In this homework I was able to collaborate with **Kylie, Nate** and **Parker**. It was quite the endeavor, but I really like keeping track of my indices and bracketing elementary matrices.

Exercise 1 (Exercise 3)**.** Determine the dimension of the adjoint representation of $\mathfrak{so}_{2n+1}$.

Answer

The dimension of the adjoint representation can be seen to be the same as the dimension of the original space by considering the isomorphism $X \mapsto [X, -]$. Then the dimension of the adjoint representation is the same as $\mathfrak{so}_{2n+1}$'s so it's

$$\binom{2n+1}{2} = (2n+1)n.$$

Exercise 2 (Exercise 4)**.** Write out a basis for the adjoint representation of $\mathfrak{so}_5$ and show how it corresponds to the root system in type B.

Answer

We may take advantage of the relationship $X^\intercal S + SX = 0$ where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0_2 & I_2 \\ 0 & I_2 & 0_2 \end{pmatrix}$$

to determine the basic elements. Assume $X = (x_{ij}) \in \mathfrak{so}_5$, then SX permutes rows 2 with 4 and 3 with 5. Observe that transposing SX returns $X^\intercal S$. So the relation $X^\intercal S + SX = 0$ allows us to rewrite the matrix X as

$$\begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{14} & x_{22} & x_{23} & 0 & x_{25} \\ -x_{15} & x_{32} & x_{33} & -x_{25} & 0 \\ -x_{12} & 0 & x_{43} & -x_{22} & -x_{32} \\ -x_{13} & -x_{43} & 0 & -x_{23} & -x_{33} \end{pmatrix}$$

Now, let us use the notation E_{ij} to denote the matrix whose $(i,j)^{\text{th}}$ entry is 1 and 0 otherwise. The basis for $\mathfrak{so}_5$ may be written as:

- ◇ $X_1 = E_{12} - E_{41}$
- ◇ $X_2 = E_{13} - E_{51}$
- ◇ $X_3 = E_{14} - E_{21}$
- ◇ $X_4 = E_{15} - E_{31}$
- ◇ $X_5 = E_{22} - E_{44}$
- ◇ $X_6 = E_{23} - E_{54}$
- ◇ $X_7 = E_{25} - E_{34}$
- ◇ $X_8 = E_{32} - E_{45}$
- ◇ $X_9 = E_{33} - E_{55}$
- ◇ $X_{10} = E_{43} - E_{52}$

The Cartan subalgebra is generated by the matrices X_5 and X_9 and in order to identify our elements with type B root system we must take the Lie bracket of them with $H = aX_5 + bX_9$. We will apply the formula

$$[E_{ij}, E_{k\ell}] = \delta_{jk} E_{i\ell} - \delta_{i\ell} E_{kj}$$

and with this we obtain

$$\begin{aligned}
[H, X_1] &= [aX_5 + bX_9, X_1] \\
&= a\,[X_5, X_1] + b\,[X_9, X_1] \\
&= a\,[E_{22} - E_{44}, E_{12} - E_{41}] + b\,[E_{33} - E_{55}, E_{12} - E_{41}] \\
&= a(-E_{12} - 0 - 0 + E_{41}) + 0 \\
&= -a(E_{12} - E_{41}) = -aX_1
\end{aligned}$$

from which we deduce that X_1 matches with $-L_1$. As another example we compute

$$\begin{aligned}
[H, X_6] &= [aX_5 + bX_9, X_6] \\
&= a\,[X_5, X_6] + b\,[X_9, X_6] \\
&= a\,[E_{22} - E_{44}, E_{23} - E_{54}] + b\,[E_{33} - E_{55}, E_{23} - E_{54}] \\
&= a(E_{23} - 0 - 0 + (-E_{54})) + b(-E_{23} - 0 - 0 + E_{54}) \\
&= a(E_{23} - E_{54}) - b(E_{23} - E_{54}) = (a - b)X_6
\end{aligned}$$

and this tells us that X_6 corresponds to $L_1 - L_2$. Similarly

- ◇ $[H, X_2] = -bX_2 \Rightarrow X_2 \leftrightarrow -L_2$
- ◇ $[H, X_3] = aX_3 \Rightarrow X_3 \leftrightarrow L_1$

- $\diamond$ $[H, X_4] = bX_4 \Rightarrow X_4 \leftrightarrow L_2$
- $\diamond$ $[H, X_7] = (a+b)X_7 \Rightarrow X_7 \leftrightarrow L_1 + L_2$
- $\diamond$ $[H, X_8] = (-a+b)X_8 \Rightarrow X_8 \leftrightarrow -L_1 + L_2$
- $\diamond$ $[H, X_8] = (-a-b)X_{10} \Rightarrow X_{10} \leftrightarrow -L_1 - L_2$

With this, we have our desired correspondence.

