The purpose of the whole exercise 34 is to prove that the Vandermonde determinant is equal to the product of binomials. My initial thought when I saw part (a) was, "*oh, that expression is Vandermonde's determinant, I can just try to relate the tournament to something about powers of monomials.*" But then I saw the last part and understood that the objective was not to use the fact that the determinant of the matrix of monomials is what it is, instead we see that *that* matrix is related to our problem and that the product is also related and therefore the matrix's determinant is what it should be. So without further ado:

Exercise 1 (Exercise 3, Stanley 2.34(a))**.** Show that $\sum_T w(T) = \prod_{i,j\in[n]}(x_j - x_i)$ where the sum is taken over all $2^{\binom{n}{2}}$ tournaments on $[n]$.

Answer

As every pair of vertices is joined by exactly one edge and there are no loops, every tournament on $[n]$ has $\binom{n}{2}$ edges. This means that the sum $\sum_T w(T)$ has $2^{\binom{n}{2}}$ terms.
On the right hand side, the product in question has $\binom{n}{2}$ factors of the form $(x_j - x_i)$ and when expanded as a sum of monomials we find $2^{\binom{n}{2}}$ summands. Each of the monomials in the expanded result of the product corresponds to a weight of a tournament.
The reasoning for that conclusion is as follows. Pick two vertices $i < j$, there are two possibilities to pick a weighted edge between these two:

- ◇ Either we have $i \to j$ with weight x_j,
- ◇ or we have $i \leftarrow j$ with weight $-x_i$.

This determines the choice of monomial in the $(x_j - x_i)$ factor. In the same way, picking either x_j or $-x_i$ in the expansion determines the orientation of the edge between i and j. No tournaments are missing nor we have more than necessary as the amount matches up. Since each of the $2^{\binom{n}{2}}$ possible weights is also counted in $\prod_{i,j\in[n]}(x_j - x_i)$, it holds that

$$\sum_T w(T) = \prod_{i,j\in[n]} (x_j - x_i).$$

Exercise 2 (Exercise 4, Stanley 2.34(b))**.** A tournament is transitive if there's a permutation $\pi \in S_n$ such that $\pi(i) < \pi(j)$ when $i \to j$. Show that a non-transitive tournament contains a 3 cycle.

Sometimes I've got to lower my head and admit I made a mistake. **Kyle** showed me what was wrong with my initial argument and then proceeded to tell me what was the correct thing to do. This was my original proof:

This statement in question is equivalent to T is (3-cycle)-free implies T is transitive. The transitive condition is also equivalent to $i \to j,\ j \to k \in E \Rightarrow i \to k \in E$. So suppose we have three vertices $i, j, k \in T$ such that $i \to j$ and $j \to k$ are edges of T.
If it were the case that $k \to i$ was an edge in T, this would contradict the fact that T is (3-cycle)-free, which is not the case. So there's no $k \to i$ edge.
As T is a tournament, there *must* be an edge between i and k, so the only possibility is that the edge $i \to k$ is in T. And so, we have proven that T is transitive as

$$i \to j,\ j \to k \in E \Rightarrow i \to k \in E.$$

The problem with my argument is that I just handwave the fact that usual transitivity is equivalent to book-transitivity. However it is not as trivial to see that the equivalence is true. We will not be seeing that in the proof, but instead we will be proving:

T is (3-cycle)-free implies, T is book-transitive.

Answer

We must construct a permutation π such that $\pi(i) < \pi(j)$ whenever $i \to j$. This is done by induction on $|T|$, the interesting base case is $|T| = 3$.
For that tournament, at least one vertex v must have $d_{\text{in}}(v) = 2$, else there would be a 3-cycle. Assign $\pi(v) = 3$ and then consider the induced subgraph by the remaining vertices, one sends an edge to another so assign the endpoint vertex to 2 and the last one to 1. Thus we have constructed a permutation which respects the condition in question.
Suppose by induction, that all (3-cycle)-free tournaments with n vertices are book-transitive.

Exercise 3 (Exercise 4, Stanley 2.34(c)). The relation $T \leftrightarrow T'$ means that T' was obtained by reversing a 3-cycle in T. Show that if $T \leftrightarrow T'$ then $w(T') = -w(T)$.

Answer

Suppose Δijk is the cycle we will be reversing and consider the induced subgraph generated by i, j and k. There are only two possibilities for Δijk, either

$$i \to j \to k \to i \quad \text{or} \quad i \to k \to j \to i.$$

As one of the reverse of the other, we will see that their weights differ by sign. The weight of the first one is $x_j x_k(-x_i)$ and the second one is $x_k(-x_j)(-x_i)$. FINISH