda = (8, 8, 4, 1, 1)$ and $\mu = (8, 5, 2, 1)$, then $\lambda \backslash \mu$ is a skew-table in which we can fill in n's

Corollary 1.2.12. *The Schur polynomials are a basis of $\Lambda_{\mathbb{Q}}$.*

1.3 Day 3 | 20230125

Recall $\Lambda = \mathbb{Q}[e_1, e_2, \dots]$ where the e_j's are the elementary symmetric functions. So the e_j's are algebraic generators of Λ and they're algebraically independent. Equivalently, as a vector space, $\{ e_\lambda : \lambda \in \text{Par} \}$ is a basis.

Proposition 1.3.1. *A homomorphism $f : \Lambda \to \Lambda$ ($f(a+b) = f(a) + f(b),\ f(ab)f(a)f(b)$ for $a, b \in \Lambda$) is fully determined by where it sends the e_i's.*

Definition 1.3.2. The map $\omega \in \mathrm{End}(\Lambda)$ will send e_j to h_j.

Example 1.3.3. Consider $f = 3e_{(2,1)} + 2e_3$, then applying ω we get

$$\omega(f) = \omega(3e_{(2,1)} + 2e_3) = 3h_{(2,1)} + 2h_3.$$

For p_2, we can decompose to $e_1^2 - 2e_2$. So

$$\omega(p_2) = \omega(e_1^2 - 2e_2) = h_1^2 - 2h_2$$

and we can expand this last expression into

$$(x_1 + x_2 + \dots)^2 - 2(x_1^2 + x_2^2 + \dots + x_1x_2 + x_1x_3 + \dots) = -x_1^2 - x_2^2 - \dots$$

and we recognize this last term as $-p_2$. *This is not a coincidence.*

Theorem 1.3.4. *The map ω is involutive.*

Proof

It suffices to prove that $\omega(h_j) = e_j$. We will use power expansions and generating functions. We have

$$H(t) = \frac{1}{1 - x_1 t}\frac{1}{1 - x_2 t}\cdots = \sum h_n(\underline{x})t^n,$$

and this comes from expanding the $1/(1-y)$'s as geometric series. When collecting the coefficients of t^n we get exactly $h_n(\underline{x})$. Similarly, for the elementary symmetric functions,

$$E(t) = (1 + x_1 t)(1 + x_2 t)\cdots = \sum e_n t^n.$$

When multiplying to obtain the coefficient of t^n we get a plethora of different x_j's which form the e_j's. Now from this expressions we have $H(t)E(-t) = 1$

which means that

$$\left(\sum h_n(\underline{x})t^n\right)\left(\sum e_n(\underline{x})(-t)^n\right) \Rightarrow \sum_{k=0}^{n}(-1)^k e_k h_{n-k} = 0,\ n \geqslant 1.$$

Now applying the map to the equation we get

$$\omega\left(\sum_{k=0}^{n}(-1)^k e_k h_{n-k}\right) = \sum_{k=0}^{n}(-1)^k h_k \omega(h_{n-k}) = 0.$$

After reindexing, we get that both e_j's and $\omega(h_j)$'s are determined recursively by the h_j's in the same way. Thus we conclude that $\omega(h_j) = e_j$.

Lemma 1.3.5. *The following equation holds for the power-sum symmetric functions*:

$$\exp\left(\sum \frac{1}{n}p_n(\underline{x})p_n(\underline{y})\right) = \prod_{i,j=1}^{\infty}\frac{1}{1-x_iy_j} = :\Omega(\underline{x},\underline{y}).$$

It also holds that

$$\Omega(\underline{x},\underline{y}) = \sum_{l} a\frac{1}{z^{\lambda}}p_\lambda(\underline{x})p_\lambda(\underline{y})$$

where $z_\lambda = \prod k^{m_k}m_k!$ *where* m_k *is the number of parts of* λ *equal to* k.

Proof

We will prove both parts separately. For the first equation we will take the logarithm on both sides:

$$\sum \frac{1}{n}p_n(\underline{x})p_n(\underline{y}) = \log\left(\prod_{i,j=1}^{\infty}\frac{1}{1-x_iy_j}\right)$$

and after manipulating the logarithm we get

$$\sum_{i,j=1}^{\infty}(\log(1) - \log(1-x_iy_j)) = \sum_{i,j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}x_i^n y_j^n.$$

We can separate[a] into

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_i x_i^n\right)\left(\sum_j y_j^n\right).$$

Now taking $\exp$ on both sides we get equality.

By not removing the exponential we get the following expression

$$\exp\left(\sum \frac{1}{n} p_n(\underline{x}) p_n(\underline{y})\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum \frac{1}{n} p_n(\underline{x}) p_n(\underline{y})\right)^k.$$

To get a term of the form $p_\lambda(\underline{x})p_\lambda(\underline{y})$ we have to choose which parts of the λ come from each of the factors in $\sum \frac{1}{n} p_n(\underline{x}) p_n(\underline{y})$. If $\ell(\lambda) = k$ then it comes from the k^{th} term in the exponential sum. If $\lambda = (\lambda_1, dots, \lambda_1, \ldots, 2, \ldots, 2, 1, \ldots, 1)$ with m_{λ_1} λ_1's, m_1 1's, then out of k elements we have to choose m_1 1's and so on. Thus there are $\binom{k}{m_{\lambda_1}, \ldots, m_1}$ choices and each i in λ comes with a $\frac{1}{i}$. Therefore the coefficient of $p_\lambda(\underline{x})p_\lambda(\underline{y})$ is

$$\frac{1}{k!} \frac{k!}{m_1! m_2! \ldots} \frac{1}{1^{m_1}} \frac{1}{2^{m_2}} \cdots = \frac{1}{z_\lambda}.$$

[a] Are we using Fubini-Tonelli here?

Lemma 1.3.6. *We have the following identities*

$$\exp\left(\sum \frac{(-1)^{n-1}}{n} p_n(\underline{x}) p_n(\underline{y})\right) = \prod_{i,j=1}^{\infty} \frac{1}{1 + x_i y_j} = \sum_\lambda \frac{(-1)^{n-\ell(\lambda)}}{z_\lambda} p_\lambda(\underline{x}) p_\lambda(\underline{y}).$$

Lemma 1.3.7. *Another equality for $\Omega(\underline{x}, \underline{y})$ is*

$$\Omega(\underline{x}, \underline{y}) = \sum_\lambda m_\lambda(\underline{x}) h_\lambda(\underline{y})$$

Theorem 1.3.8. *It holds that $\omega(p_\lambda) = (-1)^{n-k} p_\lambda$ where k is the number of parts of λ.*

Proof

Applying ω to Ω, but *only working with $\underline{y}$ variables* we get

$$\omega(\Omega) = \omega\left(\sum_\lambda m_\lambda(\underline{x}) h_\lambda(\underline{y})\right) = \sum_\lambda m_\lambda(\underline{x}) e_\lambda(\underline{y}) = \prod_{i,j=1}^{\infty} (1 + x_i y_j) = \sum_\lambda \frac{1}{z_\lambda} (-1)^{n-k_\lambda} p_\lambda(\underline{x}) p_\lambda(\underline{y}).$$

Comparing coefficients with

$$\omega\left(\sum_l a \frac{1}{z^\lambda} p_\lambda(\underline{x}) p_\lambda(\underline{y})\right)$$

we get the result.

1.4 Day 4 | 20230127

To continue exploring the ring of symmetric functions we need a couple of tools. One of them is the involution which we have already seen. But the other one is a scalar product which is compatible with the multiplication.

Hall Inner Product

Recall an inner product is a function

$$\langle - | - \rangle \colon V \times V \to \mathbb{Q}$$

which is bilinear $\langle u+v|w\rangle = \langle u|w\rangle + \langle v|w\rangle$ and the same on the other entry. For scalars the following behavior is expected $\langle \lambda u|v\rangle = \langle u|\lambda v\rangle = \lambda\langle u|v\rangle$. Recall that if the base field is the complex numbers, then the inner product is Hermitian.

Definition 1.4.1. We say that two vectors are orthogonal when $\langle u|v\rangle = 0$.

This gives us a possible decomposition of space into several components. Suppose that $\{ u_\lambda \}_{\lambda \in \mathrm{Par}(n)}, \{ v_\lambda \}_{\lambda \in \mathrm{Par}(n)}$ are basis of Λ^n. So we would like a condition such as

$$\langle u_\lambda | v_\mu \rangle = \begin{cases} 0 \ \lambda \neq \mu, \\ 1 \ \lambda = \mu. \end{cases}$$

If we cap the dimension this says that $\langle u|v\rangle$ is the usual dot product. But in infinite dimensions we don't have matrices. We'll call this basis dual to one another. If miraculously we have the same basis, then this basis is orthonormal.

Definition 1.4.2 (Phillip Hall)**.** The Hall inner product is defined so that $\langle m_\lambda | h_\mu \rangle = \delta_{\lambda\mu}$.

By defining the product on two basis, we have defined it for all other elements by bilinearity.

Lemma 1.4.3. *The Hall inner product is symmetric.*

Theorem 1.4.4. *The Hall inner product is positive definite, this is $\langle f|f\rangle \geqslant 0$ and equality is achieved when $f = 0$.*

It's important to note that this statement is symmetric. However we are talking about an asymmetric definition. Last, before proving the statement we need a criteria for dual bases. But importantly, recall the result from last lecture: 1.3.7

Theorem 1.4.5. *If $u_\lambda, \{ v_\mu \}$ are dual, then $\sum_\lambda u_\lambda v_\lambda = \Omega$.*

Proof

Fix a partition of n, then

$$\delta_{\lambda\mu} = \langle m_\lambda | h_\mu \rangle = \left\langle \sum_{\rho \vdash n} \alpha_{\lambda_\rho} u_\rho \middle| \sum_{\tau \vdash n} \beta_{\mu_\tau} v_\tau \right\rangle = \sum_{\rho,\tau} \alpha_{\lambda_\rho} \beta_{\mu_\tau} \langle u_\rho | v_\tau \rangle .$$

We want $\langle u_\rho | v_\tau \rangle = \delta_{\rho\tau}$, to that effect name $A_{\rho\tau}$ the matrix whose entries are $\langle u_\rho | v_\tau \rangle$.

As u and v are dual bases, we have that $A = \mathrm{id}$. Thus $I = \alpha\beta^\mathsf{T}$ and now $\delta_{\rho\tau} = \sum \alpha_{\lambda_\rho} \beta_{\lambda_\tau}$. We are now going to use the hypothesis and the interpretation of m, h in the u, v basis. We have

$$\Omega = \sum \left(\sum \alpha u \right) \left(\sum \beta v \right) = \sum \left(\sum \alpha \beta \right) uv = \sum uv$$

so the inner sum must be one and thus we are done.

Corollary 1.4.6. *For the Hall inner product it holds that $\langle p_\lambda | p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$.*

The key is to recall that p_λ is an eigenfunction of ω. Also 1.3.5. By using a power-sum decomposition it is possible to prove that the Hall inner product is positive definite.

Corollary 1.4.7. *The ω involution is orthogonal with respect to $\langle - | - \rangle$. This is $\langle \omega f | \omega g \rangle = \langle f | g \rangle$.*

Once again, the idea is to transfer to power-sum and use the fact that it's an eigenfunction.

1.5 Interim 1

Theorem 1.5.1 (Fundamental Theorem of Sym. Fnc. Thry.)**.** *Every symmetric function can be written uniquely in the form $\sum_\lambda c_\lambda e_\lambda$ with $c_\lambda \in \mathbb{Q}$.*

There are at least two proofs if not more of this fact. The first comes from Maria Gillespie's blog which Mark Haiman presented to her.

Proof

It suffices to prove the transition matrix between m and e is invertible.

For proof 2 read [10] pg. 290. Proof 3 in another Maria post

1.6 Day 5 | 20230130

Exercise 1.6.1. Compute $\omega(s_{(3,1)})$.

Answer

We have that By Jacobi-Trudi

$$s_{(3,1)} = \det \begin{pmatrix} h_3 & h_4 \\ 1 & h_1 \end{pmatrix} = h_{(3,1)} - h_4.$$

Using the omega involution, we get

Recall that $\omega : h_n \leftrightarrow e_n$, $\omega p_k = (-1)^{k-1} p_k$. We have the following questions, where do m and s map to? Also

$$\langle m | h \rangle = \delta,\ \langle p | p/z \rangle = \delta,$$

but what are e and s dual to?

Definition 1.6.2. We call $\omega m_\lambda = f_\lambda$ the forgotten basis.

There's not much we could say about them, they are not Schur positive and there's no patterns.

Dual to e

Recall ω is an isometry, so $\langle \omega f | \omega g \rangle = \langle f | g \rangle$, so

$$\langle e_\lambda | ? \rangle = \langle h_\lambda | \omega ? \rangle = \delta_{\lambda\mu}.$$

Since $\langle h | m \rangle = \delta$, then applying ω again we get that $\langle e_\lambda | f_\mu \rangle = \delta_{\lambda\mu}$.

RSK algorithm

We want to show two things:

$$\omega s_\lambda = s_{\lambda^\intercal},\ \langle s_\lambda | s_\mu \rangle = \delta_{\lambda\mu}.$$

Proposition 1.6.3. *It holds that*

$$\sum_\lambda s_\lambda(\underline{x}) s_\lambda(\underline{y}) = \Omega = \sum_\lambda m_\lambda(\underline{x}) h_\lambda(\underline{y})$$

Proof

The sum on the left is
$$\sum_{(S,T) SSYT} x^S y^T$$
so we will study pairs (S, T) of SSYT of the same shape to show that they're equal to the sum on the right.

algorithm: process of doing the bijection.

The RSK bijection takes a pair (S, T) of SSYT of the same shape and it maps it to "two-line arrays" of length n.

Definition 1.6.4. A two-line array is a matrix in $\mathcal{M}_{2\times n}(\mathbb{Z}_{\geqslant 0})$ such that

i) The bottom row is weakly increasing.
ii) If $b_i = b_{i+1}$, then $a_i \leqslant a_{i+1}$, where a's are the top row and b's the bottom row.

Example 1.6.5. Consider the matrix

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{pmatrix}$$

Within "blocks", there is a weak increment. From right-to-left we will find a pair of SSYT. We will "insert" top row letters from left-to-right.

(a) Place 1st letter $\boxed{1}$

(b) For each letter, if it can go at the end of last row, put it there

$$\boxed{1}\boxed{1} \leftarrow 2, \quad \boxed{1}\boxed{1}\boxed{2} \leftarrow 1$$

but one can't go after 2.

(c) Otherwise if inserting b_1, let c be the leftmost $> b$, "bump c", then insert c into the next row.

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}$$

For the bottom row, place in a new square at each step to form a "recording tableau". The recording tableau always matches the shape of the insertion one. The first three steps lead to $\begin{ytableau} 1 & 1 & 1 \end{ytableau}$ in the recording one. But in the fourth step we get $\begin{ytableau} 1 & 1 & 1 \\ 2 \end{ytableau}$. The next step leads us to

$$\begin{ytableau} 1 & 1 & 1 & 4 \\ 2 \end{ytableau}, \quad \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 \end{ytableau}$$

then in insertion, 2 bumps 4 and 4 doesn't bump 2 on next row, so we get

$$\begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 4 \end{ytableau}, \quad \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 3 \end{ytableau}$$

The three is no problem so

$$\begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ 2 & 4 \end{ytableau}, \quad \begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 \end{ytableau}$$

then the next one bumps out the 2, the 2 bumps the 4 on the second row to get

$$\begin{ytableau} 1 & 1 & 1 & 1 & 3 \\ 2 & 2 \\ 4 \end{ytableau}, \quad \begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 \\ 4 \end{ytableau}$$

Finally

$$\begin{ytableau} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 3 \\ 4 \end{ytableau}, \quad \begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 \end{ytableau}.$$

Why do we get SSYT. The insertion tableau gives us the question, can we make a column non-increasing? No, we are always bumping something bigger. Imagine we bump $c > b$ with b, then c replaces something that goes to the left.

$$\begin{ytableau} \leqslant & b & c \\ \, & \, \\ \, \end{ytableau} \Rightarrow \begin{ytableau} \, & b & \, \\ \, & d \\ \, \end{ytableau}$$

and $d > c$ so it bumps something else. The recording tableau is also a SSYT. Let us prove it.

Lemma 1.6.6 (Key Lemma 1)**.** *The insertion path* (*sequence of squares that are bumped*) *moves up and weakly left.*

Lemma 1.6.7 (Key Lemma 2)**.** *If $a \leqslant b$ and T is a SSYT, computing*

$$T \leftarrow \boxed{a} \leftarrow \boxed{b},$$

the intersection path of a in T lies strictly left of the intersection path of b in $T \leftarrow \boxed{a}$.

Proof

We will do induction on the rows with an example.

Example 1.6.8. Consider

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&2&2&3\\\hline 2&2&3&3&4\\\cline{1-5} 3&3&5&5\\\cline{1-4} 4&4\\\cline{1-2}\end{array}$$

Inserting 1 we bump the 2, then the 3 and finally the 5. We get

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&o&2&3\\\hline 2&2&t&3&4\\\cline{1-5} 3&3&t&5&5\\\cline{1-5} 4&4&f\\\cline{1-3}\end{array}$$

so inserting the 2 we bump 3,4,5. And they will be to the side of the last sequence.

1.7 Day 6 | 20230201

Exercise 1.7.1. Apply RSK to $\begin{pmatrix} 3 & 2 & 4 & 1 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

Answer

We get $\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}$.

Notice that we got STANDARD Young tableau. So to prove it's a bijection we will begin with all different numbers.

Lemma 1.7.2. *The RSK bijection is a bijection between pairs of standard Young tableaux of the same shape and "permutations"* ($2 \times n$ *matrices whose rows are permutations.*)

To prove it's a bijection we will find an inverse by reversing the process. Look at the recording tableau, we will bump out the largest number. We will take S as the recording tableau. Then we start with the spot on S, T which corresponds to largest label in S.

- If b is the item in such a square we "un-bump" it.

- If in bottom row, just remove.
- Else, let c be the rightmost entry in row below b that is less than b. Then replace b with c and repeat the process with c until the letter that is removed is done by the just removing it.

Then we add the two letters to the matrix from right-t-left.

With the original tableau we remove the 5 and the 5 to get

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}\;,\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

then the 4 indicates that in T we must "un-bump" the 3. The three un-bumps the 2, the 2 to the 1 so that we get

$$\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}\;,\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}\;.$$

Now we get the matrix $\begin{pmatrix} x & x & x & 1 & 5 \\ x & x & x & 4 & 5 \end{pmatrix}$ and removing the 3 from S just removes the 4 from T as it is in the bottom row.

Now as this two sets are in bijection, this means that they have the same size.

Corollary 1.7.3. *Let f^λ be the number of standard Young tableau of shape λ. Then*

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!.$$

We will generalize one step at a time. Let us now assume that T is semi-standard. On the matrix, we will have that the top row is now random, but the bottom row is still from 1 to n.

Lemma 1.7.4 (Schensted)**.** *There is a bijection between (S, T), S is standard, T is SSYT, and words of length n.*

Example 1.7.5. Consider the matrix $\begin{pmatrix} 2 & 1 & 3 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ which returns the two Young tableau

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}\;,\quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}\;.$$

The proof of the inverse is similar but when un-bumping, we must bump the rightmost entry *strictly* smaller than b. But we don't need this, we will do it more creatively.

Definition 1.7.6. Suppose T is a Young tableau. Then

i) The reading word of T rw(T) is the concatenation of rows from top to bottom.
ii) The standarization of an SSYT T, std(T), is the unique SYT with same relative order of entries, ties broken with "reading order".
iii) The standarization of a word is similar

In the previous example, the reading word is

$$\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&3\\ \cline{1-2}\end{array} \to 23113.$$

The standarization are as follows:

$$\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&3\\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array}, \quad 23113 \to 34125.$$

We can standarize the matrix

$$\begin{pmatrix} 2&1&3&1&3\\ 1&2&3&4&5 \end{pmatrix} \to \begin{pmatrix} 3&4&1&2&5\\ 1&2&3&4&5 \end{pmatrix}$$

and this matrix corresponds to the pair (S,T) where T is $\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array}$. In essence, the following diagram commutes

$$\begin{tikzcd} (S,T) \arrow[d, "std"'] & 21313 \arrow[l, "RSK"'] \arrow[d] \\ (S,T') & 31425 \arrow[l, "RSK"] \end{tikzcd}$$

Definition 1.7.7. Given a content $\mu = (\mu_1, \ldots, \mu_k)$ with $\sum \mu_k = n$ (not nec. partition). Then the de-standarization with respect to μ of a SYT T is a $SSYT$ T' such that std$(T') = T$.

