Exercise 1 (Exercise 1)**.** Evaluate the following expressions using the properties of the Hall inner product discussed in class:

i) $\left\langle s_{(2,1)} \middle| h_{(1,1,1)} \right\rangle$.
ii) $\left\langle s_{(3,1,1)} \middle| s_{(3,2)} \right\rangle$.
iii) $\left\langle e_{(2,1)} \middle| h_{(2,1)} \right\rangle$.
iv) $\left\langle p_{(3,2,2,1)} \middle| p_{(3,2,2,1)} \right\rangle$.

Answer

(a) From the previous homework we have calculated

$$s_{(2,1)}(x,y,z) = x^2y + xy^2 + 2xyz + x^2z + y^2z + xz^2 + yz^2$$

which tells us in a general that $s_{(2,1)} = m_{(2,1)} + 2m_{(1,1,1)}$. Applying the inner product we get

$$\left\langle s_{(2,1)} \middle| h_{(1,1,1)} \right\rangle = \left\langle m_{(2,1)} \middle| h_{(1,1,1)} \right\rangle + 2\left\langle m_{(1,1,1)} \middle| h_{(1,1,1)} \right\rangle = 2.$$

(b) As s_λ form an orthonormal basis, it holds that $\left\langle s_{(3,1,1)} \middle| s_{(3,2)} \right\rangle = 0$.

(c) Turning $e_{(2,1)}$ to the monomial basis we obtain

$$\begin{aligned} e_{(2,1)}(x,y,z) &= (xy + yz + zx)(x + y + z) \\ &= x^2y + y^2z + z^2x + y^2x + z^2y + x^2z + 3xyz \\ &= m_{(2,1)}(x,y,z) + 3m_{(1,1,1)}(x,y,z). \end{aligned}$$

Inputting into the inner product we get

$$\left\langle e_{(2,1)} \middle| h_{(2,1)} \right\rangle = \left\langle m_{(2,1)} \middle| h_{(2,1)} \right\rangle + 3\left\langle m_{(1,1,1)} \middle| h_{(2,1)} \right\rangle = 1$$

and this is because m and h form a dual pair.

(d) As (p_λ) forms an orthogonal basis, the following holds $\langle p_\lambda | p_\lambda \rangle = z_\lambda$ where $z_\lambda = \prod k^{m_k} m_k!$ and m_k is the number of parts of λ equal to k. In this case

$$\left\langle p_{(3,2,2,1)} \middle| p_{(3,2,2,1)} \right\rangle = z_{(3,2,2,1)} = (3^1)(1!)(2^2)(2!) = 24.$$

Exercise 2 (Exercise 2)**.** Apply the ω involution to both sides of the Jacobi-Trudi formula to derive a formula for Schur functions in terms of elementary symmetric functions.

Answer

Recall that the omega involution is an homomorphism. This means that it respects sums and products which in particular means that it respects polynomials.
Now, the Jacobi-Trudi formula states that

$$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j\in[n]}.$$

As the determinant is a polynomial on its entries we can apply the omega involution on both sides of the equality to obtain

$$s_{\lambda^\intercal} = \omega\left(\det(h_{\lambda_i - i + j})\right) = \det(\omega(h_{\lambda_i - i + j})) = \det(e_{\lambda_i - i + j})_{i,j\in[n]}.$$

Applying a transposition we get the dual Jacobi-Trudi identity:

$$s_\lambda = \det\left(e_{\lambda_i^\intercal - i + j}\right)_{i,j\in[n]}.$$

Exercise 3 (Exercise 3)**.** Write out the six permutations of $1, 2, 3$ and the pairs of standard Young tableaux corresponding to each under the RSK bijection.

