Exercise 1 (Exercise 5)**.** A **binary tree** of length n constructed recursively as follows.

- ⋄ The empty set is a binary tree of length 0.
- ⋄ Otherwise a binary tree has a *root vertex* v, a *left subtree* T_1 and a *right subtree* T_2, each of which is also a binary tree having a root vertex.
  We draw the root vertex at the top with an edge going down to the root vertices of T_1, T_2. Then draw each tree recursively in the same manner.

Prove that the number of binary trees on n vertices is the n^{th} Catalan number C_n.⟦ Hint: Show that they satisfy the recursion for the Dyck paths ⟧

Answer

Let us call $f(n)$ the number of binary trees on n vertices. The initial condition is $f(0) = 1$ because the empty set is a binary tree.
To create a binary tree with $n+1$ vertices we choose the root and then we still have n vertices to go.
Fix ℓ to be the number of vertices we assign to the left tree then, the the remaining $n-\ell$ vertices go to the right tree. The number of ways to build right and left subtrees this way is $f(\ell)f(n-\ell)$.
However, running ℓ through all possible options of n gives us a plethora of disjoint events. We can sum those possibilities to get the total number of binary trees on $n+1$ vertices which is

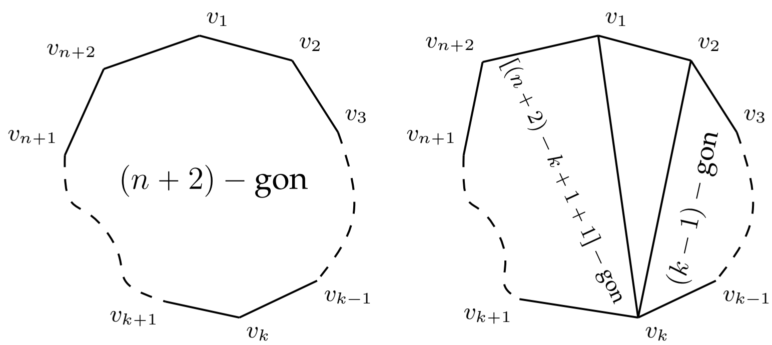
$$f(n+1) = \sum_{\ell=0}^{n} f(\ell)f(n-\ell).$$

It follows that $f(n) = C_n$.

Exercise 2 (Exercise 6)**.** A **triangulation** of a convex $(n+2)$-gon is a collection $(n-1)$ diagonals that do not intersect each other. Show that the number of triangulations of a convex $(n+2)$-gon is the n^{th} Catalan numbers C_n. ⟦ Hint: Show that they satisfy the recursion for the Dyck paths ⟧

Answer

In the same way we chose a *left* and *right* trees, here we will chose L-and-R triangulations.
The initial conditions don't match up until $T(2) = 2 = C_2$. Nonetheless we will form the recurrence for $n \geqslant 2$. Suppose we want to find $T(n)$, for that effect let us construct an $(n+2)$-gon labeling the vertices 1 through $n+2$:



We take the first two vertices of our polygon, note that this choice is independent of the actual number of triangulations, and from them we pick vertex k to draw a triangle.
Given an *nice orientation* we can see that we have a *left*-gon and a *right*-gon. The right one contains vertices from 2 through k which amount to $k-2+1=k-1$ vertices. While the left one runs from k to $n+2$ and 1. The number of vertices on the left is $(n+2)-k+1+1$. So the number of triangulations given that vertex k we chose is

$$T[(k-1)-2]T[(n-k+4)-2].$$

Summing through all the possible choices of k, we run from 3 through $n+2$. This means that

$$T(n)=\sum_{k=3}^{n+2}T(k-3)T(n-k+2)\xrightarrow[\substack{k\to 3\\ \Rightarrow \ell\to 0\\ k\to n+2\\ \Rightarrow \ell\to n-1}]{\ell=k-3}\sum_{\ell=0}^{n-1}T(\ell)T(n-\ell-1)=\sum_{\ell=0}^{n-1}T(\ell)T[(n-1)-\ell].$$

This is precisely the recurrence which defines the Catalan numbers and so $T(n)=C_n$ for $n\geqslant 2$.