In this case

$$\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array} \xrightarrow[\text{std}^{-1}(2,1,2)]{} \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&3\\ \cline{1-2}\end{array}.$$

Recall now lemma 1.6.7 about consecutive insertions.

The Full RSK

We are now going to prove that there is an inverse to the original RSK function. Consider the following example

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} \to \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array} \quad , \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 7 \\ \hline 4 & 6 & 9 \\ \cline{1-3} 8 \\ \cline{1-1} \end{array}$$

The matrix $\begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{pmatrix}$ can be standarized to our word matrix. Then the table $\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}$ also standarizes to the word table.

1.8 Day 7 | 20230203

Exercise 1.8.1. Expand $h_{(3,2)}$ in Schur basis.

Answer
This is $s_{(3,2)} + s_{(4,1)} + s_{(5)}$.

Recall that (s_λ) form an orthonormal basis and m and h are dual basis. This means that if f is a symmetric function then

$$f = \sum_\lambda c_\lambda s_\lambda \Rightarrow c_\lambda = \langle f | s_\lambda \rangle, \ f = \sum_\lambda a_\lambda m_\lambda \Rightarrow a_\lambda = \langle f | h_\lambda \rangle .$$

Lets suppose now that f is any homogenous symmetric function. We will calculate the coefficient of s_λ in an h_μ expansion:

$$\langle h_\mu | s_\lambda \rangle = \langle s_\lambda | h_\mu \rangle$$

and we can interpret this as the coefficient of m_μ in s_λ. This amount is precisely the Kostka coefficient $K_{\lambda\mu}$. Thus we have the formula $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$.

Properties of the Schur functions

We wish to show that $\langle s_\lambda | s_\mu \rangle = \delta_{\lambda\mu}$ and $\omega s_\lambda = s_{\lambda^\intercal}$.

Proposition 1.8.2. $\sum_\lambda s_\lambda(\underline{x}) s_\lambda(\underline{y}) = \sum_\lambda m_\lambda(\underline{x}) h_\lambda(\underline{y})$

Proof

Expanding the sum on the left we obtain

$$\sum_\lambda s_\lambda(\underline{x})s_\lambda(\underline{y}) = \sum \lambda \left(\sum_{T\in SSYT(\lambda)} x^T\right)\left(\sum_{S\in SSYT(\lambda)} y^S\right) = \sum_{(T,S),\ SSYT\text{same shape}} x^T y^S$$

This is basically an RSK pair and this correspond to two-line arrays, so this sum could be the same as summing over them. Thus this is

$$\sum_{\text{2line arrays}} x_{a_1}\dots x_{a_n} y_{b_1}\dots y_{b_n}.$$

We will now find the coefficient of $m_\lambda(\underline{y})$ in this expansion and show that it is $h_\lambda(\underline{x})$

What are all the ways to obtain $y_1^{\lambda_1}\dots y_k^{\lambda_k}$?

$$\begin{pmatrix} a_1^{(1)} & \dots & a_k^{(1)} & a_1^{(2)} & \dots & a_k^{(2)} & \dots \\ 1 & 1 & 1 & 2 & 2 & 2 & \dots \end{pmatrix}$$

And note that $a_1^{(i)} \leqslant \dots \leqslant a_{\lambda_i}^{(i)}$ for all i, so the coefficient is

$$\sum_{(a^{(i)}) validtuples} x_{a_1^{(1)}}\dots x_{a_{\lambda_k}^{(k)}}$$

but this factors as

$$\prod_{i=1}^k \sum_{a_1^(i)\leqslant\dots\leqslant a_{\lambda_k}^{(i)}} x_{a_1^{(i)}}\dots x_{a_{\lambda_k}^{(i)}}.$$

We can split this because the choices are independent of the blocks and then multiply the functions together. The last term is h_{λ_i} and the product is h_λ.

If (T,S) RSKs inverse to $\begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ then $x^T y^S$ is $x_1x_3x_2y_1y_1y_2$.

Corollary 1.8.3. $\omega s_\lambda = s_{\lambda^\intercal}$.

Proof

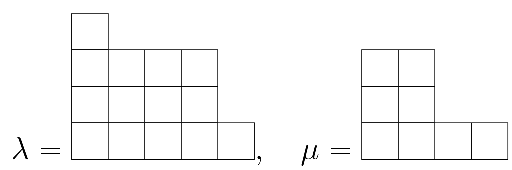
It suffices to show $\langle s_{\lambda^\intercal} | e_\mu \rangle = K_{\lambda\mu}$ because $\langle s_\lambda | h_\mu \rangle = K_{\lambda\mu}$ which implies that $\langle \omega s_\lambda | e_\mu \rangle = K_{\lambda\mu}$.
In other words, we wish to show that the coefficient of s_λ in e_μ is $K_{\lambda^\intercal \mu}$, the number of $SSYT$ shape $\lambda^\intercal$, content μ.
CONT

Pieri Rule

Definition 1.8.4. A skew shape is a diagram formed by subtracting a smaller Young diagram from a larger one.

A horizontal strip is a skew shape where no two boxes are in the same column. Similar a vertical strip doesn't have boxes in the same row.

Example 1.8.5. Suppose $\lambda = (5, 4, 4, 1)$ and $\mu = (4, 2, 2)$. Then



so la/μ is INSERT DIAG. Not horizontal nor vertical.

In a Young tableau, the biggest number forms a horizontal strip, so in general Young tableaux are made up of horizontal strips.

$$\begin{array}{|c|c|c|c|c|c|}
\hline 4 \\
\hline 3 & 4 & 4 \\
\hline 2 & 2 & 3 & 3 \\
\hline 1 & 1 & 1 & 2 & 3 & 4 \\
\hline
\end{array}.$$

Theorem 1.8.6 (Pieri). *Let $r \in \mathbb{N}$, then*

$$e_r s_\lambda = \sum_{\rho/\lambda \text{vert. strip size } r} s_\rho$$

$$h_r s_\lambda = \sum_{\rho/\lambda \text{horiz. strip size } r} s_\rho$$

This is basically all the ways to fill up the shapes.

Proof

$$h_r s_l a = s_{(r)} s_\lambda = \left(\sum_{T \in SSYT((r))} x^T \right) \left(\sum_{S \in SSYT(\lambda)} x^S \right)$$

Example 1.8.7. $h_3 s_{(3,1)}$ is $x^T x^S$ is inserting the boxes of T one at a time in S.

$$\begin{array}{ccc} 3 & & \\ 1 & 2 & 3 \end{array} \leftarrow \begin{array}{ccc} 1 & 1 & 2 \end{array} = \begin{array}{cccc} 3 & & & \\ 2 & 3 & & \\ 1 & 1 & 1 & 2 \end{array}$$

so by 1.6.6 about insertion path, the new squares are a horizontal strip which is the s_ρ in the Pieri rule. Unbumping we recover something.

1.9 Day 8 | 20230206

Exercise 1.9.1. Apply RSK to 82357146 and 62235124.

Answer

$$\begin{array}{cccc} 8 & & & \\ 2 & 3 & 5 & 7 \end{array}, \begin{array}{cccc} 2 & & & \\ 1 & 3 & 4 & 5 \end{array}$$

then 1 bumps the 2, 2 bumps 8

$$\begin{array}{cccc} 8 & & & \\ 2 & 5 & 7 & \\ 1 & 3 & 4 & 6 \end{array}, \begin{array}{cccc} 6 & & & \\ 2 & 7 & 8 & \\ 1 & 3 & 4 & 5 \end{array}$$

The next one standarizes to the last string. The same recording table but we get for insertion

$$\begin{array}{cccc} 6 & & & \\ 2 & 3 & 5 & \\ 1 & 2 & 2 & 4 \end{array}.$$

Consequences of RSK

We will talk about increasing and decreasing subsequences.

Definition 1.9.2. A longest increasing subsequence of a word $w \in \mathbb{N}^n$ is a subsequence $w_{i_1} \leqslant \ldots \leqslant w_{i_\ell}$ with $i_1 < \cdots < i_\ell$ such that ℓ is as large as possible. We will write $\ell(w)$ to be the length of the longest increasing subsequence.

A longest decreasing subsequence of a word is $w_{i_1} > \cdots > w_{i_d}$ with $i_1 < \cdots < i_d$. In this case $d(w)$ is the longest decreasing.

Example 1.9.3. In the case of 82357146, we have $2357, 2356, 146, 2346$. Notice that this is the length of ?? of the Young tableau. For decreasing we have $821, 831, \ldots$, the height of the Young tableau is the longest decreasing subsequence.

Theorem 1.9.4. *Suppose w is a word, $S = ins(w)$ is the insertion tableau through RSK and $\lambda = sh(S)$ is the shape of the table. Then $\ell(w) = \lambda_1$ and $d(w) = \lambda_1^{\mathsf{T}}$.*

To prove this we will develop some tools.

Lemma 1.9.5. *For a tableau T, $ins(rw(T)) = T$.*

The reading word of $\begin{array}{|c|c|c|c|} \hline 8 \\ \cline{1-1} 2 & 5 & 7 \\ \cline{1-3} 1 & 3 & 4 & 6 \\ \hline \end{array}$ is 82357146 which inserts to the same table precisely.

Remark 1.9.6. The column reading word also works! For this table it's 82153746. We get a bunch of decreasing subsequences. 821 creates the first column by bumping, then 53 creates the second column and so on.

Let's analyze the longest increasing subsequence of the reading word. Clearly we can get the bottom row as a longest subsequence, but looking in the reading order we need to go to the right. Going down decreases!

Lemma 1.9.7. *If $\lambda = sh(T)$ then $\ell(rw(T)) = \lambda_1$ and $d(rw(T)) = \lambda_1^{\mathsf{T}}$.*

Proof

Given an entry $a \in T$, let $b \in T$ such that $a <_{\mathrm{ro}} b$. Then b is in a column to the right of a, this means that

$$\ell(\mathrm{rw}(T)) \leqslant \#\text{columns} = \lambda_1.$$

The bottom row is an example of a subsequence where the length is achieved. So equality holds.

For decreasing it's equivalent. Now, how do we tell when two words have the same insertion tableau?

Example 1.9.8. In the case of all permutations in S_3 we have that some are equivalent FILL

Knuth equivalence

Definition 1.9.9. A Knuth move on a permutation swaps two letters a, c if $a < b < c$ (reading order) and one of consecutive subsequences acb, cab, bac, bca appears in the word.

Two words are Knuth-equivalent if they differ by a sequence of Knuth-moves.

In the first case, b is between a, c and those are always together.

Proposition 1.9.10. *Knuth equivalence defines an equivalence relation on S_n.*

Theorem 1.9.11. *Two words π, w are Knuth-equivalent iff ins(w) = ins(π).*

Example 1.9.12. In size 4, 1234 is in its own class because we don't have any Knuth moves available. Same thing happens with 4321.

Consider 1243, if we apply Knuth moves we can get

- 1423
- 4123

All of these have the insertion tableau $\begin{array}{|c|c|c|}\cline{1-1} 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$ whose reading word is 4123.

For the tableau $\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$, its reading word is 2413. Applying Knuth moves we get only 2143, which is the column reading word.

The tableau $\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & 1 \\ \hline \end{array}$'s equivalence class also has size 2.

Proposition 1.9.13. *If two tableau have the same shape, their equivalence classes have the same size.*

We are seeking to prove $\ell(w)$ is invariant under Knuth moves. This will imply the theorem 1.9.4 because once we know that things have the same insertion tableau and the reading word has the same longest increasing subsequence length.

1.10 Day 9 | 20230208

Exercise 1.10.1. Insert f, g and then c into

$$\begin{array}{|c|c|c|c|c|}\cline{1-1} k \\ \cline{1-3} e & i & j \\ \hline a & b & d & h & l \\ \hline\end{array}$$

and then f, c and then g.

Example 1.10.2. The Knuth equivalence class of words whose insertion tableau is

$$\begin{array}{|c|c|c|}\cline{1-2} 3 & 4 \\ \hline 1 & 2 & 5 \\ \hline\end{array}.$$

The reading word is 34125 and we can Knuth-move it. The 341 can switch into 314 (this has the form bac). From that one we can switch 2 and 5 to get 31452. Once again with 314 we get 34152 and 34512.

In total we have 5 elements.

Proposition 1.10.3. *The size of the Knuth equivalence class whose insertion tableau is T with shape λ is $\#SYT(\lambda)$.*

Proof

We have one permutation in the Knuth equivalence class for every recording tableau S that can be paired with T.

The Knuth equivalence class of $\begin{array}{|c|c|c|}\cline{1-2} 3 & 4 \\ \hline 1 & 2 & 5 \\ \hline\end{array}$ can be identified by RSK with the pairs (T, S) and S varies through all SYT of corresponding shape.

Also, recall that by that hook-length formula we have that

$$\#SYT(\lambda) = \frac{|\lambda|!}{\prod_{\text{hooks}\subseteq T} \text{size hooks}}.$$

Theorem 1.10.4. *Two permutations π, w have the same insertion tableau if and only if π is Knuth-equivalent to w.*

Proof

By induction on the length, we can assume π, w differ by a single Knuth-move on the last 3 letters. We separate into cases:

i) Want
$$T' \leftarrow b \leftarrow c \leftarrow a = T' \leftarrow b \leftarrow a \leftarrow c$$
Note that $\texttt{IP}(b) < \texttt{IP}(c)$ by lemma 1.6.7 of consecutive insertions and $\texttt{IP}(a)$ is *weakly left* of $\texttt{IP}(b)$ from which holds $\texttt{IP}(a)$ is strictly left of c's. So we can switch order.

ii) In the other case we want
$$T' \leftarrow c \leftarrow a \leftarrow b = T' \leftarrow a \leftarrow c \leftarrow b.$$
$\texttt{IP}(a)$ is *weakly left* of c's. If it's *strictly*, then we can switch, but otherwise the insertion paths of a and c collide. CHECK NOTES

Now on the other direction, we wish to show that two permutations with the same insertion tableau are Knuth-equivalent.
It suffices to show that they are Knuth-equivalent to the reading word. By induction of the size of the word, suppose $\text{ins}(w') = T'$. Then $w' \sim \text{rw}(T')$ for w' of length $n-1$.
Let $w \in S_n$ with $b = w_n$. If $T' = \text{ins}(w_1, \ldots, w_{n-1})$, by induction $w_1 \ldots w_{n-1} \sim \text{rw}(T') = (first\ row) \ldots (last\ row)$.

Example 1.10.5. For the second case consider the table

$$\begin{array}{|c|c|c|} \hline 3 & 8 & 9 \\ \hline 1 & 4 & 7 \\ \hline \end{array}$$

and we insert $6, 2$ then 5 but then $2, 6$ and then 5. In the first case, DUNNO

In the second case consider

$$T' = \begin{array}{|c|c|c|c|} \hline 6 \\ \cline{1-1} 4 & 7 \\ \cline{1-2} 1 & 2 & 5 & 8 \\ \hline \end{array}$$

1.11 Day 10 | 20230210

Lemma 1.11.1. *The length of the longest increasing subsequence, $\ell(w)$ is invariant under Knuth moves.*

Proof

Given an increasing subsequence, if a Knuth move changes two of its entries $a < c$, we have two cases:

i) Either b is to the right of ac so we get

$$\dots acb \dots d \dots \to \dots cab \dots \dots d \dots$$

then replacing c with b gives an increasing subsequence of same length in a new word $a < b < c$ by assumption so $\underline{a} < \underline{b} < c < \underline{d}$ where d is the next element of the subsequence after c.

ii) We have b to the left so we have

$$\dots bac \dots d \dots \to \dots bca \dots d \dots$$

and the same proof shows that replacing a and b gives s anew subsequence and a new subword of the same length.

Knuth equivalence is natural when it comes to increasing and decreasing subsequences. With the lemmas we have, we can now prove that d is the height of the insertion tableau.

Remark 1.11.2. Dual equivalence is finding 3 values acb so we can switch bca. For example

$$615342 \to 625341 \to 624351$$

and dual equivalent words have the same *recording tableau*.

The result we've been aiming for is

$$\ell(w) = \text{width of ins(w)} = \lambda_1$$

Proof

Suppose $T = \text{ins}(w)$ with $\lambda = \text{sh}(T)$. Then $w \sim \text{rw}(T)$, by the previous lemma we have

$$\ell(w) = \ell(\text{rw}(T)) = \lambda_1$$

and the last equality comes from Monday class. Add references

For decreasing subsequences we have the same argument. To wrap it up we have a theorem we have a theorem from Stanley [10]:

Theorem 1.11.3. *A longest i-chain of increasing subsequences of w consists of:*

- *An increasing subsequence s_1 of w.*
- *An increasing subsequence s_2 of $w \backslash s_1$.*
- $(\dots)$*An increasing subsequence s_i of $w \backslash s_1 \dots s_{i-1}$.*

Then the length of the longest i-chain is $\lambda_1 + \lambda_2 + \cdots + \lambda_i$.

Jeu de taquin

The phrase *jeu de taquin* means "teasing game". This process is equivalent to RSK and insertion. As motivation, inserting $\varnothing \leftarrow w$ and then insert ρ and finally π, then this is the same as

$$w \leftarrow \mathrm{rw}(\mathrm{ins}(\rho) \leftarrow \pi).$$

This means that there's some *associativity* in this operation.

Definition 1.11.4. A skew-SSYT is a filling of the boxes of a skew shape (skew Ferrers diagram) with $n \in \mathbb{N}$ such that rows are weakly increasing and columns are strictly increasing.

This is analogous to SSYT, so let's see an example

Example 1.11.5. $(9, 9, 5, 5, 3)/(7, 6, 3, 3)$, but the initial partition could've been $(10, 10, 6, 6, 4)$ or $(100, 9, 9, 5, 5, 3)$.

o									
2	2	4	o						
		x	3	4					
			1	3	o				
					x	2	2	3	
						x	1	1	o

Notice that all rows and columns are adjacent, and there are no leftover squares.

Definition 1.11.6. A corner of a Young diagram μ is a square of μ at the top of its column, right of its row.

An inner corner of λ/μ is a corner of μ. While an outer corner of λ/μ is a square outside λ that is just above its column (poss. 0) just right of a row (poss. 0).

Remark 1.11.7. Notice that adding an outer corner makes a valid shape for a partition.

The JdT game is defined through the following moves

Definition 1.11.8. An <u>inner slide</u> into an inner corner x of a SSYT T of skew-shape λ/μ is given by the following process:

i) Compare squares a, b in the following shape

$$\begin{array}{cc} a & \\ x & b \end{array}$$

If $a \leqslant b$ or there's no b, slide a down. Else slide b left.

ii) Repeat for the new location of x until it becomes an outer corner.

Example 1.11.9. In the previous example we can consider

$$\begin{array}{ccccc} x & 2 & 2 & 3 & \\ & x & 1 & 1 & o \end{array}$$

and slide the 1 because if we slide 2 it's no longer SSYT. But then we get an empty square, so we keep going until it's a valid shape. The end result is

$$\begin{array}{ccccc} x & 2 & 2 & o & \\ & 1 & 1 & 3 & o \end{array}$$

For a bigger example consider

$$\begin{array}{ccccc} 2 & 4 & 4 & 5 & \\ & 3 & 3 & 3 & 6 \\ & 1 & 2 & 2 & 3 \\ & x & 1 & 1 & 2 \end{array} \to \begin{array}{ccccc} 2 & 4 & 4 & o & \\ & 3 & 3 & 5 & 6 \\ & 2 & 2 & 3 & 3 \\ & 1 & 1 & 1 & 2 \end{array}$$

Remark 1.11.10. A slide sends an SSYT to an SSYT. A case like

$$\begin{array}{ccc} & 3 & \\ 4 & x & 5 \end{array}$$

because there should be an entry there $\leqslant 3$ and > 4.

We can continue applying the process to the following inner corners to get a SSYT.

Definition 1.11.11. The <u>rectification</u> of a skew SSYT is the result of performing JDT slides until we don't have inner corners left. We say that the shape is <u>straight</u>.

Remark 1.11.12. The rw of the skew-shape is the same as the rectification.

