MATH619 — Complex Geometry

Based on the lectures by Renzo Cavalieri

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This is a topics course on this stuff

Requirements

Knowledge on stuff

TO DO:

- Write 10.1
- Write 11.3
- Write 12.4

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Chapter 1

Renzo's Complex Projective Exercises

1.1 Set of points of the projective line

Exercise 1.1.1. Show that there is a bijection between the set $\mathrm{Set}\mathbb{CP}^1$ and a quotient set of a disjoint union of two copies of $\mathbb{C}$.

Answer

Indeed consider our copies of $\mathbb{C}$ embedded into $\mathbb{C}^2$ as the lines $\{x = 1\}$ and $\{y = 1\}$. Then for a line $\ell \in \mathbb{CP}^1$ our map is

$$\ell \mapsto \ell \cap (\text{corresponding line}) \mapsto (\text{corresponding coordinate}).$$

Explicitly, if our line is $[X : Y]$, then the map is $[X : Y] \mapsto X/Y$ on one chart while Y/X on the other.
Observe that this map is surjective as every point in each copy of $\mathbb{C}$ is hit by a line of a different slope. The only points which are not hit twice are the origins of both lines. From this, we define the quotient by identifying the coordinates as $x \sim y$ whenever $y = 1/x$. Thus our map becomes a bijection at the level of the quotient as we can now properly trace back each point to a particular line.

1.2 The projective line as a topological space

Exercise 1.2.1 (Hopf Fibration)**.** Show there is a fibration of topological spaces:

$$S^1 \to S^3 \to S^2$$

meaning that there is a surjective continuous function from the three-dimensional sphere to the two-dimensional sphere, and the inverse image of any point is homeomorphic to a circle. This is called the Hopf fibration; notice that while the construction of these maps is rather mysterious in terms of spheres, it becomes transparent when thinking of the two-dimensional sphere as the complex projective line.

Answer

We have shown that the map

$$\pi_2 : S^3_{\mathbb{R}} \to \mathbb{CP}^1, (\text{pt. in } S^3) \mapsto (\text{corresponding line through origin in } \mathbb{C}^2)$$

is surjective. Also, we have that

$$\mathbb{CP}^1 \simeq \mathbb{C} \cup \{ \infty \} \simeq S^2$$

where the first homeomorphism comes from previous discussion and the second one from stereographic projection. This means that we have a map $S^3 \to S^2$ which is our candidate for the Hopf map. It remains to be seen that this map is continuous and that the fibers are homeomorphic to S^1.
It suffices to show π_2 is continuous as the rest of the maps are homeomorphisms. To that effect, take an open set $U \subseteq \mathbb{CP}^1$. This means that in the quotient topology induced by the π_2 map, U is open whenever $\pi_2^{-1}(U)$ is open. But this proves immediately that π_2 is continuous as it takes open sets back to open sets.
Now let $\ell \in \mathbb{CP}^1$, we'll analyze what the fibers are:

$$\pi_2^{-1}(\ell) = \{ \lambda z_0 : \ z_0 \in \ell \cap S^3, \ \lambda \in \mathbb{C} \}$$

but when restricting to S^3, we get the condition that λ can only vary an S^1's worth of values:

$$\pi_2^{-1}(\ell) = \{ \lambda z_0 : \ z_0 \in \ell \cap S^3, \ |\lambda| = 1 \} \simeq S^1.$$

This means that fibers of our map are homeomorphic to S^1 and thus we have the desired fibration structure.

1.3 The projective line as a complex manifold

Exercise 1.3.1. Compute

$$\phi_{21} := \varphi_2 \circ \varphi_1^{-1} \mid_{\varphi_1(U_1 \cap U_2)} : (\varphi_1(U_1 \cap U_2), x) \to (\mathbb{C}, y)$$

and show that it is a holomorphic function on its domain of definition. Show that its inverse is also holomorphic on its domain of definition.

These exercises show that $\mathbb{CP}^1$ has the structure of a complex analytic manifold. The pairs (U_i, φ_i) are called complex charts, the biholomorphic map ϕ_{21} a transition function, and the coordinates x and y are called local (or affine) coordinates.

