Anisotropic fluid collapsing matter

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Abstract

The generalized Vaidya geometry is investigated for collapsing matter. Compared to other studies, we take into account an anisotropic stress tensor as the source of curvature with an equation of state resembling the MIT bag model form. The energy conditions are generally satisfied. For $r < M(v, r)$ the gravitational field becomes repulsive and the true singularity at $r = 0$ is never reached.

1 Introduction

The problem of gravitational collapse in general relativity leading eventually to black hole (BH) formation is of fundamental interest. Geometries with special symmetry are necessary to build gravitational collapse models, due to the complexity of Einstein's equations. Husain [1] gives exact non-static spherically symmetric solutions of the Einstein equations for a collapsing null fluid. He firstly determined the stress-energy tensor from the metric. Then the equation of state and the dominant energy condition are imposed on its eigenvalues, so obtaining an equation for the metric function. A similar investigation on a regular Vaidya-type metric is given in [2], where the author proposed a metric with no singularities and an anisotropic stress tensor as the source of curvature.

Vertogradov et al. [3] found exact solutions to Einstein's field equations that describe singularity-free spacetimes, through the gravitational collapse of baryonic matter characterized by a time- and radius-dependent coefficient of equation of state. The solutions have interesting physical properties but the pressure increases with radius and violates the dominant energy condition beyond some critical value. Vertogradov and Ovgun [4] investigated gravitational collapse scenarios for regular BH models in the transition from baryonic matter to quark matter. Certain strong interaction theories, particularly quark bag models, conjecture a phase of vacuum symmetry breaking within hadrons. This leads to significant differences between vacuum energy densities inside and outside hadrons, resulting in a vacuum-induced pressure on the quark confinement

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boundary (bag wall) [5] (seed also [6]), which balances internal quark pressures and stabilizes the hadronic structure.

The geometric units $c = 8\pi G = 1$ are used, unless otherwise specified, where c is the velocity of light in *in vacuo* and G is the Newton constant.

2 Anisotropic stress tensor

To start with, we write down the generalized spherically-symmetric Vaidya line-element in advanced time-coordinate v [7, 8, 2]

$$ds^2 = -\left[1 - \frac{2M(v,r)}{r}\right] dv^2 + 2dvdr + r^2 d\Omega^2, \tag{2.1}$$

where $M(v,r)$ is the mass function, $v > 0$ is the Vaidya ingoing null coordinate and $d\Omega^2$ stands for the metric of the unit two-sphere.

The Einstein equations

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab} \tag{2.2}$$

give us the connection between the geometry (the Einstein tensor) and matter (the stress tensor T_{ab}). For the metric (2.1) one obtains the following non-zero mixed components of G_{ab} [4, 2]

$$G^v{}_v = G^r{}_r = -\frac{2}{r^2} M', \quad G^r{}_v = \frac{2}{r^2} \dot{M}, \quad G^\theta{}_\theta = G^\phi{}_\phi = -\frac{1}{r} M'' \tag{2.3}$$

where $M' \equiv \partial M(v,r)/\partial r$ and $\dot{M} \equiv \partial M(v,r)/\partial v$. As the source of the geometry (2.1) we employ an energy-momentum tensor corresponding to an imperfect fluid with energy flux [9, 2]

$$T_{ab} = (p_t + \rho) u_a u_b + p_t g_{ab} + (p_r - p_t) n_a n_b + u_a q_b + u_b q_a, \tag{2.4}$$

where ρ is the energy density of the fluid, p_r is the radial pressure, p_t are the transversal (tangential) pressures, q^a is the energy flux 4-vector and n^a is a unit spacelike vector orthogonal to u^a. We have $u_a n^a = 0$, $u_a u^a = -1$, $n_a n^a = 1$ and $u_a q^a = 0$. In the geometry (2.1) one chooses a 4-velocity of the form

$$u^a = \left(1, -\frac{M}{r}, 0, 0\right), \tag{2.5}$$

whence, keeping in mind the above relations

$$n^a = \left(1, 1 - \frac{M}{r}, 0, 0\right). \tag{2.6}$$

One observes that we obtained from (2.1) and (2.2) four equations and five unknown functions, namely ρ, p_r, p_t, q and M. Usually, an equation of state connecting the energy density and pressure is introduced for to solve the system of equations. We stick to the Vinogradov and Ovgun equation of state for the MIT bag model, which is suitable for strange quark matter. However, our fluid is anisotropic ($p_r \neq p_t$) and, therefore, we propose a slightly modified form of it

$$p_r = \frac{1}{3}(\rho - 4b) \tag{2.7}$$

From the Einstein equations $G^a_{\ b} = T^a_{\ b}$, using $u^a q_a = 0$ and following the steps from [2], we get

$$\begin{aligned} \frac{M}{r}(\rho + p_r) - \rho - \left(1 - \frac{2M}{r}\right) q^v &= -\frac{2M'}{r^2} \\ -\frac{M}{r}(\rho + p_r) + p_r + \left(1 - \frac{2M}{r}\right) q^v &= -\frac{2M'}{r^2} \\ \frac{M}{r}\left(1 - \frac{M}{r}\right)(\rho + p_r) - \left(1 - \frac{2M}{r} + \frac{2M^2}{r^2}\right) q^v &= \frac{2\dot{M}}{r^2} \\ \rho + p_r + 2q^v &= 0. \end{aligned} \tag{2.8}$$