1.12 Day 11 | 20230213

Recall we have talked about the Jeu de Taquin. We observed that JDT slides send skew SSYT to other skew SSYT. To show compatibility with Knuth-equivalence, we want will require a more general definition of reading word because for example in the middle of a JDT we may have

$$\begin{array}{cccccc} 3 & 4 & 7 & 8 & & \\ & & 2 & 4 & & \\ & & 1 & 1 & 3 & 3 \end{array}.$$

Definition 1.12.1. A reading word of any labeled set of boxes in the 1^{st} quadrant is formed by reading the rows from top to bottom L to R with each row.

Lemma 1.12.2. *If a skew SSYT S is obtained from another skew SSYT T by a sequence of JDT slides, then $rw(S) \sim_K rw(T)$.*

Proof

Assume S, T differ by one inner-slide. There are a couple of cases, we'll show that at each step of the slide the Knuth equivalence class of the reading word is unchanged. (Even when we haven't finished the slide.)
The first case is when we do a horizontal slide

$$\begin{array}{cc} a & \\ x & b \end{array} \to \begin{array}{cc} a & \\ b & x \end{array}$$

and nothing happens here because the reading word is still ab. The interesting case is the vertical slide.
PROOF BY EXAMPLE

Example 1.12.3. Consider the tableau

$$\begin{array}{ccccccc} s & t & u & f & f & & \\ 1 & 2 & 5 & 6 & 7 & T & \\ & & 3 & 4 & x & 8 & 9 \end{array} \to \begin{array}{ccccccc} 1 & 2 & 5 & 6 & x & T & \\ & & 3 & 4 & 7 & 8 & 9 \end{array}$$

the reading word changes from $12567T3489$ to $1256T34789$. We will apply Knuth moves to the reading word of the original tableau.

We need an algorithm to guarantee that we get from one point to another. It suffices to consider the subword $567T348$ on the smaller window

$$\begin{array}{cccc} 5 & 6 & 7 & T \\ 3 & 4 & x & 8 \end{array}$$

Thinking in steps we want to move the 3 backwards past the 7, then 4 and finally pull the 10 back. We have

$$567T348 \to 5673T48$$

Notice that $6 > 3$ because $6 > 4 > 3$ and it's also less than 7, there's still something still past the 7 which we can use to move the 3. We get

$$5637T48 \to 56374T8$$

We use the 5 to pull the 3 out of the way, once again the 5 is there due to semi-standardness. We get

$$53674T8 \to 53647T8$$

and to move the $T = 10$ we use the 8 which is below 10 due to semi-standardness. Applying a Knuth move with $7T8$ we obtain

$$5364T78 \to 5634T78 \to 563T478 \to 56T3478$$

Remark 1.12.4. If we had more *stuff* we would wave to take a longer subword.

We said the rectification was the end goal of a JDT, with this in hand we can *well*-define it.

Definition 1.12.5. The <u>rectification</u> of a SSYT is formed by performing inner JDT slides until we have a straight shape.

This is well defined because no matter the order of slides, the reading word of the rectification is equivalent to the reading word of the original tableau.

As the Knuth equivalence determines the insertion tableau (rect(T)=ins(rw(T))) (knuth class=ins tableau thm.) So putting it all together we see it works.

Example 1.12.6. Rectify the tableau

$$\begin{array}{|c|c|c|} \hline 2 & 4 & \\ \hline & 1 & 5 \\ \hline & & 3 \\ \hline \end{array} \to \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 4 & 5 \\ \hline & & 3 \\ \hline \end{array} \to \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 4 & \\ \hline & 3 & 5 \\ \hline \end{array} \to \begin{array}{|c|c|c|} \hline 2 & 4 & \\ \hline 1 & 3 & 5 \\ \hline \end{array}.$$

However we can also do

$$\begin{array}{|c|c|c|} \hline 2 & 4 & \\ \hline & 1 & 5 \\ \hline & & 3 \\ \hline \end{array} \to \begin{array}{|c|c|c|} \hline 2 & & \\ \hline & 4 & 5 \\ \hline 1 & 3 & \\ \hline \end{array} \to \begin{array}{|c|c|c|} \hline 2 & 4 & 5 \\ \hline & 1 & 3 \\ \hline \end{array} \to \begin{array}{|c|c|c|} \hline 2 & 4 & \\ \hline 1 & 3 & 5 \\ \hline \end{array}.$$

The Knuth equivalence of the reading words of all tableau is the same! The other method is to take the reading word of the original tableau 24153 and then insert it to the corresponding tableau.

Now we can completely replace insertion with rectification because we can make any word with skew-tableau.

Corollary 1.12.7. *The rectification of the skew tableau where w is any word is ins(w).*

Example 1.12.8. Consider the tableau

$$\begin{array}{ccccc} \boxed{5} & & & & \\ \boxed{4} & \boxed{6} & \boxed{7} & & \\ \boxed{1} & \boxed{2} & \boxed{4} & \boxed{5} & \\ & & & & \boxed{3} \end{array},$$

we will rectify it and see it is the same as bump-inserting the 3 into the tableau. We get

$$\begin{array}{ccccc} \boxed{5} & & & & \\ \boxed{4} & \boxed{6} & \boxed{7} & & \\ \boxed{1} & \boxed{2} & \boxed{4} & \boxed{5} & \\ & & & & \boxed{3} \end{array} \longrightarrow\!\!\longrightarrow \begin{array}{ccccc} \boxed{5} & & & & \\ \boxed{4} & \boxed{6} & \boxed{7} & & \\ \boxed{1} & \boxed{2} & \boxed{4} & & \\ & & \boxed{3} & \boxed{5} & \end{array} \longrightarrow\!\!\longrightarrow \begin{array}{cccc} \boxed{5} & \boxed{6} & & \\ \boxed{4} & \boxed{4} & \boxed{7} & \\ \boxed{1} & \boxed{2} & \boxed{3} & \boxed{5} \end{array}$$

DO THE SLIDING

Then doing the diagonal process is intuitively inserting by this process.

Definition 1.12.9. We define the product of two tableau T, U as the rectification of the skew tableau formed by connecting the lower right corner of T to the upper left of U.

Equivalently

$$T \circ U = \text{rect}(T \leftarrow \text{rw}(U)).$$

This gives us an associative operation because of the Knuth-equivalence. In consequence the set of tableaux is a monoid, as the identity element is the empty tableau.

We define it as the <u>Plactic monoid</u>, the set of SSYT's of straight shape with $\circ$.

It's interesting to look at the Plactic monoid in terms of words as well.

Definition 1.12.10. The Plactic monoid is the $\{\text{ words }\}/\sim$ with the concatenation of words as the operation and $\sim$ is Knuth-equivalence.

Example 1.12.11. If $w = [2131]\,, v = [2213]$, then

$$wv = [21312213] = [21132213]\,.$$

For the next class we will talk about skew Schur functions. We will build up to writing skew functions in terms of the ordinary ones.

1.13 Day 12 | 20230215

Skew Schur functions

Definition 1.13.1.

$$s_{\lambda/\mu} = \sum_{*} x^T$$

where $x^T = x^{\#1's} x^{\#2's} \ldots$. This is still a symmetric function by the same proof as for s_λ.

Example 1.13.2. Compute $s_{(3,2)/(1)}$ in terms of m basis PHOTO

But instead we can see that this function is also $s_{(2,2)} + s_{(3,1)}$. It turns out that skew Schur functions are Schur positive. Recall that this means that in the Schur basis expansion, all of its coefficients are positive integers.

Theorem 1.13.3. *$s_{\lambda/\mu}$ is Schur-positive,*

$$s_{\lambda/\mu} = \sum_{\nu \vdash |\lambda| - |\mu|} c^{\lambda}_{\mu,\nu} s_\nu$$

The coefficients are called the Littlewood-Richardson coefficients.

We can use the Littlewood-Richardson coefficients to count certain Young tableaux. However another way to compute coefficients which uses a similar rule is the *Knutson-Tao puzzles*.

Littlewood-Richardson rule

Definition 1.13.4. A word w of positive indices is (reverse) ballot, Yamanouchi, or lattice if every suffix $w_i w_{i+1} \ldots w_n$ has partition content. This is that reading the word from right-to-left the number of $1's$ is greater the $2's$ and so on.

$$\#1's \geqslant \#2's \geqslant \#3's \geqslant \ldots$$

Example 1.13.5. Consider the word 341231211, this is Yamanouchi, while 21433231211 is not Yamanouchi.

Definition 1.13.6. A skew tableau is called Littlewood-Richardson if its reading word is a Yamanouchi word.

Theorem 1.13.7. *The Littlewood-Richardson coefficient is the number of Littlewood-Richardson tableaux of shape λ/μ and content ν.*

Example 1.13.8. Let us find some Littlewood-Richarson coefficients. Consider

$$\lambda/\mu = \begin{ytableau} \; & \; \\ \none & \; & \; \end{ytableau}, \ \nu = (2,2)$$

so we need to find L-R tableaux of shape λ/μ with content $(2,2)$ (2 ones and 2 twos). The only possibilities are

$$\begin{ytableau} 2 & 2 \\ \none & 1 & 1 \end{ytableau}, \quad \text{and} \quad \begin{ytableau} 1 & 2 \\ \none & 1 & 2 \end{ytableau}$$

and the latter's reading word is 1212 which is not Yamanouchi because it ends in a 2 and not a 1.

Let us now find the number of L-R tableaux with shape

$$\begin{ytableau} \; & \; \\ \; & \; \\ \none & \none & \; & \; & \; \\ \none & \none & \none & \none & \; & \; \end{ytableau}$$

and content $(4,3,2)$. We have the following tableaux

$$\begin{ytableau} 3 & 3 \\ 1 & 2 \\ \none & \none & 1 & 2 & 2 \\ \none & \none & \none & \none & 1 & 1 \end{ytableau}, \quad \text{and} \quad \begin{ytableau} 3 & 3 \\ 2 & 2 \\ \none & \none & 1 & 1 & 2 \\ \none & \none & \none & \none & 1 & 1 \end{ytableau}$$

This means that

$$c^{(6,5,2,2)}_{(4,2),(4,3,2)}.$$

Notice that by proving this result, we immediately get that the skew Schur functions are Schur positive. We will use crystal-base theory. This comes from the representation theory of $\mathcal{U}_q(\mathfrak{sl}_n)$.

A crystal of tableau sorts the monomials into a graph, let us for example consider

$$\begin{ytableau} 2 \\ 1 & 1 \end{ytableau}$$

1.14 Day 13 | 20230217

Crystals on words

Operating on words means we can operate on reading words and thus on tableau.

Definition 1.14.1. For a word w of 1's and 2's, we define the raising operator E_1 and lowering operator F_1 by:

- $\diamond$ Replacing all ones with right parenthesis and twos with left parenthesis.
- $\diamond$ Then we pair off matching parenthesis.
- $\diamond$ E_1 makes the first unmatched 2 to a 1 and F_1 the last unmatched 1 to a 2.

Example 1.14.2. Consider the word

$$112212111122212 \to))(()()))))((()($$

Applying E_1 we get

$$112212111112212 \to 112212111111212 \to 112212111111211$$

and applying once again we get the empty word because there's no more match. We can see that applying F_1 reverses this process. While applying F_1 to our original word several times we get

$$222212112222212$$

before we get to the empty word.

Definition 1.14.3. For a word $w \in \mathbb{Z}^n_{\geqslant 0}$, then E_i is formed looking at the $(i, i+1)$, treating them as $(1,2)$ and performing E_1.

This allows us to draw graphs on words where we connect after applying the operations.

The reason they are called raising and lowering is because in terms of content, the raise and lower the content:

$$\begin{aligned} 222212112222212 &\to (4,11) \\ 112212111122212 &\to (8,7) \\ 112212111111211 &\to (11,4) \end{aligned}$$

This also correspond to weight spaces in representations of $\mathfrak{sl}_n$.

Remark 1.14.4. Recall the Lie algebra of SL_2 is

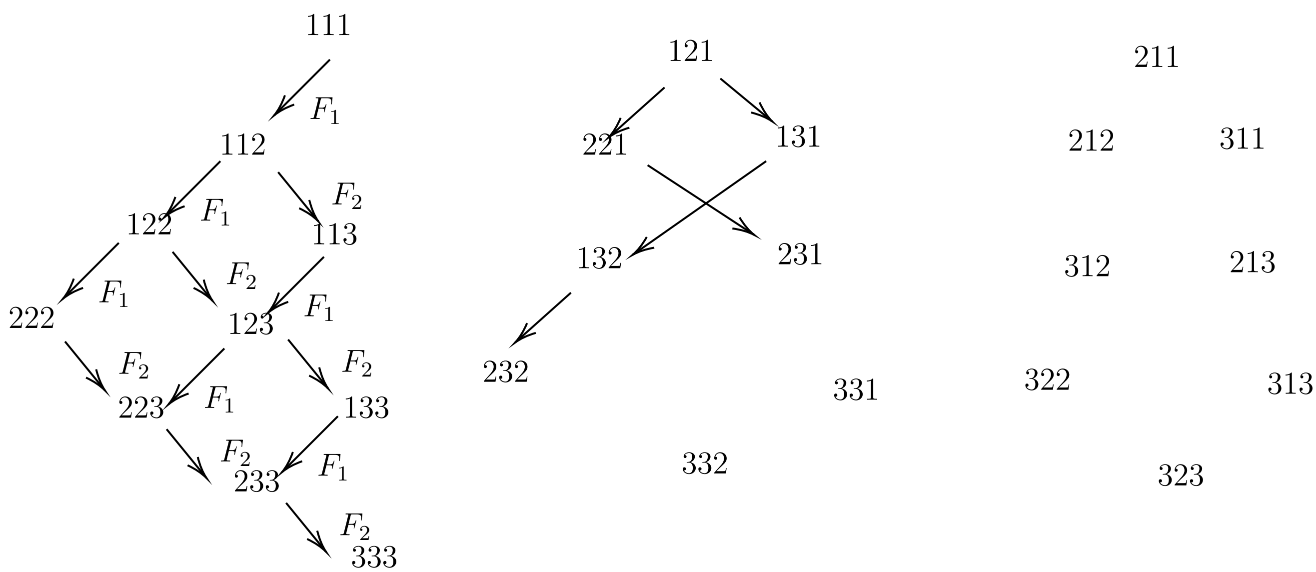
$$\mathfrak{sl}_2 = \{\, A \in \mathcal{M}_2 : \operatorname{tr}(A) = 0 \,\}.$$

This is an additive vector space with a Lie bracket, it's not closed under multiplication. The matrices with generate this space are

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In general in $\mathfrak{sl}_n$, E_i has a 1 on top of the i^{th} position on the diagonal.

Example 1.14.5. Consider the word 111 we get the following graph



Note that the words 121 and 211 both insert to

$$\mathrm{ins}(121) = \mathrm{ins}(211) = \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}.$$

Also look at the fact that we have listed all the words of length 3. As an alternative definition, write all the words, match them by $F's$ then we get disjoint graphs. Each connected component gives us a Schur function.

How many connected components do we have?

If we have length n, the question is how many (reverse-)ballot words of length n are there? (Why ballot words?)

Lemma 1.14.6 (Homework)**.** *For a word w*

$$E_i(w) = \varnothing \iff w \text{ is Yamanouchi.}$$

Assuming this, on length 4 we have the following Yamanouchi words:

(a) 1111

(b) 2111

(c) 3211

(d) 3121

(e) 1121

(f) 2211

(g) 1321

(h) 4321

(i) 1211

(j) 2121

There are $n \leqslant x \leqslant n!$ components. With the lemma comes the following definition:

Definition 1.14.7. We say a word w has highest weight if $E_i(w) = \varnothing$ for all i.

Theorem 1.14.8. *By acting on the reading word we have that E_i, F_i are well defined on skew tableaux (and therefore on tableaux.)*

We can always act on the reading word, but this action doesn't necesarilly guarantee that the result is an SSYT.

Proof

Look at the $(i, i+1)$ in a tableau T. This will "look like"

$$\begin{ytableau} i & i & j & j \\ \none & \none & \none & i & i & j & j & j \\ \none & \none & \none & \none & \none & \none & i & i \\ \none & \none & \none & \none & \none & \none & \none & \none & i & i & j & j \end{ytableau}$$ [a][b].

The columns "cancel" and some others do to, we can change the left most two $i's$ and the rightmost $i(i+1)$.

[a]We used this tableau when proving the Schur functions were symmetric.
[b]j is i+1, have to change ytableau package.

Compatibility with Knuth moves and Jeu de Taquin

Example 1.14.9. Consider a skew tableau and an inner slide:

$$\begin{ytableau} 1 & 1 & 2 & 2 \\ \none & \none & 1 & 1 & 2 \end{ytableau} \to \begin{ytableau} 1 & 2 & 2 \\ \none & 1 & 1 & 1 & 2 \end{ytableau}$$

But applying F_1 means that we get

$$\begin{ytableau} 1 & 2 & 2 & 2 \\ \none & \none & 1 & 1 & 2 \end{ytableau} \to \begin{ytableau} 1 & 2 & 2 \\ \none & 1 & 1 & 2 & 2 \end{ytableau}.$$

DRAW COMMDIAG

Compatibility means commutation at the level of diagrams.

1.15 Day 14 | 20230220

Today we will continue working on the Littlewood-Richardson rule. As a warm-up lets work on an example.

Exercise 1.15.1. Compute E_3 of the tableau

$$\begin{ytableau} 5 \\ 3 & 3 & 4 & 6 \\ 2 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{ytableau}.$$

Answer

We can focus only on the sub skew tableau

$$\begin{ytableau} 3 & 3 & 4 \\ \none & \none & 3 & 4 \end{ytableau}$$

with reading word 33434, the leftmost un-paired 4 is the last one so we change the word to 33433 and then replace on the tableau to get

$$\begin{ytableau} 5 \\ 3 & 3 & 4 & 6 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{ytableau}.$$

Notice we do get a SSYT as we showed last time. We will now work on crystal operators.

Theorem 1.15.2. *E_i, F_i commute with JDT slides.*

It suffices to check this on $1-2$ tableaux and E_1, F_1. This is because E_i, F_i only affect $i, i+1$. (Look at example from last time) This example is the only case that we have to worry about because we can only have at most two rows. The only problematic case is a vertical slide like in the example.

But what about

$$\begin{ytableau} 1 & 2 & 2 & 2 \\ \none & \none & \none & 1 & 2 \end{ytableau} \to \begin{ytableau} 1 & 2 & 2 \\ \none & \none & 1 & 2 & 2 \end{ytableau},$$

it slides the same way after applying F_1:

$$\begin{ytableau} 2 & 2 & 2 & 2 \\ \none & \none & \none & 1 & 2 \end{ytableau} \to \begin{ytableau} 2 & 2 & 2 \\ \none & \none & 1 & 2 & 2 \end{ytableau}.$$

Theorem 1.15.3. *The operations E_i, F_i preserve the "being in the same Knuth-equivalence class". In other words*

$$E_i(w) \sim E_i(v), \quad F_i(w) \sim F_i(v).$$

Since rectification is unique and that determines the Knuth-equivalence class, we can use the previous result.

Answer

Consider the skew tableau $D_w = \operatorname{diag}(w)$ and D_v. Then

$$\operatorname{ins}(w) = \operatorname{ins}(v) \Rightarrow \operatorname{rect}(D_w) = \operatorname{rect}(D_v)$$

so

$$\begin{aligned} \operatorname{rect}(E_i(D_w)) &= E_i(\operatorname{rect}(D_w)) \\ &= E_i(\operatorname{rect}(D_v)) \\ &= \operatorname{rect}(E_i(D_w)) \end{aligned}$$

Therefore $E_i(D_w) \sim E_i(D_v)$.

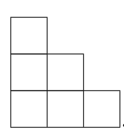
Highest Weight

We will now work with highest-weight elements.

Lemma 1.15.4. *For a straight shape λ, the SSYT's with $\operatorname{sh}(\lambda)$ all lie in a single connected component of the crystal graph.* (*i.e. connecting via E_i, F_i we get a connected component.*)

Moreover we get a unique tableau with highest weight ($E_i(T) = \varnothing$). We claim that such a tableau has row i full of i's.

Take any shape

,

as the reading word is Yamanouchi, the first entry is a 1. Then all entries to the left are 1's by semi-standardness. The next rightmost element of the next row can't be a 3 by Yamanouchi so it must be a 2 and so on.

