Exercise 1. Suppose $\mathcal{F}$ is a presheaf and $\mathcal{G}$ is a sheaf, both of sets, on X. Let $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) \colon= \mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Show that this is a sheaf of sets on X.

Answer

We first need to show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a presheaf, this requires a sensible notion of restriction mapping which satisfies the following:

i) $\mathrm{res}_{U,U} = \mathrm{id}_{(*)}$ where the identity map is over the object $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$.
ii) If $U \subseteq V \subseteq W$ then $\mathrm{res}_{W,U} = \mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}$.

Let us consider two objects $\mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$ and $\mathrm{Mor}(\mathcal{F}|_V, \mathcal{G}|_V)$ with $U \subseteq V$. A restriction mapping acts on sections, and sections on these sets are morphisms of sheaves. Our restriction mapping takes $\varphi \in \mathrm{Mor}(\mathcal{F}|_V, \mathcal{G}|_V)$ to $\mathrm{res}_{V,U}(\varphi) \in \mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$, but recall vf is a collection of maps of objects of the form

$$\varphi(W) : \mathcal{F}(W) \to \mathcal{G}(W), \quad \text{with} \quad W \subseteq V.$$

In this sense, it suffices to only consider the open sets contained in U. We declare that $\mathrm{res}_{V,U}(\varphi)$ is the collection of maps

$$\varphi(W) : \mathcal{F}(W) \to \mathcal{G}(W), \quad \text{with} \quad \underline{W \subseteq U}.$$

i) The map $\mathrm{res}_{U,U}(\varphi)$ acts as follows, every map of the form $\varphi(W)$ with $W \subseteq U$ is sent to the map $\varphi(W)$ between the same objects because $W \subseteq U$ is still itself.

   This means that $\mathrm{res}_{U,U}$ is the identity map in $\mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$.
ii) Now suppose $U \subseteq V \subseteq W$ are open sets, then $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}$ acts on φ