Exercise 3 (Exercise 6)**.** Show that the Killing form on $\mathfrak{sl}_n$ satisfies $\langle X|Y\rangle = 2n\,\mathrm{tr}(XY)$ in general for any elements of $\mathfrak{sl}_n$.

Answer

For this problem we will find the Killing form on $\mathfrak{gl}_n$ and then specialize to the case of $\mathfrak{sl}_n$. This will be done to only deal with one type of basic element E_{ij} instead of E_{ij} and H_k.
Take $Y = \sum_{k,\ell=1}^n y_{k\ell}E_{k\ell}$ and $X = \sum_{r,s=1}^n x_{rs}E_{rs}$, then it suffices to view the action of $[X, [Y, -]]$ on a basic element E_{ij}. So let us fix (i, j) and observe that

$$\begin{aligned}
[Y, E_{ij}] &= \left[\sum_{k,\ell=1}^n y_{k\ell}E_{k\ell}, E_{ij}\right] \\
&= \sum_{k,\ell=1}^n y_{k\ell}\left[E_{k\ell}, E_{ij}\right] \\
&= \sum_{k,\ell=1}^n y_{k\ell}(\delta_{i\ell}E_{kj} - \delta_{kj}E_{i\ell}) \\
&= \sum_{k,\ell=1}^n y_{k\ell}\delta_{i\ell}E_{kj} - \sum_{k,\ell=1}^n y_{k\ell}\delta_{kj}E_{i\ell} \\
&= \sum_{k=1}^n y_{ki}E_{kj} - \sum_{\ell=1}^n y_{j\ell}E_{i\ell}
\end{aligned}$$

Reindexing the second sum we get

$$[Y, E_{ij}] = \sum_{k=1}^n y_{ki}E_{kj} - y_{jk}E_{ik}$$

In a similar fashion we apply $\operatorname{Ad}(X)$ to this matrix in order to obtain

$$[X,[Y,E_{ij}]] = \sum_{k=1}^{n} y_{ki}\,[X,E_{kj}] - y_{jk}\,[X,E_{ik}]$$

and to not lose track of indices we will do the brackets separately

$$\begin{cases} [X,E_{kj}] = \displaystyle\sum_{r,s=1}^{n} x_{rs}\,[E_{rs},E_{kj}] = \sum_{r=1}^{n} x_{rk}E_{rj} - \sum_{s=1}^{n} x_{js}E_{ks} = \sum_{r=1}^{n} x_{rk}E_{rj} - x_{jr}E_{kr}, \\ [X,E_{ik}] = \displaystyle\sum_{r,s=1}^{n} x_{rs}\,[E_{rs},E_{ik}] = \sum_{r=1}^{n} x_{ri}E_{rk} - \sum_{s=1}^{n} x_{ks}E_{is} = \sum_{r=1}^{n} x_{ri}E_{rk} - x_{kr}E_{ir}. \end{cases}$$

Putting everything back together we get

$$\begin{aligned} &[X,[Y,E_{ij}]] \\ &= \sum_{k=1}^{n} y_{ki}\left(\sum_{r=1}^{n} x_{rk}E_{rj} - x_{jr}E_{kr}\right) - y_{jk}\left(\sum_{r=1}^{n} x_{ri}E_{rk} - x_{kr}E_{ir}\right) \\ &= \sum_{k=1}^{n}\sum_{r=1}^{n} y_{ki}x_{rk}E_{rj} - \sum_{k=1}^{n}\sum_{r=1}^{n} y_{ki}x_{jr}E_{kr} - \sum_{k=1}^{n}\sum_{r=1}^{n} y_{jk}x_{ri}E_{rk} + \sum_{k=1}^{n}\sum_{r=1}^{n} y_{jk}x_{kr}E_{ir} \end{aligned}$$

Now looking for the coefficient of E_{ij} in order to later ask for the trace we get

$$\begin{cases} \displaystyle\sum_{k=1}^{n} y_{ki}x_{ik} & \text{by looking at the } r=i \text{ term of the first sum.} \\ -\,y_{ii}x_{jj} & \text{from the second sum} \quad , \\ -\,y_{jj}x_{ii} & \text{from the third, and} \\ \displaystyle\sum_{k=1}^{n} y_{jk}x_{kj} & \text{on the } j^{\text{th}} \text{ term of the last sum.} \end{cases}$$