But why is everything in the same connected component?

If we have a tableau T, then applying E_i we *must* reach T_λ. Reversing this process for any T with F_i, we get all the tableau.

Remark 1.15.5. This method works with no restriction on the letters of the word.

Corollary 1.15.6. *Every connected component of a crystal of all tableaux of a given skew shape is isomorphic as a directed graph to a straight shape crystal obtained via JDT.*

With this key step we can compute a Schur function decomposition.

Example 1.15.7. Let us compute $s_{(3,1)/(1)}$. The crystal of this SSYT is obtained by writing the highest weight fillings of the skew shape:

$$\begin{ytableau} 2 \\ \none & 1 & 1 \end{ytableau}, \qquad \begin{ytableau} 1 \\ \none & 1 & 1 \end{ytableau}.$$

We get the following crystals:

aaaaa

By rectifying with JDT we get INSERT DIAGRAMS

What this means is that adding monomials corresponding to crystal one we get $s_{(2,1)}$ and the other crystal $s_{(3)}$.

As a shortcut, we draw the Yamanouchi ones, then rectify and that's it.

Corollary 1.15.8. *The coefficient of s_ν is $s_{\lambda/\mu}$ is the number of heighest weight SSYT with $\operatorname{sh}(T) = \lambda/\mu$ that rectify to $\operatorname{sh}(\nu)$.*

$$\#hw \operatorname{sh}(\lambda/mu) \quad \textit{content} \quad \nu$$

this is exactly $c^\lambda_{\mu\nu}$.

This completes a proof of the Littlewood-Richardson rule.

Products of Schur Functions

There's another Littlewood-Richarson Rule:

Theorem 1.15.9. *Suppose μ, ν are straight shapes, then*

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda.$$

There's a proof with Knuth-equivalence and concatenation.

We know that $s_\nu \cdot s_\mu$ can be obtained by finding s_T where T is the skew shape formed by concatenating right down of μ with LEFT UPν. Then the product is

$$\sum_\lambda c^\rho_{R\lambda} s_\lambda.$$

To prove it we use Fulton's techniques [3].

We want to show $c^{\rho}_{R\lambda} = c^{\lambda}_{\mu\nu}$. This means that the number of Yamanouchi tableaux of shape ρ/R and content λ is the same as $\mathrm{sh}(\lambda/\mu)$ and content ν.

Remark 1.15.10. Consider

$$\begin{ytableau} 3 & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & 4 \\ \none & \none & \none & \none & 3 \\ \none & \none & \none & \none & 2 & 2 & 2 & 2 \\ \none & \none & \none & \none & 1 & 1 & 1 & 1 & 1 \end{ytableau}.$$

the lower one has to be T_ν in any Yamanouchi filling.

We first de-RSK it to get

$$\begin{ytableau} 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{ytableau} \to \begin{pmatrix} 123331222111 \\ 111112222333 \end{pmatrix}$$

if r is the first row and b the second one, then we insert r into U. But when inserting we are keeping track with b, so we label the new squares with b as a skew recording tableau.

Name Q such tableau, we claim that Q is a ballot SSYT shape λ/μ content ν.

Putting this together in the example we get

$$U = \begin{ytableau} 3 & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & 4 \end{ytableau} \leftarrow 123331222111$$

READ IN FULTON AND UNDERSTAND BIJECTion

Chapter 2

Representation Theory: A Crash Course

2.1 Day 15 | 20230222

Representation is a big and not unified area, so we will talk only about what we need for combinatorics.

Definition 2.1.1. A group is a set $(G, *)$ such that $*$ is a binary operation which is associative, possesses an identity element and every element possesses an inverse.

We say a matrix representation of a group is an assignment $g \mapsto M_g \in \mathrm{GL}_n(\mathbb{C})$ for $g \in G$ with $M_g M_h = M_{g*h}$.

If the assignment is injective we say the representation is faithful.

Example 2.1.2. Some groups that we will use are $(S_n, \circ)$, the general linear group $(\mathrm{GL}_n(\mathbb{C}), \cdot)$, the Multiplicative subgroup of the real numbers $(\mathbb{R}\backslash\{0\}, \cdot)$.

On the other hand group representations of S_3 we have that the composition is the operation. We can trivially represent it as $\sigma \mapsto (1)$ for every $\sigma \in S_3$. This representation is not faithful.

We also have the *sign representation*

- $\mathrm{id} \mapsto (1)$.
- $(12) \mapsto (-1)$.
- $(13) \mapsto (-1)$.
- $(23) \mapsto (-1)$.
- $(123) \mapsto (1)$.
- $(132) \mapsto (1)$.

These are the only 1-dimensional representations of S_3. In fact for S_n this is true, there are only 2 one-dimensional representation.

We can represent our permutations as permutation matrices!

$\diamond$ $\mathrm{id} \mapsto \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$. $\diamond$ $(13) \mapsto \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$. $\diamond$ $(123) \mapsto \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$.

$\diamond$ $(12) \mapsto \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$. $\diamond$ $(23) \mapsto \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$. $\diamond$ $(132) \mapsto \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$.

In general $\sigma \mapsto M_\sigma$ where $(M_\sigma)_{i,j} = \delta_{i,\pi(j)}$. This representation is faithful, but it's *too big*.

We can notice that $v = \begin{pmatrix}1\\1\\1\end{pmatrix}$ is an eigenvector with eigenvalue 1 of all of these matrices. The eigenspace $\mathrm{gen}(v)$ is fixed by all of these matrices. The orthogonal complement to $\mathrm{gen}(v)$ is rotated about the normal axis when acted by this matrices, such plane is

$$\mathrm{gen}\left(\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right).$$

If we add our eigenvector to the basis then

$$[\mathrm{id}]^{\mathcal{C}}_{\mathcal{B}} = \begin{pmatrix}1&1&0\\1&-1&1\\1&0&-1\end{pmatrix} \Rightarrow [\mathrm{id}]^{\mathcal{B}}_{\mathcal{C}} = \frac{1}{3}\begin{pmatrix}1&1&1\\2&-1&-1\\1&1&-2\end{pmatrix}.$$

Lemma 2.1.3. *If $g \mapsto M_g$ is a representation, then $g \mapsto A^{-1}M_gA$ is also a representation.*

Proof

$A^{-1}M_gAA^{-1}M_hA = A^{-1}M_{g*h}A$.

Conjugating with our matrix we get the following representation:

$\diamond$ $\mathrm{id} \mapsto \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$. $\diamond$ $(12) \mapsto \begin{pmatrix}1&0&0\\0&-1&1\\0&0&1\end{pmatrix}$. $\diamond$ $(13) \mapsto \begin{pmatrix}1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$.

The fourth representation of S_3 comes from the two-by-two components of the block diagonal matrices in question. Knowing where the generators go is sufficient to know where every element maps. This means that

$$(132) = (12)(13) \mapsto \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$
$$(123) = (13)(12) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$
$$(23) = (13)(132) \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

But how would we have found this representation without recurring to the basis? We can think of our representation as we did, every group element is a matrix.

Another way is a homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$. Also we can consider it as a vector space space action

$$G \cdot V : G \times V \to V.$$

With this understanding 3 and $3'$ aren't fundamentally different.

Definition 2.1.4. Two representations $g \mapsto M_g$ and $g \mapsto N_g$ are isomorphic whenever there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $A^{-1}M_gA = N_g$.

In terms of the vector spaces, two representations V, W are isomorphic if there's an isomorphism that preserves the group action.

- ⋄ Can we have isomorphic representations between dimensions? No, but,
- ⋄ can we have non-isomorphic representations of the same dimension? Yes! The one dimensional ones aren't isomorphic.

Operations on representations

The direct sum acts on vector spaces as $V \oplus W$ or on maps as

$$(g \mapsto M_g) \oplus (g \mapsto N_g) = g \mapsto \begin{pmatrix} M_g & \\ & N_g \end{pmatrix}.$$

Example 2.1.5. The permutation representation for S_3 is the direct sum of the standard one and the trivial one.

Definition 2.1.6. A representation is <u>irreducible</u>[1] if it is not the direct sum of smaller dimensional representation.

[1] indecomposable means the same as irreducible when working over $\mathbb{C}$.

Any number is a unique product of primes, so in an analogous way, any representation is a direct sum of irreducibles.

Theorem 2.1.7 (Maschke/Schur). *Every complex representation of a finite group G is uniquely (up to isomorphism) a direct sum of irreducibles.*

We will see that irreducible representations of the symmetric group correspond to Schur functions.

2.2 Day 16 | 20230224

The main goal of a representation is to break it down into irreducible representations. We want to classify into irreducible representations.

Proposition 2.2.1. *The number of irreducible representations of a finite group G is the number of conjugacy classes.*

Recall $g, h \in G$ are conjugate if $h = kgk^{-1}$ for some $k \in G$.

Example 2.2.2. If G is abelian, the conjugacy classes are singletons. This means that the number of irreducible representations of G is $|G|$. As we represent over complex numbers, we map each element of G to a $|G|^{\text{th}}$ root of unity.

Example 2.2.3. Recall that conjugacy classes in the symmetric group are determined by cycle type. Notice that the number of conjugacy classes is $5 = p(4)$.

Partition	Cycle type	Elements of conjugacy class
$(1,1,1,1)$	4 1-cycles	() (identity element)
$(2,1,1)$	1 transposition	$(12), (13), (14),$ $(23), (24), (34)$
$(2,2)$	2 transpositions	$(12)(34), (13)(24), (14)(23)$
$(3,1)$	1 3-cycle	$(123), (132), (234), (243),$ $(341), (314), (412), (421)$
(4)	1 4-cycle	$(1234), (1243), (1324),$ $(1342), (1423), (1432)$

Symmetric Group S_4 cycle structure

We now have 2 irreducible representations, the trivial and sign one. The rest we will find with tableaux.

Proposition 2.2.4. *There is one irreducible representation V_λ of S_n for each partition $\lambda \vdash n$.*

Specht's construction of V_λ

Sometimes this V_λ's are called Specht modules, this means irreducible representation of S_n. First remember that S_n acts on $\mathbb{C}[x_1, x_2, \ldots, x_n]$.

We can think of $(\mathbb{C}[x_1, \ldots, x_n], +)$ as an infinite dimensional S_n representation. This in the sense that

$$\mathbb{C}[x_1, x_2, \ldots, x_n] = \bigoplus_{d=0}^{\infty}\{\text{ degree d homogenous component }\}$$

We will construct V_λ as a sub-representation of $\mathbb{C}[x_1, \ldots, x_n]$.

Definition 2.2.5. Let T be a standard filling (1 to n no restrictions on order) of a shape λ. The Garnir polynomial is

$$F_T = \prod_{(*)}(x_j - x_i)$$

where we sum through i "below" j in the same column of T.

Example 2.2.6. Consider the partition $(2, 2, 1)$ and a filling

$$\begin{array}{|c|c|}\hline 4 & \\ \hline 1 & 3 \\ \hline 5 & 2 \\ \hline\end{array}$$

so we look at the first column. We begin by adding factors

$$(x_4 - x_1)(x_4 - x_5)(x_1 - x_5)$$

and in the second column, only get $(x_3 - x_2)$. So F_T is the product

$$F_T = (x_4 - x_1)(x_4 - x_5)(x_1 - x_5)(x_3 - x_2).$$

Definition 2.2.7. The Specht module is $V_\lambda = \mathrm{gen}\{\, F_T : \, T$ is a standard filling of $\lambda \,\}$.

Lemma 2.2.8. *S_n acts on V_λ by permuting variables.*

Example 2.2.9. In the previous example, (12) acts on F_T as

$$(12)\cdot F_T = (x_4 - x_2)(x_4 - x_5)(x_2 - x_5)(x_3 - x_1)$$

so $(12)T$ is

$$\begin{array}{|c|c|}\hline 4 & \\ \hline 2 & 3 \\ \hline 5 & 1 \\ \hline\end{array}.$$

The proof of the lemma 2.2.8 begins by determining the action of $\pi \in S_n$ over the basis. We can see that $\pi F_T = F_{(\pi T)}$ and after that and we extend linearly to all elements, since the action on the generators determines the action on the whole space.

Proposition 2.2.10. *The set of F_T's such that T is a SYT is a basis for V_λ. In particular*

$$\dim(V_\lambda) = \#SYT(\lambda)$$

where we can use the hook-length formula.

We will see the proof next class, for now, an example:

Example 2.2.11. Consider $V_{(2,1)}$, we have the following possible fillings of $\mathrm{sh}(2,1)$:

$$\begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\,, \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\,, \quad \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\,, \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\,, \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\,, \quad \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c}{} \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\,.$$

By taking the Garnir polynomial of each filling T we can find the generators for our $V_{(2,1)}$:

$$V_{(2,1)} = \mathrm{gen}(x_3 - x_1, x_2 - x_1, x_1 - x_2, x_3 - x_2, x_2 - x_3, x_1 - x_3)$$

Clearly we don't need some of them due to signs. After eliminating the negatives, we can see that

$$x_3 - x_2 = (x_3 - x_1) - (x_2 - x_1)$$

so that we may eliminate it as well. We are only left with $x_3 - x_1$ and $x_2 - x_1$, the ones that correspond to SYT and we see that the dimension of $V_{(2,1)}$ is indeed 2, the number of SYT's of shape $(2,1)$.

Exercise 2.2.12. Compute $V_{(1,1,1)}$ and $V_{(3)}$. What is their dimension? ⟦Hint: We saw both of this represntations on last class.⟧

Answer

Both representations have dimension 1 as there's only one SYT of shape $(1,1,1)$ and (3):

$$\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}\,, \quad \text{and} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\,.$$

For $V_{(3)}$ this is the span of 1, because we can't make any Garnir polynomials. Also S_3 acts on trivially on 1. This means $V_{(3)}$ that is the trivial representation. For $V_{(1,1,1)}$ we have that it's generated by $(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$. We can see that this is the sign representation.

Let's now see why the representation $V_{(2,1)}$ is the same as the standard representation we found last time.

First, $V_{(2,1)}$ is a subspace of $\mathbb{C}[x,y,z]_{(1)}$, the homogenous linear polynomials of degree 1 and we have

$$\mathbb{C}[x,y,z]_{(1)} = \mathrm{gen}(x,y,z).$$

As $V_{(2,1)} = \mathrm{gen}(z-x, y-x) = \mathrm{gen}\left(\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix},\right)$. This is the action of S_3 on the orthogonal subspace $\begin{pmatrix}1\\1\\1\end{pmatrix}^{\mathsf{T}}$.

Lemma 2.2.13. *The permutation representation of S_n is*

$$V_{(n)} \oplus V_{(n-1,1)}.$$

On monday we will continue with tensor products and $V_\lambda \to s_\lambda$, $V\lambda \otimes V_\mu \to s_\lambda s_m u$

2.3 Day 17| 20230227

Going back to Yamanouchi words on 2 letters, the way to solve it is to consider the crystals. Each connected component contains a sequence with three 2's and two 1's which can be counted using $\binom{n}{\lfloor n/2 \rfloor}$ as this question is equivalent to asking number of words of length $2n+1$ with $n+1$ ones and n twos.

Garnir polynomials

Recall that T is a filling on λ with $[n]$ with no restrictions on rows or columns. T's Garnir polynomial is $\prod(x_i - x_j)$ where i is "above" j. From this, V_λ is S_n-invariant. This means that $\pi v \in V_\lambda$ when $v \in V_\lambda$ and $\pi \in S_n$.

We now need a *straightening algorithm* to convert F_T's to other F_T's.

Lemma 2.3.1 (Column straightening)**.** *If T is a filling of λ with $[n]$ and T' is formed from T by ordering columns, then $F_{T'} = (\pm)F_T$.*

Example 2.3.2. Consider the fillings T and T' respectively:

$$\begin{array}{|c|c|cc}\hline 8 & 7 \\ \cline{1-3} 1 & 2 & \multicolumn{1}{c|}{5} \\ \hline 4 & 9 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{6} \\ \hline\end{array} \quad \to \quad \begin{array}{|c|c|cc}\hline 8 & 9 \\ \cline{1-3} 4 & 7 & \multicolumn{1}{c|}{5} \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{6} \\ \hline\end{array}.$$

We have

$$\begin{cases} F_T = (x_8 - x_1)\dots \\ F_{T'} = (x_8 - x_1)\dots \end{cases}$$

Proof was omitted in class LOOK AT MARIA NOTES

Lemma 2.3.3 (Row straightening)**.** *Suppose T is column-increasing and we do the following:*

i) Choose the topmost row having a decrease (if it's not row increasing).
ii) Choose the rightmost decrease in that row.
iii) Define the Garnir operator

$$g_{A,B} = \sum_{(*)} \operatorname{sgn}(\sigma)\sigma$$

where the sum is taken over $\sigma \in S_{A\cup B}$ which preserve column increases.

Then

$$g_{A,B}F_T = 0 \iff \sum_{\sigma} \operatorname{sgn}(\sigma)F_{\sigma(T)} = 0.$$

Example 2.3.4. In the filling

$$\begin{array}{|c|c|cc}\hline 8 & 9 \\ \cline{1-3} 4 & 7 & \multicolumn{1}{c|}{5} \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{6} \\ \hline\end{array}$$

we have A to be the column above and including the 7, while B is the column including and below 5. Then the Garnir operator is

$$g_{A,B} = \mathrm{id} - (57) + (37)(59) + (735) + (597) - (3597).$$

The result of applying each term of the operator to the filling results in

$$\begin{array}{|c|c|cc}\hline 8 & 9 \\ \cline{1-3} 4 & 5 & \multicolumn{1}{c|}{7} \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{6} \\ \hline\end{array}, \quad \begin{array}{|c|c|cc}\hline 8 & 5 \\ \cline{1-3} 4 & 3 & \multicolumn{1}{c|}{9} \\ \hline 1 & 2 & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{6} \\ \hline\end{array}, quad \begin{array}{|c|c|cc}\hline 8 & 9 \\ \cline{1-3} 4 & 3 & \multicolumn{1}{c|}{7} \\ \hline 1 & 2 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\ \hline\end{array}, \quad \begin{array}{|c|c|cc}\hline 8 & 7 \\ \cline{1-3} 4 & 5 & \multicolumn{1}{c|}{9} \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{6} \\ \hline\end{array}, \quad \begin{array}{|c|c|cc}\hline 8 & 7 \\ \cline{1-3} 4 & 3 & \multicolumn{1}{c|}{9} \\ \hline 1 & 2 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\ \hline\end{array}.$$

and notice that (properrty is now there, which one column inc or row incr?) so that

$$F_T = F_1 - F_2 - F_3 - F_4 + F_5$$

Proposition 2.3.5. *Irreducibility of V_λ*

Tensor product on representations

The *inner* tensor product is defined on all representations whereas the *outer* tensor product is specific to the S_n case.

Proposition 2.3.6. *If V, W are vector spaces of dimension n and m, then $\dim(V \otimes W) = mn$.*

The construction of the tensor product that we will use is the one of the free vector space over $V \times W$ modulo the relations of bilinearity.

Proposition 2.3.7. *If V, W are S_n-representations, then $V \otimes W$ is an S_n representation as well given by*

$$\pi(v \otimes w) = (M_\pi v \otimes M_\pi w).$$

The Kronecker problem is that we have V_λ and V_μ and we would like to determine the decomposition of $V_\lambda \otimes V_\mu$ into irreducibles.

2.4 Day 18 | 20230301

The inner tensor product is

$$g(v \otimes w) = M_g v \otimes N_g w$$

so if V, W represent, $V \otimes W$ represents. For matrices $A \otimes B$ is a block matrix.

Outer Tensor Product

Definition 2.4.1. Suppose $H \leqslant G$ for a group G and V represents H. Then the induced representation of G is

$$\mathbb{C}G \otimes_{\mathbb{C}H} V.$$

A basis is the basis of V tensored by coset elements.

Example 2.4.2. Consider the trivial representation of S_2, let us call it 1_{S_2}. Let us now induce from S_2 to S_3, we are fixing the 3 so $S_2 = \mathrm{gen}((12))$. Thus we may find a basis

$$v_1 = 123 \otimes v_0, \quad v_2 = 132 \otimes v_0, \quad v_3 = 312 \otimes v_0,$$

where v_0 is the only basis element of 1_{S_2}. Sending this to matrices we get the permutation representation.

Definition 2.4.3. If V repreesents G and $H \leqslant G$, then restricting the representation to H means just taking the *ones in H*.

