

**Exercise 1** (3.2.E Vakil). Show that we have identified all the prime ideals of  $\mathbb{C}[x, y]$ .

[[Hint: Suppose  $\mathfrak{p}$  is a prime ideal that is not principal. Show you can find  $f, g \in \mathfrak{p}$  with no common factor. By considering the Euclidean algorithm in the Euclidean domain  $\mathbb{C}(x)[y]$ , show that you can find a nonzero  $h \in \text{gen}(f, g) \subseteq \mathfrak{p}$ . Using primality, show that one of the linear factors of  $h$ , say  $(x - a)$ , is in  $\mathfrak{p}$ . Similarly show there is some  $(y - b) \in \mathfrak{p}$ . ]]

The example in the book before the exercise describes  $\mathbb{A}_{\mathbb{C}}^2 = \text{Spec } \mathbb{C}[x, y]$ . The example shows that

- ◇  $0$  is a prime ideal.
- ◇ Ideals of the form  $\text{gen}(x - a, y - b)$  with  $a, b \in \mathbb{C}$  are prime. Even more, that they are maximal.
- ◇ And finally ideals of the form  $\text{gen}(f)$  with an irreducible  $f$  are also prime.

The hint tells us to take a prime ideal and assume it is not of the form  $\text{gen}(f)$  with an irreducible  $f$ . Then we will conclude that it is of the form  $\text{gen}(x - a, y - b)$  which is the other only non-zero possibility.

### Answer

Take a non-principal ideal  $\mathfrak{p} \in \text{Spec } \mathbb{C}[x, y]$ , we begin by wanting to find such  $f, g$  with  $\text{gcd}(f, g) = 1$ .

If this were not the case, then all polynomials in  $\mathfrak{p}$  would have a common factor. Let  $p = \text{gcd}(f)_{f \in \mathfrak{p}}$ , then  $p$  is a generator for  $\mathfrak{p}$ . As it was the case that  $\mathfrak{p}$  wasn't principal, our assumption that no such  $f, g$  exist must be false.

Assume that  $g$ 's degree in  $y$  is lower than  $f$ 's we may apply the division algorithm on  $\mathbb{C}(x)[y]$  to obtain

$$f = qg + r, \quad q, r \in \mathbb{C}(x)[y] \quad \text{and} \quad \deg_y(r) \leq \deg_y(g).$$

We may iterate this process and continue dividing with the residues in order to obtain

$$g = q_2r + r_2 \Rightarrow r = q_3r_2 + r_3 \Rightarrow \dots$$

until we reach a point where the remainder has degree zero in  $y$ . Retracing the equalities from the last point to the first equation, let us write

$$f(x, y) = \frac{q_1(x, y)}{q_2(x)}g(x, y) + \frac{r_1(x)}{r_2(x)}$$

where  $\frac{r_1}{r_2}$  is the last remainder. Homogenizing we obtain an equation of the form

$$q_2 r_2 f = q_1 r_2 g + r_1 q_2 \Rightarrow r_1 q_2 \in \text{gen}(f, g)$$

and we may also see that  $r_1 q_2$  is a polynomial depending only on  $x$ . Thus we may factor it into

$$r_1 q_2(x) = \prod_{i=1}^d (x - a_i) \Rightarrow \exists j ((x - a_j) \in \mathfrak{p}).$$

The same argument may be repeated but this time we obtain a polynomial  $(y - b) \in \mathfrak{p}$ . With this we have

$$\text{gen}(x - a, y - b) \subseteq \mathfrak{p}$$

and as  $\mathfrak{p}$  is a proper prime ideal, it must occur that  $\mathfrak{p}$  is this maximal ideal.

**Exercise 2** (3.2.K Vakil). Suppose  $S$  is a multiplicative subset of  $A$ . Describe an order-preserving bijection of the prime ideals of  $S^{-1}A$  with the prime ideals of  $A$  that don't meet the multiplicative set  $S$ .

### Answer

We will describe the bijection

$$\{\mathfrak{p} \in \text{Spec } A : \mathfrak{p} \cap S = \emptyset\} \rightarrow \text{Spec } S^{-1}A.$$

If  $\mathfrak{p} \in \text{Spec } A$  with  $\mathfrak{p} \cap S = \emptyset$  we will show that  $S^{-1}\mathfrak{p}$  is a prime ideal in  $S^{-1}A$ .