Exercise 3 (Exercise 8)**.** A **derangement** of $[n]$ is a permutation $\pi\in S_n$ with no fixed points. That is $\forall i(\pi(i)\neq i)$. Let D_n be the number of derangements of $[n]$. Prove that

$$\sum_{n=0}^{\infty}\frac{D_n}{n!}x^n=\frac{e^{-x}}{1-x}.$$

Answer

Let us begin by establishing a recurrence relation for D_n. We will do this by considering `grad students` and their `preffered place to sit at`. Then the number D_n is the number of ways not `grad student` sits at their `preffered desk`.
Suppose that the first `grad student` enters the room and sits on desk i. When the i^{th} `grad student` enters the room there are two possibilities:

- They sit on desk 1, and then the problem reduces to the case with $n-2$ `grad students`.
- Otherwise we may relabel `grad student` i as the first `grad student` and then say that desk 1 is i's preferred desk. This reduces to the case of $n-1$ `grad students`.

Since this events are disjoint, the possibilities for each must be summed. But our choice for the first one's preference was arbitrary, there are other $n-1$ possible choices. It follows that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Let us shift indices to obtain the recurrence $D_{n+2} = (n+1)(D_{n+1} + D_n)$. By taking the exponential generating function on both sides we get

$$\sum_{n=0}^{\infty} D_{n+2}\frac{x^n}{n!} = \sum_{n=0}^{\infty}(n+1)D_{n+1}\frac{x^n}{n!} + \sum_{n=0}^{\infty}(n+1)D_n\frac{x^n}{n!}$$

and if we call $\mathcal{D}(x)$, D_n's e.g.f. then this equation translates to the differential equation

$$\begin{aligned} D^2\mathcal{D}(x) &= (xD+1)D\mathcal{D}(x) + (xD+1)\mathcal{D}(x), \\ &= xD^2\mathcal{D}(x) + D\mathcal{D}(x) + xD\mathcal{D}(x) + \mathcal{D}(x). \end{aligned}$$

Let's switch notation to make this equation a bit more refreshing to the eyes, say $y = y(x) = \mathcal{D}(x)$ and we will change the differential operator by primes:

$$y'' = xy'' + y' + xy' + y.$$

Let us gather two initial conditions for this equation. By evaluating $\mathcal{D}$ and $\mathcal{D}'$ at $x = 0$ we recover the following

$$\mathcal{D}(0) = D_0 = 1,\ \mathcal{D}'(0) = D_1 = 0$$

where we take $D_0 = 1$ by convention and D_1 tells us that there are no permutations on 1 which do not fix 1.
Now we can solve the differential equation as follows:

$$\begin{aligned}
(1-x)y'' - y' = xy' + y &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left((1-x)y'\right) = \frac{\mathrm{d}}{\mathrm{d}x}(xy), \\
&\Rightarrow (1-x)y' = xy + c_1, \\
(x \to 0) &\Rightarrow (1-0)(0) = (0)(1) + c_1 \Rightarrow c_1 = 0, \\
&\Rightarrow (1-x)y' = xy, \\
&\Rightarrow \frac{y'}{y} = \frac{x}{1-x} = -\left[\frac{-x+1-1}{1-x}\right] = -\left(1 - \frac{1}{1-x}\right), \\
&\Rightarrow \log(y) = -x - \log(1-x) + c_2, \\
(x \to 0) &\Rightarrow \log(1) = 0 - 0 + c_2 \Rightarrow c_2 = 0, \\
&\Rightarrow y = e^{-x}\frac{1}{1-x}.
\end{aligned}$$

We can conclude that $\mathcal{D}(x) = \frac{e^{-x}}{1-x}$ as we wanted.