With the induced and restriced representations we can now go on to defining the outer tensor product of representations.

Definition 2.4.4. The outer tensor product of V, W representations of S_n and S_m is

$$Ind_{S_n \times S_m}^{S_{n+m}} V \otimes W.$$

Notice that $V \otimes W$ is a representation of $S_n \times S_m$ so we induce it to the larger group S_{n+m}. Notice that we may interpret a permutation in $S_n \times S_m$ as a permutation of the word

The difference is that V, W in the inner one are in the same group. Now we get an S_{n+m} representation from the outer. The Kronecker problem looks for irreducible representations of $V_\lambda \otimes_{in} V_\mu = \bigoplus g)\lambda\mu^\nu V_\nu$.

However it is possible to find the decomposition of the outer tensor product:

$$V_\mu \otimes_{out} V_\nu = \bigoplus_\lambda c_{\mu\nu}^\lambda V_\lambda$$

where $c_{\mu\nu}^\lambda$ is the Littlewood-Richardson coefficient. This is analogous to the product of two Schur functions

$$s_\mu s_\lambda = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda.$$

Example 2.4.5. Consider $V_{(2)} \otimes_{out} V_{(1)}$, this the trivial representation of $S_2 \times S_1$ which means this is the induction from $S_2 \times S_1$ to S_3.

Notice that the group $S_2 \times S_1 \simeq S_2$, so from before, we found that this was the permutation representation of S_3. We have decomposed this (REF WHERE) as

$$V_{(2,1)} \oplus V_{(3)}.$$

This is analogous to $s_{(2)}s_{(1)}$.

Also read example in notes column sign, row trivial We wish to understand the previous correspondence.

Definition 2.4.6. The Frobenius map takes a representation $V = \bigoplus c_\lambda V_\lambda$ to $\sum c_\lambda s_\lambda$. In other words, takes irreducibles V_λ to basis elements s_λ and extend by linearity.

Notice that

$$\text{Frob}(V_\lambda \otimes_o V_\mu) = s_\lambda s_m u \Rightarrow \text{Frob}(V \otimes_o W) = Frob(V)Frob(w).$$

We may extend to virtual representations which are complex formal sums of representations. Then $Frob : (Virt, \oplus, \otimes) \to \Lambda$ is a ring homomorphism.

But what's the deal with Schur positivity, this theory gives us a corollary.

Corollary 2.4.7. *A homogenous degree d symmetric function is a Frobenius character of some representation if and only if f is Schur positive.*

This means that there are two strategies when working in combinatorial represetation theory.

(a) We have a representation for which we can decompose $Frob(V)$ indto Schur functions, then we can find a decomposition of V into irreducibles.

(b) If we have a collection of symmetric functions which seem to be Schur positive, then we can show that the fact is true by finding a representation V_i such that $f_i = Frob(V_i)$.

2.5 Day 19 | 20230303

Exercise 2.5.1. Write $s_{(2,1)}$ in terms of p_λ and $p_{(2,1)}$ in terms of s_λ.

Recall that we have some formulas to go between basis of symmetric functions. The full is available in Maria's webpage. With what we know, we can't solve the previous exercise.

Murnaghan-Nakayama Rule

Definition 2.5.2. The character χ_V of a representation V of a group G is

$$\chi_V : G \to \mathbb{C},\ g \mapsto \operatorname{tr}(M_g) = \sum(\text{diagonal elements}) = \sum(\text{eigenvalues}).$$

No matter what basis we choose, the character is invariant.

Lemma 2.5.3.

χ_V is constant on conjugacy classes.

Proof

$$\chi_V(hgh^{-1}) = \operatorname{tr}(M_{hgh^{-1}}) = \operatorname{tr}\left(M_h M_g M_h^{-1}\right) = \operatorname{tr}(M_g) = \chi_V(g).$$

From this we can think of the character as a function from the conjugacy classes. As the amount of irreducible representations is the number of conjugacy classes, then we can write a character table

TABLE

Characters are also additive across representations, this means that

$$\chi_{V \oplus W} = \chi_V + \chi_W.$$

So knowing irreducibles allows us to find characters for all other representations. The other direction is much more important, if we know χ_V, we can uniquely decompose it as a linear combination of irreducible characters. So we can deduce the decomposition of V into irreducible representations.

Example 2.5.4. We will find the character table of S_2. As S_2 is abelian, every element is in its own conjugacy class:

S_2	id	(2)
$\chi_{(2)}$	1	1
$\chi_{(1,1)}$	1	-1

Character table for S_2

Example 2.5.5. We will find the character table of S_3:

S_3	id	(12)	(123)
$\chi_{(3)}$	1	1	1
$\chi_{(1,1,1)}$	1	-1	1
$\chi_{(2,1)}$	2	0	-1

Character table for S_3

The character table for $\chi_{\text{reg}S_3}$ is (600) so we may decompose the character as

$$2(20-1) + (111) + (1-11) \Rightarrow V_{\text{reg}} = \text{triv} \oplus \text{sign} \oplus \text{std}^2.$$

The regular represntation always decomposes a $dim * character$. This is an application of RSK as well AUDIO 20min

We will alternatively define the Frobenius map with characters.

Definition 2.5.6. Suppose V represents S_n, then

$$\text{Frob}(V) = \frac{1}{n!} \sum_{\pi \in S_n} \chi_V(\pi) p_{c(\pi)}$$

where $c((12)(35)(467)) = (3, 2, 2)$.

This is summing over $n!$ elements, but remember some of them share cycle type.

Lemma 2.5.7. *Under this definition, this can also be written as*

$$\text{Frob}(V) = \sum_{\lambda \vdash n} \chi_V(\pi \in \lambda \textit{conj.class}) \frac{p_\lambda}{z_\lambda}$$

where $z_\lambda = \prod k^{m_k} m_k!$.

Proof

We have that the size of the conjugacy class of cycle type λ is $n!/z_\lambda$. So rewrtiting the Frobenius map we get

$$\text{Frob} = \frac{1}{n!} \sum_{\pi \in S_n} \chi_V(\pi) p_{c(\pi)} = \frac{1}{n!} \sum_{\lambda \vdash n} \sum_{\pi \text{cyc.type} \lambda} \chi_V(\pi) p_\lambda$$

Since character is constant, we have

$$\frac{1}{n!} \sum_{\lambda \vdash n} (\#\text{conj.class cyctype}\lambda) \chi_V(\pi_\lambda) p_\lambda$$

that amount is $n!/z_\lambda$ so factorials cancel out and we get the desired formula.

Example 2.5.8. Recall that the character table for $\chi_{(2,1)}$ is $(2, 0, -1)$ so

$$\text{Frob}(V_{(2,1)}) = 2\frac{p_{(1,1,1)}}{z_{(1,1,1)}} + 0p_{(2,1)}/z - \frac{p_3}{z_3} = \frac{1}{3}p_1^3 - \frac{1}{3}p_3 = m_{(2,1)} + 2m_{(1,1,1)} = s_{(2,1)}.$$

Now for instance from the trivial character, $\text{Frob}(V_{(3)})$, the character table is $(1, -1, 1)$ and

In essence the coefficients are the character tables.

Theorem 2.5.9. *The character tables for S_n are the transition matrices from (s_λ) to the normalized power sum*: (p_λ/z_λ).

The product for this fact can be found in [10]. This theorem is also equivalent to the fact that both Frobenius maps are equivalent.

We now ask,

Is there a combinatorial way to compute $\chi_\mu(\lambda)$?

To derive it, we will use a duality. The Schur and power sum bases are self dual with respect to the Hall inner product so

$$\chi_\mu(\lambda) = \text{coeff.}\frac{p_\lambda}{z_\lambda} \quad \text{in} \quad s_\mu = \langle s_\mu | p_\lambda \rangle = \text{coeff.}s_\mu \quad \text{in} \quad p_\lambda.$$

So $p_\lambda = \sum_\mu \chi_\mu(\lambda) s_\mu$.

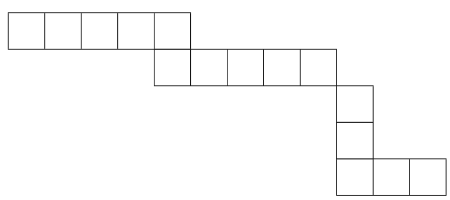
2.6 Day 20 | 20230306

The Inverse Murnaghan-Nakayama Rule

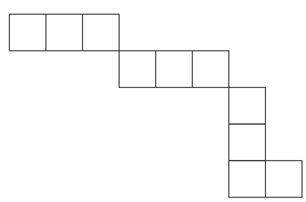
We wish to compute $\chi_U(\lambda)$, the character of V_μ on a permutation π of cycle type λ. This is the coefficient $\langle s_\mu | p_\lambda \rangle$, the coefficient of s_μ in p_λ.

Definition 2.6.1. A border strip or a ribbon is a connected skew-shape containing no 2×2 square.

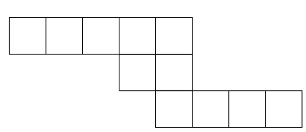
Example 2.6.2. The tableau



is a ribbon, the tableau



is not a ribbon because it's not connected. Finally the tableau



has a 2×2 square.

Definition 2.6.3. The height of a border strip λ/μ is #rows-1.

A border strip tableau with shape λ is a filling with positive integers such that

- ⋄ row and columns are weakly increasing
- ⋄ Every integer forms a border strip.

Example 2.6.4. The tableau

$$\begin{array}{|c|c|c|c|c}\hline 3 & 3 & 5 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 2 & 2 & \multicolumn{1}{c|}{2} \\ \hline 1 & 1 & 1 & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

is a border strip tableau

Theorem 2.6.5. *The character*

$$\chi_\mu(\lambda) = \langle s_\mu | p_\lambda \rangle = \sum_{(*)} \prod_{i=1}^{\ell(\lambda)} (-1)^{\mathrm{ht}(strip(i))}$$

where the sum runs through border strip tableaux T with shape μ and content λ.

Example 2.6.6. Let us find $\left\langle s_{\mathrm{sh}(3,2,1)} \middle| p_{(3,3)} \right\rangle$. We wish to fill border strip tableau with 111222. We have the following fillings

$$\begin{array}{|c|c|c|}\cline{1-1} 2 \\ \cline{1-2} 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|c|c|}\cline{1-1} 1 \\ \cline{1-2} 1 & 2 \\ \hline 1 & 2 & 2 \\ \hline \end{array}.$$

Calculating the heights we have

$$(-1)^0(-1)^1 = -1, \quad \text{and} \quad (-1)^2(-1)^2 = -1$$

so $\left\langle s_{\mathrm{sh}(3,2,1)} \middle| p_{(3,3)} \right\rangle = -1 - 1 = -2$. The coefficients might not be positive but at least they are integers.

Exercise 2.6.7. Calculate $\left\langle s_{\mathrm{sh}(2,1)} \middle| p_{(2,1)} \right\rangle$.

The tableaux are

$$\begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 1 & 2 \\ \hline \end{array}$$

so we get $1 - 1 = 0$.

Example 2.6.8. Notice that $\left\langle s_{\mathrm{sh}(3,2)} \middle| p_{(1,1,1,1,1)} \right\rangle$ is the number of standard Young tableaux because we are filling with 12345.

On the other hand

$$\left\langle s_{\mathrm{sh}(\mu)} \middle| p_{(n)} \right\rangle$$

is either 0 if μ is not a *hook* and $(-1)^{\ell(\mu)-1}$ when it actually is a hook.

The following proof comes from Egge's symmetric function, we will separate it into stages.

Lemma 2.6.9. *$p_r = \sum_{j=0}^{r-1} (-1)^j s_{(r-j,1,1,\dots,1)}$ where there are k ones.*

Proof

We have that

$$s_{hook} = \sum_{(*)} x^T$$

we wish to cancel x^T's as follows

$$\begin{ytableau} y \\ \ \\ \ \\ \ & \ & \ & \ & x \end{ytableau}, \quad \text{with} \quad \begin{ytableau} x \\ y \\ \ \\ \ \\ \ & \ & \ & \ \end{ytableau}$$

if $x > y$. Else if $x \leqslant y$ we pair with

$$\begin{ytableau} \ \\ \ \\ \ & \ & \ & \ & x & y \end{ytableau}.$$

This is a sign-reversing involution and the only edge cases are

$$\begin{ytableau} y & \ & \ & \ & \ & x \end{ytableau}, \quad \text{and} \quad \begin{ytableau} y \\ \ \\ \ \\ \ \\ x \end{ytableau}$$

so the fixed points are tableau of the form

$$\begin{ytableau} i & i & \ & \ & \ & i \end{ytableau}$$

which contribute monomials of the form x_i^r. So monomials that don't cancel out are the power sum symmetric functions.

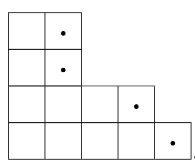
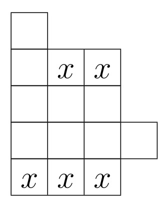
Let's see an example

Example 2.6.10. What about p_4?

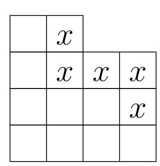
Lemma 2.6.11. *We have*
$$s_\mu p_r = \sum_{(*)} (-1)^{ht(\lambda/\mu)} s_\lambda$$
where the sum runs through λ such that λ/μ is a border strip of size r.

Recall the Pieri rules which say
$$s_\mu h_r = \sum_{(*)} s_\lambda$$
where the sum is through λ such that λ/μ is a horizontal strip of size r. The e Pieri rule instead goes
$$s_\mu e_r = \sum_{(*)} s_\lambda$$
where the sum now is through vertical strips of size r.

The tableaux

, and 

represent the veritcal and horizonal strips. But the tableau



represents the border strip.

Proof

First notice that
$$s_{(r-j,1,1,\ldots,1)} = \sum_{k=0}^{j} (-1)^{j-k} h_{r-k} e_k.$$
The idea is that hook shapes are similar to vertical and horizontal strips.

Example 2.6.12. Consider $s_{\mathrm{sh}(3,1,1)}$. By the formula this should be

$$h_{\mathrm{sh}(5)}e_0 - h_4e_1 + h_3e_2$$

and we stop here because the height here is $j = 2$. This is the same as

$$s_{\mathrm{sh}(5)}e_0 - s_4e_1 + s_3e_2.$$

By the Pieri rule this is

$$s_{\mathrm{sh}(5)} - \big(s_{\mathrm{sh}(5)} + s_{\mathrm{sh}(4,1)}\big) + \big(s_{\mathrm{sh}(4,1)} + s_{\mathrm{sh}(3,1,1)}\big).$$

So all terms cancel except for $s_{(r-j,1,1,\dots,1)}$.

Substituting this result in the lemma

$$s_\mu p_r = \sum_{j=0}^{r-1}(-1)^j s_\mu \sum_{k=0}^{j}(-1)^{k-j}h_{r-k}e_k.$$

The (-1) gets a k power and we are left with

$$\sum_{j=0}^{r-1}\sum_{k=0}^{j}(-1)^k s_\mu h_{r-k}e_k$$

Where h adds horzontal strips and e adds vertical strips. And they are gonna cancel unless they are a border strip.

r	r	b				
		b				
				r	b	
						r

So the difference is that looking at the shape, this contributes s_λ. We can *toggle something* so it does change the height.

Chapter 3

Counting with Group Actions

3.1 Day 21 | 20230308

Definition 3.1.1. An action of a group G on a set X is a map

$$G \times X \to X,\ (g, x) \mapsto g \cdot x$$

such that $h \cdot (g \cdot x) = (hg) \cdot x$ and $\mathrm{id} \cdot x = x$.

We have already seen group actions when X is a vector space. Representations are actions: $M_g \in \mathrm{End}(V)$. This map is bijective because we can find an inverse to the action.

Example 3.1.2. A group action which is not a representation is S_n acting on $[n]$ by $\pi \cdot i = \pi(i)$. This is an action because composition does what we want it to do.

Also S_n acts on itself:

- By conjugation: $\pi.\sigma = \pi\sigma\pi^{-1}$.
- By left-multiplication: $\pi.\sigma = \pi\sigma$.

Every group acts on itself by multiplication.

Definition 3.1.3. An action $G \cdot X$ is transitive if $g \cdot x = y$ for some $g \in G$ and all $x, y \in X$.

On the previous examples, $S_n \cdot [n]$ is transitive. Left multiplication is transitive, but conjugation is not.

Example 3.1.4. The group $\mathbb{Z}/_{5\mathbb{Z}}$ can be thought of $\langle(12345)\rangle \leqslant S_5$. This acts on the set of points of a 5-pointed star by rotation.

We wish to study the orbits of the actions which tell us how *not-transitive* an action is.

Definition 3.1.5. The orbit of a point $x \in X$ under an action $G \cdot X$ is

$$\operatorname{Orb}(x) = \{ g \cdot x : g \in G \}.$$

The idea is that when we have a cyclic group, these sets look like *orbits around the center*. Also, be quick to notice that

$$\operatorname{Orb}(x) = \operatorname{Orb}(g \cdot x), \quad g \in G$$

which means that the orbits partition the set X. In particular, "being in the same orbit" is an equivalence relation.

In the previous examples:

- Conjugations's orbits are the conjugacy classes.
- Left multiplication's orbit is only one.

Example 3.1.6. For the action $(S_3 \times S_2) \cdot [5]$, we have orbits $[3]$ and $\{ 4, 5 \}$.

Our objective now is to count orbits. This count is related to stabilizers.

Definition 3.1.7. The stabilizer of $x \in X$ under $G \cdot X$ is

$$\operatorname{Stab}(x) = \{ g \in G : g \cdot x = x \}.$$

Example 3.1.8. In $S_n \cdot [n]$ we have

$$\operatorname{Stab}_{S_n}(n) = S_{n-1}$$

because we can freely move the other $n-1$ elements. Now consider the conjugation action on S_5, then

$$\operatorname{Stab}_{S_5}((123)(45)) = C_3 \times C_2.$$

Now how do the orbit and stabilizer relate? The main reference is [8] in chapter 6.

Theorem 3.1.9 (Orbit-Stabilizer Theorem). *For $G \cdot X$ a group action, and a point $x \in X$ we have*

$$|\operatorname{Stab}(x)||\operatorname{Orb}(x)| = |G|.$$

Proof

If s is the size of the stabilizer, then we consider $y \in \operatorname{Orb}(x)$. This means that $y = g \cdot x$, so we get s elements which send x to y. In other words

$$h \cdot x = g \cdot x \Rightarrow (g^{-1}h) \cdot x = x$$

so $g^{-1}h \in \operatorname{Stab}(x)$, i.e. $h = g \cdot r$ for some $r \in \operatorname{Stab}(x)$.
So there exists s elements of G sending x to y for $y \in \operatorname{Orb}(x)$.

Example 3.1.10. We will find the size of a conjugacy class of cycle type λ. We can identify this as the orbit of the conjugacy action. This is, by Orbit-Stabilizer:

$$\frac{|S_n|}{|\operatorname{Stab}(\pi_\lambda)|}.$$

For example if $\lambda = (3, 3, 2, 1, 1)$ then $\pi_\lambda = (123)(456)(78)(9)(10)$. To stabilize this we can still do cyclic group actions on the cycles. This accounts for $C_3 \times C_3 \times C_2$. We add two $2!$ reswitching. If m_i is the number of i's in λ, then $m_1 = 2$, $m_2 = 1$ and $m_3 = 2$. So we have:

$$\frac{|S_n|}{|\operatorname{Stab}(\pi_\lambda)|} = \frac{n!}{\prod_i i^{m_i} m_i!} = \frac{n!}{z_\lambda}.$$

We can also find the size of the group with this.

Example 3.1.11. We ask, how many rotation in 3D fix a cube? If we fix x to be a corner of the cube, then

$$|\operatorname{Stab}(x)| = 3, \quad |\operatorname{Orb}(x)| = 8,$$

then we have 24 elements in our group.

We could also fix an edge to get 2 and 12 or fixing faces 4 and 6.

Burnside's Lemma

It's actually not Burnside but Frobenius' lemma. With this result we will count orbits. The question now is *in how many ways can we 2-color the faces of a cube up to rotation?* It's 10, not 8. What did I overcount?

Lemma 3.1.12. *The number of orbits of an action is*

$$\frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|$$

where $\operatorname{Fix}(g) = \{\, x \in X : \ g \cdot x = x \,\}$.