The new equation is (2.7) and so the system of equations is determined, giving us the possibility to find all unknown functions. Adding the 1st two equations (2.8) and using the 3rd one, we get

$$p_r - \rho = -\frac{4M'}{r^2}, \qquad p_r + \rho = \frac{4\dot{M}}{r^2}. \tag{2.9}$$

We get rid of p_r and ρ from (2.7) and (2.9) and obtain an equation for $M(v,r)$

$$2M' - \dot{M} = br^2. \tag{2.10}$$

The above partial differential equation can be immediately solved and gives us

$$M(v,r) = \frac{br^3}{6} + g(a(\frac{r}{2} + v)), \tag{2.11}$$

where $g(r/2 + v)$ is an arbitrary function and a is a constant of integration. We consider the simplest choice for g and take $g(v,r) = a(r/2 + v)$. The final expression for the mass M appears as

$$M(v,r) = \frac{br^3}{6} + \frac{ar}{2} + av. \tag{2.12}$$

The Eqs. (2.3) and (2.9) yield

$$\rho = b + \frac{3a}{r^2}, \quad p_r = -b + \frac{a}{r^2}, \quad p_t = T^\theta_{\ \theta} = -\frac{M''}{r} = -b. \tag{2.13}$$

Eqs. (2.8) gives us the energy flux density 4-vector

$$q^b = \left(-\frac{2a}{r^2}, -\frac{2a}{r^2}(1 - \frac{M}{r}), 0, 0\right) \tag{2.14}$$

The constant of integration a from(2.12) may be determined from initial conditions. If one considers that at $v = 0$ the collapsing object has some radius R and mass m, one obtains

$$a = \frac{2m}{R} - \frac{bR^2}{3}, \tag{2.15}$$

where a has units of a force (b is an energy density). For instance, if we consider the mass of the Sun $m_\odot \approx 2{\cdot}10^{33}$ g, its radius $R_\odot \approx 7{\cdot}10^{10}$ cm, $b \approx 10^{21} erg/cm^3$, Eq.2.15 yields $a \approx 10^{44}$ dynes. If we take instead the case of a neutron star or a quark star with $m = m_\odot, R = 10Km, b \approx 10^{35} ergs/cm^3$, we get $a \approx 4 \cdot 10^{48}$ dynes, a value superior to the previous one. That is due to the different roles played by R in the two terms of the expression of a from (2.15). One notes that $a > 0$, in general.

Let us investigate now the energetic conditions applied to the stress tensor (2.4). From(2.13) it is clear that $\rho \geq 0, \rho + p_r \geq 0$ and $\rho + p_t \geq 0$, so that the weak and null energy conditions (WEC, respectively NEC) are satisfied. The dominant energy condition (DEC) is also obeyed because $\rho > |p_r|,\ \rho > |p_t|$ but the strong energy condition (SEC), $\rho + p_r + 2p_t \geq 0$ is respected only for $r < \sqrt{2a/b}$. From the previous numerical examples one sees that $a >> b$ such that all energy conditions are generally obeyed.

3 Kinematical quantities

The velocity field from (2.5) is not tangent to a geodesic. Indeed, the corresponding covariant acceleration is given by

$$a^b = u^a \nabla_a u^b = \left(\frac{1}{r}(\frac{M}{r} - M'),\ \frac{1}{r}(1 - \frac{M}{r})(\frac{M}{r} - M'), 0, 0 \right), \tag{3.1}$$

with $A \equiv \sqrt{a^b a_{\ b}} = (1/r)|M/r - M'|$ (noting that a^r is the acceleration of the particle required to preserve its static location). It can be shown that $M/r - M' > 0$ when it is expressed in terms of v, r, a, b. It is worth observing that a^r becomes negative when $r < M$ and the field is repulsive in this regime, such that the collapsing matter is rejected. This property is also clear from (2.14), where q^r becomes positive (we have outgoing particles). If the above arguments are valid, the collapsing particles never reach the origin. Therefore, the scalar curvature

$$R^a_{\ a} = \frac{4M'}{r^2} + \frac{M''}{r} = 3b + \frac{2a}{r^2} \tag{3.2}$$

remains finite everywhere, a property valid also for the Kretschmann scalar

$$K = \frac{4}{r^6}\left(m''^2 - \frac{4m'm''}{r} + \frac{4mm''}{r^2} + \frac{8m'^2}{r^2} - \frac{16mm'}{r^3} + \frac{12m^2}{r^4} \right). \tag{3.3}$$

4 Concluding remarks

We propose a model for the gravitational collapsing matter having an imperfect fluid as the source (with an equation of state similar to that one from the

MIT Bag Model) and nonzero energy flux density. The acceleration of a static observer becomes negative once the sphere of radius $r = M(v,r)$ is crossed, signaling a repulsive gravitational field. Since the true singularity is never reached, the scalar curvature and the Kretschmann scalar remain finite.

References

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