Homework 4

Due: Friday, February 17

1. In this problem, we'll use the Cauchy integral formula to show that an analytic function has a power series representation.

   Suppose f is analytic on an open set containing $\overline{N}_R(0)$. We will show that, on $N_R(0)$, there is an equality of functions

   $$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

   where

   $$a_n = \frac{f^{(n)}(0)}{n!}.$$

   Let $C = C_R(0)$ be the circle around 0 of radius R, oriented in the positive direction, and fix some z with $|z| = r < R$.

   (a) Show that

   $$f(z) = \frac{1}{2\pi i} \int_C g(w) f(w)\, dw$$

   where

   $$g(w) = \frac{1}{w} \frac{1}{1 - z/w}.$$

   (b) Let N be a natural number. Show that

   $$g(w) = \sum_{j=0}^{N-1} \frac{z^j}{w^{j+1}} + \frac{z^N}{(w-z)w^N}.$$

   (c) Show that

   $$f(z) = \sum_{j=0}^{N-1} \frac{f^{(j)}(0)}{j!} z^j + \rho_N(z)$$

   where

   $$\rho_N(z) = \frac{z^N}{2\pi i} \int_C \frac{f(w)}{(w-z)w^N}\, dw.$$

2. *Continuation of* (1).

(a) Let M be the maximum value of $|f(z)|$ on the contour C. Show that

$$|\rho_N(z)| \leq \frac{rM}{R-r}\left(\frac{r}{R}\right)^{N-1}.$$

(HINT: *If $w \in C$, then $|w-z| \geq R-r$.*)

(b) Show that $\lim_{N\to\infty} \rho_N(z) = 0$.

3. For $0 \leq r \leq n$, the binomial coefficient $\binom{n}{r}$ is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

(a) Let $\gamma = \gamma_1(0)$, the positive, circular arc around 0 with radius 1. Show

$$\binom{n}{r} = \frac{1}{2\pi i}\int_\gamma \frac{(1+z)^n}{z^{r+1}}\,dz.$$

In fact, this works for any simple closed curve around the origin.

(b) Use (a) to show that

$$\binom{n}{r} \leq 2^n.$$

Remark: (b) can also be shown directly by computing $(1+1)^n$.

4. [SS]2.7. *Make sure to justify the hint given in [SS].*

5. [SS]2.9. *You may assume you have already translated the fixed point to the origin, so that $z_0 = 0$.*