To solve the previous question we have that the key is the phrase "up to rotation". A coloring up to rotation is an orbit under F of 2-colorings of a labeled cube. There are 2^6 such colorings. Now we have that the number of orbits is

$$\frac{1}{24}\left(2^6 + 8\cdot 2 + 6\cdot 2^3 + 6\cdot 2^3 + 3\cdot 2^4\right)$$

identity, 120, 90 deg through face, 6 edge rotations, and the last one is through the face.

3.2 Day 22 | 20230310

Orbit Counting

We have asked in how many ways we can 2-color faces of a cube up-to-rotation.

FIGURE

Alternatively with Burnside's lemma, if G is the rotation group of a cube and X is the set of 2-colorings of faces of *labeled* cubes, then the number we are looking for is the number of orbits of $G\cdot X$. The elements of G are

- ⋄ identity
- ⋄ 8 120 vtx
- ⋄ 6 +-90 throu faces
- ⋄ 3 180 thru faces
- ⋄ 6 180 thru opp edges

Then adding

$$\frac{2^6 + 8\cdot 2^3 + 6\cdot 2^3 + 3\cdot 2^4 + 6\cdot 2^3}{24} = \frac{2^3 + 4 + 6 + 6 + 6}{3} = 10.$$

The proof of Burnside's lemma is a double counting argument.

Proof

We will count in two ways the set

$$\{\, (g, x) :\; g\cdot x = x \,\}.$$

For each g, there's $|\operatorname{Fix}(g)|$ pairs (g, x) such that $g \cdot x = x$. So counting through all $g \in G$,

$$|\{\, (g, x) : \ g \cdot x = x \,\}| = \sum_{g \in G} |\operatorname{Fix}(g)|.$$

Now for $x \in X$, there are $|\operatorname{Stab}(x)|$ g's such that $g \cdot x = x$, then

$$|\{\, (g, x) : \ g \cdot x = x \,\}| = \sum_{x \in X} |\operatorname{Stab}(x)|.$$

The sum of the stabilizers can be interpreted as

$$\sum_{x \in X} |\operatorname{Stab}(x)| = \sum_{x \in X} \frac{|G|}{|\operatorname{Orb}(x)|} = |G| \sum_{x \in X} \frac{1}{|\operatorname{Orb}(x)|}.$$

We may break the sum as

$$\sum_{x \in X} \frac{1}{|\operatorname{Orb}(x)|} = \sum_{O \text{ orbit}} \sum_{x \in O} \frac{1}{|O|} = \sum_{O \text{ orbit}} \frac{|O|}{|O|} = \#(\text{orbits}).$$

This proves the result.

Example 3.2.1. Consider the action $S_3 \times S_2 \cdot [5]$. In this case

$$\sum_{x \in [5]} |\operatorname{Stab}(x)| = \sum_{x \in [5]} \frac{6}{|\operatorname{Orb}(x)|} = 6 \sum_{x=1}^{5} \frac{1}{|\operatorname{Orb}(x)|} = 6 \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} \right).$$

Example 3.2.2. Consider the action of S_n on binary sequences of length n. The number of orbits is $n + 1$, because orbits are determined by amount of ones of a string.

By Burnside's lemma

$$\#\text{orbits} = \frac{1}{n!} \sum_{\pi \in S_n} |\operatorname{Fix}(\pi)|$$

By considering something ASK this sum becomes

$$\frac{1}{n!} \sum_{\lambda \vdash n} (\#\pi_\lambda' s) 2^{\ell(\lambda)}$$

and we may conclude

$$n + 1 = \sum_{\lambda \vdash n} \frac{2^{\ell(\lambda)}}{z_l la}.$$

We may also derive this from row-shapes! We have

$$s_{\text{row}} = \sum \frac{p_\lambda}{z_\lambda} \Rightarrow s_{\text{row}}(x,y) = \sum \frac{p_\lambda(x,y)}{z_\lambda}$$

This begs us to ask the question, how many SSYT have only ones and twos with shape (n)? This is $s_{(n)}(1,1) = n+1$. So we get

$$n+1 = \sum_{\lambda \vdash n} \frac{p_\lambda(1,1)}{z_\lambda} = \sum_{\lambda \vdash n} \frac{2^{\ell(\lambda)}}{z_\lambda}.$$

Examples like this can be worked with Polya theory. Look at [8].

Cyclic Sieving Phenomenom

This topic was discovered in the early 2000's, it can be read in Sagan's [8]. The big idea is summing roots of unity, we will say that $\omega_n = e^{\frac{2\pi i}{n}}$. If $\omega = \omega_n$ then

$$1 - \omega^n = (1-\omega)(1+\omega+\cdots+\omega^{n-1}) = 0 \Rightarrow 1+\omega+\cdots+\omega^{n-1} = 0.$$

Definition 3.2.3. A triple (X, G, f) exhibits the cyclic sieving phenomenom if $G \cdot X$ and f is a polynomial with integer coefficients such that

$$|\operatorname{Fix}(g)| = f(\omega_{\operatorname{ord}(g)}), \quad \text{where} \quad \operatorname{ord}(g) = \min\{\, k : \, g^k = \mathrm{id}\,\}.$$

Many times, the polynomials $f(q)$ are q-analogues.

Example 3.2.4. Consider $G = C_n \leqslant S_n$ generated by $(123\ldots n)$ acting on $X = \left(\!\!\binom{[n]}{k}\!\!\right)$, this is the set of all multisets of k chosen from $[n]$.

If $f(q) = \left(\!\!\binom{n}{k}\!\!\right)_q$, then this makes a triple (X, G, f) exhibing the cyclic sieving phenomenom.

For $n = 4$ and $k = 2$, then

$$X = \{\, 11, 12, 13, 14, 22, 23, 24, 33, 34, 44 \,\} \Rightarrow |X| = \left(\!\!\binom{4}{2}\!\!\right) = \binom{5}{2} = 10.$$

Now $f(q) = \frac{(5)_q(4)_q}{(2)_q}$, simplifying we obtain

$$(1+q+q^2+q^3+q^4)(1+q^2)$$

Then plugging in $q = i$ we get zero, which coincides with $|\operatorname{Fix}(1234)|$. In the case of $(13)(24)$, this only fixes 13 and 24 also $f(-1) = 2$ as -1 is a second root of unity.

Now notice that when taking the identity, every element is fixed and $f(1) = 10$.

Chapter 4

Combinatorial Designs

4.1 Day 23 | 20230320

Suppose we have different varieties of grain and different varieties of soil. We wish to find the best grain growing conditions so we design an experiment with a certain number of blocks to plant our grain.

We are constrained by time or any other condition so we want to extract the most amount of information.

Example 4.1.1. If we had 7 types of grain and 7 types of soil we could design an experiment as follows:

$$(123)(345)(561)(147)(257)(367)(246),$$

every pair is tested at least. To have *all* of the information we need $\binom{7}{3}$, but that might be too much information. Our design is symmetric in some sense which makes it *efficient*.

We can visualize this as a Fano Plane graph. INCLUDE GRAPH

Definition 4.1.2. A $\underline{t-(v,k,\lambda)\text{ design}}$ is a pair $(X,\mathcal{B})$ where $|X|=v$ and $\mathcal{B}\subseteq\mathcal{P}(X)$ where the elements of $\mathcal{B}$ are called blocks. For $B\in\mathcal{B}$, $|B|=k$ and every t-element subset of X appears in λ blocks.

Based on the definition, the previous example is a $2-(7,3,1)$ and a $1-(7,3,3)$ design. When every pair is in one block, we call it a Steiner system.

Example 4.1.3. We have that $\left([n], \binom{[n]}{k}\right)$ is a $t-\left(n,k,\binom{n-t}{k-t}\right)$ design, where the binomial coefficient counts the number of k-element subsets of $[n]$ contining a given t-element subset.

Lemma 4.1.4. *Every t design is an s design for $s \leqslant t$ with possibly different parameters.*

Proof

Consider $(X, \mathcal{B})$ a $t-(v, k, \lambda)$ design and let $S \subseteq X$ with s elements. Then we wish to show that $(X, \mathcal{B})$ is a $s-(v, k, \mu)$ design. FINISH

Lemma 4.1.5. *Let r be the number of blocks each point $x \in X$, then $bk = vr$ where b is the number of each block.*

Proof

The quantity bk is the number of pairs (x, B) such that $x \in B$. This same count can be done by fixing the element x and letting FINISH

Definition 4.1.6. The dual $\mathcal{D}$ of a design $(X, \mathcal{B})$ is $(X^\intercal, \mathcal{B}^\intercal)$ such that $X^\intercal = \mathcal{B}$ and $\mathcal{B}^\intercal$ is the set of $\mathcal{B}$'s such that $x \in X$.

Example 4.1.7. The dual of a $2-(9, 3, 1)$ design is a $1-(12, 4, 3)$ design. We can see their incidence matrices: DO

Lemma 4.1.8. *If M is the incidence matrix of $\mathcal{B}$ then $M^\intercal$ is the incidence matrix of $\mathcal{D}$ the dual of $\mathcal{B}$.*

Isomorphisms

Definition 4.1.9. An isomorphism of designs $(X, \mathcal{B})$, $(X', \mathcal{B}')$ is a bijection between X and X' which preserves the block structure.

Example 4.1.10. The Fano plane has non-trivial automorphisms, rotations, polarities, reflections and so on. In general these transformations form the automorphism group of this plane.

4.2 Day 24 | 20230322

Recall that a $t-(v, k, \lambda)$ design is a pair $(X, \mathcal{B})$ such that $v = |X|$, $k = |B|$ for all $B \in \mathcal{B}$ and λ is the number of blocks which contain a fixed set T with size t. We will relate this with the incidence matrix.

Theorem 4.2.1 (Fisher's Inequality). *In a 2-design with $k < v$ we have $b \geqslant v$.*

Proof

Consider the matrix $M^\intercal M$, the (i,j) entry is the dot product of columns i and j. Recall column i codifies the incidence of x_i in each block, so the dot product is equal to the number of block containing both x_i and x_j. In other words we get

$$(M^\intercal M)_{ij} = \begin{cases} \lambda, & i \neq j \\ r, & i = j \end{cases}$$

where r is the number of blocks containing x_i. So $M^T M$ is matrix with diagonal entries r and off-diagonal entries λ. The vector $(1, \ldots, 1)^\intercal$ is an eigenvector with eigenvalue $(v-1)\lambda + r$ and we have that for a two design

$$(v-1)\lambda + = r + r(k-1) = rk.$$

The remaining eigenvectors are of the form $(0, \ldots, 1, \ldots, -1, \ldots, 0)^\intercal$ with eigenvalues $r - \lambda$. Then the determinant of $M^\intercal M$ is

$$rk(r-\lambda)^{n-1}$$

so we require $r \neq \lambda$ to show $M^\intercal M$ is not signular. So from our lemma we have

$$r(k-1) = (v-1)\lambda \Rightarrow (r-\lambda)(k-1) = \lambda(v-k) \Rightarrow r - \lambda > 0$$

because the remaining quantities in the expression are all positive.

The question now is, when are incidence matrices square?

Theorem 4.2.2 (Square 2-designs). *The following are equivalent for 2-designs with $k < v$:*

i) $b = v$
ii) $r = k$
iii) *Any two blocks have λ points in common.*
iv) *Any two blocks have a constant number of common points.*

Proof

Notice that the first two conditions are equivalent by $bk = vr$.
Now if $r = k$, then

$$MJ_v = J_r M = kJ_{r \times v}$$

where J_s is the ones matrix with size s. So

$$MM^\intercal = MM^\intercal MM^{-1} = M((r-\lambda)I_V + \lambda J_v)M^{-1} = (r-\lambda)I_V + \lambda J_v$$

so any two blocks have λ common points. As the third condition implies the last one we are left with proving the blocks having constant number of common points implies $b = v$. This is a consequence of Fisher's inequality because the dual design is now a 2-design then $b \geqslant v$ and originally $v \geqslant b$ so we have $b = v$.

Hadamard Designs

Definition 4.2.3. A Hadamard matrix is a matrix with entries in $\{1, -1\}$ such that $HH^\intercal = nI$.

This means that for rows u, v we have $\langle u|u\rangle = n$ and $\langle u|v\rangle = 0$.

Example 4.2.4. In dimension 1 we have (1) and (-1). In dimension 2 we only have $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. None exist in dimension 3 but in dimension 4 we have

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Now Hadamard matrices are the equality case in the Hadamard inequality which says $|\det M| \leqslant n^{n/2}$ whenever $|M_{ij}| \leqslant 1$.

Observe that for a Hadamard matrix, we can negate a row or a column and still get another Hadamard matrix and also rearrange rows to get other Hadamard matrices. Applying this operations we can get an *normalized* version of a Hadamard matrix with ones on the top row and first column.

Proposition 4.2.5. *Let H be a normalized Hadamard matrix. If we consider the $(1,1)$ minor and replace the -1 by zeroes we obtain the incidence matrix of a $2-(4k-1, 2k-1, k-1)$ design. When replacing the ones with zeroes we get a matrix of a $2-(4k-1, 2k, k)$.*

Example 4.2.6. From the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

we extract $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ which codify the designs $\{1\}, \{2\}, \{3\}$ and $\{1,3\}, \{2,3\}, \{1,2\}$.

The Fano plane gives us a Hadamard matrix CONJECTURE

4.3 Day 25 | 20230324

Lemma 4.3.1. *If H is an $n \times n$ Hadamard matrix then $n = 1, 2$ or $4t$ for some t.*

Proof

If $n \geqslant 3$, we consider the first three rows of H. We may normalize the first row to convert it to only ones. And in the second row we may rearrange columsn so that it looks like $11\ldots1-1\cdots-1$. As the first two rows are orthogonal we deduce that the amount of $1'$s and $-1'$s must be the same.
The third row is ones and minus ones arranged in four blocks of lengths p, q, r, s. We have

$$\begin{aligned} \langle u|v\rangle = 0 &\Rightarrow p+q = r+s \\ \langle u|w\rangle = 0 &\Rightarrow p+r = q+s \\ \langle v|w\rangle = 0 &\Rightarrow p+s = q+r \end{aligned}$$

IN PIC $v = u, w = v$ and $u = w$. Solving the system of equations we get $p = q = r = s$ so $4 \mid n$.

Corollary 4.3.2. *There exists a Hadamard matrix of order $4t$ for all t.*

Dxuality in Desgins

Recall a design $\mathcal{D}$ is dual when $\mathcal{D} \simeq \mathcal{D}^{\mathsf{T}}$.

Definition 4.3.3. A duality of designs is a pair of morhpisms (σ, τ) such that

$$\tau(B) \in \sigma(X).$$

If $\sigma\tau = \mathrm{id}_B$ and $\tau\sigma = \mathrm{id}_X$ we say that the duality is a polarity.

Example 4.3.4. In the Fano Plane, $X = [7]$, and

$$\mathcal{B} = \{ 123, 345, 561, 147, 257, 367, 246 \}.$$

We will form a bijection between these sets. Our function σ for example maps as follows:

$$1 \mapsto 345, \quad 3 \mapsto 561, \quad 5 \mapsto 123.$$

The vertex 7 goes to the circle 246 and the midpoints go to the heights of the triangle. There are points which are on their own polarity like $2, 4$ and 6 while the others aren't.

Example 4.3.5. This motivates the following design, consider $X = [16]$ as labels of a 4×4 grid. The blocks look like blocks to all the sides of our pick. For example 10's block is $\{ 2, 6, 9, 11, 12, 14 \}$. This gives us a $2 - (16, 6, 2)$ design.

The polarity in this case is literally $x \mapsto B_x$. In this case no point is contained in its own block.

Strongly Regular Graphs

Recall regular graphs have same degree across all vertices.

Definition 4.3.6. A strongly regular graph with parameters (n, k, λ, μ) is a graph with n vertices, not complete nor empty such that for every pair of vertices x, y, the number of **common neighbors** is

$$\begin{cases} k & \text{if} \quad x = y \\ \lambda & \text{if} \quad x \quad \text{is adjacen to} \quad y \\ \mu & \text{if} \quad x \quad \text{isn't adjacen to} \quad y \end{cases}$$

Example 4.3.7. The Peterson graph is strongly regular. Its parameters are $(10, 3, 0, 1)$. The Clebsch and Schlafi graphs are strongly regular.

We can construct a graph from a self-dual design. The vertices will be the set X and edges are xy such that $x \neq y$ and $y \in \sigma(x)$. The Fano Plane's graph is PICTURE We can see that this graph isn't strongly regular as some points are in their own polarity and some aren't.

Example 4.3.8. In the grid design example we have

Each point is connected to everything in its block, so for example

$$N(1) = \{2, 3, 4, 5, 9, 13\}$$

Every vertex is connected to every other vertex in its row or column. The resulting graph is called $L_2(4)$, we claim it's strongly regular. Recall the parameters of the design is $2 - (16, 6, 2)$. So in the case of n, k we still have $16, 6$. For λ, μ we have that they are $2, 2$. Observe that $\lambda = \mu$!

This gives rise to a strongly regular family wth $\lambda = \mu$.

Lemma 4.3.9. *If in a polarity σ, we never (or always) have $x \in \sigma(x)$, then the associated graph is strongly regular. If the design is a $2(v, k, \lambda)$ we get a strongly regular graph with parameters (v, k, λ, λ) (in the never case). While in the other it's $(v, k-1, \lambda-1, \lambda-1)$.*

Proof

For the never case,

$$|\sigma(x) \cap \sigma(y)| = \lambda$$

for $x \neq y$. So whether or not xy is an edge, they have λ common neighbors. Also $|\sigma(x)| = k$ gives us the degree parameter.

From the graph, we can construct a symmetric 2-design. The vertices are the points, and the blocks are the neighborhoods of each point.

Chapter 5

Matroids

5.1 Day 26 | 20230329

- ⋄ Groups are an abtracion of operations
- ⋄ Topological spaces are an abstraction on continuity
- ⋄ And in the same fashion, matroids are an abstraction of linear independence.

June Huh and Chris Eur both investigate Matroid theory. Hodge theory studies vector bundles over matroids.

Definition 5.1.1. A matroid is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I} \subseteq \mathcal{P}(E)$, called he set of independent sets, satisfying the following properties:

- ⋄ $\varnothing \in \mathcal{I}$.
- ⋄ If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.
- ⋄ (Independence Exchange) If $I_1, I_2 \in \mathcal{I}$ with $|I_1| \leqslant |I_2|$, then

$$\exists x \in I_2 \backslash I_1 (I_1 \cup \{ x \}) \in \mathcal{I}.$$

Example 5.1.2. Consider $\{ v_1, \ldots, v_n \} \subseteq k^n$, this will be our set. The independent sets are

$$\mathcal{I} = \{ I \subseteq E : \ I \text{is linearly independent} \}.$$

Example 5.1.3. The set of vectors $\{ \pm e_1, \pm e_2 \}$ is also a matroid when considering the independent sets as pairs of these vectors and their subsets. We can represent this as a poset: DRAWING In this case we have a dependence relation among $v_1, \ldots, v_n$ which means that

$$\sum a_i v_i = 0 \quad \text{with not all} \quad a_i = 0.$$

Example 5.1.4. If we consider now the set of edges of a graph as our finite set, we declare the independent sets to be the cycle free subsets of the edges. The first two axioms are clear, the empty set is independent and subgraphs of forests are forests.

Now take $|I_1| < |I_2|$ and assume on contradiction that all red[1] edges connect vertices with a green component. DRAWING

If somehow the condition holds, then $|I_2| \leqslant |I_1|$ because graphs are a forest. So there exists an edge $vw \in I_2$ where v, w are not in the same component of I_1. $I_1 \cup \{ x \}$ doesn't contain a cycle.

There's much more matroids than desgins! The issue instead of finding is classifying.

Graphical Matroids

Example 5.1.5. Consider the graph C_4 plus a diagonal. The maximal independent sets are

$$\{ abc, abd, abe, acd, ace, bcd, bde, cde \}.$$

Contrasting with vectors, where maximal independent sets are bases, here the maximal independent are spanning forests.

In general this holds for abstract matroids.

Definition 5.1.6. A <u>basis of a matroid</u> is a maximal independent set under the inclusion relation.

Lemma 5.1.7. *In any matroid $M = (E, \mathcal{I})$, all bases have the same size.*

Proof

Assume on the contrary that there are two bases B_1, B_2 with different sizes. Assume $|B_1| < |B_2|$, then by the independent Exchange axiom we can add x to B_1 so that it's still independent.
This contradicts the maximality of B_1.