Answer

Recall that the open set $U_1 \cap U_2$ is the collection of lines in $\mathbb{CP}^1$ which are not x or y axes. The image then is all the non-zero x coordinates of the intersection of those lines with $x = 1$. Taking those lines through $\varphi_2 \circ \varphi_1^{-1}$ gives us the y-coordinates of the intersections of those lines with the $y = 1$ line. We get all except the y-axis. Computing this for a particular line, if x_0 is the intersection with $x = 1$, then $\frac{1}{x_0}$ will be the intersection with $y = 1$. Therefore, the map $\varphi_2 \circ \varphi_1^{-1}$ is $x_0 \mapsto \frac{1}{x_0}$ of non-zero x_0. This function and its inverse are holomorphic as the vertical and horizontal lines are excluded from this.

1.4 Functions on the projective line

Exercise 1.4.1. Show that meromorphic functions $f : \mathbb{CP}^1 \to \mathbb{C}$ may be described in two equivalent ways:

(a) As the ratio of two homogeneous polynomials of the same degree in the homogeneous coordinates:
$$f(X : Y) = \frac{P_d(X, Y)}{Q_d(X, Y)}$$

(b) As a rational function in one of the affine coordinates (with no restrictions on the degrees of the polynomials)
$$f(x) = \frac{p(x)}{q(x)}$$

How do you go from one perspective to the other?

Answer

We begin with the second item by claiming that if f is meromorphic has a zero of degree m at $z = z_0$ then we may write $f = (z - z_0)^m g$ where g is meromorphic but has no zeroes at z_0. Similarly for poles. This means that we may write f as a product of possibly repeated linear factors over another product of linear factors. These products are the desired polynomials.
We may homogenize to obtain the first characterization.
I don't recall how to construct the function in the homogeneous way :(

1.5 Automorphisms of the projective line

Exercise 1.5.1. Prove that, given any two ordered triple of points P_1, P_2, P_3 and Q_1, Q_2, Q_3 of the projective line, there exists a unique automorphism Φ of the projective line such that $\Phi(P_i) = Q_i$. Show that it follows that the only automorphism that fixes three points is the identity. Describe the subgroups of $\mathrm{Aut}(\mathbb{CP}^1)$ consisting of automorphisms that fix one, or two points in the projective line.

Answer

In affine coordinates, we can map any triple to $0, 1, \infty$ by considering the function

$$z \mapsto \frac{z - p_1}{z - p_3}\left(\frac{p_2 - p_3}{p_2 - p_1}\right).$$

This maps p_1, p_2, p_3 to $0, 1, \infty$ respectively. This is a Möbius transformation, so it is an automorphism of $\mathbb{CP}^1$. Call it φ_P and then create φ_Q, the desired function Φ is $\varphi_Q^{-1}\varphi_P$.

From this we immediately see that is P is fixed then the function is

$$\Phi = \varphi_P^{-1}\varphi_P = \mathrm{id}\,.$$

If Φ fixed two points we get rotations about the axis passing through those two points. Be it, for example, $0, \infty$ with scalings $z \mapsto \alpha z$ or $1, -1$ with $z \mapsto 1/z$.

If only one point is fixed, then it is a translation of the line leaving that point fixed. Say for example maps of the form $z \mapsto \frac{az+b}{d}$ leave infinity fixed.

1.6 Maps to projective spaces

Exercise 1.6.1. We define the degree of $F(\mathbb{CP}^1)$ to be the number of intersections with a general hyperplane in $\mathbb{CP}^r$. Prove that if the degree of the polynomials $P_i(X, Y)$ is equal to d, then the degree of $F(\mathbb{CP}^1)$ is less than or equal to d. When does the strict inequality hold?

