Exercise 1 (4.1.A Vakil)**.** Show that the natural map $A_f \to \mathcal{O}_{\operatorname{Spec}(A)}(D(f))$ is an isomorphism. ⟦Hint: Exercise 3.5.E Vakil.⟧

Answer

Exercise 2 (Restrictions)**.** Do the following:

i) Explain, using Definition 4.1.1 (and not exercise 4.1.A) what the restriction map is.
ii) Explain, using exercise 4.1.A what the restriction map is.

Answer

Exercise 3 (4.1.D Vakil)**.** Suppose M is an A-module. Show that the following construction describes a sheaf $\widetilde{M}$ on the distinguished base. Define $\widetilde{M}(D(f))$ to be the localization of M at the multiplicative set of all functions that do not vanish outside of $V(f)$.

Define restriction maps $\operatorname{res}_{D(f),D(g)}$ in the analogous way to $\mathcal{O}_{\operatorname{Spec}(A)}$.

Show that this defines a sheaf on the distinguished base, and hence a sheaf on $\operatorname{Spec}(A)$. Then show that this is an $\mathcal{O}_{\operatorname{Spec}(A)}$-module.

Answer

Exercise 4. Let $A = \mathbb{C}[x, y]$ and let $\mathfrak{p} = \operatorname{gen}(y)$, viewed as a point of $X = \operatorname{Spec}(A)$. What is $\mathcal{O}_{X,p}$?

Recall that $\mathcal{O}_{X,p}$ is a local ring, that is, it has a unique maximal ideal, $\mathfrak{m}_p$.

What is the residue field $\kappa_{\mathfrak{p}} = \mathcal{O}_{X,p}/\mathfrak{m}_p$?

Answer

Exercise 5 (4.4.A Vakil)**.** Show that you can glue an arbitrary collection of schemes together. Suppose we are given:

- schemes X_i (as i runs over some index set I, not necessarily finite),
- open subschemes $X_{ij} \subseteq X_i$ with $X_{ii} = X_i$,
- isomorphisms $f_{ij} : X_{ij} \to X_{ji}$ with f_{ii} the identity

such that

the isomorphisms "agree on triple intersections", i.e.,

$$f_{ik} \mid_{X_{ij} \cap X_{ik}} = f_{jk} \mid_{X_{ji} \cap X_{jk}} \circ f_{ij} \mid_{X_{ij} \cap X_{ik}} \circ$$

(so implicitly, to make sense of the right side, $f_{ij}(X_{ik} \cap X_{ij}) \subseteq X_{jk}$*).*

This *cocycle condition* ensures that f_{ij} and f_{ji} are inverses. In fact, the hypothesis that f_{ii} is the identity also follows from the cocycle condition.

Show that there is a unique scheme X (up to unique isomorphism) along with open subsets isomorphic to the X_i respecting this gluing data in the obvious sense.⟦ Hint: what is X as a set? What is the topology on this set? In terms of your description of the open sets of X, what are the sections of this sheaf over each open set? ⟧

Answer