The Independent Exchange axiom captures the nice algebraic properties!

Equivalently we may define the matroids through bases as follows.

Definition 5.1.8. A <u>matroid</u> is a pair $(E, \mathcal{B})$, where E is finite and $\mathcal{B} \subseteq \mathcal{P}(E)$ such that

[1] red is I_2 and green is I_1.

- ◇ $\mathcal{B}$ is nonempty.
- ◇ (Basis Exchange) If $B_1 \neq B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$, then $\exists y \in B_2$ such that $B_1 - x \cup y \in \mathcal{B}$.

To prove the equivalence of definitions we need some lemmas.

Lemma 5.1.9. *If $M = (E, \mathcal{I})$ satisfies the first set of axioms and $\mathcal{B}$ is taken to be the maximal elements, then the structure $(E, \mathcal{B})$ satisfies the basis axioms instead.*

On the other hand we have:

Lemma 5.1.10. *If $M = (E, \mathcal{B})$ satisfies the basis axioms then*

$$\mathcal{I} = \{ I \subseteq E : \exists B \in \mathcal{B},\ I \subseteq B \}$$

then this structure $(E, \mathcal{I})$ satisfy the first set of actions.

Matroids can be looked at by more than 20 ways, dependent sets, rank functions, closure operations, lattices and posets. There's geometric lattices which are matroids. We are going to look a many cryptomorphisms of Matroids.

5.2 Day 27 | 20230331

5.3 Day 28 | 20230403

Connection with posets

Definition 5.3.1. Suppose $M = (E, \mathcal{I})$ is a matroid and $X \subseteq E$. The restriction of M from E to X is

$$M \mid_X = (X, \mathcal{I} \mid_X) \quad \text{where} \quad \mathcal{I} \mid_X = \{ I \in \mathcal{I} :\ I \subseteq X \}.$$

The rank of a matroid is the size of any basis. For a subset X, the rank is $\operatorname{rk}(M \mid_X)$, the size of the largest independent set contained in X.

In graphs, rank is largest spanning tree. With this we can rank-define a matroid.

Definition 5.3.2. A matroid is a finite set E with a rank function $\operatorname{rk} : \mathcal{P}(E) \to \mathbb{N}$ which satisfies

i) If $X \subseteq E$, then $0 \leqslant \operatorname{rk}(X) \leqslant |X|$.
ii) If $X \subseteq Y$, then $\operatorname{rk}(X) \leqslant \operatorname{rk}(Y)$.
iii) If $X, Y \subseteq E$, then $\operatorname{rk}(X \cup Y) + \operatorname{rk}(X \cap Y) \leqslant \operatorname{rk}(X) + \operatorname{rk}(Y)$.

Notice that the last condition looks like upper semi-modularity. As a quick example

$$\dim(V + W) + \dim(V \cap W) \leqslant \dim(V) + \dim(W).$$

Recall that in the matroid $\{ \pm e_1, \pm e_2 \}$ we have rank 2. The subsets $\{ e_1, e_2 \}$, $\{ e_1 \}$ and $\{ e_1, -e_1 \}$ have ranks $2, 1$ and 1. So we may recover the independent sets as

$$\mathcal{I} = \{ I \subseteq E : \operatorname{rk}(I) = |I| \}.$$

Definition 5.3.3. If $M = (E, \operatorname{rk})$ is a matroid, then for $X \subseteq E$ we define the closure as

$$\operatorname{cl}(X) = \{ x \in E : \operatorname{rk}(X \cup x) = \operatorname{rk}(X) \}.$$

In representable matroids, the closure is the span.

Example 5.3.4. Consider the graph

GRAPH

We can add edges to make cycles because they will have our graph as a spanning tree.

Definition 5.3.5. A matroid is a pair (E, cl) if $\operatorname{cl} : \mathcal{P}(E) \to \mathcal{P}(E)$

i) For $X \subseteq E$, $X \subseteq \operatorname{cl}(X)$.
ii) If $X \subseteq Y$, then $\operatorname{cl}(X) \subseteq \operatorname{cl}(Y)$.
iii) $\operatorname{cl}(\operatorname{cl}(X)) = \operatorname{cl}(X)$.
iv) If $y \in \operatorname{cl}(X \cup x) \backslash \operatorname{cl}(X)$, then $x \in \operatorname{cl}(X \cup y)$.

Example 5.3.6. Illustration of last axiom with vectors

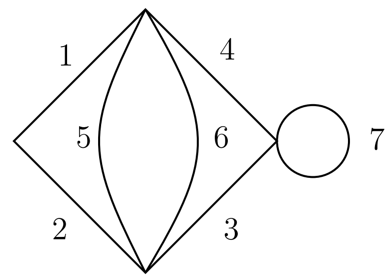
The basis is the minimal set whose closure is the whole set.

Definition 5.3.7. A flat of a matroid M is a set $X \subseteq E$ such that $X = \operatorname{cl}(X)$. A hyperplane is a flat of rank $\operatorname{rank}(M) - 1$.

Definition 5.3.8. The lattice of flats of a matroid M is the poset of all flats ordered by inclusion.

This is a very important construction, besides bases, this is the other way that we think about matroids.

Example 5.3.9 (Ex. 1.7.7 [6] (pg. 64))**.** Consider the graphical matroid M



We have a loop and a 2-cycle. The lattice is denoted $\mathcal{L}(M)$, the closure of the empty set in this matroid is $\{ 7 \}$, the full rank of the matroid is 3 so we may write the poset as follows:

GRAPh

If we were going to build a flat of rank 3, we could only obtain the whole set. There's gonna be some rank 1 and 2.

This is a lattice! Every pair of elements has a *meet* and a *join*.

Recall that the meet is the greatest lower bound and join is the smallest upper bound.

If we replace the graph with its associated simple graph, by deleting loops and simplifying multiple edges, gives us the *same* lattice of flats. Simple graphs determine *simple* matroids.

5.4 Day 29 | 20230405

Lattice of flats

We can naively call flat sets closed because X is flat if $X = \mathrm{cl}(X)$. The poset of flats is graded by the rank rk function.

The meet $X \wedge Y$ of two sets is $X \cap Y$. However the join $X \vee Y$ is different. On the past example we had

$$137 \wedge 147 = 17 \quad \text{and} \quad 17 \vee 567 = 12567$$

from this we extract $X \vee Y = \mathrm{cl}(X \cup Y)$. The fact that these are indeed the join and the meet will be left as homework. Today we will use the fact a matroid is a finite lattice.

Since the lattice is finite we have a 0 and a 1. In a matroid the 0 is the closure of the empty set which corresponds to the *loops* of M.

Definition 5.4.1. A <u>loop</u> is a circuit of size 1, where a circuit is minimal dependent set.

Simple matroids

Now we want to simplify matroids.

Definition 5.4.2. The parallel class of $x \in E$ is $\{ y : \{ x, y \} \in \mathcal{C} \}$.

In a representable matroid, the parallel class are the linearly dependent elements.

Definition 5.4.3. A matroid M is simple if it has no loops or 2-circuits.
The simplification of M, $M_{\sim}$ is formed by

- ⋄ Deleting all loops.
- ⋄ Choose 1 representative from each parallel class.

If $M = (E, \mathcal{B})$, then $M_{\sim} = (E_{\sim}, \mathcal{B}_{\sim})$. $E_{\sim}$ is formed as before, and for $B \in \mathcal{B}_{\sim}$, if $y \in B$ and $y \parallel x$ where $x \in E_{\sim}$, then $B_{\sim} = B \backslash y \cup x$.

Example 5.4.4. The simplification of a matroid is still a matroid.

xy, yz are 2-circuits, we want to show that xz is a 2-circuit. By the circuit exchange axiom there has to be $C3$ if C_1, C_2 have $e \in C_1 \cap C_2$, then there is a $C_3 \subseteq C_1 \cup C_2$.

Lemma 5.4.5. $\mathcal{L}(M) = \mathcal{L}(M_{\sim})$.

Theorem 5.4.6. *$\mathcal{L}$ is injective on simple matroids.*

This theorem we will not prove but this means that lattices of flats determine simple matroids.

Theorem 5.4.7. *A finite poset is the lattice of flats of a matroid if and only if it is a geometric lattice.*

Recall atoms on posets are the elements just above 0, the rank 1 elements of a poset. A lattice is atomic if every element is made up of atoms. In other words every element is a join of atoms.

Definition 5.4.8. A geometric lattice is a graded, atomic and upper semi-modular lattice.

In terms of matroids, we have rk as a grading and this is already upper semi-modular.

Example 5.4.9. As we have no multiple edges, then the closure of any element doesn't contain any other element, this means that atoms are singletons.

We will also have closures of 13 which don't close up to anything.

We can check upper semi-modularity with 13 and 24. Then $x \wedge y = \varnothing$ their join is the whole set. So we get

$$0 + 3 \leqslant 2 + 2.$$

We went down further than what we went in the join.

The atoms are *elements* of our matroid. So as the flats are unions of elements, we see that it is atomic!

Application to geomtric lattices

Assuming all the facts we will prove something about geometric lattices.

Definition 5.4.10. A coatomic lattice is a lattice where every element is the meet of coatoms (elements below 1).

Example 5.4.11. In matroids, the coatoms of $\mathcal{L}(M)$ are the hyperplanes. Those with rank $\operatorname{rk}(M) - 1$.

Theorem 5.4.12. *Finite geometric lattices are coatomic.*

Proof

Suppose $\mathcal{L}$ is a finite geometric lattice and M is a matroid such that $\mathcal{L} = \mathcal{L}(M)$.
If X is a flat, we want to show it's an intersection of hyperplanes.
If X has rank $\operatorname{rk}(M) - k$ (corank k) we will show that X is such an intersection by induction.
If $k = 1$, then X is a hyperplane. So it's trivially an intersection of hyperplanes.
If the fact is true for $k = n$ and X has corank $n + 1$ then M has an element $y \notin X$, so $Y = \operatorname{cl}(X \cup y)$ covers X. Thus the corank of Y is n.
By induction hypothesis $Y = H_1 \cap H_2 \cap \cdots \cap H_n$ where H_i are hyperplanes. Let H_{n+1} be the maximal flat containing X but not y, so $\operatorname{cl}(H_{n+1} \cup y) = E$ so H_{n+1} is a hyperplane. Now $Y \cap H_{n+1}$ contains X but not Y so

$$Y \cap H_{n+1} = X \Rightarrow X = H_1 \cap \cdots \cap H_{n+1}.$$

Using only the upper semi-modularity, proving this would be very tough! But this connection to matroid really eases the job.

fig1

fig 2

A Review

- ⋄ Matroids can be defined on: Independent sets, bases, circuits, rank, closure, Lattice of flats (up to simplicity) and by the greedy algorithm.

5.5 Day 30 | 20230407

The Greedy Algorithm

Recall that for weighted graphs we have Kruskal's algorithm. Consider the graph If each graph has a weight, how do you find the minimum weight spanning tree? We just take the lowest weight edge that doesn't form a cycle and then continue. Sometimes the greedy algorithm doesn't work but in this case it does.

We can't pick edges that form cycles, alternatively we could've also done this trees which *also* have the same weight. We will prove that Kruskal's algorithm works in matroids.

Lemma 5.5.1. *Suppose we have a weighted matroid $(E, \mathcal{I})$ where the weight is a function $w : E \to \mathbb{R}$.*

Applying the greedy algorithm to find a basis produces a basis of minimal or maximal weight.

We will basically go to a spanning forest which is an independent set. If we try to add one more edge, then we notice that things go wrong with the weights. Graphical matroid satisfies (I3).

The greedy algorithm can be used to define a matroid.

Theorem 5.5.2. *A pair $(E, \mathcal{I})$ is a matroid if and only if it satisfies the first two independence axioms and (G): for all weight functions $w : E \to \mathbb{R}$, greedy algorithm produces a maximal member of $\mathcal{I}$ with maximal or minimal weight.*

The fact that the algorithm finds the right thing is equivalent to the axiom $(I3)$. The lemma proves the forward direction, that if the pair is a matroid then the greedy algorithm works. We will prove that given the greedy algorithm, then we get the independent sets back.

figure

Proof

We must show that $(E, \mathcal{I})$ satisfies the axiom $I3$. If we assume the contrary that there are I_1, I_2 independent with $|I_1| < |I_2|$ but no element of I_2 can augment I_1. Let us compare $I_2 \backslash I_1$ and $I_1 \backslash I_2$. We have

$$|I_1 - I_2| < |I_2 - I_1|$$

so let $\varepsilon < 1$ such that $\frac{|I_1 - I_2|}{|I_2 - I_1|} < \varepsilon$. Consider the weight function

FIGURE

By the (G) axiom we can find a maximal element of $\mathcal{I}$ with *maximal weight*. The greedy algorithm does the following

- ⋄ Picks all 1 elements, all of I_1.
- ⋄ Picks an ε if it can. *It can't*.
- ⋄ Keeps picking 0's

So the total weight of max is I_1.
By $(I2)$, there exists a basis element $B_2 \supseteq I_2$, so

$$w(B_2) \geqslant w(I_2) \Rightarrow |I_1 \cap I_2| \geqslant w(B_2) \geqslant |I_1 \cap I_2| + \varepsilon |I_2 - I_1|$$

by our choice of ε we have that

$$|I_1| > |I_1 \cap I_2| + |I_1 \backslash I_2| = |I_1|.$$

This is absurd, so our assumption must've been false from the beginning.

Deletion and Contraction

Similar to talking about minors of a graph, we may consider a similar operation on matroids.

Definition 5.5.3. For a graph G, the <u>contraction</u> of G at e is G/e is The <u>deletion</u> is $G \backslash e$. A graph that can be formed from deletion and contraction from G is a <u>minor</u> of G.

this graph are planar/not planar

Graph with red spanning tree

graph

In general there exist k-minors which allow us to only do k steps. Ulam's conjecture says that every graph can be uniquely determined by its 1-vertex deletions (DIFFERENT THATN THIS OPERATION). This is a reconstructibility problem.

There's other question on reconstructibility for families of graphs.

Definition 5.5.4. A planar graph is a graph that can be embedded into the plane and has no crossing edges.

Theorem 5.5.5 (Kuratowski)**.** *A graph is planar if and only if it has no minors isomorphic to K_5 or $K_{3,3}$.*

For matroids, deleting edges from matroid gives us restriction, independent sets are everything defined not containing that edge.

Definition 5.5.6. The dual matroid of $M = (E, \mathcal{B})$ is $M^* = (E, \{ E - B : B \in \mathcal{B} \})$.

Example 5.5.7. Suppose we have a planar graphical matroid M represented by a planar graph. We wish to find its dual: The dual of a planar graph is planar and the dual is an involutive operation. We may complement the basis on the original graph to get a new one

5.6 Day 31 | 20230410

Recall G/e is contraction and $G\backslash e$ is deletion. For matroids recall that the dual matroid is $M^* = (E, \{ E\backslash B \})$ where $B \in \mathcal{B}$ the original basis.

Example 5.6.1. For a graph we can take the vertices as the faces and the edges are drawn when two faces meet.

Loops in the dual graph look like leaves in the original graph.

We now claim that $(M_G)^* = M_{G^*}$. In terms of the graph, the complement of the spanning tree determines a spanning tree of the dual graph. What's happening with the dual, contraction corresponds with deleting an edge on the dual graph.

Characteristic polynomial of a matroid

We will use the to construct the characteristic polynomial of the matroid which generalizes the chromatic polynomial of a graph.

Definition 5.6.2. A proper coloring of a graph G is a coloring such that no two adjacent vertices is the same color.

Generalizing this definition to matroids is a bit hard. How do we generalize it? We use the chromatic polynomial of the graph. Recall $\chi_G(q)$ is the number of proper colorings of G with q colors.

Example 5.6.3. Consider the graph P_2, then $\chi_{P_2}(q) = q(q-1)$. For two lone vertices, $\chi = q^2$.

If we have a cycle, a C_3 we have $q(q-1)(q-2)$. But for a P_3 we obtain $q(q-1)^2 = q(q-1)(q-2) + q(q-1)$.

There's a recursive way to see that the chromatic polynomial is ineed a polynomial.

Proposition 5.6.4. *For all graphs,*

$$\chi_G = \chi_{G \setminus e} - \chi_{G/e}.$$

Example 5.6.5. With this, we may compute the chromatic polynomial of C_4. We have

$$\chi_{C_4} = \chi_{P_4} - \chi_{C_3} = (\chi_{P_2 \circlearrowright P_2} - \chi_{P_3}) - \chi_{C_3} = \chi^1_{P_2} - \chi_{P_3} - \chi_{C_4}.$$

This quantity is $q(q(q-1)^2) - q(q-1)^2 - q(q-1)(q-2)$. In another way we can count directly as

$$q(q-1)(q-2) + q(q-1)^2.$$

Observe the absolute values of the coefficients! We can see that they are unimodal! Not only that, they are log-concave.

Definition 5.6.6. A sequence (a_n) is log-concave if $(\log(a_n))$ is a concave sequence. Or Equivalently $a_n^2 \geqslant a_{n-1}a_{n+1}$.

Similar q-analogues seem to exhibit this property. The problem of showing this for chromatic polynomials was open until 2012 when proven by June Huh. He also extended the result for matroids.

Theorem 5.6.7 (Huh *et al*, 2012). *The coefficients in absolute value of χ_G forms a log-concave sequence.*

Example 5.6.8. In the previous example, expanding the polynomial we obtain coefficients $1, 4, 6, 3$ and we can see that it is indeed log-concave.

Strategies to work about this include Hodge theory and Tropical Geometry.

Definition 5.6.9. For a matroid M, we define the characteristic polynomial as $\chi_{\text{loop}} = 0$ and $\chi_{P_2} = q(q-1)$ then recursively:

$$\chi_M = \chi_{M\setminus e} - \chi_{M/e}.$$

The same result of Huh applies to matroids.

Chapter 6

Geometries

Finite Geometries

We will be working on finite fields of the form $\mathbb{F}_{p^r}$ where p is a prime number. In classical geoemtry we have points, lines hyperplanes, circles and more. Hyperplanes can be interpreted as $\langle a|x\rangle = c$ and then lines are intersections of hyperplanes.

These are all types of equations which can work on finite fields! In this cases the curves will be finite sets of points.

Example 6.0.1. In $\mathbb{F}_3^2$ we want to look for solutions of $x^2 + y^2 = 1$. We are looking for solutions of $x^2 + y^2 \equiv 1 \mod 3$. Solutions are $(0, 1), (1, 0), (0, 2)$ and $(2, 0)$.

6.1 Day 32 | 20230412

Example 6.1.1. In $\mathbb{F}_3^2$ we have:

- Solutions to $x + 2y = 1$ are $(1, 0), (0, 2)$ and $(2, 1)$.
- For $x = 1$ we have $(1, 0), (1, 1)$ and $(1, 2)$.

Both this equations are linear. But if we draw them we don't obtain exactly lines.

Example 6.1.2 (SET game)**.** The game SET is essentially a game of finding lines in $\mathbb{F}_3^4$. The coordinates are shape, color, number, and shade. Each of this coordinates has three possibilities so a winning set of 3 cards must have all the same attribute or all different.

PICTURE In this example a line is determined by 3 linear equations.

DRAWING

DRAWING OF P2

Example 6.1.3 (FOURMATION game)**.** We are now looking for planes in $\mathbb{F}_2^6$. Planes here look like "faces" or the set generated by two vectors.

Cards have 4 shapes, number of shapes, color of the shapes but this is coded in binary.

Each attribute is ALL SAME, ALL DIFF, or Equal in Pairs.

Projective Spaces

Definition 6.1.4. For a field $\mathbb{F}$, $\mathbb{P}^n_{\mathbb{F}}$ can be seen as the set $\mathbb{F}^{n+1} \backslash \{ 0 \}$ modded with the relation $x \sim \lambda x$ where λ is a scalar.

Example 6.1.5. The projective plane over $\mathbb{R}$ is $(\mathbb{R}^3 \backslash \{ 0 \})/ \sim$ and this looks like half a sphere in a funny way. Points are roughly unit vectors in the upper hemisphere. Algebraically this looks like $[x : y : z]$ with x, y, z not all zero and $[x : y : z] = [\lambda x : \lambda y : \lambda z]$. Intuitively we can see it as the space of directions in $\mathbb{R}^3$.

