Regarding my project, I plan to work on understanding the basics of the Moduli Space of n marked points. For this, I wish to do the following

◇ First, show that isomorphisms of the projective line are determined by $0, 1$ and ∞. Then conclude that $\overline{\mathcal{M}}_{0,3}$ is only one point.

◇ Draw examples of $\overline{\mathcal{M}}_{0,n}$ for $n \geqslant 4$ and see exactly how moving points does correspond to the continuous transformations.

◇ With these examples, interpret what they parametrize as in the case of

$$\overline{\mathcal{M}}_{0,5} = \{\, \text{degree } 3 \text{ curves passing through } 5 \text{ points in } \mathbb{P}^3 \,\}.$$

The work will based on **your** papers, the tournaments one and the one with **Renzo**. I will also use *the green book* (I still don't know what that book is, but **Jae** and **Mark** recommended it to me).

Exercise 1 (Exercise 2)**.** Compute the following:

i) $s_{(5,3,3,1)}s_{(4)}$.
ii) $s_{(2,1)}s_{(2,1)}$.
iii) The decomposition of $V_{(2,1)} \otimes_o V_{(2,1)}$ into irreducible S_6 representations.

Answer

Recall that the product of Schur functions can be calculated using Littlewood-Richardson coefficients as follows:

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda$$

where $c_{\mu\nu}^\lambda$ is the number of Littlewood-Richardson tableaux of shape λ/μ with content ν.

In order to fill out λ/μ with content ν it must hold that $|\lambda| - |\mu| = |\nu|$.

i) In this first case we have $\mu = (5, 3, 3, 1)$ and $\nu = (4)$ which means that λ should partition 16.

This means that we must append 4 new blocks to our partition $(5, 3, 3, 1)$ to obtain a partition of 16 which we will fill with only ones.

As the tableaux must be semi-standard, it can't happen that we append more than one block in the same column. In other words, we must append

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a horizontal strip of ones. The following tableaux correspond to partitions λ such that λ/μ is a horizontal strip:
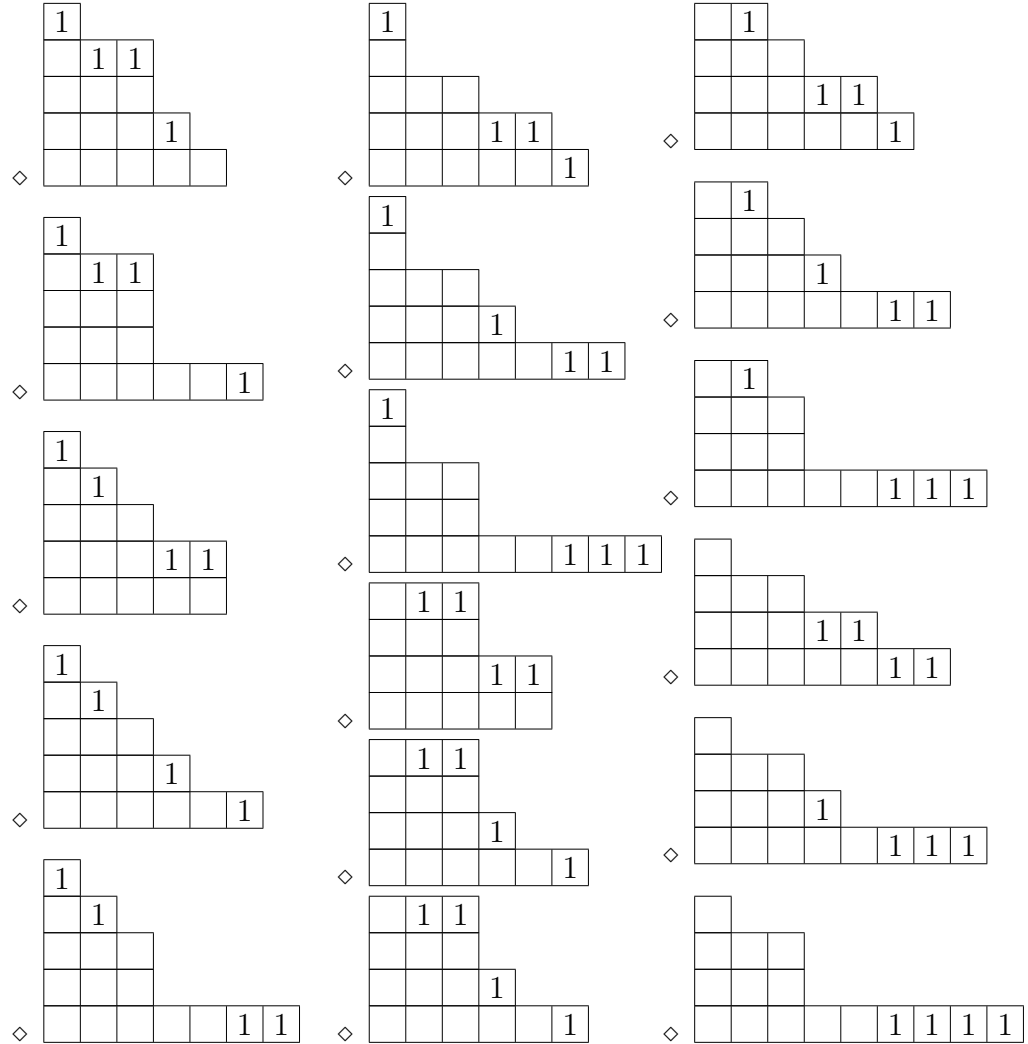
As there is only one possible tableau for each shape we have that each coefficient is one. Therefore we have

