Homework
Due: Friday, April 28

1. [SS]7.8. *In (a), just show that $F(s) = F(-s)$, and accept the statement that this implies that there is some G such that $F(s) = G(s^2)$.*

2. Read [SS]7.6, assume its result, and proceed as follows. Let δ be the function defined in [SS]7.6:
$$\delta(a) = \begin{cases} 1 & 1 < a \\ \frac{1}{2} & a = 1 \\ 0 & 0 \leq a < 1 \end{cases}$$
Fix a positive real number X which is not an integer.
   (a) Show that
   $$\psi(X) = \sum_{n \geq 1} \Lambda(n)\delta(\frac{X}{n}).$$
   (b) Consider the function
   $$G(s) = \frac{X^s}{s}\left(\frac{-\zeta'(s)}{\zeta(s)}\right).$$
   Show that
   $$\psi(X) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(s)\,ds.$$
   Assume you can exchange summation and integration; you will need to use our formula from class for $L(\zeta(s)) = \frac{\zeta'(s)}{\zeta(s)}$, which is also in [SS] Chapter 7, section 2.

3. One uses the results of the previous problems in the following way.
   (a) Show that $\mathrm{res}_{s=1}\, G(s) = X$. (HINT: *Use the fact that $\zeta(s)$ has a pole at $s = 1$ of order* 1.)
   (b) Show that $\mathrm{res}_{s=0}\, G(s) = \mathrm{res}_{s=0} \frac{-\zeta'(s)}{\zeta(s)}$. *It turns out that this is $-\log 2\pi$.*
   (c) Show that $\sum_{\rho<0} \mathrm{res}_{s=\rho}\, G(s) = -\frac{1}{2}\log(1 - X^{-2})$, where the sum is over the trivial zeros of $\zeta(s)$.

   From here, moving c "all the way to the left" means that we pick up all the residues of $G(s)$, and we are left with von Mangoldt's explicit formula:
   $$\psi(X) = X - \sum_{\rho} \frac{X^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - X^{-2}),$$
   where the sum is over all critical zeros of $\zeta(s)$.