Answer

As recording tableau are customarily read from the bottom row, we will write the list notation permutations from bottom to top. The following are the elements of S_3:

i) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	iii) $\begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$	v) $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$
ii) $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	iv) $\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$	vi) $\begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

Applying the algorithm from $\mathcal{M}_{2\times n}$ to (S, T) we get the following pairs of tableaux:

i) $\begin{ytableau} 1 & 2 & 3 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 \end{ytableau}$

ii) $\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$

iii) $\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}, \begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}$

iv) $\begin{ytableau} 3 \\ 1 & 2 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$

v) $\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$

vi) $\begin{ytableau} 3 \\ 1 & 2 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$

These pairs of tableaux are ordered respectively with the matrices on top.

Exercise 4 (Exercise 4)**.** Write out all rearrangments of the letters $1, 1, 2, 3$, and the pairs (S, T) of a semi-standard and standard Young tableau, respectively, corresponding to each word under the RSK bijection.

Answer

[a] The following are the possible words given our alphabet:

i) $\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

ii) $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

iii) $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

iv) $\begin{pmatrix} 1 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

v) $\begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

vi) $\begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

vii) $\begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

viii) $\begin{pmatrix} 3 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

ix) $\begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

x) $\begin{pmatrix} 3 & 2 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

xi) $\begin{pmatrix} 1 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

xii) $\begin{pmatrix} 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

After applying the algorithm we obtain the following tableaux

i) $\ytableaushort{1123}$, $\ytableaushort{1123}$

ii) $\ytableaushort{2,113}$, $\ytableaushort{3,124}$

iii) $\ytableaushort{2,113}$, $\ytableaushort{4,123}$

iv) $\ytableaushort{3,112}$, $\ytableaushort{4,123}$

v) $\ytableaushort{2,113}$, $\ytableaushort{2,134}$

vi) $\ytableaushort{23,11}$, $\ytableaushort{24,13}$

vii) $\ytableaushort{23,11}$, $\ytableaushort{34,12}$

viii) $\ytableaushort{3,112}$, $\ytableaushort{2,134}$

ix) $\ytableaushort{3,2,11}$, $\ytableaushort{4,2,13}$

x) $\ytableaushort{3,2,11}$, $\ytableaushort{3,2,14}$

xi) $\ytableaushort{3,112}$, $\ytableaushort{3,124}$

xii) $\ytableaushort{3,2,11}$, $\ytableaushort{4,3,12}$

[a] **Ian** and **Kelsey** helped me out with the verification of the algorithms.

Exercise 5 (Exercise 5)**.** What two-line array corresponds to the following pair of semi-standard Young tableaux under the RSK bijection?

$$\ytableaushort{4,223,11134}, \quad \ytableaushort{3,223,11112}.$$

Answer

We[a] obtain the the array:

$$\begin{pmatrix} 2 & 2 & 3 & 4 & 1 & 3 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \end{pmatrix}$$

[a] **Clare** and myself worked through this problem on Wednesday after class.

Quick questions on problems 6, and 7.

- For problem 6 I've seen the result that if $\begin{pmatrix} a \\ b \end{pmatrix}$ is a two-line array which under RSK maps to (S, T), then $\begin{pmatrix} b \\ a \end{pmatrix}$ maps to (T, S). Assuming that result, proving question 6 is as follows:

  As permutations correspond to SYT's we have that RSK$(\pi) = (S, T)$ and RSK$(\pi^{-1}) = (T, S)$. When $S = T$ it holds that

$$\text{RSK}(\pi) = (T, T) = \text{rsk}(\pi^{-1}) \Rightarrow \pi = \pi^{-1}$$

which means π is an involution. However I don't understand how to prove the result I used because I'm unable to adapt the proof we saw in class to the case of π and π^{-1}.

◇ I can't quite put my head around the last problem. What I'm interpreting is that $s = Km$ then $h = K^{\mathsf{T}}m$ where K is the respective matrix of Kostka coefficients. But I'm unsure on how to proceed or how to use the inner product.

Also, going a bit ahead, I've seen the statements of the Pieri rules and that they are used to prove that h and e are Schur positive. So the last sum looked strikingly similar to the statement of $h_\mu = \sum_\lambda K_{\lambda\mu} s_\mu$. However it might be my mind playing me some tricks.