Answer

1.7 Line bundles on the projective line

Exercise 1.7.1. For $x_0 \neq 0$, let $i_1 : \{ x = x_0 \} \rightarrowtail \mathcal{O}_{\mathbb{CP}^1}(d)$ and $i_2 : \{ y = \frac{1}{x_0} \} \rightarrowtail \mathcal{O}_{\mathbb{CP}^1}(d)$ be the two inclusions of vertical lines. Show that $i_2^{-1} \circ i_1 : \mathbb{C} \to \mathbb{C}$ is a linear isomorphism. Observe that the collection of these linear isomorphisms defines a holomorphic function $c_{12} : \mathbb{CP}^1 \backslash \{ 0, \infty \} \to \mathbb{C} \backslash 0$.

Answer

Let's concretely analyze the $i_2^{-1} \circ i_1$ map. This comes out of the line $\{ x = x_0 \}$ and gives us

$$\operatorname{Im} i_1 = \{ (x_0, u) : u \in \mathbb{C} \} \subseteq (\mathbb{C}^2, (x, u)).$$

In order to see what i_2^{-1} does, we translate $\operatorname{Im} i_1$ into $(\mathbb{C}^2, (y, v))$ via

$$x \mapsto \frac{1}{y}, \qquad u \mapsto \frac{v}{y^d} \Rightarrow v = y^d u.$$

So $\operatorname{Im} i_1$ on the other chart is

$$\operatorname{Im} i_1 = \left\{ \left(\frac{1}{x_0}, \frac{u}{x_0^d} \right) : u \in \mathbb{C} \right\}$$

and i_2^{-1} returns us $\frac{u}{x_0^d}$. This means that the composition in question is that map $u \mapsto \frac{u}{x_0^d}$ which means that the map is multiplication by x_0^{-d}. For a fixed non-zero x_0, this is a linear isomorphism of $\mathbb{C}$.
The collection of such isomorphisms is obtained when we let x_0 vary and the function c_{12} given by $x_0 \in \mathbb{CP}^1 \backslash \{ 0, \infty \} \mapsto x_0^{-d} \in \mathbb{C} \backslash \{ 0 \}$ is indeed holomorphic.

1.8 Sections of line bundles

Exercise 1.8.1. Show that if s_0, s_1 are two sections of the same line bundle, then their ratio is a (meromorphic) function on $\mathbb{CP}^1$. Show that if $s_0, s_1, \ldots, s_r$ are $(r+1)$ sections of the same line bundle, then they define a map $\mathbb{CP}^1 \to \mathbb{CP}^r$.

Answer

Observe that if $z \in \mathbb{CP}^1$, then $s_0(z), s_1(z)$ lie on the fiber $\pi^{-1}(z)$ which is isomorphic to $\mathbb{C}$. So taking their ratio on this fiber does produce a complex number. However, we must verify that the ratio is well-defined. Assume we picked another element of the base, $w \in \mathbb{CP}^1$ and asked about the ratio of $s_0(w)$ with $s_1(w)$.

In this case, observe that there's a linear isomorphism between $\pi^{-1}(z)$ and $\pi^{-1}(w)$ which scales all vectors by the same length. This means that $s_i(w) = \alpha s_i(z)$ for some $\alpha \in \mathbb{C}$ and therefore their ratios are the same.
So s_0/s_1 does define a meromorphic function on $\mathbb{CP}^1$ thanks to the linear isomorphisms.
In the same fashion, if we instead have $r+1$ sections, via a same argument we can see that when changing fibers, the sections only change by a scaling which is the same on all entries. So this means that we may write $[s_0 : \cdots : s_r]$ as a function to $\mathbb{CP}^r$.

1.9 Divisors

Exercise 1.9.1. When you multiply two meromorphic functions, what happens to their divisors? If two meromorphic functions produce the same divisor, what can you say about them?