Canonically we can decompose into

$$\mathbb{P}^2_{\mathbb{R}} = \{ [x : y : 1] \} \cup \{ [x : 1 : 0] \} \cup \{ [1 : 0 : 0] \}.$$

In other words we can identify this as

$$\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \mathbb{R} \cup \{ \text{pt.} \}.$$

Given this, we ask, *what are lines in* $\mathbb{P}^2$? Once again they are solutions to equations, but in this case equations must be *homogenous*.

Example 6.1.6. Consider the equation $2x - y = z$, we have two cases, when $z = 1$ or $z = 0$. We get all the affine solutions and also $[\frac{1}{2} : 1 : 0]$ which is the point at infinity.

The equation $z = 0$ describes the line at infinity. Keep in mind that $\mathbb{P}^2$ is compact!

Example 6.1.7. The projective plane over $\mathbb{F}_2$ can be constructed in an analogous way. We have 7 points, triplets of points either zero or one except all zeroes.

The line through $[0 : 0 : 1]$ and $[0 : 1 : 1]$ also touches infinity at $x = 0$. Notice that lines are defined as $x = 0$, $x + z = 0$, $x + y + z = 0$

6.2 Day 33 | 20230414

Lemma 6.2.1. $\mathbb{F}_q^r$ *is a matroid under linear independence over* $\mathbb{F}_q$*, and* $\mathbb{P}^{r-1}_{\mathbb{F}_q}$ *is its associateed simple matroid.*

Latiz en foto

Proof

Recall that

$$\mathbb{P}^{r-1}_{\mathbb{F}_q} = (\mathbb{F}_q^r \backslash \{ 0 \}) / \sim .$$

Now loops in matroids are circuits of size 1, the only loop in $\mathbb{F}_q^r$ is the zero vector. So when removing it we remove the loops.

Example 6.2.2. The lattice of flats of $\mathbb{F}_2^3$, where we can visualize the set as the unit cube's vertices, can be seen as the lattices of spanning sets.

The rank 1 flats are $\{ 0, v \}$, so 7 in total. For the rank two flats we get $\binom{7}{2}$ in total choices but we divide by the ways to form 3 out of 2. This number is $\binom{3}{2}$. This is because if we generate with $1, 4$ we get 5, but if we generate with $1, 5$ we get 4. So that's why we divide. There's only one rank 3 flat.

This is the same lattice of flats of $\mathbb{P}^2_{\mathbb{F}^2}$ after deleting 0. We don't have to worry about scaling because the only scalar is 1.

Let's build designs with this information. From the last picture we can observe a 2-design.

Lemma 6.2.3. *The matroid* $(\mathbb{P}^2_{\mathbb{F}_q}, \mathcal{B})$ *where* $\mathcal{B}$ *is the set of lines in* $\mathbb{P}^2_{\mathbb{F}_q}$ *forms a* $2 - (q^2 + q + 1, q + 1, 1)$ *design.*

Proof

The number of vertices is

$$v = |\mathbb{P}^2_{\mathbb{F}_q}| = \frac{|\mathbb{F}_q^2| - 1}{|\mathbb{F}_q| - 1} = \frac{q^3 - 1}{q - 1} = q^2 + q + 1.$$

The size of the blocks is

$$k = |\text{lines in } \mathbb{P}^2_{\mathbb{F}_q}| = |\mathbb{F}_q^2 \backslash \{ 0 \} / \sim | = |\mathbb{P}_q^1| = q + 1.$$

And finally any two points determine a line.

A similar exercise can be done for k-flats, to show it is a collection of 2-designs. The annoying part is finding λ so we need to use overcounting and dividing.

We can make an interesting space out planes called the Grassmannian which generalizes projective space.

Definition 6.2.4. The Grassmannian is the set of k-dimensional subspaces of $\mathbb{F}^n$. We denote it $Gr_{\mathbb{F}}(k, n)$.

Example 6.2.5. The case $k = 1$ is projective space.

We wish to count the number of points in $Gr_{\mathbb{F}_q}(k, n)$ in general. We will find that it is $\binom{n}{k}_q$, the q-analogue.

For this effect, recall

$$\mathbb{P}^2 = \{ [x : y : 1] \} \cup \{ [x : 1 : 0] \} \cup \{ [1 : 0 : 0] \},$$

this is called the Schubert decomposition of $\mathbb{P}^2$. For te analogue for the Grassmannian we must describe the coordinates inside it. We have

$$Gr(k, n) = \{ \text{full rank } k \times n \text{ matrices} \} / \sim$$

where matrices are similar if row-equivalent.

Example 6.2.6. Via row-reduction we have the Schubert decomposition of $Gr(2, 4)$: VER FOTO This paramatrizes all the two planes in four space seen as the row-spaces of the matrices. In $\mathbb{F}_q$ we have that the size of the Grassmannian is

$$q^4 + q^3 + q^2 + q^2 + q + 1 = \binom{4}{2}_q$$

which is supporting evidence for our claim.

Observe that if we delete the pivot columns and form the diagrams, we can see that the empty squares form the sub-partitions of the shape in question. In general, deleting the k pivot columns from $k \times n$ gives us partitions.

Proposition 6.2.7. *We have*

$$Gr(k, n) = \bigcupdot_{(*)} \Omega^0_\lambda$$

where the disjoint union is taken over partitions formed in the previous way ASK HOW lambda subset box??? and Ω^0_λ is the set of row-reduced $k \times n$ matrices with pivots in columns $\lambda_k + 1$, $\lambda_{k-1} + 2$ all through $\lambda_1 + k$.

6.3 Day 34 | 20230417

Last time we said that the size of the Grassmannian was $\binom{n}{k}_q$.

Projective Transformations

We say projective space has dimension n as a manifold.

Definition 6.3.1. We define the projective linear group as

$$\mathrm{PGL}_n(\mathbb{F}) = {}^{\mathrm{GL}_n(\mathbb{F})}\big/_{\sim}$$

where the relation is scalar multiplication.

Example 6.3.2. Consider the linear transformation $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. This acts the same as $\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$ in projective space.

Proposition 6.3.3. $\mathrm{PGL}_n(\mathbb{F}) = \mathrm{Aut}(\mathbb{P}^n_{\mathbb{F}})$.

Since we are doing finite geometries, we would like to count how many automorphisms does $\mathbb{P}^n_{\mathbb{F}_q}$ have. In other words, we wish to find $|\,\mathrm{PGL}_2(\mathbb{F}_q)|$.

Example 6.3.4. Let us find $|\,\mathrm{PGL}_2(\mathbb{F}_2)|$. We are looking for invertible matrices of size 3×3 over $\mathbb{F}_2$. Consider counting by the columns, the first column has 7 possibilities, the next one can't be the same so 6 possibilities. And the last columnn can't be the sum of the first two so 4 possibilities. This means that

$$|\,\mathrm{PGL}_2(\mathbb{F}_2)| = 7 \cdot 6 \cdot 4 = 168,$$

which is also the number of automorphisms of the Fano plane.

The next natural question is to generalize the size of the field or the space's dimension.

Theorem 6.3.5. *For any q we have*

$$|\,\mathrm{GL}_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q)(q^n - q^2)\ldots(q^n - q^{n-1}).$$

Proof

Inductively, the first column can be filled in $q^n - 1$ ways. At the i^{th} step we can have columns not in the span of the previous $i - 1$ columns. This is an $i - 1$ dimensional space, so we subtract q^{i-1}.

Corollary 6.3.6. $|\,\mathrm{PGL}_n(\mathbb{F}_q)| = \frac{1}{q-1}|\,\mathrm{GL}_n(\mathbb{F}_q)|$

We would like to count this with q-analogues, but we won't for now.

Flags

Definition 6.3.7. A flag is a chain of subspaces

$$\{ 0 \} = V_0 \leqslant V_1 \leqslant \ldots \leqslant V_n = \mathbb{F}^n$$

where $\dim(V_i) = i$.

We ask, how many flags are there in $\mathbb{F}_q^n$.

Example 6.3.8. In $\mathbb{F}_3^2$, then the number of flags is the number of lines through the origin.

For $\mathbb{F}_3^3$ it's just a bit tougher, we have to pick the origin, a line and a plane which contains that line. For V_0 we have only one choice, for V_1 there's 13 choices. We can find this by considering $Gr_{\mathbb{F}_3}(1,3)$ whose cardinality is $\binom{3}{1}_3$. Counting by hand we get $(3^3-1)/(3-1)$.

For the case of V_2 we count 52. Finally for V_3 there's only one choice.

Theorem 6.3.9. *The number of flags in $\mathbb{F}_q^n$ is $(n)_q!$.*

Proof

Inductively, there's 1 choice for V_0. For V_1 there are $\frac{q^n-1}{q-1}$, because choose a nonzero point in q^n-1 ways, and each line is overcounted $q-1$ times.
The number of choices of a $V_2 \supseteq V_1$ is $\frac{q^{n-1}-1}{q-1}$ becuase it's the number of choices of $1D$ space in $\mathbb{F}_q^n/V_1 \simeq \mathbb{F}_q^{n-1}$. In general this process returns

$$\left(\frac{q^n-1}{q-1}\right)\left(\frac{q^{n-1}-1}{q-1}\right)\ldots 1 = (n)_q!$$

This proof gives us an alternate proof for the size of the Grassmannian. As a corollary we have that the size of the Grassmannian is $\binom{n}{k}_q$.

Counting k-dimensional subspaces, we overcount by counting flags, but that overcounts each V_k a certain number of times. It overcounts by the number of flags $0 \leqslant V_1 \leqslant V_2 \leqslant \ldots \leqslant V_k$ and $V_{k+1} \leqslant \ldots \leqslant V_n$. So we divide by $(k)_q!$ and $(n-k)_q!$.

Schubert Decomposition

A flag can be represented as a list of n vectors, each one represents a "new direction" which we obtain at each step.

Definition 6.3.10. A Flag variety is the set of flags in $\mathbb{F}^n$. This is denoted:

$$Fl_n(\mathbb{F}) = \{ \text{flags in } \mathbb{F}^n \}.$$

Example 6.3.11. We can consider the vectors $u_1 = (1,-1,3)$, $u_2 = (2,-2,5)$ and $u_3 = (8,2,1)$. This represents the flag

$$\langle u_1 \rangle \leqslant \langle u_1, u_2 \rangle \leqslant \langle u_1, u_2, u_3 \rangle .$$

To simplify our matrix we can rescale our matrix and combine rows with *lower* rows. The obtained matrix is said to be in normal form.

This makes it easy to count points in the flag variety in a more systematic way, becuase if π is a permutation and Ω_π is the set of flags with normal form π, then

$$|\Omega_q(\mathbb{F}_q)| = q^{inv(\mathrm{rev}(\pi))}.$$

For the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & x & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

we have 4132 which reverses to 2314.

6.4 Day 35 | 20230419

We will continue talking about Grassmannians over real and complex numbers, but first we divert to topology.

Topology of the Projective Space

We can cover projective space by the distinguished open sets:

$$\mathbb{P}^2 = D(x) \cup D(y) \cup D(z)$$

and even if the sets intersect, we can still cover by these sets. We may also call these sets, *affine patches*.

We may *glue* the topologies in order to say when a set is open in projective space.

Limits in Projective Space

How would we begin to calculate $\lim_{t\to\infty}[1 : t]$? As t is growing, we have that it's not zero. So this means that this point is $[1/t : 1]$. But this tends to $[0 : 1]$.

In the same fashion $[1 : t^2 : t]$ *tends* to $[0 : 1 : 0]$ and $[2t : 3t : 1]$ tends to $[2 : 3 : 0]$. Observe the connection with the slope of the line!

In the same fashion as metric topology, the closure of set in projective space will include all of its limit points. With this we ask

What is the closure of points in projective space of the form $\{ [0 : 0 : 1 : x : y : z] \}$*?*

This is a Schubert cell and we would like to find the limit points of this. Before we tackle it, we may ask an easier question.

Example 6.4.1. Consider the set $\{ [x : y : 1] \}$, its closure is $\mathbb{P}^2$. We have all of the affine plane and the limit points of lines which form the line at infinity.

If we try closing the line $\{ [x : 1 : 0] \}$ we just have to add the point at infinity of this line which is $[1 : 0 : 0]$.

In general we have the following lemma:

Lemma 6.4.2. *Call Ω_j the union of $D(x_i)$'s up to j, intersecting with $V(x_i)$ for $i > j$. Then its closure is the disjoint union of Ω_k for $k \leqslant j$.*

Proof

To obtain $[y_0 : y_1 : \cdots : y_{k-1} : 1 : 0 : \cdots : 0]$ as a limit we take

$$\lim_{t\to\infty}[y_0 t : y_1 t : \cdots : y_{k-1}t : t : 0 : \cdots : 0].$$

Its complement is open because it's a union of affine patches.

For the Grassmannian, the affine patches are defined by minors of our matrix with determinant 1.

Theorem 6.4.3. *For the Schubert cell Ω^o_λ, its closure is*

$$\bigcup_{(*)} \Omega^o_\mu$$

where the union is taking over partitions μ which contain λ.

6.5 Day 36 | 20230421

Recall the Grassmannian is the set of k-planes in n-space, the set of k-dimensional subspaces of $\mathbb{F}^n$. Recall the Schubert cells are

$\Omega^o_\lambda = \{\,$points in $Gr_k(n)$ represented by matrices in RR form with pivots in cols.$\lambda_k+1, \ldots, \lambda_1+k\,\}$.

Example 6.5.1. Consider the matrix

$$\begin{pmatrix} 1 & x & y & 0 & z & 0 & a \\ 0 & 0 & 0 & 1 & w & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 1 & c \end{pmatrix}$$

which represents something, this is associated to the tableau

x	y	z	a
		w	k
			c

and the empty spaces correspond to the partition $\lambda = (3, 2)$. An element of $\Omega^o_{3,2}$ would be the row span of the matrix when substituting the variables by some values.

This cell is contained in $Gr_3(7)$ and itself the cell is isomorphic to $\mathbb{F}^7$ because we have 7 degrees of freedom.

Today we wish to show that the closure of the Schubert cell is the Schubert variety. Recall that for example $\overline{\{\,[0:0:1:x:y:z]\,\}}$ contains

$$\{\,[0:0:0:1:a:b]\,\} \cup \{\,[0:0:0:0:1:w]\,\} \cup \{\,[0:0:0:0:0:1]\,\}.$$

We will see how it works with the example in question. This means that we want to show some limit of points in $\Omega^o_{\ydiagram{2,3}}$ gives any chosen point of

$$\Omega^o_{\ydiagram{2,4}} \cup \Omega^o_{\ydiagram{3,3}} \cup \Omega^o_{\ydiagram{1,2,3}}.$$

So we would like to see how

$$\begin{pmatrix} 0 & 1 & x & 0 & y & 0 & z \\ 0 & 0 & 0 & 1 & w & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 1 & s \end{pmatrix}$$

is a limit of our original matrix. Observe that

$$\lim_{t\to\infty}\begin{pmatrix}1 & t & xt & 0 & yt & 0 & zt\\ 0 & 0 & 0 & 1 & w & 0 & b\\ 0 & 0 & 0 & 0 & 0 & 1 & c\end{pmatrix} = \lim_{t\to\infty}\begin{pmatrix}1/t & 1 & x & 0 & y & 0 & z\\ 0 & 0 & 0 & 1 & w & 0 & b\\ 0 & 0 & 0 & 0 & 0 & 1 & c\end{pmatrix}$$

and this tends to the new Schubert cell. For the case of $\Omega^o_{(3,3)}$ we now have the other way. We creatively form

$$\begin{pmatrix}1 & x & y & 0 & -tz & 0 & w-trz\\ 0 & 0 & 0 & 1 & t & 0 & tr\\ 0 & 0 & 0 & 0 & 0 & 1 & s\end{pmatrix}$$

and the limit of this goes to a Schubert cell in $\Omega^o_{(3,3)}$. Let us work backwards from a Schubert cell in $\Omega^o_{(4,2)}$, we have

$$\begin{pmatrix}1 & x & y & 0 & z & w & 0\\ 0 & 0 & 0 & 1 & a & b & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

Observe that this is the limit of

$$\begin{pmatrix}1 & x & y & 0 & z & w & 0\\ 0 & 0 & 0 & 1 & a & b & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & t\end{pmatrix} \to \begin{pmatrix}1 & x & y & 0 & z & 0 & -wt\\ 0 & 0 & 0 & 1 & a & 0 & -bt\\ 0 & 0 & 0 & 0 & 0 & 1 & t\end{pmatrix}$$

and this last matrix is a Schubert cell of $\Omega^o_{(3,2)}$. This can be seen in the following diagram:

PICTURE

Poset of Schubert cells for $Gr(2,4)$

All the partitions we can obtain are the subpartitions of $(2,2)$. The empty partition is at the top, which corresponds as

$$\Omega^o_{\varnothing} = \left\{ \begin{pmatrix}1 & 0 & x & y\\ 0 & 1 & z & w\end{pmatrix} \right\} \simeq \mathbb{R}^4.$$

On this partition we can find (1) which is

$$\Omega^o_{\varnothing} = \left\{ \begin{pmatrix}1 & x & 0 & y\\ 0 & 0 & 1 & w\end{pmatrix} \right\} \simeq \mathbb{R}^3.$$

PICTURE

Partitions which contain these are $(1, 1)$ and (2) and they correspond to $\mathbb{R}^2$. We have that elements of the Schubert cell of (2) look like

$$\left\{ \begin{pmatrix} 1 & x & y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \simeq \mathbb{R}^2.$$

Given the poset, finding the closure is just taking the union of objects below it!

Example 6.5.2. Now consider the poset $Gr_2(5)$. At the top there is an empty set one. This is a subposet of Young's lattice!

Definition 6.5.3. An algebraic variety is the set of points defined by polynomial equations. When it's projective, the polynomials must be homogenous.

Example 6.5.4. The variety of $xy - 1$ is associated to the projective variety $xy - z^2$.

6.6 Day 37 | 20230424

Rational Equivalence

This concept is similar to homotopic equivalence. Recall that projective varieties are zero loci of sets of homogenous polynomials. Rational equivalence wants to smoothly deform points from one variety to the other.

Definition 6.6.1. Two varieties V, W are rationally equivalent if there exist homogenous (in $\underline{x}$) polynomials $f_i(\underline{x}, t)$ such that

$$V = \{\, \underline{x} : \forall i(f_i(\underline{x}, 0) = 0) \,\}, \quad \text{and} \quad W = \{\, \underline{x} : \forall i(f_i(\underline{x}, 1) = 0) \,\}.$$

Example 6.6.2. In $\mathbb{P}^2$ consider the polynomial

$$f(x, y, z, t) = xy - tz^2.$$

At $t = 0$ we have xy and the zero locus of this polynomial is the variety $\mathbb{V}(xy)$ in the affine patch U_z. This forms two circles which touch at the origin.

When $t = 1$ on the other hand, we have $xy - z^2$. This variety looks like an hyperbola on the affine patch U_z and at infinity we get $xy = 0$.

Basically the couple of intersecting lines is rationally equivalent to an hyperbola.

This idea turns into an equivalence relation, so we would like to study projective varieties up to rational equivalence.

Index

Bibliography

[1] François Bergeron, Gilbert Labelle, and Gilbert Leroux. *Combinatorial Species and Tree-like Structures*. Cambridge University Press, 1998.

[2] Peter Jephson Cameron and Jacobus Hendricus van Lint. *Designs, Graphs, Codes and their Links*. London Mathematical Society Student Texts. Cambridge University Press, 1991.

[3] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. London Mathematical Society Student Texts #35. Cambridge University Press, 1999.

[4] Grayson Graham. The ring of symmetric polynomials. *REUsChicago*, dunno.

[5] Ronald Lewis Graham, Donald Ervin Knuth, and Oren Patashnik. *Concrete mathematics: a foundation for computer science*. Addison-Wesley, 2nd ed edition, 1994.

[6] James Oxley. *Matroid Theory*. Oxford University Press, USA, 2nd edition, 2011.

[7] Bruce Eli Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. Graduate Texts in Mathematics №203. Springer, 2 edition, 2001.

[8] Bruce Eli Sagan. *Combinatorics: The Art of Counting*. Graduate Studies in Mathematics. American Mathematical Society, 2020.

[9] Richard Peter Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.

[10] Richard Peter Stanley and Sergey Fomin. *Enumerative Combinatorics: Volume 2*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.