$$s_{(5,3,3,1)}s_{(4)} = s_{(5,4,3,3,1)} + s_{(6,3,3,3,1)} + s_{(5,5,3,2,1)} + s_{(6,4,3,2,1)} + s_{(7,3,3,2,1)}$$

$$+ s_{(6,5,3,1,1)} + s_{(7,4,3,1,1)} + s_{(8,3,3,1,1)} + s_{(5,5,3,3)} + s_{(6,4,3,3)} + s_{(7,3,3,3)}$$

$$+ s_{(6,5,3,2)} + s_{(7,4,3,2)} + s_{(8,3,3,2)} + s_{(7,5,3,1)} + s_{(8,4,3,1)} + s_{(9,3,3,1)}.$$

ii) For this next case we have $\mu = \nu = (2,1)$. This means that $\lambda \vdash 6$ and we must append two 1's and one 2. As our tableau must be Littlewood-

Richardson, the reading word must be one of either

$$211, \quad \text{or} \quad 121.$$

The following tableaux correspond to $\lambda \vdash 6$ such that λ/μ is a Littlewood-Richardson tableau with content $(2,1)$:

We see that $\mathrm{sh}(\lambda) = (3,2,1)$ is repeated once so we account for a coefficient of 2 with that tableau. The decomposition is thus:

$$s^2_{(2,1)} = s_{(2,2,1,1)} + s_{(3,1,1,1)} + 2s_{(3,2,1)} + s_{(4,1,1)} + s_{(4,2)} + s_{(2,2,2)} + s_{(3,3)}.$$

iii) By the correspondence of the outer tensor product with the product of Schur functions we have

$$V_{(2,1)} \otimes_o V_{(2,1)} = V_{(2,2,1,1)} \oplus V_{(3,1,1,1)} \oplus 2V_{(3,2,1)} \oplus V_{(4,1,1)} \oplus V_{(4,2)} \oplus V_{(2,2,2)} \oplus V_{(3,3)}.$$

Exercise 2 (Exercise 3)**.** Use the Straightening Algorithm (Garnir Relations) to express the Garnir polynomial F_T where T is the filling

2	4
3	1

in terms of F_S's where each S is a standard Young tableau. Write out each polynomial in your formula and check that your answer works.

Answer

We first column straighten T by ordering its columns, we obtain

$$\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 2 & 1 \\ \hline \end{array}.$$

To row-straighten we find the topmost row with a descent and consider rightmost decrease. The block in question is

$$\begin{array}{|c|c|} \hline 3 \\ \hline 2 & 1 \\ \hline \end{array}$$

and for this block we will find permutations of S_3 which preserve the columns-increasing condition withing the blocks. We have the permutations

$$\text{id}, \quad (12), \quad (123)$$

which correspond to

$$\begin{array}{|c|c|} \hline 3 \\ \hline 2 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 \\ \hline 1 & 2 \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|c|} \hline 2 \\ \hline 1 & 3 \\ \hline \end{array}.$$

This means that

$$(-1)^0(x_3 - x_2)(x_4 - x_1) + (-1)^1(x_3 - x_1)(x_4 - x_2) + (-1)^2(x_2 - x_1)(x_4 - x_3) = 0$$

and from this relation we obtain

$$(x_3 - x_2)(x_4 - x_1) = (x_3 - x_1)(x_4 - x_2) - (x_2 - x_1)(x_4 - x_3).$$

Similarly just by calculating the Garnir polynomial of T we obtain $(x_2 - x_3)(x_4 - x_1)$ which corresponds to our formula, just with a sign change.

Exercise 3 (Exercise 5)**.** Decompose the inner tensor product $V(2,1) \otimes_i V(2,1)$ into irreducible representations of S_3.

Answer

Each of the factors in the product is generated by the pair of polynomials

$$(x_2 - x_1), (x_3 - x_1), \quad \text{and} \quad (y_2 - y_1), (y_3 - y_1)$$

and therefore the tensor product is generated by the pairwise products of these polynomials. Let us call

$$\begin{cases} v_1 = (x_2 - x_1)(y_2 - y_1) = x_2y_2 - x_2y_1 - x_1y_2 + x_1y_1 \\ v_2 = (x_2 - x_1)(y_3 - y_1) = x_2y_3 - x_2y_1 - x_1y_3 + x_1y_1 \\ v_3 = (x_3 - x_1)(y_2 - y_1) = x_3y_2 - x_3y_1 - x_1y_2 + x_1y_1 \\ v_4 = (x_3 - x_1)(y_3 - y_1) = x_3y_3 - x_3y_1 - x_1y_3 + x_1y_1 \end{cases}$$

We now consider the action of S_3 on each of our basic elements. It suffices to consider the action per representative of cycle type, so we only consider $\mathrm{id}, (12)$ and (123).