Answer

If we consider our functions as sections of $\mathcal{O}(d)$ then,

$$s_0(x) = \frac{\prod_{i=1}^m (x-\alpha_i)}{\prod_{j=1}^n (x-\beta_j)} \quad \text{and} \quad s_1(x) = \frac{\prod_{i=1}^r (x-\gamma_i)}{\prod_{j=1}^s (x-\delta_j)}$$

then their divisors are

$$\operatorname{div}(s_0) = \sum_{i=1}^m [\alpha_i] - \sum_{j=1}^n [\beta_j] + (n-m+d)[\infty], \ \operatorname{div}(s_1) = \sum_{i=1}^r [\gamma_i] - \sum_{j=1}^s [\delta_j] + (s-r+d)[\infty].$$

Where we see the degree at infinity by transitioning via $s(x) \mapsto y^d s(1/y)$. Their product is precisely

$$s_0(x)s_1(x) = \frac{\prod_{i=1}^{m+r} (x-\tilde{\alpha}_i)}{\prod_{j=1}^{n+s} (x-\tilde{\beta}_j)},$$

where $\tilde{\alpha}_i$ corresponds to α_i when i is between 1 and m and γ_i from $m+1$ onwards. The same happens for $\tilde{\beta}_j$. In this case the divisor is

$$\operatorname{div}(s_0) = \sum_{i=1}^{m+r} [\tilde{\alpha}_i] - \sum_{j=1}^{n+s} [\tilde{\beta}_j] + ((n+s)-(m+r)+d)[\infty]$$

and the $x = \infty$ still gets the correct coefficient after using the transition function. ¿But why don't we get $2[\infty]$? ¿Is it because this only works in $\mathcal{O}(0)$, the trivial bundle?
So all non-infinite coefficients do get added and the divisors are summed.
When two meromorphic functions have the same divisor, they should be scalar multiples of each other at most.

Exercise 1.9.2. Let s be a meromorphic section of a line bundle $\pi : L \to \mathbb{CP}^1$, we call the support of $\operatorname{div}(s)$ the set of points that appear with non-zero coefficients in $\operatorname{div}(s)$. Show that there is a natural bijection

$$T_s : \pi^{-1}\left[\mathbb{CP}^1 \setminus \operatorname{supp}(\operatorname{div}(s))\right] \to \mathbb{CP}^1 \setminus \operatorname{supp}(\operatorname{div}(s)) \times \mathbb{C}.$$

Meditate on the following fact: the function T_s is an isomorphism of complex manifolds, and in fact an isomorphism of line bundles on the punctured $\mathbb{CP}^1$ (we of course did not precisely define these notions, so try and make a guess of what these things should mean). It is called a *trivialization* of the line bundle $\pi : L \to \mathbb{CP}^1$ on the complement of the support of $\operatorname{div}(s)$.

Answer

The first intuitive way to define the function is take an isomorphism φ of the fiber $\pi^{-1}(x)$ with $\mathbb{C}$. This gives us the map

$$T_s(x) = (\pi(x), \varphi(x)),$$

but this map is very non-canonical and ALSO, doesn't depend on s. So let us take advantage of a couple of points that we know in the fiber: x and wherever s maps $\pi(x)$ to, $s(\pi(x))$. Outside the support of s, s is never zero, so it makes sense to define the quotient $\varphi(x)/\varphi(s(\pi(x)))$. The desired function is then

$$T_s(x) = \left(\pi(x), \frac{\varphi(x)}{\varphi(s(x))}\right)$$

which is independent of φ But, ¿why was this?, making it natural in that sense. This function is bijective as

$$\left(\pi(x), \frac{\varphi(x)}{\varphi(s(x))}\right) = \left(\pi(y), \frac{\varphi(y)}{\varphi(s(y))}\right)$$

implies x, y lie in the same fiber via the first component. This immediately gives us $s(x) = s(y)$ and from this

$$\frac{\varphi(x)}{\varphi(s(x))} = \frac{\varphi(y)}{\varphi(s(x))} \Rightarrow \varphi(x) = \varphi(y) \Rightarrow x = y.$$

1.10 Distinguished line bundles on the projective line

Exercise 1.10.1. Recall that the very first definition of the set of points of the projective line is that each point corresponds to a line in $\mathbb{C}^2$. We now want to construct a space $\mathrm{Taut}(\mathbb{CP}^1)$ whose points correspond to the choice of line in $\mathbb{C}^2$ together with a point on it:

$$\text{Set}\left(\mathrm{Taut}(\mathbb{CP}^1)\right) = \{(\ell, P) : \ell \text{ is a line through the origin in } \mathbb{C}^2,\ P \in \ell\}.$$

(a) Realize $\mathrm{Taut}(\mathbb{CP}^1)$ as a subspace of $\mathbb{CP}^1 \times \mathbb{C}^2$.

