MATH519 — Complex Analysis

Based on the lectures by Jeff Achter

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Please note that these notes were not provided or endorsed by the lecturer and have been significantly altered after the class. They may not accurately reflect the content covered in class and any errors are solely my responsibility.

This course is an introduction to analytic functions of a single complex variable. The subject is beautiful Links to an external site.– it turns out that a function with a complex derivative is highly structured – and enjoys a give and take with many other areas of mathematics.

Requirements

Knowledge of convergence of sequences, series: limits, continuity, differentiation, integration of one-variable functions is required.

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Chapter 1

First Midterm

1.1 Interim | HW1

Exercise 1.1.1 (1.1 Stein & Shakarchi)**.** Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$.
(b) $1/z = \overline{z}$.
(c) $\mathrm{Re}(z) = 3$
(d) $\mathrm{Re}(z) > c$, (resp.,$\geqslant c$) where $c \in \mathbb{R}$.
(e) $\mathrm{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$.
(f) $|z| = \mathrm{Re}(z) + 1$.
(g) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$.

Answer

i) The first set is the set of points at the same distance from z_1 and z_2. If we consider the line segment $z_1 z_2$, then the set in question is the bisector of that line segment.

ii) Note that

$$1/z = \overline{z} \iff 1 = \overline{z}z \iff 1 = |z|^2 \iff 1 = |z|,$$

thus the set is the unit circle.

iii) The set is a perpendicular line to the real axis at $z = 3$.

iv) This infinite set is an infinite half plane to the right (but not including) of the line $z = c$. In the other case, we do include the line in question.

v) DO

vi) The equation in question is equivalent to

$$\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 = (\mathrm{Re}(z) + 1)^2.$$

To ease the notation, assume $z = x + iy$. Then the equation reads

$$x^2 + y^2 = x^2 + 2x + 1 \iff y^2 = 2x + 1 \iff x = (y^2 - 1)/2.$$

It holds the the parabola in question contains the points which satisfy the equation.

vii) This set is a line parallel to the real axis at $z = c$

Exercise 1.1.2. Do the following:

i) Show that the complex conjugation map $\kappa : \mathbb{C} \to \mathbb{C},\ z \mapsto \overline{z}$ is an involution, i.e., a ring homomorphism such that $\kappa \circ \kappa = \mathrm{id}$.
ii) Suppose $a \in \mathbb{R},\ z \in \mathbb{C}$. Show that

$$\mathrm{Re}(az) = a\,\mathrm{Re}(z), \quad \text{and} \quad \mathrm{Im}(az) = a\,\mathrm{Im}(z).$$

Answer

Let us take $z = x + iy$ with $x, y \in \mathbb{R}$.

i) We have $\overline{z} = x + i(-y) = x - iy$. Once more we get $\overline{\overline{z}} = x - i(-y) = x + iy = z$. Thus $\overline{\overline{z}} = z$ for any $z \in \mathbb{C}$. In conclusion $\overline{\overline{\cdot}} = \mathrm{id}$.
ii) It holds that

$$\begin{aligned}\mathrm{Re}(az) &= \mathrm{Re}(ax + aiy) = ax = a\,\mathrm{Re}(z),\\ \mathrm{Im}(az) &= \mathrm{Im}(ax + aiy) = ay = a\,\mathrm{Im}(z).\end{aligned}$$

Exercise 1.1.3. Do the following:

i) Prove that $|z + w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\overline{w})$.
ii) Use this to prove the parallelogram rule: $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$.

Answer

i) Note that

$$|z + w|^2 = (z + w)\overline{(z + w)} = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + w\overline{z} + z\overline{w} + w\overline{w}.$$

The number $w\bar{z}$ is the conjugate of $z\overline{w}$, and summing a number and its conjugate returns twice its real part. Thus we get the desired identity.

ii) As the past identity holds for all complex numbers, it holds when $w = -w$. This means that $|z-w|^2 = |z|^2+|-w|^2+2\operatorname{Re}(z(\overline{-w})) = |z|^2+|w|^2-2\operatorname{Re}(z\overline{w})$ and summing this together with the first identity gives us the parallelogram law.

Exercise 1.1.4 (1.5 Stein & Shakarchi)**.** A set Ω is said to be pathwise connected if any two points in Ω can be joined by a (piecewise-smooth) curve entirely contained in Ω. The purpose of this exercise is to prove that an open set Ω is pathwise connected if and only if Ω is connected.

i) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a parametrization $z : [0,1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let
$$t_* = \sup_{0\leqslant t\leqslant 1} \{\, t : \forall s[(0 \leqslant s < t) \Rightarrow (z(s) \in \Omega_1)] \,\}.$$
Arrive at a contradiction by considering the point $z(t_*)$.

ii) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subseteq \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subseteq \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega 1$ as desired.

Answer

i) Following the idea, we consider the point $z(t_*)$. We have two options to place $z(t_*)$, either in Ω_1 or Ω_2.

Let's start by definition of supremum

Exercise 1.1.5 (1.7 Stein & Shakarchi)**.** The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

i) Let $z, w \in \mathbb{C}$ such that $\bar{z}w \neq 1$. Prove that
$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1$$

if $|z| < 1$ and $|w| < 1$, and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1$$

if $|z| = 1$ or $|w| = 1$. ⟦ Hint: Why can one assume that z is real? I then suffices to prove that $(r-w)(r-\overline{w}) \leqslant (1-rw)(1-r\overline{w})$ with equality for appropriate r and $|w|$. ⟧⟦ Here is an alternate approach, which you may use if you like. Fix $w \in \mathbb{C}$ with $w < 1$, and consider the function $z \mapsto \frac{w-z}{1-\overline{w}z}$. What is $\overline{f(z)}$? By computing $f(z)\overline{f(z)}$, show that $|z| = 1$ implies $|f(z)| = 1$. Find a point z with $|z| < 1$ such that $|f(z)| < 1$. Since f is continuous, this shows that f takes the unit disc to itself. (Why?) ⟧

ii) Prove that for a fixed $w \in \mathbb{D}$, the mapping $F : z \mapsto \frac{w-z}{1-\overline{w}z}$ satisfies the following:

   a) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.
   b) F interchanges 0 and w.
   c) $|F(z)| = 1$ if $|z| = 1$.
   d) F is bijective. ⟦ Hint: Calculate $F \circ F$. ⟧

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I dunno lol