Applying the identity to each of our elements returns it as it was. So we have

$$M_{\mathrm{id}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, we apply the transposition (12) to our elements to obtain

◇ $(12)v_1 = (x_1 - x_2)(y_1 - y_2) = v_1$

◇ $(12)v_2 = (x_1 - x_2)(y_3 - y_2) = x_1y_3 - x_1y_2 - x_2y_3 + x_2y_2$

◇ $(12)v_3 = (x_3 - x_2)(y_1 - y_2) = x_3y_1 - x_3y_2 - x_2y_1 + x_2y_2$

◇ $(12)v_4 = (x_3 - x_1)(y_3 - y_1) = x_3y_3 - x_3y_2 - x_2y_3 + x_2y_2$

On the other hand notice that

$$\begin{aligned} v_1 - v_2 &= x_2y_2 - x_2y_1 - x_1y_2 + x_1y_1 - (x_2y_3 - x_2y_1 - x_1x_3 + x_1y_1) \\ &= x_2y_2 - x_2y_3 - x_1y_2 + x_1y_3 \\ v_1 - v_3 &= x_2y_2 - x_2y_1 - x_1y_2 + x_1y_1 - (x_3y_2 - x_3y_1 - x_1y_2 + x_1y_1) \\ &= x_2y_2 - x_2y_1 - x_3y_2 + x_3y_1 \\ v_1 - v_2 - v_3 + v_4 &= x_2y_2 - x_2y_3 - x_1y_2 + x_1y_3 - (x_3y_2 - x_3y_1 - x_1y_2 + x_1y_1) \\ &\quad + x_3y_3 - x_3y_1 - x_1y_3 + x_1y_1 \\ &= x_2y_2 - x_2y_3 - x_3y_2 + x_3y_3 \end{aligned}$$

We can identity the image of the transposition with this elements to build the matrix

$$
M_{(12)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}
$$

And in a similar fashion we may calculate the image of our basic elements through the 3-cycle:

    $\diamond$ $(123)v_1 = (x_3 - x_2)(y_3 - y_2) = v_1 - v_2 - v_3 + v_4$

    $\diamond$ $(123)v_2 = (x_3 - x_2)(y_1 - y_2) = v_1 - v_3$

    $\diamond$ $(123)v_3 = (x_1 - x_2)(y_3 - y_2) = v_1 - v_2$

    $\diamond$ $(123)v_4 = (x_1 - x_2)(y_1 - y_2) = v_1$

With this information we may build

$$
M_{(123)} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

The traces of these matrices allow us to build character table for $V(2,1) \otimes_i V(2,1)$:

$$
\chi_{\text{inner}} = \begin{pmatrix} 4 & 0 & 1 \end{pmatrix}
$$

To find the decomposition we solve a system of linear equations by row reducing the following matrix:

$$
\begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}
$$

which means that

$$
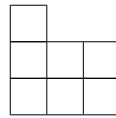V(2,1) \otimes_i V(2,1) = V_{(3)} \oplus V_{(1,1,1)} \oplus V_{(2,1)}.
$$

Exercise 4 (Exercise 6)**.** Use the Murnaghan-Nakayama rule to compute $\left\langle s_{(3,3,1)} \middle| p_{(3,3,1)} \right\rangle$.
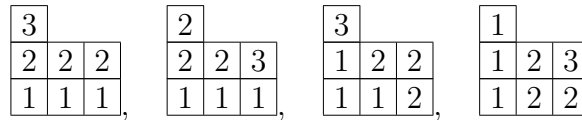
Answer

Recall that the Murnaghan-Nakayama rule has the following statement:

$$\chi_\mu(\lambda) = \langle s_\mu | p_\lambda \rangle = \sum_{(*)} \prod_{i=1}^{\ell(\lambda)} (-1)^{\text{ht}(\text{strip}(i))}$$

where the sum runs through border strip tableaux T with shape μ and content λ. In this case we have $\mu = \lambda = (3,3,1)$ which means we must fill the diagram

with the word 1112223 only forming border strips. The only possible fillings are

3		
2	2	2
1	1	1

2		
2	2	3
1	1	1

3		
1	2	2
1	1	2

1		
1	2	3
1	2	2

Also, recall that the height of the strip is one less than the *actual height*. With this we have the corresponding height vectors:

$$(0,0,0), \quad (0,1,0), \quad (1,1,0), \quad (2,1,0)$$

and we may calculate $(-1)^{\mathbf{v}}$ for each of these to obtain

$$\langle s_{(3,3,1)} | p_{(3,3,1)} \rangle = 1 - 1 + 1 - 1 = 0.$$

Then $s_{(3,3,1)}$ and $p_{(3,3,1)}$ form a dual pair. *Is this true in general for any λ?*