(b) Show that the first projection restricts to $\mathrm{Taut}(\mathbb{CP}^1)$ to make it into a line bundle on $\mathbb{CP}^1$.

(c) Describe the fiber over a point $[\ell] \in \mathbb{CP}^1$.

(d) Show that $\mathrm{Taut}(\mathbb{CP}^1) \simeq \mathcal{O}(-1)$.

(e) Show that $\mathrm{Taut}(\mathbb{CP}^1) \simeq \mathcal{B}\ell_{(0,0)}\,\mathbb{C}^2$.

Answer

(a) As mentioned, elements of Set $\mathrm{Taut}(\mathbb{CP}^1)$ are $A = (\ell, P)$. This immediately defines a map

$$\mathrm{Taut}(\mathbb{CP}^1) \hookrightarrow \mathbb{CP}^1 \times \mathbb{C}^2,\ A \mapsto (\ell, P).$$

I had this question for a bit, but, how can I show that this map is a map of projective varieties?

(b) The map

$$\mathrm{Taut}(\mathbb{CP}^1) \to \mathbb{CP}^1,\ (\ell, P) \mapsto \ell$$

describes a structure of a line bundle. To show this we must see that for an open neighborhood U of $\ell \in \mathbb{CP}^1$, we have that $\pi^{-1}(U) \simeq U \times \mathbb{C}^1$. We define the map as

$$U \times \mathbb{C}^1 \to \pi^{-1}(U), \ (\ell, z) \mapsto (\ell, zP), \ P \in \ell.$$

1.11 Serre Duality

We have observed that

$$\dim(H^0(\mathbb{P}^1, \mathcal{O}(d))) = \dim(H^1(\mathbb{P}^1, \mathcal{O}(-d-2))).$$

This follows from the fact that $H^0(\mathbb{P}^1, \mathcal{O}(d))$ is always isomorphic to the dual of $H^1(\mathbb{P}^1, \mathcal{O}(-d) \otimes T^*_{\mathbb{P}^1})$. This is also sometimes denoted the canonical divisor of $\mathbb{P}^1$: And a modification if we have X of dimension greater than 1, then we want to write this as

$$H^i(X, \mathcal{L}) \simeq H^{n-i}(X, \mathcal{L}' \otimes K_X)$$

where $K_X \simeq \bigwedge^n T^*X$ is the canonical sheaf. This amounts to having a perfect pairing (non-degenerate bilinear form), a bilinear map

$$H^0(\mathbb{P}^1, \mathcal{O}(d)) \times H^1(\mathbb{P}^1, \mathcal{O}(-d) \otimes T^*_{\mathbb{P}^1}) \to \mathbb{C}$$

which means that if we represent this map by a basis and a matrix, the matrix is invertible. There is no vector on the left which maps to zero with a vector on the right unless it's zero. It's precisely like a dot product.

We have two ingredients to show this:

(a) We can identity $\mathbb{C}$ with $H^1(\mathbb{P}^1, T^*_{\mathbb{P}^1})$. The particular cohomology group is one-dimensional.

(b) We have a natural map from the cartesian product to that H^1.

Spelling this out carefully will give us our natural map. Even if we have $T^*_{\mathbb{P}^1} \simeq \mathcal{O}(-2)$ here, we will see it with differential forms in order to apply it to other spaces.

