Homework

Due: Friday, April 21

1. [SS]5.10(a).

2. [SS]5.11.

3. Here is another approach to the problem of meromorphic continuation of the Riemann zeta function. Assume $\mathrm{Re}(s) = \sigma > 0$.

   For natural numbers n and N, define functions

$$\begin{aligned} \delta_n(s) &= \frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s} \\ &= \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \\ F_N(s) &= \sum_{1 \le n \le N} \delta_n(s). \end{aligned}$$

   (a) Show that $|\delta_n(s)| \le \frac{|s|}{n^{\mathrm{Re}(s)+1}}$. (HINT: *Represent the integrand in the definition of δ_n using the observation that $\int_n^x \frac{du}{u^{s+1}} = \frac{1}{-s}(x^{-s} - n^{-s})$.*)

   (b) Show that $\{F_N(s)\}$ converges uniformly on any half-plane of the form $\mathrm{Re}(s) \ge \alpha > 0$.

   (c) Show that $\zeta(s) - \frac{1}{s-1}$ is bounded and holomorphic near $s = 1$. (HINT: *Use the fact that $\frac{1}{s-1} = \int_1^\infty \frac{1}{x^s} dx$.*)