Suppose that  $\frac{a_1}{s_1} \frac{a_2}{s_2} \in S^{-1}\mathfrak{p}$ . Then there exist  $p \in \mathfrak{p}$  and  $s \in S$  such that

$$\frac{a_1}{s_1} \frac{a_2}{s_2} = \frac{p}{s} \Rightarrow u(a_1 a_2 s - s_1 s_2 p) = 0, \quad \text{for some } u \in S.$$

Now

$$u s_1 s_2 p \in \mathfrak{p} \Rightarrow u a_1 a_2 s \in \mathfrak{p} \Rightarrow a_1 a_2 \in \mathfrak{p} \Rightarrow a_1 \in \mathfrak{p} \vee a_2 \in \mathfrak{p}$$

from which we conclude that either  $\frac{a_1}{s_1}$  or  $\frac{a_2}{s_2}$  is in  $S^{-1}\mathfrak{p}$ , which means that  $S^{-1}\mathfrak{p}$  is a prime ideal.

On the other hand if  $\mathfrak{q} \in \text{Spec } S^{-1}A$  we can take its preimage through the mapping:

$$\phi^{-1}[\mathfrak{q}] = \left\{ x \in A : \frac{x}{1} \in \mathfrak{q} \right\}$$

and we will show that this ideal doesn't intersect  $S$ .

On the contrary, if it did, if there was  $s_0 \in S \cap \phi^{-1}[\mathfrak{q}]$  then  $\frac{s_0}{1} \in \mathfrak{q}$ . Then

$$\left(\frac{1}{s_0}\right) \left(\frac{s_0}{1}\right) = 1 \in \mathfrak{q} \quad \text{because } \mathfrak{q} \text{ is prime}.$$

This means that  $\mathfrak{q}$  must be the whole ring, but if  $\mathfrak{q}$  were proper, we would have a contradiction. This means that our assumption was wrong and therefore  $\phi^{-1}[\mathfrak{q}] \cap S = \emptyset$ .

This is the bijection in question. It preserves order because preimages of sets preserve order.

**Exercise 3** (3.6.M Vakil). Verify that  $[\text{gen}(y - x^2)] \in \mathbb{A}_{\mathbb{C}}^2$  is a generic point for  $V(y - x^2)$ .

**Answer**

Call  $\mathfrak{p} = \text{gen}(y - x^2)$ , the closure of  $\{\mathfrak{p}\}$  is

$$\overline{\{\mathfrak{p}\}} = \bigcap_{\substack{F \supseteq \{\mathfrak{p}\} \\ F \text{ closed}}} F = \bigcap_{\mathfrak{p} \in V(S)} V(S).$$

*I'm confused still on treating ideals as points. Couldn't finish on time.*

**Exercise 4** (3.6.P Vakil). Show that  $\mathbb{A}_{\mathbb{C}}^2$  is a Noetherian topological space: any decreasing sequence of closed subsets of  $\mathbb{A}_{\mathbb{C}}^2 = \text{Spec } \mathbb{C}[x, y]$  must eventually stabilize. Note that it can take arbitrarily long to stabilize. (The closed subsets of  $\mathbb{A}_{\mathbb{C}}^2$  were described in 3.4.5.).

Show that  $\mathbb{C}^2$  with the classical topology is not a Noetherian topological space.

**Answer**

Let  $(F_k)$  be a descending chain of closed subsets of  $\mathbb{A}_{\mathbb{C}}^2$ . Every  $F_k$  can be seen to be  $V(I_k)$  where  $I_k \triangleleft \mathbb{C}[x, y]$  is an ideal. So by the Nullstellensatz:

$$\begin{aligned} F_1 &\supseteq F_2 \supseteq \dots \\ \Rightarrow V(I_1) &\supseteq V(I_2) \supseteq \dots \\ \Rightarrow I_1 &\subseteq I_2 \subseteq \dots \end{aligned}$$

Now  $(I_k)$  is an ascending chain of ideals in  $\mathbb{C}[x, y]$ , and as this ring is Noetherian, ascending chains stabilize. Which means that there exists  $r$  with the property

that

$$I_r = I_{r+1} = \dots$$

and therefore  $V(I_r) = V(I_{r+1}) = \dots$  from which we extract that  $\mathbb{A}_{\mathbb{C}}^2$  is Noetherian as a topological space.

On the other hand if we consider the collection of closed balls  $(\overline{B}(0, \frac{1}{n}))_{n \in \mathbb{N}}$ , we see that this collection doesn't stabilize. If it did, there would exist an  $r$  such that

$$\overline{B}\left(0, \frac{1}{r}\right) = \overline{B}\left(0, \frac{1}{r+1}\right) = \dots$$

but there are no points in  $\overline{B}(0, \frac{1}{r}) \setminus \overline{B}(0, \frac{1}{r+1})$  which are limits of sequences inside  $\overline{B}(0, \frac{1}{r+1})$  because any point in the annulus has positive distance to the smaller ball.