In the more general case what we have is a map

$$H^i(X, \mathcal{L}_1) \times H^j(X, \mathcal{L}_2) \to H^{i+j}(X, \mathcal{L}_1 \otimes \mathcal{L}_2).$$

Let's rewrite the Čech complex for the cotangent bundle:

$$T^*_{\mathbb{P}^1}(U_0) \oplus T^*_{\mathbb{P}^1}(U_\infty) \to T^*_{\mathbb{P}^1}(U_0 \cap U_\infty)$$

This is

$$\mathbb{C}[x]\mathrm{d}x \oplus \mathbb{C}[y]\mathrm{d}y \to \mathbb{C}[x, 1/x]\mathrm{d}x$$

and the map acts on monomials as

$$\begin{cases} (x^k \mathrm{d}x, 0) \mapsto x^k \mathrm{d}x \\ (0, y^\ell \mathrm{d}y) \mapsto -(-x^{-\ell}\mathrm{d}x/x^2) = x^{-2-\ell}\mathrm{d}x. \end{cases}$$

At the end of the day $H^1(\mathbb{P}^1, T^*_{\mathbb{P}^1})$ is represented by the cocycle $\mathrm{d}x/x$. So $H^1(\mathbb{P}^1, T^*_{\mathbb{P}^1}) = \langle \mathrm{d}x/x \rangle_{U_0 \cap U_\infty}$. The only monomial we are not catching is $1/x$, every other we've caught.

If we did this whole process tensoring with $\mathcal{O}(-d) \otimes -$, then on left we still have functions times $\mathrm{d}x$ and functions times $\mathrm{d}y$. The y now transtitions as

$$(0, y^\ell \mathrm{d}y) \mapsto -(-x^{-\ell-d}\mathrm{d}x/x^2) = x^{-2-\ell-d}\mathrm{d}x.$$

So in this case, for example in $d = 5$, we will not catch things between -1 and $-5-1$:

$$H^1(\mathbb{P}^1, \mathcal{O}(-5) \otimes T^*_{\mathbb{P}^1}) \simeq \left\langle \frac{\mathrm{d}x}{x}, \frac{\mathrm{d}x}{x^2}, \ldots, \frac{\mathrm{d}x}{x^6} \right\rangle.$$

We don't get $\mathrm{d}x/x^7$ because it's in the image of the map!

Now we would like to show that we have a map

$$H^0(\mathbb{P}^1, \mathcal{L}_1) \times H^1(\mathbb{P}^1, \mathcal{L}_2) \to H^1(\mathbb{P}^1, \mathcal{L}_1 \otimes \mathcal{L}_2)$$

at the level of the Čech complex:

$$C^0(\mathbb{P}^1, \mathcal{L}_1) \times C^1(\mathbb{P}^1, \mathcal{L}_2) \to C^1(\mathbb{P}^1, \mathcal{L}_1 \otimes \mathcal{L}_2).$$

Elements of the first group are $\{ s_0, s_\infty \}$ where $s_0 \in L_1(U_0)$ and $s_\infty \in L_1(U_\infty)$ and on the other we have $u_{0\infty} \in L_2(U_0 \cap U_\infty)$. Given this we want to produce a section $v_{0\infty}$ of $\mathcal{L}_1 \otimes \mathcal{L}_2(U_0 \cap U_\infty)$.

We can take s_0 and restrict it: $s_0 \mid_{U_0 \cap U_\infty} \cdot u_{0\infty}$. This gives us a $v_{0\infty}$.

Our choice was biased! ¿Why didn't we choose s_∞? The fact that

$$\{ s_0, s_\infty \} \in \ker d$$

means that $s_0 \mid_{U_0 \cap U_\infty} - s_\infty \mid_{U_0 \cap U_\infty} = 0$ which means that on the intersection they're the same.

So basically we have

$$\begin{gathered} H^0(\mathcal{O}(d)) \times H^1(\mathcal{O}(-d) \otimes T^*) \to H^1(T^*) \\ \Rightarrow \langle 1, x, \ldots, x^d \rangle_{U_0} \otimes \left\langle \frac{\mathrm{d}x}{x}, \frac{\mathrm{d}x}{x^2}, \ldots, \frac{\mathrm{d}x}{x^{d+1}} \right\rangle \to \left\langle \frac{\mathrm{d}x}{x} \right\rangle. \end{gathered}$$

The matrix of this pairing is the identity matrix and then it's obviously the *dot product*. All the other $x^k \mathrm{d}x = 0$.

Chapter 2

Higher Genus

Bibliography

[1] D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry*. Graduate Studies in Mathematics. American Mathematical Society, 2021.