Now, the trace of $\operatorname{Ad}(X)\operatorname{Ad}(Y)$ would be obtained by summing over the diagonal entries of the matrix. But observe that we haven't *flattened* any matrices at any point (in the sense that we haven't converted a matrix $A = (A_i j)$ into a vector $a_{(i-1)n+j} = A_{ij}$). So, even if

$$\operatorname{Ad}(X)\operatorname{Ad}(Y) : \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$$

and we could calculate the trace by summing the diagonal entries of the expression, this will instead be done by summing across all (i,j). We could also

interpret $\mathrm{Ad}(X)\,\mathrm{Ad}(Y)$ as a rank 3 tensor and then take the trace by summing across 2 dimensions.
From the previous discussion, we have that

$$\begin{aligned}
\mathrm{tr}(\mathrm{Ad}(X)\,\mathrm{Ad}(Y)) &= \sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} y_{ki}x_{ik} - y_{ii}x_{jj} - y_{jj}x_{ii} + \sum_{k=1}^{n} y_{jk}x_{kj}\right)\\
&= \sum_{i,j,k=1}^{n} y_{ki}x_{ik} - \sum_{i,j=1}^{n} y_{ii}x_{jj} - \sum_{i,j=1}^{n} y_{jj}x_{ii} + \sum_{i,j,k=1}^{n} y_{jk}x_{kj}\\
&= n\sum_{i,k=1}^{n} y_{ki}x_{ik} - \sum_{i=1}^{n} y_{ii}\sum_{j=1}^{n} x_{jj} - \sum_{j=1}^{n} y_{jj}\sum_{i=1}^{n} x_{ii} + n\sum_{j,k=1}^{n} y_{jk}x_{kj}\\
&= n\,\mathrm{tr}(YX) - \mathrm{tr}(Y)\,\mathrm{tr}(X) - \mathrm{tr}(Y)\,\mathrm{tr}(X) + n\,\mathrm{tr}(YX)\\
&= 2n\,\mathrm{tr}(YX) - 2\,\mathrm{tr}(Y)\,\mathrm{tr}(X)
\end{aligned}$$

which gives us the identity

$$\mathrm{tr}(\mathrm{Ad}(X)\,\mathrm{Ad}(Y)) = 2n\,\mathrm{tr}(YX) - 2\,\mathrm{tr}(Y)\,\mathrm{tr}(X)$$

and now, specializing to the case of $\mathfrak{sl}_n$, we have that X, Y have zero trace so that the identity becomes

$$\langle X|Y\rangle = \mathrm{tr}(\mathrm{Ad}(X)\,\mathrm{Ad}(Y)) = 2n\,\mathrm{tr}(XY).$$

Exercise 4 (Exercise 5)**.** Generalize the computation we did in class to show that the Killing form for $\mathfrak{sl}_n$, when restricted to the Cartan subalgebra $\mathfrak{h}$, satisfies

Answer

I sadly didn't see the computation in class and thought that doing it the hard way around would be the best approach. The previous identity holds in all of $\mathfrak{sl}_n$ so in particular it holds for $\mathfrak{h}^{\mathfrak{sl}} \leqslant \mathfrak{sl}_n$.

Exercise 5 (Exercise 9)**.** What is the size of the Weyl group of type G_2? Write out its elements as reduced words in the two simple reflections s_1, s_2 corresponding to the two simple roots of G_2.

Answer

This Weyl group can be presented as

$$\operatorname{gen}(s_1, s_2)\Big/\left\langle s_i^2, (s_1 s_2)^6 \right\rangle.$$

We may thus enumerate the elements of the group as

- $e = (s_1 s_2)^6 = (s_2 s_1)^6$
- s_1
- s_2
- $s_1 s_2$
- $s_2 s_1$
- $s_1 s_2 s_1$
- $s_2 s_1 s_2$
- $s_1 s_2 s_1 s_2$
- $s_2 s_1 s_2 s_1$
- $s_1 s_2 s_1 s_2 s_1$
- $s_2 s_1 s_2 s_1 s_2$
- $s_1 s_2 s_1 s_2 s_1 s_2 = (s_2 s_1)^3$

Observe that the remaining elements are determined by the braid relation.